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Curvature Bounds in Riemannian Geometry

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Abstract

Curvature Bounds in Riemannian Geometry

by

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We first review a number of well known theorems in Riemannian geometry, and we discuss in detail some of their proofs. We then present, in chapters 2, 3 and 4, proofs of three results: a local L_p bound on $||\mathrm{Ric}||$ for $p < \frac{1}{2}$ under a lower Ricci curvature bound, the lower semicontinuity of volume on surfaces of bounded Euler characteristic, and a construction for metrics of nonpositive sectional curvature that develop a positive sectional curvature somewhere with respect to the Ricci flow.

Contents

Chapter 1

Review

1.1 Background

We begin with a survey of a number of results in Riemannian geometry that relate bounds on curvature of various sorts to geometric and topological control of Riemannian manifolds. The first collection of results, section 1.1.2 below, focuses primarily on the bound $\mathrm{Ric} \geq K$, and the second, section 1.1.3, on bounds of the form $|\sec| \leq K$, $|\mathrm{Ric}| \leq K$ etc. Section 1.1.1 reviews background material and discusses some fundamental ideas that support the subsequent discussion.

1.1.1 General Preliminaries and Motivation

We define a Riemannian manifold as a pair (M, g), where M is a smooth manifold and g is a Riemannian metric on M, i.e. a smoothly varying inner product

$$g : T_pM \times T_pM \to \mathbb{R}$$

on the tangent spaces T_pM of M. g induces a connection

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM),$$

where $\Gamma(TM)$ denotes the set of all vector fields on M, that satisfies

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

for all vector fields X, Y and Z, where $[X, Y]$ denotes the Lie bracket of the vector fields X and Y. We define the Riemann curvature tensor in terms of the connection as,

$$\mathrm{Rm}(X, Y, Z, W) := g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W),$$

We define the sectional curvature by normalizing Rm,

$$\sec(x, y) := \frac{\mathrm{Rm}(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}.$$

We say $\sec \geq K$ if, for any pair x, y we have $\sec(x, y) \geq K$.

We define the Ricci curvature as the trace,

$$\mathrm{Ric}(X,Y) := \mathrm{Rm}(e_i, X, Y, e_i),$$

where $\{e_1, ..., e_n\}$ are locally defined orthonomral vector fields. We say $\mathrm{Ric} \geq K \in \mathbb{R}$ if all of the eigen values of Ric are greater than or equal to K.

We lastly define the scalar curvature,

$$\mathrm{R} := \mathrm{Ric}(e_i, e_i).$$

Example 1.1.1. Let $(\mathbb{S}^2, g_{\mathrm{rnd}})$ denote the 2-sphere with standard metric g_{rnd} of sectional curvature $\sigma_{rnd} = 1$ and (N^2, g_{hyp}) a genus 2 surface with uniform sectional curvature $\sigma_{\mathrm{hyp}} = -1$. Let $M = \mathbb{S}^2 \times N$ denote their product, with metric $g = g_{\mathrm{rnd}} + g_{\mathrm{hyp}}$. Taking $s \in \mathbb{S}^2$, $n \in N$, $u \in T_s\mathbb{S}^2$ and $v \in T_nN$, it follows from the general case for product metrics (see Chapter 3 of citepetersen) defined this way that

$$\mathrm{Rm}(u, v, v, u) = 0$$

and thus that

$$\mathrm{R} = 1 + 0 + 0 - 1 = 0.$$

As such, we see that (M, g) has zero scalar curvature while Rm does not vanish identically. We also have

$$\mathrm{Ric}(u, u) = 1$$

and

$$\mathrm{Ric}(v, v) = -1.$$

Thus, the condition $\mathrm{R} \equiv 0$ is not restrictive enough to provide control over Rm or Ric. It is also the case that there exist Ricci flat (M, g), i.e. $\mathrm{Ric} \equiv 0$, such that Rm does not vanish identically, although constructions are more subtle. See [18], for instance, for examples. □

The metric g induces a length l of curves, defined as

$$l(\gamma) := \int_a^b g(\gamma'(t), \gamma'(t))dt,$$

where $\gamma : [a, b] \to M$, as well as a distance function d satisfying, for $p, q \in M$,

$$d(p, q) := \inf_{\gamma} l(\gamma),$$

where this infimum is taken over curves γ satisfying $\gamma(a) = p,\ \gamma(b) = q$. We say γ is a minimizing geodesic if

$$l(\gamma) = d(\gamma(a), \gamma(b)).$$

The distance d gives rise to the standard notion of the diameter of M,

$$\mathrm{diam}M := \sup\{d(p, q) \mid p, q \in M\}$$

as well as of the injectivity radius of $p \in M$,

$$\mathrm{inj}(p) := \sup\{r \mid B(p, r) \text{ is diffeomorphic to } B(r) \subset \mathbb{R}^n\},$$

where $B(p,r)$ is the $r-$ball in M and $B(r)$ denotes the r-ball in $\mathbb{R}^n$. Note that, because we only require the two balls to be diffeomorphic, these two radii need not be equal. We then define the injectivity radius of M as

$$\text{inj}M := \inf\{\text{inj}(p) \mid p \in M\}.$$

See Chapter 5 of [24] for a thorough presentation of the injectivity radius.

The metric g also induces, for a point $p \in M$, a volume form dvol_g that by definition satisfies, for any positively oriented orthonormal basis $\{e_1, ..., e_n\} \subset T_pM$,

$$\text{dvol}_g(e_1, ..., e_n) = 1,$$

which gives rise to a volume for open sets $U \subset M$:

$$\text{Vol}(U) := \int_U 1 \cdot \text{dvol}_g.$$

Finally, recall that g induces a norm $| \cdot |$ on tensor products of T_pM and its dual T_p^*M in the standard way.

Gromov Hausdorff metric and convergence

In studying the influence of curvature on various geometric quantities, it is useful to have in place the notion of Gromov-Hausdorff convergence of metric spaces, as it is often the case that geometric restrictions on a sequence of Riemannian manifolds results in the existence of convergent subsequences in the Gromov-Hausdorff sense, where the limit is, in general, only a metric space. To that end, we define, for metric spaces X and Y,

$$d_{GH}(X,Y) = \inf\{d_H(f(X), g(Y))\}$$

where this infimum is taken over all metric spaces M and isometric embeddings $f : X \to M$ and $g : Y \to M$, and where $d_H(\ , \ )$ denotes the Hausdorff distance between subsets of a metric space. We then say a sequence of metric spaces X_i has the Gromov-Hausdorff limit X if

$$d_{GH}(X_i, X) \to 0.$$

If in fact the sequence of metric spaces and the limit space are smooth manifolds then we can talk about stronger notions of convergence. If, for instance, there exist, for large enough i, diffeomorphisms $\phi_i : X \to X_i$, then we can examine the pullbacks $\phi^* g_i$ of the Riemannian metrics g_i on X_i to X, and define the C^m, C^∞, $C^{m,\alpha}$ etc. sense of convergence of $(X_i, g_i) \to (X, g)$ by requiring that these pullback metrics converge to g in the appropriate topology. Because we do not always assume (X_i, g_i) and their limits are closed, it is also useful to have the notion of pointed convergence, which, among other things, allows for the convergence of a sequence of closed Riemannian manifolds to something unbounded. We fix a sequence of points $p_i \in X_i$, $p \in X$ and we say $(X_i, g_i, p_i) \to (X, g, p)$ in the pointed Gromov-Hausorff (or C^m etc.) sense if for all $R > 0$

$$(B(p_i, R), g_i) \to (B(p, R), g)$$

in the Gromov-Hausdorff (or C^m etc.) sense. Note that the existence of diffeomorphisms between $B(p_i, R)$ and $B(p, R)$ for large i does not imply that X is diffeomorphic to any of the manifolds X_i.

We briefly review some fundamental results related to Gromov-Hausdorff convergence. For the material here and a more in depth discussion, see [petersen].

The strongest bounds we consider take the form $|\text{sec}| \leq K$ paired with a lower bound $\text{inj} \geq i_0$, in which setting the resulting subclasses of Riemannian manifolds are in fact compact. To show this, one defines a $C^{m,\alpha}$ norm at the scale of r on subsets A of a Riemannian manifold (M^n, g) as follows: we say

$$||A||_{C^{m,\alpha},r} < Q$$

if we can find charts $\phi_s : B(0, r) \subset \mathbb{R}^n \to U_s \subset M$ such that

1. Every ball $B(p, \frac{1}{10}e^{-Qr})$, $p \in A$ is contained in some U_s
2. $|D\phi_s| \leq e^Q$ on $B(0, r)$ and $|D\phi_s^{-1}| \leq e^Q$ on U_s
3. $r^{|j|+\alpha}||D^j g_{s\cdot}||_\alpha \leq Q$ for all multi indices j with $0 \leq |j| \leq m$.
4. $||\phi_s^{-1} \circ \phi_t||_{C^{m+1,\alpha}} \leq (10 + r)e^Q$

Now consider the following theorem [24]

Theorem 1.1.2. *(Fundamental Theorem of Convegence Theory) For given $Q > 0$, $n \geq 2$, $m \geq 0$ $\alpha \in (0, 1]$ and $r > 0$ consider the class $\mathcal{M}^{m,\alpha}(n, Q, r)$ of complete, pointed Riemannian manifolds (M, p, g) with $||(M, g)||_{C^{m,\alpha},r} < Q$. $\mathcal{M}^{m,\alpha}(n, Q, r)$ is compact in the pointed $C^{m,\beta}$ topology for all $\beta < \alpha$.*

Utilizing (1.1.2), it is then possible to show [24]

Theorem 1.1.3. *For every $Q > 0$ there exists $r > 0$ depending only on i_o and K such that any complete (M, g) with* $|\text{sec}M| \leq K$, $\text{inj}M \geq i_o$ *has $||(M, g)||_{C^0,r} \leq Q$. Furthermore, if (M_i, p_i, g_i) satisfy* $\text{inj}M_i \geq i_o$ *and* $|\text{sec}M_i \leq K_i \to 0$, *then a subsequence will converge in the pointed Gromov-Hausdorff topology to a flat manifold with* $\text{inj} \geq i_o$.

See [24] for generalizations of Theorem (1.1.3) as well as other compactness results which follow from Theorem (1.1.2).

The spaces in Theorem (1.1.3) are too restrictive for most of our discussion, but it turns out the less restrictive bounds $\text{Ric} \geq K$ lead to spaces which are not compact. Nonetheless we have the following [11]

Theorem 1.1.4. *(Gromov) Let $(\mathcal{M}, d_{GH})$ denote the space of isometry classes of metric spaces equipped with the Gromov-Hausdorff distance. Let $\mathcal{M}(n, K, D)$ denote the space of all n dimensional Riemannian manifolds with* $\text{Ric} \geq K$ *and* $\text{diam} \leq D$. *Then $\mathcal{M}(n, K, D)$ is precompact in $(\mathcal{M}, d_{GH})$.*

Theorems (1.1.3) and (1.1.4) provide fundamental compactness results necessary to begin a study of spaces of Riemannian manifolds defined by various curvature bounds. As we describe in various theorems below, Gromov-Hausdorff convergence paired with appropriate geometric constraints on the sequence of Riemannian manifolds induces a smooth structure on the limit space, or some subset thereof, and implies some form of stronger convergence of the metrics as well. This can also lead to other implications for the structure of limits of Riemannian manifolds belonging to certain classes, and this in turn leads to implications about the classes themselves.

We say (M, g) is non collapsed if $\text{Vol}(B(p, 1)) > \nu > 0$ for some ν and all balls $B(p, 1) \subset M$ of radius 1. We say a sequence of noncollapsed Riemannian manifolds (M_i, g_i) is uniformly non-collapsed if ν can be chosen independently of i.

Example 1.1.5. To observe the significance of this condition, consider the sequence of cylinders

$$C_i := (\epsilon_i \mathbb{S}^1) \times \mathbb{R}$$

where $\epsilon\mathbb{S}^1$ denotes the circle of circumference ϵ and ϵ_i is any sequence of positive numbers tending to zero. Then it is immediate that

$$C_i \underset{GH}{\to} \mathbb{R}$$

and we therefore have an example where $\dim(C_i) = 2$ for all i but their limit is a one dimensional manifold. Observe however that the sequence C_i is not uniformly non-collapsed. It turns out that the added assumption of a non-collapsed condition coupled with bounds on the curvature has a number of implications. $\square$

Ricci curvature as an "elliptic" operator

The coordinate expressions for Rm and its traces can be simplified by choosing coordinate functions that behave well with respect to the underlying metric. In particular, if locally defined coordinate functions x^i are harmonic with respect to g, i.e. for each i

$$\Delta x^i = 0,$$

where $\Delta = \nabla \cdot \nabla$ is the Laplace-Beltrami operator, then with respect to these coordinate we have

$$\text{Ric}_{ij} = -\frac{1}{2}\Delta(g_{ij}) + \text{lower order terms}$$

and, while the general expression for Ric is more complicated and the operator $g \to \text{Ric}_g$ taking a metric to its Ricci curvature is not elliptic and therefore general regularity results do not apply, it is often thought of as a multiple of the Laplacian and this thinking motivates a number results. To that end, we might consider bounds either of the form

$$\text{Ric} \geq K \text{ or } \text{Ric} \leq K$$

with the hope that we are led to some kind of control on g, and this is broadly speaking what we investigate below.

There are a plethora of results that demonstrate how the Riemann tensor and these traces determine the shape of a Riemannian manifold, and therefore how apriori restrictions on their behavior lead to restrictions on various geometric quantities. To motivate the kind of results we discuss here,

To motivate the restriction of our attention to either Ric or R, we observe that there are geometric interpretations for these traces that lend intuition to how they control the geometry of a manifold, and we describe some of these now.

Over any set $U \subset \mathbb{R}^n$ where the exponential map (see Chapter 5 of [24]) is a diffeomorphism, the volume form can be expressed as

$$\mathrm{dvol} = v dr \wedge d\theta$$

for some function $v : U \to \mathbb{R}$, and where $r > 0$, $\theta \in S^{n-1}$. Letting $f(x) := d(x,p)$, in these coordinates, v and f satisfy

$$\frac{\partial v}{\partial r} = \Delta f \cdot v \tag{1.1.1}$$

$$\frac{\partial}{\partial r}\Delta f + \frac{(\Delta f)^2}{n-1} \leq \frac{\partial}{\partial r}\Delta f + (\nabla^2 f)^2 = -\mathrm{Ric}(\nabla f, \nabla f) \tag{1.1.2}$$

A lower bound on the Ricci curvature $\mathrm{Ric} \geq K$ therefore gives an upper bound on Δf, as we can solve

$$\frac{\partial}{\partial r} m + \frac{m^2}{n-1} = -K \tag{1.1.3}$$

directly to show that

$$\Delta f(r,\theta) \leq m_K(r) := (n-1)\frac{\mathrm{sn}_K'(r)}{\mathrm{sn}_K(r)}, \tag{1.1.4}$$

where $\mathrm{sn}_K(r) = \sin(\sqrt{K}r)$ for $K > 0$, $\mathrm{sn}_0(r) = r$ and $\mathrm{sn}_K(r) = \sinh(\sqrt{K}r)$ for $K < 0$. Observe that $m_0(r) = \frac{n-1}{r}$ and as such, is asymptotic to ∞ at the origin and to 0 at ∞. Consider below the graphs of m_{-1} and m_1, which demonstrate fundamentally different behavior for the laplacian in negatively and positively curved manifolds respectively.

We remark that in this setting, Δf is the mean curvature m of the level sets of f. Since we have $\lim\limits_{r\to 0}(m - m_K) = 0$, it follows that $m(r) = m_K(r) \to m(r') = m_K(r')$ for all $r' < r$. We collect this into the following theorem [24]

Theorem 1.1.6. *(mean curvature comparison) If* $\mathrm{Ric}M \geq K$*, then along any minimal geodesic segment from* p*,*

$$m(r) \leq m_K(r).$$

Moreover, equality holds if and only if all radial sectional curvatures are equal to $\frac{1}{n-1}K$*.*

More generally, we have the following theorem [24],

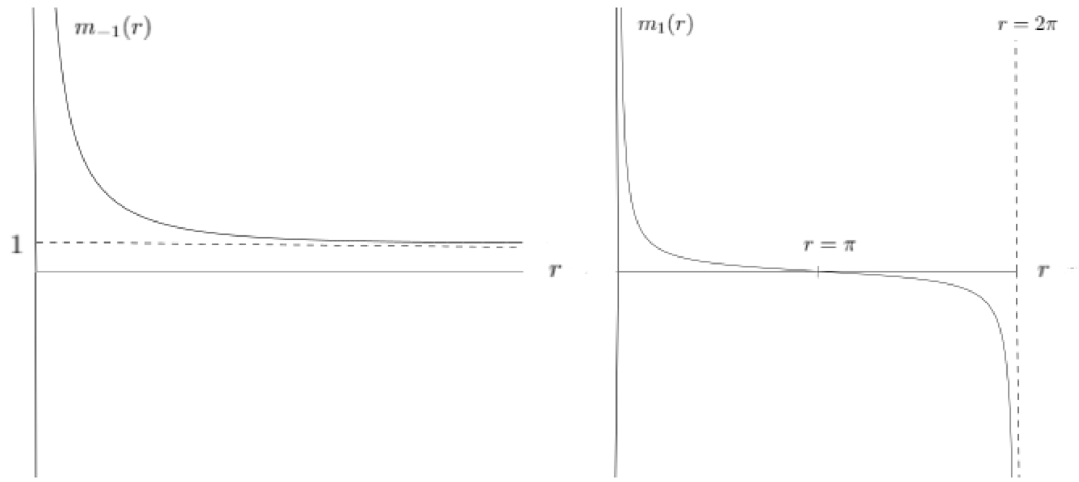

Figure 1.1.1: Unlike in the case of $K = 0$, we can see from the graph of m_{-1} that, for $K < 0$, $\lim_{r\to\infty} m_K > 0$. Furthermore, from the graph of m_1 we can see that, for $K > 0$, m_K approaches a singularity at finite distance. This second fact corresponds to the phenomenon that lines of longitude emanating from the north pole on a sphere converge on a single point at the south pole, and demonstrates the kind of collapsing that positive curvature can induce.

Theorem 1.1.7. *(Bochner) For a smooth function f on a complete Riemannian manifold (M, g),*

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + g(\nabla f, \nabla(\Delta f)) + \mathrm{Ric}(\nabla f, \nabla f).$$

Observe that in the special case that f is a distance function and $\mathrm{Ric} \geq K$, because where f is smooth we have,

$$|\nabla f| = 1 \text{ and } \nabla f = \partial_r,$$

where ∂_r is coordinate direction for r in exponential coordinates, and therefore at such places (1.1.7) reduces to (1.1.3).

