Title
Limiting Evolution of Families of Partial Differential Equations

Author
Alexander, Damon

Publication Date
2014

University of California
Los Angeles

Limiting Evolution of Families of Parabolic Differential Equations

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Damon Spiegel Alexander

2014

Abstract of the Dissertation

Limiting Evolution of Families of Parabolic Differential Equations

by

Damon Spiegel Alexander

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2014

Professor Christina Kim, Chair

We study the relationships between several families of parabolic partial differential equations as we take limits of physical parameters. We first consider the porous medium equation with a drift term, and how it relates to Hele-Shaw equations with drift and equations describing congested crowd motion . We use viscosity solution arguments to prove that the porous medium equation solutions converge to the Hele-Shaw solutions, provided the drift is strictly subharmonic. Next, we prove that the porous medium equation also converges to a congested crowd motion model by exploiting the gradient flow structure. The combination of these results leads to a proof that, provided the initial data is a patch, or characteristic function, the crowd motion evolves with Hele-Shaw dynamics.

Next, we consider the heat equation on a bounded domain with no-flow Neumann boundary conditions. We show that we can approximate the solution by writing the equation on all of $\mathbb{R}^n$ by introducing a convection term that pushes mass back inside the domain. As we take the strength of the convection to infinity, we obtain locally uniform convergence to the heat equation solution, with an optimal error rate. This is done by building suitable barriers to control the solution behavior. We then generalize the result to quasi-linear parabolic differential equations with oblique boundary conditions by applying approximation arguments and viscosity techniques.

The dissertation of Damon Spiegel Alexander is approved.

Jeff D. Eldredge

Rowan Killip

Chris Anderson

Christina Kim, Committee Chair

University of California, Los Angeles

2014

To my mom, for our adventures making quinzees and gingerbread houses and toothpick boats.

To my dad, for many games of pool and tennis, and the most boring take-your-child-to-work-days ever.

Table of Contents

1 Introduction **1**

2 Quasi-static evolution and congested crowd transport **5**

2.1 Introduction 5

2.1.1 A model in congested crowd motion 7

2.1.2 Summary of results 10

2.1.3 An outline of the chapter 12

2.1.4 Remarks on possible extensions 14

2.2 On the continuum solutions 15

2.2.1 Inf- and Sup-convolutions 17

2.2.2 Comparison principle 20

2.3 Approximation by degenerate diffusion with drift 22

2.3.1 Properties of u_m at the free boundary 23

2.3.2 Characterization of the half relaxed limits of u_m as $m \to \infty$ . . . 31

2.3.3 Convergence of u_m as $m \to \infty$ 39

2.4 Convergence of the gradient flow solution as $m \to \infty$ 43

2.4.1 Definition of the gradient flow solution and the discrete scheme . 43

2.4.2 One-step estimate for large m 47

2.4.3 Convergence of the continuum solutions as $m \to \infty$ 54

2.5 Comparison principle and long-time behavior for gradient flow solutions . 57

2.5.1 Comparison principle for the discrete-time solutions 57

2.5.2 Confinement result and long-time behavior. 63

2.6 Prior results on gradient flows 66

3 Approximating parabolic PDEs with oblique boundary data 68

3.1 Introduction 68

3.2 PDEs of divergence form 74

3.2.1 The heat equation in one dimension 75

3.2.2 The general linear divergence form equation 80

3.3 Finite difference approximation results 91

3.4 PDEs of non-divergence form 94

3.5 Additional remarks 109

3.5.1 Examples in one dimension 109

3.5.2 Stability of the drift potential 111

3.5.3 Constructing $A(x, t)$ 112

3.5.4 Why the method of viscosity solutions fails 113

4 Closing Remarks **115**

4.1 Future work 115

4.2 Conclusion 116

References **117**

List of Figures

1.1 Evolution of the positive phase of Hele-Shaw flow, converging toward the equilibrium . . . 2

2.1 This diagram is a summary of the results of Theorems 2.1.1 and 2.1.2. Here ρ_∞ denotes the gradient flow solution in the continuum limit, which in particular is a solution of (2.1.5). . . . 9

2.2 The motivation for ξ: it crosses u_1 at an earlier time than ϕ does . . . 37

2.3 Illustration of the tree structure. . . . 55

2.4 Illustration of the set $B, T_1(B)$ and $T_2(B)$. Recall that T_i is the optimal map between ρ_{0i} and ρ_i for $i = 1, 2$. Moreover, the set B is chosen such that $T_1(B) \subset \{\frac{1}{\delta} \geq \rho_1 > \rho_2 + \delta\}$, while $T_2(B) \subset \{\rho_1 \leq \rho_2 \leq \frac{1}{\delta}\}$. . . . 61

3.1 A sample u, u_ϵ, and φ, where $\Omega = [0, 1]$. . . . 78

3.2 An illustration of how S functions . . . 81

3.3 Calculating an upper bound on $\tilde{d}$ . . . 83

3.4 The true solution u at various time points . . . 92

3.5 The approximating solution v with $N = 100$ . . . 93

3.6 The approximating solution v with $N = 1000$ . . . 93

3.7 How the different problems and their solutions relate as we vary the parameters. . . . 96

3.8 Showing $\{(x - x_r) \cdot \vec{\nu}_r \leq -3r\}$ is contained in Ω_{2r} 108

List of Tables

3.1 Estimates of the rate of convergence N^{-p} as we vary N 94

Acknowledgments

I am deeply grateful for the boundless support and guidance offered by Professor Inwon Kim over her five years of advising. Her deep intuition and experience in the field were critical to helping me develop my research ability, starting all the way back in her differential equations class.

I would like to thank and acknowledge my committee members, Professors Chris Anderson, Jeffery Eldredge, and Rowan Killip. In particular, I had the privilege of taking many excellent courses from Professor Anderson, who taught me everything I know about scientific computing. Likewise, I had the honor of taking very insightful courses on functional analysis taught by Professor Killip.

I owe my sanity to all the friends I made in Los Angeles, in particular my roommates of the last four years for many late night discussions.

Thanks to my sister Kate for her love, incredible creativity, and support.

Lastly, I am eternally thankful for my parents, Gordon Alexander and Arlene Spiegel. Every last step of my development stems from the environment of creativity and learning that they took every measure to provide.

Vita

2005 - 2008	B.A., Mathematics and Physics University of Wisconsin - Madison
2009	M.S., Mathematics University of Wisconsin - Madison

Publications

Damon Alexander, Inwon Kim, Yao Yao. Quasi-static evolution and congested crowd transport. Accepted by *Nonlinearity*, February 2014.

Damon Alexander, Inwon Kim. Approximating parabolic PDEs with oblique boundary data. Submitted to *Transactions of the AMS*, February 2014.

CHAPTER 1

Introduction

We set out to prove convergence between several families of partial differential equations as we take limits of parameters. The first set of partial differential equations we consider consists of the Hele-Shaw equation and porous medium equation. Hele-Shaw flow with a convex drift term Φ is modeled as follows:

$$(HS) \qquad \begin{cases} -\Delta u(\cdot,t) = \Delta\Phi & \text{in} \quad \{u>0\}; \\ V = -\nabla u\cdot\vec{\nu} - \nabla\Phi\cdot\vec{\nu} & \text{on} \quad \partial\{u>0\}. \end{cases}$$

Here V is the free boundary velocity of the set $\{u > 0\}$ and $\vec{\nu}$ is the outward normal. $u(x,t)$ models the density of a fluid, which occupies the moving region $\{u > 0\}$; a diagram of the positive phase is displayed in Figure 1.1. Our interest was in proving that the solution to Hele-Shaw flow can be realized as a limit of solutions to the porous medium equation with corresponding drift,

$$(PME-D)_m \qquad \rho_t = \nabla\cdot[\nabla(\rho^m) + \rho\nabla\Phi)],$$

as the porosity scalar $m \to \infty$.

Our results also prove a relationship of Hele-Shaw flow to the crowd motion model proposed in [RV11] and [MRS10]. These papers consider crowds that move with a desired velocity given by a prescribed $\nabla\Phi$, with constraint $\rho \leq 1$ representing maximal crowding.

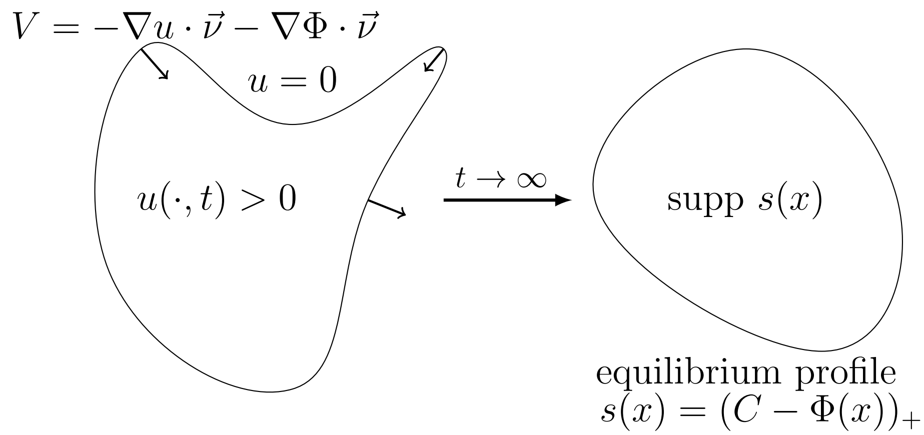

Figure 1.1: Evolution of the positive phase of Hele-Shaw flow, converging toward the equilibrium

By exploiting the relationship of the gradient flow structures of (PME-D)$_m$ and the crowd motion model, we prove that the crowd location must in fact be equal to the positive phase $\{u > 0\}$ in the case that the initial density is a characteristic function.

The second set of equations we consider starts with a quasi-linear second order parabolic equation with zero oblique boundary data given by a prescribed vector field $\vec{v}_0$:

$$(P_g) \qquad \begin{cases} u_t - F(D^2u, Du, u, x) & = 0 \quad \text{in} \quad \Omega \times (0, \infty); \\ \nabla u \cdot \vec{v}_0(x) & = 0 \quad \text{on} \quad \partial\Omega \times (0, \infty); \\ u(x, 0) & = u_0(x) \quad \text{in } \Omega. \end{cases}$$

In this case, we prove that we can approximate the boundary conditions by solving the equation on a bigger domain Ω' with Dirichlet conditions, using a penalizing drift term.

This gives the equation

$$(P_N) \qquad \begin{cases} v_t - F(D^2v, Dv, v, x) - N\nabla\cdot[vA(x,t)\nabla\Phi] = 0 & \text{in } \Omega' \times (0,\infty); \\ v = 0 & \text{on } \partial\Omega' \times (0,\infty); \\ v(x,0) = v_0(x) & \text{in } \Omega. \end{cases}$$

The approximation is valid in that the solutions v converge uniformly as we take $N \to \infty$, with the rate of convergence

$$\|u - v\|_{L^\infty(\Omega\times[0,T])} < CTN^{-1/3}.$$

We prove that this rate is in fact sharp for our approximation by considering the size of the stationary solution $e^{-N\Phi(x)}$ near $\partial\Omega$. This was inspired in part by the Orhnstein-Uhlenbeck process, which has probability density f satisfying

$$f_t = \theta[xf]_x + \sigma^2 f_{xx}/2.$$

The elliptic case was proven in [BGJ13], where the penalty term is written in non-divergence form as $N\nabla v\cdot\nabla\Phi$. This form makes the standard viscosity solution arguments work in a rather straightforward fashion. However, that form cannot work for parabolic PDEs since the scheme no longer conserves mass. The required addition of a $Nv\Delta\Phi$ term creates substantial problems for the standard viscosity argument, which is why we were required to construct barriers by extension.

An almost identical penalization scheme was used in [LS84] to prove the existence of solutions for stochastic differential equations with reflecting or oblique boundary conditions. In this setting, the penalization scheme is used to prove existence for the Skorokhod problem of finding a solution to a reflecting ordinary differential equation. There is also an extensive body of literature on the fictitious domain method, a similar scheme tailored to the finite element method, They were used in [GPW92] and [BBZ10] for alleviating difficulties arising from complicated interface conditions.

Both of these projects made extensive use of viscosity solution techniques and the comparison principle to show convergence. That is, by constructing suitable barrier functions, we could show that the approximating solutions must be near the original functions.

CHAPTER 2

Quasi-static evolution and congested crowd transport

We consider the relationship between Hele-Shaw evolution with drift, the porous medium equation with drift, and a congested crowd motion model originally proposed by Maury, Roudneff-Chupin and Santambrogio. We first use viscosity solutions to prove that the porous medium equation solutions converge to the Hele-Shaw solution as $m \to \infty$ provided the drift potential is strictly subharmonic. Next, using of the gradient flow structure of both the porous medium equation and the crowd motion model, we prove that the porous medium equation solutions also converge to the congested crowd motion as $m \to \infty$. Combining these results lets us deduce that in the case where the initial data to the crowd motion model is given by a *patch*, or characteristic function, the solution evolves as a patch that is the unique solution to the Hele-Shaw problem. While proving our main results we also obtain a comparison principle for solutions to the minimizing movement scheme based on the Wasserstein metric, of independent interest.

2.1 Introduction

Let Ω_0 be a compact set in $\mathbb{R}^d$ with locally Lipschitz boundary, and let $\Phi(x) : \mathbb{R}^d \to \mathbb{R}$ be a C^2 function which satisfies

$$(\mathbf{A1}) \qquad \Delta\Phi > 0 \text{ in } \mathbb{R}^d.$$

For Ω_0 and Φ as given above, we consider a function $u : \mathbb{R}^d \to \mathbb{R}$, $u(x,t) \geq 0$ solving the following free boundary problem:

$$(HS) \qquad \begin{cases} -\Delta u(\cdot,t) = \Delta\Phi & \text{in} \quad \{u > 0\}; \\ \\ V = -\partial_\nu u - \partial_\nu \Phi & \text{on} \quad \partial\{u > 0\}. \end{cases}$$

Here $\nu_{x,t}$ is the outward normal vector of the set $D_t(u) := \{x : u(x,t) > 0\}$ at $x \in \Gamma_t(u) := \partial D_t(u)$, and V denotes the outward normal velocity of $D_t(u)$ at $x \in \Gamma_t(u)$. The strict inequality in (**A1**) is given to rule out the scenario where $\Delta\Phi \equiv 0$ in $\{u(\cdot, T_0) > 0\}$ in (HS) for some time $T_0 > 0$, which necessitates $u(\cdot, T_0)$ to be identically zero.

In terms of u, $\nu = -\nabla u/|\nabla u|$ and thus one can write down the second condition of (HS) as

$$u_t = |\nabla u|^2 + \nabla u \cdot \nabla \Phi \quad \text{ on } \partial\{u > 0\},$$

given that $|\nabla u| \neq 0$ at the boundary point. Note that the free boundary velocity V may be positive or negative depending on the behavior of Φ on $\Gamma(t)$. Consequently $D_t(u)$ may expand or shrink over time (see Figure 1). Indeed formal calculations based on (HS) yield that $D_t(u)$ preserves its volume over time. The initial data $u(x,0) = u_0$ is the unique function satisfying

$$-\Delta u_0 = \Delta\Phi \text{ in the interior of } \Omega_0, \quad u_0 = 0 \text{ on } \Omega_0^C. \tag{2.1.1}$$

Note that, due to (**A1**), u_0 is positive in Ω_0 and thus (2.1.1) is well-defined. Still, even starting from a smooth domain Ω_0, the solution of (HS) can develop finite-time singularities as its support goes through topological changes such as pinching and merging, and thus it is necessary to consider a notion of weak solutions. We will use the notion of viscosity solutions for (HS), see section 2 for definitions and properties of u. Let us mention that the usual variational inequality formulation for weak solutions of Hele-Shaw flow, introduced by [EJ81], does not apply here due to the non-monotonicity of solutions in time variable.

In the context of fluid dynamics, the problem (HS) describes a flow in porous media. Indeed if we denote by $u = u(x,t)$ the density of a fluid and define the velocity of the fluid as

$$\vec{U} = -\nabla\Phi - \nabla u, \tag{2.1.2}$$

where $\nabla\Phi$ is the external velocity field given by Φ, then (2.1.2) and the incompressibility condition

$$\nabla \cdot \vec{U} = 0 \tag{2.1.3}$$

yields (HS).

When $\Phi = 0$ and there is a fixed boundary in the positive phase through which the fluid is injected, (2.1.2) and (2.1.3) yield the classical one-phase Hele-Shaw problem [Hel98]. In this article however, our goal is to derive (HS) from a model problem in crowd motion with hard congestion, as described below.

2.1.1 A model in congested crowd motion

Let us recall the transport problem with density constraint, introduced in [MRS10]-[RV11]. Formally the problem can be written as the following: we look for a solution $\rho : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ satisfying

$$\rho_t + \nabla \cdot (\rho\nabla\Phi) = 0 \text{ if } \rho < 1, \text{ and } \rho \leq 1 \text{ for all times.} \tag{2.1.4}$$

The density constraint is natural in many settings, and it describes motion of congested individuals. We refer to the articles [MRS10, RV11, San12] for applications and mathematical formulations of the problem (2.1.4). More rigorously, the problem can be written as

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \quad \mathbf{u} = P_{C_\rho} \nabla \Phi, \tag{2.1.5}$$

where P is the projection operator and C_ρ is the space of L^2-admissible velocity fields which do not increase ρ on the saturated zone $\{\rho = 1\}$. We refer to [RV11] for further description of C_ρ. Due to the low regularity of the velocity field $\mathbf{u}$ and the non-continuous dependence of the operator P_{C_ρ} with respect to ρ, classical methods to study transport equations do not apply to (2.1.5).

In [RV11], the authors study the connection between the PDE (2.1.5) with ρ_∞, which is the gradient flow of the following functional E_∞ with respect to the 2-Wasserstein distance:

$$E_\infty[\rho] := \begin{cases} \int_{\mathbb{R}^d} \rho(x)\Phi(x)dx & \text{for } \|\rho\|_\infty \leq 1 \\ +\infty & \text{for } \|\rho\|_\infty > 1. \end{cases} \tag{2.1.6}$$

Further, they prove that when Φ is λ-convex, the gradient flow solution ρ_∞ is a weak solution for (2.1.5). However, the full characterization of the solution and further qualitative properties of the solution remain open due to the lack of available methods to study (2.1.5). The connection between ρ_∞ and (HS) has been hinted, but only formally in the context of particle velocity.

∘ *Our contribution:* In this article, our main focus is on establishing the connection between the free boundary problem (HS) and the gradient flow of E_∞ in the setting of *patches*, i.e. when the initial data is given as a characteristic function of a compact set Ω_0, which we denote by χ_{Ω_0}. Note that since Φ is assumed to have a positive Laplacian, solutions tend to aggregate and thus we expect that the gradient flow $\rho_\infty(\cdot, t)$ will stay as a characteristic function at all times $t > 0$.

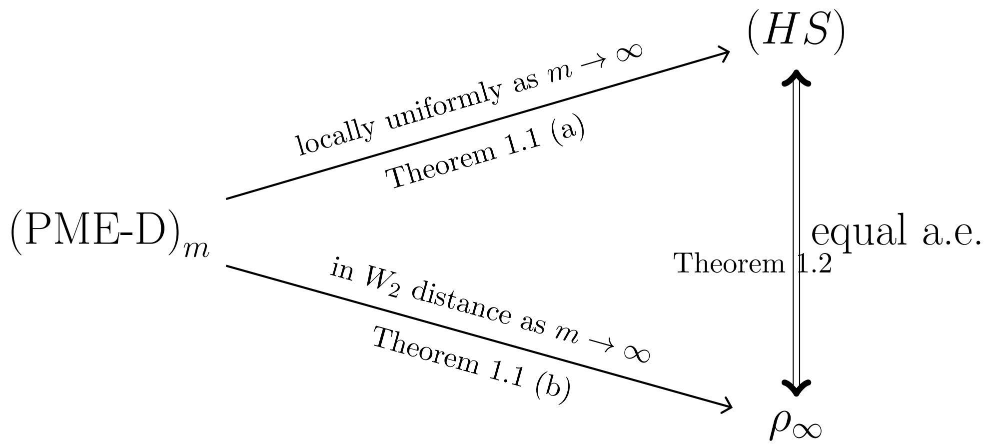

Figure 2.1: This diagram is a summary of the results of Theorems 2.1.1 and 2.1.2. Here ρ_∞ denotes the gradient flow solution in the continuum limit, which in particular is a solution of (2.1.5).

We show that the preservation of patches is indeed the case, and moreover the gradient flow solution $\rho_\infty(\cdot, t)$ indeed coincides with the characteristic function of the set Ω_t, which evolves according to our problem (HS) with the initial support Ω_0 (see Theorem 2.1.2 below). This result enables us to characterize the evolution of ρ_∞ in a unique way and also helps to understand the geometric behavior of ρ_∞. A summary of our results is shown in Figure 2.1.

In our analysis, the main challenge is the low regularity of ρ_∞, since a priori we only know that it is in $C_W([0,T], \mathcal{P}_2(\mathbb{R}^d))$ (for the definition of C_W, see the end of Theorem 2.4.1(b)). Thus it is rather difficult to directly study the geometric property of ρ_∞. Instead of trying to directly show the link between the free boundary problem (HS) with the gradient flow ρ_∞, we use an approximation with degenerate diffusion. It has been formally suggested in [MRS10] and [San12] that one could consider approximating the gradient flow of E_∞ by the unconstrained gradient flow problem with the energy

$$E_m[\rho] := \int \left(\frac{1}{m}\rho^m + \rho\Phi\right) dx. \tag{2.1.7}$$

It is well known (for example, see [Ott01]) that the gradient flow ρ_m associated with E_m solves the porous medium equation with drift

$$\rho_t - \nabla \cdot (\nabla(\rho^m) + \rho\nabla\Phi) = 0. \tag{2.1.8}$$

Let us denote ρ_m as the viscosity solution to (2.1.8) with initial data χ_{Ω_0}. We will prove that as $m \to \infty$, ρ_m on the one hand converges to χ_{Ω_t} locally uniformly, and on the other hand converges to $\rho_\infty(\cdot, t)$ in 2-Wasserstein distance. Thus it follows that χ_{Ω_t} and ρ_∞ must be equal to each other almost everywhere. The main ingredients of the proof consist of stability results from viscosity solution theory and optimal transport theory, both of which rely strongly on the convexity-type conditions on Φ. We also obtain comparison results and qualitative rates of convergences; see section 1.2 for precise statements.

2.1.2 Summary of results

We are now ready to state our main results. The relevant assumptions, besides (**A1**) in the introduction, are stated in the beginning of section 4.

Theorem 2.1.1. *Let Ω_0 be a compact set in $\mathbb{R}^d$ with locally Lipschitz boundary, and consider the initial data u_0 as given in* (2.1.1). *Then the following holds:*

(a) (Theorem 2.3.4) Assuming (**A1**), *there exists a unique family of compact sets Ω_t in $\mathbb{R}^d$ starting from Ω_0 such that any viscosity solution u of (HS) satisfies $\overline{\{u(\cdot, t) > 0\}} = \Omega_t$ for all $t > 0$. Furthermore, let ρ_m denote the viscosity solution to* (2.1.8) *with initial data χ_{Ω_0}. Then as $m \to \infty$, ρ_m converges to $\bar{\rho} := \chi_{\Omega_t}$ locally*

uniformly in $\mathbb{R}^d \backslash \partial\Omega_t$ *at each time* $t > 0$.

(b) *(Theorem 2.4.2) Assume* (**A2**) *and* (**A3'**)*, and consider* $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ *with* $\|\rho_0\|_\infty \leq 1$ *and* $\int \rho_0 \Phi dx \leq M$*. Let* $\rho_m(x,t)$ *denote the viscosity solution of* (2.1.8) *with initial data* ρ_0*. Then there exists* $\rho_\infty \in C_W([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ *such that for any* $T > 0$*, as* $m \to \infty$*,* $\rho_m(\cdot,t)$ *converges to* $\rho_\infty(\cdot,t)$ *in 2-Wasserstein distance, uniformly in* t *for* $t \in [0,T]$*, with the following convergence rate:*

$$\sup_{t\in[0,T]} W_2(\rho_m(t), \rho_\infty(t)) \leq \frac{C(M,T,\|\Delta\Phi\|_\infty)}{m^{1/24}}.$$

Combining Theorem 2.1.1 (a) and (b), we immediately draw the following conclusion for the identification of ρ_∞.

Theorem 2.1.2. *[Characterization of* ρ_∞*] Let* Ω_0*,* ρ_∞ *and* $\bar{\rho}$ *as given in Theorem 2.1.1. If* (**A1**)*,* (**A2**) *and* (**A3'**) *hold and if* $\rho_0 = \chi_{\Omega_0}$*, then* $\rho_\infty = \bar{\rho}$ *a.e.*

As a by-product of our analysis, we also show that a version of comparison principle holds between solutions to the discrete Jordan-Kinderlehrer-Otto (JKO) steepest descent scheme:

Theorem 2.1.3 (Comparison principle, see Theorem 2.5.1)**.** *Let* Φ *satisfy* (**A3**)*. For* $2 < m \leq \infty$*, consider the two densities* $\rho_{01} \in \mathcal{P}_{2,M_1}(\mathbb{R}^d)$*,* $\rho_{02} \in \mathcal{P}_{2,M_2}(\mathbb{R}^d)$ *(*$\mathcal{P}_{2,M_i}$ *is as defined in section 2.5.1) with the property* $M_1 \leq M_2$ *and* $\rho_{01} \leq \rho_{02}$ *a.e. (In the case* $m = \infty$*, we require in addition that* $\|\rho_{0i}\|_\infty \leq 1$ *for* $i = 1,2$*). For given* $h > 0$*, let* ρ_1, ρ_2 *be the respective minimizers of the following schemes:*

$$\rho_i := \underset{\rho\in\mathcal{P}_{2,M_i}(\mathbb{R}^d)}{\operatorname{argmin}} \left[E_m[\rho] + \frac{1}{2h} W_2^2(\rho, \rho_{0i}) \right] \quad \text{for } i = 1,2, \tag{2.1.9}$$

Then $\rho_1 \leq \rho_2$ *a.e..*

This comparison result is new in the context of Wasserstein distances and might be of independent interest (see section 2.5.1 for more discussions). As a consequence one obtains geometric properties of the discrete solutions such as the confinement property (Corollary 2.5.1).

Lastly, making use of this confinement result, for strictly convex Φ (but not necessarily uniformly convex), we have the following result concerning the long time behavior of ρ_∞ starting from general initial data:

Theorem 2.1.4 (Convergence to the stationary solution, see Theorem 2.5.2)**.** *Let* $2 < m \leq \infty$*. Let* Φ *be strictly convex and satisfy* (**A2**) *and* (**A3'**)*. Assume the initial data* $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ *has compact support, and in addition satisfies* $\|\rho_0\|_\infty \leq 1$ *in the case* $m = \infty$*. For* $2 < m \leq \infty$*, let* ρ_m *be given as the gradient flow for* E_m *with initial data* ρ_0*, as defined in Theorem 2.4.1(b). Then as* $t \to \infty$*,* $\rho_m(\cdot, t)$ *converges to the unique global minimizer* ρ_S *of* E_m *exponentially fast in 2-Wasserstein distance.*

2.1.3 An outline of the chapter

In section 2 we introduce the notion of viscosity solutions for (HS) and state basic properties of solutions. This part is largely parallel to [Kim03]. In section 3 we show Theorem 2.3.4. A key ingredient in this section is Theorem 2.3.3, which identifies properties of the half-relaxed limits of ρ_m as $m \to \infty$. We point out that such convergence is previously known without the presence of the drift ([GQ01], [Kim03]), but the presence of the drift and the resulting non-monotonicity of the support $\{\rho(\cdot, t) > 0\}$ causes new challenges. In particular the weak formulation used in [GQ01] based on variational inequalities no longer applies, and thus we proceed with the viscosity solutions approach similar to those taken in [Kim03]. The argument presented in Theorem 2.3.3 is of independent interest: it presents a strong stability argument which would apply to a general class of non-monotone free boundary problems. Let us point out that the assumption (**A1**) not only justifies (HS) but also ensures the non-generacy of solutions of (HS) near the free boundary which leads to stability properties (see e.g. the proof of Theorem 2.3.3.)

In section 4 we introduce the corresponding discrete-time schemes with free energy E_m and E_∞ respectively, and we study the convergence of the discrete solutions (and continuous gradient flow solutions) as $m \to \infty$. There are new difficulties in handling the singular limit $m \to \infty$, since the discrete solutions $\rho_{m,h}$ corresponding to free energy (2.1.7) are not necessarily less than 1. Lemma 2.4.2 ensures that $\rho_{m,h}$ can be approximated with a density less than 1 which is close to the original solution in W_2 distance and has similar energy E_m. This approximation as well as estimates between $\rho_{m,h}$ and $\rho_{\infty,h}$ obtained in Proposition 2.4.3 enable us to prove Theorem 2.4.2. Finally, by combining the uniform convergence results obtained in Theorem 2.3.4 and Theorem 2.4.2, we conclude with Theorem 2.1.2. Let us mention that the Γ-convergence approach ([Dal93] - [Ser11]) may apply here to derive the convergence of ρ_m to ρ_∞ in 2-Wasserstein distance. On the other hand our approach is more quantitative and thus provides, for example, convergence rates in terms of m.

Finally, in section 5, for any fixed $2 < m \leq \infty$, we present a comparison principle between solutions $\rho_{m,h}$ of the discrete-time scheme corresponding to free energy E_m when Φ is semi-convex (Theorem 2.5.1). As mentioned above this result is new for the (discrete) gradient flow solutions in the setting of Wasserstein distances. As applications of the comparison principle, we discuss some confinement results and the long time behavior of ρ_m for convex Φ in section 5.2 (Theorem 2.5.2).

2.1.4 Remarks on possible extensions

For simplicity of the presentation we did not consider the most general setting our approach could handle. Below we discuss some situations where our approach (partially) extends.

1. Our approach would apply, with little modification, to the problem confined in a convex domain $\Sigma \subset \mathbb{R}^d$ with Neumann boundary data. On the other hand our approach would not apply, at least in its direct form, if one puts an exit (e.g. Dirichlet) condition on parts of $\partial\Sigma$. The challenge is in showing the convergence of discrete solutions, due to the fact that the λ-convexity of the associated energy no longer holds. On the other hand, the analysis in section 2 and 3 should still go through to yield that the solution of (2.1.8) converges to the solution of (HS) in domain Σ, with corresponding boundary conditions.

2. In the case that $\Delta\Phi$ is not necessarily positive, and for general initial data $0 \leq \rho_0 \leq 1$, the results in sections 4 and 5 are still valid and one can conclude that the solutions ρ_m of (2.1.8) uniformly converges to a limiting profile ρ_∞ in 2-Wasserstein distance. In this case, the jammed region $\{\rho_\infty(\cdot, t) = 1\}$ no longer satisfies finite speed of propagation and may nucleate at times. Due to this reason further characterization of ρ_∞ beyond as a weak solution of (2.1.5) remains open.

2.2 On the continuum solutions

In section 2 and 3 we assume that Φ satisfies (**A1**). As mentioned before, we do not expect classical solutions to exist globally either for (HS) or (2.1.8). Hence to investigate qualitative behavior of solutions we begin by introducing the notion of weak solutions for (HS), in our case the *viscosity solutions.* This notion of solutions is particularly useful when we are interested in the stability properties of interface problems. Let us point out that solutions of (HS) may be discontinuous due to the quasi-static nature of the evolution, and due to the singularity of the free boundary. Therefore in the definition of viscosity solutions we need to consider semi-continuous functions, in contrast to the definitions of viscosity solutions in section 3. We introduce a definition using comparison with smooth functions similar to the one in [Kim03] and [Poz13].

Definition 2.2.1. *A nonnegative upper-semicontinuous function u defined in $Q := \mathbb{R}^d \times [0,\infty)$ is a viscosity subsolution of (HS) with compactly supported initial data u_0 if the following hold:*

(a) $u = u_0$ *at* $t = 0$ *and* $\{u_0 > 0\} = \overline{\{u(x,t) > 0\}} \cap \{t = 0\}$*;*

(b) $\{u > 0\} \cap \{t \le \tau\} \subset \overline{\{u > 0\} \cap \{t < \tau\}}$ *for every* $\tau > 0$ *;*

(c) For every $\phi \in C^{2,1}(Q)$ *that has a local maximum of* $u - \phi$ *in* $\overline{\{u > 0\}} \cap \{t \le t_0\}$ *at* (x_0, t_0)*,*

(i) if $(x_0, t_0) \in \{u > 0\}$*,* $-\Delta\phi(x_0, t_0) \le \Delta\Phi(x_0)$*, and*

(ii) if $(x_0, t_0) \in \partial\{u > 0\}$*,* $u(x_0, t_0) = 0$*, and if* $|\nabla\phi(x_0, t_0)| \ne 0$*, then*

$$\min(-\Delta\phi - \Delta\Phi, \phi_t - |\nabla\phi|^2 - \nabla\phi \cdot \nabla\Phi)(x_0, t_0) \le 0.$$

Note that the condition (c)(ii) is to ensure that limits of viscosity solutions are viscosity solutions, since the boundary can collapse in a limit and then boundary points of the limiting functions becomes interior points of the limit.

