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THREE DIMENSIONAL CONSTITUTIVE VISCOELASTIC LAWS WITH FRACTIONAL ORDER TIME DERIVATIVES

BY

NICOS MAKRIS
Assistant Professor

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DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

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Nicos Makris
Assistant Professor

Department of Civil and Environmental Engineering, SEMM
University of California at Berkeley
Berkeley, CA 94720

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ABSTRACT

In this report the three-dimensional behavior of constitutive models containing fractional order time derivatives in their strain and stress operators is investigated. Assuming isotropic viscoelastic behavior it is shown that when the material is *incompressible*, then the one-dimensional constitutive law calibrated either from shear or elongation tests can be directly extended in three dimensions, and the order of fractional differentiation is the same in all deformation patterns. When the material is elastically compressible, the constitutive laws during elongation and shear are different; however the order of fractional differentiation remains the same. It is shown that for an *elastically compressible* material, the *four-parameter fractional solid* model (RTG model) which has been used extensively to approximate the *elongation* behavior of various polymers, can be constructed from the *three-parameter fractional Kelvin* model (RT model) in *shear* and the *bulk modulus* of the material. Some of the analytical results obtained herein with operational calculus are in agreement with experimental observations reported in the literature. Results on the viscoelastic Poisson behavior of materials described with the fractional solid model are presented and it is shown that at early times the Poisson function reaches negative values. Finally, it is shown that when the material is *viscoelastically compressible*, the constitutive law in elongation involves additional orders of fractional differentiation that do not appear in the constitutive law in shear. Nevertheless, the orders of differentiation appearing in the stress and strain operators are the same.

ACKNOWLEDGEMENTS

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I. INTRODUCTION

In the last two decades several viscoelastic models with fractional order time derivatives have been proposed to approximate the frequency dependent response of various viscoelastic solids and fluids such as synthetic fibers (Smit and deVries 1970), polubutane (Bagley and Torvik 1983a), poly(methymethacrylate) (Rogers 1983), nitrile rubber (Bagley and Torvik 1986), Hevea (natural) rubber (Koh and Kelly 1990) and silicone gel (Makris et al. 1993, 1995), among others. All these phenomenological models have been validated by predicting the measured material response either from one dimensional shear oscillatory tests or one dimensional elongation oscillatory tests. Within the context of rheology, fractional-derivative constitutive models have been used as early as 1938, by Gemant (1938); however, it appears that little is known on the three-dimensional behavior of these models.

Bagley (1979), proposed the RT (Rubbery-Transition regions) model (a three parameter fractional derivative Kelvin model) to capture the behavior in *shear* of the elastomer 3M-467. The fractional derivative Kelvin model (RT model) proposed by Bagley (1979) was subsequently used by Koh and Kelly (1990) to approximate the oscillatory shear stress-strain response of lightly vulcanized Hevea (natural) rubber. An extended version of the RT model is the RTG (Rubbery, Transition and Glassy regions) model, which is a four parameter fractional derivative model, named the *fractional solid model.* This model has been used extensively to approximate the elongation behavior of various polymers (Bagley and Torvik 1983a, 1983b, 1986; Rogers 1983). Herein it is shown that the parameters of the RTG model in elongation can be directly obtained from the parameters of the RT model in shear and the compressibility modulus of the material.

When the material is assumed compressible, the viscoelastic law in shear deformation is different to the viscoelastic law in elongation-contraction. For instance, it is shown that when the viscoelastic behavior in shear is described with the fractional derivative

Kelvin model (RT model) and the material behaves like an elastic solid during a pressure test, then during an elongation test the material behaves according to the fractional solid model (RTG model) in which the order of differentiation in the stress and strain operators is the same. This analytical result is confirmed from experimental observations reported by Bagley and Torvik (1983b, 1986). At the limit where the material is assumed incompressible, the one dimensional constitutive law obtained either from shear or elongation tests can be directly generalized in three dimensions. Results on the viscoelastic Poisson behavior of materials which are described with the fractional solid model are presented. Finally, it is shown that when the material is viscoelastically compressible, the constitutive laws during shear and elongation are not only different, but additional orders of fractional differentiation appear in the elongation constitutive law. Nevertheless, the orders of differentiation appearing in the stress and strain operators are the same.

This work was motivated from the need to compute the multidimensional dynamic response of seismic protection devices that involve various elastomers and polymers that deform in all three directions.

II. INCOMPRESSIBLE MATERIALS

First, the case of an incompressible material is examined. In this case the volumetric change, ΔV, is zero.

$$\frac{\Delta V}{V} = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0, \tag{1}$$

where, $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the small-displacement-gradient strain tensor. As an example, the silicone gel used within the viscoelastic fluid dampers investigated by Makris et al. (1993, 1995) is nearly incompressible and equation (1) is a good approximation.

The small-amplitude dynamic behavior of this silicone gel in shear can be approximated satisfactorily over a wide range of frequencies with the following generalized derivative Maxwell model (Makris et al 1993),

$$\sigma_{12}(t) + \lambda \frac{d^r \sigma_{12}(t)}{dt^r} = \eta \frac{d\gamma_{12}(t)}{dt}, \tag{2}$$

where $\sigma_{12}(t)$ is the time dependent shear stress, $\gamma_{12}(t) = 2\varepsilon_{12}(t)$, λ is the generalized relaxation parameter with units $\sec^r$ and η is the zero-shear-rate viscosity of the material. $d^r f(t)/dt^r = \mathcal{D}^r f(t)$ is the generalized (fractional) derivative of order r of the time dependent function f(t) (Oldham and Spanier 1974, Miller and Ross 1993). The definition of the fractional derivative is given through the Riemann-Liouville integral,

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_{-\infty}^{t} f(\xi)\,(t-\xi)^{r-1} d\xi, \tag{3}$$

where $\Gamma(r)$, is the Gamma function. The above integral converges only for r>0, or in the case where r is complex, the integral converges for $\mathcal{R}(r) > 0$. However, by a proper analytic continuation across the line $\mathcal{R}(r) = 0$, and provided that the function f(t) is n times differentiable, it can be shown that the integral exists for $n - \mathcal{R}(r) > 0$ (Riesz 1949). In this case the generalized derivative of order r is defined as

$$\frac{d^r f(t)}{dt^r} = \mathcal{D}^r f(t) = I^{-r} f(t), \qquad \mathcal{R}(r) > 0. \tag{4}$$

With the help of (4) equation (2) can be expressed in terms of linear differential operators (Davis 1934, Tschoegl 1989).

$$(1 + \lambda \mathcal{D}^r)\,\sigma_{12} = (2\eta \mathcal{D}^1)\,\varepsilon_{12}. \tag{5}$$

Herein the behavior of the fractional derivative Maxwell model given by (2) or (5) is illustrated with its shear creep compliance function, $J(t)$, which is defined as the resulting

shear strain due to a unit amplitude step stress function, $\sigma_{12}(t) = h(t)$. With this stress history in (5), the resulting creep compliance function is

$$J_{12}(t) = \frac{1}{\eta}\left[\frac{\lambda}{\Gamma(2-r)}t^{1-r} + t\right]. \tag{6}$$

Figure 1 plots the quality $\eta J(t)$ as a function of time for $\lambda = 1s^r$ and different values of r. When r=1 equation (6) reduces to the creep compliance of the classical Maxwell model.

