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ABSTRACT

Factorization of residues of poles of the S matrix is derived from the requirements of unitarity for partial wave helicity amplitudes. Careful attention is given to questions of spin and the kinematic singularities of the relevant amplitudes, especially at $t = 0$. Residues β of a pole in the full partial wave amplitude satisfy factorization in the simple form, $\beta_{ba}^2 = \beta_{aa}\,\beta_{bb}$. In general, β can be written as $\beta = K\,\gamma$, where K contains the standard kinematic singularities of the Hara-Wang type, plus threshold behavior, and γ is a reduced residue. The K's for various mass classes are exhibited in a compact and consistent form and the corresponding factorization statements for the reduced residues are derived. These factorization relations are of the form, $t^x\,\gamma_{ba}^2 = \gamma_{aa}\,\gamma_{bb}$, where x is an integer. The reduced residues are analytic in the neighborhood of thresholds and pseudothresholds, but may, in the case of conspiracies, contain poles at $t = 0$. Various examples are presented to illustrate the use of factorization. These include LeBellac's argument on the behavior of the pion residue at $t = 0$ and its circumvention with a type II conspiracy. Mandelstam's treatment of

Adler's self-consistency condition and PCAC using an $M = 1$ pion is discussed from the viewpoint of factorization. It is shown that factorization for an $M = 1$ pion seems to imply smallness of both "soft pion" and "hard pion" amplitudes. The smallness of the latter casts some doubt on the $M = 1$ assignment for the pion. The final section considers the nature of the relations between amplitudes and the behavior of the reduced residues at $t = 0$ for conspiracies with unequal masses.

I. INTRODUCTION

The concept of factorization of the residues of Regge poles was discussed soon after the introduction of Regge pole ideas into high energy physics by Gell-Mann,[1] Gribov and Pomeranchuk,[2] and Charap and Squires.[3] Factorization of pole residues follows from unitarity and is well known in nuclear physics (e.g., Breit-Wigner resonance amplitudes involving partial widths for the initial and final states). Early applications in high energy physics included relations among the total cross sections for NN, πN, and $\pi\pi$ interactions at very high energies.[1,2] Recently factorization has entered the general discussion of Regge pole exchange in inelastic processes,[4] and in conspiracy and evasion,[5] as well as for specific reactions, such as the pion trajectory in hadronic processes,[6] pion production[7,8] and photoproduction[9] of vector mesons.

The presence of spin and unequal mass kinematics complicates the discussion of factorization considerably. Rules of thumb deduced from simple examples (e.g., always attach a $(t)^{\frac{1}{2}}$ factor to the $\pi\bar{N}N$ vertex) fail to hold in general. Questions arise as to exactly what part of the amplitude satisfies factorization, etc. The purpose of this paper is to present an elementary, but thorough, derivation of factorization and discussion of a number of examples. The separation of the kinematic singularities at thresholds and at $t = 0$ and the definition of meromorphic reduced residues receive careful attention. Our approach here is not the only one possible. But it is one logically correct and consistent way to handle the complications of spin. The examples illustrate the

interplay of factorization and conspiracy, and the complexities that arise with daughter trajectories. No great originality is claimed; our purpose is mainly pedagogical.

II. DEFINITIONS AND UNITARITY FOR PARTIAL WAVE AMPLITUDES

We begin by giving definitions, a statement of unitarity and the optical theorem. Then we specialize to two-body channels, introduce helicity amplitudes and their partial wave expansions, and finally obtain a statement of unitarity for partial waves.

A. S-Matrix and the Feynman Amplitude

The S-matrix is realted to the invariant Feynman amplitude by

$$S_{ba} = \delta_{ba} - i(2\pi)^4 \delta^{(4)}(p_b - p_a)\, N_b\, N_a\, \mathcal{M}_{ba} \qquad (1)$$

where a, b are labels containing all necessary information for specification of the initial and final states, $N_a = [\Pi_a (2E)]^{-\frac{1}{2}}$ is the reciprocal square root of the product of factors of $2E$, one for each particle in state a, and similarly for state b. The word, "state", is used to denote a certain number of various types of particles with definite momenta and spin projections, while the word, "channel", is used to denote the particle composition and their spin projections.

B. Unitarity and the Optical Theorem in Terms of $\mathcal{M}$

Unitarity of S implies that

$$i(\mathcal{M}_{ba} - \mathcal{M}^*_{ab}) = (2\pi)^4 \sum_c N_c^2\, \mathcal{M}^*_{cb}\, \mathcal{M}_{ca}\, \delta^{(4)}(p_c - p_a) \qquad (2)$$

where the states a and b now satisfy the energy-momentum conservation requirement, $p_b = p_a$, and $\sum_c$ means integration over $d^3p/(2\pi)^3$

for each particle in the channel c and sum over all channels c.

If the state a consists of two particles of masses m_1 and m_2 and 4-momenta p_1 and p_2 and the state b is chosen equal to a (i.e., forward elastic scattering), then Eq. (2) becomes the optical theorem,

$$-\mathrm{Im}\ \mathcal{M}_{aa} = 2\sqrt{(p_1 \cdot p_2)^2 - m_1^2\, m_2^2}\ \sigma_{\mathrm{total}} \tag{3}$$

where σ_{total} is the total cross section in channel a. With

$$\mathcal{M}_{aa} = -8\pi W f_{CM}(0^\circ),$$

Eq. (3) takes the more familiar form,

$$\frac{4\pi}{p_{CM}}\ \mathrm{Im}\ f_{CM}(0^\circ) = \sigma_{\mathrm{total}} \tag{4}$$

where p_{CM}, W and $f_{CM}(0^\circ)$ are the center of mass initial momentum, total energy, and forward non-spin flip scattering amplitude, respectively.

C. Two-Body Channels

If we restrict consideration to two-body processes, the sum over states c in Eq. (2) can be written

$$\sum_c N_c^2 = \sum_c \int \frac{(d^3p\ d^3p')_c}{(2\pi)^6 4(EE')_c} \tag{5}$$

where the sum over c on the right is over distinct two-body channels (including, for the present, sums over spin projections). Then the right-hand side of Eq. (2) becomes

$$\sum_c \frac{p_c}{16\pi^2 W} \int d\Omega_c \, \mathcal{M}^*_{cb} \, \mathcal{M}_{ca} \tag{6}$$

where p_c is the center of mass momentum of the particles in state c with total energy W. The statement of unitarity for two-body channels then reads

$$4\pi i(\mathcal{M}_{ba} - \mathcal{M}^*_{ab}) = \sum_c \frac{p_c}{W} \cdot \int \frac{1}{4\pi} \, d\Omega_c \, \mathcal{M}^*_{cb} \, \mathcal{M}_{ca} \tag{7}$$

where the sum on the right is over all open channels.

D. Unitarity for Helicity Amplitude Partial Waves

Conservation of angular momentum allows transformation of Eq. (7) into a separate equation for each partial wave of angular momentum j. We will choose the helicity representation and identify the channel label a as standing for a definite pair of particles (1, 2) with masses (m_1, m_2), spins (s_1, s_2) and helicities (λ_1, λ_2). Similarly b represents particles (3, 4) with masses, spins and helicities, (m_3, m_4), (s_3, s_4), (λ_3, λ_4). Where necessary more specific labeling will be used. The helicity amplitudes can be expanded in partial waves as[10]

$$\mathcal{M}_{ba} = \sum_j (j + \tfrac{1}{2}) \langle b|F^j|a\rangle \, D^{j\,*}_{\lambda_a \lambda_b}(R_{ba}) \tag{8}$$

where $\lambda_a = \lambda_1 - \lambda_2$, $\lambda_b = \lambda_3 - \lambda_4$ and the rotation R_{ba} transforms a unit vector in the direction of $\vec{p}_1$ in the center of mass into a unit vector in the direction of $\vec{p}_3$. Customarily, the Euler angles of R_{ba} are chosen to be $(\varphi, \theta, -\varphi)$,[10] or $(\varphi, \theta, 0)$,[11] or $(0, \theta, 0)$. But

we keep it as R_{ba} here in order to exploit the group property of rotations in integrating over angles.

On the left-hand side of Eq. (7) we also need an expansion of $\mathcal{M}^*_{ab}$:

$$\mathcal{M}^*_{ab} = \sum_j (j + \tfrac{1}{2}) \langle a|F^j|b\rangle^* D^j_{\lambda_b \lambda_a}(R^{-1}_{ba}) \tag{9}$$

Since R is unitary, $D^j_{\lambda\mu}(R^{-1}) = D^{j*}_{\mu\lambda}(R)$. Therefore the left-hand side of Eq. (7) can be written

$$\text{LHS} = 4\pi i \sum_j (j + \tfrac{1}{2}) \left[\langle b|F^j|a\rangle - \langle a|F^j|b\rangle^*\right] D^{j\,*}_{\lambda_a \lambda_b}(R_{ba}) \tag{10}$$

Substitution of partial wave series like Eqs. (8) and (9) into the right-hand side of Eq. (7) gives

$$\text{RHS} = \sum_{c,\lambda_c} \frac{p_c}{W} \sum_{j,j'} (j + \tfrac{1}{2})(j' + \tfrac{1}{2}) \langle c|F^{j'}|b\rangle^* \langle c|F^j|a\rangle \times I \tag{11}$$

where I is the angular integral,

$$I = \frac{1}{4\pi} \int d\Omega_c \, D^{j'}_{\lambda_b \lambda_c}(R_{cb}) \, D^{j\,*}_{\lambda_a \lambda_c}(R_{ca}) \tag{12}$$