Definition 2.2.2. *A nonnegative lower-semicontinuous function v defined in $Q := R^d \times [0,\infty)$ is a viscosity supersolution of (HS) with initial data v_0 if the following hold:*

(a) $v = v_0$ *at* $t = 0$,

(b) For every $\phi \in C^{2,1}(Q)$ *that has a local minimum zero of* $v - \phi$ *in* $\mathbb{R}^d \times (0, t_0]$ *at* (x_0, t_0),

(i) if $(x_0, t_0) \in \{v > 0\}$, $-\Delta\phi(x_0, t_0) \geq \Delta\Phi(x_0)$, *and*

(ii) if $(x_0, t_0) \in \partial\{v > 0\}, v(x_0, t_0) = 0$ *and if*

$$|\nabla\phi(x_0, t_0)| \neq 0 \text{ and } \{\phi > 0\} \cap \{v > 0\} \cap B(x_0, t_0) \neq \emptyset \text{ for some ball } B \quad (2.2.1)$$

then

$$\max(-\Delta\phi - \Delta\Phi, \phi_t - |\nabla\phi|^2 - \nabla\phi \cdot \nabla\Phi)(x_0, t_0) \geq 0.$$

The condition (2.2.1) is to ensure that ϕ touches v from below in a non-degenerate way.

Let us define, for a function h in Q, the upper and lower semi-continuous envelopes of h:

$$h^*(x,t) := \lim_{\epsilon\to 0} \sup_{\{|x-y|,|t-s|\leq\epsilon\}} h(y,s), \quad h_*(x,t) := \lim_{\epsilon\to 0} \inf_{\{|x-y|,|t-s|\leq\epsilon\}} h(y,s). \qquad (2.2.2)$$

Definition 2.2.3. *u is a viscosity solution of (HS) with initial data u_0 if u_* and u^* are respectively viscosity sub- and supersolutions of (HS) with initial data u_0.*

We will discuss several properties of viscosity solutions which will be used in the main theorem of the article.

2.2.1 Inf- and Sup-convolutions

Next we introduce regularizations for viscosity solutions of (HS), which is by now standard for free boundary problems (see e.g. [CV99]). Given a viscosity subsolution u and $r > 0$, we define

$$\overline{u}_r = \sup_{B_r(x,t)} u(y,\tau) \quad \text{for } t \geq r, \tag{2.2.3}$$

and likewise given a viscosity supersolution v, and $r, \delta > 0$ with $\delta << r$, we define

$$\underline{v}_r = \inf_{B_{r-\delta t}(x,t)} v(y,\tau) \quad \text{for } t \geq r. \tag{2.2.4}$$

These are called the sup- and inf- convolutions, respectively, and serve to smooth out viscosity solutions to help analyze the speed of the free boundary. The following properties of $\overline{u}_r$ and $\underline{v}_r$ are direct consequences of their definitions.

Lemma 2.2.1. *(a) $\overline{u}_r$ is a viscosity subsolution of (HS). Moreover, at each point $(x_0, t_0) \in \partial\{\overline{u}_r > 0\}$ there exists a space-time* interior ball B *such that*

$$B \subset \{\bar{u}_r > 0\} \text{ and } \overline{B} \cap \{\bar{u}_r = 0\} = \{(x_0, t_0)\}.$$

(b) $\underline{v}_r$ is a viscosity supersolution of (HS). Moreover, at each point $(x_0, t_0) \in \partial\{\underline{v}_r > 0\}$ there exists a space-time exterior ball B *such that*

$$B \subset \{\underline{v}_r = 0\} \text{ and } \overline{B} \cap \overline{\{\underline{v}_r > 0\}} = \{(x_0, t_0)\}.$$

Let e_{n+1} denote the vector $(0, ..., 1)$ in Q. The following two lemmas will prove useful in our analysis later. The first lemma can be proven with a parallel proof to that of Lemma

2.5 in [Kim03] and thus we omit the proof. The second lemma is more interesting and involves ruling out the case of local total collapse of the solution, that is, the solution completely vanishing at a given time. The proof relies on (**A1**) to build a quadratic barrier subsolution.

Lemma 2.2.2 ($\{\bar{u}_r > 0\}$ cannot expand with infinite speed)**.** *Suppose* $(x_0, t_0) \in \partial\{\overline{u}_r > 0\}$*. Then the corresponding interior ball cannot have its outward normal as* e_{n+1} *at* (x_0, t_0)*.*

Lemma 2.2.3 ($\{\underline{v}_r > 0\}$ cannot shrink with infinite speed)**.** *Suppose* $(x_0, t_0) \in \partial\{\underline{v}_r > 0\}$*. Then the corresponding exterior ball cannot have its outward normal as* $-e_{n+1}$ *at* (x_0, t_0)*.*

Proof. 1. Suppose that $\{\underline{v}_r > 0\}$ has an exterior ball with outward normal $-e_{n+1}$ at a point (x_0, t_0). Then at (x_0, t_1), v will have an interior ball B_1 centered at (x_0, t_0) where $t_1 - t_0 = r - \delta t_0$ and B_1 has outward normal e_{n+1} at (x_0, t_1).

2. Fix a number λ satisfying

$$\lambda < \frac{1}{5 \max_{B_2(x_0,t_0)} |\nabla\Phi|}, \quad \lambda < 1, \quad \lambda << r - \delta t_0.$$

3. We define

$$\omega(x,t) := v\left(\lambda x + x_0, \lambda^2(t-1) + t_1\right).$$

This serves to map the cylinder

$$C_0 := \{(x,t) : |x - x_1| < \lambda, t_1 - \lambda^2 < t < t_1\}$$

to the cylinder $C := \{|x| < 1\} \times [0,1]$. Then ω is a viscosity solution of a re-scaled version of (HS):

$$\begin{cases} \Delta\omega + \lambda^2 \Delta\Phi_1 = 0 & \text{in} \quad \{\omega > 0\}; \\ \\ V = -\partial_\nu \omega - \lambda \partial_\nu \Phi_1 & \text{in} \quad \partial\{\omega > 0\}, \end{cases}$$

where Φ_1 is a rescaled and recentered version of Φ. By our choice of λ, the bottom of C is strictly contained in B_1, and so by lower semi-continuity we can find $\epsilon > 0$ satisfying $\omega > \epsilon$ at $t = 0$.

4. We construct our barrier. Define

$$\varphi := \alpha(1 - t/5 - x^2/2),$$

where we choose $\alpha > 0$ so that $\alpha < \min(\inf_C \Delta\Phi_1, \epsilon)$. Then $-\Delta\varphi = \alpha < \Delta\Phi_1$, and on the bottom of C, $\varphi < \epsilon < \omega$. On the sides of C, $\varphi < 0 < \omega$, so $\varphi < \omega$ on the parabolic boundary of C. However, $\varphi(0,1) = 4\alpha/5 > 0 = \omega(0,1)$, so they eventually cross.

5. We examine their crossing. To this end, we define T to be the first crossing time of φ and ω:

$$T := \inf\{t \geq 0 | \text{ there exists } x \in C \text{ s.t. } \omega(x,t) - \varphi(x,t) < 0\}$$

Then we can find a sequence (x_n, t_n) with $t_n \downarrow T$ and

$$\omega(x_n, t_n) - \varphi(x_n, t_n) \leq 0.$$

Now we are in a compact set so we can suppose that $x_n \to \bar{x} \in C$, and since ω is lower semi-continuous, we must have that

$$\omega(\bar{x}, T) - \varphi(\bar{x}, T) = -\beta \leq 0.$$

Then $(\bar{x}, T)$ must be in the parabolic interior of C.

The fact that this is a local minimum of $\omega - \phi$ follows since it is the first time ω and ϕ cross. We are done now because $-\Delta\varphi(\bar{x}, T) = \alpha < \Delta\Phi_1(\bar{x}, T)$ and

$$\varphi_t - |\nabla\varphi|^2 - \lambda\nabla\varphi \cdot \nabla\Phi_1 = -\alpha/5 - \alpha^2 x^2 - \lambda\alpha x \cdot \nabla\Phi_1 < 0,$$

where the final inequality comes from our assumption on λ. $\square$

2.2.2 Comparison principle

The central property of the viscosity solution theory is in the comparison principle, which we state below. The proof is mostly parallel to that of [Kim03], and thus we only sketch the outline of the proof.

We say two functions $u, v : \mathbb{R}^d \to \mathbb{R}^+$ are *strictly separated*, denoted by $u \prec v$, if

$$u < v \text{ in } \overline{\{u > 0\}} \quad \text{and } \overline{\{u > 0\}} \text{ is a compact subset of } \{v > 0\}.$$

Theorem 2.2.4. *Let u and v be respectively viscosity sub- and supersolutions of (HS). If $u(\cdot, 0) \prec v(\cdot, 0)$ then $u(\cdot, t) \leq v(\cdot, t)$ for all $t > 0$.*

Sketch of the proof

1. Due to the fact that $u_0 \prec v_0$, applying Definition 2.1 (a)-(b) and the semi-continuities of u and v, we have $\bar{u}_r(\cdot, r) \prec \underline{v}_r(\cdot, r)$ for sufficiently small $r > 0$.

2. We claim that $\bar{u}_r \leq \underline{v}_r$ for all times bigger than r, which yields our theorem. Hence suppose not, and define

$$t_0 := \sup\{t : u_r(\cdot, s) \prec v_r(\cdot, s) \text{ for } s \leq t\} < \infty.$$

One can then proceed as in [Kim03], using the above lemmas to exclude the possibility that $\bar{u}_r$ and $\underline{v}_r$ cross over each other discontinuously in time, to show that at $t = t_0$, there is a point x_0 such that

$$(x_0, t_0) \in \partial\{\bar{u}_r > 0\} \cap \partial\{\underline{v}_r > 0\}.$$

Moreover, there exists an interior ball B_1 to $\{\bar{u}_r > 0\}$ and an exterior ball B_2 to $\{\underline{v}_r > 0\}$ at (x_0, t_0) such that

$$\overline{B_1} \cap \overline{B_2} \cap \{t \leq t_0\} = (x_0, t_0).$$

Let (ν, m_1) be the interior normal to the interior ball B_1 and (ν, m_2) be the exterior normal to the exterior ball B_2 at (x_0, t_0), with $|\nu| = 1$. Due to the Lemmas 2.2.2 and 2.2.3, m_1 and m_2 are both finite. In particular at $t = t_0$ both the sets $\{\bar{u}_r > 0\}$ and $\{\underline{v}_r > 0\}$ have the interior space ball $B_1 \cap \{t = t_0\}$ with interior normal ν. Since $\bar{u}_r$ crosses $\underline{v}_r$ from below at (x_0, t_0), we have $m_1 \geq m_2$. Moreover at (x_0, t_0), the support of $\bar{u}_r$ propagates faster than normal velocity m_1, and the support of $\underline{v}_r$ slower than normal velocity m_2. Formally speaking, we would like to claim from the fact that $\bar{u}_r$ and $\underline{v}_r$ are respectively sub- and (strict) supersolutions of (HS) that

$$|\nabla \bar{u}_r| \geq m_1 + \mu \text{ and } |\nabla \underline{v}_r| < m_2 + \mu, \text{ where } \mu = \nabla\Phi(x_0, t_0) \cdot \nu. \tag{2.2.5}$$

From the claim, we deduce a contradiction since $\bar{u}_r \leq \underline{v}_r$ at $t = t_0$ and $m_1 \geq m_2$.

3. To prove (2.2.5) in the viscosity sense, we can use appropriate barriers to compare with $\bar{u}_r$ and $\underline{v}_r$, to measure the growth of these functions at x_0. This part of the proof is parallel to that of Theorem 2.2 in [Kim03]. Indeed the barriers corresponding to our problem (HS) are constant multiples of the ones constructed in Appendix A of [Kim03].
□

Remark 2.2.1. Let us point out that, due to the restriction on the strict separation of the initial data, the above comparison principle does not immediately yield the uniqueness of the solutions for (HS). Later in the chapter we will derive the uniqueness result (see Theorem 2.3.4), by showing that L^1-contraction holds between the characteristic functions of the positive sets of the viscosity solutions.

2.3 Approximation by degenerate diffusion with drift

As in section 2 we continue to assume (**A1**). Let ρ be a weak, continuous solution of (2.1.8), as given in [Vaz06]. We define the pressure variable u by

$$u := \frac{m}{m-1}\rho^{m-1}. \tag{2.3.1}$$

Then u formally solves

$$\text{(PME-D)}_m \qquad u_t = (m-1)u(\Delta u + \Delta\Phi) + |\nabla u|^2 + \nabla u \cdot \nabla\Phi.$$

In [CV99] (for $\Phi = 0$) and in [KL10] it was shown that u is a viscosity solution of (PME-D)$_m$. For completeness we review the definitions. First we define a *classical solution* of (PME-D)$_m$ as a nonnegative function $u \in C^{2,1}(\overline{\{u>0\}})$ that

(a) solves (PME-D)$_m$ in $\{u > 0\}$,

(b) has a free boundary $\Gamma = \partial\{u > 0\}$ which is a $C^{2,1}$ hypersurface, and

(c) Γ evolves with the outer normal velocity $|\nabla u| + \eta \cdot \nabla\Phi$, where η is the inward normal of Γ.

We then use the classical solutions as test functions to define viscosity solutions of (PME-D)$_m$.

Definition 2.3.1. *A non-negative continuous function u defined in $Q := \mathbb{R}^d \times (0,\infty)$ is a viscosity subsolution of (PME-D)$_m$ if for every $\phi \in C^{2,1}(Q)$ that has a local maximum* *zero of* $u-\phi$ *in* $\{t \le t_0\}$ *at* (x_0,t_0),

$$(\phi_t - (m-1)\phi(\Delta\phi + \Delta\Phi) - |\nabla\phi|^2 - \nabla\phi\cdot\nabla\Phi)(x_0,t_0) \le 0.$$

Definition 2.3.2. *A continuous function* $v : Q \to \mathbb{R}_+$ *is a viscosity supersolution of (PME-D)*$_m$ *if:*

(a) For every $\phi \in C^{2,1}(Q)$ *that has a local minimum zero of* $v-\phi$ *in* $\{v>0\}\cap\{t\le t_0\}$ *at* (x_0,t_0),

$$\big(\phi_t - (m-1)\phi(\Delta\phi + \Delta\Phi) - |\nabla\phi|^2 - \nabla\phi\cdot\nabla\Phi\big)(x_0,t_0) \ge 0.$$

(b) Any classical solution of (PME-D)$_m$ *that lies below* v *at time* $t_1 \ge 0$ *cannot cross* v *at a later time.*

Finally, u *is a viscosity solution of (PME-D)*$_m$ *with compactly supported initial data* u_0 *if it is both a viscosity subsolution and supersolution of (PME-D)*$_m$ *and both* $u(\cdot,t)$ *and* $\{u(\cdot,t)>0\}$ *uniformly converge to* u_0 *and* $\{u_0>0\}$ *as* $t\to 0$, *respectively, in uniform norm and in Hausdorff distance.*

Let us point out that the above definitions, based on comparison with classical solutions, are essentially in the same spirit as the definition of viscosity solutions of (HS) introduced in section 2.

2.3.1 Properties of u_m at the free boundary

Next we show that the support of (PME-D)$_m$ solutions have bounded jumps.

Lemma 2.3.1. *Suppose* u_m *is a solution to (PME-D)*$_m$ *in* $\mathbb{R}^d$. *Then for* $K>0$, *there exist constants* $r_{\max}, T>0$ *only depending on* $K, d,$ *and* Φ *near* x' *such that the following* *holds for any $r_0 < r_{\max}$: suppose $u_m(\cdot, t') = 0$ in $B_{r_0}(x')$ and $u_m \le K$ on the parabolic boundary of $B_{2r_0}(x') \times [t', t'+T]$. Then we have that $u_m = 0$ in $B_{r_0/4}(x') \times [t', t'+T]$.*

Proof. 1. We may assume that $(x', t') = 0$. Now to prove this theorem, first we construct a supersolution of (PME-D)$_m$ in $B_{2r_0}(0) \times [0, T]$ where T is yet to be determined. We start by constructing $u(r)$ on $B_{2r_0}(0)$ satisfying $u(r_0) = K$ and $u(r) = 0$ for $r \le r_0/2$. We take r_0 small enough so that $\sup_{B_{4r_0}(0)} |\nabla\Phi(x) - \nabla\Phi(0)| < 1$. Define $\alpha = \sup_{B_{4r_0}(0)} \Delta\Phi(x)$. Then we solve that $\Delta u = -\alpha$ in the annulus $B_{2r_0}(0) \backslash B_{r_0/2}(0)$. This yields u given by

$$u = \begin{cases} \frac{C}{r^{d-2}} - \frac{\alpha r^2}{2d} + D & \text{if } d \neq 2 \\ \\ C \ln r - \alpha r^2/4 + D & \text{if } d = 2. \end{cases}$$

We proceed assuming $d > 2$; the $d = 2$ case is similar. We choose C and D so that $u(r_0) = K, u(r_0/2) = 0$:

$$C = -r_0^{d-2} \frac{K + 3\alpha r_0^2/8d}{2^{d-2} - 1}, \quad D = K + \alpha r_0^2/2d + \frac{K + 3\alpha r_0^2/8d}{2^{d-2} - 1}.$$

By taking derivatives it can be seen that u has the largest derivative at $r_0/2$, and we then estimate:

$$u'(r_0/2) \le C(d, K)/r_0 + C(d, \Phi) d r_0^2.$$

Further, we notice that $u' \ge O(1/r_0^{d-1})$ and so if we take r_0 small enough we have $u'(r) \ge 1$ when $r_0/2 \le r \le 4r_0$. Then this entails that $u(r) \ge u(r_0) = K$ if $r \in [r_0, 4r_0]$, and we are done finding $r_{\max}$ which is the largest value of r_0 that makes the desired estimates hold.

2. Now let us define

$$\tilde{u}(r, t) := u(R(t) r),$$

where $R(t)$ is a function to be determined with $R(0) = 1$, $1 \leq R(t) \leq 3/2$. Then by construction of u,

$$\Delta\tilde{u} = R(t)^2(\Delta u)(R(t)r) = -\alpha R(t)^2 \leq -\alpha \leq 0.$$

Further straightforward computation yields that $\tilde{u}_t \geq 2|\nabla\tilde{u}|^2$ holds if

$$\frac{R'(t)}{R(t)^2} > 2\frac{u'(R(t)r)}{r}.$$

To this end, let us choose $R(t) = 1/(1 - Lt)$, where

$$L := C(N, \Phi, K)/r_0^2 = \frac{8}{r_0} \sup_{r\in[r_0/2,4r_0]} u'(r) \geq 2 \sup_{r\in[r_0/2,8r_0/3]} \frac{u'(R(t)r)}{r}.$$

3. Lastly we define

$$v(x, t) := \tilde{u}(x + \vec{b}t, t),$$

where $\vec{b} = \nabla\Phi(0, 0)$. We claim that v is a (PME-D)$_m$ supersolution in $B_{2r_0}(0) \times [0, T(d, \Phi, K)]$ for any choice of m. To see this, note that

$$v_t - \vec{b} \cdot \nabla v \geq 2|\nabla v|^2 \geq (m - 1)v(\Delta v + \Delta\Phi) + |\nabla v|^2 + |\nabla v|^2.$$

Now if $r_0 < r_{\max}$, we have that

$$|\nabla v|^2 - \nabla v \cdot (\nabla\Phi - \vec{b}) \geq |\nabla v|(|\nabla v| - |\vec{b} - \nabla\Phi| \geq |\nabla v|(|\nabla v| - 1).$$

Since we know that $|\nabla v| = u'(r) \geq R(t)u' \geq u' \geq 1$, the above quantity must be positive. Thus

$$v_t \geq (m - 1)v(\Delta v + \Delta\Phi) + |\nabla v|^2 + \nabla v \cdot \nabla\Phi,$$

and since $v_t \geq |\nabla v|^2 + \vec{b} \cdot \nabla v$ in general this holds at the boundary too. Thus v is a classical free boundary supersolution, and so by Lemma 2.6 in [KL10] it is a viscosity supersolution.

Lastly, T is chosen so that both $|\vec{b}|T \leq 2r_0/3$ and $T < 1/(3L)$, and lastly so that $T < r_0/(12|\vec{b}|)$. The first condition ensures that the bounds on $\tilde{u}$ in $B_{8r_0/3}$ hold for v in B_{2r_0} and the second ensures that $R(t) \leq 3/2$. The last one ensures that if $|x| \leq r_0/4$, $|x + \vec{b}t| \leq |x| + r_0/12 \leq r_0/3 \leq r_0/2R(t)$ and hence

$$v(x,t) = u(R(t)(x + \vec{b}t), t) = 0 \text{ in } |x| \leq r_0/4.$$

Now by construction $u_m \leq v$ on the parabolic boundary of $B_{2r_0}(0) \times [0, T(d, \Phi, K)]$, so we can apply a comparison principle, Theorem 2.25 in [KL10], to find that

$$u_m \leq v \text{ in } B_{2r_0}(0) \times [0, T(d, \Phi, K)].$$

Then observing the properties of v, we are done. □

Our next lemma shows that we can construct a (PME-D)$_m$ subsolution that is almost a supersolution.

Lemma 2.3.2. *Fix $\epsilon > 0$, $m > 0$, a number γ, a point x', and vector $\vec{n}$. Then if $\gamma > \nabla\Phi(x') \cdot \vec{n}$, there exists a positive constant η depending on ϵ so that we can construct a classical subsolution S of $(PME - D)_m$ in $E_\eta := B_\eta(x') \times [-\eta, \eta]$ with $(x', 0)$ on its free boundary with outward normal $\vec{n}$, which moves with normal velocity γ. Further, S will "almost" be a supersolution near $(x', 0)$ in the following sense:*

$$\gamma \geq |\nabla S| + \nabla S \cdot \nabla\Phi - \epsilon \text{ at } (x', 0).$$

Proof. Recall that the Barenblatt profiles are given by

$$B(x, t; \tau, C) = \frac{(C(t+\tau)^{2\lambda} - K|x - x_0|^2)_+}{(t + \tau)},$$

where $C > 0$ and $\lambda = ((m-1)n + 2)^{-1}$, $K = \lambda/2$, and they are solutions of (PME)$_m$. C and τ are parameters that control the free boundary speed and initial support.

Now we change variables so that $\vec{n}$ is colinear with x', and take $x_0 = 0$. Then it suffices to take $x_0 = 0$. We start with $B(x,t)$: a Barenblatt solution with initial support $B_R(0)$, and initial free boundary advancement speed ξ where R and ξ will be determined later. We fix $r(t) = \mu - \nu t$, with μ, ν as yet unspecified. Then we define

$$\tilde{S}(x,t) = \sup_{y \in B_{r(t)}(x)} B(y,t) = B((1 - r(t)/|x|)x, t) \text{ in } E_\eta,$$

where η is for now much smaller than $R/2$.

Note that

$$\tilde{S}_t = B_t - r'(t)\nabla B \cdot \frac{x}{|x|} = B_t + r'(t)|\nabla B| = B_t - \nu|\nabla B|.$$

Moreover in E_η, since $\eta < R/2$, then E_η is bounded away from the origin and we get $1/|x| \leq 2/R$. Thus we find that

$$\frac{\partial \tilde{S}}{\partial x_j} = \frac{\partial B}{\partial x_j} + \mu|\nabla B|O(1/R).$$

Thus $\nabla\tilde{S} = \nabla B + O(\mu)$ and since $|\nabla B|$ does not vary fast in E, we can repeat and find $\Delta\tilde{S} = \Delta B + O(\mu)$. Using that B is a $(PME)_m$ solution then gives

$$\tilde{S}_t = B_t - \nu|\nabla B| = (m-1)\tilde{S}\Delta\tilde{S} + |\nabla\tilde{S}|^2 - \nu|\nabla\tilde{S}| + O(\mu).$$

Next let us define

$$S(x,t) = \tilde{S}(x + \vec{b}t, t), \text{ where } \vec{b} = \nabla\Phi(x', 0).$$

Then $S_t = \tilde{S}_t + \nabla\tilde{S}\cdot\vec{b}$, and we conclude that

$$S_t = (m-1)S(\Delta S + \Delta\Phi) + |\nabla S|^2 + \nabla S\cdot\nabla\Phi - (m-1)S\Delta\Phi - \nu|\nabla S| + O(\mu).$$

Now in E_η, $S(x,t) \leq 2\eta\sup_{E_\eta}|\nabla S| = O(\eta)$, so $(m-1)S\Delta\Phi = O(\eta)$. Therefore

$$S_t = (m-1)S(\Delta S + \Delta\Phi) + |\nabla S|^2 + r'(t)|\nabla S| + \nabla S\cdot\nabla\Phi + O(\mu) + O(\eta).$$

At this point we have to start picking our parameters carefully. First, we can assume that ϵ is small enough so that $\epsilon < \inf_{E_\eta \cap \{S>0\}} |\nabla S|/6$ for some small value of η. Then we take

$$\nu = \epsilon/3, \quad \xi = \gamma - \vec{b} \cdot x' + \nu > 0.$$

Now we take η, μ small enough so that in E_η,

$$|O(\mu) + O(\eta)| < \epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S|/3,$$

and we set $R = |x'| - \mu = 1 - \mu$. Now we refine η so that

$$\sup_{E_\eta \cap \{S>0\}} |\nabla S| - \inf_{E_\eta \cap \{S>0\}} |\nabla S| < \epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S|.$$

Then our choice of ν gives us the estimates

$$\nu |\nabla S| \geq \frac{\epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S|}{3},$$

while also

$$\nu |\nabla S| \leq \frac{\epsilon \sup_{E_\eta \cap \{S>0\}} |\nabla S|}{3} < \frac{\epsilon (\inf_{E_\eta \cap \{S>0\}} |\nabla S| + \epsilon)}{3} = \epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S|/3 + \epsilon^2/3 \leq \epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S|/2,$$

where we used our assumption on ϵ small. Thus we find that

$$-\epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S| \leq r'(t)|\nabla S| + O(\eta) + O(r(t)) \leq 0,$$

and so finally

$$(m-1)S(\Delta S + \Delta \Phi) + |\nabla S|^2 + \nabla S \cdot \nabla \Phi - \epsilon \inf_{E_\eta \cap \{S>0\}} |\nabla S| \leq S_t \leq (m-1)S(\Delta S + \Delta \Phi) + |\nabla S|^2 + \nabla S \cdot \nabla \Phi.$$

Then we are done, since it is clear that $(x', 0)$ is on the free boundary of S and the free boundary has initial velocity γ.

$\square$

We remark that the definition of viscosity supersolutions of (PME-D)$_m$ only applies in $\{v > 0\}$ in order to make the viscosity solution notion be equivalent to the idea of weak solutions. This has the consequence of needing extra effort to analyze the behavior at the free boundary, which is provided by the following lemma. Its proof is analogous to Lemma 1.7 in [Kim03], with the difference in the construction of barriers.

Lemma 2.3.3. *Let v be a viscosity supersolution (subsolution) of (PME-D)$_m$, and suppose that ϕ is a smooth function where $v - \phi$ has a local minimum (maximum) zero in $\overline{\{v > 0\}}$ at $(x_0, t_0) \in \partial\{v > 0\}$ with $t_0 > 0$. If ϕ satisfies* (2.2.1) *at (x_0, t_0), then*

$$(\phi_t - |\nabla\phi|^2 - \nabla\phi \cdot \nabla\Phi)(x_0, t_0) \geq (\leq) \quad 0. \tag{2.3.2}$$

Proof. First note that the subsolution case by definition is trivial as discussed above, since $\phi(x_0, t_0) = 0$. Thus we proceed to the supersolution case. We may set $t_0 = 0$ after a translation.

Let us fix constants $r, \delta > 0$ and prove the lemma for the inf-convolution of v,

$$W(x,t) = \inf_{B_{r-\delta t}(x,t)} v(y, \tau).$$

Then the lemma follows by taking $\delta \to 0$ and then $r \to 0$.

Now suppose that for a smooth ϕ, $W - \phi$ has a local minimum in $\overline{\{W > 0\}}$ at $(x_0, 0) \in \partial\{W > 0\}$, with ϕ satisfying (2.2.1). By perturbing ϕ we may assume that the minimum is strict. Let H be the hyperplane tangent to $\{\phi > 0\}$ at $(x_0, 0)$, with (ν, γ) the inward normal to H with $|\nu| = 1$. Note that $\gamma > -\infty$ from Corollary 2.16 in [KL10]. Let $\alpha = |\nabla\phi|(x_0, 0) = \phi_\nu(x_0, 0) > 0$. Towards a contradiction, we assume that (2.3.2) fails, and so it follows that for some $\sigma > 0$

$$\gamma = V_\phi = \frac{\phi_t}{\phi_\nu}(x_0, 0) < (\alpha - \sigma) + \nu \cdot \nabla\Phi(x_0). \tag{2.3.3}$$

Hence γ is finite. Moreover we have

$$W(x,t) \geq \phi(x,t) \text{ in } B_\eta(x_0) \times [-\eta, 0] \text{ for } \eta << 1. \tag{2.3.4}$$

Due to the regularity of ϕ, there exists a space ball B_0 interior to the set $\{x : \phi(x,0) > 0\}$ with $x_0 \in \partial B_0$. We define γ_1 as follows:

$$\gamma_1 := \begin{cases} \gamma + \sigma/4 & \text{if } \gamma \geq \nabla\Phi(x_0)\cdot\nu \\ \alpha/2 + \nabla\Phi(x_0)\cdot\nu & \text{otherwise.} \end{cases}$$

Then we use the result of Lemma 2.3.2 to find a classical subsolution S of $(PME-D)_m$ in a neighborhood $B_\eta(x_0)\times[-\eta,\eta]$ that firstly has initial support inside B_0, secondly has advancing speed γ_1 at $(x_0, 0)$, and lastly has a parameter $0 < \epsilon < \min(\sigma/4, \alpha/4)$ such that S satisfies

$$\gamma_1 \geq |\nabla S| + \nu\cdot\nabla\Phi - \epsilon \text{ at } (x_0, 0). \tag{2.3.5}$$

This condition helps us to show that it initially lies under ϕ.

We now claim that S lies under W in $B_\eta(x_0)\times[-\eta,0]$ for sufficiently small η , which will yield the desired contradiction to the fact that S is a subsolution and W is a supersolution, since S will cross W at $(x_0, 0)$.

Due to (2.3.3) and (2.3.5), we have

$$|\nabla S|(x_0,0) < \begin{cases} \alpha - \frac{\sigma}{2} & \text{if } \gamma \geq \nabla\Phi(x_0)\cdot\nu \\ 3\alpha/4 & \text{otherwise} \end{cases} \tag{2.3.6}$$

$$< \alpha = |\nabla\phi|(x_0,0). \tag{2.3.7}$$

On the other hand, observe that the support of S propagates with the normal speed faster than that of ϕ at $(x_0, 0)$ due to (2.3.5). Due to the regularity of ϕ and S and their ordering at $t = 0$ it then follows that

$$\overline{\{S > 0\}} \subset \{\phi > 0\} \text{ in } B_\eta(x_0) \times [-\eta, 0] \qquad (2.3.8)$$

if η is sufficiently small. From the above two inequalities it follows that $S \leq \phi$ in $B_\eta(x_0) \times [-\eta, 0]$ if η is sufficiently small. We can now conclude using the fact that $\phi \leq W$ in that neighborhood.

$\square$

2.3.2 Characterization of the half relaxed limits of u_m as $m \to \infty$

Let Ω_0 and u_0 as given in the introduction, and let u_m be the unique viscosity solution to (PME-D)$_m$ with the initial data u_0. Recall that u_m is given as the pressure variable of ρ_m by (2.3.1), where ρ_m assumes the corresponding initial data $(\frac{m-1}{m}u_0)^{1/(m-1)}$. Let us then define

$$\begin{aligned} u_1(x,t) &= \inf_{n \geq 0} \sup_{\substack{m \geq n \\ |(x,t)-(y,s)|<1/n}} u_m(y,s); \\ u_2(x,t) &= \sup_{n \geq 0} \inf_{\substack{m \geq n \\ |(x,t)-(y,s)|<1/n}} u_m(y,s). \end{aligned}$$

Note that the $\{u_m\}$ are uniformly bounded in m, as a consequence with comparison with stationary solutions of the form $(C - \Phi(x))_+$ with sufficiently large $C > 0$. Hence u_1 and u_2 are both finite.