Example 1.1.8. Consider the manifold $M := \mathbb{S}^2 \times \mathbb{R}$ equipped with the product metric

$$g = g_{rnd} + dr^2.$$

Let r denote the projection onto $\mathbb{R}$. On the set $\{p \in M \mid r(p) \geq 0\}$ the function r thus represents the distance from the sphere $r^{-1}(0)$ and thus satisfies, by (1.1.3),

$$\frac{\partial}{\partial r}\Delta r \leq -\frac{|\Delta r|^2}{2} - \mathrm{Ric}(\nabla r, \nabla r).$$

Now, this also follows from the fact that $\Delta r \equiv 0$, but let us observe what happens if we perturb g to

$$g' := g + a \tag{1.1.5}$$

such that a has support compactly contained within the set $\{p \in M \mid r(p) > 0\}$ and the resulting metric g' is smooth over all of M with $\mathrm{Ric}_{g'} \geq 0$. Assume for simplicity that the resulting distance function $r'(\,\cdot\,) := d_{g'}(r^{-1}(0),\,\cdot\,)$, where $r^{-1}(s)$ denotes the preimage of s with respect to r, remains smooth and that there exists a point p with $r(p) > 0$ such that

$$\mathrm{Ric}_{g'}(\nabla r', \nabla r')(p) > 0.$$

In this case we have

$$\frac{\partial}{\partial r'}\Delta r'(p) \leq -\frac{|\Delta r'(p)|^2}{2} - \mathrm{Ric}_{g'}(\nabla r', \nabla r')(p) < 0.$$

Now, for any s there exists $q(s)$ such that $d(p, q(s)) = d(p, (r')^{-1}(s))$. If γ_s is a minimizing geodesic with $\gamma_s(0) = q(s)$ and $\gamma_s(r'(p) - s) = p$, then the concatenation, for $s > r'(p)$, $\gamma_0 \cdot \gamma_s^{-1}$ is a minimizing geodesic connecting $q(0)$ and $q(s)$. Taking s to ∞, these curves converge to a geodesic ray γ with $\gamma(0) = q(0)$, $\gamma(r'(p)) = p$ and $\gamma' = \partial_{r'}$, and because r and r' agree in a neighborhood of $r^{-1}(0)$, we have $\Delta r'(q(0)) = 0$. Furthermore, by (1.1.4), $\Delta r' \leq 0$ everywhere on γ. Now, moving along γ, we see that

$$\frac{\partial}{\partial r}\Delta r(\gamma(r'(p))) \leq -\frac{|\Delta r(\gamma(r'(p)))|^2}{2} - \mathrm{Ric}_{g'}(\nabla r, \nabla r)(\gamma(r'(p))) < 0,$$

and thus, by comparing to solutions to (1.1.3) for $K < 0$, we see that $\Delta r'$ becomes singular at $\gamma(\bar{t})$ for some finite $\bar{t}$. By (1.1.1), $d\mathrm{vol}_{g'}$ becomes singular at this point as well and we reach a contradiction to the assumption that g' defines a smooth metric over all of M. This shows that metrics of the form (1.1.5) cannot satisfy $\mathrm{Ric} \geq 0$. See also Theorem (1.1.19) below, which takes this idea significantly further. $\square$

1.1.2 lower bounds

To begin, we recall some classical theorems, proven for instance in [24], which lend a basic intuition to the geometric influence of the Ricci curvature

Theorem 1.1.9. *(Myers) Suppose (M, g) is a complete Riemannian manifold with Ric $\geq (n-l)k > 0$. Then $\mathrm{diam}(M, g) \leq \frac{\pi}{\sqrt{k}}$. Furthermore, (M, g) has finite fundamental group.*

Observe in particular that the universal cover of any complete (M, g) satisfying $\mathrm{Ric}(M) \geq (n-l)k > 0$ must also satisfy this diameter bound, and in particular must be compact. It follows that the fundamental group of M is finite.

Cheng showed in the rigidity case for (1.1.9) that we recover the sphere

Theorem 1.1.10. *(Cheng) Suppose (M, g) is a complete Riemannian manifold with Ric $\geq (n-l)k > 0$ and $\mathrm{diam}(M, g) = \frac{\pi}{\sqrt{k}}$. Then (M, g) is isometric to $\mathbb{S}^n_k$.*

We remark that Theorem (1.1.9) is a consequence of the inequality (1.1.4). We also have the following theorem, which relies on (1.1.4) as well

Theorem 1.1.11. *(Bishop-Cheeger-Gromov) Suppose (M, g) is a complete Riemannian manifold with Ric $\geq (n-1)k$. Then*

$$r \to \frac{Vol(B(p, r))}{V^n_k(r)}$$

is a non-increasing function whose limit as $r \to 0$ is 1.

Remark 1.1.12. (1.1.11) is in fact a key element in the proof of Theorem (1.1.4). To illustrate this, we define the capacity of the metric space X at scale $r > 0$ as

$$\mathrm{Cap}_X(r) := |\{B(x, \frac{r}{2}) \mid B(x, \frac{r}{2}) \subset X \text{ are pairwise disjoint}\}|.$$

For any $M \in \mathcal{M}(n, K, D)$, and $x_1, ..., x_N \in M$ such that $B(x_i, \frac{r}{2})$ are pairwise disjoint. We then have

$$\mathrm{Vol}(M) \geq \sum_i \mathrm{Vol}(B(x_i, \frac{r}{2})),$$

and, because for each x_i we have, by Theorem (1.1.11),

$$\frac{\mathrm{Vol}(B(p, \frac{r}{2}))}{V_k^n(\frac{r}{2})} \geq \frac{\mathrm{Vol}(B(p, D))}{V_k^n(D)} = \frac{\mathrm{Vol}(M)}{V_k^n(r)}$$

it follows that

$$\mathrm{Vol}(M) \geq N \frac{V_K^n(\frac{r}{2})}{V_K^n(D)} \mathrm{Vol}(M)$$

and thus that

Lemma 1.1.13. *There exists $N(n, K, D, r) > 0$ such that*

$$\mathrm{Cap}_M(r) \leq N(n, K, D, r)$$

for any $M \in \mathcal{M}(n, K, D)$.

Utilizing this upper bound it is possible to argue by a diagonalization argument that any sequence $M_i \in \mathcal{M}(n, K, D)$ has a convergent subsequence, and thus Theorem (1.1.4) follows. $\square$

The preceding theorems illustrate how negativity in Ricci curvature allows a manifold locally to have more volume relative to a given radius and, conversely, that positive Ricci curvature forces a certain amount of boundedness of volume locally. This leads to a general insight that certain pathological behaviors can be avoided if one assumes a lower bound on the Ricci curvature. Consider the following theorems that illustrate this [16]

Theorem 1.1.14. *(Lohkamp) Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 3$ and $K > 0$ a constant. Then there is a sequence of Riemannian metrics g_i on M such that $Ric_{g_i} \leq -K g_i$ for all i and*

$$(M, g_i) \to (M, g)$$

in the Gromov-Hausdorff sense.

Theorem (1.1.14) demonstrates in particular that a restriction like $\mathrm{Ric} \leq K$ leads to a set of Riemannan metrics on M that is dense in the set of Riemannian metrics on M, and therefore does not restrict the possible large scale geometry. Coonsider in contrast the following [9]

Theorem 1.1.15. *(Colding) For $r > 0$, consider all metric balls of radius r in all complete n-dimensional Riemannian manifolds with $Ric \geq -(n-1)$. Equip this space with the Gromov-Hausdorff topology. Then the volume function is continuous.*

Discussion of theorem (1.1.15)

To illustrate how Theorem (1.1.15) shows that $\mathrm{Ric} \geq -(n-1)$ does in fact restrict the local volume, we describe here a well-known counter example in the case that the curvature assumption is dropped and $n = 2$, i.e. we describe a sequence of pointed Riemannian manifolds (M_i, g_i, p_i) such that $B_{M_i}(p_i, 1)$, the ball of radius 1 at p_i, Gromov-Hausdorff converges to $B(0,1)$, the Euclidian 2-ball, but such that the volumes of these balls approach 0:

Example 1.1.16. For $\epsilon > 0$, consider the set in $\mathbb{R}^3$ of points $(x, y, z) = (n\epsilon, m\epsilon, 0)$, with $n, m \in \mathbb{Z}$, and consider the subset of such points contained in the unit ball in $\mathbb{R}^3$. connect each pair of these points by the unique line segment they determine. The union of such line segments is a lattice L_ϵ. "Fatten" L_ϵ by replacing it with

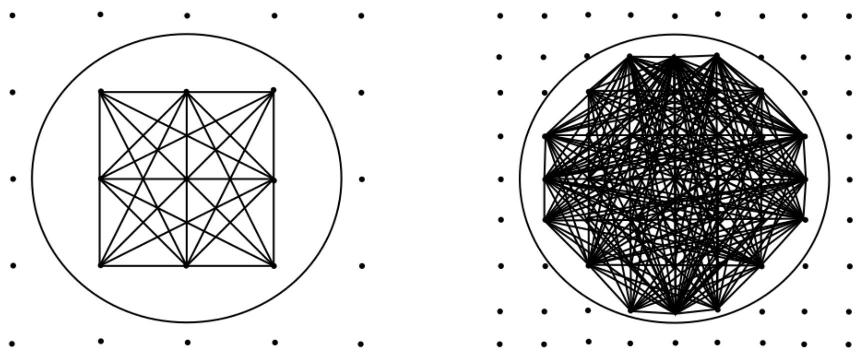

Figure 1.1.2: Pictured are L_ϵ for $\epsilon = \frac{8}{15}, \frac{4}{15}$. As ϵ is chosen smaller, L_ϵ approaches, in the Gromov-Hausdorff sense, the unit disc in the xy-plane. Any suitably chosen tubular neighborhood in $\mathbb{R}^3$ of L_ϵ will also approach the unit disc.

the tubular neighborhood of points $\{x \in \mathbb{R}^3 | \ d(x, L_\epsilon) = \delta\}$ for some $\delta > 0$, $\delta << \epsilon$. For small δ, we can smooth out this set to create a Riemannian manifold T_ϵ^δ, with induced metric from $\mathbb{R}^3$. The volume T_ϵ^δ at each step can be made arbitrarily small by choosing δ appropriately. Note, however, that the L^∞-bound on the curvature of T_ϵ^δ approaches ∞ as $\epsilon \to 0$ and in fact, so does the genus. It is clear, however, that these manifolds Gromov-Hausdorff converge to the intersection of $B(0,1)$ with the plane $z = 0$.

While the surfaces constructed above do contain neighborhoods where the curvature is very large, intuitively, they seem to have small area because they contain a large number of holes, i.e. the genus grows without bound as the manifolds converge in the Gromov-Hausdorff sense. By [colding], this is only possible on account of the presence of arbitrarily negative curvature in the sequence above, and so it serves to illustrate more concretely how allowing for arbitrarily negative curvature allows for more freedom than we see in the case of a bound like $\mathrm{Ric} \geq k \in \mathbb{R}$.

We discuss here the proof of Theorem (1.1.15) that appears in [9], and we mention that our result Theorem (1.1.18) below and its proof are similar.

Firstly, consider the n vectors $\{re_1, ..., re_n\} \subset \mathbb{R}^n$, where e_i form an orthonormal in $\mathbb{R}^n$ with respect to the Euclidian metric. Let

$$b_i^r(\cdot) := r - d(re_i, \cdot)$$

and consider the map $\Phi^r(x) = (b_1^r(x), ..., b_n^r(x))$. If we fix a bounded neighborhood $U \subset \mathbb{R}^n$ of the origin, then for large r, $\Phi^r : U \to \mathbb{R}^n$ is a near isometry. This follows from the fact that on U

$$\lim_{r\to\infty} b_i^r = x_i,$$

where x_i are the standard coordinates with respect to the basis vectors e_i. Note also that similarly, if we restrict attention for instance to $\Phi := \Phi^1$ and restrict our attention to neighborhoods $U^\epsilon := B(0, \epsilon)$, then, for small ϵ, $\Phi : U^\epsilon \to \mathbb{R}^n$ is a near isometry. This approach provides a way of mapping a neighborhood U of a general

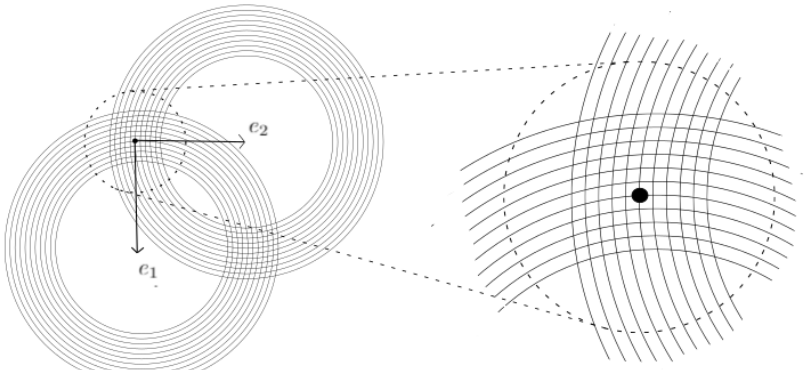

Figure 1.1.3: Concentric circles about e_1, e_2 in $\mathbb{R}^2$ determine coordinates in a small neighborhood of the origin. At the scale of smaller neighborhoods, these coordinates are closer to standard Euclidian coordinates. This technique generalizes to give local coordinates on an arbitrary Riemannian manifold, but only at sufficiently small scales, depending on the metric.

Riemannian manifold M to $\mathbb{R}^n$ provided functions b_i on M can be found that behave sufficiently like the coordinate functions x_i, which in particular sastisfy the following expression:

$$x_i(\gamma_v(r)) = \langle \nabla x_i, v\rangle \cdot r$$

which holds for all $v \in \mathbb{R}^n$ and $r > 0$. This equation states that we can predict the coordinates of the end point of a curve given the values v and r. If, in general, we were to try to construct a map in this way, we might start with functions

$$b^+(\cdot) = d(q, \cdot) - d(q, p)$$

and ask when the quantity

$$|b^+(\gamma_v(r)) - b^+(\gamma_v(0)) - \langle v, \nabla b^+\rangle r| \tag{1.1.6}$$

is small. We in fact have the following, from [9]

Theorem 1.1.17.

$$\frac{1}{Vol(SB(p,R))}\int_{SB(p,R)} |b^+(\gamma_v(r)) - b^+(\gamma_v(0)) - \langle v, \nabla b^+\rangle r| < \epsilon R$$

and

$$\frac{1}{Vol(SB(p,R))}\int_{SB(p,R)} |(b^+ \circ \gamma_v)'(r) - \langle v, \nabla b^+\rangle| < \epsilon$$

Here SM denotes the unit tangent bundle of a manifold M (See the preliminaries of Chapter 2 below for a precise definition).

Theorem (1.1.17) states that (1.1.6) is small in a certain integral sense when we have a lower Ricci curvature bound. This in turn provides a measure of how well we can predict the local behavior of distance functions on (M, g) given a function's linearization at a point, and it is by the existence of such near isometries that Theorem (1.1.15) is proved.

Returning to the counter example described above, one can imagine how Theorem (1.1.17) implies Theorem (1.1.15). The fact that on average we can estimate, in comparison to Euclidian geometry, where the end point of a geodesic parametrized over an interval lies heuristically tells us that, if there were something like the holes in the example above, we could reach a contradiction by aiming toward these holes and finding points within them. This motivates the question of whether a restriction on the topology of M in place of a curvature condition also results in (at least partial) continuity of the volume function. We have the following

Theorem 1.1.18. *For $r > 0$, consider all metric balls of radius r in all complete 2-dimensional Riemannian manifolds with Euler characteristic uniformly bounded. Equip this space with the Gromov-Hausdorff topology. Then the volume function is lower semi-continuous. i.e. for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $B(p, r) \subset M$ and $B(p_i, r) \subset M_i$ are as described and satisfy*

$$\lim_{i\to\infty} d_{GH}(B(p_i, r), B(p, r)) = 0$$

then

$$\lim_{i\to\infty} \mathrm{Vol}(B(p_i, r)) \geq \mathrm{Vol}(B(p, r))$$

Note, if the surfaces in question are closed, then the Euler characteristic can be bounded in terms of the genus, and the statement of Theorem (1.1.18) can be simplified to reflect that. We provide a proof of Theorem (1.1.18) in chapter 3.

Now, recall Example (1.1.8) above. We saw there that certain perturbations of the standard metric on $\mathbb{S}^2 \times \mathbb{R}$ cannot satisfy $\mathrm{Ric} \geq 0$ without forcing some kind of singularity. It turns out that this is an example of a more general phenomenon described by the following theorem, where we define a line to be a curve $\gamma : (-\infty, \infty) \to M$ such that its restriction to any finite sub-interval of its domain is a minimizing geodesic [6]

Theorem 1.1.19. *(Cheeger Gromoll) Let M be a complete manifold of nonnegative Ricci curvature. Then M is the isometric product $\bar{M} \times \mathbb{R}^k$ where $\bar{M}$ has no lines and $\mathbb{R}^k$ has its standard flat metric.*

Discussion of theorem (1.1.19)

The idea of Example (1.1.8) is that, by (1.1.4), the presence of positive Ricci curvature curbs the spread of a family of geodesics, and ultimately can cause them to begin to converge on one another. Along with the condition $\mathrm{Ric} \geq 0$, we then see that Theorem (1.1.19) illustrates the incompatibility of the presence of geodesic lines γ in spaces satisfying $\mathrm{Ric} \geq 0$ unless they result from the trivial case of a product with $\mathbb{R}$, which in particular implies

$$\mathrm{Ric}(\gamma', \gamma') = 0$$

everywhere, and thus that such curves do not experience any of the influence of positive Ricci curvature.

Example 1.1.20. similarly to Example (1.1.8), consider the scenario where we know that M splits isometrically as

$$M \cong \bar{M} \times \mathbb{R}$$

and observe that local coordinates $\{x^1, ..., x^{n-1}\}$ on $U \subset \bar{M}$ extend in an obvious way to coordinates on $U \times \mathbb{R} \subset M$ by the introduction of $x^n = \pi_{\mathbb{R}}$, where $\pi_{\mathbb{R}}$ is projection onto $\mathbb{R}$, and that the function x^n satisfies, for any $p, q \in M$,

$$d(S(p), S(q)) = |x^n(p) - x^n(q)|$$

where the sets $S(p) := \{y \in M \mid x^n(y) = x^n(p)\}$ are the level sets of x^n. Furthermore, we have that ∇x^n is parallel, i.e. $\nabla_{(\ \cdot\ )} \nabla x^n \equiv 0$, and in particular $\Delta x^n \equiv 0$. While this all follows immediately from the decomposition of M, we illustrate it because the proof of Theorem (1.1.19) relies on the introduction of a superharmonic function, which we denote $b : M \to \mathbb{R}$, that is in fact a Riemannian projection onto $\mathbb{R}$ exactly like x^n, and which arises as the limit of distance functions like f above. $\square$

To see the role Ricci Curvature plays in this result, let us illustrate the proof of Theorem (1.1.19) found in [6]. Recall again that $\mathrm{Ric} \geq 0$ implies that the distance function f from a point $p \in M$ satisfies

$$\frac{\partial}{\partial r} \Delta f + \frac{(\Delta f)^2}{n-1} \leq 0,$$

so that

$$\Delta f \leq (n-1) \frac{\mathrm{sn}_0'}{\mathrm{sn}_0} = (n-1) \frac{1}{r}, \tag{1.1.7}$$

which in particular tends to zero for large r. This indicates that for (M, g) with nonnegative Ricci curvature and $p \in M$ that for any $\epsilon > 0$ we can choose $D > 0$ such that $\Delta f < \epsilon$ on the (possibly empty) set $B(p, D)^c \subset M$. One way of interpreting this is that for large r the distance function f behaves almost superharmonically.