Now R_{cb} can be written as the product of two rotations, namely $R_{cb} = R^{-1}_{ba} R_{ca}$.[12] The group property of rotations thus allows us to write

$$I = \sum_\lambda D^{j'}_{\lambda_b \lambda}(R^{-1}_{ba}) \cdot \frac{1}{4\pi} \int d\Omega_c \, D^{j'}_{\lambda \lambda_c}(R_{ca}) \, D^{j\,*}_{\lambda_a \lambda_c}(R_{ca})$$

The integral is just the orthogonality integral for the D-functions and I becomes

$$I = \delta_{jj'} \frac{1}{2j+1} D^{j}_{\lambda_b \lambda_a}(R^{-1}_{ba}) = \delta_{jj'} \frac{1}{2j+1} D^{j\,*}_{\lambda_a \lambda_b}(R_{ba})$$

When this value of I is inserted into Eq. (11), the result is

$$\text{RHS} = \sum_{c,\lambda_c} \frac{p_c}{2W} \sum_{j} (j + \tfrac{1}{2}) \langle c|F^j|b\rangle^* \langle c|F^j|a\rangle D^{j\,*}_{\lambda_a \lambda_b}(R_{ba}) \qquad (13)$$

Equating the LHS from Eq. (10) to the RHS from Eq. (13), term by term in the summation over j, gives the partial wave statement of unitarity:

$$8\pi i \left[\langle b|F^j|a\rangle - \langle a|F^j|b\rangle^* \right] = \sum_{c} \frac{p_c}{W} \sum_{\lambda_c} \langle c|F^j|b\rangle^* \langle c|F^j|a\rangle \qquad (14)$$

Time-reversal invariance implies that $\langle b|F^j|a\rangle = \langle a|F^j|b\rangle$. Then Eq. (14) becomes

$$8\pi i \left[\langle b|F^j|a\rangle - \langle b|F^j|a\rangle^* \right] = \sum_{c} \frac{p_c}{W} \sum_{\lambda_c} \langle b|F^j|c\rangle^* \langle c|F^j|a\rangle \qquad (15)$$

E. Parity-Conserving Amplitudes

A further step can be taken so that parity, as well as angular momentum and helicities, are well-defined for the amplitudes entering the statement of unitarity. We define linear combinations of the partial wave amplitudes as follows:

$$\langle \lambda_3\lambda_4|F^{j\pm}|\lambda_1\lambda_2\rangle = \langle \lambda_3\lambda_4|F^{j}|\lambda_1\lambda_2\rangle \pm \eta_1\eta_2(-1)^{s_1+s_2-v}\langle \lambda_3\lambda_4|F^{j}|-\lambda_1-\lambda_2\rangle ,$$

where η_i is the intrinsic parity of particle i, s_i is its spin, and $v = 0\ (\frac{1}{2})$ for j integer (odd half integer). These amplitudes have parity $P = \pm(-1)^{j-v}$ and so are appropriate for Regge theory since a Regge pole amplitude is equivalent to a linear combination of partial waves of one or the other of the parity sequences.[13]

Inspection of Eq. (15) shows that unitarity can be expressed for the parity-conserving amplitudes as

$$8\pi i\left[\langle b|F^{j\pm}|a\rangle - \langle b|F^{j\pm}|a\rangle^*\right] = \sum_c \frac{p_c}{W}\sum_{\lambda_c}\langle b|F^{j}|c\rangle^*\,\langle c|F^{j\pm}|a\rangle \quad (16)$$

The parity-conserving amplitudes can equally well be written as linear combinations of $\langle b|F^j|a\rangle$ and $\langle -b|F^j|a\rangle$, rather than in terms of $\langle b|F^j|a\rangle$ and $\langle b|F^j|-a\rangle$, where a minus sign denotes opposite helicities. Thus it is clear that an equally acceptable form is

$$8\pi i\left[\langle b|F^{j\pm}|a\rangle - \langle b|F^{j\pm}|a\rangle^*\right] = \sum_c \frac{p_c}{W}\sum_{\lambda_c}\langle b|F^{j\pm}|c\rangle^*\,\langle c|F^{j}|a\rangle. \quad (17)$$

In fact, Eqs. (16) and (17) can be combined to give a statement of unitarity entirely in terms of parity-conserving partial-wave amplitudes:

$$16\pi i\left[\langle b|F^{j\pm}|a\rangle - \langle b|F^{j\pm}|a\rangle^*\right] = \sum_c \frac{p_c}{W}\sum_{\lambda_c}\langle b|F^{j\pm}|c\rangle^*\,\langle c|F^{j\pm}|a\rangle \quad (18)$$

The left side of (18) is proportional to the imaginary part of the partial wave amplitude, while the right-hand side itemizes the contributions from the various open channels c.

III. EXTENDED UNITARITY AND FACTORIZATION OF RESIDUES

A. Extended Unitarity

In the theory of the analytic S-matrix and Regge poles we wish to generalize to complex angular momentum and complex energies. For this purpose it is necessary to extract certain kinematic factors, e.g., $(pp')^j$, from the partial wave amplitudes before continuation in angular momentum and/or energy. We thus write

$$\langle b|F^{j\pm}|a\rangle = K^{\pm}_{ba}(j,\ t)\ \tilde{F}^{\pm}_{ba}(j,\ t+i\epsilon) \tag{19}$$

where we have exhibited the dependence on $t = W^2$, the square of the center of mass energy, explicitly, and have indicated the physical value (just above the unitarity cut) by $+i\epsilon$. The factor $K^{\pm}_{ba}$ is an explicit kinematic function of j and t, containing threshold factors like $(pp')^j$ and other kinematic singularities specified in detail in Section IV below. The tilde amplitudes $\tilde{F}$ are assumed to be Hermitean analytic and suitable for continuation in j and t. We first consider physical t values, but complex j. Then we have

$$\langle b|F^{j}|a\rangle^* = K^{*}_{ba}\left[\tilde{F}_{ba}(j^*,\ t+i\epsilon)\right]^* = K^{*}_{ba}\ \tilde{F}_{ba}(j,\ t-i\epsilon) \tag{20}$$

where the last step follows from Hermitean analyticity. In Eq. (20) and subsequently we omit the parity superscript for simplicity of notation. The meaning of $(t - i\epsilon)$ in (20) is that a path is taken from the position $(t + i\epsilon)$ above the unitarity cut to the left to beyond the lowest physical threshold branch point, around the branch point

counterclockwise and back to the right below the unitarity cut to a position just below $t + i\epsilon$. Thus $t \pm i\epsilon$ are points on the physical sheet, above and below the real axis.

We now consider complex energies. If t moves away from the real axis in an upward direction, $(t + i\epsilon)$ moves to an arbitrary position t on the physical sheet, called sheet I. But $(t - i\epsilon)$ moves through the unitarity branch cut onto another sheet which we will call sheet II. For complex t we denote a quantity ξ on sheet I and II by $(\xi)^{I}$ and $(\xi)^{II}$, respectively. With these definitions the statement of unitarity, Eq. (18), has as its generalization,

$$K_{ba}\,\tilde{F}^{I}_{ba}(j,t) - K^{*}_{ba}\,\tilde{F}^{II}_{ba}(j,t) = \frac{1}{16\pi i}\sum_{c}\frac{p_c}{W}\sum_{\lambda_c} K^{*}_{bc}\,\tilde{F}^{II}_{bc}(j,t)\,K_{ca}\,\tilde{F}^{I}_{ca}(j,t) \qquad (21)$$

Because of our definition of sheets I and II the right-hand side of Eq. (21) contains contributions from all open channels. But it can be shown that the discontinuity obtained by a counterclockwise circuit around one arbitrary threshold branch point is given by the appropriate term from the sum.[14]

B. Factorization of Residues of a Pole

In perturbation theory factorization is an automatic consequence of the concepts of vertices and propagators, with a definite coupling constant attached to each vertex. Thus, for the processes, $a \to a$,

$b \to b$, and $a \to b$, the contributions from a single particle intermediate state (pole terms) will be proportional respectively to g_a^2, g_b^2, and $g_a g_b$. Furthermore, the spin structure of the vertices at each end of the intermediate particle line determines the helicity dependence for the initial and final states separately. This means that a factorization property will also hold for the pole contributions to the amplitudes of an elastic reaction with different helicities.