Since we cannot guarantee that the support of u_1 traces those of u_m, we need to define an auxiliary function. Let us define the function

$$\eta(x,t) := \limsup_{\substack{m \to \infty \\ (y,s) \to (x,t)}} \chi_{\{\text{supp}(u_m)\}}(y,s),$$

and the closure of the support of η:

$$\Omega = \overline{\{(x,s) : \eta(x,s) > 0\}}, \quad \Omega(t) := \Omega \cap \{s = t\}.$$

Finally, let us define the largest subsolution of the Poisson equation $-\Delta w = \Delta\Phi$ supported in Ω:

$$\tilde{u}_1 := [\sup\{v : \mathbb{R}^d \times (0,\infty) \to \mathbb{R} \text{ such that } -\Delta v \le \Delta\Phi \text{ and } v = 0 \text{ outside of } \Omega\}]^*.$$

Here f^* denotes the upper semicontinuous envelope of f, as defined in (2.2.2).

Note that then $\tilde{u}_1 = 0$ outside of Ω and for each $t > 0$, $\tilde{u}_1$ satisfies

$$-\Delta\tilde{u}_1(\cdot,t) \le \Delta\Phi \text{ in } \mathbb{R}^d, \quad -\Delta\tilde{u}_1(\cdot,t) = \Delta\Phi \text{ in the interior of } \Omega(t).$$

This auxiliary function $\tilde{u}_1$ is indeed the new component of the proof compared to the corresponding theorem in [Kim03]. We point out that $\tilde{u}_1$ is positive in the interior of $\Omega(t)$ due to (**A1**).

Theorem 2.3.3. *Let u_1, u_2 and $\tilde{u}_1$ be as given above, and let Φ satisfy (**A1**). Then $\tilde{u}_1$ is a viscosity subsolution of (HS) in Q, and u_2 is a viscosity supersolution of (HS) in Q with initial data u_0.*

Proof. First note that u_2 is lower semicontinuous by its definition. Likewise, u_1 is upper semicontinuous.

A. u_2 is a supersolution:

1. Suppose we have a smooth function ϕ and $u_2 - \phi$ has a local minimum at (x_0, t_0) in $\overline{\{u_2 > 0\}} \cap \{t \le t_0\}$. By adding $\epsilon(t - t_0) - \epsilon(x - x_0)^2 + c$ to ϕ one may assume that the minimum is zero, and is strict in $C_r \cap \overline{\{u_2 > 0\}}$, where $C_r := B_r(x_0) \times [t_0 - r, t_0]$ for small $r > 0$.

If (x_0, t_0) is in $\{u_2 > 0\}$, by lower-semicontinuity of u_2, we can make r smaller and assume that $C_r \subset \{u_2 > 0\}$. On the other hand if $(x_0, t_0) \in \partial\{u_2 > 0\}$, we can assume that (2.2.1) holds for ϕ. In particular, $|\nabla\phi| \neq 0$ so that $u_2 - \phi > 0$ in $C_r \cap \{u_2 > 0\}^c$ away from (x_0, t_0). Thus in either case we can find that $u_2 - \phi$ has a strict local minimum zero in all of C_r.

2. We now claim the following: if r is sufficiently small, along a subsequence $u_m - \phi$ has a minimum at points $(x_m, t_m) \in C_r$ with $(x_m, t_m) \to (x_0, t_0)$ and $(x_m, t_m) \in \overline{\{u_m > 0\}}$.

To show this, define $(x_m, t_m) = \text{argmin}_{C_r}\,(u_m - \phi)$; we can assume that the sequence only ranges over m that achieve the infimum of u_2 at (x_0, t_0). Let (x', t') be a limit point of $\{(x_m, t_m)\}_m$.

First let us show that upon further refinement of our sequence we have $(x_m, t_m) \in \overline{\{u_m > 0\}}$. Clearly this is true if $(x_0, t_0) \in \{u_2 > 0\}$, and thus suppose $(x_0, t_0) \in \partial\{u_2 > 0\}$ and (x_m, t_m) lies outside of the support of u_m. Then we can assume that (2.2.1) holds for ϕ, so in particular we can assume that $|\nabla\phi| \neq 0$ in C_r. This rules out the possibility that (x_m, t_m) lies in the interior of C_r. Also, in this case $\phi(x_0, t_0) = 0$, and so we can find $\alpha > 0$ so that $\phi < -\alpha < 0$ on $\partial C_r \cap \{u_2 > 0\}^c$. This rules out the possibility that (x_m, t_m) lies on the boundary of C_r. Thus we conclude that $(x_m, t_m) \in \overline{\{u_m > 0\}}$ for sufficiently large m.

Next let us verify that $(x', t') = (x_0, t_0)$. By definition for arbitrary (y, s) in C_r

$$(u_m - \phi)(y, s) \geq (u_m - \phi)(x_m, t_m). \tag{2.3.9}$$

Since $(x_m, t_m) \to (x', t')$, for each n there is $M(n)$ so that $|(x_m, t_m) - (x', t')| < 1/n$ if

$m \geq M$, and we may assume that $M(n) \geq n$. Then

$$\inf_{m\geq M(n)} u_m(x_m,t_m) \geq \inf_{\substack{m\geq M(n)\\ |(x',t')-(y,s)|<1/n}} u_m(y,s) \geq \inf_{\substack{m\geq n\\ |(x',t')-(y,s)|<1/n}} u_m(y,s).$$

Taking $\sup_n$ on both sides we find $\liminf_{m\to\infty} u_m(x_m,t_m) \geq u_2(x',t')$. Then, taking $\liminf$ of both sides of (2.3.9) as $(y,s) \to (x_0,t_0)$ and $m \to \infty$, we find

$$(u_2-\phi)(x_0,t_0) \geq (u_2-\phi)(x',t'),$$

which contradicts that (x_0,t_0) is the strict minimum of $u_2-\phi$ in C_r. This proves our claim.

3. To finish showing that u_2 is a viscosity supersolution, take ϕ and (x_m,t_m) as given above. When $(x_0,t_0) \in \{u_2 > 0\}$, a straightforward computation using the properties of u_m as viscosity solutions of (PME-D)$_m$ gives

$$-\Delta\phi(x_0,t_0) \geq \Delta\Phi(x_0,t_0),$$

as needed. Next suppose $(x_0,t_0) \in \partial\{u_2 > 0\}$, and that (2.2.1) holds for ϕ. Suppose towards a contradiction that there is $\alpha > 0$ so that

$$\max(-\Delta\phi - \Delta\Phi, \phi_t - |\nabla\phi|^2 - \nabla\phi\cdot\nabla\Phi)(x_0,t_0) = -\alpha < 0. \tag{2.3.10}$$

Let us define $\phi_m := \phi + C(m)$ so that $(u_m - \phi_m)(x_m,t_m) = 0$. Since $(x_m,t_m) \to (x_0,t_0)$, (2.3.10) yields that

$$\big((\phi_m)_t - (m-1)\phi_m(\Delta\phi_m + \Delta\Phi) - |\nabla\phi_m|^2 - \nabla\phi_m\cdot\nabla\Phi\big)(x_m,t_m) < 0,$$

which contradicts with the fact that u_m is viscosity solution of (PME-D)$_m$. Thus we have $(x_m,t_m) \in \partial\{u_m > 0\}$. But then the inequality

$$((\phi_m)_t - |\nabla\phi_m|^2 - \nabla\phi_m\cdot\nabla\Phi)(x_m,t_m) < -\alpha/2 < 0$$

contradicts Lemma 2.3.3, which applies since ϕ is smooth and so satisfies (2.2.1) at (x_m, t_m) for large m.

B. $\tilde{u}_1$ is a subsolution

The subsolution part of our theorem is harder to prove, since a smooth test function touching $\tilde{u}_1$ at a free boundary point (x_0, t_0) from above in $\overline{\Omega}$ cannot be extended smoothly to outside of Ω so that the order is preserved. Thus the proof of B requires a careful study of the behavior of the free boundary of $\tilde{u}_1$, which is achieved by studying the properties of u_1 and u_m. First note that Definition 2.2.1 (b) is satisfied due to Lemma 2.3.1. We proceed to show the property given in Definition 2.2.1 (c).

1. It is straightforward from the definition of $\Omega(t)$ that

$$\{u_1(\cdot, t) > 0\} \subset \Omega(t). \tag{2.3.11}$$

Parallel arguments to the supersolution case yield that $-\Delta u_1 \leq \Delta\Phi$ in $\{u_1 > 0\}$ in the viscosity sense. Thus it follows that

$$u_1 \leq \tilde{u}_1. \tag{2.3.12}$$

Suppose that we have a smooth function ϕ and $\tilde{u}_1 - \phi$ has a strict local maximum at (x_0, t_0) in $\Omega \cap \{t \leq t_0\}$. As mentioned before $\tilde{u}_1$ satisfies $-\Delta\tilde{u}_1(\cdot, t) \leq \Delta\Phi$, indeed with equality in the interior of $\Omega(t)$. Thus to check that $\tilde{u}_1$ is a subsolution, it is enough to consider the case when $x_0 \in \partial\Omega(t_0)$ and $\tilde{u}_1(x_0, t_0) = 0$. Note that in this case (2.3.12) yields that $u_1 - \phi$ also has a local maximum at (x_0, t_0) in $\Omega \cap \{t \leq t_0\}$. Now suppose towards a contradiction that

$$\alpha := \min(-\Delta\phi - \Delta\Phi, \phi_t - |\nabla\phi|^2 - \nabla\phi \cdot \nabla\Phi)(x_0, t_0) > 0.$$

Then since ϕ and Φ are smooth, it follows that for a small $r > 0$

$$\min(-\Delta\phi - \Delta\Phi, \phi_t - |\nabla\phi|^2 - \nabla\phi \cdot \nabla\Phi) > 2\alpha/3 \quad \text{in } C_r := B_r(x_0) \times [t_0 - r, t_0].$$

2. Let Γ be the parabolic boundary of C_r. We claim that $\Gamma \cap \Omega \subset \{\phi \geq \delta_0\}$ for some $\delta_0 > 0$.

To see this, suppose the claim is false: this means that we can find $(y, s) \in \Gamma \cap \Omega \cap \{\phi > 0\}^c$. But then $\phi(y, s) \leq 0$ and so $(\tilde{u}_1 - \phi)(y, s) \geq -\phi(y, s) \geq 0$ which violates the assumption that $\tilde{u}_1 - \phi$ is strictly negative in $\Gamma \cap C_r$.

3. Now we proceed to show that $u_m < \phi$ on the relevant part of the parabolic boundary, that is, there exists some $\epsilon > 0$ independent of m such that

$$u_m < \phi - \epsilon \text{ on } \Gamma \cap \operatorname{supp}(u_m) \text{ for sufficiently large } m. \tag{2.3.13}$$

To show (2.3.13), suppose not. Then we can find $(x_k, t_k) \in \Gamma \cap \operatorname{supp}(u_{m_k})$ where $u_{m_k}(x_k, t_k) \geq \phi(x_k, t_k) - \frac{1}{k}$, and by compactness we can assume $(x_k, t_k) \to (x', t') \in \Gamma \cap \Omega$. Then we have that for each n, there is $K(n)$ so that $|(x_k, t_k) - (x', t')| < 1/n$ and $m_k \geq k$ if $k \geq K(n)$, where we can assume $K(n) \geq n$. Then

$$\sup_{k \geq K(n)} u_{m_k}(x_k, t_k) \leq \sup_{\substack{k \geq K(n) \\ |(x',t')-(y,s)|<1/n}} u_k(y, s) \leq \sup_{\substack{k \geq n \\ |(x',t')-(y,s)|<1/n}} u_k(y, s).$$

Taking the infimum over both sides, we find

$$u_1(x', t') \geq \limsup_{k \to \infty} u_{m_k}(x_k, t_k) \geq \limsup_{k \to \infty} \phi(x_k, t_k) = \phi(x', t') > \delta_0,$$

which contradicts that $u_1 - \phi < 0$ on $\Gamma \cap \Omega$.

4. Now we define

$$\xi(x, t) = \phi(x + \gamma\nu, t), \text{ where } \nu = -\frac{\nabla\phi}{|\nabla\phi|}(x_0, t_0).$$

Here $\gamma > 0$ is chosen small enough to satisfy first that

$$\min(-\Delta\xi - \Delta\Phi, \xi_t - |\nabla\xi|^2 - \nabla\xi \cdot \nabla\Phi) \geq 2\alpha/3 + O(\gamma) > \alpha/3 \text{ in } C_r,$$

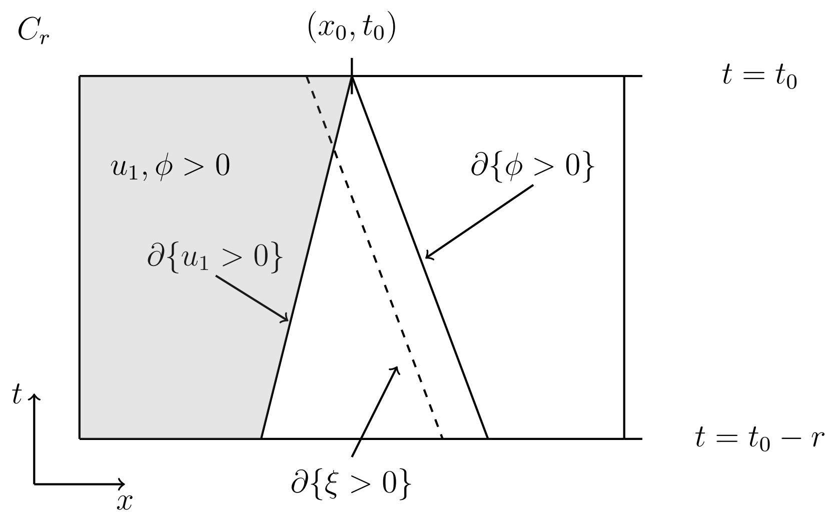

Figure 2.2: The motivation for ξ: it crosses u_1 at an earlier time than ϕ does

and secondly that for m large, $u_m < \xi$ on $\Gamma \cap \text{supp}(u_m)$. This is possible since $\xi - \phi = O(\gamma)$ and $u_m - \phi$ is bounded away from zero on $\Gamma \cap \text{supp}(u_m)$.

This justifies the following definition:

$$\tau_m := \sup\{t : (u_m - \xi)(x,t) < 0 \text{ for all } x \in \overline{\{u_m(\cdot,t) > 0\}} \cap C_r\}.$$

Then τ_m will be the first crossing time of u_m and ϕ, provided they cross (since u_m is continuous, we need not worry about jumps inside its support).

5. We now wish to show that, along a subsequence, u_m crosses ξ in C_r. To do this, we first prove that there is a subsequence $\{m_k\}$ so that

$$C_\delta \cap \{t < t_0\} \cap \text{supp}(u_{m_k}) \neq \emptyset \text{ for all } \delta > 0.$$

To show this first observe that, since $u_1(x_0, t_0) = 0$, there exists M so that

$$\sup_{\substack{m \geq M \\ |(x_0,y_0)-(y,s)|<1/M}} u_m(y,s) < 1. \tag{2.3.14}$$

Now assume towards a contradiction that there is a δ where our claim fails; we can take δ small enough so that $\delta < M^{-1}$. We use Lemma 2.3.1 to derive our contradiction. First, we use the theorem to find positive numbers $r_{\max}, T$ that depend on $K = 1$, the behavior of Φ near (x_0, t_0), and dimension, and we may assume $T < \delta/8$. Now set r_0 a number smaller than $\min(r_{\max}, \delta/8)$ and $\mu = \min(T/8, r_0/8)$. By definition of Ω, we can find (x', t') within distance μ of (x_0, t_0) where $\eta(x', t) = 1$. Thus we can find a subsequence $\{m_k\}_{k=1}^{\infty}$ all bigger than M and points $(y_{m_k}, s_{m_k}) \in \operatorname{supp} u_{m_k}$ within distance μ of (x', t').

Consider a specific m_k. Then by the assumption that $u_{m_k} = 0$ in $C_\delta \cap \{t < t_0\}$, since r_0 and T_0 are chosen much smaller than δ, we find $u_{m_k} = 0$ in $B_{r_0}(x') \cap \{t = t_0 - T/2\}$. Further, by (2.3.14),

$$u_{m_k} \leq 1 \text{ on the parabolic boundary of } B_{2r_0}(x') \times [t', t' + T].$$

Thus we apply Lemma 2.3.1 to find that

$$u_{m_k} = 0 \in B_{r_0/4}(x') \times [t_0 - T/2, t_0 + T/2],$$

and by the size of r_0, we have that $y_{m_k} \in B_{r_0/4}(x')$ and $s_{m_k} < t_0 + T/2$. This yields $u_{m_k} = 0$ in a neighborhood of (y_{m_k}, s_{m_k}). This is a contradiction, so we find our claim holds for every m_k, giving us our desired subsequence.

6. We will use the subsequence from the previous step to show that $\tau_{m_k} < t_0$. Indeed note that, for $|\alpha|, \beta > 0$,

$$\begin{aligned}\xi(x_0 + \alpha, t_0 - \beta) &= -\gamma|\nabla\phi(x_0, t_0)| + \alpha \cdot \nabla\phi(x_0, t_0) \\ &\quad -\beta\phi_t(x_0, t_0) + O(|\alpha|^2 + \beta^2 + \alpha\beta + \gamma^2).\end{aligned}$$

Thus there exists $\delta = \delta(\gamma) > 0$ so that if $|\alpha|^2 + \beta^2 < \delta^2$ then $\xi(x_0 + \alpha, t_0 - \beta) < 0$. Due to the previous claim we can now find points $(y_{m_k}, s_{m_k}) \in C_\delta \cap \{t < t_0\} \cap \operatorname{supp}(u_{m_k})$, hence

$$(u_{m_k} - \xi)(y_{m_k}, s_{m_k}) > 0.$$

Thus $\tau_{m_k} \leq s_{m_k} < t_0$.

7. Consequently there exists a crossing point $(x_{m_k}, \tau_{m_k}) \in C_r \cap \{t < t_0\} \cap \mathrm{supp}(u_{m_k})$ where $(u_{m_k} - \xi)(x_{m_k}, \tau_{m_k}) = 0$. Further, since $u_{m_k} - \xi < 0$ on $\Gamma \cap \mathrm{supp}\,(u_{m_k})$ from step 3, we have that (x_{m_k}, τ_{m_k}) is on the parabolic interior of C_r. Then we have that

$$\min(-\Delta\xi - \Delta\Phi, \xi_t - |\nabla\xi|^2 - \nabla\xi \cdot \nabla\Phi)(x_{m_k}, \tau_{m_k}) > \alpha/3,$$

which forces that $(x_{m_k}, \tau_{m_k}) \in \partial\{u_{m_k} > 0\}$. But then the inequality

$$\left[\xi_t - |\nabla\xi|^2 - \nabla\xi \cdot \nabla\Phi\right](x_{m_k}, \tau_{m_k}) > 0$$

contradicts Lemma 2.3.3, which applies since ξ is smooth and thus satisfies (2.2.1) at (x_{m_k}, τ_{m_k}).

C. $\tilde{u}_1, u_2$ converges to u_0 at $t = 0$

It is not hard to check via comparison with radial barriers of $(\mathrm{PME})_m$, based on the local Lipschitz geometry of $\partial\Omega_0$, that $\Omega(t)$ and $\{u_2(\cdot, t) > 0\}$ converges to Ω_0 in Hausdorff distance as $t \to 0^+$. From this fact and that u_0 solves $-\Delta u_0 = \Delta\Phi$ in the interior of Ω_0, we have $\lim_{\tau\to 0} \tilde{u}_1(\cdot, \tau) = u_0$ from the definition of $\tilde{u}_1$. On the other hand u_2 satisfies $-\Delta u_2 > \Delta\Phi$ in $\{u_2 > 0\} \cap \{t > 0\}$ and thus we have $\liminf_{\tau\to 0} u_2(\cdot, \tau) \geq u_0$. Since $u_2 \leq \tilde{u}_1$ by definition, it follows that $u_2(\cdot, \tau)$ converges to u_0 as $\tau \to 0$ as well. □

2.3.3 Convergence of u_m as $m \to \infty$

Now let us fix a compact set Ω_0 in $\mathbb{R}^d$ with Lipschitz boundary, and let u_0 be as given in (2.1.1). Let u_m be the viscosity solution of $(\mathrm{PME})_m$ with initial data u_0. If we knew that $\{u_m\}$ locally uniformly converges to a function u as $m \to \infty$, then Theorem 2.3.3 would yield that u is a viscosity solution of (HS). Unfortunately we do not know whether such convergence is true: due to the quasi-static nature of (HS), u may not be continuous over time and this may complicate the convergence of u_m. Thus we take the alternative approach to show the convergence of the support of $\{u_m > 0\}$ (see Theorem 2.3.4 (b)). The proof relies on the fact that $\{u_m\}$ has a stability property obtained from the L^1 contraction of the corresponding density function ρ_m given by (2.3.1). Using this stability as well as the comparison principle (Theorem 2.2.4) we will obtain that the support of u_m converges to the set Ω_t, which is characterized as the unique support for solutions of (HS). From this result we then obtain the uniform convergence of ρ_m to the characteristic function of Ω_t away from the boundary of Ω_t (Corollary 2.3.1).

Theorem 2.3.4. *Take Ω_0 and u_0 as given above and let Φ satisfy ($\boldsymbol{A1}$). Then the following hold:*

(a) There exists a unique evolution of compact sets $\{\Omega_t\}_{t>0}$ such that any viscosity solution u of (HS) satisfies $\Omega_t = \overline{\{u(\cdot,t)>0\}}$ for each $t>0$.

(b) For each $t>0$, the Hausdorff distance $d_H(\Omega_t, \overline{\{u_m(\cdot,t)>0\}})$ goes to zero as $m \to \infty$, and $\limsup_{m\to\infty} u_m(\cdot,t)$ is uniformly bounded.

Proof. 1. The proof is based on the L^1-contraction property (see e.g. section 3.5 of [Vaz06]), which states that for two weak solutions ρ_1, ρ_2 of (2.1.8), the L^1 norm of their differences decreases in time. In terms of the pressure variable $p_i = \frac{m}{m-1}\rho_i^{m-1}$, this reads

$$\|(p_1^{1/(m-1)} - p_2^{1/(m-1)})(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|(p_1^{1/(m-1)} - p_2^{1/(m-1)})(\cdot,0)\|_{L^1(\mathbb{R}^d)}. \tag{2.3.15}$$

Let us fix the initial data u_0 and v_0 so that $u_0 \prec v_0$. For each m, let u_m and v_m be respectively the viscosity solutions of (PME-D)$_m$with their respective initial data u_0 and v_0. Let us consider u_1, u_2, $\tilde{u}_1$ as given in Theorem 2.3.3, and let $v_1, v_2, \tilde{v}_1$ denote the corresponding functions given in Theorem 2.3.3 defined with $\{v_m\}$ instead of $\{u_m\}$.

Since $u_0 \prec v_0$, Theorem 2.2.4 applies to $\tilde{u}_1$ and v_2, and so using Theorem 2.3.3 yields that

$$u_1 \leq \tilde{u}_1 \leq v_2.$$

On the other hand, (2.3.15) yields that

$$\|(u_m^{1/(m-1)} - v_m^{1/(m-1)})(\cdot, t)\|_{L_1} \leq \|(u_0^{1/(m-1)} - v_0^{1/(m-1)})(\cdot, 0)\|_{L^1}.$$

The above inequality and the fact that $u_m \leq v_m$ and $\tilde{u}_1 \leq v_2$ imply that

$$|\{v_2(\cdot,t) > 0\} - \{\tilde{u}_1(\cdot,t) > 0\}| \leq \limsup_{m\to\infty} \|(u_m^{1/(m-1)} - v_m^{1/(m-1)})(\cdot,t)\|_{L_1} \leq |\{v_0 > 0\} - \{u_0 > 0\}|. \tag{2.3.16}$$

2. Take u_0 as given above, and let us consider

$$V(x,t) := (\inf\{v : v \text{ is a viscosity supersolution of } (HS) \text{ with } u_0 \prec v(\cdot,0)\})_*$$

and

$$U(x,t) := \sup\{u : u \text{ is a viscosity subsolution of } (HS) \text{ with } u(\cdot,0) \prec u_0\}.$$

Here f_* denotes the lower semicontinuous envelop of f, as defined in (2.2.2). Due to Theorem 2.2.4, U (V) then has the property of being below (above) any viscosity supersolution (subsolution) of (HS) with initial data u_0.

Let us consider a sequence of initial data $v_{0,n}^-$ and $v_{0,n}^+$ such that

(a) $v_0^{-,n} \prec u_0 \prec v_0^{+,n}$ for each n;

(b) $v_{0,n}^\pm$ uniformly converges to u_0 and $\{v_{0,n}^\pm > 0\}$ converges $\{u_0 > 0\}$ uniformly in Hausdorff distance.

Such $v_{0,n}^{\pm}$ can be constructed using the fact that $\partial\Omega_0$ is locally Lipschitz. Now let $u_1^{\pm,n}$ and $u_2^{\pm,n}$ be the corresponding versions of $\tilde{u}_1$ and u_2 with the initial data $v_0^{\pm,n}$. Then due to Theorem 2.3.3 and the definition of U and V we have

$$\tilde{u}_1^{-,n} \leq U \leq V \leq u_2^{+,n} \text{ for any } n.$$

Using these approximations of initial data, the fact that $\{V(\cdot,t) > 0\}$ is open, and (2.3.16), we conclude that

$$\Omega_t := \overline{\{V(\cdot,t) > 0\}} = \overline{\{U(\cdot,t) > 0\}}. \tag{2.3.17}$$

Now for any viscosity solution u of (HS) with initial data u_0, we have $U \leq u \leq V$. This yields that $\overline{\{u(\cdot,t) > 0\}} = \Omega_t$, and we showed (a).

3. By Theorem 2.3.3 and the definition of U and V, we have

$$U \leq u_2 \leq (\tilde{u}_1)_* \leq V.$$

Hence we have

$$\Omega_t = \overline{\{u_2(\cdot,t) > 0\}} = \overline{\{\tilde{u}_1(\cdot,t) > 0\}}.$$

The above inequality and the fact that $\tilde{u}_1$ is a viscosity subsolution of (HS) with initial data u_0 yield (b). □

In terms of $\rho_m = (\frac{m-1}{m}u_m)^{1/(m-1)}$ the convergence results can be stated as follows:

Corollary 2.3.1. *Let $(\Omega_t)_{t>0}$ be the family of compact sets in $\mathbb{R}^d$ as given in Theorem 2.3.4, and let ρ_m solve* (2.1.8) *with initial data $\rho_m(\cdot,0) = (\frac{m-1}{m}u_0)^{1/(m-1)}$. Then for each $t > 0$,*

(a) $\limsup \rho_m \leq 1$*;*

(b) $\overline{\{\rho_m(\cdot,t) > 0\}}$ *uniformly converges to* Ω_t *in Hausdorff distance;*

(c) $\rho_m(\cdot,t)$ *locally uniformly converges to* 1 *in* $Int(\Omega_t)$*, and to 0 in* $(\Omega_t)^C$*.*

The same result holds for ρ_m *with initial data* χ_{Ω_0}*.*

This concludes our analysis on the limiting profile of ρ_m. In the next two sections we study the gradient flow solution ρ_∞ of the crowd transport equation (2.1.4). Among other things, we show that ρ_m converge to ρ_∞ as $m \to \infty$ in the Wasserstein distance (see Theorem 2.4.2), and hence ρ_∞ must coincide with χ_{Ω_t}.

2.4 Convergence of the gradient flow solution as $m \to \infty$

2.4.1 Definition of the gradient flow solution and the discrete scheme

For section 4 we introduce more assumptions:

(**A2**) $\inf_{\mathbb{R}^d} \Phi$ is finite, and without loss of generality we assume $\inf_{\mathbb{R}^d} \Phi = 0$.

(**A3**) Φ is semi-convex, i.e. there exists $\lambda \in \mathbb{R}$ such that $D^2\Phi(x) \geq \lambda (Id)_{d\times d}$ for all $x \in \mathbb{R}^d$.

(**A3'**) In addition to (**A3**), $\|\Delta\Phi\|_\infty \leq C$ for some finite C.

The semi-convexity assumption (**A3**) guarantees the well-posedness of the discrete-time JKO solution. When we prove convergence results as $m \to \infty$, we will replace (**A3**) by the stronger assumption (**A3'**). It ensures that $\Delta\Phi(x)$ cannot be too large, which makes it possible for us to obtain some quantitative estimates on the difference between ρ_m and ρ_∞ for large m. (**A2**) is a technical assumption, and will be used explicitly in the proof of Lemma 2.4.1 in section 2.4. The assumption (**A1**) will be only used in section 4 to link ρ_∞ with the free boundary problem (HS), and in section 5 to obtain convergence results as $t \to \infty$.

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the space of Borel probability measures on $\mathbb{R}^d$ with finite second moment, i.e., the set of probability measures $\rho(x)$ such that $\int_{\mathbb{R}^d} \rho(x)|x|^2 dx < \infty$. For a probability density $\rho \in \mathcal{P}_2(\mathbb{R}^d)$, we define its "free energy" $E_m[\rho]$ as

$$E_m[\rho] := \mathcal{S}_m[\rho] + \int_{\mathbb{R}^d} \rho(x)\Phi(x)dx \quad \text{for } 1 < m \leq \infty, \tag{2.4.1}$$

where $\int_{\mathbb{R}^d} \rho(x)\Phi(x)dx$ corresponds to the potential energy of ρ, and $\mathcal{S}_m[\rho]$ is its "internal energy", given by

$$\mathcal{S}_m[\rho] := \int_{\mathbb{R}^d} \frac{1}{m}\rho^m(x)dx \quad \text{for } 1 < m < \infty, \tag{2.4.2}$$

while S_∞ is defined as

$$\mathcal{S}_\infty[\rho] := \begin{cases} 0 & \text{for } \|\rho\|_{L^\infty(\mathbb{R}^d)} \leq 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4.3}$$

Next let us introduce the following discrete-time scheme (also called a minimizing movement scheme) introduced by [JKO98]. We consider a time-step $h > 0$, and initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\rho_0\|_{L^\infty(\mathbb{R}^d)} \leq 1$. For $1 < m \leq \infty$, the sequence $(\rho^n_{m,h})_{n\in\mathbb{N}}$ is recursively defined by $\rho^0_{m,h} = \rho_0$ and

$$\rho^{n+1}_{m,h} \in \arg\min \left\{ E_m[\rho] + \frac{1}{2h} W_2^2(\rho^n_{m,h}, \rho) : \rho \in \mathcal{P}_2(\mathbb{R}^d) \right\}, \tag{2.4.4}$$

where $W_2(\cdot,\cdot)$ is the 2-Wasserstein distance (for definition, see e.g. [AGS06].) We then define $\rho_{m,h}(x,t)$ as a function piecewise constant in time, given by

$$\rho_{m,h}(x,t) := \rho^n_{m,h}(x) \text{ for } t \in [nh, (n+1)h). \tag{2.4.5}$$

Under the assumption in (**A3**) that Φ is semi-convex, one can check that for all $m > 1$, the free energy $E_m[\rho]$ is λ-convex along the generalized geodesics with respect to 2-Wasserstein distance, where λ is as given in (**A3**) (For the definition of generalized geodesics and λ-convexity, we refer to Section 2.6). One can then apply the theory of gradient flow solution developed in [AGS06], which gives the following existence and uniqueness results of the discrete solution, as well as a convergence result as $h \to 0$.