The elongation-contraction constitutive law can be constructed from the shear constitutive law given by (5) with the help of operational calculus (Davis 1934, Tschoegle 1989, Shames and Cozzarelli 1992). In general, the one dimensional elongation-contraction and shear deformation laws of a linear viscoelastic material are

$$P^E\sigma_{11} = Q^E\varepsilon_{11}, \tag{7}$$

$$P^G\sigma_{12} = Q^G\varepsilon_{12}, \tag{8}$$

where, P^E, Q^E, P^G, and Q^G are linear differential operators containing generalized time derivatives. As an example, in the case of the fractional Maxwell model given by (5), $P^G = 1 + \lambda\mathcal{D}^r$ and $Q^G = 2\eta\mathcal{D}^1$. In equation (7), σ_{11}, is the total normal stress and its deviatoric component, τ_{11}, is given by

$$\tau_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} = \sigma_{ij} - p\delta_{ij}, \tag{9}$$

where, p, is the pressure and δ_{ij} is the Kronecker delta.

The viscoelastic Poisson effect is expressed as

$$P^\nu\varepsilon_{22} = -Q^\nu\varepsilon_{11}, \tag{10}$$

where P^ν and Q^ν are also linear differential operators. Under isotropic behavior the fundamental analogy between the theories of linear elasticity and linear viscoelasticity, known

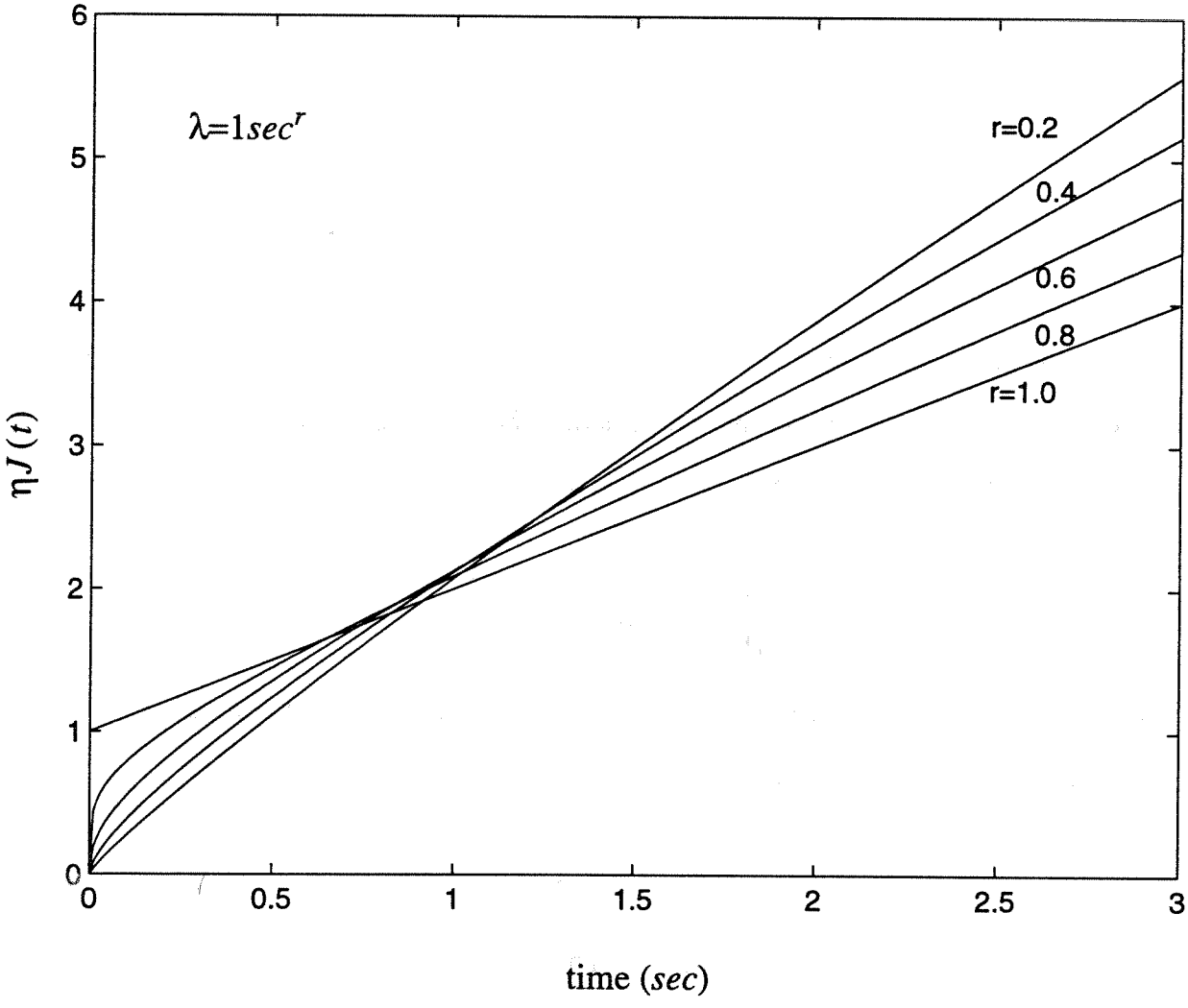

Figure 1. Shear creep compliance of the fractional derivative Maxwell model.

as the correspondence principle (Flugge 1975, Tschoegl 1989, Shames and Cozzaralli 1992), reveals the following analogies

$$\frac{Q^E}{P^E} \sim E \tag{11}$$

$$\frac{Q^G}{P^G} \sim 2G \tag{12}$$

$$\frac{Q^\nu}{P^\nu} \sim \nu. \tag{13}$$

The analogies given by (11), (12) and (13) allow for the development of relations between differential operators by extrapolating the relations between elastic constants obtained from elasticity theory. Accordingly, the relation $E = 2G(1+\nu)$, corresponds

$$\frac{Q^E}{P^E} = \frac{Q^G}{P^G}\left(1 + \frac{Q^\nu}{P^\nu}\right). \tag{14}$$

For an incompressible material, $P^\nu = 1$ and $Q^\nu = 1/2$; and equation (14) shows that for the generalized derivative Maxwell model given by (5),

$$P^E = 1 + \lambda \mathcal{D}^r, \tag{15}$$

$$Q^E = 3\eta \mathcal{D}^1, \tag{16}$$

and the resulting elongation constitutive law for the fractional derivative Maxwell model is

$$\sigma_{11}(t) + \lambda \frac{d^r \sigma_{11}(t)}{dt^r} = 3\eta \frac{d\varepsilon_{11}(t)}{dt}. \tag{17}$$

Equation (17) shows that when the material is assumed incompressible the same form of constitutive law describes the shear and the elongation behavior; and that the order of fractional differentiation, r, and generalized relaxation parameter, λ, are the same. The zero-

elongation-rate viscosity, $\eta_E = 3\eta$; which is the expected result for the Trouton viscosity (Bird et al. 1987).