A corresponding statement of factorization follows in S-matrix theory from unitarity. Our discussion parallels the original one of Gribov and Pomeranchuk,[2] but with generalization to include spin. We begin with the unitarity equation (21), but with sheet II defined by a circuit around the branch point of a definite channel c. Then the sum over channels in (21) reduces to a single term and only the sum over the helicities of channel c remains. A slight distinction needs to be made between the unitarity equation when one of the channels (a,b) is equal to c and when it is not. We first consider a process, $c \to d$, where d is arbitrary. Then Eq. (21) can be written as

$$K_{dc} \tilde{F}^{I}_{dc} = \sum_{c'} K^{*}_{dc'} \tilde{F}^{II}_{dc'} (\delta_{c'c} - i\rho_c K_{c'c} \tilde{F}^{I}_{c'c})$$

where we have used the subscripts c and c' to indicate different helicities in channel c, and we have introduced the phase space factor $\rho_c = p_c/16\pi W$. The amplitudes on sheet II can be written in terms of those on sheet I as

$$K^*_{dc}\,\tilde{F}^{II}_{dc} = \sum_{c'} K_{dc'}\,\tilde{F}^{I}_{dc'}\,(S^{-1})_{c'c} \tag{22}$$

where S^{-1} is the inverse of the finite-dimensional symmetric matrix (the S-matrix!),

$$S_{c'c} = \delta_{c'c} - i\rho_c\,K_{c'c}\,\tilde{F}^{I}_{c'c}\,. \tag{23}$$

For a process, $a \to b$, not directly involving channel c, the unitarity equation (21), plus Eq. (22), can be used to express the amplitudes on sheet II in terms of those on sheet I:

$$K^*_{ba}\,\tilde{F}^{II}_{ba} = K_{ba}\,\tilde{F}^{I}_{ba} + i\rho_c \sum_{c'}\sum_{c''} K_{bc''}\,\tilde{F}^{I}_{bc''}\,(S^{-1})_{c''c'}\,K_{c'a}\,\tilde{F}^{I}_{c'a} \tag{24}$$

We first assume that channel c is spinless. Then Eqs. (22) and (24) do not involve any sums over helicities. Now suppose there is a Regge pole of definite quantum numbers on sheet II for $j = \alpha(t)$. This pole will occur in S^{-1} on the right-hand side of (24) and the residue β^{II}_{ba} of $K^*_{ba}\,\tilde{F}^{II}_{ba}$ will be of the factorized form,

$$\beta^{II}_{ba} = i\rho_c\,(K_{bc}\,\tilde{F}^{I}_{bc})\,\sigma_c\,(K_{ca}\,\tilde{F}^{I}_{ca})$$

where σ_c is the residue of S^{-1}. The label c merely denotes what threshold branch point has been encircled and therefore defines sheet II. By analytic continuation the residues can be obtained on the physical sheet. They satisfy the typical factorization equation,[1,2,3]

$$\beta^2_{ba} = \beta_{bb}\,\beta_{aa} \tag{25}$$

We have thus established factorization of Regge pole residues provided that there exists at least one spinless two-body channel which communicates to the others.

If the particles in channel c possess spin the threshold is degenerate, with the sub-channels of different helicity all having the same mass. The sums over helicities in (22) and (24) now remain and the pole in S^{-1} at $j = \alpha(t)$ comes from $\det S = 0$. In order to prove factorization of pole residues it is necessary to assume that the zero of $\det S$ at $j = \alpha(t)$ is simple. This assumption, sometimes described as the absence of "accidental degeneracy", is eminently plausible - it holds in potential theory, and corresponds to the pole occurring in only one eigenamplitude of the S-matrix.[15] By imagining S in diagonal form, it is easy to see that a simple zero of $\det S$ implies that only one element of the diagonal S^{-1} has the pole. Consequently the singular part of the nondiagonal S^{-1} can be written in factorized form:

$$(S^{-1})_{c'c} = \frac{\xi_{c'}\, \xi_c}{j-\alpha} + \text{regular}$$

From Eqs. (22) and (24) the residues of the pole are found to be

$$\begin{aligned} \beta^{II}_{dc} &= C_d\, \xi_c \\ \beta^{II}_{ba} &= i\rho_c\, C_b\, C_a \end{aligned} \tag{26}$$

where

$$C_x = \sum_{c'} K_{xc'}\, \tilde{F}^{I}_{xc'}\, \xi_{c'}$$

These residues satisfy the general factorization statement,

$$\beta_{ab}\,\beta_{cd} = \beta_{ac}\,\beta_{bd} \tag{27}$$

where now channel c is not necessarily the channel whose branch point was encircled to get on to sheet II.

Note that the β's are the residues of the pole in the full partial wave amplitudes, i.e., the coefficients of the Wigner D-functions in Eq. (8). In order to be explicit and careful about their kinematic singularities we write

$$\beta_{ba} = K_{ba}\,\gamma_{ba}$$

where the kinematic factor K_{ba} is defined in Eq. (19) and specified in detail in the next Section, and γ_{ba} is a reduced residue of the Regge pole. The γ's are residues in the tilde amplitudes and so are supposed to have good analytic properties. We will see that they are normally analytic, but may contain poles at $t = 0$.

Specialization to the two common types of factorization can be made easily. For clarity we now exhibit helicities explicitly, using (α, α') for the helicities in channel a, etc. With $c = a$, $d = b$, $(\alpha, \alpha') = (\gamma, \gamma')$, $(\delta, \delta') = (\beta, \beta')$, we obtain the familiar result relating the residues for the processes, $a \to b$, $a \to a$ and $b \to b$:

$$\left[K^{ba}_{\beta\beta';\alpha\alpha'}\,\gamma^{ba}_{\beta\beta';\alpha\alpha'}\right]^2 = K^{aa}_{\alpha\alpha';\alpha\alpha'}\,\gamma^{aa}_{\alpha\alpha';\alpha\alpha'}\,K^{bb}_{\beta\beta';\beta\beta'}\,\gamma^{bb}_{\beta\beta':\beta\beta'} \tag{28}$$

With $a = b = c = d$, but helicities $(\alpha, \alpha') = (\gamma, \gamma') = (\alpha_1, \alpha_2)$ and $(\beta, \beta') = (\delta, \delta') = (\alpha_3, \alpha_4)$, we get the relation for elastic scattering amplitudes of different helicities:

$$\left[K^{aa}_{\alpha_3\alpha_4;\alpha_1\alpha_2}\ \gamma^{aa}_{\alpha_3\alpha_4;\alpha_1\alpha_2}\right]^2 = K^{aa}_{\alpha_1\alpha_2;\alpha_1\alpha_2}\ \gamma^{aa}_{\alpha_1\alpha_2;\alpha_1\alpha_2}$$

$$\times\ K^{aa}_{\alpha_3\alpha_4;\alpha_3\alpha_4}\ \gamma^{aa}_{\alpha_3\alpha_4;\alpha_3\alpha_4} \qquad (29)$$

It should be emphasized that the factorization statements, (28) - (29) hold for the products of the kinematic singularity factors K and the reduced residues γ. For consistency, the threshold kinematic singularity structure on both sides of the equation must be the same, and, after cancellation of these factors, the reduced residues must satisfy "analytic factorization". In practice this means that all powers of the momenta (and helicity-independent factors intrinsic to the pole itself, e.g., $\Gamma(\alpha + \frac{3}{2})$) will cancel, leaving only integral powers of t, or possibly W for fermion poles, in the relation for reduced residues. To accomplish these ends care must be taken to include all kinematically necessary singularities and zeros in the K's. This is spelled out in detail in the next Section.

IV. KINEMATIC SINGULARITY FACTORS AND FACTORIZATION FOR REDUCED RESIDUES

The kinematic singularity structure of helicity amplitudes has been investigated thoroughly from a number of points of view[16,17,18,19] since the original work of Hara[20] and Wang.[21] We will follow the notation of Ref. 18 for the specification of the singularity structure; it corresponds closely to the conventions of Gell-Mann et al.[13]

A. Restricted Range of Helicity Values

We consider normal helicity amplitudes with definite helicities in the initial and final states. This means that the amplitudes are linear combinations of so-called parity conserving amplitudes:

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t, \theta_t) = \left[\sqrt{2}\cos\frac{\theta_t}{2}\right]^{|\lambda+\mu|}\left[\sqrt{2}\sin\frac{\theta_t}{2}\right]^{|\lambda-\mu|} \times\frac{1}{2}\left\{F^{+}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t, z_t) + F^{-}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(t, z_t)\right\} \qquad (30)$$

where F^{+} and F^{-} are given by Eq. (15) of Ref. 18 or Eq. (2.7) of Ref. 13, and are functions of the energy and $z_t = \cos\theta_t$; $\lambda = \lambda_1 - \lambda_2$ and $\mu = \lambda_3 - \lambda_4$. The amplitudes $F^{\pm}$ are dominated for large z_t by contributions from natural and unnatural parity sequences, respectively ($\eta = \pm 1$). If either λ or μ is equal to zero, the correlation of $\pm$ with parity sequence is exact, and not only an asymptotic property.

The pole whose residues enter the factorization equations has a definite parity, and so occurs in either F^{+} or F^{-}, but not in

both (to leading order in z_t). This means that it is sufficient to choose both λ and μ non-negative. Amplitudes or residues with negative values of λ and/or μ can be obtained from those with $\lambda, \mu \geq 0$ by symmetry considerations, as is discussed in Appendix D of Ref. 18. This restriction on the ranges of λ and μ makes the discussion of the kinematic singularities simpler and less confusing.

B. Different Kinematic Classes

The discussion of the kinematic factors $K^{ba}_{\beta\beta';\alpha\alpha'}$ is conveniently divided into classes, according to the locations of the various thresholds. For the t-channel process, $a \to b$ with masses $m_a + m'_a \to m_b + m'_b$, kinematic singularities occur in general at the two normal thresholds, $t = (m_a + m'_a)^2$, $(m_b + m'_b)^2$, at the two pseudothresholds, $t = (m_a - m'_a)^2$, $(m_b - m'_b)^2$, and at $t = 0$. If there are equalities among the masses, some of the five singular points can coincide. We will not consider "chance" equalities, such as $(m_a - m'_a)^2 = (m_b + m'_b)^2$. Furthermore, we assume that two particles of the same mass have the same spin, although the parities may be different (e.g., NN and $\bar{N}N$). We distinguish the following four classes (E and U stand for equal and unequal masses, respectively):

1. U → U' $[m_a \neq m'_a, m_b \neq m'_b]$

   The point $t = 0$ is distinct from the thresholds. It is possible, however, that the initial and final thresholds might coincide.

2. $\underline{E \to U}$ $[m_a = m'_a,\ m_b \neq m'_b]$

   The initial pseudothreshold is at $t = 0$.