Theorem 2.4.1 ([AGS06])**.** *Let $1 < m \leq \infty$ and suppose Φ satisfies* (**A3**)*. Moreover suppose $E_m[\rho_0] < \infty$, where E_m be as given in* (2.4.1)*. Then for given $h > 0$ the following holds for the sequence $(\rho^n_{m,h})_{n\in\mathbb{N}}$ as defined in* (2.4.4)*:*

(a) Existence & Uniqueness for discrete solutions (Section 2-3 of [AGS06]): Let λ be as defined in (**A3**)*, and let $h_0 = -\frac{1}{\lambda}$ for $\lambda < 0$, $h_0 = \infty$ for $\lambda \geq 0$. Then for $0 < h < h_0$, $\rho^n_{m,h}$ is uniquely defined for all $n \in \mathbb{N}$.*

(b) Uniform convergence as $h \to 0$ (Theorem 4.0.7 – 4.0.10 in [AGS06]): Assume that Φ satisfies (**A2**) *in addition to* (**A3**)*, and consider initial data ρ_0 such that $E_m[\rho_0] \leq A$ for some constant A. Let $\rho_{m,h}$ be as defined in* (2.4.5)*. Then for any $T > 0$ and step size $0 < h < 1$, there exists some $\rho_m(t)$ (and $\rho_\infty(t)$ in the case $m = \infty$) in $C_W([0,T], \mathcal{P}_2(\mathbb{R}^d))$ such that*

$$W_2(\rho_{m,h}(\cdot,t), \rho_m(\cdot,t)) \leq C(\lambda)\sqrt{Ah}e^{-\lambda T} \quad \text{for all } t \in [0,T],$$

where λ is given by (**A3**)*. Here we say $\rho \in C_W([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ if $\rho(\cdot,t) \in \mathcal{P}_2(\mathbb{R}^d)$ for each $0 \leq t \leq T$ and*

$$\rho(\cdot,t) \to \rho(\cdot,t_0) \text{ weakly in } \mathcal{P}_2(\mathbb{R}^d) \text{ as } t \to t_0 \text{ in } [0,T].$$

Moreover, for finite m, $\rho_m(x,t)$ coincides with the viscosity solution of (2.1.8)*.*

(c) *Contraction in Wasserstein distance (Theorem 4.0.4 (iv) in [AGS06]): For a given m, consider the initial data $\rho_{01}, \rho_{02} \in \mathcal{P}_2(\mathbb{R}^d)$, with $E_m[\rho_{0i}] < \infty$ for $i = 1, 2$. Let $\rho_1(x,t)$ and $\rho_2(x,t)$ denote the limit solutions as defined in part (b), with initial data ρ_{01} and ρ_{02} respectively. Then we have the following stability result, where λ is as given in* (**A3**)*:*

$$W_2(\rho_1(\cdot,t), \rho_2(\cdot,t)) \leq e^{-\lambda t} W_2(\rho_{01}, \rho_{02}) \text{ for all } t \geq 0.$$

The above theorem yields the gradient flow solutions $\rho_m(\cdot,t)$ and $\rho_\infty(\cdot,t)$. In this section, our main goal is to prove that as $m \to \infty$, $\rho_m(\cdot,t)$ converges to $\rho_\infty(\cdot,t)$ uniformly in $t \in [0,T]$ in 2-Wasserstein distance. Convergence rates will also be obtained in terms of m. Although the rate is not optimal, to the best of our knowledge, our result is the first that gives some explicit convergence rate as the exponent $m \to \infty$ in the porous medium equation. More precisely, our main theorem in this section is as follows:

Theorem 2.4.2. *Let Φ satisfy* (**A2**) *and* (**A3'**)*, and consider $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\rho_0\|_\infty \leq 1$ and $\int \rho_0 \Phi dx \leq M$. Let $\rho_m(t)$ and $\rho_\infty(t)$ be as given in Theorem 2.4.1(b) with the initial data ρ_0. Then for any $T > 0$, we have*

$$\lim_{m \to \infty} \sup_{t \in [0,T]} W_2(\rho_m(t), \rho_\infty(t)) = 0.$$

More precisely, we have the following convergence rate:

$$\sup_{t \in [0,T]} W_2(\rho_m(t), \rho_\infty(t)) \leq \frac{C(M, T, \|\Delta\Phi\|_\infty)}{m^{1/24}}$$

We point out that under the additional assumption (**A1**) and the assumption that $\rho_0 = \chi_{\Omega_0}$, we can combine the results in Theorem 2.4.2 with Theorem 2.3.4, and immediately obtain that ρ_∞ must coincide with χ_{Ω_t} almost everywhere, which gives Theorem 2.1.2. Without these two additional assumptions, Theorem 2.4.2 still holds, but our approach fails to yield the connection between ρ_∞ with the free boundary problem (HS). Thus a further characterization of ρ_∞ beyond as a weak solution of (2.1.5) remains open in the general context.

The rest of this section will be devoted to proving Theorem 2.4.2. In section 2.4.2, we consider the discrete JKO scheme (2.4.4) for E_m and E_∞ respectively, with the same initial data $\|\rho_0\|_\infty \leq 1$. We show that if we run the JKO scheme for one step only, then their Wasserstein distance is small. Once we have the one-step estimate, we are finally ready to prove Theorem 2.4.2 in section 2.4.3, which says the Wasserstein distance between the continuous gradient flow solutions ρ_m and ρ_∞ also goes to zero as $m \to \infty$, with an explicit rate in terms of m.

2.4.2 One-step estimate for large m

We consider the initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\rho_0\|_\infty \leq 1$ and with finite potential energy, and let h be some fixed small time step. Then for any $2 < m \leq \infty$, we define μ_m (and μ_∞ in the case $m = \infty$) as follows:

$$\mu_m := \underset{\rho \in \mathcal{P}_2(\mathbb{R}^d)}{\operatorname{argmin}} \left[E_m[\rho] + \frac{1}{2h} W_2^2(\rho_0, \rho) \right]. \tag{2.4.6}$$

Our main result in this subsection is Proposition 2.4.3, which says that the Wasserstein distance between μ_m and μ_∞ is of order $O(m^{-1/8})$ for large m. To show that we first establish the following two technical lemmas concerning μ_m for $2 < m < \infty$.

Lemma 2.4.1. *Let $2 < m < \infty$, and let Φ satisfy* (**A2**) *and* (**A3**)*, and consider the initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\rho_0\|_\infty \leq 1$ and $\int \rho\Phi \leq M$. Letting μ_m be defined as in* (2.4.6)*, the following estimate holds (where $a_+ := \max\{a, 0\}$):*

$$\int_{\mathbb{R}^d} (\mu_m - 1)_+ dx \leq 2\sqrt{\frac{M+1}{m}}.$$

Proof. Our proof is based on the following crude estimate: $\frac{1}{m}\int_{\mathbb{R}^d}(\mu_m)^m dx \le M+1$. This inequality directly comes from the fact that $E_m[\mu_m] \le E_m[\rho_0]$, together with the assumption (**A2**) that $\inf \Phi \ge 0$:

$$\int_{\mathbb{R}^d} \frac{1}{m}(\mu_m)^m dx \le \int_{\mathbb{R}^d} \rho_0 \Phi dx + \int_{\mathbb{R}^d} \frac{1}{m}(\rho_0)^m dx \le M+1, \tag{2.4.7}$$

which upon rearranging gives

$$\int_{\mathbb{R}^d} (\mu_m)^m dx \le m(M+1). \tag{2.4.8}$$

Note that for $m > 2$, we have

$$\int_{\{\mu_m \ge 1\}} (\mu_m)^m dx \ge \int_{\{\mu_m \ge 1\}} \Big(1 + m(\mu_m - 1) + \frac{m(m-1)}{2}(\mu_m - 1)^2\Big) dx. \tag{2.4.9}$$

Combining the inequalities (2.4.8) and (2.4.9) together, we have

$$\int_{\mathbb{R}^d} (\mu_m - 1)_+^2 dx \le \frac{2(M+1)}{m-1} \le \frac{4(M+1)}{m}.$$

Finally, note that $|\{x : \mu_m(x) \ge 1\}| \le 1$, and so the Cauchy-Schwarz inequality yields that

$$\int_{\mathbb{R}^d} (\mu_m - 1)_+ dx \le 2\sqrt{\frac{M+1}{m}}.$$

□

The following lemma says that for large m, we can find a probability density $\tilde{\mu}_m$ that is close to μ_m in Wasserstein distance, has maximum density bounded by one, and has potential energy not much larger than μ_m.

Lemma 2.4.2. *Let Φ satisfy* (**A2**) and (**A3'**). *Under the conditions of Lemma 2.4.1, there exists a probability density $\tilde{\mu}_m \in \mathcal{P}_2(\mathbb{R}^d)$, such that $\|\tilde{\mu}_m\|_{L^\infty(\mathbb{R}^d)} \le 1$,*

$$\int_{\mathbb{R}^d} \tilde{\mu}_m \Phi dx \le \int_{\mathbb{R}^d} \mu_m \Phi dx + 2\|\Delta\Phi\|_\infty \sqrt{\frac{M+1}{m}}, \tag{2.4.10}$$

and $\tilde{\mu}_m$ is "close" to μ_m in the sense that

$$W_2(\mu_m, \tilde{\mu}_m) \leq \frac{2(M+1)^{1/4}}{m^{1/4}}. \tag{2.4.11}$$

Proof. Due to the previous lemma, $\int_{\mathbb{R}^d} (\mu_m - 1)_+ dx \leq 2\sqrt{\frac{M+1}{m}}$ for all $2 < m < \infty$ and $h > 0$. We denote by $a := 2\sqrt{\frac{M+1}{m}}$ for short, and note that a is small for large m. Next we will give a explicit construction of $\tilde{\mu}_m$, such that it satisfies all the requirements.

We begin with breaking μ_m into the sum

$$\mu_m(x) = \mu_m^1(x) + \mu_m^2(x),$$

where

$$\mu_m^1(x) := \min\{\mu_m(x), 1-a\}, \qquad \mu_m^2(x) := \big(\mu_m(x) - (1-a)\big)_+.$$

The idea is to construct $\tilde{\mu}_m$ by keeping μ_m^1 and modifying μ_m^2. We first make the observation that μ_m^2 only contains a small amount of mass: more precisely,

$$\int_{\mathbb{R}^d} \mu_m^2(x) dx \leq 2a. \tag{2.4.12}$$

This is due to the following two facts. First, due to Lemma 2.4.1, the mass of μ_m above 1 cannot exceed a. Second, we claim $|\{\mu_m > 1-a\}| \leq 1$. To show the claim, suppose not, then we have $\int \min\{\mu_m, 1-a\} dx > 1-a$. As a result, $\int \mu_m dx > (1-a) + a = 1$, where the $(1-a)$ corresponds to the mass below $(1-a)$, and a corresponds to the mass squeezed between $(1-a)$ and 1 due to our (false) assumption.

Let us now construct $\tilde{\mu}_m$ as follows:

$$\tilde{\mu}_m(x) := \mu_m^1(x) + (g * \mu_m^2)(x), \tag{2.4.13}$$

where * denotes convolution and $g(x) := \frac{1}{2}\chi_{B(0,R)}$, where $R(d)$ is the dimensional constant chosen such that $\int_{\mathbb{R}^d} g(x)dx = 1$. Note that although $R(d)$ depends on d, we indeed have $R(d) \leq 1$ for all $d \geq 1$.

We claim that $\tilde{\mu}_m$ constructed in (2.4.13) satisfies all the requirements stated in the theorem. First note that the facts $\int g = 1$ and $g \geq 0$ imply that $\tilde{\mu}_m$ is nonnegative and has the same mass as μ_m. To show that $\|\tilde{\mu}_m\|_\infty \leq 1$, it suffices to check $\|g * \mu_m^2\|_\infty \leq a$. Since the mass of μ_m^2 is less than $2a$, this inequality is a direct consequence of Young's inequality:

$$\|g * \mu_m^2\|_\infty \leq \|\mu_m^2\|_1 \|g\|_\infty \leq 2a \cdot \frac{1}{2} = a.$$

Next we verify that the inequality (2.4.10) holds, which is equivalent to

$$\int_{\mathbb{R}^d} (g * \mu_m^2)\Phi dx \leq \int_{\mathbb{R}^d} \mu_m^2 \Phi dx + \|\Delta\Phi\|_\infty a.$$

This can be rewritten as

$$\int_{\mathbb{R}^d} \mu_m^2(x)\Big[(g * \Phi)(x) - \Phi(x)\Big] dx \leq \|\Delta\Phi\|_\infty a.$$

Since μ_m^2 has mass less than $2a$, it suffices to show that

$$\Big|\varnothing_{B(x,R(d))} \Phi(y)dy - \Phi(x)\Big| \leq \frac{1}{2}\|\Delta\Phi\|_\infty \text{ for all } x \in \mathbb{R}^d,$$

where we used the fact that $R(d) \leq 1$ to get the right hand side, and note that $\|\Delta\Phi\|_\infty$ is finite due to (**A3'**). The proof of this inequality is similar to the proof of the mean value property for harmonic functions, and hence is omitted here.

Finally it remains to show (2.4.11), which is equivalent to

$$W_2(\mu_m, \tilde{\mu}_m) \leq \sqrt{2a}. \qquad (2.4.14)$$

We now heuristically describe a transport plan, which is not necessarily optimal. First, we keep the mass of μ_m^1 at its original location, so that no transportation cost is induced. Second, for for every "particle" located at x in μ_m^2, the transport plan is to distribute it evenly in the disk $B(x, R(d))$. (Again recall that $R(d) \leq 1$ for any dimension $d \geq 1$.) Since the mass of μ_m^2 is no more than $2a$, the total cost of the transportation plan is bounded by $2aR(d)^2$, which immediately implies (2.4.14). $\square$

Now we are ready to state the following one-step estimation, which controls the Wasserstein distance between μ_m and μ_∞:

Proposition 2.4.3. *Let Φ satisfy* (**A2**) *and* (**A3'**)*, and consider the initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\rho_0\|_\infty \leq 1$ and $\int \rho_0 \Phi dx \leq M$. Let λ be as given in* (**A3**)*, and $\lambda^- := -\min\{0, \lambda\}$. For any $0 < h < \frac{1}{32\lambda^-}$, let μ_m and μ_∞ be as defined in (2.4.6) for the cases m finite and $m = \infty$ respectively. Then the following inequality holds:*

$$W_2(\mu_m, \mu_\infty) \leq \frac{1}{m^{1/8}} C(M).$$

Proof. Let us fix M and d. Suppose the statement is false. Then for an arbitrarily large $A_0 > 0$, there exist $m > 2$, $0 < h < \frac{1}{32\|\Delta\Phi\|_\infty}$ such that

$$W_2(\mu_m, \mu_\infty) = Am^{-1/8}, \text{ where } A > A_0. \tag{2.4.15}$$

To get a contradiction, we will construct a new probability measure $\eta \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\eta\|_{L^\infty(\mathbb{R}^d)} \leq 1$ such that the following inequality holds if A_0 is chosen to be sufficiently large:

$$\Big[E_m[\eta] + \frac{1}{2h} W_2^2(\rho_0, \eta)\Big] + \Big[E_\infty[\eta] + \frac{1}{2h} W_2^2(\rho_0, \eta)\Big] < \Big[E_m[\mu_m] + \frac{1}{2h} W_2^2(\rho_0, \mu_m)\Big] + \Big[E_\infty[\mu_\infty] + \frac{1}{2h} W_2^2(\rho_0,\tag{2.4.16}$$

This means that η would beat at least one of the minimizers in (2.4.6) for some m (m may either be finite or $+\infty$), contradicting the definition of μ_m and μ_∞.

The probability density η is constructed as follows. Let $\tilde{\mu}_m$ be the probability density constructed in Lemma 2.4.2, and we denote by $\tilde{T}_m$ the optimal transport map such that $(\tilde{T}_m)_\# \rho_0 = \tilde{\mu}_m$. Similarly, let T_∞ be the optimal transport map such that $(T_\infty)_\# \rho_0 = \mu_\infty$. Then η is defined as

$$\eta = \Big(\frac{1}{2}\tilde{T}_m + \frac{1}{2} T_\infty\Big) \# \rho_0. \tag{2.4.17}$$

η is thus the midpoint between $\tilde{\mu}_m$ and μ_∞ on their generalized geodesics, as defined in Sec 9.2 of [AGS06].

Next we will prove that η satisfies the inequality (2.4.16). This is done by proving the inequalities (2.4.18)–(2.4.20):

$$\mathcal{S}_m[\eta] + \mathcal{S}_\infty[\eta] \leq \mathcal{S}_m[\mu_m] + \mathcal{S}_\infty[\mu_\infty] + \frac{1}{m}, \tag{2.4.18}$$

$$2\int_{\mathbb{R}^d} \eta\Phi dx \leq \int_{\mathbb{R}^d} \mu_m \Phi dx + \int_{\mathbb{R}^d} \mu_\infty \Phi dx + \frac{C(M)(\|\Delta\Phi\|_\infty - \lambda)}{\sqrt{m}} - \frac{2A^2\lambda}{m^{1/4}}, \tag{2.4.19}$$

$$\frac{1}{h} W_2^2(\rho_0, \eta) \leq \frac{1}{2h} W_2^2(\rho_0, \mu_m) + \frac{1}{2h} W_2^2(\rho_0, \mu_\infty) + \frac{C(M) - A^2/8}{hm^{1/4}}. \tag{2.4.20}$$

If A_0 is chosen to be a sufficiently large number depending only on M and d, since $A > A_0$, the sum of these three inequalities implies (2.4.16) (where we make use of the assumption that $h \leq \frac{1}{32\lambda^-}$), thereby yielding a contradiction.

To show (2.4.18), it suffices to prove that $\|\eta\|_\infty \leq 1$, since then $S_m[\eta] = \frac{1}{m}\int \eta^m dx \leq \frac{1}{m}$ and $S_\infty[\eta] = 0$. It was first shown by McCann in [McC97] that that the L^p norm with $p > 1$ is convex along the generalized geodesics, and thus

$$2\|\eta\|_p \leq \|\tilde{\mu}_m\|_p + \|\mu_\infty\|_p \text{ for all } p > 1.$$

Sending $p \to \infty$ in the above inequality yields $\|\eta\|_\infty \leq 1$, since both $\|\tilde{\mu}_m\|_\infty$ and $\|\mu_\infty\|_\infty$ are bounded by 1.

(2.4.19) comes from the semi-convexity of Φ given by (**A3**) (which is a consequence of (**A3'**)). Let $\lambda \in \mathbb{R}$ be as given in (**A3**). Proposition 9.3.2 in [AGS06] yields that $\int \rho\Phi dx$ is a λ-convex functional of ρ along any generalized geodesic, and thus

$$\begin{aligned} 2\int_{\mathbb{R}^d} \eta\Phi dx &\leq \int_{\mathbb{R}^d} \tilde{\mu}_m \Phi dx + \int_{\mathbb{R}^d} \mu_\infty \Phi dx - \frac{1}{4}\lambda W_2^2(\tilde{\mu}_m, \mu_\infty) \\ &\leq \int_{\mathbb{R}^d} \mu_m \Phi dx + \int_{\mathbb{R}^d} \mu_\infty \Phi dx + \frac{C(M)\|\Delta\Phi\|_\infty}{m^{1/2}} - \frac{1}{4}\lambda W_2^2(\tilde{\mu}_m, \mu_\infty), \end{aligned} \tag{2.4.21}$$

where the last line comes from (2.4.10). Next let us estimate $W_2^2(\tilde{\mu}_m, \mu_\infty)$. Due to our assumption (2.4.15) in the beginning of this proof and the inequality (2.4.11), we have

$$W_2(\tilde{\mu}_m, \mu_\infty) \le W_2(\mu_m, \mu_\infty) + W_2(\mu_m, \tilde{\mu}_m) \le Am^{-1/8} + C(M)m^{-1/4}.$$

We take the square of the above inequality and apply the inequality $(a+b)^2 \le 2a^2 + 2b^2$, and obtain

$$W_2^2(\tilde{\mu}_m, \mu_\infty) \le 2A^2 m^{-1/4} + C(M)m^{-1/2}.$$

Plugging this inequality into (2.4.21) yields (2.4.19).

Finally it remains to show (2.4.20). Due to Lemma 9.2.1 of [AGS06] $W_2^2(\rho_0, \cdot)$ is 1-convex along generalized geodesics, and thus

$$W_2^2(\rho_0, \eta) \le \frac{1}{2} W_2^2(\rho_0, \tilde{\mu}_m) + \frac{1}{2} W_2^2(\rho_0, \mu_\infty) - \frac{1}{4} W_2^2(\tilde{\mu}_m, \mu_\infty). \tag{2.4.22}$$

Now, by the triangle inequality, we have

$$\begin{aligned} W_2^2(\rho_0, \eta) &\le \frac{1}{2}\Big(W_2(\rho_0, \mu_m) + W_2(\mu_m, \tilde{\mu}_m)\Big)^2 + \frac{1}{2} W_2^2(\rho_0, \mu_\infty) - \frac{1}{4}\Big(W_2(\mu_m, \mu_\infty) - W_2(\mu_m, \tilde{\mu}_m)\Big)^2 \\ &\le \frac{1}{2} W_2^2(\rho_0, \mu_m) + \frac{1}{2} W_2^2(\rho_0, \mu_\infty) - \frac{1}{4} W_2^2(\mu_m, \mu_\infty) + \frac{C(M)}{2m^{1/4}}(1 + W_2(\mu_m, \mu_\infty)) \\ &\le \frac{1}{2} W_2^2(\rho_0, \mu_m)^2 + \frac{1}{2} W_2^2(\rho_0, \mu_\infty) + \frac{C(M) - A^2/8}{m^{1/4}}. \end{aligned}$$

For the second inequality we used the fact that $W_2(\mu_m, \tilde{\mu}_m) \le C(M)m^{-1/4}$ due to Lemma 2.4.2 as well as that $W(\rho_0, \mu_m) \le C(M)$ for all m and h (otherwise μ_m would fail to be a minimizer of Ψ_m). For the third inequality we use the assumption that $W_2(\mu_m, \mu_\infty) = Am^{-1/8}$. Finally, dividing both sides of the above inequality by h yields (2.4.20). □

2.4.3 Convergence of the continuum solutions as $m \to \infty$

In this subsection, we give a proof of Theorem 2.4.2. The proof is done by combining the one-step estimation results in section 2.4.2 with the convergence results for discrete solutions as $h \to 0$.

Proof of Theorem 2.4.2. 1. Note that the assumptions on ρ_0 immediately imply that $E_m[\rho_0] \le M + 1$ for all $2 < m \le \infty$. This enables us to apply Theorem 2.4.1(b): For all time steps h satisfying $0 < h < h_0$ (where h_0 is a small constant depending on $M, T, \|\Delta\Phi\|_\infty$), we have

$$W_2(\rho_m(t), \rho_{m,h}(t)) \le C(\lambda)\sqrt{M}e^{-\lambda T}\sqrt{h} =: C\sqrt{h} \text{ for all } t \in [0,T], \tag{2.4.23}$$

and this inequality holds for both finite m and $m = \infty$. For notational simplicity, the various constants C appearing in this proof may depend on $M, \|\Delta\Phi\|_\infty, T$, and the value of C may differ from line to line.

2. Now we fix the small time step h such that $0 < h < h_0$, and our goal is to show that

$$W_2(\rho_{m,h}(t), \rho_{\infty,h}(t)) \le C\sqrt{h} \text{ for all } t \in [0,T] \text{ when } m \ge Ch^{-12}. \tag{2.4.24}$$

Before we prove this inequality, let us point out the proof is finished once we obtain this: by combining (2.4.24) with the inequality (2.4.23) (and note that (2.4.23) holds for both finite m case and $m = \infty$), one immediately has

$$W_2(\rho_m(t), \rho_\infty(t)) \le C\sqrt{h} \text{ for all } t \in [0,T] \quad \text{given that } m \ge Ch^{-12},$$

which concludes the proof and would give the rate $W_2(\rho_m(t), \rho_\infty(t)) \le Cm^{-1/24}$ for $t \in [0,T]$.

3. To prove the inequality (2.4.24) in step 2, note that it is equivalent to prove

$$W_2(\rho^n_{m,h}, \rho^n_{\infty,h}) \le C\sqrt{h} \quad \text{for all } n \le \frac{T}{h}. \tag{2.4.25}$$

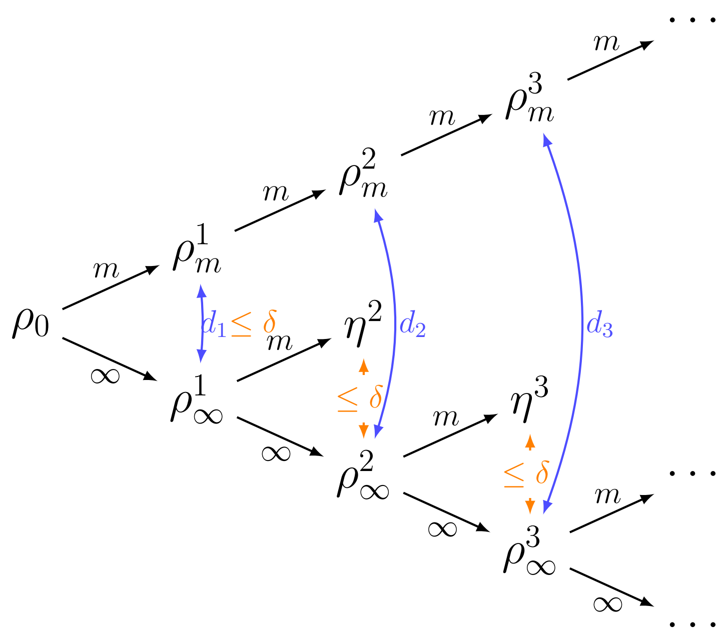

Figure 2.3: Illustration of the tree structure.

From now on we will denote $\rho_{m,h}^n$ by ρ_m^n (and denote $\rho_{\infty,h}^n$ by ρ_∞^n) for notational simplicity. Proposition 2.4.3 then shows that $W_2(\rho_m^1, \rho_\infty^1)$ is small for sufficiently large m. To deal with the case $n > 1$ we consider the tree structure as illustrated in Figure 2.3.

Here for $n \geq 2$, η^n is defined as below:

$$\eta^n := \operatorname*{argmin}_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} \left\{ E_m[\rho] + \frac{1}{2h} W_2^2(\rho_\infty^{n-1}, \rho) \right\} \quad \text{for } n \geq 2.$$

We point out that for $n \geq 2$, $\|\rho_\infty^{n-1}\|_\infty \leq 1$ holds by definition, and in addition we have $\int \rho_\infty^{n-1} \Phi dx \leq \int \rho_0 \Phi dx \leq M$. Hence by taking ρ_∞^{n-1} as the initial data, the one-step estimate in Proposition 2.4.3 yields that

$$W_2(\rho_\infty^n, \eta^n) \leq \delta := C(M) m^{-1/8} \text{ for all } n \geq 2. \tag{2.4.26}$$

Let us denote $d_n := W_2(\rho_m^n, \rho_\infty^n)$, which satisfies

$$d_n \leq W_2(\rho_m^n, \eta^n) + W_2(\eta^n, \rho_\infty^n) \leq W_2(\rho_m^n, \eta^n) + \delta, \tag{2.4.27}$$

where δ is as defined in (2.4.26). Now it remains to control $W_2(\rho_m^n, \eta^n)$. Note that ρ_m^n and η^n are minimizers given by the discrete-time scheme (2.4.4) with the same free energy functional E_m, but with different initial data ρ_m^{n-1} and ρ_∞^{n-1}. To estimate d_n in terms of d_{n-1}, we use Lemma 4.2.4 of [AGS06] which states that the Wasserstein distance between two discrete solutions does not grow too fast. More precisely, it gives the following inequality

$$\begin{aligned} W_2^2(\rho_m^n, \eta^n) &\leq e^{-2\lambda^- h}\left[W_2^2(\rho_m^{n-1}, \rho_\infty^{n-1}) + h\big(E_m[\rho_m^{n-1}] - E_m[\rho_m^n]\big)\right] \\ &\leq e^{-2\lambda^- h}\left(d_{n-1}^2 + h a_{n-1}\right), \end{aligned} \tag{2.4.28}$$

where $\lambda^- := -\min\{\lambda, 0\}$, with λ as given in (**A3**). We denote $a_{n-1} := E_m[\rho_m^{n-1}] - E_m[\rho_m^n]$, which satisfies the following properties:

$$a_n \geq 0 \text{ for all } n \in \mathbb{N}^+, \text{ and } \sum_{n=0}^{\infty} a_n \leq M + 1. \tag{2.4.29}$$

Finally, we plug (2.4.28) into (2.4.27) to obtain the following family of inequalities:

$$\begin{aligned} d_1 &\leq \delta \\ d_n &\leq e^{-2\lambda^- h}\sqrt{d_{n-1}^2 + h a_{n-1}} + \delta \quad \text{for } n = 2, 3, \ldots. \end{aligned} \tag{2.4.30}$$

4. We next focus on the inequality (2.4.30), and our goal is to show that $d_{\frac{T}{h}} \leq C\sqrt{h}$ for δ sufficiently small (more precisely, $\delta \leq h^{3/2}$ would be enough). By taking the square of (2.4.30) and applying the inequality $2ab \leq ha^2 + \frac{b^2}{h}$, we obtain

$$\begin{aligned} d_n^2 &\leq (1+h)e^{-4\lambda^- h}(d_{n-1}^2 + h a_{n-1}) + (1 + \frac{1}{h})\delta^2 \\ &\leq (1 + Ch)d_{n-1}^2 + h\underbrace{(2a_{n-1} + 2h)}_{:=b_{n-1}}, \end{aligned} \tag{2.4.31}$$

where in the last line we let $\delta \leq h^{3/2}$ so that $(1+\frac{1}{h})\delta^2 \leq 2h^2$. Also note that $b_n := 2a_n + 2h$ satisfies $\sum_{n=0}^{T/h} b_n \leq 2(M+T+1)$. Now by dividing by $(1+Ch)^n$ on both sides of (2.4.31) and summing the inequality from 2 to n, we obtain that

$$d_n^2 \leq d_1^2(1+Ch)^{n-1} + \sum_{k=1}^{n} hb_k(1+Ch)^{n-k} \text{ for all } n.$$

Hence as a result, we see that $W_2(\rho_m^{T/h}, \rho_\infty^{T/h}) = d_{T/h}$ satisfies

$$d_{T/h} \leq \sqrt{e^{CT}h^3 + 2(M+T+1)e^{CT}h} \leq C\sqrt{h} \tag{2.4.32}$$

as long as $\delta \leq h^{3/2}$ (recall that $\delta = Cm^{-1/8}$, hence it is equivalent with $m > h^{-12}$), and so we are done. □

2.5 Comparison principle and long-time behavior for gradient flow solutions

2.5.1 Comparison principle for the discrete-time solutions

In the beginning of section 2.4, we have defined the discrete-time scheme (2.4.4) for the porous medium equation with drift (2.1.8). Since the comparison principle for the viscosity solutions of (2.1.8) is well-known (see e.g. [KL10]), it is natural to ask whether the comparison principle holds for the discrete-time solutions generated by (2.4.4) as well. In this section we prove that this is indeed true, but the proof is quite different from the continuous case. In fact comparison principle-type results have been shown between discrete gradient flow solutions with L^2 distances, for instance in [Cha04], [GK11], etc. The novelty in our result is that we address the discrete gradient flow solutions with W_2 distances, for which nonlocal perturbation arguments are necessary.

In order to define the scheme for two ordered initial data, we need to consider non-negative measures which do not necessarily integrate to 1. We denote by $\mathcal{P}_{2,A}(\mathbb{R}^d)$ the set of non-negative measures which integrate to $A > 0$ and have finite second moment.

We also generalize the Wasserstein distance W_2 as follows: For two regular measures $\rho_1, \rho_2 \in \mathcal{P}_{2,A}(\mathbb{R}^d)$, we define $W_2(\rho_1, \rho_2)$ as

$$W_2^2(\rho_1, \rho_2) := \inf_{T\#\rho_1=\rho_2} \int_{\mathbb{R}^d} |T(x) - x|^2 \rho_1(x) dx.$$

Next we state the comparison result.

Theorem 2.5.1. *Let Φ satisfy (**A3**). For $2 < m \leq \infty$, consider the two densities $\rho_{01} \in \mathcal{P}_{2,M_1}(\mathbb{R}^d)$, $\rho_{02} \in \mathcal{P}_{2,M_2}(\mathbb{R}^d)$ with the property $M_1 \leq M_2$ and $\rho_{01} \leq \rho_{02}$ a.e. (In the case $m = \infty$, we require in addition that $\|\rho_{0i}\|_\infty \leq 1$ for $i = 1, 2$). For given $h > 0$, let ρ_1, ρ_2 be the respective minimizers of the following schemes:*

$$\rho_i := \underset{\rho\in\mathcal{P}_{2,M_i}(\mathbb{R}^d)}{\operatorname{argmin}} \mathcal{F}_i(\rho) := \underset{\rho\in\mathcal{P}_{2,M_i}(\mathbb{R}^d)}{\operatorname{argmin}} \left[E_m[\rho] + \frac{1}{2h} W_2^2(\rho, \rho_{0i})\right] \quad \text{for } i = 1, 2. \tag{2.5.1}$$

Then $\rho_1 \leq \rho_2$ almost everywhere.

Remark 2.5.1. *The proof of above theorem does not directly use the semi-convexity of Φ, except to guarantee the existence of the ρ_i.*

Before we prove Theorem 2.5.1, we first state and prove the following simple lemma, which can be informally stated as follows: Given that ρ_1 is the minimizer for the discrete scheme in (2.5.1) and T_1 is the optimal map between ρ_{01} and ρ_1, if a part of ρ_{01} is forced to be transferred by the map T_1, then T_1 is still the optimal map for the rest of ρ_{01}.