From equations (17), (2) and (6) one immediately obtains that the elongation creep compliance, $J_{11}(t) = J_{12}(t)/3$. Moreover, from equations (7) and (8) the transverse normal strain, $\varepsilon_{22}(t)$, during an elongation test along the 1-1 direction is given by

$$Q^E P^\nu \varepsilon_{22} = -Q^\nu P^E \sigma_{11}. \tag{18}$$

With equation (18) one can define the transverse elongation creep compliance, $J^1_{22}(t)$, as the resulting negative strain along the transverse 2-2 direction for a unit amplitude step stress in the 1-1 direction. Equation (18) gives that for an incompressible material, $J^1_{22}(t) = J^1_{11}(t)/2$, and the viscoelastic Poisson function, $\nu(t)$, defined as (Shames and Cozarelli 1992)

$$\nu(t) = \frac{J^1_{22}(t)}{J^1_{11}(t)}, \tag{19}$$

is a constant with time equal to 0.5.

It is interesting to express equation (17) in terms of deviatoric normal stresses. This is possible by using the relation from elasticity theory

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}. \tag{20}$$

Application of the correspondence principle gives

$$\varepsilon_{ij} = \frac{1+Q^\nu/P^\nu}{Q^E/P^E}\sigma_{ij} - \frac{Q^\nu/P^\nu}{Q^E/P^E}\sigma_{kk}\delta_{ij}. \tag{21}$$

Using the expressions given by (15) and (16) and that for an incompressible material $P^\nu = 1$, and $Q^\nu = 1/2$, equation (21) becomes

$$\frac{3}{2}(1+\lambda\mathcal{D}^r)\,\sigma_{ij}-\frac{1}{2}(1+\lambda\mathcal{D}^r)\,\sigma_{kk}\delta_{ij}\;=\;3\eta\,\mathcal{D}^1\varepsilon_{ij}. \tag{22}$$

When i is different than j, equation (22) reduces to (5). When i=j, $\sigma_{kk} = 3p$, and in this case (say i=1)

$$\frac{1}{2}(1+\lambda\mathcal{D}^r)\;(\sigma_{11}-p)\;=\;\eta\,\mathcal{D}^1\varepsilon_{11}. \tag{23}$$

From equation (9), $\sigma_{11}-p = \tau_{11}$, and equation (23) also takes the form of (5). Consequently, the three dimensional constitutive law of the generalized derivative Maxwell model is

$$\tau_{ij}(t)+\lambda\frac{d^r\tau_{ij}(t)}{dt^r}\;=\;\eta\dot{\gamma}_{ij}(t)\,, \tag{24}$$

where $\tau_{ij}(t)$, is the deviatoric stress tensor and $\dot{\gamma}_{ij}(t)$ is the rate of strain tensor (Bird et al. 1987). The expression offered by (24) is attractive in computational schemes, since deviatoric stresses appear directly in the balance of momentum equation (Bird et al. 1987). The 3-dimensional fractional derivative constitutive law given by (24) with r=0.6 was recently implemented in a boundary element formulation to predict the dynamic response of viscoelastic fluid dampers used in seismic protection of structures (Makris et al. 1993, 1995).

III. COMPRESSIBLE MATERIALS

In the case of a compressible material equation (1) becomes

$$\varepsilon_{kk}\;=\;\frac{p}{K} \tag{25}$$

where, p, is the pressure and, K, is the bulk modulus of the material. This study concentrates on the fractional derivative Kelvin model (RT model) in shear used by Bagley 1979, Koh and Kelly (1991) and others.

$$\sigma_{12}(t) = 2G\varepsilon_{12}(t) + 2\zeta \frac{d^q \varepsilon_{12}(t)}{dt^q}, \tag{26}$$

where $0 \le q \le 1$. According to equation (8) the shear stress and strain operators of the RT model are: $P^G = 1$ and $Q^G = 2G + 2\zeta \mathcal{D}^q$. Moreover, since the material is now compressible, there are linear operators, P^K and Q^K, such that

$$P^K \sigma_{kk} = Q^K \varepsilon_{kk}. \tag{27}$$

Bulk deformations are fundamentally different than shear deformations since they are accompanied by quite different molecular processes. They are far fewer experimental data available on bulk viscoelastic properties; however, a much narrower range of behavior among various types of polymers may be expected, since volumetric changes are dominated by local configurational rearrangements which are scarcely affected by molecular weight, entanglements or cross-links in moderate numbers (Ferry 1980).

Figure 2 shows qualitatively the dependance with frequency of the bulk compliance, $B(\omega) = B'(\omega) - iB''(\omega)$. The real part, $B'(\omega)$, and imaginary part, $-B''(\omega)$, shown on Figure 2 are computed with the generalized solid model discussed in the next section. The storage bulk compliance, $B'(\omega)$, falls from a low-frequency limiting value, B_{el}, to a high-frequency limiting value, B_{gl}, but the change is less than a factor of two, instead of the many-powers-of-ten change that the shear compliance experiences. The loss bulk compliance, $B''(\omega)$, is zero within experimental error at both low and high frequencies and passes through a maximum in the region of transition. However, the loss tangent is a small number of the order of 0.1 or so. Accordingly, because of this weak viscoelastic bulk behavior, the simplest case of a purely elastic bulk behavior is first analyzed.