3. $\underline{E \to E'}$ $[m_a = m'_a,\ m_b = m'_b]$

   Both pseudothresholds are at $t = 0$, but the mass, spin and parities of the equal mass particles in the initial state may be different from those in the final state.

4. $\underline{E \to E}$

   This class is the same as 3, but with the same masses, spins and parities initially and finally.

These four classes include all the common situations. For a general discussion of the kinematic singularities, including "chance" equalities of thresholds, see the paper by Kotanski.[19]

For the process, $a \to b$, we introduce the following notation:

(a) Kinematics: t, p_a, p_b are the square of the total energy, the magnitudes of the initial momenta and the final momenta in the center of mass frame, respectively.

(b) Parities: $\eta_a = \eta_1\eta_2$, $\eta_b = \eta_3\eta_4$ are the products of the intrinsic parities of the particles in the initial and final states, respectively. $\eta = \pm 1$ denotes the parity sequence ($P = (-1)^{j-v}\eta$) of the pole. $\eta = +1$ (-1) is called natural (unnatural) parity.

(c) Spins: (s_a, s'_a) and (s_b, s'_b) are the intrinsic spins of the particles in the channels a and b, respectively.

(d) Helicities: (α, α') and (β, β') are the helicities in the initial and final states, respectively. $\lambda_a = \alpha - \alpha'$,

$\lambda_b = \beta - \beta'$. $m = \max[\lambda_a, \lambda_b]$, $n = \min[\lambda_a, \lambda_b]$. (Remember that $\lambda_a, \lambda_b \geq 0$, by choice.)

C. Kinematic Factors and Factorization of Reduced Residues for $U \to U'$

The factors K^{ba} contain the standard kinematic singularities[16-21] and also the threshold behavior, $(p_a p_b)^{\alpha - m}$, where $\alpha(t)$ is the trajectory of the Regge pole. For $U \to U'$, we have

$$K^{ba}_{\beta\beta';\alpha\alpha'} = \left(\frac{1}{\sqrt{t}}\right)^{m-n} (t p_a p_b)^m (p_a p_b)^{\alpha - m} \; K_o{}^a \; K_o{}^b \tag{31}$$

where $K_o{}^a \; K_o{}^b$ is the kinematic singularity factor for $\lambda_a = \lambda_b = 0$, and is clearly factorizable. Explicitly,[18]

$$K_o{}^a = \left[t - (m_a + m'_a)^2\right]^{-\frac{A_N}{2}} \left[t - (m_a - m'_a)^2\right]^{-\frac{A_P}{2}}$$
$$K_o{}^b = \left[t - (m_b + m'_b)^2\right]^{-\frac{B_N}{2}} \left[t - (m_b - m'_b)^2\right]^{-\frac{B_P}{2}} \tag{32}$$

where

$$A_N = s_a + s'_a - \tfrac{1}{2}\left(1 - \eta\eta_a(-1)^{s'_a + s_a - v}\right)$$
$$A_P = s_a + s'_a - \tfrac{1}{2}\left(1 - \eta\eta_a(-1)^{s'_a - s_a - v}\right)$$
$$B_N = s_b + s'_b - \tfrac{1}{2}\left(1 - \eta\eta_b(-1)^{s'_b + s_b - v}\right) \tag{33}$$
$$B_P = s_b + s'_b - \tfrac{1}{2}\left(1 - \eta\eta_b(-1)^{s'_b - s_b - v}\right)$$

In writing these expressions we have assumed that $m_a < m'_a$, $m_b < m'_b$. In Eq. (31) the $\sqrt{t}$ singularity is different from that generally quoted,[16-21] namely $(1/\sqrt{t})^{m+n}$. The reason for the difference is discussed in Ref. 18, above their Eq. (24). For $U \to U'$ the normal helicity amplitude, Eq. (30), in general is regular and finite at $t = 0$. The $\sqrt{t}$ singularity in K^{ba} is present merely to compensate for the $\sqrt{t}$ behavior coming from $\left[\sin(\theta_t/2)\right]^{|\lambda-\mu|}$ as a consequence of $\cos\theta_t \to 1$ as $t \to 0$. This is spelled out in detail in Sect. VD.

Inspection of Eq. (31) shows that K^{ba} itself is factorizable:

$$K^{ba}_{\beta\beta';\alpha\alpha'} = \left[(\sqrt{t})^{\lambda_a} p_a^{\alpha} K_o^a\right] \cdot \left[(\sqrt{t})^{\lambda_b} p_b^{\alpha} K_o^b\right] \qquad (34)$$

This means the kinematic factors in Eq. (28) and in (29) cancel out and leave the reduced residues for $U \to U'$ satisfying the factorization relation,

$$\left[\gamma^{ba}_{\beta\beta';\alpha\alpha'}\right]^2 = \gamma^{aa}_{\alpha\alpha';\alpha\alpha'}\ \gamma^{bb}_{\beta\beta';\beta\beta'} \qquad (35)$$

Eq. (35) holds, of course, in the absence of spin, regardless of the masses. It was first explicitly derived for $U \to U'$ processes with spin by Frautschi and Jones.[7]

It is worthwhile to note here that the treatment of this Section can be applied almost without change to unequal mass processes involving photons, e.g., $\bar{N}\Delta \to \gamma\pi$, $\bar{Y}N \to \gamma K^*$, or in the following Section to the unequal mass side of processes like $\bar{N}N \to \gamma V$, $\gamma\pi$. The kinematic singularities for photon amplitudes are discussed by Ader, Capdeville

and Navelet.[22] One finds that, if channel b consists of a photon and a massive particle of spin s_b, the structures of Eqs. (31) and (36) are the same, with K_o^b given by Eq. (32) with $B_N = B_P = s_b$ and $m'_b = 0$. That is, $K_o^b(\text{photon + particle}) = (\sqrt{t}\, p_b)^{-s_b}$. This singularity can be understood simply in terms of the multipolarity of photon transitions.[23]

D. Kinematic Factors and Factorization of Reduced Residues for $E \to U$

For $E \to U$ we must consider separately K^{ba}, K^{aa} and K^{bb}. With $m_a = m'_a$, the kinematic factors are[4,16,21]

$$\left.\begin{aligned} K^{ba}_{\beta\beta';\alpha\alpha'} &= (\sqrt{t})^{\xi+m}\ (p_a p_b)^{\alpha} \left(\frac{1}{p_a}\right)^{A_N} K_o^b \\ K^{aa}_{\alpha\alpha';\alpha\alpha'} &= \left(p_a^2\right)^{\alpha - A_N} \\ K^{bb}_{\beta\beta';\beta\beta'} &= t^{\lambda_b} \left(p_b^2\right)^{\alpha} \left(K_o^b\right)^2 \end{aligned}\right\} \quad (36)$$

where $\xi = \frac{1}{2}\left(1 - \eta\eta_a(-1)^{\lambda_a - \lambda_b + m}\right) - 2s_a$

and A_N is given by Eq. (33) with $s_a = s'_a$; K_o^b is given by Eq. (32). It should be remarked that K^{aa} does not follow from the singularity structure given by Cohen-Tannoudji, Morel and Navelet[16] in their Table X . They give the maximum singularity, independent of parity. For a definite η, their regularized amplitudes may have zeros at $p_a = 0$. The correct kinematic behavior at a normal threshold is always given by the first factors on the right in Eq. (32), regardless

of the masses. Another point worthy of note is that, for $E \to E$, K^{aa} has the same form whether or not the helicities are the same initially and finally. For $U \to U$, K^{bb} can be generalized to the situation of different helicities initially and finally by inspection of Eq. (34) with $a = b$.

When the kinematic factors, (36), are inserted into Eq. (28) the result for $E \to U$ is

$$\left.\begin{aligned} t^x \left[\gamma^{ba}_{\beta\beta';\alpha\alpha'}\right]^2 &= \gamma^{aa}_{\alpha\alpha';\alpha\alpha'}\ \gamma^{bb}_{\beta\beta';\beta\beta'} \\ \text{where} \qquad & \\ x &= m - \lambda_b - 2s_a + \frac{1}{2}\left(1 - \eta\eta_a(-1)^n\right) \end{aligned}\right\} \quad (37)$$

We see that the powers of momenta all cancel, but that there remains an integral power of t on the left-hand side.[24]

E. Kinematic Factors and Factorization of Reduced Residues for $E \to E'$ and $E \to E$

For the situation with $m_a = m'_a$, $m_b = m'_b$, but $m_a \neq m_b$, the kinematic factor K^{ba} is given by[16]

$$\left.\begin{aligned} K^{ba}_{\beta\beta';\alpha\alpha'} &= (\sqrt{t})^y\ (p_a)^{\alpha - A_N}\ (p_b)^{\alpha - B_N} \\ \text{where} \qquad & \\ y &= \frac{1}{2}\left[1 - \eta_a\eta_b(-1)^{\alpha+\alpha'+\beta+\beta'}\right] \end{aligned}\right\} \quad (38)$$

and A_N, B_N are given by Eq. (33) with $s_a = s'_a$, $s_b = s'_b$.

The factors K^{aa} and K^{bb} are $K^{aa} = \left(p_a^2\right)^{\alpha - A_N}$, $K^{bb} = \left(p_b^2\right)^{\alpha - B_N}$. Combining these expressions with Eq. (28), we obtain the reduced residue relation for $E \to E'$:

$$t^y \left[\gamma^{ba}_{\beta\beta';\alpha\alpha'}\right]^2 = \gamma^{aa}_{\alpha\alpha';\alpha\alpha'}\, \gamma^{bb}_{\beta\beta';\beta\beta'} \tag{39}$$

Note that the limit $m_a = m_b$ (all four masses equal) is allowable here, and the case of the same particles initially and finally is included in (38) with $\eta_a \eta_b = 1$.