Now, assume that M contains a line. Similarly to the discussion for Theorem (1.1.15), define the functions

$$b_{\gamma,t}(q) = d(q, \gamma(t)) - t.$$

By the triangle inequality,

$$|b_{\gamma,t}(q_1) - b_{\gamma,t}(q_2)| = |d(q_1, \gamma(t)) - d(q_2, \gamma(t))| \leq d(q_1, q_2)$$

and so these functions are uniformly equicontinuous, and furthermore $t \to b_t(q)$ is decreasing and bounded below by $-d(\gamma(0), q)$ for any fixed q. Thus these functions converge to a continuous function

$$b_\gamma := \underset{t\to\infty}{b_{\gamma,t}}\,.$$

Remark 1.1.21. In the case of $\mathbb{R}^n$ with the Euclidian metric for instance, it is clear that the function thus defined is a linear function, and thus splits the space in an obvious way,

$$\mathbb{R}^n = \{x \mid b(x) = 0\} \times \{s\nabla b \mid s \in \mathbb{R}\} \cong \mathbb{R}^{n-1} \times \mathbb{R}. \qquad \square$$

Now consider the following theorem, also proven in [6]

Theorem 1.1.22. *If M has non-negative Ricci curvature, then the functions b_γ are superharmonic.*

Heuristically, Theorem (1.1.22) follows from our discussion regarding the implications of (1.1.7), but there are a number of technical details arising from the fact that distance functions cannot be assumed to be differentiable everywhere on M (see [6] for a proof).

Now, assuming Theorem (1.1.22), we proceed by defining

$$b_+ := b_\gamma \text{ and } b_- := b_{-\gamma}$$

and observing that, again by the triangle inequality and recalling that γ is a line,

$$d(q, \gamma(t)) - t + d(\gamma(-s), q) - s \geq 0,$$

and thus,

$$b := b_+ + b_- \geq 0,$$

with equality holding for all points on γ. We have furthermore that b is superharmonic, and thus for any neighborhood intersecting γ we have $b \equiv 0$. By the strong maximum principle, this implies $b \equiv 0$ over M and thus

$$b_+ = -b_-.$$

Thus b_+ is subharmonic as well, and in particular it is harmonic. This constitutes the contribution to the structure of M resulting from the condition $\mathrm{Ric} \geq 0$ that we hope to convey. With a little more work, it also follows that

$$||\nabla b|| = 1$$

and in particular, $\nabla_{\nabla b_+} \nabla b_+ \equiv 0$, and subsequently that

$$\nabla(\nabla b_+) \equiv 0.$$

The properties of the function b_+ then imply, by the De Rham decomposition theorem [23], that, locally, M decomposes isometrically as the product of an $n-1$ dimensional manifold and an interval in $\mathbb{R}$. This splitting is in fact global, given by the level sets of b_+, and the proof is complete.

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1.1.3 two sided bounds

For some stronger upper bound conditions on curvature, $\sec < 0$ for instance, there is much that can be said, but we remark here that the condition $\mathrm{Ric} < K$ offers much less. There are, however, a number of results that follow from the combination of upper and lower bounds.

Sphere Theorems

We briefly mention the well-known sphere theorems, the first proven by Berger [2] and Klingenberg [15]

Theorem 1.1.23. *(Berger, Klingenberg) If M is a complete, simply-connected, n-dimensional Riemannian manifold with sectional curvature taking values in the interval $(1, 4]$ then M is homeomorphic to the n-sphere.*

and its differentiable analogue, proven by Chen [7]

Theorem 1.1.24. *(Chen) If M is a complete, simply-connected, n-dimensional Riemannian manifold with sectional curvature taking values in the interval $(1, 4]$ then M is diffeomorphic to the n-sphere.*

We also remark in passing that the upper bound in Theorem (1.1.23) can be replaced by a lower diameter bound:

Theorem 1.1.25. *Let M be a compact Riemannian manifold with sectional curvature greater than 1. If the diameter of M is greater than $\frac{\pi}{2}$, then M is homeomorphic to $\mathbb{S}^n$.*

Integral Bounds

It has already been made clear in the introduction that, without restrictions, a limit of a sequence of manifolds of dimension n may have dimension $d < n$. On the other hand, consider the following 2 dimensional example of a sequence of non-collapsed positively curved manifolds, where the limit space is a 2 dimensional *orbifold*, i.e. a topological space that is locally a finite group quotient of two dimensional Euclidian space:

Example 1.1.26. Let

$$g_{c,k} = dr^2 + r\sqrt{\frac{1 + (ckr)^2}{1 + (kr)^2}}d\theta^2.$$

For all k, $g_{c,k}$ is a smooth metric on $\mathbb{R}^2$, and we have the limit metric

$$g_c := \lim_{k\to\infty} g_{c,k} = dr^2 + crd\theta^2$$

which defines a smooth metric over all of $\mathbb{R}^2$ when $c = 1$. Taking instead $c = \frac{1}{2}$ for instance, $g_{\frac{1}{2}}$ is the quotient metric of the Euclidian metric on $\mathbb{R}^2$ by the action of

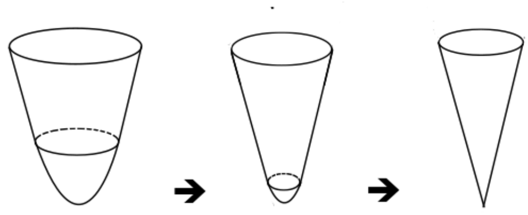


Figure 1.1.4: As k increases, the curvature at the apex $r = 0$ grows without bound, producing a singularity in the limit.

rotation by 180^o, i.e. $v = -v$ for all $v \in \mathbb{R}^2$. The pair $(\mathbb{R}^2, g_{\frac{1}{2}})$ is then an orbifold, and $g_{\frac{1}{2}}$ is not smooth at the origin $\vec{0}$.

In general, we have

$$(\mathbb{R}^2, g_c) = C(S^1_{2\pi c}),$$

where S^1_β denotes the circle of radius β and $C(Z)$ denotes the metric cone over a metric space Z. Recall that $C(S^1_{2\pi c})$ is only smooth at its vertex when $c = 1$. $\square$

Thus, while the dimension is not reduced in these examples, we see that limits are not Riemannian manifolds in general. Geometrically, it is clear that these examples form a singularity at the cone vertex because the curvature of the metrics $g_{c,k}$ becomes arbitrarily large at this point. To qualify this further, and following the material presented in [5], for a metric space (X, d) we define a *tangent cone* of X at x as any limit point, for $t \to \infty$, of (X, x, td) in the pointed Gromov-Hausdorff sense, where td denotes the rescaled distance by the parameter t. This generalizes the notion of a tangent plane in the sense that every tangent cone to a point in an n dimensional Riemannian manifold is isometric to $\mathbb{R}^n$. Observe also that $C(S^1_\beta)$ occurs as a tangent cone at its own vertex.

It turns out that if (X, d) occurs as the Gromov-Hausdorff limit of (M^n_k, g_k) with $\mathrm{Ric}_{g_k} \geq K$, then, if a tangent cone at $x \in X$ is isometric to $\mathbb{R}^n$, every tangent cone at x is isometric to $\mathbb{R}^n$. Thus, we call a point x *regular* if $\mathbb{R}^n$ occurs as a tangent cone at x, and *singular* otherwise. This creates a dichotomy between points near which the metric is smooth and those at which it isn't. Thus, every point in a Riemannian manifold is regular.

Now, for a limit (X, d), let $\mathcal{S} \subset X$ denote the set of singular points. We have the following, from [4]

Theorem 1.1.27. *(Cheeger, Naber) Let* $(M^n, g_i, p_i) \underset{GH}{\to} (X, d, p)$ *be a Gromov-Hausdorff limit of manifolds with* $||Ric_{g_i}|| \leq n - 1$ *and* $Vol(B(p_i, 1)) > v > 0$. *Then the singular set* $\mathcal{S}$ *satisfies*

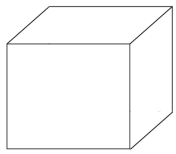
$$dim \mathcal{S} \leq n - 4.$$

The dimension can be taken to be the Hausdorff or Minkowski dimension.

Discussion of theorem (1.1.27)

Regarding the structure of the set $\mathcal{S}$, let the stratum $\mathcal{S}^k \subset \mathcal{S}$ be defined as the set of points for which no tangent cone splits off isometrically a factor $\mathbb{R}^{k+1}$. Intuitively, this decomposes $\mathcal{S}$ into isolated vertices, edges, etc. similarly to a simplicial complex.

Example 1.1.28. In the example C^2 of the boundary of the unit cube $\mathbb{I}^3$ embedded in $\mathbb{R}^3$,



the set of 8 corners comprise $\mathcal{S}^0$. Their union is $\mathcal{S}$, which has $\text{codim}(\mathcal{S}) = 2$. Observe how a neighborhood of $p \in C^2 \backslash \mathcal{S}^0$ is isometric to a neighborhood of $q = (x, z^*) \in \mathbb{R}^1 \times C(\mathbb{S}^0)$, where z^* denotes the vertex in $C(\mathbb{S}^0)$, and no neighborhood of $p \in \mathcal{S}^0$ can be decomposed in a similar way. In the analagous example of the boundary C^4 of $\mathbb{I}^5 \subset \mathbb{R}^5$, we have $\mathbb{I}^5 = \mathbb{I}^3 \times \mathbb{I}^2$, so

$$C^2 \times \mathbb{I}^2 \subset C^4$$

and thus $\text{codim}(\mathcal{S}) \leq 2$. It then follows that the C^4 does not arise as the limit of a sequence of surfaces with Ricci curvature bounded from below.

$\square$

it is shown in [5] that, in the non-collapsed setting and with a lower Ricci curvature bound, not only is $\dim \mathcal{S} \leq n-2$, but the set $\mathcal{S}$ can be stratified in the following way:

$$\mathcal{S}^0 \subset \mathcal{S}^1 ... \subset \mathcal{S}^{n-2} = \mathcal{S} \tag{1.1.8}$$

$$\dim(\mathcal{S}^k) = k \text{ in the sense of Hausdorff dimension} \tag{1.1.9}$$

These statements show that $\mathcal{S}$ can be controlled so that it does in fact behave in a geometric way similarly to (1.1.28). Now, (1.1.26) suggests a relationship between unbounded curvature and the set $\mathcal{S}$. Specifically, the singular region in the limit arises through the presence of unbounded curvature in the sequence. It follows that if $\mathcal{S}$ is suitably sparse in X, then one may hope that this in turn implies a bound on some form of curvature in an integral sense for the sequence of manifolds.

Pursuing this, we review the ideas of quantitative stratification, introduced in [5], which provide local control over the measure of tubular neighborhoods of $\mathcal{S}$. Firstly, denote by $(\underline{0}, z^*)$, a vertex of the metric cone with isometric splitting $\mathbb{R}^{k+1} \times C(Z)$. Define the k-th effective singular stratum $\mathcal{S}^k_{\eta,r} \subset X$ by

$$\mathcal{S}^k_{\eta,r} := \{x \mid d_{GH}(B(x,s), B((\underline{0}, z^*), s)) > \eta s \text{ for all } \mathbb{R}^{k+1} \times C(Z) \text{ and } r \leq s \leq 1\}$$

Thus, $\mathcal{S}^k_{\eta,r}$ is those $x \in X$ locally, down to the scale of r, are not η close to splitting off a factor of $\mathbb{R}^{k+1}$. We have the following [5]

Theorem 1.1.29. *There exists $c(n,\nu,\eta) > 0$ such that if $M_i^n \underset{GH}{\to} X$, and the (M_i^n, g_i) satisfy the lower Ricci curvature bound $Ric \geq -(n-1)$, and ν-noncollapsing condition $\frac{Vol(B(p,1))}{V_{-1}} > \nu > 0$, then for all $x \in X$ and $\eta > 0$,*

$$Vol(T_r(\mathcal{S}^k_{\eta,r}) \cap B(x, \frac{1}{2})) < c(n,\nu,\eta) r^{n-k-\eta}$$

The combination of Theorems (1.1.27) and (1.1.29) are used in [4] to prove

Theorem 1.1.30. *There exists $C = C(n,v,q)$ such that if M^n satisfies $||\mathrm{Ric}|| \leq n-1$ and $Vol(B(p,1)) > v > 0$, then for each $q < 2$,*

$$\int_{B(p,1)} ||Rm||^q d\mathrm{vol} \leq C$$

Example 1.1.31. Similarly to Example (1.1.1), let $(\mathbb{S}^2, g_{\mathrm{rnd}})$ and $(\mathbb{H}^2, g_{\mathrm{hyp}})$ denote the two dimensional round sphere and hyperbolic space respectively. Using (1.1.1) and (1.1.2) it can be calculated that the area of a disc of radius r in two dimensional hyperbolic space with sectional curvature K is given by

$$V_K(r) = 2\pi \int_0^r \frac{\sinh \sqrt{K}t}{\sqrt{K}} dt$$

which in particular grows exponentially as $K \to \infty$. On the other hand, we have

$$\mathrm{Vol}_{\sqrt{K}g_{rnd}}(\mathbb{S}^2) = \frac{4\pi}{K}.$$

Combining these, the volume of a ball of a given radius in (M, g_K), where $M = \mathbb{S}^2 \times \mathbb{H}^2$ and $g_K = \sqrt{K}(g_{rnd} + g_{hyp})$, can be made arbitrarily large by choosing K sufficiently large. As in (1.1.1), we have $\mathrm{R}_{g_K} = 0$, but $||\mathrm{Ric}_{g_K}|| = \sqrt{K^2 + K^2 + K^2 + K^2} = 2K$ and thus, for $x \in M$, the quantity

$$\int_{B(x,1)} ||\mathrm{Ric}||^p d\mathrm{vol}_{g_K}$$

can be made arbitrarily large, for any $p \geq 1$. $\square$

Example (1.1.31) is significant in that it shows that pointwise or integral control over R in the L_p sense for $1 \leq p \leq \infty$ are insufficient to bound $||\mathrm{Ric}||_{L_p}$ or $||\mathrm{Rm}||_{L_p}$ in the limit, in contrast to Theorem (1.1.27).

Regarding integral bounds in general, we remark here that the following is proven in [21]

Theorem 1.1.32 (Petrunin)**.** *Let (M^n, g) be a Riemannian manifold satisfying $sec \geq k$ for some $k \in \mathbb{R}$ and $p \in M$. There exists $C = C(n,k)$ such that*

$$\int_{B(p,1)} |R| dvol < C.$$

Lastly, we have the following

Theorem 1.1.33. *Let (M^n, g) be a complete Riemannian manifold and $p_0 \in M$. Assume the Ricci curvature satisfies $Ric \geq -1$ on the ball $B(p_0, 5)$. Then for any $0 < q < 1/2$ there exists $C(q, n)$ such that*

$$||Ric||^q_{L^q(B(p_0,1))} \leq C(q, n)Vol(B(p_0, 1))^{(1-2q)}$$

Which we discuss and prove in chapter 2.

1.2 Santaló's Formula

We include in this section a proof of the well-known Santaló's Formula, as it plays a role in some of our work below, but its proof does not often appear. Before stating the theorem we review some background and notation.

The tangent bundle TM splits locally so that, for small neighborhoods $U \subset M$, we have $\pi^{-1}(U) \cong U \times \mathbb{R}^n$, where $\pi : TM \to M$ is the projection onto M. Thus, we freely use the notation $(p, v) \in TM$, where $p \in M$ and $v \in T_pM$. The connection gives a unique way to define a lift

$$d\pi^{-1}_{(p,v)} : T_pM \to T_{(p,v)}(TM)$$

by the property that, if $Y \in T_pM$ and we extend v to a section $v \in \Gamma(TM)$ such that $\nabla_Y v = 0$, then, recalling that a section v is a map $v : M \to TM$, we have

$$d\pi^{-1}_{(p,v)}(Y) = dv(Y).$$

We then say a vector field $Y \in \Gamma(T(TM))$ is horizontal if for any (p, v),

$$d\pi^{-1}_{(p,v)} \circ d\pi(Y(p, v)) = Y(p, v).$$

Now define the horizontal vector field $X \in T(TM)$ by the further property that $d\pi(X(p, v)) = v$ for all $(p, v) \in TM$. There exists a smoothly varying family of diffeomorphisms $\Phi_t : TM \to TM$ such that

$$\frac{d}{dt}\Phi_t(p, v) = X(\Phi_t(p, v))$$

and

$$\Phi_0(p, v) = (p, v).$$

Φ is called the geodesic flow, as the curves $t \to \pi \circ \Phi_t(p, v)$ are geodesics for any initial (p, v).

Now consider local coordinates $x^i : U \subset M \to \mathbb{R}$. These give rise pointwise to a basis $\{\partial x_1(p), ..., \partial x_n(p)\} \subset T_pM$. Using these vectors, x^i lift to coordinates $\tilde{x}^i, y^i$ on $TU \subset TM$ by the properties

$$\tilde{x}^i(p, v) = x^i(\pi(p, v))$$

and

$$v = y^i(p, v)\partial x_i.$$

We extend g to a metric, which we also denote by g, on TM so that in these coordinates

$$g(\partial\tilde{x}_i, \partial\tilde{x}_j) = g(\partial y_i, \partial y_j) = g(\partial x_i, \partial x_j)$$

and, at an arbitrary point (p, v),

$$g(\partial\tilde{x}_i, \partial y_j) = y^k(p, v)g(\nabla_{\partial x_i}\partial x_k, \partial x_j),$$

where the right-hand side is evaluated at $p \in M$.