Lemma 2.5.1. *Let $2 < m < \infty$, and let h and ρ_{01} be as given in Theorem 2.5.1. We denote by ρ_1 the minimizer of $\mathcal{F}_1$ as given by (2.5.1), and let T_1 be the optimal mapping such that $T_1\#\rho_{01} = \rho_1$. Consider an arbitrary function $\eta : \mathbb{R}^d \to \mathbb{R}$ such that $0 \leq \eta(x) \leq 1$ for all $x \in \mathbb{R}^d$, and let $\varphi(x) := T_1\#((1-\eta)\rho_{01})$. Then $T_1\#(\eta\rho_{01})$ minimizes*

$$\tilde{\mathcal{F}}(\rho) := \int_{\mathbb{R}^d} \left(\frac{1}{m}(\varphi + \rho)^m + \rho\Phi\right) dx + \frac{1}{2h} W_2^2(\eta\rho_{01}, \rho)$$

among all $\rho \in \mathcal{P}_{2,\tilde{M}}(\mathbb{R}^d)$, where $\tilde{M} = \int_{\mathbb{R}^d} \eta\rho_{01} dx$.

Proof. Suppose that the minimum of $\tilde{\mathcal{F}}$ is achieved by another measure $\tilde{\rho} \in \mathcal{P}_{2,\tilde{M}}(\mathbb{R}^d)$, such that $\tilde{\mathcal{F}}(\tilde{\rho}) < \tilde{\mathcal{F}}(T_1\#(\eta\rho_{01}))$. We denote by $\tilde{T}$ the optimal map such that $\tilde{T}\#(\eta\rho) = \tilde{\rho}$. The claim is then that we can find a better transfer plan of ρ_{01} than ρ_1 in (2.5.1), yielding a contradiction. We construct the transfer plan as follows.

First, we separate ρ_{01} into two parts: $\eta\rho_{01}$ and $(1-\eta)\rho_{01}$. Then we use T_1 to push forward $(1-\eta)\rho_{01}$, and use $\tilde{T}$ to push forward $\eta\rho_{01}$. The resulting measure would be equal to $\varphi + \tilde{\rho}$. Then it follows that

$$\begin{aligned}
\mathcal{F}_1(\varphi+\tilde{\rho}) &= \int_{\mathbb{R}^d} \Big(\frac{1}{m}(\varphi+\tilde{\rho})^m + (\varphi+\tilde{\rho})\Phi\Big)dx + \frac{1}{2h}W_2^2(\rho_{01},(\varphi+\tilde{\rho}))\\
&\le \int_{\mathbb{R}^d} \Big(\frac{1}{m}(\varphi+\tilde{\rho})^m + (\varphi+\tilde{\rho})\Phi\Big)dx + \frac{1}{2h}W_2^2(\eta\rho_{01},\tilde{\rho}) + \frac{1}{2h}W_2^2((1-\eta)\rho_{01},\varphi)\\
&= \tilde{\mathcal{F}}(\tilde{\rho}) + \int_{\mathbb{R}^d} \varphi\Phi dx + \frac{1}{2h}W_2^2((1-\eta)\rho_{01},\varphi)\\
&< \tilde{\mathcal{F}}(T_1\#(\eta\rho)) + \int_{\mathbb{R}^d} \varphi\Phi dx + \frac{1}{2h}W_2^2((1-\eta)\rho_{01},\varphi)\\
&= \mathcal{F}_1(T_1\#\rho) = \mathcal{F}_1(\rho_1),
\end{aligned} \tag{2.5.2}$$

which contradicts the fact that ρ_1 is the minimizer of (2.5.1) and so we are done. □

Proof of Theorem 2.5.1. First we point out that once we prove the comparison result for all

$2 < m < \infty$, it will be automatically true for the case $m = \infty$ as well, due to the one-step estimate in Proposition 2.4.3. To see this, let us denote by $\rho_{i,m}$ the minimizer ρ_i when the free energy is E_m. Then Proposition 2.4.3 gives us $\rho_{i,m} \to \rho_{i,\infty}$ as $m \to \infty$ in Wasserstein distance. Therefore if we know that $\rho_{1,m} \le \rho_{2,m}$ a.e. for all $2 < m < \infty$ then it directly follows that $\rho_{1,\infty} \le \rho_{2,\infty}$ a.e.

Due to the above discussion, it suffices to prove the comparison principle for any fixed

m satisfying $2 < m < \infty$. Let T_i denote the optimal map such that $T_i\#\rho_{0i} = \rho_i$ for $i = 1, 2$. We prove by contradiction and suppose $\Omega := \{\rho_1 > \rho_2\}$ has non-zero measure. We first claim that

$$|T_1^{-1}(\Omega)\backslash T_2^{-1}(\Omega)| > 0, \tag{2.5.3}$$

which directly follows from the inequality below:

$$\begin{aligned}
\int_{T_1^{-1}(\Omega)} \rho_{01} dx &= \int_\Omega \rho_1 dx \quad (\text{since } T_1\#\rho_{01} = \rho_1) \\
&> \int_\Omega \rho_2 dx \quad (\text{from the definition of } \Omega \text{ and the assumption that } |\Omega| > 0) \\
&= \int_{T_2^{-1}(\Omega)} \rho_{02} dx \quad (\text{since } T_2\#\rho_{02} = \rho_2) \\
&\geq \int_{T_2^{-1}(\Omega)} \rho_{01} dx \quad (\text{since } \rho_{01} \leq \rho_{02}).
\end{aligned} \tag{2.5.4}$$

Let $\Omega_\delta = \{x \in \mathbb{R}^d : \rho_1(x) > \rho_2(x) + \delta\}$, and let $A_\delta = \{x \in \mathbb{R}^d : \rho_1(T_1(x)) \leq \frac{1}{\delta}, \rho_2(T_2(x)) \leq \frac{1}{\delta}\}$. Since $\cup_{\delta>0}\Omega_\delta = \Omega$ and $\cup_{\delta>0}A_\delta = \mathbb{R}^d$, (2.5.3) yields that

$$\left|\big(T_1^{-1}(\Omega_\delta) \cap A_\delta\big)\backslash T_2^{-1}(\Omega)\right| > 0 \text{ for sufficiently small } \delta > 0. \tag{2.5.5}$$

From now on we fix δ such that the above inequality is true, and denote

$$B := \big(T_1^{-1}(\Omega_\delta) \cap A_\delta\big)\backslash T_2^{-1}(\Omega).$$

By definition of the set B, it immediately follows that T_1 maps B into the set where $\rho_2 + \delta < \rho_1 \leq \frac{1}{\delta}$, while T_2 maps B into the set where $\rho_1 \leq \rho_2 \leq \frac{1}{\delta}$. (Note that these inequalities hold in the a.e. sense). These facts are illustrated in Figure 2.4.

Let $\rho_\epsilon := \epsilon\rho_{01}\chi_B$, where $0 < \epsilon \ll \delta$ is a sufficiently small number to be determined later. Let $\varphi_1 := T_1\#(\rho_{01} - \rho_\epsilon) = (1-\epsilon)\rho_1\chi_{T_1(B)}$, then by applying Lemma 2.5.1 to the

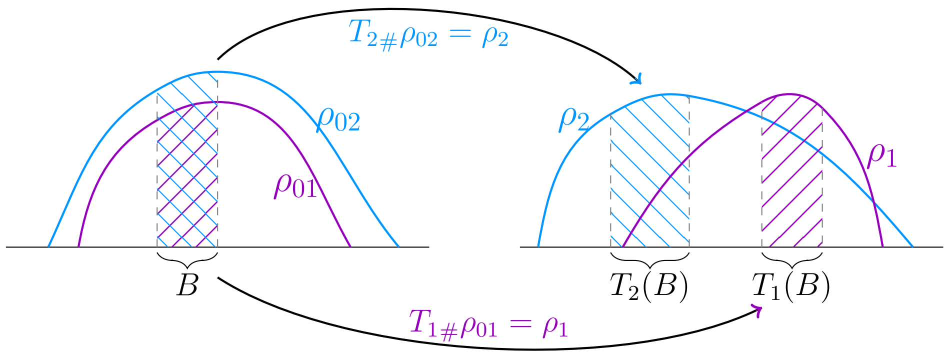

Figure 2.4: Illustration of the set $B, T_1(B)$ and $T_2(B)$. Recall that T_i is the optimal map between ρ_{0i} and ρ_i for $i = 1, 2$. Moreover, the set B is chosen such that $T_1(B) \subset \{\frac{1}{\delta} \geq \rho_1 > \rho_2 + \delta\}$, while $T_2(B) \subset \{\rho_1 \leq \rho_2 \leq \frac{1}{\delta}\}$.

optimal plan T_1 in comparison to T_2, and subtracting $\frac{1}{m}\varphi_1^m$ on both sides, we arrive at the following inequality:

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(\frac{1}{m}(\varphi_1 + T_1\#\rho_\epsilon)^m - \frac{1}{m}\varphi_1^m\right) dx + E[T_1] \\ \leq &\int_{\mathbb{R}^d} \left(\frac{1}{m}(\varphi_1 + T_2\#\rho_\epsilon)^m - \frac{1}{m}\varphi_1^m\right) dx + E[T_2], \end{aligned} \tag{2.5.6}$$

where

$$E[T_i] := \int_{\mathbb{R}^d} \Big((T_i\#\rho_\epsilon)\Phi + \frac{1}{2h}|T_i(x) - x|^2\rho_\epsilon\Big)dx, \quad i = 1, 2.$$

Next we state a simple algebraic inequality without proof. For all real numbers a and b satisfying $0 < b < a < \frac{1}{\delta}$ and $m > 2$, we have

$$a^{m-1}b \leq \frac{1}{m}(a+b)^m - \frac{1}{m}a^m \leq a^{m-1}b + Cb^2, \tag{2.5.7}$$

where the constant C only depends on m and δ. Using (2.5.7), (2.5.6) yields that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \varphi_1^{m-1}(T_1\#\rho_\epsilon)dx + E[T_1] &\le \int_{\mathbb{R}^d} \varphi_1^{m-1}(T_2\#\rho_\epsilon)dx + E[T_2] + \int_{T_2(B)} C(m,\delta)\epsilon^2\rho_2^2 dx \\
&\le \int_{\mathbb{R}^d} \varphi_1^{m-1}(T_2\#\rho_\epsilon)dx + E[T_2] + C(m,\delta)\epsilon^2 \quad (\text{since } \rho_2 \le \frac{1}{\delta} \text{ in } T_2(B)) \\
&\le \int_{\mathbb{R}^d} \rho_1^{m-1}(T_2\#\rho_\epsilon)dx + E[T_2] + C(m,\delta)\epsilon^2 \quad (\text{since } \varphi_1 \le \rho_1) \\
&\le \int_{\mathbb{R}^d} \rho_2^{m-1}(T_2\#\rho_\epsilon)dx + E[T_2] + C(m,\delta)\epsilon^2
\end{aligned}
\tag{2.5.8}
$$

where the last inequality holds since $\rho_1 \le \rho_2$ in $T_2(B)$, and $\text{supp}(T_2\#\rho_\epsilon) \subset T_2(B)$.

Similarly, we define $\varphi_2 := T_2\#(\rho_{02} - \rho_\epsilon)$, and note that ρ_2 is the minimizer to (2.5.1). We then apply Lemma 2.5.1 to the optimal plan T_2 in comparison to T_1, and an argument parallel to that above yields the following inequality for φ_2:

$$
\begin{aligned}
\int_{\mathbb{R}^d} \varphi_2^{m-1}(T_2\#\rho_\epsilon)dx + E[T_2] &\le \int_{\mathbb{R}^d} \varphi_2^{m-1}(T_1\#\rho_\epsilon)dx + E[T_1] + C(m,\delta)\epsilon^2 \\
&\le \int_{\mathbb{R}^d} \rho_2^{m-1}(T_1\#\rho_\epsilon)dx + E[T_1] + C(m,\delta)\epsilon^2 \quad (\text{since } \varphi_2 \le \rho_2)
\end{aligned}
\tag{2.5.9}
$$

Note that in the set $T_2(B)$, ρ_2 is bounded above by $\frac{1}{\delta}$, hence φ_2 is just smaller than ρ_2 by order ϵ in this set, namely $\rho_2 < \varphi_2 + \epsilon/\delta$ in $T_2(B)$. Combining this with the fact that the integral of $T_2\#\rho_\epsilon$ is also of order ϵ, and the assumption that $m > 2$, we have the following:

$$
\int_{\mathbb{R}^d} \rho_2^{m-1}(T_2\#\rho_\epsilon)dx \le \int_{\mathbb{R}^d} \varphi_2^{m-1}(T_2\#\rho_\epsilon)dx + C(m,\delta)\epsilon^2. \tag{2.5.10}
$$

(2.5.10) provides us a link between the RHS of (2.5.8) and the LHS of (2.5.9), and so we arrive at

$$
\int_{\mathbb{R}^d} \varphi_1^{m-1}(T_1\#\rho_\epsilon)dx \le \int_{\mathbb{R}^d} \rho_2^{m-1}(T_1\#\rho_\epsilon)dx + C(m,\delta)\epsilon^2. \tag{2.5.11}
$$

Next we show that (2.5.11) leads to a contradiction if ϵ is chosen to be small enough. First, recall that $\phi_1 = (1-\epsilon)\rho_1\chi_{T_1(B)}$, and $\rho_1 > \rho_2 + \delta$ in $T_1(B)$. Hence if we let ϵ be sufficiently small, we would have $\phi_1 > \rho_2 + \frac{\delta}{2}$ in $T_1(B)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi_1^{m-1}(T_1\#\rho_\epsilon)dx &\geq \int_{\mathbb{R}^d} (\rho_2 + \frac{\delta}{2})^{m-1}(T_1\#\rho_\epsilon)dx \\ &\geq \int_{\mathbb{R}^d} \left(\rho_2^{m-1} + (\frac{\delta}{2})^{m-1}\right)(T_1\#\rho_\epsilon)dx \\ &\geq \int_{\mathbb{R}^d} \rho_2^{m-1}(T_1\#\rho_\epsilon)dx + \epsilon(\frac{\delta}{2})^{m-1}\|\rho_1\|_{L^1(B)},\end{aligned} \tag{2.5.12}$$

which contradicts (2.5.11) when we fix δ and let ϵ be sufficiently small. This concludes the proof. □

Remark 2.5.2. *By sending the time step $h \to 0$, the comparison principle for discrete solutions immediately leads to a comparison principle for gradient flow solutions. Also, although we only prove the comparison principle for the energy $\int \rho^m dx$ with $2 < m \leq \infty$, the proof can indeed be easily extended for $1 < m \leq \infty$, and also the case when the entropy part is given by$\int \rho \log \rho dx$.*

2.5.2 Confinement result and long-time behavior.

In this subsection, we show some applications of the comparison principle for discrete JKO solutions. The first application is the following confinement result for discrete solutions (hence continuous gradient flow solutions as well), given that $\Phi \to +\infty$ as $|x| \to \infty$.

Corollary 2.5.1. *Let $2 < m \leq \infty$ and let $\Phi(x)$ satisfy* (**A3**) *and the additional assumption that $\lim_{|x|\to\infty} \Phi(x) = +\infty$. Assume the initial data $\rho_0 \in L^\infty(\mathbb{R}^d)$ has compact support, and if $m = \infty$ we assume in addition that $\|\rho_0\|_\infty \leq 1$. Then the support for the discrete solution $\rho_{m,h}^n$ will stay bounded for all n, where the bound of the support does not depend on n or h.*

Proof. ◦ *Case 1:* $2 < m < \infty$. For any $A > 0$, let us look for the global minimizer ρ_A of the energy E_m as defined by (2.1.7) among $\mathcal{P}_{2,A}(\mathbb{R}^d)$. Due to [CJM01, Lemma 6], the global minimizer ρ_A is given by

$$\rho_A = \Big(\frac{m-1}{m}(C_A - \Phi(x))_+\Big)^{\frac{1}{m-1}},$$

where C_A is chosen such that the total mass of ρ_A is equal to A. Observe that for any $A > 0$, such ρ_A is also a stationary solution for the discrete JKO scheme, and it has a compact support.

Therefore for any $\rho_0 \in L^\infty(\mathbb{R}^d)$ with compact support, one can choose A to be sufficiently large such that $\rho_0 \le \rho_A$ a.e. Then one can apply Theorem 2.5.1 and obtain that $\rho^n_{m,h} \le \rho_A$ a.e., hence the support of $\rho^n_{m,h}$ stays within the support of ρ_A for all time steps.

∘ *Case 2:* $m = \infty$. In this case, we first point out that for any mass size $A > 0$, the global minimizer of E_∞ among $\mathcal{P}_{2,A}$ is given by some characteristic function χ_{S_A}, where S_A is the level set of the function Φ, i.e.

$$S_A = \{x \in \mathbb{R}^d : \Phi(x) \ge C_A\},$$

and C_A is chosen so that χ_{S_A} has mass A. Moreover, since χ_{S_A} is the global minimizer of E_∞, it must be a stationary solution as well.

Recall that ρ_0 has compact support, $\|\rho_0\|_\infty \le 1$, and $\Phi(x) \to \infty$ as $|x| \to \infty$. Therefore if we let A be sufficiently large, we will have supp $\rho_0 \subset S_A$, which implies that $\rho_0 \le \chi_{S_A}$. Since χ_{S_A} is a stationary solution, the comparison result in Theorem 2.5.1 immediately implies that supp $\rho^n_{\infty,h} \subset S_A$ for all n and h, and we are done. □

Lastly we briefly discuss the long time behavior of the gradient flow solution ρ_m for $2 < m \le \infty$, when Φ is bounded below in $\mathbb{R}^d$, and $D^2\Phi(x)$ is positive definite for all x. In this case, one can easily obtain that the global minimizer for E_∞ in $\mathcal{P}_2(\mathbb{R}^d)$ is $\rho_S := \chi_{\mathcal{O}}$, where $\mathcal{O} = \{x \in \mathbb{R}^d : \Phi(x) \le C\}$, and C is chosen such that $\chi_{\mathcal{O}}$ has mass 1.

Theorem 2.5.2. *Let $2 < m \leq \infty$. Let Φ satisfy* (**A2**) *and* (**A3'**)*, and in addition assume that $D^2\Phi(x)$ is positive definite for all x. Assume the initial data $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ has compact support, and in addition satisfies $\|\rho_0\|_\infty \leq 1$ in the case $m = \infty$. For $2 < m \leq \infty$, let ρ_m be given as the gradient flow for E_m with initial data ρ_0, as defined in Theorem 2.4.1(b). Then as $t \to \infty$, $\rho_m(\cdot, t)$ converges to the unique global minimizer ρ_S of E_m exponentially fast in 2-Wasserstein distance.*

Proof. If Φ is uniformly convex in $\mathbb{R}^d$, then there exists some $\lambda > 0$, such that $D^2\Phi(x) \geq \lambda I$ for all $x \in \mathbb{R}^d$. In this case we can directly apply the contraction result in Theorem 2.4.1(c) between $\rho_\infty(x,t)$ and $\rho_S(x)$ (where ρ_S is the global minimizer for the free energy $E_m[\rho]$ in $\mathcal{P}_2(\mathbb{R}^d)$), which gives

$$W_2(\rho_\infty(\cdot,t), \rho_S(\cdot)) \leq W_2(\rho_0, \rho_S)e^{-\lambda t},$$

and hence the 2-Wasserstein distance between $\rho_\infty(x,t)$ and $\rho_S(x)$ decays exponentially fast in t.

On the other hand, if $D^2\Phi(x)$ is positive definite for all x, but Φ is not uniformly convex, we will make use of the confinement result in Corollary 2.5.1. As long as ρ_0 is compactly supported, the proof of Corollary 2.5.1 shows the support of $\rho_\infty(\cdot,t)$ will stay in some compact set $\mathcal{O}_A$ for all time, and indeed one can find an $\mathcal{O}_A$ such that it is independent of m for all $2 < m \leq \infty$. This confinement result allows us to apply the contraction result in Theorem 2.4.1 (c), which gives that

$$W_2(\rho_\infty(\cdot,t), \rho_S(\cdot)) \leq W_2(\rho_0, \rho_S)e^{-\tilde{\lambda} t},$$

where $\tilde{\lambda} = \inf\{\lambda : D^2\Phi(x) \geq \lambda I \text{ for all } x \in \mathcal{O}_A\}$ is a strictly positive constant depending on ρ_0 and Φ. □

Finally we remark that for finite m and ρ_m, the corresponding result is shown in [CJM01], where they use entropy dissipation methods. We suspect the convergence rate to be exponential in stronger norms instead of Wasserstein distance, but this issue is not pursued here.

2.6 Prior results on gradient flows

We now state some results from [AGS06], concerning the existence and uniqueness of the discrete solution $\rho^n_{m,h}$ as defined in (2.4.4), and the convergence as the time step $h \to 0$.

The key step leading to these results is the λ-convexity of $E_m[\rho]$ for all $1 < m \leq \infty$ along the generalized geodesics. Thus we first digress a little bit to state some definition and results from the optimal transport theory (see e.g. section 9.2 in [AGS06]). Recall that $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is regular if $\mu \in L^p(\mathbb{R}^d)$ with some $p > 1$.

Definition 2.6.1 (generalized geodesics)**.** *Let the reference measure $\mu^1 \in \mathcal{P}_2(\mathbb{R}^d)$ be regular. Let $\mu^2, \mu^3 \in \mathcal{P}_2(\mathbb{R}^d)$; then we can find two optimal transport maps $\boldsymbol{t}^2$ and $\boldsymbol{t}^3$ such that $\boldsymbol{t}^i_{\#}\mu^1 = \mu^i$ and $W_2^2(\mu_1, \mu^i) = \int_{\mathbb{R}^d} |\boldsymbol{t}^i(x) - x|^2 d\mu^1(x)$ for $i = 2, 3$. The generalized geodesics joining μ^2 to μ^3 (with base μ^1) is defined as*

$$\mu_t^{2\to 3} = (\boldsymbol{t}_t^{2\to 3})_{\#}\mu^1 \quad \textit{where} \quad \boldsymbol{t}_t^{2\to 3} := (1-t)\boldsymbol{t}^2 + t\,\boldsymbol{t}^3, \quad t \in [0,1]. \tag{2.6.1}$$

Using the notion of generalized geodesics, one can define a notion of semi-convexity (or λ-*convexity*) for energy functionals on $\mathcal{P}_2(\mathbb{R}^d)$:

Definition 2.6.2 (λ-convexity along generalized geodesics)**.** *Given $\lambda \in \mathbb{R}$, a functional E is called* λ-convex along the generalized geodesics *if for any μ_1, μ_2 and μ_3 satisfying the conditions in Definition 2.6.1, the following inequality holds*

$$E[\mu_t^{2\to 3}] \leq (1-t)E[\mu_2] + tE[\mu_3] - \frac{\lambda}{2} t(1-t) \int_{\mathbb{R}^d} |\boldsymbol{t}^2 - \boldsymbol{t}^3|^2 d\mu_1 \textit{ for all } 0 \leq t \leq 1,$$

where $\mu_t^{2\to3}$, $\boldsymbol{t}^2$ *and* $\boldsymbol{t}^3$ *are as defined in Definition 2.6.1.*

The following Lemma is a direct consequence of [AGS06, Sec 9.3], which says that as long as Φ is semi-convex, the functional E_m would be convex for all $1 < m \leq \infty$. Since the case $m = \infty$ is not directly covered in the book, we provide a short proof below for the sake of completeness.

Lemma 2.6.1 ([AGS06])**.** *Let* Φ *satisfy* (**A3**) *, and let* $E_m : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ *be as defined as in* (2.4.1)*. Then* E_m *is* λ*-convex along general geodesics for all* $1 < m \leq \infty$*.*

Proof. Due to (**A3**), Proposition 9.3.2 of [AGS06] gives the λ-convexity of the functional $\int_{\mathbb{R}^d} \rho\Phi dx$ along generalized geodesics.

For a finite $m > 1$, let S_m be given by (2.4.2). One can directly apply Proposition 9.3.9 in [AGS06] to obtain the convexity of $\mathcal{S}_m$ along generalized geodesics. Since the sum of two λ-convex functionals is still λ convex, we obtain the λ-convexity of E_m for any finite $m > 1$.

It remains to check that the functional $\mathcal{S}_\infty$ defined in (2.4.3) is also λ-convex along generalized geodesics. To do this, let μ^i, $i = 1, 2, 3$ be as given in Definition 2.6.1. It suffices to show that if $\|\mu^i\|_{L^\infty} \leq 1$ for $i = 2, 3$, then $\|\mu_t^{2\to3}\|_{L^\infty} \leq 1$ for all $0 < t < 1$ as well. Note that due to the λ-convexity of S_m for all $m > 1$, we obtain

$$\|\mu_t^{2\to3}\|_{L^m} \leq \min\{\|\mu^2\|_{L^m}, \|\mu^3\|_{L^m}\} \text{ for all } m > 1,$$

and sending $m \to \infty$ immediately yields the desired result. □

Once we have the λ-convexity of E_m, Lemma 9.2.7 in [AGS06] guarantees that the Assumption 4.0.1 in [AGS06] is satisfied, which leads to the existence, uniqueness and convergence results in Theorem 2.4.1.

CHAPTER 3

Approximating parabolic PDEs with oblique boundary data

We consider solutions of a quasi-linear parabolic PDE with zero oblique boundary data in a bounded domain. Our main result states that the solutions can be approximated by solutions of a PDE in the whole space with a penalizing drift term. The convergence is locally uniform and optimal error estimates are obtained.

3.1 Introduction

Consider the following parabolic problem with oblique boundary data:

$$(P_g) \qquad \begin{cases} u_t - F(D^2u, Du, u, x, t) = 0 & \text{in} \quad \Omega \times (0,\infty); \\ Du \cdot \vec{v}(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,\infty); \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain, $u_0 \in C(\bar{\Omega})$, $\vec{v} \in \mathbb{R}^n$ is smooth, and F is a quasi-linear operator with smooth coefficients given by

$$F(D^2u, Du, u, x, t) = \sum_{i,j} q^{ij}(u,x,t)u_{x_ix_j} + b(Du,u,x,t). \tag{3.1.1}$$

We use D and ∇ interchangeably to denote the spacial gradient. We assume that $q^{ij}(z,x,t)$ satisfies a uniform ellipticity condition, that is, there exists constants $0<\lambda<\Lambda$ such that for all $(z,x,t)\in\mathbb{R}\times\mathbb{R}\times[0,\infty)$,

$$\lambda Id_{n\times n}\leq (q^{ij})\leq \Lambda Id_{n\times n}. \tag{3.1.2}$$

For a given matrix M, we write M^+ and M^- to denote its positive and negative parts, that is, $M=M^+-M^-$ with $M^+,M^-\geq 0$ and $M^+M^-=0$. Using this notation, (3.1.2) is equivalent to the condition

$$\mathcal{P}^-(M)\leq \sum q^{ij}(z,x,t)M_{ij}\leq \mathcal{P}^+(M),$$

where $\mathcal{P}^{\pm}$ are the extremal Pucci operators defined by

$$\mathcal{P}^+(M):=\Lambda\,\mathrm{tr}(M^+)-\lambda\,\mathrm{tr}(M^-),\quad \mathcal{P}^-(M):=\lambda\,\mathrm{tr}(M^-)-\Lambda\,\mathrm{tr}(M^-). \tag{3.1.3}$$

We also assume that $q^{ij}(z,x,t)$ and $b(p,z,x,t)$ are smooth and

$$q^{ij} \text{ and } b \text{ are uniformly Lipschitz with respect to } p,z \text{ in } \mathbb{R}^n\times\mathbb{R}. \tag{3.1.4}$$

We assume that $\vec{v}(x,t)$ given in the boundary condition of (P_g) is a smooth vector field which satisfies

$$\vec{v}(x,t)\cdot\vec{\nu}(x)\geq c_0, \tag{3.1.5}$$

for some $c_0>0$, where $\vec{\nu}(x)$ denotes the outward normal vector of Ω at $x\in\partial\Omega$.

As we show in Lemma 3.5.3, for a given $\vec{v}(x,t)$ satisfying (3.1.5) by possibly adjusting the size of λ and Λ, one can always find a symmetric matrix $A(x,t)$ defined on $\mathbb{R}^n\times[0,\infty)$ that is smooth, satisfies (3.1.2) and

$$\vec{v}(x,t)=A(x,t)\cdot\vec{\nu}(x) \text{ on } \partial\Omega.$$

With this representation of $\vec{v}$ using A, our goal is to approximate the above problem by introducing a penalizing drift. First let us discontinuously extend F onto all of $\mathbb{R}^n$ by taking

$$F(D^2v, Dv, v, x, t) = \begin{cases} F(D^2v, Dv, v, x, t) & \text{if } x \in \Omega \\ \nabla \cdot (A(x,t)\nabla v) & \text{if } x \in \Omega^c. \end{cases}$$

Now consider

$$(P_N) \qquad \begin{cases} v_t - F(D^2v, Dv, v, x, t) - N\nabla \cdot [vA(x,t)\nabla\Phi] = 0 & \text{in } \mathbb{R}^n \times (0,\infty); \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

Here v_0 is an extension of u_0 to $\mathbb{R}^n$ to be defined in (3.2.11). Φ is a potential whose gradient is zero inside of Ω and is proportional to the inward normal of Ω outside of Ω. More precisely, we start with $d(x,\Omega)$ which is C^2 provided x is in an outer ball of Ω, and we consider a smooth extension $d(x)$ onto all of $\mathbb{R}^n$ that goes to infinity as $|x| \to \infty$. Then we write

$$\Phi(x) := d(x)^3 \tag{3.1.6}$$

See Theorem 3.4.1(a) for the well-posedness of (P_N) with the discontinuous operator F. Alternatively one can consider a regularized version of F (see section 4).

The approximating problem (P_N) can be viewed in the framework of stochastic particles, where the added drift represents an external force that pushes back the particles which diffused out of the domain Ω. In the context of stochastic differential equations, relevant results have been established in the classical paper of Lions and Snitzman [LS84], where a similar method of introducing a drift term was used to derive existence of solutions to the Skorokhod problem.

Showing the validity of this approximation is the goal of the chapter. The main result is the following:

Theorem 3.1.1. *Let u and v respectively solve (P_g) and (P_N) as given above, and let v_0 be given by* (3.2.11)*. Then for any $T > 0$, v uniformly converges to u in $\bar{\Omega} \times [0, T]$ as $N \to \infty$. Moreover we have*

$$|v(x,t) - u(x,t)| \leq CN^{-1/3} \text{ in } \bar{\Omega} \times [0, T], \tag{3.1.7}$$

where C depends only on n, λ, Λ, T and the regularity of the coefficients and the domain Ω.

While (P_N) is a natural approximation of the original problem (P_g), the convergence result does not appear to be previously proven, even for the case of the heat equation with Neumann boundary data. Let us briefly discuss the main challenges in the analysis.

Remarks

1. It is not clear to us whether the above theorem holds with the original F in (P_N) without extending it to have a diffusion term that corresponds to the boundary conditions given $A(x,t)$ outside of Ω. For our analysis this extension was necessary for the rather technical reason of constructing an appropriate barrier of the form $e^{-N\Phi} f$, based on the stationary solution of the divergence-form equation outside of Ω.

2. The rate in (3.1.7) is optimal in some sense for our choice of Φ in (3.1.6), which we show in Section 5.1. Φ is chosen to have cubic growth for the technical reason that Φ then is C^2 across $\partial\Omega$. See Theorem 3.5.3 for a result on different choices of potentials.

3. The result is limited to quasi-linear PDEs of the form (3.1.1). This is due to the nature of our argument, which is based on approximating (P_g) by switching the

operator F near the boundary of Ω with the diffusion operator associated with $A(x,t)$, as explained in the outline of the chapter below. To guarantee stability of such an approximation we need uniform regularity of the approximate solutions. This corresponds to the regularity of parabolic PDEs with leading coefficients discontinuous in one variable; see Theorem 3.4.1. It remains open whether the theorem holds for general nonlinear operators that go beyond (3.1.1).