V. EXAMPLES OF FACTORIZATION AND ITS CONSEQUENCES

In this Section we present some examples of factorization that hopefully clarify the interrelationships of factorization, kinematic singularities, different conspiracy schemes and daughter trajectories.

A. LeBellac's Example

LeBellac[6] demonstrated the use of factorization for statements about the behavior of residues of the pion trajectory in various processes at $t = 0$. The basic idea is that anomalous behavior, e.g., conspiracy instead of evasion, in one process will, because of factorization, propagate and give anomalous behavior (of an inverse type) in some other process. The reactions considered by LeBellac are $\bar{N}N \to \pi\rho$, $\bar{N}N \to \bar{N}N$, $\pi\rho \to \pi\rho$, $\bar{N}\Delta \to \pi\rho$ and $\bar{N}\Delta \to \bar{N}\Delta$. The first three processes are related by factorization of the $E \to U$ class (Sect. IVD), while the last three are of the $U \to U'$ class (Sect. IVC). We consider helicities $\lambda_{\bar{N}} = \lambda_N = \lambda_\Delta = \frac{1}{2}$ and $\lambda_\pi = \lambda_\rho = 0$. Furthermore, we consider in this Section that the pion trajectory is a leading trajectory, i.e., a parent. This was LeBellac's assumption, and seems likely in nature. In the next Section we will show how LeBellac's conclusions are altered if the pion is assumed to be the daughter of an A_1-like parent with which it conspires.

With $\lambda_a = \lambda_b = m = n = 0$ and $\eta = -1$ we find from Eqs. (37) that the pion's reduced residues for $\bar{N}N \to \pi\rho$, $\bar{N}N \to \bar{N}N$ and $\pi\rho \to \pi\rho$ are related by

$$\frac{1}{t}\left[\gamma^{\pi\rho \leftarrow \bar{N}N}_{00;++}\right]^2 = \gamma^{\bar{N}N \leftarrow \bar{N}N}_{++;++}\ \gamma^{\pi\rho \leftarrow \pi\rho}_{00;00} \qquad (40)$$

Under our assumptions the reduced residues are analytic in the neighborhood of $t = 0$. This means that $\gamma^{\pi\rho\leftarrow\bar{N}N}_{oo;++}$ must be proportional to some positive power of t. We thus write

$$\gamma^{\pi\rho\leftarrow\bar{N}N}_{oo;++} = t\,\bar{\gamma}^{\pi\rho\leftarrow\bar{N}N}_{oo;++}$$

Note that factorization has converted a $1/\sqrt{t}$ kinematic singularity (see Eqs. (36)) into a $\sqrt{t}$ kinematic singularity. The kinematic structure of the residues of L. L. Wang,[4] shown in her Eqs. (18a) and (18b), have this factorization requirement built in. Eq. (40) now becomes

$$t\left[\bar{\gamma}^{\pi\rho\leftarrow\bar{N}N}_{oo;++}\right]^2 = \gamma^{\bar{N}N\leftarrow\bar{N}N}_{++;++}\;\gamma^{\pi\rho\leftarrow\pi\rho}_{oo;oo} \qquad (41)$$

There is evidence from np charge exchange scattering[25] that the residue $\gamma^{\bar{N}N\leftarrow\bar{N}N}_{++;++}$ does not vanish at $t = 0$, as expected from perturbation theory or normal (evasive) Regge poles. Within the framework of Regge poles alone, it is necessary to invoke some type of "conspiracy," e.g., to postulate the existence of another trajectory $\alpha_c(t)$ with all the same quantum numbers as the pion, except parity, and to demand that $\alpha_c(0) = \alpha_\pi(0)$. This is called type III conspiracy in the notation of Freedman and Wang[26] and $M = 1$ conspiracy in the notation of Toller.[27] If $\gamma^{\bar{N}N\leftarrow\bar{N}N}_{++;++}$ does not vanish at $t = 0$, factorization, (41), forces $\gamma^{\pi\rho\leftarrow\pi\rho}_{oo;oo}$ to be proportional to t.

Now consider factorization for $\bar{N}\Delta \to \pi\rho$, $\bar{N}\Delta \to \bar{N}\Delta$ and $\pi\rho \to \pi\rho$. From Eq. (35) we have

$$\left[\gamma^{\pi\rho\leftarrow\bar{N}\Delta}_{oo;++}\right]^2 = \gamma^{\bar{N}\Delta\leftarrow\bar{N}\Delta}_{++;++}\,\gamma^{\pi\rho\leftarrow\pi\rho}_{oo;oo} \qquad (42)$$

With the conclusion that $\gamma^{\pi\rho\leftarrow\pi\rho}_{oo;oo} \propto t$, and the requirement of analyticity of the reduced residues near $t = 0$, we see that both the other residues in (42) must also be proportional to t. We have thus reached the conclusion of LeBellac: The non-spin-flip residue for the reaction $\bar{N}\Delta\rightarrow\pi\rho$ must vanish at $t = 0$. Since the pion trajectory is believed to be important for this reaction and $\lambda_\rho = 0$ is known from the decay density matrix elements to be dominant, we are led to expect a dip in the cross section in the forward direction.

This powerful conclusion stemming from factorization (the lack of a dip at $t = 0$ in np charge exchange forces a dip in the inelastic reaction, $\pi N \rightarrow \rho\Delta$!) is especially curious because the data on $\pi N \rightarrow \rho\Delta$ do not support it. There are difficulties and ambiguities associated with the finite widths of the ρ and the Δ and the inaccessibility of the point $t = 0$. But a recent analysis at 8 GeV/c[28] shows, not a dip in the cross section at small momentum transfers, but rather a rise for $\Delta^2 \lesssim 2\, m_\pi^2$.

B. If the Pion Were a Daughter; the A_1 and its Daughter

Because of the basic nature of factorization of pole residues it is important to find loopholes in LeBellac's argument if the data do not support its conclusions. There are, of course, many ways out. The pion may not dominate processes like $\pi N \rightarrow \rho N$ and $\pi N \rightarrow \rho\Delta$ at

high energies and very small momentum transfers; the contributions of Regge cuts may mask the behavior of the pole residues, etc. But we discuss here circumstances by which the above conclusions are modified, still within the framework of Regge poles alone.

We consider the residues of a Regge pole with the quantum numbers of the pion, denoted by D, and also its parent, denoted by A. The parent trajectory $\alpha_A(t)$ and the daughter trajectory $\alpha_D(t)$ are related at $t = 0$ by $\alpha_A(0) \equiv \alpha$, $\alpha_D(0) = \alpha - 1$. Both trajectories contribute to various amplitudes in the processes involved in LeBellac's argument. Consequently we must look in more detail at the different reactions in order to determine the interconnections among the parent and daughter residues at $t = 0$.

Consider first $\bar{N}N \to \bar{N}N$. The D-pole contributes to the two amplitudes, $\varphi_1 = \langle ++|T|++\rangle$ and $\varphi_2 = \langle ++|T|--\rangle$, while the A-pole contributes to $\varphi_3 = \langle +-|T|+-\rangle$ and $\varphi_4 = \langle +-|T|-+\rangle$. The kinematic structure of the amplitudes is given by K^{aa} in Eq. (36) with $A_N = 0$. The asymptotic forms of the amplitudes are therefore

$$\varphi_1 \longrightarrow \frac{\gamma_D^{(1)}(t)}{\Gamma(\alpha_D(t)+1)} \left[\frac{1 + e^{-i\pi\alpha_D(t)}}{\sin \pi\, \alpha_D(t)}\right] \left(\frac{s}{s_o}\right)^{\alpha_D(t)}$$

$$\varphi_2 \longrightarrow -\varphi_1 \tag{43}$$

$$\varphi_3 \longrightarrow \frac{\gamma_A^{(1)}(t)}{\Gamma(\alpha_A(t)+1)}\, \alpha_A(t) \left(\alpha_A(t) + \frac{1}{z_t}\right) \left[\frac{1 - e^{-i\pi\alpha_A(t)}}{\sin \pi\, \alpha_A(t)}\right] \left(\frac{s}{s_o}\right)^{\alpha_A(t)}$$

$$\varphi_4 \rightarrow \left[\frac{\alpha_A(t) - \frac{1}{z_t}}{\alpha_A(t) + \frac{1}{z_t}}\right] \varphi_3$$