We denote the unit tangent bundle on M by $SM \subset TM$, where $(p, v) \in SM \iff g(v, v) = 1$ and we remark that the metric g restricts to SM in an obvious way. We similarly define, for $s \geq 0$, the sets $sSM \subset TM$, where $(p, v) \in sSM \iff g(v, v) = s$. Because X is horizontal, it is easy to check that for any t, $(p, v) \in sSM \to \Phi_t(p, v) \in sSM$, and thus that Φ_t restricts to a diffeomorphism on sSM for any s as well. Furthermore, $TM = \bigcup_{s \geq 0} sSM$ and, by Fubini's theorem,

$$\int_{TM} f \; d\mathrm{vol}_{TM} = \int_0^\infty \int_{sSM} f \; d\mathrm{vol}_{sSM} ds$$

where f is any integrable function, $d\mathrm{vol}_{TM}$ denotes the induced volume form on TM and $d\mathrm{vol}_{sSM}$ denotes its restriction to sSM. We denote the pullback of $d\mathrm{vol}_{TM}$ through Φ_t by $\Phi_t^* d\mathrm{vol}_{TM}$.

recall finally that for any vector field F on a smooth manifold N and corresponding flow $\Phi_{F,t}$, for $Y \in \Gamma(TN)$, if we define the new vector fields Y_t by the relation $Y_t(\Phi_{F,t}(p)) = d\Phi_{F,t}(Y(p))$, then

$$\frac{d}{dt}\Big|_{t=0} (Y_t(\Phi_{F,t}(p)) - Y(\Phi_{F,t}(p))) = [F, Y], \tag{1.2.1}$$

where $[\cdot, \cdot]$ denotes the Lie bracket on vector fields. Furthermore, since

$$\frac{d}{dt}\Big|_{t=0} Y(\Phi_{F,t}(p)) = (\nabla_F Y)(p), \tag{1.2.2}$$

we have for $Y_1, ..., Y_n \in \Gamma(TN)$, with $Y_{i,t}$ defined as above and using (1) and (2), that

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} & d\mathrm{vol}_N(Y_{1,t}(\Phi_{F,t}(p)), ..., Y_{n,t}(\Phi_{F,t}(p))) \\
&= (F \cdot d\mathrm{vol}_N)(Y_1, ..., Y_n) + \\
&\qquad d\mathrm{vol}_N(\nabla_F Y_1, ..., Y_n) + ... + d\mathrm{vol}_N(Y_1, ..., \nabla_F Y_n) + \\
&\qquad\qquad d\mathrm{vol}_N([[F, Y_1], ..., Y_n) + ... + d\mathrm{vol}_N(Y_1, ..., [F, Y_n]) \qquad (1.2.3) \\
&= F \cdot (d\mathrm{vol}_N(Y_1, ..., Y_n)) + \\
&\qquad d\mathrm{vol}_N([[F, Y_1], ..., Y_n) + ... + d\mathrm{vol}_N(Y_1, ..., [F, Y_n]).
\end{aligned}$$

We can now state Santaló's formula, :

Theorem 1.2.1. *The induced volume form* $d\mathrm{vol}_{TM}$ *on* TM *(or* $d\mathrm{vol}_{SM}$ *on* SM*) is preserved with respect to* Φ_t*, i.e. for any* $t > 0$

$$d\mathrm{vol}_{TM} = \Phi_t^* d\mathrm{vol}_{TM}$$

or equivalently, for any $U \subset TM$ *(or* SM*) we have*

$$\mathrm{vol}(U) = \mathrm{vol}(\Phi_t(U)).$$

Proof. Firstly, observe that if we prove the theorem for TM, then it follows for SM as well, since, by Fubini's theorem,

$$\int_0^\infty \int_{U\cap sSM} d\mathrm{vol}_{sSM} ds = \int_{U\subset TM} d\mathrm{vol}_{TM} = \int_{\Phi_t(U)} d\mathrm{vol}_{TM} = \int_0^\infty \int_{\Phi_t(U)\cap sSM} d\mathrm{vol}_{sSM} ds,$$

where $U \subset TM$ is any measurable set. This implies that $\mathrm{vol}(U\cap SM) = \mathrm{vol}(\Phi_t(U)\cap SM)$ measured as subsets of SM and, since U is arbitrary, that the theorem holds.

Proceeding with the proof for the case of TM, observe that it is sufficient to show that

$$\frac{d}{dt}\Big|_{t=0} \Phi_t^* d\mathrm{vol}_{TM} = 0 \tag{1.2.4}$$

since, for any $0 < s < t$, $\Phi_t = \Phi_s \circ \Phi_{t-s}$ implies that

$$\frac{d}{dt}\Big|_{t=s} \Phi_t^* d\mathrm{vol}_{TM} = \Phi_s^* \left(\frac{d}{dt}\Big|_{t=0} \Phi_{s+t}^* d\mathrm{vol}_{TM} \right).$$

Thus, let $p \in M$ and a neighborhood $U \subset M$ with $p \in U$ be given. we shall show that

$$\frac{d}{dt}\Big|_{t=0} d\mathrm{vol}_{TM}(d\Phi_t(\partial\tilde{x}_1), ..., d\Phi_t(\partial\tilde{x}_n), d\Phi_t(\partial y_1), ..., d\Phi_t(\partial y_n)) = 0$$

holds for any element of $\pi^{-1}(p)$. Since p is arbitrary this is equivalent to (4). To do so, we lift coordinates x^i on U to coordinates $\tilde{x}^i, y^i$ on TU as above. Because X is horizontal and $d\pi(X(p,v)) = v$, we have the following:

$$X(p,v) = y^i \partial\tilde{x}_i - \frac{g(\partial\tilde{x}_i, \partial y_i)}{g(\partial y_i, \partial y_i)} \partial y_i.$$

Thus, if we have chosen our coordinates (see [24] for a description of exponential coordinates for instance) so that at p we have $g_{ij} = \delta_{ij}$ and $\Gamma^i_{jk} = 0$, then the above implies for any $v \in T_pM$ that

$$X(p,v) = y^i(p,v)\partial\tilde{x}_i$$

and furthermore by direct computation that

$$(\nabla_{\partial\tilde{x}_i} X)(p,v) = 0, \quad (\nabla_{\partial y^i} X)(p,v) = \partial\tilde{x}_i(p,v).$$

It then follows that

$$[X, \partial\tilde{x}_i](p,v) = 0, \quad [X, \partial y_i](p,v) = -\partial\tilde{x}_i(p,v)$$

and thus that, at p,

$$g\Big(\partial\tilde{x}_i, [X, \partial\tilde{x}_i]\Big) = g\Big(\partial y_i, [X, \partial y_i]\Big) = 0.$$

Combining all of this, using (3), and restricting to points $(p, v) \in \pi^{-1}(p)$, we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} &\mathrm{dvol}_{TM}(d\Phi_t(\partial\tilde{x}_1), ..., d\Phi_t(\partial\tilde{x}_n), d\Phi_t(\partial y_1), ..., d\Phi_t(\partial y_n)) \\
&= X \cdot (\mathrm{dvol}_{TM}(\partial\tilde{x}_1, ..., \partial\tilde{x}_n, \partial y_1, ..., \partial y_n)) \\
&\qquad + d\mathrm{vol}_{TM}([X, \partial\tilde{x}_1], ..., \partial\tilde{x}_n, \partial y_1, ..., \partial y_n)+ \\
&\qquad ... + d\mathrm{vol}_{TM}(\partial\tilde{x}_1, ..., [X, \partial\tilde{x}_n], \partial y_1, ..., \partial y_n) \\
&\qquad ... + d\mathrm{vol}_{TM}(\partial\tilde{x}_1, ..., \partial\tilde{x}_n, [X, \partial y_1], ..., \partial y_n) \\
&\qquad ... + d\mathrm{vol}_{TM}(\partial\tilde{x}_1, ..., \partial\tilde{x}_n, \partial y_1, ..., [X, \partial y_n]), \\
&= 0 + 1 \cdot \Big(g\Big(\partial\tilde{x}_1, [X, \partial\tilde{x}_1]\Big)+ \\
&\qquad ... + g\Big(\partial\tilde{x}_n, [X, \partial\tilde{x}_n]\Big)+ \\
&\qquad ... + g\Big(\partial y_1, [X, y_1]\Big)+ \\
&\qquad ... + g\Big(\partial y_n, [X, y_n]\Big)\Big) \\
&= 0.
\end{aligned}$$

Since coordinates as above can be chosen for any $(p, v) \in TM$, we are done. $\square$

Chapter 2

An Integral Curvature Bound

2.1 Introduction

The main result of this section is the following:

Theorem 2.1.1. *Let (M^n, g) be a complete Riemannian manifold and $p_0 \in M$. Assume the Ricci curvature satisfies $Ric \geq -1$ on the ball $B(p_0, 5)$. Then for any $0 < q < 1/2$ there exists $C(q, n)$ such that*

$$||Ric||^q_{L^q(B(p_0,1))} \leq C(q,n)Vol(B(p_0,1))^{(1-2q)}$$

Where we define the above norm

$$||f||_{L^q(B(p_0,1))} := \left(\int_{B(p_0,1)} |f|^q dvol\right)^{\frac{1}{q}}$$

Note that the bound does not deteriorate as volume collapses. We are unable to immediately push the result to the case $q = 1/2$ for the obvious reason that the constant $C(q, n)$ in Theorem (2.1.1) blows up too quickly as $q \to 1/2$. In the course of the proof of Lemma (2.1.2) below, we show that $C(q, n)$ behaves roughly like $(\frac{1}{2} - q)^{-1}$ as $q \to \frac{1}{2}$. It is possible that with other techniques the result extends beyond $q = 1/2$. Our proof of Lemma (2.1.2) relies on Hölder's inequality in a way that requires $q < 1/2$, and it is for this reason that we can't directly extend the result. We also note that we have no examples to demonstrate whether these bounds are optimal.

To get an idea of what is happening we sketch a few examples. Firstly we consider $(\mathbb{S}^n, \lambda^2 g_{\mathbb{S}^n})$, the n-dimensional sphere with round metric $g_{\mathbb{S}^n}$ scaled to have diameter λ. Of course Theorem (2.1.1) is most interesting for small λ, where the curvature is large. Because volume scales like λ^n and ricci curvature scales like λ^{-2}, for any center point p_0 the term on the left in Theorem (2.1.1) is roughly λ^{n-2q} for small λ and the term on the right is roughly $\lambda^{n(1-2q)}$, and we just check that

$$\lambda^{n-2q} \leq \lambda^{n-2q}\lambda^{-2(n-1)q} = \lambda^{n(1-2q)}$$

for small λ. Regarding the question of optimality, note that the term on the left approaches 0 even for $q = \frac{1}{2}$ but that the term on the right is simply equal to 1.

Next we look at non-collapsed, rotationally symmetric metrics $g_a = dr^2 + \phi_a(r) g_{\mathbb{S}^{n-1}}$ on $\mathbb{R}^n$ that behave like the the round sphere near the origin, but which flatten out, approximating a cone. If we choose $\phi_a(r) = a^{-1}\sin(ar)$ for $r \le \frac{\pi}{4a}$ and for ϕ_a to equal its tangent line at $r = \frac{\pi}{4a}$ for all $r > \frac{\pi}{4a}$, we achieve such a metric. If we choose p_0 to be the apex of the approximate cone, similarly to above, the term on the left in Theorem (2.1.1) is ka^{n-2q} for some k but the term on the right does not deteriorate. As an inequality this says,

$$ka^{n-2q} < C$$

Which is also clearly not optimal.

Lastly we note that, by taking a product of a manifold as above with $\mathbb{R}^k$, we can find examples of dimension $n+k$ exhibiting the behavior expressed in the inequalities above (i.e. not depending on k).

Petrunin [21] obtains an integral bound for the scalar curvature assuming a lower bound on the sectional curvature which also holds in the collapsed setting, and Cheeger and Naber [4] obtain integral bounds for the full Riemann curvature tensor for $q < 2$ assuming both upper and lower bounds on the Ricci curvature, but with the assumption that the volume is sufficiently noncollapsed.

The idea of the proof is the following: we show that we can achieve a similar bound to the one in Theorem (2.1.1) for the Ricci curvature along a geodesic. i.e.

Lemma 2.1.2. *Let M^n a Riemannian manifold. Let $\gamma : [0,l] \to M$ a minimizing geodesic parametrized by arc-length with $l \le 2$, and assume that $Ric \ge -1$ on the image of γ. Then*

$$\int_0^l |Ric(\gamma'(t), \gamma'(t))|^q dt \le C(q)$$

This bound passes immediately to a similar bound over an open collection of such geodesics in the tangent bundle. Our main goal is to show that we can then pass to Theorem (2.1.1) if the collection of geodesics suitably "covers" $B(p_0, 1)$. The technicalities essentially involve relating integrals over different sets (i.e. on subsets of TM versus M) by showing that, for manifolds with $Ric \ge -1$, the measures of the various sets are comparable.

2.2 Notation and Preliminaries

Let (M^n, g) be an n-dimensional complete Riemannian manifold and TM its tangent bundle. Let SM be the sub-bundle of unit length vectors. We use the notation $u = (p, v) \in TM$ (or SM), where $p \in M$ and $v \in T_pM$, to identify an arbitrary point in TM. Furthermore, we take for granted the inclusion $T_pM \subset TM$, so that we can for example take $U \in TM$, define $V_p := U \cap T_pM$ and think of V_p as a set of vectors in the tangent space at p, rather than a set in the tangent bundle. Ignoring this detail simplifies notation. We let $\pi : TM(or\ SM) \to M$ be the standard projection and also take for granted expressions like

$$U = \bigcup_{p \in \pi(U)} V_p$$

for U and V_p above.

We define the geodesic flow, $\phi : \mathbb{R} \times TM \to TM$, and the flow restricted to time t, $\phi_t : TM \to TM$, as follows: there is an obvious identification $T_{(p,v)}TM = T_pM \oplus T_pM$ and we let X be the vector field on TM such that $X(p,v) = (v,0)$. Let ϕ and ϕ_t satisfy

$$\frac{d}{dt}\phi(t,u) = X(\phi(t,u))$$

and

$$\phi_t(u) = \phi(t,u)$$

Then geodesics are smooth curves in M that satisfy

$$\gamma(t) = \pi\Big(\phi_t(u_\gamma(0))\Big)$$

where $u_\gamma(t) = (\gamma(t), \gamma'(t))$. We will take the liberty of sometimes referring to curves in TM as geodesics rather than their projections into M, again to reduce the need for more definitions and notation, but because it is a natural correspondence we don't expect it to cause any confusion.

Santaló's formula states that, with respect to the product measure on SM, the geodesic flow preserves volume. This means we can equivalently integrate a function f over the image of a set $U \subset SM$ under ϕ_t or pull back f through ϕ_t and integrate over U itself, i.e. let $f : SM \to \mathbb{R}$. For an open set $U \subset SM$ and $t \in \mathbb{R}$ the following holds

$$\begin{aligned}\int_{\phi_t(U)} f(u)du &= \int_U (\phi_t^* f)(u)du \\ &= \int_{\pi(U)} \int_{U \cap S_pM} f(\phi_t(u))d_pu\ dp \qquad (2.2.1)\end{aligned}$$

The notation in the third expression is more cumbersome but it allows us to work on each tangent plane separately, which will be necessary below. Above du is the product measure on SM, dp expresses the volume form in M, and d_pu is the Lebesgue measure on S_pM, i.e. it locally satisfies $d_pu \times dp = du$.

To make use of lemma (2.1.2), we must restrict our attention to geodesics which are minimizing. We define the segment domain:

$$seg(p) := \{v \in T_pM |\ exp_p(tv) : [0,1] \to M \textit{ is minimizing}\}$$

and its interior:

$$seg_0(p) := \{tv |\ 0 < t < 1, v \in seg(p)\}$$

It turns out that we will need to restrict our attention even further, to geodesics which are both minimizing and at least of a specified length. This motivates the definitions of the following sets. Let

$$W_{t,p} := \{v \in S_pM |\ tv \in seg_0(p)\}$$

Strictly speaking, this is the collection of unit vectors based at a point p that determine geodesics that, when parameterized by unit speed, are minimizing on the interval $[0,t]$.

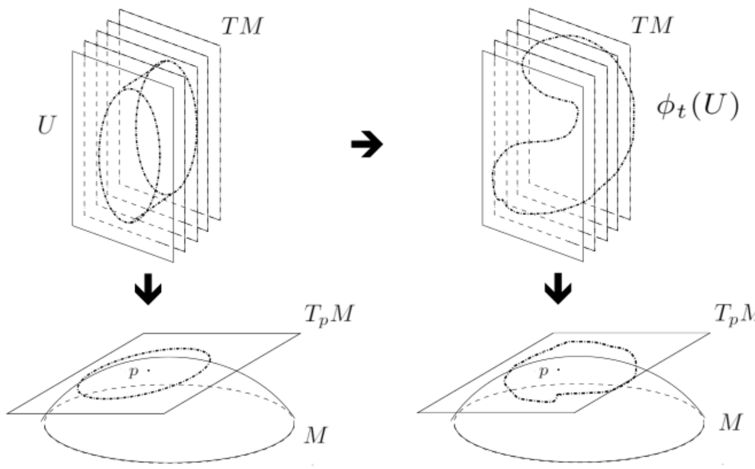


Figure 2.2.1: Given an open set $U \subset TM$, By Santaló's formula, the volumes of U and $\phi_t(U)$ are equal. If t is not too large, their projections onto M will have significant overlap. Thus, we can determine a basic relationship between volume and integrals in subsets of TM and in sets they cover in M. This idea is central to our proof of Theorem (2.1.1). See also Lemmas (2.2.1) and (2.2.2) below.

We will actually be more interested in unit vectors as above based at any of an open set of points in M. Furthermore it turns out that the only open sets we need are balls of a fixed radius and we now fix that radius to 3. Thus, let

$$W_t(p_0) := \bigcup_{p \in B(p_0,3)} W_{t,p} \tag{2.2.2}$$

(We remind the reader that we take for granted the inclusion $W_{t,p} \subset SM$) The significance of the radius 3 comes from the following statement: if we move along a unit speed geodesic for $t < 2$ and end at a point in $B(p_0, 1)$ then our starting point must have been within $B(p_0, 3)$. This significance will become more precise in the proof.

Observe that with these new sets defined, (1) gives

$$\begin{aligned} \int_{\phi_t(W_t(p_0))} f(u)du &= \int_{W_t} (\phi_t^* f)(u)du \\ &= \int_{B(p_0,3)} \int_{W_{t,p}} f(\phi_t(u))\ d_p u\ dp \end{aligned} \tag{2.2.3}$$

Let Ric_p denote the Ricci tensor at $p \in M$ and $|Ric|(p)$ denote its magnitude, i.e.

$$|Ric|(p) :=< Ric_p, Ric_p >^{\frac{1}{2}}$$

Ultimately, our goal is to find an integral bound for $|Ric|$ in the manifold M. When we apply (3) we will use powers of the following function, which evaluates the magnitude of the Ricci curvature at points along a geodesic *in the direction of the geodesic* (compare with the integrand of Lemma (2.1.2)): for $u = (p, v) \in SM$, let

$$R(u) := |Ric(v, v)|$$

We clarify that R is a function defined on the unit tangent bundle SM, and does not denote, as R often does, the scalar curvature on M. Observe also that this function extends naturally to a homogeneous function of degree zero (i.e. constant on rays) over TM that we also denote R.

For every $p \in M$, let $v_p \in S_pM$ be an eigenvector corresponding to the largest (in absolute value) eigenvalue of Ric_p. Keep in mind the relation

$$|Ric_p(v_p, v_p)| \geq \frac{1}{n}|Ric|(p)$$

Now let $\mu > 0$ be a small constant whose exact value will be determined later. let

$$S_{p,\mu} := \{v \in S_pM \ ||g(v, v_p)| > \mu\}$$

which we think of as the set of those unit vectors who point "roughly" in the same direction as v_p, i.e. it can easily be shown that for $v \in S_{p,\mu}$

$$|Ric_p(v, v)| \geq \frac{\mu^2}{n}|Ric|(p) \tag{2.2.4}$$

It follows from finite dimensional analysis that for all $p \in M$ we can control the measure of the complement of $S_{p,\mu}$ in S_pM

$$d_pu(S_pM \backslash S_{p,\mu}) \leq C\mu \tag{2.2.5}$$

for some constant C depending only on the dimension.

Finally, let S_μ be the union of $S_{p,\mu}$ over all $p \in M$. S_μ is a particular set of based vectors in SM that, up to a multiplicative constant that depends only on μ and the dimension, realize the magnitude of Ric at the corresponding base point.

With the notation clarified, we now state two lemmas. They provide the control over volume that we need to pass between integrals over open collections of geodesics in TM and open balls in M.

