

Lawrence Berkeley National Laboratory

Recent Work

Title

FREQUENCY PERTURBATION METHOD OF FIELD MEASUREMENT IN AN ELECTROMAGNETIC RESONATOR

Permalink

<https://escholarship.org/uc/item/7qn8j7zc>

Author

Waddell, John F.

Publication Date

1951-07-18

UCRL-1392

UNCLASSIFIED

UNIVERSITY OF CALIFORNIA - BERKELEY

TWO-WEEK LOAN COPY

*This is a Library Circulating Copy
which may be borrowed for two weeks.
For a personal retention copy, call
Tech. Info. Division, Ext. 5545*

RADIATION LABORATORY

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

UNIVERSITY OF CALIFORNIA

Radiation Laboratory

Contract No. W-7405-eng-48

FREQUENCY PERTURBATION METHOD OF FIELD
MEASUREMENT IN AN ELECTROMAGNETIC RESONATOR

John F. Waddell

July 18, 1951

Berkeley, California

FREQUENCY PERTURBATION METHOD OF FIELD
MEASUREMENT IN AN ELECTROMAGNETIC RESONATOR

John F. Waddell

Radiation Laboratory, Department of Physics
University of California, Berkeley, California

July 18, 1951

Introduction

The method outlined here provides a means of deducing the magnitude of the field vectors \underline{E} and \underline{H} in a cavity resonator by measurement of the perturbation in frequency resulting from an adiabatic perturbation in stored energy of the system which occurs when a foreign body is introduced into the cavity. If the body is small, and its material constitution and geometry are such that the energy perturbation is readily calculable, the method provides a means by which the fields in the deep interior of a cavity may be explored without the introduction of additional perturbing elements other than a small dielectric thread providing support for the body. It is only in certain highly restricted cases that the interior of a cavity may be explored by means of electric and magnetic probe dipoles which directly sample the field. The use of such a device is restricted by the need for some sort of waveguide ("transmission line") for removal of the output of such a device in order that it be compared with a standard. The introduction of a probe into the field in general produces very large perturbations in the very fields to be measured. In some cases there are surfaces in which it is possible to introduce a probe without causing excessive perturbation; in general, however such surfaces either do not exist, are of insufficient number, or are so curved as to make impossible the exploration of the fields with a probe. In some cases the bounding surface of

a cavity may be resolved into a set of constant coordinate surfaces of an orthogonal coordinate system of practical utility.¹ Probe dipoles may then be used to explore the fields immediately adjacent to the cavity surface, the fields within being deduced from these observed at the boundaries. Since such cases are rarely encountered in particle accelerator cavities, it may be readily seen that the frequency perturbation method is a powerful experimental tool. This method provides direct measurement of electric field, and hence complements the probe magnetic dipole method in cases where both can be used. One disadvantage of the perturbation method is that magnetic field is deduced by taking the difference between two functions having values over most of the range of the independent variables. These values are of so nearly the same magnitude that the difference is in general less than either by an order of magnitude or more. This situation is a result of the relationship between the maximum values of \underline{E} and \underline{H} which makes any experiment for direct measurement of \underline{E} have much decreased sensitivity as an \underline{H} -measuring method.

In the discussion to follow the units to be used will be rationalized MKS with charge as the fourth dimension.

Principle of the Method

This method involves a relationship between frequency of resonance and energy stored at resonance in a harmonic oscillator. It is assumed that the relationship may be described by a linear differential equation with constant coefficients in the range of variable of interest; thus

$$\ddot{x} + g\dot{x} + \Omega^2 x = Ae^{-i\omega t}, \quad (2.1)$$

¹ Remarks pertaining to this problem may be found in Stratton, "Electromagnetic Theory" (McGraw, 1941), pp 349 and 392-3. The matter will not here be pursued further.

where x is a coordinate of whatever nature required to describe

the excursions of some physical quantity from a mean value

g is the damping constant

Ω^2 is the stiffness constant

A is a parameter, the amplitude of the applied driving force
of angular frequency ω

It is understood that the driving function is physically the real part of $Ae^{-i\omega t}$. Ω is the angular frequency of free vibrations in the undamped case ($g = 0$).