∘ *Heuristics and difficulties*

For the elliptic case, arguments from the standard viscosity solution theory were applied in [BGJ13] to prove that the solution of

$$-F(D^2v, Dv, v, x) - N\nabla v \cdot (A(x)\nabla\Phi) = 0 \text{ in } \mathbb{R}^n \tag{3.1.8}$$

uniformly converges to the stationary version of (P_g) for nonlinear, uniformly elliptic F. Heuristically, this result can be justified by observing that $N\nabla\Phi$ approximates a singular measure concentrated on the boundary of Ω with the normal direction, thus leading to the boundary condition $\nabla v \cdot A(x)\vec{\nu} = 0$. However, for the parabolic problem the above approximation fails, due to additional challenges created by the time variable. For example, we prove in Theorem 3.5.1 that for $F = \Delta u$, replacing the divergence-form drift term in (P_N) by the non-divergence drift term in (3.1.8) causes the solution to converge to zero over time as $N \to \infty$. On the other hand, the zeroth order term $Nv\nabla \cdot (A(x,t)\nabla\Phi)$ in (P_N) causes a problem in the above heuristics to yield the oblique boundary condition in the limit $N \to \infty$. Indeed the standard viscosity theory argument fails to show the approximation of (P_g) with (P_N), due to the zeroth order term in the divergence-form drift in (P_N). Thus one concludes that there is a delicate balance between the two terms coming out of the penalizing drift term in (P_N), which must be handled carefully. The main observation that enables our analysis is that the solution of (P_N) outside of Ω can be bounded by the quickly vanishing barriers of the form $e^{-N\Phi}f$, where f is a smooth function. Our actual argument is built on estimates for the barriers (see section 2.1.1) and does not involve direct estimates on v, which suffices for our convergence result, but further asymptotic analysis on v may reveal information on the dynamics of the penalizing drift leading to the boundary condition in (P_g).

∘ *Outline of the chapter*

Due to the difficulties described above, we were not able to produce a direct argument to show Theorem 3.1.1. Instead, we prove the theorem first for linear operators where the diffusion matches the boundary flux condition in Section 3.2, and then build on these results to address the general case in Section 3.4.

The general idea in Section 3.2 is to use the comparison principle, by testing against barriers created by extending a particular perturbation of the true solution for (P_g). To illustrate the construction of barriers done in Section 3.2, we first carry out the argument in one dimension in Section 3.2.1, in the special case where $F = \Delta u$ and $A = 1$. Then we proceed to the more general linear case in higher dimensions in Section 3.2.2, still in the case where F and A correspond. One important ingredient in the proof is a decomposition argument which eliminates the zeroth order term in the penalizing drift in (P_N), as shown in (3.2.2) and (3.2.13). In Section 3.3 we will show results from basic numerical experiments which verify the rate of convergence for the heat equation in one dimension.

In section 3.4, we introduce an additional approximation to let us utilize the results of the previous section to prove the main theorem. Roughly speaking, we will interpolate the diffusion term of F in (P_g) with the one matching the boundary condition near

$\partial\Omega$; see (P_r) in Section 3.4. We then consider approximating the modified problem with the penalizing drift term. The important estimate in this section is the uniform rate of convergence between (P_r) and its penalizing approximation $(P_{r,N})$ which is independent of r (see Theorem 3.4.2), based on the uniform regularity of solutions of (P_r) (see Lemma 3.4.1). The uniform regularity estimate for (P_r) draws from the result of Kim and Krylov [KK07b], and is of independent interest. We finish with remarks and examples in Section 3.5.

3.2 PDEs of divergence form

We first consider the case when F is in linear, in divergence form and matches the co-normal boundary condition, in the following way:

$$(D) \qquad \begin{cases} u_t - \nabla\cdot(A(x,t)\nabla u) = 0 & \text{in} \quad \Omega\times(0,\infty); \\ \nabla u^T A(x,t)\vec{\nu} = 0 & \text{on} \quad \partial\Omega\times(0,\infty); \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

For simplicity we rescale so that $\lambda = 1$, so that

$$Id_{n\times n} \leq A(x,t) \leq \Lambda Id_{n\times n} \text{ for all } x\in\mathbb{R}^n, t\geq 0. \tag{A1}$$

In this case, the approximating problem is written as

$$(D_N) \qquad \begin{cases} v_t - \nabla\cdot[A(x,t)\nabla v] - N\nabla\cdot[vA(x,t)\nabla\Phi] = 0 & \text{in} \quad \mathbb{R}^n\times(0,\infty); \\ v(x,0) = v_0(x) & \text{on} \quad \mathbb{R}^n. \end{cases}$$

Here $\Phi(x)$ is defined in (3.1.6), and v_0 is an extension of u_0 onto $\mathbb{R}^n$ which will be defined in more detail in Section 3.2.2. We will prove

Theorem 3.2.1. *Suppose Ω is C^2 and that A is C^2, symmetric, and satisfies* (A1)*. Then if u solves (D) and v solves (D_N) with initial data v_0 given in* (3.2.11)*, we have that*

$$\|u - v\|_{L^\infty(\Omega \times [0,T])} < C(u_0, \Omega, A) T N^{-1/3}.$$

3.2.1 The heat equation in one dimension

Before handling the problem in multiple dimensions, we illustrate the proof technique on a simpler example, the one dimensional heat equation with Neumann data:

$$(H) \qquad \begin{cases} u_t = u_{xx} & \text{in } (a,b) \times [0,\infty); \\ u_x(a,t) = u_x(b,t) = 0 & \text{for all } t > 0; \\ u(x,0) = u_0(x) & \text{for all } x \in [a,b]. \end{cases}$$

We define the approximating problem

$$(H_N) \qquad \begin{cases} v_t = v_{xx} + N v_x \Phi_x + N v \Phi_{xx} & \text{in } \mathbb{R} \times (0,\infty); \\ v(x,0) = v_0(x) & \text{for all } x \in \mathbb{R}. \end{cases}$$

Here v_0 is defined as

$$v_0(x) := \begin{cases} u_0(x) & \text{if } x \in [a,b] \\ u_0(b) e^{-N\Phi(x)} & \text{if } x > b \\ u_0(a) e^{-N\Phi(x)} & \text{if } x < a, \end{cases} \tag{3.2.1}$$

and Φ is defined as follows:

$$\Phi(x) := \begin{cases} |x-a|^3 & \text{if } x \le a \\ 0 & \text{if } a < x < b \\ |x-b|^3 & \text{if } x \ge b. \end{cases}$$

In words, Φ grows cubically outside the original region, which makes it C^2 at the boundary.

Theorem 3.2.2. *Assume $u_0 \in C([a,b])$, and let u and v solve (H) and (H_N) respectively with initial data u_0 and v_0. Then for any $T > 0$, v uniformly converges to u in $[a,b]\times[0,T]$ as $N \to \infty$. In particular we have that for all $T > 0$,*

$$\|u - v\|_{L^\infty([a,b]\times[0,T])} < C(u_0, a, b)(T+1)N^{-1/3}.$$

For the proof we will perturb the true solution and then extend it to get super- and subsolutions of (H_N) on all of $\mathbb{R}$. The super- and subsolutions will serve as barriers to show that v is close to u in Ω. For the specific v_0 given by (3.2.1), the minimal size of the perturbation can be estimated by the barriers and we obtain the rate of convergence.

Building a supersolution

The first step is to create a supersolution to extend u off Ω, taking the form

$$\varphi(x,t) = f(x,t)e^{-N\Phi(x)}. \qquad (3.2.2)$$

Without loss of generality, we will only show the details of the extension to the right of $x = b$. Note that φ satisfies

$$\begin{aligned}\varphi_t - \varphi_{xx} - N\varphi_x\Phi_x - N\varphi\Phi_{xx} &= e^{-N\Phi}\Big(f_t - f_{xx} + 2Nf_x\Phi_x - N^2 f\Phi_x^2 \\ &\quad + Nf\Phi_{xx} - Nf_x\Phi_x + N^2 f\Phi_x^2 - Nf\Phi_{xx}\Big) \\ &= e^{-N\Phi}\left(f_t - f_{xx} + Nf_x\Phi_x\right).\end{aligned}$$

Thus we need only verify that

$$f_t - f_{xx} + Nf_x\Phi_x > 0. \tag{3.2.3}$$

We want φ to match u at the boundary, and go above it to the left, that is, $(\varphi_x - u_x)|_{x=b} < 0$. This would let us create a supersolution extension by taking the infimum of φ and u. However, since $u_x = 0$ at b, this requires $\varphi_x = f_x < 0$ which makes (3.2.3) difficult to satisfy. To avoid this, consider

$$u_\epsilon(x,t) := u(x,t) + \frac{5\alpha}{b-a}\epsilon(x-(a+b)/2)^2 + \frac{10\alpha}{b-a}\epsilon t.$$

Here $\alpha := \|u_t(b,\cdot)\|_{L^\infty([0,\infty))}$, and ϵ is a perturbation parameter. Then u_ϵ will satisfy the heat equation except with boundary condition $u_{\epsilon,x}(b,t) = 10\alpha\epsilon/2 = 5\epsilon\alpha$.

Now we construct f so that it matches u_ϵ at the boundary. For simplicity, we assume $b = 0$ and write

$$f(x,t) := u_\epsilon(0,t) + \alpha\frac{(x-\epsilon)^3 + \epsilon^3}{\epsilon} + \alpha\epsilon x.$$

A sample φ is shown in Figure 3.1. The cubic term in f is designed to help for x small, while the linear term will help for larger x. We calculate:

$$\begin{aligned}f_t(x,t) &= u_{\epsilon,t}(0,t) = u_t(0,t) + 10\epsilon\alpha/(0-a), \\ f_x(0,t) &= \epsilon(3\alpha + \alpha) < u_{\epsilon,x}(0,t).\end{aligned}$$

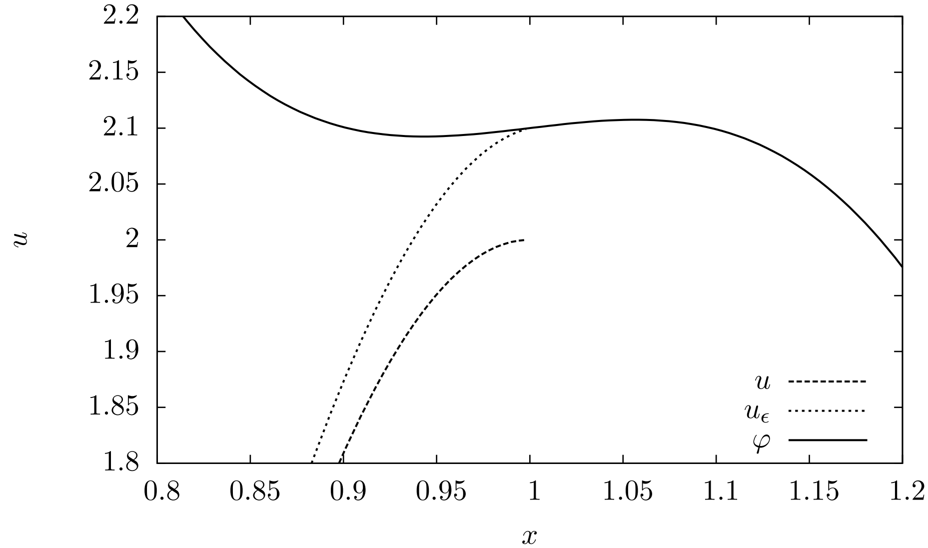

Figure 3.1: A sample u, u_ϵ, and φ, where $\Omega = [0, 1]$.

Then $f_t(x,t) > -2\alpha$ if $\epsilon < (b-a)/10$.

Now for $x \in [-\epsilon, \epsilon/2]$, we find

$$f_x = \alpha \frac{3(x-\epsilon)^2}{\epsilon} + \alpha\epsilon > 0, \qquad f_{xx} = 6\alpha \frac{x-\epsilon}{\epsilon} \leq -3\alpha.$$

Thus we find

$$f_t - f_{xx} + N f_x \Phi_x > -2\alpha + 3\alpha > 0.$$

Next, if $x > \epsilon/2$, we find

$$f_x = 3\alpha(x-\epsilon)^2/\epsilon + \alpha\epsilon \geq \alpha\epsilon, \qquad f_{xx} = 6\alpha(x-\epsilon)/\epsilon < 6\alpha x/\epsilon.$$

This gives the result

$$\begin{aligned} f_t - f_{xx} + N f_x \Phi_x &> -2\alpha - 6\alpha x/\epsilon + 3x^2\alpha\epsilon N \\ &\geq (N\alpha\epsilon^3/4 - 2\alpha) + \alpha\epsilon^{-1}x\left(2x\epsilon^2 N - 6\right). \end{aligned}$$

Then if $N > 8\epsilon^{-3}$, both terms will be positive in this region, letting us conclude that φ is a supersolution of (H_N) on $[-\epsilon, \infty) \times [0, \infty)$.

The full supersolution

Our final supersolution will be as follows:

$$w(x,t) = \begin{cases} u_\epsilon(x,t) & \text{if } a < x < 0 \\ \\ \varphi(x,t) & \text{if } x \geq 0. \end{cases} \tag{3.2.4}$$

This is a supersolution of (H_N) because it can be written as the infimum of a smooth extension of u_ϵ and φ. This works since they touch at $x = 0$ and are ordered appropriately because as shown above, $u_{\epsilon,x}(0,t) > f_x(0,t)$. Then for $x > 0$,

$$w(x,0) \geq u_\epsilon(0,t)e^{-N\Phi(x)} \geq u_0(0)e^{-N\Phi(x)} = v(x,0).$$

Since we can extend w in an analogous way to the left of a, applying the comparison principle ensures that $v \leq w$ in $\mathbb{R} \times [0, \infty)$ and hence $v \leq u_\epsilon$ in $[a,b] \times [0, \infty)$.

The proof of Theorem 3.2.2

From the supersolution u_ϵ constructed above setting $N = 10\epsilon^{-3}$, we obtain $v \leq u_\epsilon \leq u + CT\epsilon \leq u + CTN^{-1/3}$. Next, we construct a subsolution as follows. Let

$$g(x,t) = u_{-\epsilon}(0,t) - \alpha\frac{(x-\epsilon)^3 + \epsilon^3}{\epsilon} - \alpha\epsilon x.$$

Then we have that $g_x = -f_x$ and $g_{xx} = -f_{xx}$, and so by similar estimates we find $\psi(x,t) := g(x,t)e^{-N\Phi(x)}$ will be a subsolution on $[-\epsilon, \infty) \times [0, \infty)$. This lets us extend $u_{-\epsilon}$ to a subsolution $\tilde{w}$ on all of $\mathbb{R} \times [0, \infty)$. Then by construction, $\tilde{w} \leq v$ at $t = 0$. Hence

$\tilde{w} \leq v$ for all time by the comparison principle, so in particular $u_{-\epsilon} \leq v$ in $[a,b] \times [0,\infty)$. This lets us conclude that for $(x,t) \in [a,b] \times [0,\infty)$,

$$u_{-\epsilon}(x,t) \leq v(x,t) \leq u_{\epsilon}(x,t).$$

Thus provided $N > 10[(b-a)/10]^{-3}$, we have

$$\|u - v\|_{L^\infty(\Omega\times[0,\infty))} < C(u_0, a, b)(T+1)N^{-1/3}.$$

□

Remark 3.2.1. *Perhaps the most natural choice for v_0 is $v_0 = u_0$ inside Ω and zero outside. In this case the convergence rate can be obtained in L^1 norm. Observe that for w as given in (3.2.4), $v \leq w \leq u + CN^{-1/3}(T+1)$ in $\Omega \times [0,T]$. Moreover, since $0 \leq v \leq w$ one can show that $\int_{\mathbb{R}\setminus\Omega} v(x,t)dx \leq CN^{-1/3}$. The above estimates as well as conservation of mass yields that*

$$\|v(\cdot,t) - u(\cdot,t)\|_{L^1(\Omega)} \leq CN^{-1/3}(T+1).$$

3.2.2 The general linear divergence form equation

Now we consider the divergence form parabolic equation (D), and the approximating problem (D_N). Generalizing the extension process used in the one-dimensional case requires using the distance function, which is only smooth if we are close to Ω. To this end, we will require an intermediate domain Ω' that contains Ω. For γ a lower bound on the radius of interior and exterior balls to $\partial\Omega$, we define Ω' as

$$\Omega' := \{x : d(x,\Omega) < d_0\}, \tag{3.2.5}$$

$$\text{where } d_0 = \frac{1}{2}\min\left[\gamma, \frac{\gamma}{\sqrt{\Lambda^2-1}}\left(\Lambda - \sqrt{\Lambda^2-1}\right)\right]. \tag{3.2.6}$$

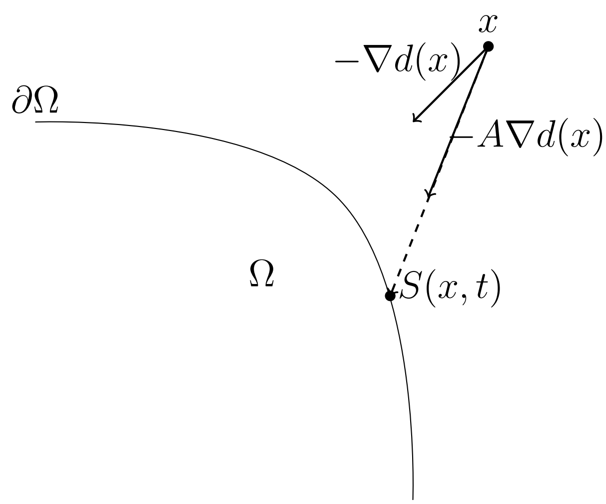

Figure 3.2: An illustration of how S functions

Then we have that $d(x,\Omega)$ is C^2 inside Ω', so we can find a C^2 extension $d(x)$ that matches $d(x,\Omega)$ inside Ω', and goes to infinity as $|x| \to \infty$. Before we prove Theorem 3.2.1, we prove two lemmas that will help us extend u off Ω. We define the mapping $S : \Omega' \to \partial\Omega$ to tell us what boundary point our extension takes data from. We define S in formula as

$$S(x,t) := x - \tilde{d}(x,t)A(x,t)\nabla d(x).$$

Here $\tilde{d}(x,t)$ is defined so that $S(x,t) \in \partial\Omega$, and in the case $A = Id_{n\times n}$ simply equals $d(x)$. In words, S maps x to the closest point in Ω in direction $-A(x,t)\nabla d(x)$, whereas the closest point is actually in direction $-\nabla d(x)$; see Figure 3.2.

Our first lemma shows basic properties of S, d, and $\tilde{d}$; the proof employs basic geometry and the implicit function theorem.

Lemma 3.2.1. *Suppose that* (3.2.5) *holds and A is C^2. Then in Ω', the distance function $d(x,\Omega)$ is C^2, $S(x,t)$ is well defined, and $\tilde{d} \lesssim d$. Further, for $x \in \Omega'\backslash\Omega$, $A\nabla d|_x \notin$*

$T_{S(x,t)}\partial\Omega$. *Lastly,* $\tilde{d}$ *is also* C^2*, and hence* S *is* C^2 *as well, with*

$$\nabla\tilde{d}(x,t)^T = \frac{\nabla d(S(x,t))^T\left[I - \tilde{d}(x,t)\nabla A(x,t)\nabla d(x) - \tilde{d}(x,t)A(x,t)D^2 d(x)\right]}{\nabla d(S(x,t)^T)A(x,t)\nabla d(x)}. \quad (3.2.7)$$

Proof. The regularity of the distance function is proven in [GT01]. For the second part, consider a point $x \in \Omega'\backslash\Omega$. Then since x is contained in an exterior ball of Ω, we must have that there is a unique nearest point $y \in \partial\Omega$ at distance d, and at y, there is an interior ball $B_\gamma(z)$. We show that starting at x, going in direction $-A\nabla d$ we will wind up in this interior ball, which will show that $S(x,t)$ is well-defined. The worst case scenario is when the angle between ∇d and $A\nabla d$ is maximal, and we note we can get an upper bound since it satisfies

$$\cos\theta = \frac{\nabla d^T A\nabla d}{|\nabla d||A\nabla d|} \geq \frac{1}{\Lambda}.$$

We wish to show that a ray starting from x, deflected by a maximal θ, will hit $B_\gamma(z)$ provided d is small enough. Projecting into the plane containing ∇d and $A\nabla d$, we can consider this in two dimensions; see Figure 3.3. Solving for the intersection, we find that the distance from where we hit $B_\gamma(z)$ is given by

$$d' = \frac{d(d+2\gamma)\Lambda}{d+\gamma+\sqrt{(d+\gamma)^2 - d(d+2\gamma)\Lambda^2}} \approx C(\gamma,\Lambda)d \quad (3.2.8)$$

for d small. Then in particular we hit if

$$(d+\gamma)^2 - d(d+2\gamma)\Lambda^2 \geq 0, \quad (3.2.9)$$

and it can be checked this is equivalent to requiring

$$d < \frac{\gamma}{\sqrt{\Lambda^2-1}}\left(\Lambda - \sqrt{\Lambda^2-1}\right).$$

Thus $S(x,t)$ is well-defined if we hit $B_\gamma(z)$, which is guaranteed if Ω' satisfies (3.2.5). Also, since $\tilde{d} \leq d'/|A\nabla d|$, we find $\tilde{d} \lesssim C(A,\Omega)$.

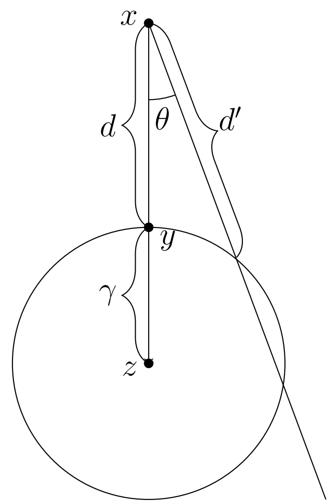

Figure 3.3: Calculating an upper bound on $\tilde{d}$

Next, we check that $A\nabla d|_x \notin T_{S(x,t)}\partial\Omega$. Consider the line from x along $A\nabla d$ to where it hits $B_\gamma(z)$. If $A\nabla d \in T_{S(x,t)}\partial\Omega$, then there would be an interior ball $B_\gamma(w)$ that is perpendicular to $A\nabla d$ at $S(x,t)$. We claim $d(w,x) < d(z,x)$, which would contradict that y is the nearest point to x since then $d(x,\Omega) \leq d(w,x) - \gamma < d(z,x) - \gamma = d(x,y)$. This claim follows by showing that

$$d(x,w)^2 = d(x,S(x,t))^2 + \gamma^2 < (d+\gamma)^2 = d(x,z)^2.$$

But since $d(x,S(x,t)) \leq d' < d(d+2\gamma)\Lambda/(d+\gamma)$ by (3.2.8), this is ensured if

$$\frac{d^2(d+2\gamma)^2\Lambda^2}{(d+\gamma)^2} < (d+\gamma)^2 - \gamma^2 = d(d+2\gamma).$$

Rearranging this yields that we need

$$d(d+2\gamma)\Lambda^2 < (d+\gamma)^2,$$

which is the same as (3.2.9), and hence true by our assumptions on the maximal size of d.

Finally, we use the implicit function theorem to show that $\tilde{d}$ is continuous. This is because it can be given implicitly as the solution λ to

$$f(x,t,\lambda) = d(x - \lambda A(x,t)\nabla d(x,\Omega),\Omega) = 0.$$

Then for a given (x_0,t_0), a minimal solution λ must exist since the solution space is non-empty and everything is continuous. Then we compute that

$$\frac{\partial f}{\partial \lambda} = -\nabla d(x_0 - \lambda A(x_0,t_0)\nabla d(x_0))^T A(x_0,t_0)\nabla d(x_0).$$

It can be proven that (3.2.9) implies that at the first point of contact to Ω along $A\nabla d$, the angle to Ω is not tangent, and hence $\frac{\partial f}{\partial \lambda} \neq 0$. Thus we find an implicit solution exists as we vary around (x,t) around (x_0,t_0), there is locally a solution $\tilde{d}(x,t)$, and since everything else in f is C^2, so is $\tilde{d}$. Then (3.2.7) follows from the implicit function as well. □

Lemma 3.2.2. *We can control the size of the directional derivative of S in the drift direction $A\nabla d$ in the following sense:*

$$|DSA\nabla d|\Big|_{(x,t)} \lesssim d(x),$$

for all $t \geq 0$ and $x \in \Omega' \backslash \Omega$.

Proof. We remark that as we move slightly in direction $A\nabla d$, S is affected by both A and ∇d changing. Since those changes are small, the deflection of S will be proportional to the length we have to travel to get back to $\partial\Omega$, which is proportional to $\tilde{d} \lesssim d$.

Fix x and t. Define $\eta := A(x,t)\nabla d(x)/|A(x,t)\nabla d(x)|$. As proven in Lemma 3.2.1,

$$\begin{aligned}
\nabla \tilde{d}(x,t)^T\eta &= \frac{\nabla d(S(x,t))^T\left[I - \tilde{d}(x,t)\nabla A(x,t)\nabla d(x) - \tilde{d}(x,t)A(x,t)D^2 d(x)\right]}{\nabla d(S(x,t))^T A(x,t)\nabla d(x)}\eta \\
&= \frac{1}{|A(x,t)\nabla d(x)|} - \tilde{d}(x,t)\left[\nabla d(S(x,t))^T\frac{\nabla A(x,t)\nabla d + A(x,t)D^2 d(x)}{\nabla d(S(x,t))^T A(x,t)\nabla d(x)}\eta\right] \\
&=: \frac{1}{|A(x,t)\nabla d(x)|} + \beta\tilde{d}(x,t).
\end{aligned}$$

Here β is defined this way for brevity and $\nabla A\nabla d$ is the matrix whose (i,j) entry is $(\nabla a^{ij})^T\nabla d$. We Taylor expand the quantities in S in direction η to find

$$\begin{aligned}
\tilde{d}(x+h\eta,t) &= \tilde{d}(x,t) + \frac{h}{|A(x,t)\nabla d(x)|} + \beta\tilde{d}h + O(h^2), \\
\nabla d(x+h\eta) &= \nabla d(x) + hD^2 d\eta + O(h^2), \\
(A(x+h\eta,t)) &= A(x,t) + h(\nabla A\eta) + O(h^2).
\end{aligned}$$

Calculating the directional derivative directly yields

$$\begin{aligned}
S(x+h\eta,t) - S(x,t) &= x + h\eta - \tilde{d}(x+h\eta,t)A(x+h\eta,t)\nabla d(x+h\eta) - x + \tilde{d}(x)A(x)\nabla d(x) \\
&= h\eta - \left(\tilde{d} + \frac{h}{|A\nabla d|} + h\beta\tilde{d}\right)(A + h(\nabla A\eta))(\nabla d + hD^2 d\eta) + \tilde{d}A\nabla d + O(h^2) \\
&= h\left(\eta - \tilde{d}AD^2 d\eta - \eta - \beta\tilde{d}\eta - \tilde{d}(\nabla A\eta)\nabla d\right) + O(h^2).
\end{aligned}$$

Dividing by h and taking the limit as $h \to 0$, we see that

$$DSA\nabla d = -|A\nabla d|\left(\tilde{d}AD^2 d\eta + \beta\tilde{d}\eta + \tilde{d}(\nabla A\eta)\nabla d\right). \tag{3.2.10}$$

Then by Lemma 3.2.1, $A\nabla d|_x \notin T_{S(x,t)}\partial\Omega$, that is, it is not tangent to $\partial\Omega$ at the point $S(x,t)$. Thus by compactness, we find $\nabla d(S(x,t))^T A(x,t)\nabla d(x)$ can be bounded away from zero, and so $\beta < C(A,\Omega)$. Then factoring out $\tilde{d}$ from (3.2.10) and using that $\tilde{d} \lesssim d$ from Lemma 3.2.1, we find

$$|DSA\nabla d| < C(A,\Omega)\tilde{d} < C(A,\Omega)d.$$

□

With this lemma in hand, we are ready to prove Theorem 3.2.1. We define v_0 as follows:

$$v_0(x) := \begin{cases} u_0(x) & \text{in } \Omega \\ e^{-N\Phi(x)}\mu(x)u_0(S(x,0)) & \text{in } \Omega^c. \end{cases} \tag{3.2.11}$$

Here $\mu(x) : \mathbb{R}^n \to [0,1]$ is a smooth function that is one when $d(x,\Omega) < d_0/2$ and zero when $d(x,\Omega) > d_0$. This smoothing factor μ is necessary since the map S is only defined when $d(x,\Omega) < d_0$.

3.2.2.1 Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. We proceed in a similar fashion to the heat equation case, with the difference being more care is required in the extension process. In particular, the extension used previously now only works on Ω', and we have to patch it to another solution to create a supersolution on all of $\mathbb{R}^n$.