The idea of Lemma (2.2.1) is that many minimizing geodesics emanating from points within $B(p_0, 5)$ should indicate that the ball has sufficient volume.

The idea of Lemma (2.2.2) is slightly subtler. It says that for *every* point $p \in B(p_0, 2)$ and every time $1 < t < 2$, there are sufficiently many minimizing geodesics passing through p in a direction where $|Ric_p(\cdot, \cdot)|$ is comparable to $|Ric|_p$ in the sense of (2.2.4). This means that we can choose an open collection of geodesics so that, along each geodesic, the integral in Lemma (2.1.2)

$$\int_0^l |Ric_{\gamma(t)}(\gamma'(t), \gamma'(t))|^q dt$$

is comparable to

$$\int_0^l |Ric|^q(\gamma(t))dt$$

We save their proofs for after the proof of Theorem (2.1.1). It may be useful throughout to keep in mind that, by the Bishop Gromov volume comparison [3 ch. 9, lemma 36], for any $r_1, r_2 > 0$, a term $C_1 Vol(B(p_0, r_1)$ appearing in an inequality can always be replaced by a term $C_2 Vol(B(p_0, r_2))$ for suitable C_2 depending on the ratio r_1/r_2. In particular it is more or less inconsequential that the radii in lemmas (2.2.1) and (2.2.2) are different, so long as their ratio is bounded away from zero and infinity.

Lemma 2.2.1. *Let (M^n, g) be a Riemannian manifold. Assume $p_0 \in M$ is given. Then*

$$Vol(W_1(p_0)) \leq C(n) Vol(B(p_0, 5))^2$$

for some $C(n)$ depending only on the dimension.

Lemma 2.2.2. *Let (M^n, g) be a Riemannian manifold. Assume $p_0 \in M$ is given and that the diameter $diam(M, g)$ of M satisfies $diam(M, g) \geq 6$. Then there exists a choice of μ such that for all $p \in B(p_0, 1)$ and all $1 < t < 2$,*

$$d_p u\Big(\phi_t\big(W_t(p_0)\big) \cap S_{p,\mu}\Big) > c_1 Vol(B(p_0, 1))$$

where

$$\mu = c_2 vol(B(p_0, 1))$$

for c_1, c_2 depending only on the dimension.

2.3 Proof of Theorem (2.1.1) assuming lemmas (2.1.2), (2.2.1), (2.2.2)

Proof. To begin the proof, we first argue why we can assume $diam(M, g) \geq 6$. Assume $R := \frac{6}{diam(M,g)} > 1$. Define the rescaled manifold (M, g_R), where $g_R(\cdot, \cdot) := R^2 g(\cdot, \cdot)$. Let Ric_R, $B_R(p_0, r)$ and $d_R p$ denote the Ricci curvature, r-ball and volume form respectively, each with respect to g_R. Note that, because $R > 1$, we preserve the condition $Ric \geq -1$ for the rescaled manifold. Because of the way the Ricci curvature and the volume form scale with respect to a scaling of the metric, we have the following:

$$\begin{aligned} ||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))} &= R^{-n+1-2\epsilon} ||Ric_R||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B_R(p_0,R))} \\ &\leq R^{-n+1-2\epsilon} ||Ric_R||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B_R(p_0,6))} \end{aligned}$$

Where the last inequality follows because the diameter of the rescaled manifold is 6. Now, we can cover $B_R(p_0, 6)$ with K-many balls $B_R(p_i, 1)$, where K only depends on the dimension and a lower bound on the Ricci curvature, and so assuming we have proven the theorem for the case $diam(M, g) \geq 6$, and using the Bishop-Gromov volume comparison, it follows that

$$
\begin{aligned}
||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))} &\leq R^{-n+1-2\epsilon} \cdot \sum ||Ric_R||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B_R(p_i,1))} \\
&\leq R^{-n+1-2\epsilon} \cdot KC(n) \cdot Vol_R(B_R(p_i,1))^{2\epsilon} \\
&\leq R^{-n+1-2\epsilon} \cdot C'(n) \cdot Vol_R(B_R(p_0,1))^{2\epsilon} \\
&\leq R^{-n+1-2\epsilon} \cdot C''(n) \cdot Vol_R(B_R(p_0,R))^{2\epsilon} \\
&= R^{-n+1-2\epsilon} \cdot C''(n) \cdot R^{2\epsilon n} Vol(B(p_0,1))^{2\epsilon} \\
&\leq C''(n) Vol(B(p_0,1))^{2\epsilon}
\end{aligned}
$$

where the last line follows because $R > 1$, $n \geq 2$ and $\epsilon < 1/2$. Now, with the assumption that the diameter of M satisfies $diam(M,g) \geq 6$, we assume lemmas (2.1.2), (2.2.1), and (2.2.2) hold. Firstly, for $1 < t < 2$ we have

$$
\begin{aligned}
\int_{B(p_0,3)} \int_{W_{t,p}} R(\phi_t(u))^{\frac{1}{2}-\epsilon} d_p u \; dp &= \int_{W_t} R(\phi_t(u))^{\frac{1}{2}-\epsilon} du \\
&= \int_{\phi_t(W_t)} R(u)^{\frac{1}{2}-\epsilon} du \\
&\geq \int_{\phi_t(W_t) \bigcap S_\mu} R(u)^{\frac{1}{2}-\epsilon} du \\
&= \int_M \int_{\phi_t(W_t) \cap S_{p,\mu}} R(p,v)^{\frac{1}{2}-\epsilon} d_p u \; dp \\
&\geq \int_{B(p_0,1)} \int_{\phi_t(W_t) \cap S_{p,\mu}} R(p,v)^{\frac{1}{2}-\epsilon} d_p u \; dp \\
&\geq \int_{B(p_0,1)} \int_{\phi_t(W_t) \cap S_{p,\mu}} (\frac{\mu^2}{n} |Ric|(p))^{\frac{1}{2}-\epsilon} d_p u \; dp \\
&\geq \int_{B(p_0,1)} c_1 Vol(B(p_0,1)) (\frac{\mu^2}{n})^{\frac{1}{2}-\epsilon} (|Ric|(p))^{\frac{1}{2}-\epsilon} dp \\
&\geq c_1 Vol(B(p_0,1)) (\frac{\mu^2}{n})^{\frac{1}{2}-\epsilon} ||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))} \\
&\geq c Vol(B(p_0,1))^{2-2\epsilon} ||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))} \qquad (2.3.1)
\end{aligned}
$$

where we have used μ as in Lemma (2.2.2).

Because this holds for all $1 < t < 2$, we similarly obtain

$$
\int_1^2 \int_{B(p_0,3)} \int_{W_{t,p}} R(\phi_t(u))^{\frac{1}{2}-\epsilon} d_p u \; dt \; dp \geq \int_1^2 c Vol(B(p_0,1))^{2-2\epsilon} ||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))} dt \qquad (2.3.2)
$$

$$
= c Vol(B(p_0,1))^{2-2\epsilon} ||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))}
$$

We obtain an upper bound for the term on the left as follows. By Fubini's theorem, lemmas (2.1.2) and (2.2.1) and the assumption that $Ric \geq -1$ on $B(p_0, 5)$,

$$\begin{aligned}
\int_1^2 \int_{B(p_0,3)} \int_{W_{t,p}} R(\phi_t(u))^{\frac{1}{2}-\epsilon} d_p u \; dt \; dp &= \int_{B(p_0,3)} \int_{W_{1,p}} \int_1^{\beta_2(u)} R(\phi_t(u))^{\frac{1}{2}-\epsilon} dt \; d_p u \; dp \\
&\leq \int_{B(p_0,3)} \int_{W_{1,p}} C(q) \;\; d_p u \; dp \\
&\leq C(q) Vol(W_1) \\
&\leq C(q) Vol(B(p_0,5))^2 \\
&\leq C(q,n) Vol(B(p_0,1))^2
\end{aligned}$$

Where $q = \frac{1}{2} - \epsilon$. It finally follows that

$$cVol(B(p_0,1))^{2-2\epsilon} ||Ric||^{\frac{1}{2}-\epsilon}_{L^{1/2-\epsilon}(B(p_0,1))} \leq C(q,n) Vol(B(p_0,1))^2$$

which, after redefining $C(q,n)$, is equivalent to theorem (2.1.1). $\square$

2.4 Proofs of Lemmas (2.1.2), (2.2.1), (2.2.2)

Proof of Lemma (2.1.2). Along a minimising geodesic γ parametrized by arc length, choose $E_1(t), ..., E_{n-1}(t)$ to be orthonormal parallel vector fields along γ that are perpendicular to γ'. If h is a nonnegative continuously differentiable function vanishing at 0 and l, by the second variation formula,

$$0 \leq \int_0^l (h'(t))^2 dt - \int_0^l (h^2 sec(\gamma'(t), E_i(t)))dt$$

and by approximation the same holds if h is only continuous and piece-wise differentiable. Summing over $i = 1, ..., n-1$ gives

$$\int_0^l h(t)^2 Ric(\gamma'(t), \gamma'(t))dt \leq (n-1) \int_0^l (h'(t))^2 dt \tag{2.4.1}$$

Let $Ric_+(\cdot,\cdot) := max\{Ric(\cdot,\cdot), 0\}$ and $Ric_-(\cdot,\cdot) := -min\{Ric(\cdot,\cdot), 0\}$. By assumption $Ric_- \leq 1$. Therefore, we only need to show

$$\int_0^l Ric_+(\gamma'(t), \gamma'(t))^{\frac{1}{2}-\epsilon} dt \leq C(\epsilon)$$

to complete the proof. It follows from (2.4.1) that

$$\begin{aligned} 0 &\leq \int_0^l h(t)^2 Ric_+(\gamma'(t), \gamma'(t))dt \\ &\leq (n-1) \int_0^l (h'(t))^2 dt + \int_0^l Ric_-(\gamma'(t), \gamma'(t))dt \\ &\leq (n-1) \int_0^l (h'(t))^2 dt + 2 \end{aligned} \tag{2.4.2}$$

since $l \leq 2$. Define $h_l(t) := t$ on $[0, l/2]$, and $h_l(t) := (l-t)$ on $[l/2, l]$. For fixed $a > 0$, using (2.4.2) and the construction of h_l

$$\begin{aligned} \int_0^l h_l(t)^{1+a} Ric_+(\gamma'(t), \gamma'(t))dt &= \int_0^l \left(h_l(t)^{\frac{1+a}{2}}\right)^2 Ric_+(\gamma'(t), \gamma'(t))dt \\ &\leq (n-1) \int_0^l \left((\frac{1+a}{2}) h_l^{\frac{-1+a}{2}} h_l'(t)\right)^2 dt + 2 \\ &= (n-1) \int_0^{l/2} \left((\frac{1+a}{2}) t^{\frac{-1+a}{2}}\right)^2 dt \\ &+ (n-1) \int_{l/2}^l \left((\frac{1+a}{2})(\frac{l}{2} - t)^{\frac{-1+a}{2}}\right)^2 dt + 2 \\ &= 2(n-1)\frac{(1+a)^2}{4} \int_0^{l/2} t^{-1+a} dt + 2 \\ &= \frac{(n-1)(1+a)^2}{2a} \left(\frac{l}{2}\right)^a + 2 \end{aligned}$$

Let $0 < \epsilon < 1/2$ be given. Using Hölder's inequality with exponents $p = \frac{2}{1-2\epsilon}$ and $q = \frac{2}{1+2\epsilon}$ gives

$$\begin{aligned}
\int_0^l Ric_+(\gamma'(t), \gamma'(t))^{\frac{1}{2}-\epsilon} &= \int_0^l \frac{h_l(t)^{\frac{1-\epsilon}{2}}}{h_l(t)^{\frac{1-\epsilon}{2}}} Ric_+(\gamma'(t), \gamma'(t))^{\frac{1}{2}-\epsilon} dt \\
&\leq \left(\int_0^l h_l(t)^{\frac{1-\epsilon}{1-2\epsilon}} Ric_+(\gamma'(t), \gamma'(t)) dt \right)^{1/p} \left(\int_0^l h_l(t)^{\frac{\epsilon-1}{1+2\epsilon}} dt \right)^{1/q} \\
&\leq \left(\frac{(n-1)(1+a)^2}{2a} \Big(\frac{l}{2}\Big)^a + 2 \right)^{1/p} \left(\int_0^l h_l(t)^{\frac{\epsilon-1}{1+2\epsilon}} dt \right)^{1/q} \\
&= k_1(\epsilon) \Big(\frac{1-2\epsilon}{\epsilon}\Big)^{\frac{1}{p}} \Big(\frac{1-2\epsilon}{\epsilon}\Big)^{\frac{1}{q}} \\
&= k_2(\epsilon) \Big(\frac{1}{\epsilon}\Big) \\
&= C(\epsilon)
\end{aligned}$$

Where a satisfies $1 + a = \frac{1-\epsilon}{1-2\epsilon}$ and $k_i(\epsilon)$ are functions bounded away from both zero and infinity. Noting that, if we take $q = \frac{1}{2} - \epsilon$, $C(\epsilon)$ can be re-expressed as $C(q)$, the result follows.

□

Fix $u = (p, v) \in SM$. For $u \in SM$, let $\beta(u)$ denote the distance to the cut locus in the direction of u, i.e.

$$\beta(u) := \sup\{t > 0 | tv \in seg(p)\}$$

For $k > 0$ let $\beta_k(u) := min\{\beta(u), k\}$. u determines a minimizing geodesic $\gamma_v : [0, l] \to M$ parameterized by unit speed for some $l > 0$. In exponential coordinates based at $p = \gamma_v(0)$, the volume form can be expressed as

$$f(t, v) dt \wedge dv$$

where t represents the radial coordinate and $v \in SM$ determines a direction in S^{n-1}. Restricting to our fixed v, this determines a function $f_v(t)$ for $0 < t < l$, which we think of as the magnitude of the volume form along γ starting from p. By volume comparison and our curvature assumption, this function cannot be larger than the corresponding function for a manifold of constant curvature equal to -1. It can, however, be much smaller.

If we consider instead a new starting point along γ_v, say $\gamma_v(s)$ for some $s < l$, we can let $\bar{v} := \gamma_v'(s)$ and consider the magnitude of the volume form in exponential coordinates along $\gamma_{\bar{v}}$ based at $\gamma_{\bar{v}}(0) = \gamma_v(s)$. This defines a new function $f_{\bar{v}}(t)$ for $0 < t < l - s$ in the same fashion as above. In this way, we can examine the magnitude of the volume form along any minimizing geodesic starting from any base point along that geodesic. Accordingly, we define $F : SM \times \mathbb{R}_+ \to \mathbb{R}$ so that it satisfies, for a given $u = (p, v) \in SM$ and $t \in (0, \beta(u))$

$$F(u, t) := f_v(t)$$

We will use the following, lemma 9 from [10], which says that on average, the function determined in this way cannot be too small.

Lemma 2.4.1. *Let (M^n, g) be a complete Riemannian manifold and $u \in SM$. Then for every $l \leq \beta(u)$ (the distance to the cut locus in the direction of u):*

$$\int_0^l \int_0^{l-t} F(\phi_t(u), s) ds\ dt \geq C(n) \frac{l^{n+1}}{\pi^{n+1}}$$

where $F(u, s)$ restricted to $S_p M \times \mathbb{R}_+$ is the magnitude of the volume form in exponential coordinates on M, i.e. it satisfies

$$\int_{S_p M} \int_0^{\beta(u)} F(u, s) ds\ dv = Vol(M) \tag{2.4.3}$$

Proof of Lemma (2.2.1). The following inequality follows directly from the lemma,

$$\begin{aligned}
&\int_0^1 \int_{B(p_0,3)} \int_{W_{1,p}} \int_0^{1-t} F(\phi_t(u), s) ds\ d_p u\ dp\ dt \\
&= \int_{B(p_0,3)} \int_{W_{1,p}} \int_0^1 \int_0^{1-t} F(\phi_t(u), s) ds\ dt\ d_p u\ dp \\
&\geq \int_{B(p_0,3)} \int_{W_{1,p}} c\ d_p u\ dp \\
&= c Vol(W_1)
\end{aligned}$$

We also have

$$\int_{B(p_0,4)} \int_{S_p M} \int_0^{\beta_1(u)} F(u, s) ds\ d_p u\ dp \leq Vol(B(p_0, 5)^2$$

which is clear from the definition of F and β_1, and from (2.4.3). It then follows that

$$\begin{aligned}
&\int_0^1 \int_{B(p_0,3)} \int_{W_{1,p}} \int_0^{1-t} F(\phi_t(u), s) ds\ d_p u\ dp\ dt \\
&= \int_0^1 \int_{W_1} \int_0^{1-t} F(\phi_t(u), s) ds\ du\ dt \\
&= \int_0^1 \int_{\phi_t(W_1)} \int_0^{1-t} F(u, s) ds\ du\ dt \\
&\leq \int_0^1 \int_{B(p_0,4)} \int_{S_p M} \int_0^{\beta_1(u)} F(u, s) ds\ d_p u\ dp\ dt \\
&\leq C Vol(B(p_0, 5))^2
\end{aligned}$$

The second to last line follows since all elements $u = (p, u_p) \in W_1$ correspond to vectors v of magnitude $|v| = 1$ based at some $p \in B(p_0, 3)$. Therefore, the image under the geodesic flow at time t of such an element is (q, u_q) for some u_q with $|u_q| = 1$ based at q for some $q \in B(p_0, 4)$ if $t < 1$.

The two inequalities give the desired result. $\square$

Proof of Lemma (2.2.2). Notice that for all $p \in B(p_0, 1)$ and $v \in W_{t,p}$, for $1 < t < 2$, $(p, -v) \in \phi_t(W_t) \cap S_pM$, which is clear by reparameterizing the relevant geodesics to go in the opposite direction. Therefore for any $\delta > 0$,

$$d_pu(W_{t,p}) \geq 2\delta$$

implies

$$d_pu(\phi_t(W_t) \cap S_pM) \geq 2\delta$$

Therefore, if we let $\mu := \frac{\delta}{C}$ for C satisfying (2.2.5), then

$$d_pu(\phi_t(W_t) \cap S_{p,\mu}) = d_pu(\phi_t(W_t) \cap S_pM) - d_pu(\phi_t(W_t) \cap (S_pM \backslash S_{p,\mu})) \geq 2\delta - \delta \geq \delta$$

Therefore we only need to show

$$d_pu(W_{t,p}) \geq \delta$$

with δ on the order of $Vol(B(p_0, 1))$ to complete the proof. Furthermore, observe that if we show the result for $t = 2$, it immediately follows for $t < 2$.
For any $p \in B(p_0, 4)$

$$Vol(B(p, 8)) \geq vol(B(p_0, 4))$$

and therefore, by Bishop-Gromov volume comparison,

$$Vol(B(p, 1)) \geq C(n)vol(B(p_0, 1))$$

By the diameter assumption on M, for any $p \in B(p_0, 1)$ there exists $q \in B(p_0, 4)$ such that $d(p, q) = 3$. We must similarly have

$$Vol(B(q, 1)) \geq C(n)vol(B(p_0, 1))$$

But, once more using Bishop-Gromov and letting V_{-1}^n denote the volume of the sphere of radius 1 in n-dimensional hyperbolic space, this implies

$$\begin{aligned} Vol(B(q, 1)) &\leq \int_1^{\beta_3(u)} \int_{W_{2,p}} F(u, s) d_pu \; ds \\ &\leq \int_1^3 \int_{W_{2,p}} F(u, s) d_pu \; ds \\ &\leq 2V_{-1}^n vol(W_{2,p}) \end{aligned}$$

implying

$$d_pu)(W_{2,p}) \geq CVol(B(p_0, 1))$$

where F is defined in the proof of Lemma (2.2.1).