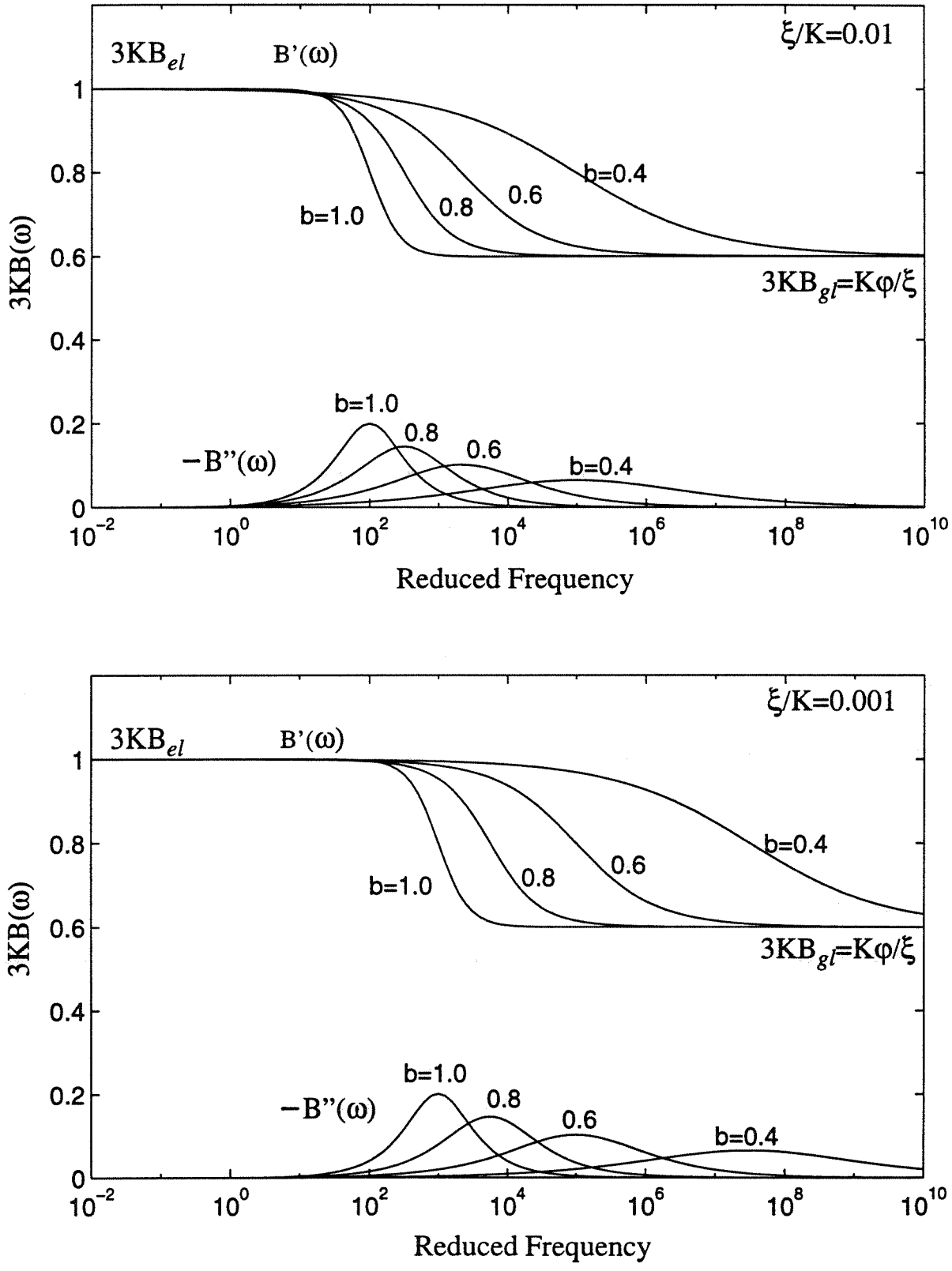

Figure 2. Typical viscoelastic behavior of the dynamic bulk compliance. The curves have been obtained with the generalized solid model given by (52) with $\phi K/\xi$=0.6

A. Elastic bulk behavior

In this section it is assumed that the material is purely linearly elastic during a pressure test along the entire frequency spectrum of interest. In this case the differential operators in (27) are, $P^K = 1$ and $Q^K = 3K$. Knowledge of the differential operators during a shear and a pressure test is sufficient to construct the differential operators during an elongation test. From elasticity theory,

$$E = \frac{9KG}{3K+G}, \tag{28}$$

$$\nu = \frac{1}{2}\left(\frac{3K-2G}{3K+G}\right), \tag{29}$$

and the correspondence principle (Flugge 1975) yields

$$P^E = 6K + 2G + 2\zeta \mathcal{D}^q \tag{30}$$

$$Q^E = 9K(2G + 2\zeta \mathcal{D}^q) \tag{31}$$

$$P^\nu = 6K + 2G + 2\zeta \mathcal{D}^q = P^E \tag{32}$$

$$Q^\nu = 3K - 2G - 2\zeta \mathcal{D}^q. \tag{33}$$

Substitution of the expressions given by (30) and (31) into (7) gives

$$\sigma_{11}(t) + \frac{\zeta}{3K+G}\frac{d^q\sigma_{11}(t)}{dt^q} = \frac{9KG}{3K+G}\varepsilon_{11}(t) + \frac{9K\zeta}{3K+G}\frac{d^q\varepsilon_{11}(t)}{dt^q}. \tag{34}$$

Consequently, an elastically compressible material that is described with a fractional derivative Kelvin model in shear (RT model) is described during elongation with a four parameter fractional derivative constitutive law given by (34). This law is named the *fractional solid model* and is precisely the RTG model used by Bagley and Torvik (1983b, 1986) and Rogers (1983) to approximate the one dimensional elongation behavior of various materials. When q approaches the value of one, the fractional solid model tends to the

standard solid model of the classical theory of viscoelasticity (Tschoegl 1989, Shames and Cozzaralli 1992).

Initially, Bagley and Torvik (1983b) proposed a five parameter fractional derivative model

$$\sigma_{11}(t) + \lambda \frac{d^{\beta}\sigma_{11}(t)}{dt^{q}} = E\varepsilon_{11}(t) + \theta \frac{d^{\alpha}\varepsilon_{11}(t)}{dt^{q}} \tag{35}$$

to approximate the dynamic behavior of several viscoelastic materials during oscillatory elongation tests. However they observed that in many cases the orders of differentiation α and β, that resulted from the nonlinear regression analysis were practically the same ($\alpha \approx \beta = q$). In a subsequent paper, Bagley and Torvik (1986) demonstrated that their initial observation, $\alpha = \beta = q$, is indeed a necessary condition for the RTG model to be consistent with the laws of thermodynamics. This result is recovered here analytically by using operational calculus. For an elastically compressible isotropic material with compressibility modulus, K, the four parameters of the RTG model in elongation are related to the three parameters of the RT model in shear via the expressions

$$E = \frac{9KG}{3K+G}, \qquad \lambda = \frac{\zeta}{3K+G}, \qquad \theta = \frac{9K\zeta}{3K+G} \tag{36}$$

$$K = \frac{\theta}{9\lambda}, \qquad G = E\frac{\theta}{3(\theta-\lambda E)}, \qquad \zeta = \frac{\theta^2}{3(\theta-\lambda E)} \tag{37}$$

and the order of differentiation, q, is the same in all deformation patterns. Accordingly, the four parameter fractional solid model (RTG model) is fully described either with the bulk/shear parameter set {K, G, ζ, q} or the elongation parameter set {E, λ, θ, q}. At the incompressible limit, $K, G \to \infty$; therefore, $\lambda \to 0$, so the four-parameter fractional solid model given by (34) condenses to the three parameter incompressible fractional Kelvin model {E=3G, $\theta = 3\zeta$, q}. This model has been found appropriate to capture the elongation behavior of various nearly incompressible polymers such as that of Polybutadiene, Butyl

B252 and Butyl 70821 reported by Bagley and Torvik 1983a. Table 1 summarizes the resulting values for the parameter-sets {K, G, ζ, q} obtained from the parameter-sets {E, λ, θ, q} reported in the literature from different materials.

Of particular interest is the poly(methylmethacrylate) (PMMA) reported by Rogers (1983). It is well known (Ferry 1980 and references reported therein) that for glassy polymers, shear and elongation behavior do not give equivalent information, indicating that such materials are not isotropic. Accordingly, the expressions offered by (36) and (37) are not applicable in the case of PMMA.