The superscript (1) on the residues identifies them as residues for $\bar{N}N \rightarrow \bar{N}N$. The factors of $\alpha_A(t)$ in φ_3 correspond to choosing sense at the "nonsense-nonsense" point, $J = 0$. In Eq. (43) we have kept the first-order corrections to the leading terms for the A-pole because they have the same s-dependence at $t = 0$ as the D-pole contributions. At $t = 0$ the amplitudes satisfy a GGMW relation,[29]

$$\varphi_1 - \varphi_2 = \varphi_3 - \varphi_4 \qquad [t = 0]$$

Unless all the amplitudes vanish at $t = 0$ (evasion), the residues must be related (type II conspiracy) according to

$$\gamma_D^{(1)}(0) = \frac{2 m_N^2}{s_o} \gamma_A^{(1)}(0) \tag{44}$$

In the process $\pi\rho \rightarrow \pi\rho$ (denoted by superscript (2)) both parent and daughter poles contribute to the amplitude $f_{oo;oo}$. From Eqs. (31)-(33) we see that the kinematic singularity of this amplitude (with $\eta = -1$) is $1/(4tp_{\pi\rho}^2)$ times the threshold behavior, $(p_{\pi\rho}^2)^{\alpha}$. The asymptotic form of the contribution from the A-pole can thus be written

$$f_{oo;oo}^{(A)} \rightarrow \frac{1}{\tau^2} \frac{\gamma_A^{(2)}(t)}{\Gamma(\alpha_A(t)+1)} \left[\frac{1 - e^{i\pi\alpha_A(t)}}{\sin \pi\, \alpha_A(t)}\right] \left(\frac{2p_{\pi\rho}^2 z_t}{s_o}\right)^{\alpha_A(t)} \tag{45}$$

with corrections of order z_t^{-2}. Here

$$\tau^2 \equiv 4\, t\, p_{\pi\rho}^2 = \left[t^2 - 2(m_\rho^2 + m_\pi^2)t + (m_\rho^2 - m_\pi^2)^2\right]$$

and

$$2\, p_{\pi\rho}^2\, z_t = s + \frac{\tau^2}{2t}$$

Keeping only the $O(1/s)$ corrections to the leading power, Eq. (45) reads

$$f^{(A)}_{oo;oo} \to \frac{1}{\tau^2}\, \frac{\gamma_A^{(2)}(t)}{\Gamma(\alpha_A(t)+1)} \left[\frac{1 - e^{-i\pi\alpha_A(t)}}{\sin \pi\, \alpha_A(t)}\right] \left(\frac{s}{s_o}\right)^{\alpha_A(t)} \left[1 + \frac{\tau^2 \alpha_A(t)}{2st} + \cdots\right] \tag{46}$$

For the daughter pole we keep only the leading power of s. Its contribution is

$$f^{(D)}_{oo;oo} \to \frac{1}{\tau^2}\, \frac{\gamma_D^{(2)}(t)}{\Gamma(\alpha_D(t)+1)} \left[\frac{1 + e^{-i\pi\alpha_D(t)}}{\sin \pi\, \alpha_D(t)}\right] \left(\frac{s}{s_o}\right)^{\alpha_D(t)} \left[1 + \cdots\right] \tag{47}$$

For unequal masses, $\tau^2 \to (m_\rho^2 - m_\pi^2)^2$ as $t \to 0$ and the contribution of the A-pole diverges as $1/t$ with s-dependence, $s^{\alpha_A - 1}$. In order that the complete amplitude $f_{oo;oo}$ not be singular at $t = 0$ it is necessary that the leading contribution of the daughter pole cancel the offending part of (46). This is the daughter mechanism of Freedman and Wang[26] for ensuring Regge behavior at $t = 0$ for the $U \to U'$ class of masses. The daughter residues $\gamma_D^{(2)}(t)$ is therefore related to the parent residue $\gamma_A^{(2)}(t)$ in the following way:

$$\gamma_D^{(2)}(t) = \frac{1}{2}\frac{(m_\rho^2 - m_\pi^2)^2}{s_o\, t}\,\gamma_A^{(2)}(0) \;+\; \text{regular part} \qquad (48)$$

Here we see the first obvious difference from the discussion of the preceding Section. The residue of the pion-like D in $\pi\rho \to \pi\rho$ is now not analytic at $t = 0$, but instead has a pole. Evidently the argument leading from Eq. (40) to (41) no longer holds.

To complete the discussion we must now consider the reaction, $\bar{N}N \to \pi\rho$. The D-pole and A-pole contribute to the amplitudes $f_{oo;++}$ and $f_{oo;+-}$, respectively, as well as to others with $\lambda_\rho \neq 0$ that are of no interest to us here. With the kinematic singularity (36) we find the asymptotic form of the contribution of the D-pole to $f_{oo;++}$ to be

$$f_{oo;++} \longrightarrow \frac{1}{\tau\sqrt{t}}\,\frac{\gamma_D^{(3)}(t)}{\Gamma(\alpha_D(t)+1)}\left[\frac{1 + e^{-i\pi\alpha_D(t)}}{\sin\pi\,\alpha_D(t)}\right]\left(\frac{s}{s_o}\right)^{\alpha_D(t)} \qquad (49)$$

Similarly the A-pole contribution to $f_{oo;+-}$ is

$$f_{oo;+-} \longrightarrow \frac{p_{\bar{N}N}\,p_{\pi\rho}\sin\theta_t}{\tau\, s_o}\cdot\frac{2\alpha_A(t)\gamma_A^{(3)}(t)}{\Gamma(\alpha_A(t)+1)}\left[\frac{1 - e^{-i\pi\alpha_A(t)}}{\sin\pi\,\alpha_A(t)}\right]\left(\frac{s}{s_o}\right)^{\alpha_A(t)\,-\,1} \qquad (50)$$

At $t = 0$ the two amplitudes satisfy a conspiracy relation (actually a pseudothreshold relation) of the form,[30]

$$f_{oo;++} \;-\; \frac{1}{\sin\theta_t}\; f_{oo;+-} \;= O(\sqrt{t}\,) \qquad (51)$$

This constraint requires the residues in (49) and (50) to be related at $t = 0$:

$$\gamma_D^{(3)}(0) = \frac{m_N}{s_o} (m_\rho^2 - m_\pi^2) \gamma_A^{(3)}(0) \tag{52}$$

The factorization equations for the reduced residues of both the A-pole and the pionic D-pole can now be written down and compared. From Eq. (37), remembering that for the A-pole the $\bar{N}N$ system has helicities (+, -), we obtain

$$\left[\gamma_A^{(3)}\right]^2 = \gamma_A^{(1)} \gamma_A^{(2)} \tag{53}$$

where the superscripts (1,2,3) refer respectively to the processes, $\bar{N}N \to \bar{N}N$, $\pi\rho \to \pi\rho$ and $\bar{N}N \to \pi\rho$. Similarly for the residues of the D-pole, we have

$$\frac{1}{t}\left[\gamma_D^{(3)}\right]^2 = \gamma_D^{(1)} \gamma_D^{(2)} \tag{54}$$

Eq. (54) is just a rewriting of Eq. (40). As has already been mentioned, the pole at $t = 0$ in $\gamma_D^{(2)}$, given by Eq. (48), removes the necessity for $\gamma_D^{(3)}$ to be proportional to t, and Eq. (41) does not hold. In the second part of LeBellac's argument, the three $U \to U'$ processes, $\bar{N}\Delta \to \pi\rho$, $\bar{N}\Delta \to \bar{N}\Delta$, and $\pi\rho \to \pi\rho$, have reduced residues of the D-pole related according to Eq. (42). Now each residue has a Freedman-Wang pole at $t = 0$ and nothing crucial can be said about the "pionic" residues because of the presence of its parent.[31]

It can be verified easily that the connections, (44), (48), and (52), between parent and daughter residues at $t = 0$ make Eqs. (53)

and (54) equivalent at that point. We thus have the circumstance that conspiracy relations in $E \to E$ and $E \to U$ processes, plus factorization, determine the residues of the daughter trajectory in a $U \to U$ process. Given the existence of daughters, we can thus "prove" Regge asymptotic behavior at $t = 0$ for unequal masses. Alternatively, if an A_1 trajectory exists, then analyticity arguments for the amplitudes of $\pi\rho \to \pi\rho$ demand the existence of its daughter (not the pion), with a residue satisfying Eq. (48). Then a natural solution of Eqs. (53) and (54) is for both $\gamma_D^{(1)}$ and $\gamma_D^{(3)}$ to be finite at $t = 0$. This results from a parent-daughter conspiracy, and is no less reasonable than the "no conspiracy" solution in which $\gamma_D^{(1)} \propto t^2$, $\gamma_D^{(3)} \propto t$. In fact, the following possibility cannot be ruled out.[32] Suppose that the pion has $M = 1$ (i.e., has a parity doublet partner at $t = 0$). Then LeBellac's arguments of Sect. VA still hold for the pion's residues. Assume that the pion, the A_1, and its daughter all contribute to the reactions $\bar{N}N \to \pi\rho$ and $\bar{N}\Delta \to \pi\rho$, and that the A_1 and its daughter conspire as indicated above. Then the very forward cross section for $\pi N \to \rho N$ will be controlled by the A_1 contribution and the pion-daughter interference (the pion's contribution vanishes at $t = 0$), giving a finite cross section at $t = 0$ but peaking somewhat away from $t = 0$, as is apparently observed. For $\pi N \to \rho\Delta$, the A_1-pion interference term will dominate the forward cross section. Since the pion residue vanishes at $t = 0$ from factorization, the interference term can give the forward peaking.[28] A relatively detailed model like this can only be tested by accurate

data up to the highest momenta (in order to check the s-dependences in detail, as well as the t-dependences).

C. M = 1 Pion

Mandelstam[33] has discussed the connection between conspiracy theory and PCAC and the commutation relations for axial vector charges. He showed that for zero mass pions the assumption of a type III conspiracy for the pion (M = 1 pion in Toller's notation) is sufficient to establish the Adler self-consistency condition,[34] from which one can go on to treat the Adler-Weisberger relation and the commutation relations for axial vector charges. We consider here what factorization says about these matters. The result turns out to be unpleasant for the hypothesis of M = 1 for the pion. As was first noted by Mandelstam himself,[35] factorization for an M = 1 pion seems to force "hard" pion amplitudes as well as "soft" pion amplitudes to be small. The latter is Adler's self-consistency, but the former is not wanted.