Since interest here is in the steady state, one may write the particular integral of (2.1) whose time-dependence is harmonic in the same frequency as the driving force, i.e. let $x = X_0(\omega) e^{-i\omega t}$; then performing the substitution it yields

$$\frac{X_0(\omega)}{A} = \frac{1}{(\Omega^2 - \omega^2) - i\omega g} \tag{2.2}$$

$\frac{X_0(\omega)}{A}$ is the "admittance-function" of the system. One can of course recognize immediately this function as descriptive of an absorption-dispersion phenomenon, the dispersion-function, real part of $X_0(\omega)$, vanishing for $\omega = \Omega$, the resonance frequency in the case without dissipation. The maximum of the absorption-function differs from Ω by terms of order $\frac{1}{Q^2}$ where Q is defined as

$$Q = 2\pi \frac{\text{Energy stored in system at resonance}}{\text{Energy lost per period at resonance}} \tag{2.3}$$

Energy stored in the system is maximum at frequency $\omega = \Omega$, at which frequency energy flows only into the system, just replenishing the losses due to damping. Vibrations thus have constant amplitude, as determined by energy available

from the source, as though it had no damping at all.²

At this point laws of dynamics are applied. If T is the kinetic energy per unit mass, one may use the familiar relation

$$T = \frac{1}{2} (\dot{x})^2. \tag{2.4}$$

At resonance

$$x = k \cos \Omega t, \tag{2.5}$$

k being a constant.

Then

$$T = \frac{k^2 \Omega^2}{2} \sin^2 \Omega t$$

where T is a maximum for $\Omega t = \pi/2$, the position of neutral displacement (maximum velocity). Then

$$T_{\max} = U_0 = \Omega^2 \frac{k^2}{2}. \tag{2.6}$$

Since energy of the system in resonant vibration is constant, U_0 is just stored energy of the system. Thus, the stored energy at resonance of a vibrating system whose resonant frequency is regarded as variable is, for vibrations of amplitude independent of resonant frequency, proportional to the square of the resonant frequency.

Letting $k = \frac{K}{2}$ (2.6) may be rewritten as

$$U = k \Omega^2 \tag{2.7}$$

Since the existence of steady state conditions is being assumed, only adiabatic changes in the variables will be allowed. A perturbation in energy may be

² Further enlargement on these facts may be found in Page, "Introduction to Theoretical Physics", pp 70 et seq., in particular p. 80, and in Rayleigh, "Theory of Sound", Volume I, pp 43 et. seq., in particular p. 47

represented by differentiation of (2.7),³

$$\delta U = 2k\Omega \delta \Omega . \quad (2.8)$$

Now one divides (2.8) by (2.7)

$$\frac{1}{2} \frac{\delta U}{U} = \frac{\delta \Omega}{\Omega} \quad (2.9)$$

If one observes the natural frequency of the system the adiabatic performance of work thereon can be detected by changes in frequency according to (2.9).

Application to the Electromagnetic Case

One may now consider adaptation of the effect noted above to the case of an electromagnetic cavity resonator. Since one seeks to determine the fields as point-functions of space the desirability of using a body having a size very small compared to the cavity is obvious. Likely candidates are small physical electric and magnetic dipoles. In the interest of simplicity and to eliminate anisotropic effects spheres are chosen as desirable shapes. Since the linear dimensions of an electromagnetic oscillator are of the order of wavelength, $\lambda = \frac{c}{\nu}$, the "characteristic length" is $a \ll \lambda$. This last condition implies the scattering of only relatively small amounts of energy (i.e., insures only localized field perturbation) since the self-oscillation frequency of the dipole is as many orders removed from the actual frequency ν as a is from λ .

³ It is important to recognize that what is really being done here is to differentiate (2.7) totally -

$$\delta U = 2k\Omega \delta \Omega + \Omega^2 \delta k$$

and impose the condition $\delta k = 0$, i.e. that the amplitude of vibration be constant. This imposes a limitation on the physical nature of the driving source that is not unduly difficult to meet in practice, namely that the amplitude of its output be independent of small perturbations in frequency.

The opportunity now arises for deriving a satisfactory approximation to the perturbing effect by means of a static solution. Only the case where the perturbing body is a sphere of radius a will be discussed. Two cases will be separately but simultaneously dealt with: (1) the material of the sphere is a perfect dielectric, with electric permittivity $\epsilon = k_e \epsilon_0$, magnetic permeability μ_0 (free space value), and conductivity $\sigma = 0$; (2) the magnetic (isotropic) body for which $\mu = k_m \mu_0$, $\epsilon = \epsilon_0$, and $\sigma = 0$. The solutions may be found easily for these cases; the respective dipole moments in a uniform field are^{4,5}

$$\underline{p} = 4\pi a^3 \frac{k_e - 1}{k_e + 2} \epsilon_0 \underline{E} \quad \text{for dielectric sphere} \quad (3.1)$$

$$\underline{m} = 4\pi a^3 \frac{k_m - 1}{k_m + 2} \underline{H} \quad \text{for magnetic sphere.} \quad (3.2)$$

Now the case may be examined of a sphere having sufficiently large conductivity that field penetration at the frequency being considered is negligible. The static solution is still valid in this case. The requirement is met at frequencies of 10^8 sec^{-1} by metals of good conductivity $10^7 \text{ ohms}^{-1} \text{ mtr}^{-1}$. For the electric field the dipole moment is given by⁶

$$\underline{p} = 4\pi a^3 \epsilon_0 \underline{E} \quad (3.3)$$

which is exactly the result if k_e were allowed to approach infinity⁷ in (3.1).