The elongation compliance of the four parameter fractional solid model given by (35), in which $\alpha = \beta = q$, is the inverse Laplace transform of

$$J_{11}^{1}(s) = \frac{1}{\theta}\left[\frac{1}{s\,(s^q + \frac{E}{\theta})} + \frac{\lambda}{s^{1-q}\,(s^q + \frac{E}{\theta})}\right]. \tag{38}$$

The Laplace inversion of (38) is not obvious since involves the evaluation of the function

$$g(t) = \mathcal{L}^{-1}\{\frac{1}{s^p\,(s^q - a)}\} \tag{39}$$

which is not common in the literature. Nevertheless for the special case where q=1/k with k being an integer (k=1,2,3...) the inverse transform of (39) is possible (Miller and Ross 1993).

$$\mathcal{L}^{-1}\{\frac{1}{s^p\,(s^q - a)}\} = \sum_{j=1}^{k} a^{j-1} E_t(jq - 1 + p, a^k t)\,, \tag{40}$$

where $E_t(r, at)$, is the fractional integral of the exponential function, exp(at).

Table I: Elongation and Shear Parameters of the Fractional Solid Model

	ELONGATION				SHEAR			
Material	E	θ	λ (sec^q)	q	G	ζ	K	ν (∞)
Butyl B252 (Bagley & Torvik 1983a)	1.05 MP_a	0.244 MP_a-$sec^{0.519}$	~0	0.519	0.350 MP_a	0.081 MP_a-$sec^{0.519}$	$\sim\infty$	~0.5
Butyl 7082 (Bagley &Torvik 1983a)	0.685 MP_a	1.37 MP_a-$sec^{0.449}$	~0	0.449	0.228 MP_a	0.456 MP_a-$sec^{0.455}$	$\sim\infty$	~0.5
Nitrile Rubber 1479 (Bagley & Torvik 1986)	5.04 MP_a	0.227 MP_a-$sec^{0.64}$	0.00028	0.64	1.69 MP_a	0.076 MP_a-$sec^{0.64}$	90.08 MP_a	0.4907
Corning glass (Bagley & Torvik 1983b)	4.15 GP_a	109.0 GP_a-$sec^{0.636}$	3.50	0.636	1.596 GP_a	41.92 GP_a-$sec^{0.636}$	3.460 GP_a	0.300
Poly(methylmethacrylate) (Rogers 1983)	2.13 GP_a	3.659 GP_a-$sec^{0.195}$	0.576	0.195	-	-	-	-

$$E_t(r,at) = I^r e^{at} = \frac{1}{\Gamma(r)}\int_0^t e^{a\xi}(t-\xi)^{r-1}d\xi = t^r e^{at}\gamma^*(r,at), \quad (41)$$

where $\gamma^*(r,at)$ is the incomplete gamma function (Abramowtz and Stegun 1970). Figure 3 plots the function $E_t(r,at)$, for r=0, 1/2 and 1, and different values of a. The restriction in (40) that q=1/k constrains the analytical computation of the compliance of the fractional solid model (RTG model) only for of the fractional order of differentiation q= 1/2, 1/3, 1/4, 1/5,... and so on. Nevertheless, q=1/2 is a common value of fractional order of differentiation; while q=1/5 was used in the fractional solid model reported by Rogers (1983). Consequently, the transform given by (40) is valuable, since the value of many fractional exponents is near 1/k. As an example, for q=1/2 the compliance of the fractional solid model given by (35) is

$$J_{11}^1(t) = \frac{1}{\theta}\left[\lambda E_t\left(0,\frac{E^2}{\theta^2}t\right)+\left(1-\lambda\frac{E^2}{\theta^2}\right)E_t\left(\frac{1}{2},\frac{E^2}{\theta^2}t\right)-\frac{E}{\theta}E_t\left(1,\frac{E^2}{\theta^2}t\right)\right]. \quad (42)$$

Figure 4 plots the quantity $\theta J_{11}^1(t)$ as a function of time for $E/\theta = 1s^{-0.5}$, q=0.5 and λ=0.01, 0.1, 0.5 $\sec^{0.5}$.

The transverse elongation creep compliance, $J_{22}^1(t)$, for the fractional solid model (RTG model) can be computed by substitution of the operators, P^E, Q^E, P^ν, Q^ν, given by (30), (31), (32) and (33) into (18). Accordingly, in the Laplace domain the transverse elongation creep compliance is

$$J_{22}^1(s) = \frac{1}{9K}\left[\frac{(2G-3K)/(2\zeta)}{s\left(s^q+\frac{G}{\zeta}\right)}+\frac{1}{s^{1-q}\left(s^q+\frac{G}{\zeta}\right)}\right]. \quad (43)$$

For q=0.5 and the help of (40) the inverse Laplace transform of (43) becomes

$$J_{22}^1(t) = \frac{1}{9K\zeta}\left[-\zeta E_t\left(0,\frac{G^2}{\zeta^2}t\right)+\frac{3K}{2}E_t\left(\frac{1}{2},\frac{G^2}{\zeta^2}t\right)-\frac{G}{\zeta}\frac{3K-2G}{2}E_t\left(1,\frac{G^2}{\zeta^2}t\right)\right]. \quad (44)$$