First we establish a generalization of LeBellac's result of Section VA. Consider the process $\bar{N}N \to BB'$, where B, B' are any pair of particles of unequal mass coupled to the pion. For the pion residue the net helicity in the $\bar{N}N$ state is $\lambda_a = 0$ of necessity. We will also choose $\lambda_b = 0$ so that we deal with "sense-sense" amplitudes for the pion. For these amplitudes, with $m = n = 0$, the various reduced residues are all analytic at $t = 0$, provided the pion is a leading trajectory. For these choices of helicities, Eq. (37) becomes

$$\frac{1}{t}\left[\gamma^{BB'\leftarrow\bar{N}N}_{\lambda\lambda;++}\right]^2 = \gamma^{\bar{N}N\leftarrow\bar{N}N}_{++;++}\ \gamma^{BB'\leftarrow BB'}_{\lambda\lambda;\lambda\lambda} \tag{55}$$

For an evasive $(M = 0)$ pion and a conspiring $(M = 1)$ pion we obtain the following results for the behaviors of the residues $t = 0$:

	$\gamma^{\bar{N}N\leftarrow\bar{N}N}_{++;++}$	$\gamma^{BB'\leftarrow\bar{N}N}_{\lambda\lambda;++}$	$\gamma^{BB'\leftarrow BB'}_{\lambda\lambda;\lambda\lambda}$
$M = 0$	t	t	1
$M = 1$	1	t	t

by means of the $U \to U$ factorization equation, Eq. (35), we establish that, for a general process, $AB \to CD$, in which the masses are unequal,

$$\gamma^{CD\leftarrow AB}_{\mu\mu;\lambda\lambda} \propto t \tag{56}$$

provided the pion has $M = 1$. Note that (56) applies only to residues with $m = n = 0$. For other helicities, conspiracies can cause the residues for $U \to U$ to have poles at $t = 0$. This is discussed in Section VE.

Before commenting on the significance of (56) in a discussion of Adler self-consistency, we examine the corresponding results when B and B' have equal masses (and spins, but not the same parities). The process is $\bar{N}N \to BB'$, with $m_b = m'_b$. The relevant equations are (38) and (39). Again consider only the "sense-sense" amplitudes with $\beta = \beta' = \lambda$. We assume an $M = 1$ pion ($\gamma^{\bar{N}N\leftarrow\bar{N}N}_{++;++} \propto 1$ near $t = 0$). The results for various channels are tabulated below. The notation is $N(\frac{1}{2}^+)$, $N_p(\frac{1}{2}^-)$, $\pi(0^-)$, $\sigma(0^+)$, $V(1^-)$, $A(1^+)$, $\Delta(\frac{3}{2}^+)$, $\Delta_p(\frac{3}{2}^-)$.

Channel b (BB')	$\eta_b(-1)^{2s_b}$	$\gamma^{BB'\leftarrow\bar{N}N}_{\lambda\lambda;++}$	$\gamma^{BB'\leftarrow BB'}_{\lambda\lambda;\lambda\lambda}$
$\bar{N}N$	+1	1	1
$\bar{\Delta}\Delta$	+1	1	1
VV	+1	1	1
$\bar{N}N_p$	-1	1	t
$\bar{\Delta}\Delta_p$	-1	1	t
$\pi\sigma$	-1	1	t
VA	-1	1	t

It should be noted that if $\gamma^{BB'\leftarrow\bar{N}N}$ is taken as proportional to t^n, then $\gamma^{BB'\leftarrow BB'}$ will be proportional to the tabulated value times t^{2n}. We have assumed that there are no extra zeros.

The tabulated behaviors illustrate the general result that for $\eta_b(-1)^{2s_b} = +1\ (-1)$ the pion residue $\gamma^{BB'\leftarrow BB'}_{\lambda\lambda;\lambda\lambda} \propto 1\ (t)$ near $t = 0$ provided $m_b = m'_b$ and $M = 1$. For $M = 0$ (no conspiracy in $\bar{N}N \rightarrow \bar{N}N$), the behavior is just the opposite. The significance for the Adler self-consistency condition can be seen from Fig. 1, which shows schematically the pion pole contribution for processes like $\bar{N}N_p \rightarrow \bar{N}N_p$. The particle N_p for example, can be thought of as a nucleon plus a zero mass pion. Then the residue $\gamma^{\bar{N}N_p\leftarrow\bar{N}N_p}$ is proportional to the square of the pion-nucleon scattering amplitude (with one zero mass pion at threshold). Since the residue is proportional to t for small t, the pion-nucleon scattering amplitude must be proportional to $\sqrt{t_{pole}} = m_\pi$, and so vanish in the limit of zero mass pions. Similar conclusions about the pion-pion and other pion scattering amplitudes can be drawn

from consideration of $\pi\sigma \to \pi\sigma$, etc. This is the Adler self-consistency condition for pion scattering processes.

Mandelstam's derivation is based on detailed arguments of 4-dimensional symmetry and $M = 1$ for zero mass pions coupled to equal mass particles. We see here that invocation of a $t = 0$ conspiracy between the pion and a scalar particle in $\bar{N}N \to \bar{N}N$, plus factorization, is all that is actually necessary to obtain Mandelstam's version of the PCAC results. Furthermore, we learn that the equal mass requirement is not necessary. Indeed, Eq. (56) gives us more than we want. It implies that all pion amplitudes are in some sense small (zero in the limit of $m_\pi \to 0$). It is difficult, of course, to give quantitative meaning to the word "small" since something proportional to t as $t \to 0$ can still be very large at $t = m_\pi^2$. Until this sort of question is clarified, Eq. (56) does not destroy the significance of Mandelstam's original arguments, but it does cast a cloud over them.

D. Zeros in Residues away from $t = 0$

A related question is the so-called "moving zero" in the pion residue at small negative t. It has been argued by Arbab and Dash[25] that the zero in the pion residue at $t = 0$ for sense-sense coupling in $\bar{N}N \to \bar{N}N$, required of a zero-mass $M = 1$ pion, moves slightly away from $t = 0$ when the symmetry is broken by the small finite mass of the actual pion. Empirically such a zero is found necessary at $t \approx -(1.5 - 2.0)\, m_\pi^2$ in the phenomenological fitting of n-p charge exchange[25] and pion photoproduction[36] at high energies. It has also been deduced from

finite energy sum rules for photoproduction.[37]

We wish to make a relatively trivial observation concerning this and other possible zeros in residues away from $t = 0$. The various factorization relations, Eqs. (35), (37), (39), involve only the reduced residues and perhaps powers of t. This means that a linear zero in the residues for $\bar{N}N \to \bar{N}N$, i.e., $\gamma^{\bar{N}N \leftarrow \bar{N}N}_{++;++} \propto (t - t_o)$, will propagate into all reactions, whether elastic or inelastic, whether equal or unequal mass. But if the zero in $\bar{N}N \to \bar{N}N$ is quadratic, then it can appear as a linear zero in a process like $\bar{N}N \to \pi\gamma$, and be absent in $U \to U$ reactions. The latter possibility is perhaps more reasonable and is apparently supported by data on inelastic reactions believed to be dominated by pion exchange, although one can question whether other contributions might not mask the effect.[32] The possible relation between "moving zeros" that might result from a breaking of the 4-dimensional symmetry (by either the finite mass of the pion or the inequality of the masses of the external particles) and the fixed zeros at $t = 0$ that result from factorization is a topic beyond the scope of these notes. It is perhaps of significance that for an $M = 1$ pion <u>all</u> residues seem to vanish at or near $t = 0$.

Another example of zeros in residues at physical t values is afforded by the ω-trajectory. The cross-over phenomenon in the differential cross sections for pp and $\bar{p}p$ elastic scattering is interpreted in terms of a linear zero in the non-helicity flip residues of the ω-trajectory at $t \approx -0.15$ (GeV/c)^2.[38] A linear zero in pp <u>elastic</u> scattering implies, via factorization, a linear zero in all ω residues,

as first noted by Phillips and Rarita.[39] The difficulties with such consequences can be traced in the literature.[40]

E. Conspiracies at $t = 0$ for $U \to U'$ Processes[41]

For equal mass processes ($E \to E$ or $E \to U$) conspiracies at $t = 0$ can involve either trajectories of the same parity sequence (type II, e.g., parent and daughter, as in Sect. VB) or trajectories of opposite parity sequences (type III, e.g., $np \to pn$[25] and $\gamma N \to \pi N$[36]). For unequal masses, the most obvious conspiracy is the parent-daughter conspiracy of Freedman and Wang[26], with the daughter residues having poles at $t = 0$ (e.g., Eq. (48)). But there is still another type of conspiracy for $U \to U'$ processes, also resulting in residues singular at $t = 0$, but involving trajectories of opposite parities.

Our starting point is the fact that for $U \to U'$ the full helicity amplitudes are regular in the neighborhood of $t = 0$. The connection between the full amplitudes and the so-called parity-conserving amplitudes is given by Eq. (30). With the conventions of Sect. IVA on the ranges of helicities the two helicity amplitudes $f_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ and $f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}$ can be written

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2} = \frac{1}{2}\left(\sqrt{2}\cos\frac{\theta_t}{2}\right)^{m+n}\left(\sqrt{2}\sin\frac{\theta_t}{2}\right)^{m-n}\left[F^+ + F^-\right]$$

$$f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} = \xi\,\frac{1}{2}\left(\sqrt{2}\cos\frac{\theta_t}{2}\right)^{m-n}\left(\sqrt{2}\sin\frac{\theta_t}{2}\right)^{m+n}\left[F^+ - F^-\right] \qquad (57)$$

where ξ is an inessential phase factor that can be read off from Eq. (15) of Ref. 18 and m and n are defined at the end of Sect. IVB. For $U \to U'$, $\cos\theta_t \to \pm 1 + O(t)$ as $t \to 0$, the sign depending on the sign of $(m_1^2 - m_2^2)(m_3^2 - m_4^2)$. For definiteness we assume that the masses are such as to give the positive sign; the argument can be changed trivially for the other choice. The half-angle factors in Eq. (57) give the following small t behavior:

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto (\sqrt{t})^{m-n}\left[F^+ + F^-\right]$$
$$f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \propto (\sqrt{t})^{m+n}\left[F^+ - F^-\right] \tag{58}$$

Without conspiracy, the various Regge poles contributing to F^+ and F^- have different s-dependences at $t = 0$. Hence the necessary t behavior must occur for each pole separately. In order that $f_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ be regular near $t = 0$ we must then have no higher singularity than

$$F^\eta \propto \left(\frac{1}{\sqrt{t}}\right)^{m-n}, \quad f_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto 1; \quad f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \propto t^n \tag{59}$$

The $t = 0$ singularity in F^η is just that given by Eq. (31). The consequence is that one helicity amplitude is finite at $t = 0$ while the other vanishes as t^n. This is called the normal behavior. Note that uniformly less singular behavior for dynamical reasons is always possible. But we do not consider that possibility here.