□

Chapter 3

Volume on Surfaces of Bounded Genus

3.1 Introduction

We first restate Theorem (1.1.15) in slightly more detail.

Theorem 3.1.1. *For $r > 0$, consider all metric balls of radius r in all complete n-dimensional Riemannian manifolds with Ricci curvature greater or equal to $-(n-1)$. Equip this space with the Gromov-Hausdorff topology. Then the volume function is continuous, i.e. for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $B(p_1, r) \subset M_1$ and $B(p_2, r) \subset M_2$ are as described and satisfy*

$$d_{GH}(B(p_1, r), B(p_2, r)) < \epsilon$$

then

$$|Vol(B(p_1, r)) - Vol(B(p_2, r))| < \delta(\epsilon).$$

Recalling our discussion in chapter 1, we restate Theorem (1.1.18), the main result of this chapter

Theorem 3.1.2. *For $r > 0$, consider all metric balls of radius r in all complete 2-dimensional Riemannian manifolds with Euler characteristic uniformly bounded. Equip this space with the Gromov-Hausdorff topology. Then the volume function is lower semi-continuous. i.e. for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $B(p, r) \subset M$ and $B(p_i, r) \subset M_i$ are as described and satisfy*

$$\lim_{\imath \to \infty} d_{GH}(B(p_i, r), B(p, r)) = 0$$

then

$$\lim_{\imath \to \infty} Vol(B(p_i, r)) \geq Vol(B(p, r)).$$

It is easy to see that, without a curvature assumption, upper semi-continuity cannot be obtained even if all spaces involved are simply connected. Furthermore, the result does not hold in higher dimensions, as can be observed by decomposing S^3

into two handlebodies M_1, M_2 with shared boundary one of the surfaces T_ϵ^δ defined in Example (1.1.16) and writing

$$S^3 \cong M_1 \# (T_\epsilon^\delta \times [a,b]) \# M_2,$$

where $\#$ denotes the gluing of each pair of surfaces along their boundary. $\pi_1(S^3)$ is trivial, but the lack of lower semicontinuity captured in Example (1.1.16) clearly holds on $T_\epsilon^\delta \times [a,b]$ as well, at least provided a, b are sufficiently large.

3.2 Preliminaries

A geodesic γ into a Riemannian manifold M will always be parametrized by the interval $[0,1]$, i.e. we don't allow for more general domains $[a,b]$. Therefore when we talk about a broken geodesic $\gamma_1 \cdot \ldots \cdot \gamma_n$, where it is assumed that $\gamma_{i+1}(0) = \gamma_i(1)$, it is understood that such a composition has been reparametrized to satisfy this condition. Furthermore, for $t_1, t_2 \in [0,1]$, $\gamma|_{[t_1,t_2]}$ defined as the restriction to $[t_1, t_2]$ of γ, has also been reparametrized in this way.
For a curve γ and $a \in \mathbb{Z}$, $a\gamma$ will mean γ composed with itself a times, with negative numbers reversing the direction of γ. We say a finite sequence of geodesics $\gamma_1, ..., \gamma_n : [0,1] \to M$ connects two points p and q, or p and q are connected by $\gamma_1, ..., \gamma_n$, if $\gamma_1(0) = p, \gamma_n(1) = q$. A geodesic loop based at $p \in M$ means a geodesic curve $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p$.

$[\gamma]$ will represent the homotopy class in $\pi_1(M,p)$ of a geodesic loop $\gamma : [0,1] \to M$ based at p.
For a metric space M and a point $p \in M$, $B_M(p,r)$ and $\bar{B}_M(p,r)$ will denote the open and closed balls, respectively, of radius r at p.

We refer to chapter 1 for definitions of the Hausdorff and Gromov-Hausdorff distances, d_H, d_{GH}, between sets induced by a metric d.

We restate Theorem (3.1.2) in the alternate form in which we prove it:

Theorem 3.2.1. *Let (M_i, g_i) be a sequence of complete 2-dimensional Riemannian manifolds of Euler characteristic $\chi_i \le h < \infty$. Consider corresponding 1-balls $B_{M_i}(p_i, 1)$ centered at points $p_i \in M_i$ and let $B(0,1)$ denote the unit ball at $0 \in \mathbb{R}^2$. If*

$$\lim_{i\to\infty} d_{GH}(B_{M_i}(p_i,1), B(0,1)) = 0 \tag{3.2.1}$$

then

$$\liminf_{i\to\infty} \mathrm{Vol}_i B_{M_i}(p_i,1) \ge \mathrm{Vol} B(0,1) \tag{3.2.2}$$

where Vol_i, Vol *denote the volume of a subset on the corresponding manifolds.*

We state some results relating to the Gromov-Hausdorff distance, Chapter 10 section 1.1 of [24]:

Lemma 3.2.2. *If compact metric spaces (M_1, d_1), (M_2, d_2) satisfy*

$$d_{GH}(M_1, M_2) \leq \epsilon$$

then there exist finite subsets $\{x_1, ..., x_n\} \subset M_1$ and $\{y_1, ..., y_n\} \subset M_2$ such that

1. $\forall x \in M_1$, $\exists k$ *such that* $d_1(x, x_k) < \epsilon$
2. $\forall y \in M_2$, $\exists k$ *such that* $d_2(y, y_k) < \epsilon$
3. $|d_i(x_j, x_k) - d(y_j, y_k)| < \epsilon$ *for all* $1 \leq j, k \leq n$

Furthermore if 1, 2, and 3 hold, then

$$d_{GH}(M_1, M_2) \leq 3\epsilon.$$

I.e. we need only consider how the metrics compare on finite, suitably dense subsets of the relevant metric spaces to understand the Gromov-Hausdorff metric. We refer to subsets satisfying the property of 1 or 2 in the lemma above as being ϵ-dense in the corresponding metric space.

From Lemma (3.2.2), we have the following characterization of compact metric spaces (M_i, d_i) converging in the Gromov-Hausdorff metric to (M, d), which is our main tool in much of what follows

Lemma 3.2.3. *If compact metric spaces (M_i, d_i), (M, d) satisfy*

$$\lim_{i \to \infty} d_{GH}(M_i, M) = 0$$

then there exists a metric $\bar{d}$ on $\bigsqcup_{i \in \mathbb{N}} M_i \bigsqcup M$, restricting to the appropriate metric on each subspace, such that $\bar{d}_H(M_i, M) \to 0$. Furthermore, for any $\{x_1, ..., x_n\} \subset M$ there exist subsets $\{x_1^i, ..., x_n^i\} \subset M_i$ such that for all $1 \leq j, k \leq n$,

$$\bar{d}(x_k^i, x_k) < i^{-1}$$

and

$$|\bar{d}(x_j^i, x_k^i) - \bar{d}(x_j, x_k)| < i^{-1}$$

and lastly, if for some $\epsilon > 0$ $\{x_1, ..., x_n\}$ is ϵ-dense in M, the sets $\{x_1^i, ..., x_n^i\}$ can be chosen so that, for all but finitely many i, they are ϵ-dense in M_i.

From now on, we will only be concerned with the metric space

$$X := \bigsqcup_{i \in \mathbb{N}} B_{M_i}(p_i, 1) \bigsqcup B(0, 1)$$

where $B_{M_i}(p_i, 1)$ are as in Theorem (3.2.1), and the metric $\bar{d}$ as in Lemma (3.2.3). We will write B_{M_i} to denote $B_{M_i}(p_i, 1)$. There is an obvious correspondence between $p \in (B_{M_i}, d_i) \subset (M_i, d_i)$ and $p \in (B_{M_i}, \bar{d}) \subset (X, \bar{d})$, and because the metric we use on

X restricts to the corresponding metric on each B_{M_i}, the distinction is insignificant in what follows.

As our first step, we show that the minimizing geodesics in $B_{M_i}(p_i,1)$ for large i are L^∞ close in $\bar{d}$ to geodesics in $B(0,1)$ (each considered as subsets of X):

Proposition 3.2.4. *Let $X := \bigsqcup_{i\in\mathbb{N}} B_{M_i}(p_i,1) \bigsqcup B(0,1)$ and $\bar{d}$ the metric on X guaranteed by Lemma (3.2.3). For each i, assume $\gamma_i : [0,1] \to B_{M_i}$ is a minimizing geodesic with $\gamma_i(0) = p_i$, $\gamma_i(1) = q_i$ for some $q_i \in B_{M_i}$. Assume further that $\lim_{i\to\infty} l(\gamma_i) = \lambda > 0$. Then there exists a subsequence γ_{i_k} and $q \in B(0,1)$ such that $\sup_{t\in[0,1]} \bar{d}(\gamma_{i_k}(t), \gamma(t)) \underset{k\to\infty}{\to} 0$, where $\gamma : [0,1] \to \mathbb{R}^2$ is the unique minimizing geodesic connecting 0 and q in $B(0,1)$.*

Proof. We first show that $\lim_{i\to\infty} \bar{d}(p_i, 0) = 0$ (where the first $0 \in B(0,1) \subset \mathbb{R}^2$). If this were not true, the sequence p_i would contain a subsequence satisfying $\bar{d}(p_i, x) \to 0$ for some $x \in B(0,1) \subset X$, $x \neq 0$. Choose a sequence $p_i' \in B_{M_i}$ converging to $y \in B(0,1)$ with $\bar{d}(x,y) > 1$. Then for large i, $\bar{d}(p_i, p_i') > 1$, which is not possible. Furthermore, there clearly exists $q \in B(0,1)$ and a subsequence q_{i_k} such that $\bar{d}(q_{i_k}, q) \to 0$. Let $\gamma : [0,1] \to \mathbb{R}^2$ be the unique minimizing geodesic connecting 0 and q in $B(0,1)$ and assume there exists $t \in (0,1)$ and $\epsilon > 0$ such that $\bar{d}(\gamma_{i_k}(t), \gamma(t)) > \epsilon$ for arbitrarily large k. Since each γ_{i_k} is minimizing, we have $\bar{d}(p_i, \gamma_{i_k}(t)) = tl(\gamma_{i_k}) \to t\lambda = \; tl(\gamma)$. But, after possibly passing to a further subsequence, we also must have $\bar{d}(\gamma_{i_k}(t), x) \to 0$ for some $x \in B(0,1)$, $x \neq \gamma(t)$, satisfying $\bar{d}(0,x) = tl(\gamma)$. Since $\gamma(t)$ uniquely satisfies both $\bar{d}(0,\gamma(t)) = t\lambda$ and $\bar{d}(\gamma(t), q) = \lambda - t\lambda$, it must be that $\bar{d}(x,q) > \lambda - t\lambda$. This implies that $\liminf_{k\to\infty} l(\gamma_{i_k}) > \lambda$ which is a contradiction. □

3.3 The Function ϕ

Next, we construct a map $\phi : X \to \mathbb{R}^2$ that we will use to map certain subsets of $B_{M_i}(p_i,1)$ into $B(0,1)$, and use the images of these subsets to determine lower bounds on their volumes. We collect the properties of ϕ in the following lemma

Lemma 3.3.1. *There exists $\phi : X \to \mathbb{R}^2$ such that*

1. *when restricted to $B_{M_i}(p_i,1)$ or $B(0,1)$, the map is $\sqrt{2}$-Lipschitz and volume non-increasing*
2. *for some small $\epsilon > 0$ and $x \in B(0,\epsilon)$, $|\phi(x) - x| < C\epsilon^2$*
3. *for any $\delta > 0$ there exists i_δ such that $\bar{d}(\phi(x), \phi(x')) \leq \sqrt{2}\bar{d}(x,x') + \delta$ for any $x, x' \in \bigsqcup_{i>i_\delta} B_{M_i}(p_i,1) \bigsqcup B(0,1)$,*
   i.e. on the tail of X, ϕ satisfies an almost Lipschitz condition.

Let $\hat{x}, \hat{y} \in B(0,1)$ $\hat{x} := (1,0)$ and $\hat{y} := (0,1)$ and choose $\hat{x}^i, \hat{y}^i \in B_{M_i}$ such that

$$\lim_{i\to\infty} \bar{d}(\hat{x}^i, \hat{x}) = 0, \; \lim_{i\to\infty} \bar{d}(\hat{y}^i, \hat{y}) = 0$$

and define for $x \in B_{M_i}$

$$\phi(x) := (\tilde{d}(p_i, \hat{x}^i) - \tilde{d}(x, \hat{x}^i), \tilde{d}(p_i, \hat{y}^i) - \tilde{d}(x, \hat{y}^i))$$

ϕ is 1-Lipschitz in each coordinate, and so the $\sqrt{2}$ bound follows. Where the distance functions in the definition of ϕ are smooth, we can think of their differentials as 1-forms on vectors in the tangent space of B_{M_i}, each of norm 1, and by direct calculation the jacobian of ϕ is their wedge product, which therefore has norm less than or equal to 1. Because the union of the sets where these functions are not smooth is the union of their cut loci, which has measure zero, we can use their differentials to integrate the volume form almost everywhere, and we therefore have

$$\mathrm{Vol}(\phi(U)) \leq \mathrm{Vol}(U)$$

whenever $U \subset B_{M_i}$ for some i. The differential of ϕ at $0 \in B(0,1)$ is the identity. This implies that there exists $C > 0$ such that for small $\epsilon > 0$ and $x \in B(0, \epsilon)$,

$$|\phi(x) - x| < C\epsilon^2. \tag{3.3.1}$$

Furthermore, assume $\{x_1, ..., x_n\}$ is δ-dense in $B(0,1)$ and $\{x_1^i, ..., x_n^i\} \subset B_{M_i}(p_i, 1)$ are as in Lemma (3.2.3). We can assume

$$x_1 = \hat{x}, \quad x_2 = \hat{y}$$

and

$$x_1^i = \hat{x}^i, \quad x_2^i = \hat{y}^i$$

Finally, because the pairwise distances between points $\bar{d}(x_j^i, x_k^i)$ for $1 \leq j, k \leq n$ converge to $\bar{d}(x_j, x_k)$ as $i \to \infty$, it follows that $\phi(x_k^i) \to \phi(x_k)$. Since for large enough i, the sets $\{x_1^i, ..., x_n^i\}$ are δ-dense and ϕ is Lipschitz on any $B_{M_i}(0,1)$ and $B(0,1)$, we reach that for any $\delta > 0$ there exists i_δ such that

$$\bar{d}(\phi(x), \phi(x')) \leq \sqrt{2}\bar{d}(x, x') + \delta \tag{3.3.2}$$

3.4 Closures of Surfaces with boundaries

In examining the sets $B_{M_i}(P_i, \epsilon) \subset M_i$., observe that there is no general relationship between $\chi(M_i)$ and $\chi(B_{M_i}(p_i, \epsilon))$, because $B_{M_i}(p_i, \epsilon)$ may be incomplete, with its boundary containing any number of components. To control this, we introduce, for an orientable surface S, the set S_{closed}, denoting the closed surface obtained by gluing a disc at each boundary component of S. It is a classical fact that this construction is unique up to homotopy. Observe also that the Euler characteristic and genus of S_{closed} never exceed those of S.

We require some basic facts pertaining to S_{closed}, that we compile into a lemma.

Lemma 3.4.1. *For S and S_{closed} as above,*

1. *If $\phi : S \to \mathbb{R}^2 \backslash B(0, r)$ is continuously differentiable, there exists an extension $\phi' : S_{closed} \to \mathbb{R}^2 \backslash B(0, r)$ that is continuously differentiable.*

2. If (S, g) and (S_{closed}, g') are such that g' restricts to g on S, then $d_{g'}(p, x) \geq \min\{d(p, y) \mid y \in \partial S\}$ for any $p \in S$ and $x \in S_{closed} \backslash S$.

Proof. 1 follows from the fact that $S_{closed} \backslash S$ is a union of discs whose boundaries are mapped to

$$\mathbb{R}^2 \backslash B(0, r)$$

by ϕ. Because $\mathbb{R}^2 \backslash B(0, r)$ is simple connected, 1 follows immediately.

2 follows from the fact that any curve connecting p and x must intersect ∂S. □

3.5 Proof of Theorem (3.2.1)

The longest portion of our proof deals with the following proposition, stating the existence of certain collections of geodesic loops in a Riemannian manifold having useful properties, whose proof we postpone until the following section. In what follows, an element γ of a subset G of geodesics is *essential* if it is not homotopic to a point and *nonperipheral* if it is not homotopic to a cusp. Thus, for an essential, nonperipheral curve γ, each component of $M \backslash \gamma([0, 1])$ is neither a disc nor an annulus.

Proposition 3.5.1. *Let (M, g, p) be an oriented, complete, pointed 2-dimensional Riemannian manifold of Euler characteristic χ, $B_M(p, r)$ the open ball of radius $0 < r < \infty$ at p, and $\pi_1(B_M(p, r)_{closed}, p)$ the fundamental group of $B_M(p, r)_{closed}$, considered as its own topological space. Then there exists a (possibly empty) collection of geodesic loops $E^r \subset \{\gamma : [0, 1] \to B_M(p, r) \mid \gamma(0) = \gamma(1) = p\}$ satisfying the following:*

1. geodesics in E^r are essential, minimizing on each half of $[0, 1]$, pairwise non-homotopic and intersecting only at p

2. The group elements $[\gamma] \in \pi_1(B_M(p, r)_{closed}, p)$ corresponding to $\gamma \in E^r$ generate $\pi_1(B_M(p, r)_{closed}, p)$

3. $|E^r| < Z_\chi$ for some Z_χ which depends only on χ

Note that E^r is empty exactly when $\pi_1(B_M(p, r)_{closed}, p)$ is trivial. The following proposition follows from Proposition (3.5.1), and is the version we use in the proof of Theorem (1.1.18):

Proposition 3.5.2. *In the context of Theorem (3.2.1), for any $0 < r < 1$ there exists (possibly empty) collections of geodesic loops $E_i^\epsilon \subset \{\gamma : [0, 1] \to B_{M_i}(p_i, \epsilon) \mid \gamma(0) = \gamma(1) = p_i\}$ such that, for each i, the following holds:*

1. geodesics in E_i^ϵ are essential, minimizing on each half of $[0, 1]$, pairwise non-homotopic and intersecting only at p_i

2. The group elements $[\gamma] \in \pi(B_{M_i}(p_i, \epsilon)_{closed}, p_i)$ corresponding to $\gamma \in E_i^\epsilon$ generate $\pi(B_{M_i}(p_i, \epsilon)_{closed}, p_i)$

3. $|E_i^\epsilon| < Z_\chi$ for some Z_χ which depends only on h.