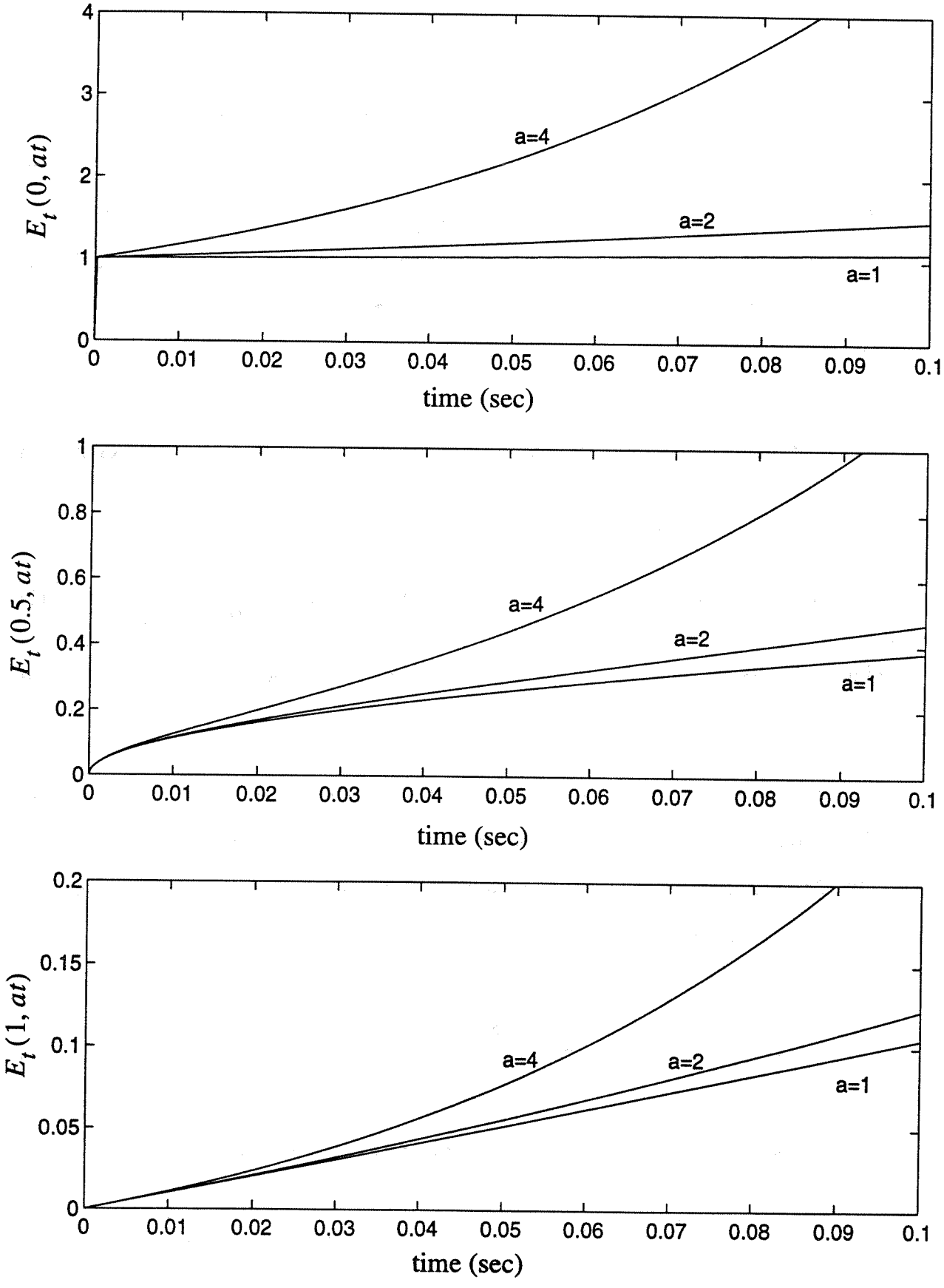

Figure 3. Graphs of the function $E_t(r, at) = t^r e^{at} \gamma^*(r, at)$.

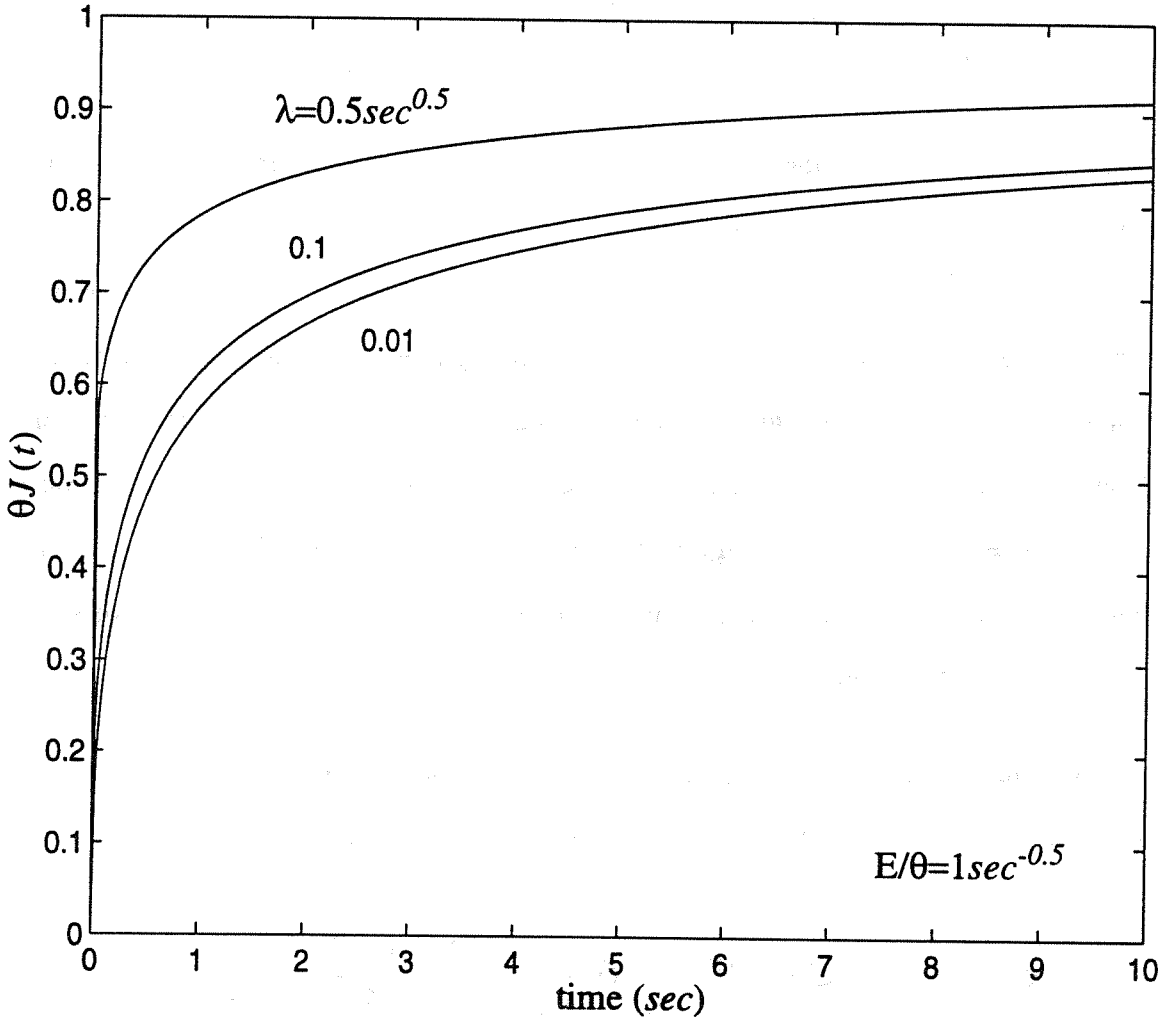

Figure 4. Elongation creep compliance of the fractional solid model.

Using the relations between the elongation parameters {E, λ, θ} and the bulk/shear parameters {K, G, ζ} given by (36) and (37), the viscoelastic Poisson function for the fractional solid model with q=0.5 is

$$\nu(t) = \frac{J_{22}^{1}(t)}{J_{11}^{1}(t)} = \frac{1}{2}\frac{3K-2G}{3K+G}\frac{-\frac{2\zeta}{3K-2G}E_t\left(0,\frac{G^2}{\zeta^2}t\right)+\frac{3K}{3K-2G}E_t\left(\frac{1}{2},\frac{G^2}{\zeta^2}t\right)-\frac{G}{\zeta}E_t\left(1,\frac{G^2}{\zeta^2}t\right)}{\frac{\zeta}{3K+G}E_t\left(0,\frac{G^2}{\zeta^2}t\right)+\frac{3K}{3K+G}E_t\left(\frac{1}{2},\frac{G^2}{\zeta^2}t\right)-\frac{G}{\zeta}E_t\left(1,\frac{G^2}{\zeta^2}t\right)}. \qquad (45)$$

At large times, $t\to\infty$, the term, $E_t\left(1,\frac{G^2}{\zeta^2}t\right)$, dominates over the first two terms in the numerator and denominator of (45), and $\nu(\infty) = 0.5(3K-2G)/(3K+G)$, which is the elastic limit. At early times, $t\to 0$ (glassy state), the two last terms in the numerator and denominator of (45) are zero, and $\nu(0) = -1$, indicating that such a material experiences lateral expansion under axial tension at the very beginning of the elongation process. This anomalous behavior is not new and is also present in the Poisson function of the standard linear solid (q=1) (Flugge1975). Figures 5 plots $\nu(t)$, as a function of time for different values of G/ζ, for a compressible material ($\nu(\infty) = 0.3$, top), and for a less compressible material ($\nu(\infty) = 0.49$, bottom).