If, on the other hand, we admit conspiracies between Regge trajectories of opposite parity sequences, the normal result, (59), need not occur. Suppose that we wish to have the small t behavior,

$$f_{\lambda_3\lambda_4;\lambda_1\lambda_2} \propto t^p, \qquad f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2} \propto t^q \tag{60}$$

where p and q are non-negative integers. Then, from (58) we conclude that

$$\left[F^+ + F^-\right] \propto (\sqrt{t})^{2p-m+n}$$

$$\left[F^+ - F^-\right] \propto (\sqrt{t})^{2q-m-n}$$

This can be arranged by having

$$F^{\pm} \propto \left(\frac{1}{\sqrt{t}}\right)^{m+n-2q}, \tag{61}$$

but demanding that

$$\left[F^+ + F^-\right] \propto \left(\frac{1}{\sqrt{t}}\right)^{m+n-2q} \left\{t^{n+p-q}\right\}, \tag{62}$$

the curly bracket giving the small t dependence of $\left[\tilde{F}^+ + \tilde{F}^-\right]$. Comparison of (61) with (31) shows that (61) is equivalent to having the reduced residue functions of the conspiring poles singular at $t = 0$:

$$\gamma^{(\pm)} \propto \left(\frac{1}{t}\right)^{n-q} \tag{63}$$

Condition (62) requires that $\tilde{F}^+ = -\tilde{F}^-$ to order $t^{n+p-q-1}$. Because of the t-dependence in $s^{\alpha(t)}$, as well as in $\gamma(t)$, this means that

$$\left.\frac{d^j \alpha^{(+)}(t)}{dt^j}\right|_{t=0} = \left.\frac{d^j \alpha^{(-)}(t)}{dt^j}\right|_{t=0}$$

$$\left.\frac{d^j}{dt^j}\left[t^{n-q}\,\gamma^{(+)}(t)\right]\right|_{t=0} = -\left.\frac{d^j}{dt^j}\left[t^{n-q}\,\gamma^{(-)}(t)\right]\right|_{t=0} \tag{64}$$

for $j = 0,1,2,\cdots,(n + p - q - 1)$. These conditions on the conspiring trajectories and their residues will guarantee the small t behavior assumed in (60). In contrast to the $E \to E$ or $E \to U$ conspiracies, the amplitudes $f_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ and $f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}$ are not related at $t = 0$ simply through a common residue value. The value of the amplitude $f_{\lambda_3\lambda_4;-\lambda_1-\lambda_2}$ is given by the residues of the conspiring poles, while $f_{\lambda_3\lambda_4;\lambda_1\lambda_2}$ depends on the $(n + p - q)$th derivatives of the trajectories, $\alpha^{(+)}$ and $\alpha^{(-)}$, and of the residues, $\gamma^{(+)}$ and $\gamma^{(-)}$. This makes the $U \to U'$ conspiracy qualitatively different from the equal mass situations and is probably responsible for the confusion on whether or not conspiracies at $t = 0$ occur for unequal masses.[42]

With this type of conspiracy the residues entering the factorization Eq. (35) need not be analytic near $t = 0$, but may possess poles as shown in (63).

Various versions of the Lorentz pole model give partial specification of the exponents, p and q in Eq. (60). The model of Cosenza, Sciarrino and Toller[43] gives $q = M - m$, 0, and $n - M$ for

$M \geq m$, $m > M \geq n$, and $n > M$, respectively, where M is the Lorentz pole quantum number. It is not clear, however, what their model predicts for p. Results in agreement with Ref. 43 have been obtained by LeBellac[44] and also by DiVecchia, Drago and Paciello,[45] using factorization arguments and the specification of a "minimal" solution of the analyticity and factorization requirements. The model of Bitar and Tindle[46] has a correlation between the small t dependence and the asymptotic s-dependence of the t-channel amplitudes. For the terms in the amplitude with the normal $s^{\alpha-m}$ behavior (in $F^{\pm}$), one finds $p = M - m$ and $p = 0$ for $M \geq m$ and $m > M$, respectively, while $q = M - m$, 0, and $n - M$ for $M \geq m$, $m > M \geq n$, and $n > M$, respectively. This corresponds to the equality of trajectories and residues in (64) for $j = 0, \cdots, (n-1)$ or $(M-1)$, whichever is smaller. In Bitar and Tindle's model there are, however, terms with $p = q = 0$ and less than the leading power of s.

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standard direction $\hat{p}_o$ and $-\hat{p}_o$, respectively, into the state $|\hat{p}_a, \lambda_x \lambda_y\rangle$, in which the momenta are parallel and antiparallel to $\hat{p}_a$. In the present case, the replacements, $\hat{p}_o \to \hat{p}_a$, $\hat{p}_a \to \hat{p}_b$, $\hat{p}_b \to \hat{p}_c$, yields the stated result.

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amplitude is as $(\omega')^L$, where ω' is the photon energy in the rest frame of the "matter" system ($\omega' = \sqrt{t}\, p_b/m_b$). The "mismatch" between J and L (see Ref. 18) is $J - L = s_b$, and thus the kinematic singularity is $(\omega')^{-s_b}$.

24. In comparing Eqs. (37) with the results of other workers, the remarks below Eq. (33) concerning the powers of $\sqrt{t}$ should be kept in mind. Differences between the present expression and the results in Ref. 9 on $\gamma V \to \bar{N}N$, for example, can be traced to their use of $(1/\sqrt{t}\,)^{m+n}$ instead of $(1/\sqrt{t}\,)^{m-n}$ in Eq. (31).

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FIGURE CAPTION

Fig. 1. Diagrams showing the pion pole contribution to the $E \to E$ process, $N_p\bar{N} \to N_p\bar{N}$, $\sigma\pi \to \sigma\pi$, and $AV \to AV$. The "particles" N_p, σ and A can be thought of composites of a zero mass pion at threshold and N, π and V, respectively. The residues of the pion pole are proportional to the square of the πN, $\pi\pi$ and πV elastic scattering amplitudes.

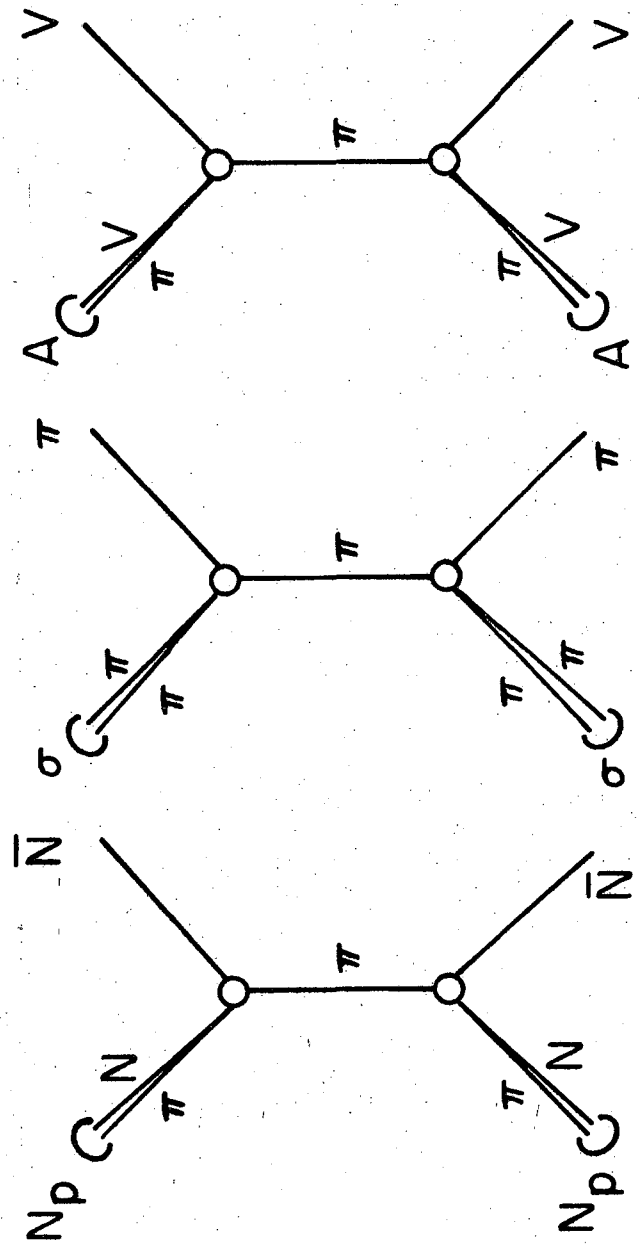

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Fig. 1.