First, we perturb u to u_ϵ which has a small positive slope at the boundary. We proceed by considering the signed distance function

$$h(x) = \begin{cases} d(x,\Omega^c) & \text{if } x \in \Omega \\ -d(x,\Omega) & \text{if } x \in \Omega' \backslash \Omega, \end{cases}$$

defined in a neighborhood of $\partial\Omega$ where this is C^2. We extend h to a C^2 function $\tilde{h}$ on all of Ω, and define

$$u_\epsilon(x,t) := u(x,t) + 5\alpha\epsilon\Lambda\left(-\tilde{h}(x) + \|\nabla\cdot(A\nabla\tilde{h})\|_{L^\infty(\Omega\times[0,\infty))}t + \|\tilde{h}\|_{L^\infty(\Omega)}\right), \tag{3.2.12}$$

where $\epsilon > 0$ is a perturbation parameter and α is a constant to be chosen later. Then u_ϵ

will be a supersolution of (D) satisfying $u_\epsilon \geq u$ and at $\partial\Omega$,

$$\nabla u_\epsilon^T A\vec{\nu} = \nabla u^T A\vec{\nu} - 5\alpha\epsilon\Lambda\nabla\tilde{h}^T A\vec{\nu} = 5\alpha\epsilon\Lambda\vec{\nu}^T A\vec{\nu} \geq 5\alpha\epsilon\Lambda.$$

We look for a supersolution of the modified equation (D_N) on $\Omega'\backslash\Omega$ of the form

$$\varphi(x,t) = f(x,t)e^{-N\Phi(x)}. \tag{3.2.13}$$

Let us calculate how this transforms the equation in detail. We have that

$$\begin{aligned}
\varphi_t &= f_t e^{-N\Phi}, \\
\nabla\varphi &= e^{-N\Phi}(\nabla f - Nf\nabla\Phi), \\
-N\nabla\cdot(\varphi A\nabla\Phi) &= e^{-N\Phi}\left(-N\nabla f^T A\nabla\Phi + N^2 f\nabla\Phi^T A\nabla\Phi - Nf\sum_{i,j} a^{ij}\Phi_{x_ix_j} - Nf\sum_{i,j} a^{ij}_{x_i}\Phi_{x_j}\right), \\
-\nabla\cdot(A\nabla\varphi) &= -e^{-N\Phi}\sum_{i,j}\left[a^{ij}_{x_i}(f_{x_j} - Nf\Phi_{x_j})\right] \\
&\quad + \sum_{i,j} a^{ij}\left[f_{x_ix_j} - Nf_{x_j}\Phi_{x_i} - Nf_{x_i}\Phi_{x_j} - Nf\Phi_{x_ix_j} + N^2 f\Phi_{x_j}\Phi_{x_i}\right] \\
&= e^{-N\Phi}\left(2N\nabla f^T A\nabla\Phi - N^2 f\nabla\Phi^T A\nabla\Phi\right) \\
&\quad + e^{-N\Phi}\sum_{i,j}\left[a^{ij}_{x_i}(Nf\Phi_{x_j} - f_{x_j}) + a^{ij}(Nf\Phi_{x_ix_j} - f_{x_ix_j})\right].
\end{aligned}$$

Summing these, we find

$$\begin{aligned}
\varphi_t - \nabla\cdot[A\nabla\varphi - N\varphi A\nabla\Phi] &= e^{-N\Phi}\left(f_t + N\nabla f^T A\nabla\Phi - \sum_{i,j}[a^{ij}_{x_i}f_{x_j} + a^{ij}f_{x_ix_j}]\right) \\
&=: e^{-N\Phi}(f_t - \mathcal{M}f),
\end{aligned}$$

where $\mathcal{M}$ is defined this way for brevity. Then we set

$$f(x,t) := u_\epsilon(S(x,t),t) + \alpha\frac{(d(x)-\epsilon)^3 + \epsilon^3}{\epsilon} + \alpha\epsilon d(x),$$

where $S(x,t)$ is the mapping onto $\partial\Omega$ defined above. We calculate:

$$
\begin{aligned}
f_t &= u_{\epsilon,t} + \sum_{i=1}^{n} u_{\epsilon,x_i} S_{i,t}, \\
\nabla f^T &= \nabla u_\epsilon^T DS + 3\alpha \frac{(d(x)-\epsilon)^2}{\epsilon} \nabla d^T + \alpha\epsilon\nabla d^T, \\
f_{x_i x_j} &= \sum_{k,l} \frac{\partial^2 u_\epsilon}{\partial x_l \partial x_k} \frac{\partial S_l}{\partial x_j} \frac{\partial S_k}{\partial x_i} + \sum_k \frac{\partial u_\epsilon}{\partial x_k} \frac{\partial^2 S_k}{\partial x_j \partial x_i} + 6\alpha \frac{(d(x)-\epsilon)}{\epsilon} d_{x_i} d_{x_j} + \alpha \left(3\frac{(d(x)-\epsilon)^2}{\epsilon} + \epsilon \right) d_{x_i x_j}, \\
\nabla\Phi &= 3d(x)^2 \nabla d.
\end{aligned}
$$

Then we can find $C(u,A,\Omega)$ so that the following bounds hold for ϵ small:

$$
\left| \sum_{i,j} a^{ij} \left[\sum_{k,l} \frac{\partial^2 u_\epsilon}{\partial x_l \partial x_k} \frac{\partial S_l}{\partial x_j} \frac{\partial S_k}{\partial x_i} + \sum_k \frac{\partial u_\epsilon}{\partial x_k} \frac{\partial^2 S_k}{\partial x_j \partial x_i} \right] + \sum_{i,j,k} a^{ij}_{x_i} \frac{\partial u_\epsilon}{\partial x_k} \frac{\partial S_k}{\partial x_j} \right| \le C(u,\Omega,A), \tag{3.2.14}
$$

$$
u_{\epsilon,t} = u_t + 5\alpha\epsilon\Lambda \|\nabla\cdot(A\nabla\tilde{h})\|_{L^\infty(\Omega\times[0,\infty))} \ge -\|u_t\|_{L^\infty(\partial\Omega\times[0,\infty))} \ge -C(u,A,\Omega), \tag{3.2.15}
$$

$$
\left| \sum_{i=1}^{n} u_{\epsilon,x_i} S_{i,t} \right| = \left| \sum_{i=1}^{n} u_{x_i} S_{i,t} - 5\alpha\epsilon\Lambda \sum_{i=1}^{n} \tilde{h}_{x_i} S_{i,t} \right| \le C(u,A,\Omega) + \alpha\epsilon C(u,A,\Omega). \tag{3.2.16}
$$

Then in particular we find that $f_t \ge -2C - \alpha\epsilon C$, so

$$
\begin{aligned}
f_t - \mathcal{M} f &= f_t + N\nabla f^T A \nabla\Phi - \sum_{i,j} [a^{ij}_{x_i} f_{x_j} + a^{ij} f_{x_i x_j}] \\
&\ge -2C - \alpha\epsilon C + N\nabla f^T A\nabla\Phi - C(u,\Omega,A) - \sum_{i,j} a^{ij}_{x_i} \alpha \left[3\frac{(d(x)-\epsilon)^2}{\epsilon} + \epsilon \right] d_{x_j} \\
&\quad + -6\alpha \frac{(d(x)-\epsilon)}{\epsilon} \nabla d^T A \nabla d - \sum_{i,j} a^{ij} \alpha \left[3\frac{(d(x)-\epsilon)^2}{\epsilon} + \epsilon \right] d_{x_i x_j}.
\end{aligned}
$$

Now let us examine the $\nabla f^T A\nabla\Phi$ term more carefully. Applying Lemma 3.2.2, we have

$$
\begin{aligned}
\nabla f^T A \nabla\Phi &= 3d(x)^2 \left(\alpha[3(d(x)-\epsilon)^2/\epsilon + \epsilon] \nabla d^T A \nabla d - \nabla u_\epsilon DSA\nabla d \right) \\
&\ge 3d(x)^2 \left(3\alpha(d(x)-\epsilon)^2/\epsilon + \alpha\epsilon - Cd \right).
\end{aligned}
$$

Then the inner polynomial has a minimum at $d = \epsilon + C\epsilon/6\alpha$, where it achieves the value of $\epsilon(\alpha - C^2/12\alpha - C)$. Thus provided $\alpha \geq 3C(u, \Omega, A)$, we have $N\nabla f^T A \nabla\Phi \geq 3N\epsilon\alpha d(x)^2/2$. Then this assumption on α yields the bound

$$\begin{aligned} f_t - \mathcal{M}f \geq & -\alpha - \alpha\epsilon C + \frac{3}{2}N\epsilon\alpha d(x)^2 - 6\alpha\frac{(d(x)-\epsilon)}{\epsilon}\nabla d^T A\nabla d \\ & - \left[3\alpha\frac{(d(x)-\epsilon)^2}{\epsilon} + \epsilon\right]\sum_{i,j}(a^{ij}_{x_i}d_{x_j} + a^{ij}d_{x_i x_j}). \end{aligned} \tag{3.2.17}$$

Now suppose $-\epsilon < d(x) < \epsilon/2$. This lets us use the estimate $-(d-\epsilon)\nabla d^T A \nabla d \geq \epsilon/2$ to bound (3.2.17) by

$$f_t - \mathcal{M}f \geq -\alpha + 3\alpha/2 + O(\epsilon).$$

Thus it follows that φ is a supersolution here if ϵ is small.

Next, suppose $\epsilon/2 \leq d(x) \leq d_0$. Then we simplify (3.2.17) to get

$$\begin{aligned} f_t - \mathcal{M}f &\geq -\alpha - \alpha\epsilon C + \frac{3}{2}N\epsilon\alpha d^2 - 6\alpha\Lambda d/\epsilon - C[3\alpha d^2/\epsilon + \epsilon] \\ &= \alpha(Nd^2\epsilon/2 - 1 - \epsilon C) + \frac{\alpha}{\epsilon}\left(Nd^2\epsilon^2 - 6\Lambda d - 3Cd^2 - C\epsilon^2\right). \end{aligned}$$

Then setting $N = 12(Cd_0 + \Lambda + 1)\epsilon^{-3}$, both terms will be positive, and so φ will be a supersolution over all of $\Omega'\backslash\Omega \times [0, \infty)$.

3.2.2.2 Creating the full supersolution

As in the one dimensional case, we define

$$w(x,t) = \begin{cases} u_\epsilon(x,t) & \text{if } x \in \Omega \\ \varphi(x,t) & \text{if } x \in \Omega'\backslash\Omega. \end{cases}$$

We need to check that w is in fact the infimum of the two supersolutions u_ϵ and φ. This is because at $\partial\Omega$ by construction we have $\varphi(x,t) = u_\epsilon(S(x,t),t) = u_\epsilon(x,t)$, while $\nabla u_\epsilon \cdot A\vec{\nu} \geq 5\alpha\epsilon\Lambda$ and

$$\nabla\varphi \cdot A\vec{\nu} = \nabla f \cdot A\vec{\nu} = \nabla u_\epsilon^T DSA\vec{\nu} + 4\alpha\epsilon\vec{\nu}^T A\vec{\nu} \leq 4\alpha\epsilon\Lambda.$$

Then since $-A\vec{\nu}$ points inside Ω, it follows that $\varphi > u_\epsilon$ immediately inside Ω, so w is in fact a supersolution. Note again that this infimum procedure works even though u_ϵ is only defined inside Ω, since it crosses φ exactly at $\partial\Omega$.

Now we extend w again from Ω' to all of $\mathbb{R}^n$. To do this, consider a stationary solution

$$\eta(x) = 2\|u_\epsilon\|_{L^\infty(\partial\Omega\times[0,\infty))}e^{-N\Phi(x)}.$$

Then at $\partial\Omega$, $\eta > w$ and at $\partial\Omega'$, if $\epsilon < d_0/2$, we have

$$\eta(x) = 2\|u_\epsilon\|_{L^\infty(\partial\Omega\times[0,\infty))}e^{-N\Phi(x)},$$
$$w(x,t) = \varphi(x,t) \geq \alpha\frac{d_0^3}{8\epsilon}e^{-N\Phi(x)}.$$

Thus w starts off below η and provided $\epsilon < d_0^3\alpha(16\|u_\epsilon\|_{L^\infty(\partial\Omega\times[0,\infty))})^{-1}$, w must cross η before $\partial\Omega'$. Thus by taking another infimum, w can be extended to be a solution on all of $\mathbb{R}^n$.

Lastly we check the ordering of w versus v at the parabolic boundary. At $t = 0$, the ordering is clear inside Ω, and since

$$f(x,0) \geq u_\epsilon(S(x,0),0) \geq \mu(x)u_\epsilon(S(x,0),0) = \mu(x)\left[u_0(S(x,0)) + 5\alpha\epsilon\Lambda\|\tilde{h}\|_{L^\infty(\Omega)}\right],$$

it follows that $\varphi(x,0) \geq v_0(x)$ outside Ω. Also, $\eta(x) \geq v_0(x)$ by construction, so $w(x,0) \geq v_0(x)$ as well. Thus we can apply the comparison principle to deduce that

$$v(x,t) \leq w(x,t) \text{ in } \mathbb{R}^n \times [0,\infty).$$

Since $w(x,t) = u_\epsilon(x,t) \in \Omega$, we see that in $[0,T]$,

$$v(x,t) - u(x,t) \lesssim \epsilon(T+1) \lesssim N^{-1/3}(T+1).$$

Now we construct the subsolution of (D_N) via a parallel procedure. As in the one-dimensional heat equation case, define

$$g(x,t) := u_{-\epsilon}(S(x,t),t) - \alpha\frac{(d(x)-\epsilon)^3+\epsilon^3}{\epsilon} - \alpha\epsilon d(x).$$

This crosses the zero solution inside Ω', since for ϵ small, $g(x,t) < 0$ when $d(x,\Omega) = d_0/2$. Then by taking a supremum with the zero stationary solution, we can extend $g(x,t)e^{-N\Phi(x)}$ to a subsolution on all of $\mathbb{R}^n$ that equals $u_{-\epsilon}$ in Ω and starts below v_0. Thus employing the comparison principle lets us deduce that $v(x,t) \geq u_{-\epsilon}(x,t)$ in $\Omega \times [0,\infty)$, and so we conclude that in $\Omega \times [0,T]$,

$$|v(x,t) - u(x,t)| \lesssim N^{-1/3}(T+1).$$

□

3.3 Finite difference approximation results

In this section we present the results of applying the Crank-Nicolson finite difference method to implement the approximation technique. First we consider the heat equation on $[0,1]$ with initial data $u_0(x) = \cos(2\pi x) + 1$, which admits solution $u(x,t) = e^{-4\pi^2 t}\cos(2\pi x) + 1$. We ran the scheme with 300 spacial panels and 2000 temporal panels, with zero Dirichlet conditions at the boundary of $\Omega' = [-1,2]$. We ran the experiment for $t \in [0,0.3]$. The true solution is shown in Figure 3.4, and two approximate solutions are shown in Figures 3.5 to 3.6.

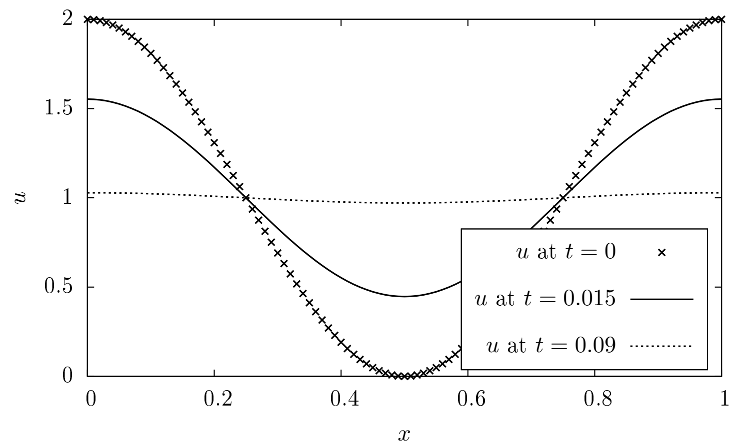

Figure 3.4: The true solution u at various time points

Estimates of the rate of convergence as we vary N are shown in Table 3.1, which are close to the analytic result of $N^{-1/3}$. To remove the inaccuracy of the underlying numerics, for each N we ran the scheme with varying space and time panels, and found that for all N tried, 6400 spacial panels and 102400 time panels ensured the results were stable for fixed N. Note that the L^∞ errors were approximately constant in time, and so it sufficed to use the final error.

The big advantage this method has for numerics is that it allows one to run finite difference schemes on domains with general geometry, which normally would require finite element methods. Since the routine winds up being done over a box, spectral methods can be used. Also, unlike the finite element methods, this method can be generalized to handle non-linear terms on the interior, at the expense of losing the converge rate estimate.

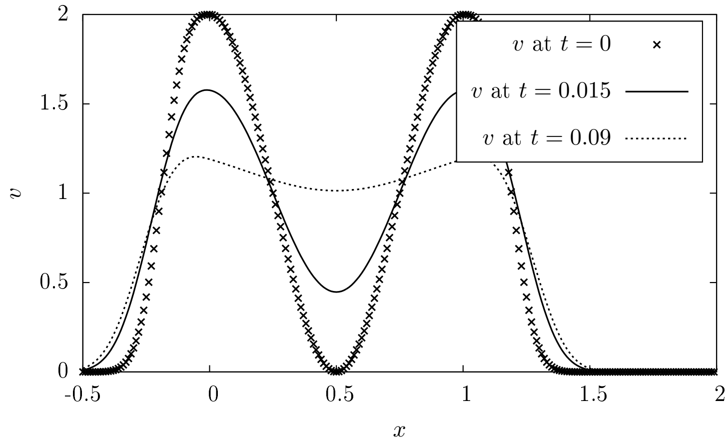

Figure 3.5: The approximating solution v with $N = 100$

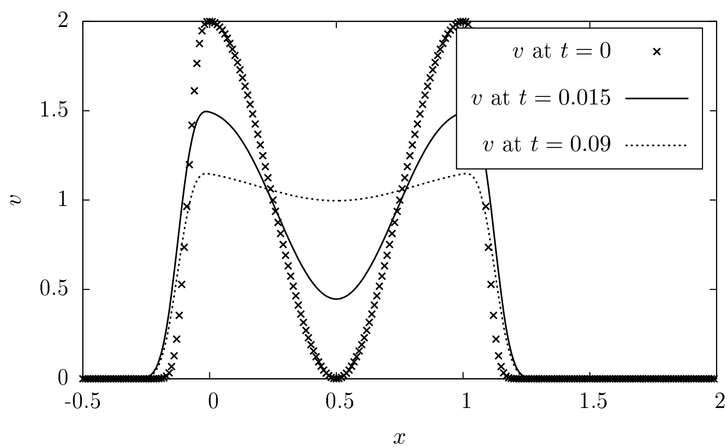

Figure 3.6: The approximating solution v with $N = 1000$

N	$\|u - v_N\|_{L^\infty([-1,2]\times\{t=0.3\})}$	p estimate
8192	0.0438889	0.2922126913041514
16384	0.0356576	0.2996465089723717
32768	0.0288452	0.30587833655159546
65536	0.0232513	0.3110198856649859
131072	0.018688	0.3151992014544745
262144	0.0149858	0.31851607677867955

Table 3.1: Estimates of the rate of convergence N^{-p} as we vary N

3.4 PDEs of non-divergence form

Building on the previous constructions of barriers, we are now ready to address the general problem given in the introduction in Theorem 3.1.1. Let u solve (P_g), and let v solve (P_N). To use the barrier argument from the previous section, the boundary operator F must correspond to the operator $\nabla\cdot(A\nabla v)$ in a neighborhood of $\partial\Omega$. This necessitates the introduction of two auxiliary problems which feature a regularized operator F_r. Towards this, we define

$$\Omega_r := \{x \in \Omega : d(x, \partial\Omega) > r\}.$$

Now take a smooth function $f(x)$ which is zero for $x < 1$ and one for $x > 2$, and write

$$g(x) := f(r^{-1}d(x, \partial\Omega)).$$

Then we define

$$F_r(D^2u, Du, u, x, t) := g(x)F(D^2u, Du, u, x, t) + (1 - g(x))\nabla \cdot (A(x, t)\nabla u), \quad (3.4.1)$$

which is smooth, satisfies (3.1.2), equals F in Ω_{2r}, and equals $\nabla \cdot (A(x,t)\nabla u)$ outside of Ω_r.

Next let w and $\tilde{v}$ solve the two auxiliary problems:

$$(P_r) \qquad \begin{cases} w_t - F_r(D^2 w, Dw, w, x, t) = 0 & \text{in} \quad \Omega \times [0,T]; \\ (A(x,t)\nabla w) \cdot \vec{\nu} = 0 & \text{on} \quad \partial\Omega \times [0,T]; \\ w(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

and

$$(P_{r,N}) \qquad \begin{cases} \tilde{v}_t - F_r(D^2\tilde{v}, D\tilde{v}, \tilde{v}, x, t) - N\nabla \cdot [\tilde{v}A(x,t)\nabla\Phi] = 0 & \text{in } \mathbb{R}^n \times [0,\infty); \\ \tilde{v}(x,0) = v_0(x) & \text{in } \mathbb{R}^n. \end{cases}$$

The proof of Theorem 3.1.1 proceeds by showing that as $r \to 0$, the solutions of the auxiliary problems (P_r) and $(P_{r,N})$ converge to the solutions of the original problems, uniformly in N. Then since the barriers constructed for the divergence form PDEs apply to the auxiliary problems, this will finish the proof (see Figure 3.4).

As a preliminary step, we will develop uniform estimates for w independent of r, using the following particular cases of Theorems 2.5 and 2.8 in Kim-Krylov [KK07b]:

Theorem 3.4.1. *Let us denote $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and define*

$$\mathcal{L}u := a^{ij}(x,t)u_{ij} + b(x,t) \cdot Du + c(x,t)u,$$

where the $\{a^{ij}\}$ satisfy (3.1.2)*, are continuous with respect to $(x_1, ..., x_{n-1})$, and measurable with respect to the x_n variable. Also, assume b and c satisfy* (3.1.4)*. Then the following holds for $2 < p < \infty$ and $T > 0$:*

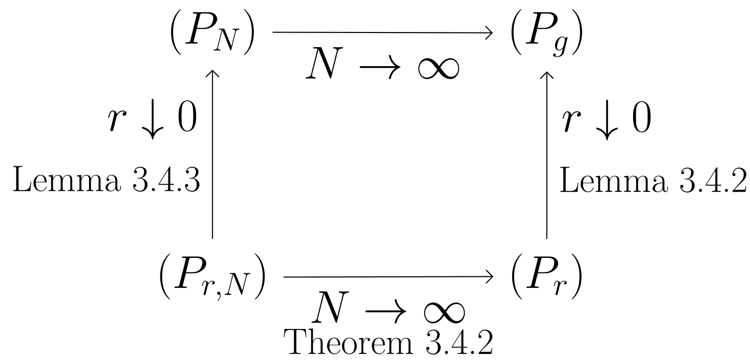

Figure 3.7: How the different problems and their solutions relate as we vary the parameters.

(a) *For any* $f \in L^p(\mathbb{R}^n \times [0,T])$, *there exists a unique* $u \in W_p^{2,1}(\mathbb{R}^n \times (0,T])$ *such that* $u_t - \mathcal{L}u = f$ *in* $\mathbb{R}^n \times (0,T]$ *with* $u(\cdot,0) = 0$. *Moreover*

$$\|u\|_{W_p^{2,1}(\mathbb{R}^n \times [0,T])} \leq C\|f\|_{L^p(\mathbb{R}^n \times [0,T])}.$$

Here C *depends only on* $n, p, T, \lambda, \Lambda$, *the bounds for* b *and* c, *and the mode of continuity of the* $\{a^{ij}\}$ *with respect to* $(x_1, ..., x_{n-1})$.

(b) *Let* H *be the half space* $\{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ *and let* $l \in H$. *Then for a given* $f \in L^p(H \times (0,T])$ *there exists a unique* $u \in W_p^{2,1}(H \times (0,T])$ *satisfying*

$$\begin{cases} u_t - \mathcal{L}u = f & \text{in} \quad H \times (0,T]; \\ l \cdot Du = 0 & \text{on} \quad \partial H \times (0,T]; \\ u(x,0) = 0 & \text{in } H, \end{cases}$$

with the estimate

$$\|u\|_{W_p^{2,1}(H \times (0,T])} \leq C\|f\|_{L^p(H \times (0,T])}.$$

Here C *depends only on* $n, p, T, l, \lambda, \Lambda$, *the bounds for* b *and* c, *and the mode of continuity of the* $\{a^{ij}\}$ *with respect to* $(x_1, ...x_{n-1})$.

The following lemma is essential to deduce an estimate, uniform in r, for the convergence of the solutions of $(P_{r,N})$ to those of (P_r).

Lemma 3.4.1. *Let* $F(D^2u, Du, u, x, t)$ *be given as in* (3.1.1)*, satisfying* (3.1.2) *and* (3.1.4)*, and let* w *solve* (P_r)*. Then for a given* $T > 0$*, the following holds for* $0 \leq t \leq T$*:*

(a) For any $0 < \alpha < 1$*,* $w(\cdot, t)$ *is uniformly* $C^{1,\alpha}$ *in* $\bar{\Omega}$ *with respect to* r*;*

(b) w_t *is bounded in* $\Omega \times [0, T]$*;*

(c) The restriction of $w(\cdot, t)$ *on* $\partial\Omega$ *is uniformly* $C^{1,1}$ *with respect to* r*.*

Proof. 0. In this proof C denotes various constants which are independent of r. Since w is C^2 up to $\bar{\Omega} \times (0, T]$, it suffices to get a uniform bound on the derivatives of w with respect to r.

1. Let us first consider the case when $\vec{v}$ is constant and the domain is a half space, i.e. when

$$\Omega = H = \{x = (x_1, ..., x_n) : x_n \geq 0\}.$$

In this case, with compactly supported u_0, Theorem 3.4.1(b) as well as Morrey's inequality yield (a) for linear PDEs. For F given as in (3.1.1) one can use Schauder's fixed point theorem with the map $\Psi : W_p^{1,0}(\bar{\Omega}_T) \to W_p^{2,1}(\bar{\Omega}_T)$ with sufficiently large p, where $u := \Psi(v)$ solves

$$\begin{cases} u_t - \sum q^{ij}(v, x, t)u_{x_i x_j} + b(Dv, v, x, t) = 0 & \text{in} \quad \Omega \times (0, T]; \\ \vec{v} \cdot Du = 0 & \text{on} \quad \partial\Omega \times (0, T); \\ u(x, 0) = u_0(x) \in C^2(\bar{\Omega}). \end{cases}$$

The argument for smooth, non-constant $\vec{v}$ is parallel to the constant case, which relies on introducing a local change of coordinates to change the problem to a Neumann problem, as written in the proof of Theorem 2.8 in [KK07b]. Also, see Remark 2.10 in [KK07a].

To generalize from H to Ω, we can apply a local change of coordinates such as in [Eva10] p. 337-339, which maps $\{x : d(x, \partial\Omega) = r\}$ to $\{x_n = r\}$, to reduce to the half-space case.

2. Note that $W := w_t$ satisfies

$$\begin{cases} W_t - \sum_{i,j} a_r^{ij}(w,x,t)W_{ij} + \partial_p b^r(Dw,w,x,t) \cdot DW + B(x,t)W + C(x,t) = 0 & \text{in} \quad \Omega \times (0,T]; \\ (A(x,t)DW) \cdot \vec{\nu} = -(A_t(x,t)Dw) \cdot \vec{\nu} \quad \text{on } \partial\Omega & \text{on} \quad \partial\Omega \times (0,T). \end{cases}$$

We write $a_r^{ij} := g(x)q^{ij} + (1 - g(x))a^{ij}$ as the second order matrix of F_r and likewise for b^r. Here

$$B(x,t) := \sum_{i,j} \partial_z a_r^{ij}(w,x,t)w_{ij} + \partial_z b^r(Dw,w,x,t),$$

and

$$C(x,t) := \sum_{i,j} (a_r^{ij})_t(w,x,t)w_{ij} + b_t^r(Dw,w,x,t).$$

Due to the uniform $W_p^{2,1}$ estimates obtained in step 1, we have that B and C are uniformly bounded with respect to r in $L^p(\Omega \times [0,T])$ for any $2 < p < \infty$. Setting $p = n+1$, [Kry76] yields the uniform L^∞ bound for W.

3. It remains to show (c). For simplicity, we will only show (c) in the case that Ω is locally a half space, that is, we will show (c) in $B_{1/2}(0)$ when

$$\Omega \cap B_1(x_0) = \{x \cdot e_1 \geq 0\} \cap B_1(0). \tag{3.4.2}$$

For general domains one can take a local change of coordinates as before to reduce to the half-space case.

Let us choose a boundary point $x_0 \in \partial\Omega \cap B_{1/2}(0)$ and a time $t = t_0$. To show (c), it is enough to show that there exists $M > 0$ which is independent of r such that for $x \in \partial\Omega$ and $t \leq t_0$, we have

$$|w(x,t) - w(x_0,t_0) - Dw(x_0,t_0) \cdot (x - x_0)| \leq M(x - x_0)^2 - C(t - t_0) \tag{3.4.3}$$

in $S := (\partial\Omega \cap B_\delta(x_0)) \times [t_0 - \tau, t_0]$. To this end we will build a supersolution and a subsolution of (P_r) and compare it with w in a parabolic neighborhood of S.

We first construct a supersolution of (P_r) to show that in S,

$$w(x,t) - w(x_0,t_0) - Dw(x_0,t_0) \cdot (x - x_0) \leq M(x - x_0)^2 - C(t - t_0). \tag{3.4.4}$$

Next, fix $\alpha \in (0,1)$. Due to (a) and (b) we know that

$$w(x,t) \leq w(x_0,t_0) + Dw(x_0,t_0) \cdot (x - x_0) + C|x - x_0|^{1+\alpha} - C(t - t_0) \tag{3.4.5}$$

in $(B_\delta(x_0) \cap \bar{\Omega}) \times [t_0 - \tau, t_0]$, where C, δ, and τ are constants independent from the choice of r.

Let us denote $(x - x_0)_T := (x - x_0) - (x - x_0) \cdot e_1$, and consider the function

$$h(x) := C\delta^{\alpha-1}\left(|(x - x_0)_T|^2 - \frac{\Lambda}{\lambda}(n+1)|(x - x_0) \cdot e_1|^2\right) - C_1\delta^\alpha (x - x_0) \cdot e_1 + CC_1\delta^\alpha a,$$

where C_1 is a constant depending only on $A(x,t)$ and $\|Dw\|_{C^\alpha}$, and a is a constant that is much smaller than δ. We work on Σ, a strip neighborhood of x_0 which is narrow in the direction of $-e_1 = \vec{\nu}$:

$$\Sigma := B_\delta(x_0) \cap \{|(x - x_0) \cdot \vec{\nu}| \leq a\} \cap \Omega.$$

Now let us define

$$\tilde{h}(x,t) := h(x) + w(x_0,t_0) + Dw(x_0,t_0) \cdot (x - x_0) - C_2(t - t_0)$$

where C_2 is to be chosen later. We claim that $\tilde{h}$ satisfies

$$\tilde{h}_t - F_r(D^2\tilde{h}, D\tilde{h}, \tilde{h}, x, t) \geq 0 \text{ in } \Sigma \times [t_0 - \tau, t_0] \tag{3.4.6}$$

if δ and τ are chosen small enough, but independently of r. To see this, observe that

$$\mathcal{P}^+(D^2 h) \leq -2\Lambda C\delta^{\alpha-1}.$$

This fact along with (3.1.2) and (3.1.4) yields

$$\begin{aligned} \tilde{h}_t - F_r(D^2\tilde{h}, D\tilde{h}, \tilde{h}, x, t) &\geq -C_2 + g(x)b(D\tilde{h}, \tilde{h}, x, t) + 2\Lambda C\delta^{\alpha-1} \\ &\geq 2\Lambda C\delta^{\alpha-1} - O(|D\tilde{h}| + |\tilde{h}| + |x - x_0| + |t - t_0|). \end{aligned}$$

From the definition of $\tilde{h}$ one can check that, in $\Sigma \times [t_0 - \tau, t_0]$ with small τ, $|\tilde{h}| \leq C$ and

$$|Dh(x)| \leq 2C\delta^\alpha + 2\frac{\Lambda}{\lambda}C\delta^{\alpha-1}a - C_1\delta^\alpha \leq C\delta^\alpha. \tag{3.4.7}$$

Hence due to (a), $|D\tilde{h}| \leq |Dw| + |Dh| \leq C + C\delta^\alpha$, and thus $|D\tilde{h}|, |\tilde{h}| \leq C$. Thus we conclude that $\tilde{h}$ satisfies (3.4.6) if δ and τ are sufficiently small.

Moreover, due to (3.1.5) we have

$$\begin{aligned} [A(x,t)D\tilde{h}(x,t)] \cdot (-e_1) &= [A(x,t)(Dh + Dw(x_0, t_0))] \cdot (-e_1) \\ &= [A(x,t)(Dh + Dw(x,t) + [Dw(x_0,t_0) - Dw(x,t)])] \cdot (-e_1) \\ &\geq [A(x,t)(C\delta^{\alpha-1}(x - x_0) - C_1\delta^\alpha e_1)] \cdot (-e_1) - \|A\|\|Dw\|_{C^\alpha} \\ &\geq C_1\delta^\alpha\lambda - \|A\|\|Dw\|_{C^\alpha} - \|A\|C\delta^\alpha \\ &\geq 0 \text{ on } (\partial\Omega \cap \Sigma) \times [t_0 - \tau, t_0], \end{aligned}$$

where the last inequality holds if C_1 is chosen sufficiently large with respect to $\|A\|$ and $\|Dw\|_{C^\alpha}$.

The above arguments let us conclude that $\tilde{h}$ is a supersolution of (P_r) in $\Sigma \times [t_0 - \tau, t_0]$. Now let us compare w and h on the parabolic boundary of the domain $\Sigma \times [t_0 - \tau, t_0]$ excluding $\partial\Omega$. Observe that on $(\partial\Sigma \backslash \partial\Omega) \times [t_0 - \tau, t_0]$, due to (3.4.5) we have

$$\begin{aligned}\tilde{h}(x,t) - w(x,t) \geq\ & C\delta^{\alpha-1}|(x-x_0)_T|^2 - C|x-x_0|^{1+\alpha} - C_1\delta^{\alpha}|(x-x_0)\cdot e_1| + \\ & + C\delta^{\alpha}a - C\delta^{\alpha-1}\Lambda(n+1)|(x-x_0)\cdot e_1|^2/\lambda \\ \geq\ & 0,\end{aligned}$$

since $|(x-x_0)_T| \sim \delta$ and $|(x-x_0)\cdot e_1| \leq a$ on $\partial\Sigma\backslash\partial\Omega$. Moreover

$$\tilde{h} - w(x,t) \geq C_2\tau - C\delta^{\alpha} \geq 0 \text{ on } \Sigma \cap \{t = t_0 - \tau\},$$

if τ is chosen to be larger than δ^{α} and if C_2 is sufficiently large. Therefore, we conclude that $w \leq \tilde{h}$ in $\Sigma \times [t_0 - \tau, t_0]$, which yields (3.4.4).

A parallel argument can be used to provide the lower bound.

□

Let us point out that the barrier constructed in the proof of Theorem 3.2.1 only relied on the $C^{1,1}$ spacial bounds of u and L^∞ bounds of u_t restricted to $\partial\Omega$ to get the bounds given in Equations (3.2.14)-(3.2.16). This is because the supersolution constructed in Theorem 2.1 was built off the behavior of the true solution u along the boundary. In addition, the space-time C^1 bound on u lets us use Taylor series to show that this supersolution in fact has the right ordering against u_ϵ at the boundary. Thus Lemma 3.4.1 lets us create a supersolution extension on the full domain that is uniformly close to w with respect to r, by taking an infimum of the candidate function and the (perturbed) true solution, which cross at $\partial\Omega$. We can then apply the comparison principle to $\tilde{v}$ and w as before and conclude:

Theorem 3.4.2. *Let $\tilde{v}$ and w respectively solve $(P_{r,N})$ and (P_r). Then for any $T > 0$ we have*

$$\sup_{\Omega\times[0,T]} |\tilde{v} - w| \leq C(\Omega, \Lambda, \lambda, T) N^{-1/3} \tag{3.4.8}$$

Next we prove by barrier arguments the following:

Lemma 3.4.2. *Let u solve (P_g) and w solve (P_r). Then*

$$\|u - w\|_{L^\infty(\Omega\times[0,\infty))} \leq C(\Omega, u_0, c_0, \frac{\Lambda}{\lambda})(1 + e^{Lt})r,$$

where L is the Lipschitz constant for F given in (3.1.4).