The general three dimensional law of the fractional solid model (RTG model) can be derived after substituting (30), (31), (32) and (33) into (21). This gives

$$\sigma_{ij}(t) = 2(G+\zeta\mathcal{D}^{q})\varepsilon_{ij}(t) + \left[1-\frac{2}{3}\frac{G+\zeta\mathcal{D}^{q}}{K}\right]p\delta_{ij} \qquad (46)$$

and in terms of deviatoric stresses equation (46) takes the form

$$\tau_{ij}(t) = 2(G+\zeta\mathcal{D}^{q})\varepsilon_{ij}(t) - \frac{2}{3}\frac{G+\zeta\mathcal{D}^{q}}{K}p\delta_{ij} \qquad (47)$$

The contribution of the term $2(G+\zeta\mathcal{D}^{q})/(3K)$, in (46) or (47) can be computed by transforming (47) in the frequency domain

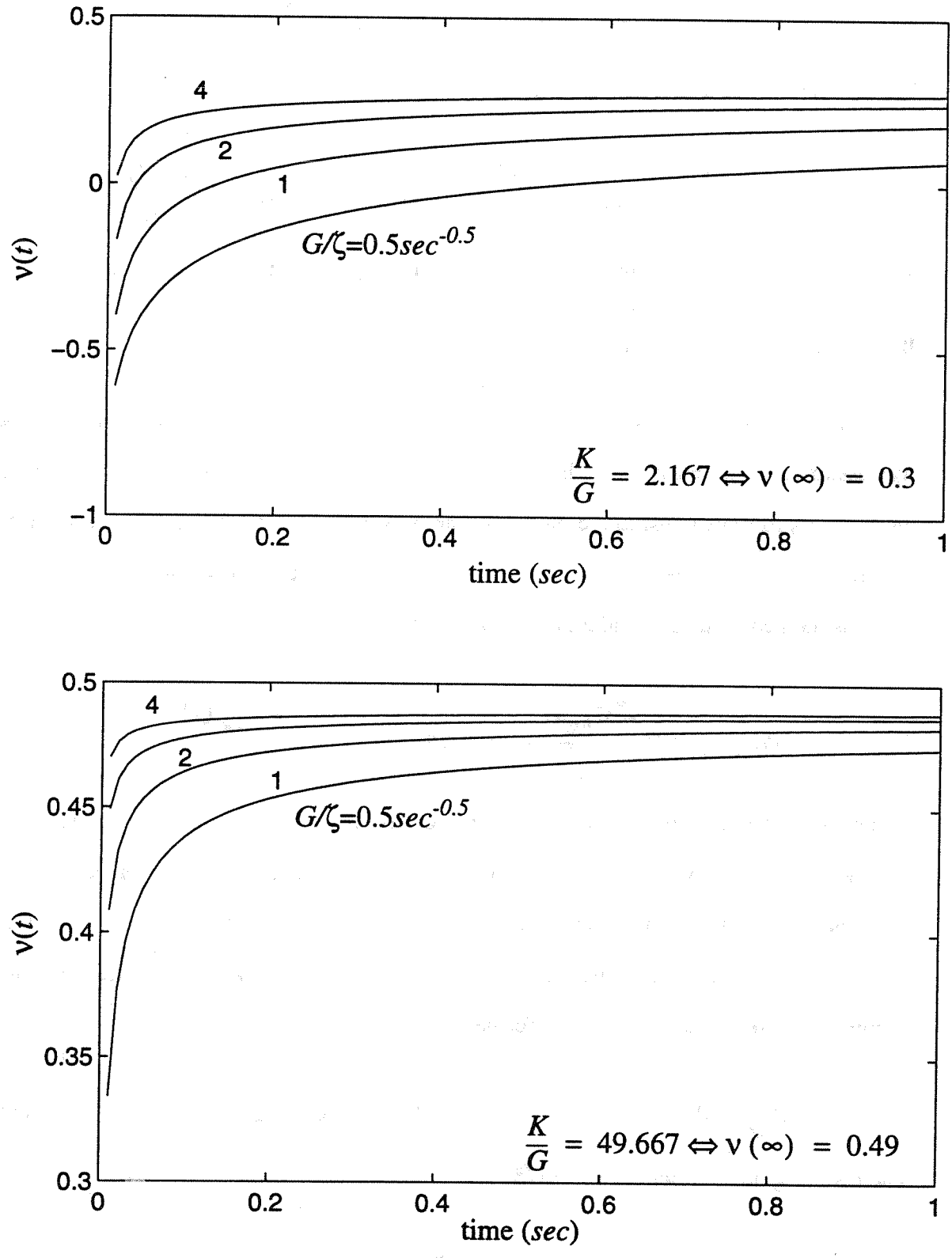

Figure 5. Poisson functions resulting from the fractional solid model in elongation, for a compressible material (top); and a less compressible material (bottom).

$$\tau_{ij}(\omega) = 2\left(G+\zeta(i\omega)^q\right)\varepsilon_{ij}(t) - \frac{2}{3}\frac{G+\zeta(i\omega)^q}{K}p\delta_{ij} \tag{48}$$

The magnitude of the pressure coefficient is

$$\left|\frac{2}{3}\frac{G+\zeta(i\omega)^q}{K}\right| = \frac{2}{3}\left|\frac{G}{K}+\frac{\zeta\omega^r}{K}\left(\cos\frac{\pi r}{2}+i\sin\frac{\pi r}{2}\right)\right| \le \frac{2}{3}\frac{G+\zeta\omega^q}{K}, \tag{49}$$

which shows that the compressible behavior is more apparent at higher frequencies.

B. Viscoelastic bulk behavior

In the forgoing analysis the bulk behavior of the fractional Kelvin model in shear was assumed purely elastic. In this section the case of viscoelastic bulk behavior is examined. An appropriate constitutive model for the bulk behavior which is consistent with the limiting constant values of the dynamic bulk compliance at the low and high frequency limits is the generalized (standard or fractional) solid model,

$$\sigma_{kk}(t) + \phi\frac{d^b\sigma_{kk}(t)}{dt^b} = 3K\varepsilon_{kk}(t) + 3\xi\frac{d^b\varepsilon_{kk}(t)}{dt^b}, \tag{50}$$

where, K, is the zero-frequency-limit bulk modulus, $3K = 1/B_{el}$, ϕ and ξ are generalized parameters where $\xi/K < \phi$, and $0 \le b \le 1$. When b=1 the model given by (50) is the standard solid model in bulk behavior. The reason for selecting the generalized solid model given by (50) as the simplest constitutive law to describe the bulk viscoelastic behavior, becomes apparent when (50) is transformed in the frequency domain,

$$\sigma_{kk}(\omega)\left[1+\phi(i\omega)^b\right] = \varepsilon_{kk}(\omega)\left[3K+3\xi(i\omega)^b\right]. \tag{51}$$