Assuming the existence of the sets E_i^ϵ in Proposition (3.5.2), we continue by showing the following

Proposition 3.5.3. *In the context of Theorem (3.2.1), for all sufficiently small $\epsilon > 0$, $E_i^\epsilon \subset \{\gamma : [0,1] \to B_{M_i}(p_i, \epsilon) | \ \gamma(0) = \gamma(1) = p_i\}$ as in Proposition (3.5.1), there exists a subsequence, which we will also call E_i^ϵ, such that we can order the elements of E_i^ϵ as $\{\gamma_1^i, ..., \gamma_{n_i}^i\}$ for each i and the following 4 properties hold:*

1. *$n_i = \bar{n}$ for some $\bar{n}$ and all i*
2. *$\lim\limits_{i\to\infty} l(\gamma_k^i) = 2\lambda_k \geq 0$ for all $1 \leq k \leq \bar{n}$*
3. *for all $1 \leq k \leq \bar{n}$ such that $\lambda_k > 0$, there exists $q_k \in B(0,1)$ and geodesics $\gamma_k : [0,1] \to B(0,1)$ connecting 0 and q_k such that $\sup\limits_{t\in[0,\frac{1}{2}]} \bar{d}(\gamma_k^i(t), \gamma_k(2t)) \to 0$ and $\sup\limits_{t\in[\frac{1}{2},1]} \bar{d}(\gamma_k^i(t), \gamma_k(2-2t)) \to 0$*
4. *For all $\delta > 0$ sufficiently small, there exists i_δ such that for any γ_k^i with $i > i_\delta$, $\phi \circ \gamma_k^i$ has winding number 0 around any $x \in B(0,1)$ with $\inf\limits_{t\in[0,1]} |\phi\circ\gamma_k^i(t) - x| \geq \delta$*

Proof. Order the elements of E_i^ϵ as $\{\gamma_1^i, ..., \gamma_{n_i}^i\}$ (where for all i, $n_i \leq Z_\chi$). We can iteratively take sub-sequences in i until the first 3 properties hold. That we ultimately can achieve properties 1 and 2 is obvious, and given 1 and 2, 3 follows from Proposition (3.2.4) and the fact that geodesics in E_i^ϵ for any i are minimizing on $[0, 1/2]$ and $[1/2, 1]$.

Now, for $\delta > 0$, because of properties 2 and 3 in Lemma (3.3.1) of ϕ as well as 3 above, it follows that there is an i_δ such that for $i > i_\delta$, $\phi \circ \gamma_k^i([0,1]) \subset \{x \in \mathbb{R}^2 | d(x, \phi(\gamma_k)) < \delta\}$. The δ-neighborhoods of $\gamma_k([0,1])$ are of course simply connected, and so the curves $\phi \circ \gamma_k^i$ can each be homotoped to a point, with these homotopies occuring completely within the sets $\{x \in \mathbb{R}^2 | d(x, \phi(\gamma_k)) < \delta\}$. This implies that there exists homotopies $F^i : [0,1] \times [0,1] \to \mathbb{R}^2$ such that $F^i(0,t) = \phi(\gamma_k^i(t))$ and $F^i(1,t) \equiv 0$ and F^i satisfies

$$F^i([0,1] \times [0,1]) \subset \{x \in \mathbb{R}^2 \mid d(x, \phi(\gamma_k)) < \delta\}$$

In particular, if $x \in \{x \in \mathbb{R}^2 | d(x, \phi(\gamma_k)) < \delta\}^c$, for $i > i_\delta$ the winding number of any curve $\phi \circ \gamma_k^i$ about x is 0. □

Property 4 of Proposition (3.5.3) immediately implies that, for fixed $i > i_\delta$ and any composition $\alpha_1\gamma_{k_1}^i \cdot ... \cdot \alpha_j\gamma_{k_j}^i$, there is a homotopy $F : [0,1] \times [0,1] \to \mathbb{R}^2$ such that $F(0,t) = \phi(\alpha_1\gamma_{k_1}^i \cdot ... \cdot \alpha_j\gamma_{k_j}^i(t))$ and $F(1,t) \equiv 0$ and F satisfies

$$F([0,1]\times[0,1])\subset S:=\bigcup_k\{x\in\mathbb{R}^2|d(x,\phi(\gamma_k))<\delta\}$$

and therefore that the winding number of any such composition about x is zero for all $x\in S^c$.

We now prove Theorem (3.2.1), under the assumption of proposition (3.5.1):

Proof of Theorem (3.2.1). We show that, for any subsequence of the balls B_{M_i}, there exists a further subsequence satisfying

$$\lim_{i\to\infty}\mathrm{Vol}_i B_{M_i}(p_i,1)\geq \mathrm{Vol}B(0,1),$$

which implies the theorem.

Choose $\epsilon>0$ satisfying Proposition (3.5.3). Fix $0<\epsilon_1<1$ and choose m large enough so that the points $\{x_1,...,x_n\}\subset B(0,1)$ with coordinates integer multiples of $1/m$ are ϵ^2-dense in $B(0,1)$. Define $\{x_1^i,...,x_n^i\}\subset B_{M_i}(p_i,1)$ as in Lemma (3.2.3). For any $\epsilon_2>2\epsilon^2$, there exists $\{x_{n_1},...,x_{n_k}\}\subset\{x_1,...,x_n\}$ such that each of the following hold:

1. each element x_{n_i} is within ϵ_2 of the circle $\{|x|=\epsilon_1\}$
2. no two distinct elements x_{n_i} and x_{n_j} lie on the same ray through the origin
3. $|x_{n_i}-x_{n_{i-1}}|<\epsilon_2$

We can furthermore assume the points are arranged in counterclockwise fashion. Therefore, we can connect x_{n_i} to $x_{n_{i+1}}$ with a minimizing geodesic, so that the composition of these geodesics forms a polygonal curve $P_{\epsilon_1}:[0,1]\to B(0,1)$ with winding number 1 about any point in its interior. Because of property 2 of Lemma (3.3.1) describing ϕ, if ϵ_1 has been chosen sufficiently small, P_{ϵ_1} defined this way does not intersect $B(0,\epsilon_1/2)$. We can form corresponding curves $P^i_{\epsilon_1}:[0,1]\to B_{M_i}$ formed from the points $\{x^i_{n_1},...,x^i_{n_k}\}$ and minimizing geodesics connecting them. Then by property 3 of Lemma (3.3.1), it follows, again if ϵ_1 has been chosen sufficiently small, that we can pick i_{ϵ_1} so that $\phi\circ P^i_{\epsilon_1}$ also does not intersect $x\in B(0,\epsilon_1/2)$ for $i>i_{\epsilon_1}$. By this fact and the closeness of $P^i_{\epsilon_1}$ to P_{ϵ_1}, it follows that, for $i>i_{\epsilon_1}$, $\phi\circ P^i_{\epsilon_1}$ winds around any point $x\in B(0,\epsilon_1/2)$ exactly once as well.

Because of the properties of the sets $\{x_1^i,....x_n^i\}$ given in Lemma (3.2.3) and the fact that $P_{\epsilon_1}([0,1])\subset B(0,\epsilon_1+\epsilon_2)$, it follows that, for any $\epsilon_3>0$ and i large enough, $P^i_{\epsilon_1}([0,1])\subset B_{M_i}(p_i,\epsilon_1+\epsilon_2+\epsilon_3)$.

Now, let $\bar{\epsilon}:=\epsilon_1+\epsilon_2+\epsilon_3$. Consider the collections $E_i^{\bar{\epsilon}}$ of Proposition (3.5.2). Fix each $B_{M_i}(p,\bar{\epsilon})_{closed}$ with ϕ' restricting to ϕ such that

$$\phi'(B_{M_i}(p,\bar{\epsilon})\backslash B_{M_i}(p,\bar{\epsilon})_{closed})\subset B(0,\frac{\bar{\epsilon}}{2})^c.$$

Because of property 2 in Proposition (3.5.2), for each i, $P^i_{\epsilon_1}$ is freely homotopic within $B_{M_i}(p_i,\bar{\epsilon})_{closed}$ to a curve of the form $\alpha_1\gamma^i_{k_1}\cdot...\cdot\alpha_j\gamma^i_{k_j}$, where $\gamma^i_k,...,\gamma^i_j\in E_i^{\bar{\epsilon}}$. This induces a homotopy of $\phi\circ P^i_{\bar{\epsilon}}$ and $\phi(\alpha_1\gamma^i_{k_1}\cdot...\cdot\alpha_j\gamma^i_{k_j})$. But for $i>i_\delta$, for

$x \in B(0, \bar{\epsilon}/2) \cap S^c$ we have shown that the winding number about x of these two induced curves is different, and therefore that there exists $y \in B_{M_i}(p_i, \bar{\epsilon})_{closed}$ with $\phi'(y) = x$. But by the choice of ϕ', we must have $y \in B_{M_i}(p_i, \bar{\epsilon})$.

We have shown that

$$B(0, \bar{\epsilon}/2) \cap S^c \subset \phi(B_{M_i}(p_i, \bar{\epsilon})).$$

Now, express

$$B(0, \bar{\epsilon}/2) = (B(0, \bar{\epsilon}/2) \cap S) \bigcup (B(0, \bar{\epsilon}/2) \cap S^c).$$

Because of property 3 in Proposition (3.5.2), S can be covered by balls of radius $\sqrt{2}\delta$ with total area less than $\sqrt{2}\delta(l(\gamma_1) + ... + l(\gamma_k)) \leq \sqrt{2}\delta(Z_\chi(2\bar{\epsilon})) \underset{\delta \to 0}{\to} 0$. This shows that

$$\lim_{i \to \infty} \mathrm{Vol}(\phi(B_{M_i}(p_i, \bar{\epsilon}))) \geq \mathrm{Vol}B(0, \bar{\epsilon}/2).$$

But note that, by property 2 of Lemma (3.3.1), we in fact have

$$\lim_{i \to \infty} \mathrm{Vol}(\phi(B_{M_i}(p_i, \bar{\epsilon}/2 + C(\bar{\epsilon}/2)^2))) \geq \mathrm{Vol}B(0, \bar{\epsilon}/2).$$

Now, ϵ was chosen arbitrarily, and then $\epsilon_2 > 2\epsilon^2$ was chosen with only this dependence. Furthermore, ϵ_1 and ϵ_3 may be chosen arbitrarily small. Ultimately we can therefore choose each constant as small as necessary to obtain that, for small enough $\bar{\epsilon} > 0$,

$$\lim_{i \to \infty} \mathrm{Vol}(\phi(B_{M_i}(p_i, \bar{\epsilon}))) \geq \mathrm{Vol}B(0, (1 - \epsilon')\bar{\epsilon}) = (1 - \epsilon')^2 \mathrm{Vol}B(0, \bar{\epsilon})$$

for some $\epsilon' > 0$ which we can make arbitrarily small by choosing ϵ small enough. Because $\mathrm{Vol}_i(B_{M_i}(p_i, \epsilon)) \geq \mathrm{Vol}(\phi(B_{M_i}(p_i, \epsilon)))$, we have

$$\lim_{i \to \infty} \mathrm{Vol}_i(B_{M_i}(p_i, \epsilon)) \geq (1 - \epsilon')^2 \mathrm{Vol}B(0, \epsilon).$$

Now, for any sequence $p_i' \in B_{M_i}(p_i, 1)$ converging with respect to $\bar{d}$ to $x \in B(0, 1)$, we can argue similarly to obtain

$$\lim_{i \to \infty} \mathrm{Vol}_i(B_{M_i}(p_i, \epsilon(x))) \geq (1 - \epsilon')^2 \mathrm{Vol}B(x, \epsilon(x))$$

for some continuous function $\epsilon : B(0, 1) \to \mathbb{R}_+$. Now, it is only possible that $\lim\limits_{i \to \infty} \epsilon(x_i) = 0$ if x_i approach the boundary of $B(0, 1)$. We therefore obtain,

Lemma 3.5.4. *Let $0 < \epsilon' < 1$ be given. For any $0 < \delta < 1$, there exists $\epsilon > 0$ such that*

$$\lim_{i \to \infty} \mathrm{Vol}_i(B_{M_i}(p_i, \epsilon)) \geq (1 - \epsilon')^2 \mathrm{Vol}B(x, \epsilon)$$

whenever $p_i \to x \in B(0, 1 - \delta)$.

Now, it follows from Lemma (3.2.3) that, if $B(x_1,\epsilon), B(x_2,\epsilon) \subset B(0,1-\delta)$ are disjoint, and $p_i^1 \to x_1$, $p_i^2 \to x_2$, then $B_{M_i}(p_i^1,\epsilon)$ and $B_{M_i}(p_i^2,\epsilon)$ are disjoint for all but finitely many i. Thus, if disjoint balls $B(x_j,\epsilon) \subset B(0,1-\delta)$ satisfy

$$\mathrm{Vol}(\bigcup_j B(x_j,\epsilon)) \geq c\mathrm{Vol}(B(0,1-\delta)) \tag{3.5.1}$$

for some $c < 1$ and we have sequences $p_i^j \to x_i$, then by Lemma (3.5.4)

$$\lim_{i\to\infty} \mathrm{Vol}_i(\bigcup_j B(x_j,\epsilon)) \geq (1-\epsilon')^2 c\mathrm{Vol}(B(0,1-\delta)).$$

Since, by the Vitali covering lemma, we can achieve (3.5.1) for any $c < 1$, and since ϵ' can be chosen arbitrarily small, it follows that

$$\lim_{i\to\infty} \mathrm{Vol}_i(B_{M_i}(p_i,1)) \geq \mathrm{Vol}B(0,1-\delta).$$

Taking $\delta \to 0$ finishes the proof. $\square$

3.6 Existence of the sets E^r

First, assume M is bounded, so that there is r large enough that $B_M(p,r) = M$, and so that every essential curve is automatically nonperipheral. Let $(\tilde{M},\pi)$ be the universal cover of M with projection map $\pi : \tilde{M} \to M$. We endow $\tilde{M}$ with the pullback metric $\tilde{g} := \pi^* g$ and define

$$F_\epsilon := \{x \in \tilde{M} | \inf_{\tilde{p}\in\pi^{-1}(p)} \tilde{d}(x,\tilde{p}) \leq \epsilon\}.$$

Note that for $\tilde{p}_1, \tilde{p}_2 \in \pi^{-1}(p)$, $B_{\tilde{M}}(\tilde{p}_1,\epsilon) \cap B_{\tilde{M}}(\tilde{p}_2,\epsilon) \neq \emptyset$ if and only if $\tilde{p}_1 = \tilde{p}_2$ or there exists a minimizing geodesic $\tilde{\gamma}$ in $\tilde{M}$ connecting $\tilde{p}_1$ to $\tilde{p}_2$ with $l(\tilde{\gamma}) \leq 2\epsilon$. Say $\tilde{p} \sim_\epsilon \tilde{q}$ if there exists a finite sequence $\tilde{p}_1, ..., \tilde{p}_n$ such that $B_{\tilde{M}}(\tilde{p}_{i+1},\epsilon) \cap B_{\tilde{M}}(\tilde{p}_i,\epsilon) \neq \emptyset$, and $\tilde{p}_1 = \tilde{p}$, $\tilde{p}_n = \tilde{q}$. This is clearly an equivalence relation and is equivalent to the condition that $\tilde{p}$ and $\tilde{q}$ can be connected by a finite sequence of geodesics each of length less than or equal to 2ϵ each starting and ending on elements of $\pi^{-1}(p)$. It is also equivalent to the condition that $\tilde{p}$ and $\tilde{q}$ are in the same component of F_ϵ:

Lemma 3.6.1. *For $\tilde{p}, \tilde{q} \in \pi^{-1}(p)$, $\tilde{p} \sim_\epsilon \tilde{q}$ if and only if they are in the same component of F_ϵ.*

Proof. Clearly, if $\tilde{p} \sim_\epsilon \tilde{q}$ then $\tilde{p}$ and $\tilde{q}$ are in the same component of F_ϵ. For the opposite direction, let $F_\epsilon(\tilde{p})$ denote the subset of F_ϵ determined by the equivalence $\sim_\epsilon \tilde{p}$, i.e.

$$F_\epsilon := \{x \in \tilde{M} | \inf_{\tilde{p}'\sim_\epsilon\tilde{p}} \tilde{d}(x,\tilde{p}') \leq \epsilon\},$$

and define $F_\epsilon(\tilde{q})$ similarly. If $F_\epsilon(\tilde{p}) \cap F_\epsilon(\tilde{q}) \neq \emptyset$, there exists $x \in F_\epsilon(\tilde{p}) \cap F_\epsilon(\tilde{q})$, $\tilde{p}' \in F_\epsilon(\tilde{p})$ and $\tilde{q}' \in F_\epsilon(\tilde{q})$ such that

$$\tilde{d}(x,\tilde{p}), \tilde{d}(x,\tilde{q}) < \epsilon.$$

Thus, $\tilde{p}' \sim_\epsilon \tilde{q}'$ and so $\tilde{p} \sim_\epsilon \tilde{q}$. Therefore, if $\tilde{p} \nsim_\epsilon \tilde{q}$, then these two points are elements of disjoint components of F_ϵ.

Now let

$$\epsilon_0 := \inf\{\epsilon \geq 0 \mid F_\epsilon \text{ is connected}\}.$$

Proposition 3.6.2. *F_{ϵ_0} is connected, so that for any $\tilde{p}, \tilde{q} \in \pi^{-1}(p)$, $\tilde{p} \sim_{\epsilon_0} \tilde{q}$, and $\epsilon_0 \leq \operatorname{diam}(M)$.*

Proof. $\operatorname{inj}(p) > 0$ implies that the distance function $\tilde{d}(\cdot, \cdot)$ on $\tilde{M} \times \tilde{M}$ restricted to $\pi^{-1}(p) \times \pi^{-1}(p)$ has discrete range. If this were not true, there would have to exist $\tilde{p}_i, \tilde{q}_i \in \pi^{-1}(p)$ with $\epsilon_i := \tilde{d}(\tilde{p}_i, \tilde{q}_i)$ satisfying $\epsilon_i \to \epsilon$ for some finite limit. Because any deck transformation of $\tilde{M}$ preserves distance, we can assume $\tilde{p}_i = p$ for all i. Then there would exist a finite radius ball $B_{\tilde{M}}(p, R)$ around $\tilde{p}$ such that each $\tilde{q}_i$ corresponds to a unique point of $\pi^{-1}(p) \bigcap B_{\tilde{M}}(\tilde{p}, R)$, which contradicts that $\pi^{-1}(p)$ is discrete. In the case of ϵ_0 this gives us that there exists $\delta > 0$ such that if some geodesic γ connects two points in $\pi^{-1}(p)$ and satisfies $|l(\gamma) - 2\epsilon_0| \leq 2\delta$, then $l(\gamma) = 2\epsilon_0$. Therefore, for any $\tilde{p}, \tilde{p}' \in \pi^{-1}(p)$, if $\tilde{p} \sim_{(\epsilon_0+\delta)} \tilde{p}'$ then $\tilde{p} \sim_{\epsilon_0} \tilde{p}'$. since $F_{\epsilon_0+\delta}$ is connected, this implies F_{ϵ_0} is connected.