⁴ Stratton, loc. cit. Sec. 3.24 (magnetic case may easily be deduced from analogy to the electric case.)

⁵ Smythe, Static and Dynamic Electricity. Sec. 12.03 (p. 420) (In these equations merely let the diagonal tensor (μ_k) be a scalar, i.e., let $\mu_1 = \mu_2 = \mu_3 = \mu$.) Note the use of e.m.u. by Smythe.

⁶ Stratton, loc. cit., Sec. 3.24, p. 205.

⁷ Stratton, loc. cit., Sec. 3.23, p. 205 (rework)

For the magnetic case we find the correct result if, for frequencies of interest we allow the permeability k_m to approach zero as the conductivity becomes very large.⁸ Bearing these facts in mind, once more the value zero in both cases will be temporarily assigned to conductivity.

One can write, for an arbitrarily shaped body having characteristic length $a \ll \lambda$ and constitutive parameters μ, ϵ, σ , that the perturbation in energy of the oscillating cavity resulting is just the energy of the dipole moments in the fields \underline{E} and \underline{H} , assuming that $\frac{\partial E_k}{\partial x_j} a \ll E_k$, k and j is any coordinate indices.

$$\delta U = 1/2(\underline{p} \cdot \underline{E} + \underline{m} \cdot \underline{B}) \quad (3.4)$$

Assuming the validity of the static approximation and making the body a sphere of radius a

$$\delta U = 2\pi a^3 \left[\frac{k_e - 1}{k_e + 2} \epsilon_0 E^2 + \frac{k_m - 1}{k_m + 2} \mu_0 H^2 \right] \quad (3.5)$$

Imposing (2.9) ($\Omega = 2\pi\nu$)

$$\delta \nu = \frac{\pi a^3 \nu}{U} \left[\frac{k_e - 1}{k_e + 2} \epsilon_0 E^2 + \frac{k_m - 1}{k_m + 2} \mu_0 H^2 \right] \quad (3.6)$$

For the metal sphere the result is obtained by evaluating the limit of (3.6) as $k_e \rightarrow \infty$ and $k_m \rightarrow 0$,

$$\delta \nu = \frac{\pi a^3 \nu}{U} \left(\epsilon_0 E^2 - \frac{\mu_0}{2} H^2 \right) \quad (3.7)$$

For the dielectric sphere $k_m = 1$ and

$$\delta \nu = \frac{\pi a^3 \nu}{U} \left(\frac{k_e - 1}{k_e + 2} \right) \epsilon_0 E^2 \quad (3.8)$$

⁸ Smythe, loc. cit., Sec. 11.05, pp. 396-8.

Merely let the constant

$$\frac{\pi a^3 \nu}{U} = C. \quad (3.9)$$

One is only concerned with its constancy during the performance of a sequence of observations.

Subject to the limitations mentioned previously, observations with a metallic sphere denoted by subscript "1", will be considered. Observations with a dielectric sphere will be denoted by subscript "2". Each sphere has been made in turn to occupy the same region in space, leading to the generation of two point-functions on the same domain; thus

$$\delta \nu_1 = C(\epsilon_0 E^2 - \frac{\mu_0}{2} H^2) \quad (3.10)$$

$$\delta \nu_2 = C \left(\frac{k_e - 1}{k_e + 2} \right) \epsilon_0 E^2 \quad (3.11)$$

From (3.11)

$$C \epsilon_0 E^2 = \delta \nu_2 \left(\frac{k_e + 2}{k_e - 1} \right) \quad (3.12)$$

Substituting (3.12) into (3.10) yields

$$H = K_1 \sqrt{\left(\frac{k_e + 2}{k_e - 1} \right) \delta \nu_2 - \delta \nu_1} \quad (3.13)$$

and from (3.11)

$$E = K_2 \sqrt{\left(\frac{k_e + 2}{k_e - 1} \right) \delta \nu_2} \quad (3.14)$$

One then has two equations in three variables, k_e , E , and H ; K_1 and K_2 are dependent constants whose magnitude is decided by the energy-level of the system. Given an equation of condition on any one of these three variables, the solution is unique for all three. The most obvious sort of condition

equation is one which specifies the value of k_e which is usually a parameter. It is obvious that such a condition may be arrived at experimentally.

It is not necessary to rely on indirect information, but rather one may find k_e if either \underline{E} or \underline{H} is known at any point; in general it is possible to find some point not in the neighborhood of a boundary at which \underline{H} vanishes.

For this point

$$\frac{k_e + 2}{k_e - 1} = \frac{\delta v_1}{\delta v_2} \tag{3.15}$$