Proof. 1. Before we begin, we remark that we will later require that the spacial operators F and F_r be decreasing in the zero-th order term. We can assume this by applying the transform $U := e^{-Lt}u, W := e^{-Lt}w$. Then we find that U satisfies

$$\begin{aligned} U_t = e^{-Lt}u_t - LU &= e^{-Lt}F(D^2u, Du, u, x, t) - LU \\ &= e^{-Lt}F(e^{Lt}D^2U, e^{Lt}DU, e^{Lt}U, x, t) - LU \\ &=: G(D^2U, DU, U, x, t), \end{aligned}$$

where now G is still uniformly elliptic for $t \in [0, T]$ and in particular is decreasing in the U argument. Likewise, W satisfies

$$\begin{aligned} W_t &= e^{-Lt}F_r(e^{Lt}D^2W, e^{Lt}DW, e^{Lt}W, x, t) - LW \\ &=: G_r(D^2W, DW, W, x, t). \end{aligned}$$

Denote the problems that U and W solve $(\tilde{P}_g)$ and $(\tilde{P}_r)$. Note that $U(\cdot, t) \in C^{2,\alpha}(\bar{\Omega})$ due to [LT86].

2. Let us define

$$M := \max_{0<r<1, x\in\Omega\setminus\Omega_{2r}, 0\leq t\leq T} (\max(G_r(D^2U, DU, U, x, t), 1)),$$

which is independent of r and finite due to the regularity of U. Let $d_r(x)$ denote the distance function $d(x, \Omega_{2r})$ and its smooth extension by $\tilde{d}(x)$, where $|\tilde{d}| \leq 1$ and $\tilde{d}(x) = d_r(x)$ in a small neighborhood of $\Omega \backslash \Omega_{2r}$. Let

$$C_0 := 2\frac{\sup_{\partial\Omega\times[0,T]} |\vec{v}(x,t)|}{\lambda c_0},$$

where c_0 is given in (3.1.5). Now consider

$$w_2(x,t) := U(x,t) + 2C_0 Mr + C_0 Mr\tilde{d}(x) \text{ in } \Omega.$$

Note that on any level set of d_r in $\Omega \backslash \Omega_{2r}$ the sum of the tangential second derivatives of d_r amounts to the mean curvature of its level set and the normal second derivative of d_r is zero. Thus, due to (3.1.4), given that r is small enough,

$$G_r(D^2 w_2, Dw_2, w_2, x, t) \leq G_r(D^2U, DU, U, x, t) + O(Mr[\|D^2\tilde{d}\|_{L^\infty} + \|D\tilde{d}\|_{L^\infty} + 1]) \text{ in } \Omega\backslash\Omega_{2r}. \tag{3.4.9}$$

From (3.4.9) and the fact that $Dd_r = \vec{\nu}(x) + O(r)$ on $\partial\Omega$ we deduce that

$$\begin{cases} (w_2)_t - G(D^2 w_2, Dw_2, w_2, x, t) \geq -C_1 Mr, & \text{in} \quad \Omega_{2r} \times (0,T]; \\ (w_2)_t - G_r(D^2 w_2, Dw_2, w_2, x, t) \geq -M - C_1 Mr & \text{in} \quad (\Omega\backslash\Omega_{2r}) \times (0,T]; \\ \nabla w_2 \cdot \vec{v}(x,t) \geq 2\sup|\vec{v}|Mr/\lambda & \text{on} \quad \partial\Omega \times (0,T]; \\ w_2(x,0) = U(x,0) + 2C_0 Mr + C_0 Mr\tilde{d}(x) \geq U(x,0) + C_0 Mr & \text{in} \quad \Omega. \end{cases}$$

Here C_1 is a constant independent of r. Now define

$$h(x) := Md_r^2(x).$$

We will show that the function defined by

$$\tilde{w}(\cdot,t) := \begin{cases} w_2 + C_1 Mrt & \text{in} \quad \Omega_{2r} \\ \\ w_2 - \lambda^{-1} h + C_1 Mrt & \text{in} \quad \Omega \backslash \Omega_{2r} \end{cases} \tag{3.4.10}$$

is a supersolution of $(\tilde{P}_r)$ if r is sufficiently small. To this end we develop estimates on h. Note that $D^2 d_r$ is bounded in $\Omega\backslash\Omega_{2r}$ since $\partial\Omega_r$ is C^2 for r small. From these facts and that $d_r \leq 2r$ in $\Omega\backslash\Omega_r$, it follows that

$$D^2(d_r^2) = 2d_r D^2 d_r + 2Dd_r(Dd_r)^T \geq 2Dd_r(Dd_r)^T - O(r)(Id_{n\times n}).$$

Thus since $|Dd_r| = 1$, we have

$$(D^2(d_r^2))^+ \geq 2 - O(r) \text{ and } (D^2(d_r^2))^- \leq O(r).$$

It follows from the uniform ellipticity of G_r with respect to r that h satisfies

$$\begin{cases} \mathcal{P}^+(D^2 h) \geq 2\lambda M - \Lambda O(r) & \text{in} \quad \Omega\backslash\Omega_{2r}; \\ \\ |Dh| = M|Dd_r|d_r \leq 2Mr & \text{on} \quad \partial\Omega; \\ \\ 0 \leq h \leq 16Mr^2 & \text{in} \quad \Omega\backslash\Omega_{2r}. \end{cases}$$

Then since G_r is decreasing in the zero-th term, we find that in Ω_{2r},

$$\begin{aligned} \tilde{w}_t &= C_1 Mr + (w_2)_t \geq G_r(D^2 w_2, Dw_2, w_2, x, t) \\ &\geq G_r(D^2\tilde{w}, D\tilde{w}, \tilde{w}, x, t). \end{aligned}$$

On the other hand, in $\Omega\backslash\Omega_{2r}$ we find

$$\begin{aligned} \tilde{w}_t &= C_1 Mr + (w_2)_t \geq G_r(D^2 w_2, Dw_2, w_2, x, t) - M \\ &\geq G_r(D^2(w_2 - \lambda^{-1}h), Dw_2, w_2, x, t) + 2M - M - O(r) \\ &\geq G_r(D^2\tilde{w}, D\tilde{w}, \tilde{w}, x, t) + M - O(r). \end{aligned}$$

Next,

$$(Dw_2 - \lambda^{-1}Dh) \cdot \vec{v}(x,t) \geq 2\lambda^{-1} \sup|\vec{v}|Mr - 2Mr|\vec{v}|\lambda^{-1} \geq 0.$$

Thus $\tilde{w}$ is a supersolution of $(\tilde{P}_r)$ in $\Omega \times (0,T]$ if r is small. Moreover, if r is sufficiently small,

$$\tilde{w}(x,0) \geq U(x,0) + Mr - 16\lambda^{-1}Mr^2 \geq U(x,0) = W(x,0).$$

Thus it follows from the comparison principle for solutions of $(\tilde{P}_r)$ that

$$W \leq \tilde{w} \text{ in } \Omega \times [0,T],$$

Then computing

$$e^{-Lt}w \leq \tilde{w} = e^{-Lt}u + 2C_0Mr + C_0Mr\tilde{d}(x) - \lambda^{-1}h + C_1Mrt$$

shows that $w \leq u + Ce^{LT}r$ in $\Omega \times [0,T]$.

A lower bound can be obtained with parallel arguments. □

Corollary 3.4.1. *For $\tilde{v}$ solving $(P_{r,N})$ and u solving (P_g),*

$$|\tilde{v} - u| \leq C(T)(r + N^{-1/3}) \quad \textit{in } \Omega \times [0,T].$$

Lastly we show the following:

Lemma 3.4.3. *For fixed N, $\tilde{v}$ locally uniformly converges to v in $\bar{\Omega} \times [0,\infty)$ as $r \to 0$.*

Proof. Let $v_r = \tilde{v}$ be the solution of $(P_{r,N})$ associated with F_r. Since N is fixed, v_r is uniformly $C^{1,\alpha}$ in space with $\alpha > 1/2$ and hence has a subsequential limit we denote by v_0. We claim v_0 is a viscosity supersolution of (P_N); the subsolution case is analogous. Suppose it is not, and so we can find a smooth function φ that crosses v_0 from below at some point (x_0, t_0) that satisfies

$$\varphi_t - F(D^2\varphi, D\varphi, x, t) - N\nabla \cdot [\varphi A(x,t)\nabla\Phi] < -\delta < 0, \tag{3.4.11}$$

and by smoothness of φ and F we can assume this holds in a neighborhood of (x_0, t_0).

Note that we must have $x_0 \in \partial\Omega$ to not get an immediate contradiction, since otherwise v_0 and v_r have the same equation for r small. Then we can find points $(x_r, t_r) \to (x_0, t_0)$ where $\varphi - v_r$ has a local maximum with value z_r. These points must all lie in $\Omega \backslash \Omega_{2r}$, as outside v_r and v_0 satisfy the same equation. The goal is to push the crossing point into Ω_{2r}. Using γ as the minimal radius of interior balls of Ω, let

$$\varphi_r := \varphi - \frac{\delta |x - x_r|^2}{4\lambda n} + \delta(t - t_r)/4.$$

Then φ_r will still be a subsolution near (x_0, t_0), but now $\varphi - v_r$ has a strict local maximum at (x_r, t_r).

Next, consider the region $B_{\sqrt{\gamma r}}(x_r) \times [t_r - \tau, t_r]$. Then if r and τ are small enough, this region will be contained in the region where $\varphi - v_0$ has a local maximum and (3.4.11) holds. We are going to use the fact that v_r is uniformly $C^{1,\alpha}$ in space, independent of r. For r small, x_r must lie within distance γ of x_0, in which case it has a unique nearest boundary point we denote by y_r. For $\vec{\nu}_r$ the outward unit vector at y_r, let

$$h(x,t) := \varphi_r(x,t) - 20Cr^{\alpha}(x - x_r) \cdot \vec{\nu}_r,$$

where C is larger than the sum of the uniform $C^{1,\alpha}$ norms of v_r and φ_r. Next, since $D\varphi_r = Dv_r$ at (x_r, t_r), by the uniform $C^{1,\alpha}$ regularity of v_r we have

$$\begin{aligned}
(\varphi_r - v_r)(x, t_r) &= \int (D\varphi_r - Dv_r)(s, t_r) \cdot ds + z_r \\
&= \int [(D\varphi_r - Dv_r)(s, t_r) - (D\varphi_r - Dv_r)(x_r, t_r)] \cdot ds + z_r \\
&\geq - \int (\|v_r\|_{C^{1,\alpha}} + \|\varphi_r\|_{C^{1,\alpha}}) |s - x_r|^{\alpha} ds + z_r \\
&\geq -C|x - x_r|^{1+\alpha} + z_r.
\end{aligned}$$

This lets us compute that

$$
\begin{aligned}
(h - v_r)(x_r - 5r\vec{\nu}_r, t_r) &\geq -5^{1+\alpha}Cr^{1+\alpha} + z_r + 20 \cdot 5Cr^{1+\alpha} \\
&\geq z_r + 75Cr^{1+\alpha}.
\end{aligned}
$$

On the other hand, if $(x - x_r) \cdot \vec{\nu}_r \geq -3r$, we have

$$
\begin{aligned}
(h - v_r)(x, t_r) &\leq (\varphi_r - v_r)(x, t_r) + 20 \cdot 3 \cdot Cr^{1+\alpha} \\
&\leq z_r + 60Cr^{1+\alpha}.
\end{aligned}
$$

Thus the maximum of $h - v_r$ in $B_{\sqrt{\gamma r}}(x_r) \times [t_r - \tau, t_r]$ occurs in $\{(x - x_r) \cdot \vec{\nu}_r \leq -3r\}$. Further, if r is small enough, it must occur on the parabolic interior because on the spacial edge of the parabolic boundary,

$$
h - v_r \leq -\frac{\gamma\delta r}{4\lambda n} - 20Cr^{\alpha}(x - x_r) \cdot \vec{\nu}_r \leq -\frac{\gamma\delta r}{4\lambda n} + 40C\gamma r^{\alpha+1/2} \leq 0.
$$

This is because $\alpha > 1/2$, and on the temporal edge, φ_r was not modified.

Now it remains to prove that this maximum occurs inside Ω_{2r}. This is because of the square root scaling we used. That is, y_r must have an interior ball $B_\gamma(x_r') \subset \Omega$ that contains x_r. Then Ω_{2r} must contain $B_{\gamma-2r}(x_r')$. We assume for simplicity the worst case scenario where $x_r = y_r$, that is, x_r is on $\partial\Omega$. Now we show that

$$
B_{\sqrt{\gamma r}}(x_r) \cap \{(x - x_r) \cdot \vec{\nu}_r \leq -3r\} \subset B_{\gamma-2r}(x_r').
$$

This follows because the hyperplane $\{(x - x_r) \cdot \vec{\nu}_r \leq -3r\}$ intersects $B_{\sqrt{\gamma r}}(x_r)$ with width $\sqrt{\gamma r - 9r^2}$ and it intersects $B_{\gamma-2r}(x_r')$ with width

$$
\sqrt{(\gamma - 2r)^2 - (\gamma - 3r)^2} = \sqrt{2r\gamma - 5r^2},
$$

as can be seen from Figure 3.8. This width is larger provided $r < \gamma/4$, and by the definition of $\vec{\nu}_r$, the hyperplane is perpendicular to the line between x_r and x_r'.

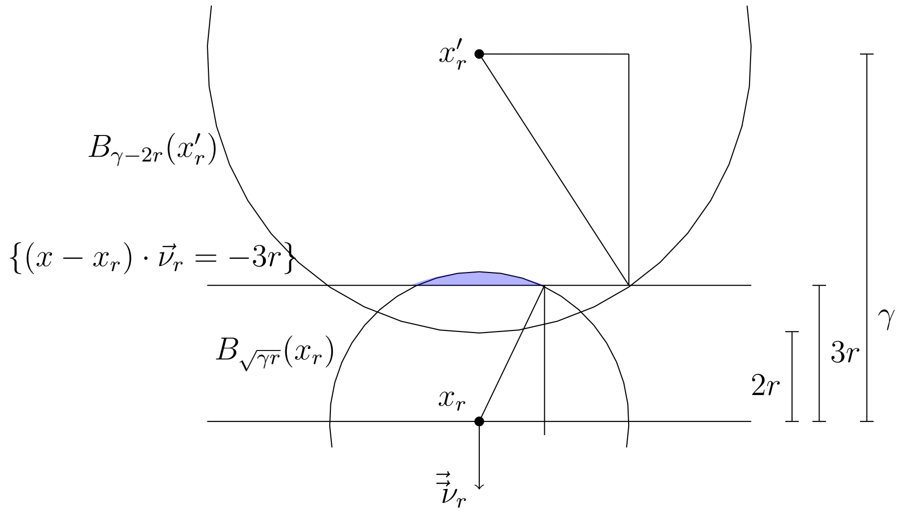

Figure 3.8: Showing $\{(x-x_r)\cdot\vec{\nu}_r \leq -3r\}$ is contained in Ω_{2r}

This entails that $\varphi_r - v_r$ has a local maximum inside Ω_{2r}, where φ_r is a subsolution of the same equation as v_r. Thus we are done, since this contradicts that v_r was a viscosity solution.

□

Theorem 3.4.2, Corollary 3.4.1 and Lemma 3.4.3 enable us to compare $\tilde{v}$ and v and conclude the following:

Theorem 3.4.3. *Let v and u respectively solve (P_N) and (P_g). Then we have, for any $T>0$,*

$$\sup_{\Omega\times[0,T]} |v-u| \leq C(T)N^{-1/3}.$$

3.5 Additional remarks

3.5.1 Examples in one dimension

First we verify that the full divergence-form drift is necessary in (P_N).

Theorem 3.5.1. *Let $v(x,0)$ be the characteristic function of $\Omega := [0,1]$, let $\Phi(x) = d^3(x,[0,1])$ and let $v(x,t)$ solve the following equation with initial data $v(x,0)$:*

$$v_t - v_{xx} - N\Phi_x v_x = 0 \text{ in } \mathbb{R}\times(0,\infty). \tag{3.5.1}$$

Then for any $\delta > 0$, there are N_0 and T_0 that only depend on δ so that for $N > N_0, v(x,T_0) < \delta$ in $[0,1]$.

The solution u of (H) with $a = 0, b = 1$ and $u_0 = v(x,0)$ is the stationary solution $u \equiv 1$. Thus the above theorem demonstrates in particular that v does not converge to u in $\mathbb{R}^n \times [0,T]$ as $N \to \infty$, if T is chosen large.

Proof. Fix $\delta > 0$. Note that $\phi(x) := e^{-N\Phi(x)}$ is a supersolution to (3.5.1), and thus $v \leq \phi$. Set N_0 so that $\phi(x) \leq \delta$ for $x \in \{-1,2\}$, and let us compare $v(x,t)$ with a barrier $h(x,t)$ in $[-1,2]\times[0,T_0]$, where

$$h(x,t) = 1 + \delta/2 - \delta(x-1/2)^2 - \delta t.$$

Here $T_0 = \delta^{-1} - 13/4$ satisfies $h(x,T_0) = \delta$ for $x \in \{-1,2\}$. Since $h_t - h_{xx} \geq 0$ and $h_x\Phi_x \leq 0$, it follows that h is a supersolution of (3.5.1), and since $h \geq \delta$ in $\{-1,2\}\times[0,T_0]$ it follows from the comparison principle that $v \leq h$ in $[-1,2]\times[0,T_0]$. Thus $v(x,T_0) \leq h(x,T_0) \leq 4\delta$.

□

Next we prove that for this penalization scheme and choice of initial data (3.2.11), the convergence rate of $N^{-1/3}$ given in Theorem 3.1.1 is optimal. The rate is connected to the cubic growth of $\Phi = d(x,\Omega)^3$. The idea is that $O(N^{-1/3})$ mass leaks out as seen by the size of mass on the outside of the stationary solution $e^{-N\Phi(x)}$. Our attempt to add additional mass in v_0 need not exactly cancel the mass loss, as the following example shows. Consider v solving (H_N) in $\mathbb{R}\times(0,\infty)$ with initial data

$$v_0(x) = \begin{cases} \cos(2\pi x)+1 & \text{if } x\in[0,1] \\ 2e^{Nx^3} & \text{if } x<0 \\ 2e^{-N(x-1)^3} & \text{if } x>1. \end{cases} \tag{3.5.2}$$

With this v we have the following theorem:

Theorem 3.5.2. *Let $u(x,t)$ solve (P_g) in $[0,1]\times(0,\infty)$, with initial condition $u_0(x) = \cos(2\pi x)+1$. Then with v as above, Then there exists a time T so that for all N,*

$$\sup_{[0,1]\times[0,T]} |u(x,t)-v(x,t)| \geq CN^{-1/3}.$$

Proof. Note that $u\to 1$ and $v\to Ce^{-N\Phi}$ exponentially as $t\to\infty$, and since Φ is a uniformly convex potential except in a compact set, this rate is uniform in N. By conservation of mass, we must have that

$$C\left[\int_{-\infty}^0 e^{Nx^3}\,dx + 1 + \int_1^\infty e^{-N(x-1)^3}\,dx\right] = 1 + \int_{-\infty}^0 2e^{Nx^3}\,dx + \int_1^\infty 2e^{-N(x-1)^3}\,dx.$$

Then denoting $I = \int_0^\infty e^{-u^3}\,du$, we have that $\int_0^\infty e^{-Nx^3}\,dx = N^{-1/3}I$, so

$$C = \frac{1+4N^{-1/3}I}{1+2N^{-1/3}I}.$$

Thus since $u(x,t)\to 1$, we find that in $[0,1]$, as $t\to\infty$,

$$v(x,t)-u(x,t)\to C-1 = \frac{1+4N^{-1/3}I}{1+2N^{-1/3}I} - 1 = \frac{2N^{-1/3}I}{1+2N^{-1/3}I} \geq N^{-1/3}I.$$

Then we can find a time T, independent of N, so that $v(x,T)-u(x,T) > N^{-1/3}I/2$. $\square$

3.5.2 Stability of the drift potential

Here we consider the potential whose gradient does not exactly line up with $\vec{\nu}$ near the boundary. For simplicity we will restrict this to the case of the heat equation and only work in Ω'; the divergence case is similar. We will write our new drift as the old drift plus a perturbation term Ψ, that is, we deal with the equation

$$(H'_N) \qquad \begin{cases} v_t = \Delta v + N\nabla \cdot (v\nabla\Phi) + N\nabla \cdot (v\nabla\Psi) \text{ in } \Omega', \\ v(x,0) = v_0(x) := \begin{cases} u_0(x) & \text{in } \Omega. \\ u_0(S(x))e^{-N(\Phi(x)+\Psi(x))} & \text{in } \Omega'\backslash\Omega \end{cases} \end{cases}$$

Here $S(x) = x - d(x)\nabla d(x)$ in the case $A \equiv Id$.

Theorem 3.5.3. *Suppose Ω is C^2, Ω' satisfies* (3.2.5) *and further $d(\Omega, \Omega') < 1$. Also, suppose $|\nabla\Psi| < d(x,\Omega)^3$. Then if u solves (D) with $A \equiv Id$ (that is, the heat equation), and v solves (H'_N), we have*

$$\|u - v\|_{L^\infty(\Omega\times[0,T])} < C(u,\Omega)(T+1)N^{-1/3}.$$

Proof. The proof is essentially the same as in Theorem 3.2.1, with the main difference being we apply the transform

$$\varphi(x,t) = f(x,t)e^{-N(\Phi(x)+\Psi(x))}.$$

Then using the fact that $|\nabla\Psi|$ is an order smaller than $|\nabla\Phi|$ makes it so that the extra terms in the transform are not problematic in the extension process. $\square$

3.5.3 Constructing $A(x,t)$

Lemma 3.5.1. *Consider a C^k domain Ω and smooth vector field $\vec{v}(x,t)$ satisfying $\vec{v}(x,t)\cdot\vec{\nu}(x) \geq c_0$ for all x and t, where $\vec{\nu}$ is the outer unit normal to Ω at x. Then there exists a symmetric C^k matrix $A(x,t)$ satisfying the property that*

$$A(x,t)\vec{\nu}(x) = \vec{v}(x,t) \text{ for all } x \in \partial\Omega.$$

Moreover, A satisfies the ellipticity condition (A1).

Proof. We start by considering an orthonormal basis of $T_x(\partial\Omega)$ written as $\{v_1(x), \ldots, v_{n-1}(x)\}$, where the v_i are C^k in x. After a rescaling $\vec{v}$ can be written as

$$\vec{v}(x,t) = \vec{\nu}(x) + \sum_{i=1}^{n-1} \alpha_i(x,t)v_i(x),$$

where the α_i are C^k and bounded. Then we define S as

$$S(x) = (\vec{\nu}(x), v_1(x), \ldots, v_{n-1}(x)).$$

Now we consider A of the form SBS^{-1}, where

$$B = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} \\ \alpha_1 & c & 0 & \ldots & 0 \\ \alpha_2 & 0 & c & \ldots & 0 \\ \vdots & 0 & & \ddots & 0 \\ \alpha_{n-1} & 0 & 0 & \ldots & c \end{pmatrix}.$$

Here c is a constant to be chosen large. We claim that if c is large enough, all the eigenvalues of B and hence A will be uniformly positive. This is because cofactor expansion gives that

$$\begin{aligned}\det(B-\lambda I) &= (1-\lambda)(c-\lambda)^{n-1} + \sum_{i=1}^{n-1}(-1)^i\alpha_i^2(c-\lambda)^{n-2} \\ &= (c-\lambda)^{n-2}\left[(1-\lambda)(c-\lambda) + \sum_{i=1}^{n-1}(-1)^i\alpha_i^2\right] \\ &= (c-\lambda)^{n-2}\left[\lambda^2 - 2\lambda c + c + \sum_{i=1}^{n-1}(-1)^i\alpha_i^2\right].\end{aligned}$$

Then this has eigenvalues $\lambda = c$ and writing $\beta := \sum_{i=1}^{n-1}(-1)^i\alpha_i^2$,

$$\lambda = \frac{2c \pm \sqrt{4c^2 - 4(c+\beta)}}{2}.$$

Thus taking c large with respect to β ensures that all eigenvalues can be bounded by

$$\lambda_0 < \lambda_i(x) < \Lambda$$

for all i and $x \in \partial\Omega$. Thus A satisfies the ellipticity condition, is symmetric, and is smooth. □

3.5.4 Why the method of viscosity solutions fails

We demonstrate why the standard viscosity solution technique fails for the heat equation in one dimension. We define u to be a viscosity subsolution of Equation 3.2.1 if for all smooth φ touching u from above,

$$\varphi_t - \Delta\varphi \le 0 \text{ in } \Omega,$$
$$\min\left(\varphi_t - \Delta\varphi, \frac{\partial\varphi}{\partial\vec{\nu}}\right) \le 0 \text{ on } \partial\Omega.$$

Then consider a sequence of approximating viscosity solutions v_N, and we investigate whether

$$v := \limsup v_N$$

is a viscosity subsolution of Equation 3.2.1. To this end, we suppose φ is a smooth function touching v from above at $\partial\Omega$, and

$$\varphi_t - \Delta\varphi > 0 \text{ and } \frac{\partial\varphi}{\partial\vec{\nu}} > 0. \tag{3.5.3}$$

Then by definition of v, eventually the v_N will touch φ nearby along a subsequence, and so we can deduce that

$$\varphi_t - \Delta\varphi - N\nabla\varphi\cdot\nabla\Phi - N\varphi\Delta\Phi \leq 0. \tag{3.5.4}$$

Then we should have that $\nabla\varphi\cdot\nabla\Phi = 3x^2\nabla\varphi\cdot\vec{\nu} > 0$. Thus we do not reach a contradiction, since the $N\nabla\varphi\cdot\nabla\Phi$ term has the wrong sign to make Equations (3.5.3) and (3.5.4) inconsistent.

However, note that if we instead considered the equation

$$\varphi_t - \Delta\varphi + N\nabla\varphi\cdot\nabla\Phi \leq 0$$

we would have gotten a contradiction: the first two terms are positive, and so is the second since $\partial\varphi/\partial\nu > 0$. This corresponds to the approximating equation of

$$v_t - \Delta v + N\nabla v\cdot\nabla\Phi = 0,$$

which converges to the heat equation with Neumann conditions in the trivial sense: the convergence we get would be to the zero solution as $N\to\infty$.

CHAPTER 4

Closing Remarks

4.1 Future work

In terms of immediately extending my doctoral research, first, when in the heat equation case of (P), we were very interested in understanding the convergence from the perspective of stochastic particles, as seen in the Ornstein-Uhlenbeck process. However, after using the comparison principle techniques directly we finally neglected this perspective. I would like to continue investigating that perspective starting by considering techniques based off those of [CR10]. We did an initial attempt at this, but ran into trouble with the growing penalization parameter N. However, it seems possible to proceed by creating estimates of the probability of particles being outside Ω as a function of N.

Second, it is an open question whether the results from Chapter 3 can be extended to the case of fully nonlinear operators. Also, it is of interest whether the techniques presented in Section 3.2.2 can be generalized to handle equations not in divergence form, possibly by modifying the drift term to look similar.

4.2 Conclusion

In this thesis we have proven the convergence of two singular limits of fundamental parabolic differential equations. In Chapter 2 we proved that porous medium equation solutions converge to Hele-Shaw solutions in the presence of drift. The drift term forces solutions to aggregate, which allowed us to prove that the support of the Hele-Shaw solutions follows a congested crowd motion model.

In Chapter 3, we proved that we can approximate quasi-linear parabolic differential equations with no-flux Neumann or oblique boundary conditions by considering a version on $\mathbb{R}^n$ with a penalizing drift that pushes mass back in. This technique of extending a problem to a larger domain is common in numerical analysis, and our proof technique shows potential to apply to many similar schemes.

References

[AGS06] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures.* Springer, 2006.

[BBZ10] J Bedrossian, J. H. von Brecht, S. Zhu, E Sifakis, and J. M. Teran. "A second order virtual node method for elliptic problems with interfaces and irregular domains." *Journal of Computational Physics*, **229**(18):6405–6426, 2010.

[BGJ13] Guy Barles, Christine Georgelin, and Espen R. Jakobsen. "On Neumann and oblique derivatives boundary conditions for nonlocal elliptic equations." *arXiv:1302.5568*, 2013.

[Cha04] Antonin Chambolle. "An algorithm for mean curvature motion." *Interfaces and free boundaries*, **6**(2):195–218, 2004.

[CJM01] JA Carrillo, Ansgar Jüngel, PA Markowich, Giuseppe Toscani, and Andreas Unterreiter. "Entropy Dissipation Methods for Degenerate ParabolicProblems and Generalized Sobolev Inequalities." *Monatshefte für Mathematik*, **133**(1):1–82, 2001.

[CR10] José A Carrillo and Jesús Rosado. "Uniqueness of bounded solutions to aggregation equations by optimal transport methods." In *European Congress of Mathematics Amsterdam, 14–18 July, 2008*, pp. 3–16, 2010.

[CV99] Luis Caffarelli and Juan Luis Vazquez. "Viscosity solutions for the porous medium equation." In *Proceedings of Symposia in Pure Mathematics*, volume 65, pp. 13–26. American Mathematical Society, 1999.

[Dal93] Gianni Dal Maso. "Introduction." In *An Introduction to* Γ*-Convergence*, pp. 1–7. Springer, 1993.

[EJ81] CM Elliott and Vladimır Janovskỳ. "A variational inequality approach to Hele-Shaw flow with a moving boundary." *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, **88**(1-2):93–107, 1981.

[Eva10] L.C. Evans. *Partial Differential Equations.* Graduate studies in mathematics. American Mathematical Society, 2010.

[GK11] Natalie Grunewald and Inwon Kim. "A variational approach to a quasi-static droplet model." *Calculus of Variations and Partial Differential Equations*, **41**(1-2):1–19, 2011.

[GPW92] Roland Glowinski, T. W. Pan, Raymond O. Wells, Jr., and Xiaodong Zhou. "Wavelet and Finite Element Solutions for the Neumann Problem Using Fictitious Domains." *J. Comp. Phys*, **126**:40–51, 1992.

[GQ01] O Gil and F Quirós. "Convergence of the porous media equation to Hele-Shaw." *Nonlinear Analysis: Theory, Methods & Applications*, **44**(8):1111–1131, 2001.

[GT01] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 2001.

[Hel98] HS Hele-Shaw. "Flow of water." *Nature*, **58**:520, 1898.

[JKO98] Richard Jordan, David Kinderlehrer, and Felix Otto. "The variational formulation of the Fokker–Planck equation." *SIAM journal on mathematical analysis*, **29**(1):1–17, 1998.

[Kim03] Inwon C Kim. "Uniqueness and existence results on the Hele-Shaw and the Stefan problems." *Archive for rational mechanics and analysis*, **168**(4):299–328, 2003.

[KK07a] Doyoon Kim and N. V. Krylov. "Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others." *SIAM Journal on Mathematical Analysis*, **39**(2):489–506, 2007.

[KK07b] Doyoon Kim and N. V. Krylov. "Parabolic equations with measurable coefficients." *Potential Analysis*, **26**(4):345–361, 2007.

[KL10] Inwon C Kim and Helen K Lei. "Degenerate diffusion with a drift potential: a viscosity solutions approach." *Dynamical Systems*, **27**(2):767–786, 2010.

[Kry76] N. V. Krylov. "Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation." *Siberian Mathematical Journal*, **17**(2):226–236, 1976.

[LS84] P.L. Lions and A.S. Sznitman. "Stochastic Differential Equations with Reflecting Boundary Conditions." *Comm. Pure Appl. Math.*, **37**:511–537, 1984.

[LT86] Gary M. Lieberman and Neil S. Trudinger. "Nonlinear oblique boundary value problems for nonlinear elliptic equations." *Transactions of the American Mathematical Society*, **295**(2):509–546, 1986.

[McC97] Robert J McCann. "A convexity principle for interacting gases." *advances in mathematics*, **128**(1):153–179, 1997.

[MRS10] Bertrand Maury, Aude Roudneff-Chupin, and Filippo Santambrogio. "A macroscopic crowd motion model of gradient flow type." *Mathematical Models and Methods in Applied Sciences*, **20**(10):1787–1821, 2010.

[Ott01] Felix Otto. "The geometry of dissipative evolution equations: the porous medium equation." *Comm. Partial Differential Equations*, **26**:101–174, 2001.

[Poz13] Norbert Pozar. "Homogenization of the Hele-Shaw problem in periodic spatiotemporal media." *arXiv preprint arXiv:1305.4079*, 2013.

[RV11] B Maury-A Roudneff-Chupin-F and Santambrogio-J Venel. "Handling congestion in crowd motion models." *Journal: Net. Het. Media*, **6**(3):485–519, 2011.

[San12] Filippo Santambrogio. "A modest proposal for MFG with density constraints." *Networks & Heterogeneous Media*, **7**(2), 2012.

[Ser11] Sylvia Serfaty. "Gamma-convergence of gradient flows on Hilbert and metric spaces and applications." *Discrete and Continuous Dynamical Systems A*, **31**(4):1427–1451, 2011.

[Vaz06] Juan Luis Vázquez. *The Porous Medium Equation: Mathematical Theory: Mathematical Theory*. Oxford University Press, 2006.