The resulting bulk compliance from (51) is

$$B(\omega) = B'(\omega) - iB''(\omega) = \frac{\varepsilon_{kk}(\omega)}{\sigma_{kk}(\omega)} = \frac{1+\phi(i\omega)^b}{3K+3\xi(i\omega)^b}. \tag{52}$$

Figure 2 plots the real and imaginary parts of (52) for different values of the fractional order of differentiation, b, $K\phi/\xi$=0.6, and ξ/K=0.01 (top), ξ/K=0.001 (bottom). At the zero-frequency limit, $B'(\omega) = B_{el} = 1/(3K)$, and $B''(\omega) = 0$, which is in agreement with experimental observations. At the high-frequency limit ($\omega \to \infty$), $B'(\omega) = B_{gl} = \phi/(3\xi)$, and $B''(\omega) = 0$, which is also in agreement with experimental observations (Ferry 1980). Note that as $\omega \to \infty$, $B'(\omega) = B_{gl}$=constant, and $B''(\omega) = 0$, only when the order of differentiation in the stress and strain operators in (50) are the same. Furthermore, note that any viscoelastic model simpler than the generalized solid model given by (50), is incapable to capture the limiting constant and zero values of the real and imaginary parts of the bulk compliance shown on Figure 2. Consequently, the most elementary admissible viscoelastic model for the bulk behavior besides the purely elastic model is the generalized solid model given by (50).

Adopting the bulk viscoelastic model given by (50) in which $P^K = 1 + \phi \mathcal{D}^b$ and $Q^K = 3K + 3\xi \mathcal{D}^b$, the corresponding elongation operators of the fractional Kelvin model in shear given by (26) are

$$P^E = 6K + 2G + 2\zeta \mathcal{D}^q + 2(3\xi + \phi G)\mathcal{D}^b + 2\phi\zeta \mathcal{D}^{b+q} \tag{53}$$

$$Q^E = 9(2KG + 2K\zeta \mathcal{D}^q + 2\xi G \mathcal{D}^b + 2\xi\zeta \mathcal{D}^{b+q}) \tag{54}$$

$$P^V = 6K + 2G + 2\zeta \mathcal{D}^q + 2(3\xi + \phi G)\mathcal{D}^b + 2\phi\zeta \mathcal{D}^{b+q} = P^E \tag{55}$$

$$Q^V = 3K - 2G - 2\zeta \mathcal{D}^q + (3\xi - 2\phi G)\mathcal{D}^b - 2\phi\zeta \mathcal{D}^{b+q}. \tag{56}$$

In deriving (53) to (56) the property of the composition rule for fractional derivatives was used, $\mathcal{D}^b[\mathcal{D}^q f(t)] = \mathcal{D}^{b+q} f(t)$ (Oldham and Spanier 1974).

Substitution of the expressions given by (53) and (54) into (7) gives the constitutive law in elongation.

$$\sigma_{11}(t) + \frac{\zeta}{3K+G}\frac{d^q\sigma_{11}(t)}{dt^q} + \frac{3\xi+\phi G}{3K+G}\frac{d^b\sigma_{11}(t)}{dt^b} + \frac{\phi\zeta}{3K+G}\frac{d^{b+q}\sigma_{11}(t)}{dt^{b+q}}$$
$$= \frac{9KG}{3K+G}\left(\varepsilon_{11}(t) + \frac{\zeta}{G}\frac{d^q\varepsilon_{11}(t)}{dt^q} + \frac{\xi}{K}\frac{d^b\varepsilon_{11}(t)}{dt^b} + \frac{\xi\zeta}{KG}\frac{d^{b+q}\varepsilon_{11}(t)}{dt^{b+q}}\right) \quad (57)$$

Note that when a standard solid model for the bulk behavior is used (b=1), the order if differentiation (1+q) in the last two terms of the stress and strain operators remains fractional.

In the special case where q=b=1 equation (57) condenses to

$$\sigma_{11}(t) + \frac{\zeta+3\xi+\phi G}{3K+G}\frac{d\sigma_{11}(t)}{dt} + \frac{\phi\zeta}{3K+G}\frac{d^2\sigma_{11}(t)}{dt^2}$$
$$= \frac{9KG}{3K+G}\left(\varepsilon_{11}(t) + \frac{\zeta K+\xi G}{KG}\frac{d\varepsilon_{11}(t)}{dt} + \frac{\xi\zeta}{KG}\frac{d^2\varepsilon_{11}(t)}{dt^2}\right) \quad (58)$$

which is a popular model in classical viscoelasticity having a relaxation and a retardation time.

The general three dimensional law of the model described in elongation by (57) and shear by (26) can be derived after substitution of (53), (54), (55) and (56) into (21). This gives

$$(1+\frac{\xi}{K}\mathcal{D}^b)\,\sigma_{ij}(t) = 2\left[G+\zeta\mathcal{D}^q+\frac{\xi}{K}(G\mathcal{D}^b+\zeta\mathcal{D}^{b+q})\right]\varepsilon_{ij}(t)$$
$$+\left[1+\frac{\xi}{K}\mathcal{D}^b-\frac{2}{3}\frac{G+\zeta\mathcal{D}^q+\phi G\mathcal{D}^b+\phi\zeta\mathcal{D}^{b+q}}{K}\right]p\delta_{ij} \quad (59)$$

and in terms of deviatoric stresses, equation (59) becomes

$$(1+\frac{\xi}{K}\mathcal{D}^b)\,\tau_{ij}(t) = 2\,(1+\frac{\xi}{K}\mathcal{D}^b)\,(G+\zeta\mathcal{D}^q)\,\varepsilon_{ij}(t) - \frac{2}{3}\frac{(1+\phi\mathcal{D}^b)\,(G+\zeta\mathcal{D}^q)}{K}p\delta_{ij} \tag{60}$$

When i is different than j equation (60) reduces to (26). Equation (60) is the three dimensional constitutive law that one should use in a finite element or boundary element formulation to compute the behavior of a material that behaves as a fractional Kelvin model in shear and as a fractional solid model in a pressure test.

This work was motivated from the need to compute the multidimensional dynamic response of seismic protection devices that involve various elastomers and polymers that deform in all three directions.

IV. CONCLUSIONS

In this paper the three-dimensional behavior of constitutive models containing fractional order time derivatives in their strain and stress operators has been investigated. Under the assumption of isotropic viscoelastic behavior it was shown that when the material is incompressible, then the one dimensional constitutive law calibrated either from shear or elongation tests can be extended in three dimensions and the order of fractional differentiation is the same in all deformation patterns. When the material is elastically compressible and its viscoelastic behavior in shear is described with a fractional derivative Kelvin model (RT model); then during an elongation test the material behavior is described with the fractional solid model (RTG model) in which the order of differentiation in the stress and strain operators is the same. This analytical result obtained herein with operational calculus is in agreement with experimental observations reported in the literature. When the material is viscoelastically compressible, the constitutive law in elongation involves additional orders of fractional differentiation that do not appear in the constitutive law in shear. Nevertheless, the orders of differentiation appearing in the stress and

strain operators are the same. Results on the viscoelastic Poisson behavior of materials described with the fractional solid model during an elongation test have been presented, and it was shown that at early times (glassy regime) the Poisson function reaches negative values.

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