For the second part, assume $\epsilon_0 > \operatorname{diam}(M) = D$ and choose $\tilde{\gamma}$ connecting them. If there exists t such that $d(\pi(\tilde{\gamma}(t)), p) > D$, then, choosing γ' a minimizing geodesic connecting p and $\pi(\tilde{\gamma}(t))$, the curves $(\gamma')^{-1} \cdot \gamma|_{[0,t]}$ and $\gamma|_{[t,1]} \cdot \gamma'$ each have shorter length than γ, and their composition equals γ in $\pi_1(M, p)$. This shows that $\tilde{p} \sim_{\epsilon_0} \tilde{q}$, and since $\tilde{p}, \tilde{q} \in \pi^{-1}(p)$ were arbitrary, this shows that ϵ_0 is not minimal. □

Proof of Proposition (3.5.1). We combine the preceding ideas to assert that there exist minimizing geodesics connecting elements of $\pi^{-1}(p)$ which have length exactly $2\epsilon_0$ and such that without them, F_{ϵ_0} is not connected. Because $F_{(\epsilon_0-\delta)}$ is not connected, there exists $\tilde{p}_1, \tilde{p}_2 \in \pi^{-1}(p)$ such that $\tilde{p}_1 \nsim_{(\epsilon_0-\delta)} \tilde{p}_2$, i.e. $\tilde{p}_1$ and $\tilde{p}_2$ cannot be connected by a finite sequence of minimizing geodesics of length less than or equal to $2(\epsilon_0 - \delta)$. But because F_{ϵ_0} is connected, and therefore $\tilde{p}_1 \sim_{\epsilon_0} \tilde{p}_2$, there does exist a finite sequence of minimizing geodesics connecting $\tilde{p}_1$ and $\tilde{p}_2$ of length less than or equal to $2\epsilon_0$, and there must be at least one minimizing geodesic γ_i in that collection satisfying $|l(\gamma_i) > 2(\epsilon_0 - \delta)$, and so $l(\gamma_i) = 2\epsilon_0$.

The Base Case

Our proof of Proposition (3.5.1) will proceed by induction, constructing a growing class of geodesic loops. We begin with an initial set defined as follows: Consider all geodesic loops γ in M based at p such that γ is the projection of some minimizing geodesic $\tilde{\gamma}$ connecting two points in $\pi^{-1}(p)$ satisfying $l(\tilde{\gamma}) = 2\epsilon_0$ and call this collection G^0. $|G^0|$ is finite, since otherwise we could lift each γ to a corresponding $\tilde{\gamma}$ with common base point $\tilde{p} \in \pi^{-1}(p)$, and again contradict that $\pi^{-1}(p)$ is discrete by considering the distinct end points of each lift.
Now define $L_\gamma \subset \tilde{M}$ as $L_\gamma := \pi^{-1}(\gamma([0, 1]))$, i.e. the union of the images of all possible lifts $\tilde{\gamma}$ of γ. We want to pick out some sub-collection $\gamma_0, ..., \gamma_n \in G^0$ having the following 2 properties:

1. there exists $\epsilon_1 < \epsilon_0$ such that $F_{\epsilon_1} \cup \bigcup_{1 \leq i \leq n} L_{\gamma_i}$ is connected

2. Any proper sub-collection $\gamma_{n_0}, ..., \gamma_{n_k} \in G^0$ does not satisfy 1.

Because any 2 points $\tilde{p}, \tilde{p}' \in \pi^{-1}(p)$ can be connected by minimizing geodesics of length either $2\epsilon_0$ or less than $2(\epsilon_0 - \delta)$, with δ defined above, condition 1 holds for the entire collection with $\epsilon_1 := \epsilon_0 - \delta$. Therefore condition 2 can be met by removing elements one at a time and asking whether 1 still holds. We now choose any subcollection satisfying 1 and 2 and call it $G^0 := \{\gamma_0, ..., \gamma_n\}$.

Note that 2 implies that no $[\gamma_i]$ corresponding to $\gamma_i \in G^0$ satisfies $[\gamma_i] = \sum_{j=1}^{n-1} a_j[\gamma_{\sigma(j)}]$ for $\gamma_j \in G^0$ and σ any permutation of the indices $\{1, ..., i-1, i+1, ..., n\}$, for if it did, then $[\gamma_i] = \sum_{j=1}^{n-1} a_j[\gamma_{\sigma(j)}]$ could be lifted to connect any two points in $\pi^{-1}(p)$ that are connected by $\tilde{\gamma_i}$, and so we could remove γ_i from G^0 and property 1 would still hold. Furthermore, by nearly the same logic, no $[\gamma_i]$ could be expressed as a finite combination of, say, $[\alpha_1], ..., [\alpha_n] \in \pi_1(M, p)$ represented by loops $\alpha_1, ..., \alpha_n$ such that for all $1 \le j \le n$, $l(\alpha_j) < 2\epsilon_0$, for the image $\tilde{\alpha}_i([0,1])$ of any $\tilde{\alpha}_i$ would have to satisfy $\tilde{\alpha}_i([0,1]) \subset F_{(\epsilon_0 - \delta)}$, which also shows that γ_i could be removed from G^0.

Recall that we parametrize any geodesic by $\gamma_i : [0,1] \to M$. We now show that each element of G^0 has the further property of being a minimizing geodesic when restricted either to $[0, 1/2]$ or $[1/2, 1]$.

Proposition 3.6.3. *For all $\gamma_i \in G^0$,*

$$l(\gamma_i|_{[0,\frac{1}{2}]}) = d(\gamma_i(0), \gamma_i(\frac{1}{2}))$$

and

$$l(\gamma_i|_{[1/2,1]}) = d(\gamma_i(1/2), \gamma_i(1)).$$

Proof. Assume $t \in (0, 1/2)$. For $t \in (1/2, 1)$ the argument is the same, and the point $\{1/2\}$ follows by continuity. If $l(\gamma_i|_{[0,t]}) > d(\gamma_i(0), \gamma_i(t))$, then there exists $\gamma_i' : [0,1] \to M$ satisfying $\gamma_i'(0) = \gamma_i(0)$, $\gamma_i'(1) = \gamma_i(t)$, and $l(\gamma_i') = d(\gamma_i(0), \gamma_i(t))$. Then the curves $(\gamma_i')^{-1} \cdot \gamma_i|_{[0,t]}$ and $\gamma_i|_{[t,1]} \cdot \gamma_i'$ each have shorter length than γ_i, and their composition equals γ_i in $\pi_1(M,p)$, which is not possible for $\gamma_i \in G^0$. $\square$

The Inductive Step

We proceed by induction to define sets G^{i+1} as follows. Assume $\{[\gamma] | \gamma \in G^i\}$ doesn't generate $\pi_1(M,p)$. Choose minimal $\epsilon_{i+1} > 0$ such that $F_{\epsilon_{i+1}} \cup \bigcup_{\gamma \in G^i} L_\gamma$ is connected. In the same way as above, there exists a set G^{i+1} of geodesic loops of length $2\epsilon_{i+1}$ such that its elements are pairwise non-homotopic and:

1. there exists $\epsilon_i + 1 < \epsilon_i$ such that $F_{\epsilon_i+1} \cup \bigcup_{j \le i} (\bigcup_{\gamma \in G^j} L_\gamma)$ is connected

2. Any proper sub-collection of G^i does not satisfy 1.

As in Proposition (3.6.3), at every step, the chosen geodesics are minimizing on each half of $[0,1]$. We now argue that this process terminates after finitely many steps.

Proposition 3.6.4. *There exists $\bar{i}$ such that the set $\{[\gamma] \mid \gamma \in G^0 \cup ... \cup G^{\bar{i}}\}$ generates $\pi_1(M,p)$.*

Proof. If for all $i \in \mathbb{N}$ this process did not terminate, we would have an infinite collection of pairwise non-homotopic geodesic loops of uniformly bounded length based at a common point, which could be lifted to $\tilde{M}$ and contradict the discreteness of $\pi^{-1}(p)$. Therefore, for some n the process terminates, which immediately implies $\pi^{-1}(p) \cup \bigcup_{j \leq n} (\bigcup_{\gamma \in G^j} L_\gamma)$ is connected. □

We now define

$$E^r := \bigcup_{1 \leq i \leq \bar{i}} G^i.$$

There is one last property of the elements of E^r that we require, described by the following proposition.

Proposition 3.6.5. *for any $\gamma_1 \neq \gamma_2 \in E^r$, γ_1 and γ_2 intersect once, at p.*

Proof. This follows from the fact that γ_1 and γ_2 can be lifted to minimizing geodesics in $\tilde{M}$ based at a common initial point, and such curves cannot intersect on their interiors. □

We have constructed a collection E^r of geodesic loops satisfying 1 and 2 of Proposition (3.5.1). We have furthermore shown that E^r is a collection of simple, pairwise non-homotopic, essential closed curves each intersecting any other at most once. The following is Theorem 1.4 in [20]:

Theorem 3.6.6. *The cardinality of a set E of nonperipheral, essential simple closed curves on a surface M of genus h and Euler characteristic χ that are pairwise non-homotopic and intersecting at most once is at most*

$$h(|\chi| + 1) + |\chi| - 1$$

where χ is the Euler characteristic of M.

This theorem guarantees that $|E^r| < Z_\chi$ for some constant Z_χ depending only on the Euler characteristic χ of M.

Until now, we have only constructed the set E^r corresponding to $B_M(p,r) = M$, but the proof is nearly identical for both unbounded M and arbitrary r, with the exception that, in general, $B_M(p,r)$ for r finite is not necessarily complete and our construction for the set F_{ϵ_0} above cannot be carried out directly. We remedy this by considering instead the sets $B_M(p,r)_{closed}$, fixed with any complete metric g' that restricts to g on $B_M(p,r)$. Then we note that any closed loop $\gamma : [0,1] \to B_M(p,r)_{closed}$ can be continuously deformed to a curve, which we will also denote γ, whose image lies entirely in $\bar{B}_M(p,r)$.

If $l(\gamma) > 2r$, we follow the argument in Proposition (3.6.2). There must be t such that $l(\gamma|_{[0,t]}) > r$. but then there exists $\gamma' : [0,1] \to M$ satisfying $\gamma'(0) = p$, $\gamma'(1) = \gamma(t)$, and $l(\gamma') = d(p, \gamma(t))$. Then the curves $(\gamma')^{-1} \cdot \gamma|_{[0,t]}$ and $\gamma|_{[t,1]} \cdot \gamma'$ each have shorter length than γ, but their composition equals γ in $\pi_1(M,p)$, which is not possible for $\gamma \in G^0$. This shows that any element of G^0 must be of length $2r$ or less, and therefore cannot intersect $B_M(p,r)_{closed} \backslash B_M(p,r) = \bar{B}_M(p,r)^c \subset B_M(p,r)_{closed}$, and we are done. □

Chapter 4

A Construction for the Development of Positive Curvature Under the Ricci Flow

The Ricci flow

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g) \tag{4.0.1}$$

is a quasilinear parabolic evolution of the metric g and, as in the discussion of the comparison of Ric to Δ above, we review here how this evolution behaves similarly to the standard heat equation. Again the distinction between *upper* and *lower* bounds is significant. We begin immediately with the evolution of curvature quantities with respect to (4.0.1):

For the curvature tensor

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{Rm}_{ijkl} &= \Delta\mathrm{Rm}_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &\quad - (\mathrm{Ric}_i^s\mathrm{Rm}_{sjkl}x + \mathrm{Ric}_j^s\mathrm{Rm}_{iskl} + \mathrm{Ric}_k^s\mathrm{Rm}_{ijsl} + \mathrm{Ric}_l^s\mathrm{Rm}_{ijks})\end{aligned} \tag{4.0.2}$$

The Ricci curvature

$$\frac{\partial}{\partial t}\mathrm{Ric}_{ij} = \Delta\mathrm{Ric}_{ij} + \mathrm{Rm}_{sjkt}\mathrm{Ric}^{st} - 2\mathrm{Ric}_i^s\mathrm{Ric}_{sj} \tag{4.0.3}$$

and the scalar curvature

$$\frac{\partial}{\partial t}\mathrm{R} = \Delta\mathrm{R} + 2|\mathrm{Ric}|^2 \tag{4.0.4}$$

(4.0.4) is particularly simple and in particular implies that

$$\frac{\partial}{\partial t}\mathrm{R} \geq \Delta\mathrm{R} + \frac{2}{n}\mathrm{R}^2$$

from which it can be shown by the scalar maximum principle that, if M is closed, R_0 denotes the scalar curvature at time $t = 0$ and $u : M \times [0, T] \to \mathbb{R}$ solves

$$\frac{\partial}{\partial t}u = \Delta u + \frac{2}{n}u^2 \tag{4.0.5}$$

$$u(\,\cdot\,,0) \le R_0$$

then $R \ge u$ for all $t \in [0,T]$. One implication of this principle is that lower bounds for the scalar curvature at time $t = 0$ extend to time dependent bounds. taking $u_K(\,\cdot\,,0) = K \in \mathbb{R}$ and solving (4.0.5) gives

$$u_K(\,\cdot\,,t) = \frac{K}{(1 - \frac{2Kt}{n})}$$

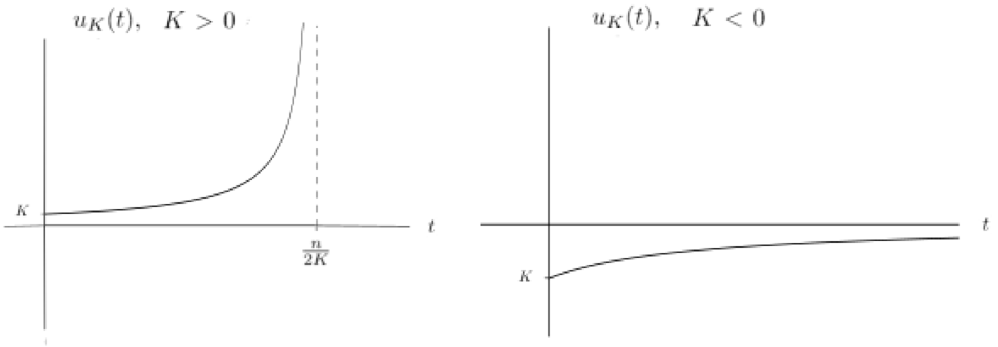


Figure 4.0.1: a positive lower bound Ric $\ge K > 0$ forces a finite time singularity at $t = \frac{n}{2K}$ and a negative lower bound Ric $\ge K < 0$ improves toward 0 as time progresses.

The images above show the behavior of solutions for $K > 0$ and $K < 0$. For $K = 0$ the solution is $u(\cdot, 0) \equiv 0$. Because the curvature of $g(t)$ is bounded from below by u_K. These bounds in particular indicate that positively curved metrics on closed manifolds develop arbitrarily positive scalar curvature everywhere as they evolve under the Ricci flow and that any (smooth) metric for which a solution to (4.0.1) exists for all $t > 0$ becomes arbitrarily almost non-negative in the sense that for any $\epsilon > 0$ there exists t_0 such that R$(t) > -\epsilon$ for all $t > t_0$.

It is well known that with respect to (4.0.1) a variety of curvature conditions are preserved, see [13], [14], [7], [3] for a number of examples. As in the case of the application of the maximum principle to (4.0.5) above these conditions often take the form of a lower bound on some curvature quantity.

A stronger example, due to Hamilton [13], states that the eigenvalues of the Ricci curvature of a metric of positive Ricci curvature on a 3-manifold do not only remain positive, but become pinched together in a certain sense under the Ricci flow:

Theorem 4.0.1. *(Hamilton) Assume $(M^3, g(t))$ satisfies the Ricci flow equation for $t \in [0,T)$, with $g(0) = g_0$. satisfying Ric$(g_0) \ge 0$. If λ, μ, ν denote the eigenvalues of* Ric*, Then there exist constants $C < \infty$ and $\delta > 0$ such that*

$$\frac{1}{3}((\lambda - \mu)^2 + (\mu - \nu)^2 + (\nu - \lambda)^2) \le CR^{2-\delta}$$

In particular, wherever the scalar curvature blows up, after renormalizing by the factor R^2 the eigenvalues must converge to a common limit. In [13] Hamilton uses this result to show that compact manifolds admitting a metric of positive Ricci curvature also admit a metric of constant positive curvature.

On the other hand, upper bounds do not tend to be preserved in general, as follows for instance from the lower bound on the scalar curvature above which implies that the scalar curvature of any compact manifold of strictly positive curvature will blow up in finite time. With this in mind we show the following

Proposition 4.0.2. *There exist compact Riemannian manifolds with negative sectional curvature which develop a positive sectional curvature at some point under the Ricci flow.*

4.1 Motivating Example

We first examine Riemannian manifolds (M^3, g_s) where $M \cong \mathbb{R}^3$ and g_s are expressed in cylindrical coordinates as follows:

$$g_s = dr^2 + \phi^2(r, z, s)d\theta^2 + \cosh^2(r)dz^2$$

Here $\phi : \mathbb{R}_+ \times \mathbb{R} \times (-s_0, s_0) \to \mathbb{R}$ is a smooth function that, for $s = 0$, we will use to interpolate between two metrics in the following way: we assume there are rectangles $C(r_0, z_0) := \{(r, z)|\ 0 \leq r < r_0, -z_0 < z < z_0\}$ and $C(R_0, Z_0)$ defined similarly for $R_0 > r_0$ and $Z_0 > z_0$ such that we have $\phi(r, z, 0) = r$ within $C(r_0, z_0)$ and $\phi(r, z, 0) = \sinh(r)$ outside of $C(R_0, Z_0)$.

We furthermore impose the condition $\phi(r, z, s) = \sinh(r)$ everywhere outside of $C(R_0, Z_0) \times (-s_0, s_0)$, and otherwise for $s \neq 0$ our only goal is that g_s has negative sectional curvature. Observe that if, for some s, over all of M we have $\phi(r, z, s) = \sinh(r)$ then $(M, g_s) = \mathbb{H}^3 = (M, g_{\mathbb{H}^3})$, where

$$g_{\mathbb{H}^3} = dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)dz^2$$

i.e. we recover hyperbolic space. This implies that we will be able to glue our interpolation metrics g_s into appropriate compact hyperbolic manifolds.

We identify restrictions on ϕ to ensure the desired behavior below, but first we examine the behavior of g_0 within $C(r_0, z_0)$.

4.2 the metric $\tilde{g}$

The Riemann curvature of the metric

$$\tilde{g} = dr^2 + r^2d\theta^2 + \cosh^2(r)dz^2$$

is diagonalized in these coordinates and is described by the matrix

$$\mathrm{Rm} = \mathrm{Rm}_{\tilde{g}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{\sinh(r)}{r\cosh(r)} \end{bmatrix}. \tag{1}$$

This can be shown using the Christoffel symbols in these coordinates [24], where

$$\tilde{g}_{rr} = 1,\ \tilde{g}_{\theta\theta} = r^2,\ \tilde{g}_{zz} = \cosh^2(r)$$

and $\tilde{g}_{ij} = 0$ otherwise. Because the functions $\tilde{g}_{ij}$ depend only on r, these formulas give $\Gamma^k_{ij} = 0$ whenever r does not appear as an index. Furthermore, because $\tilde{g}_{ij}$ is diagonal, $\Gamma^k_{ij} = 0$ whenever $i,\ j$ and k are distinct. Otherwise we have

$$\begin{aligned}
\Gamma^k_{rr} &= 0 \text{ for all } k \in \{r, \theta, z\} \\
\Gamma^r_{\theta\theta} &= -r \\
\Gamma^r_{zz} &= -\cosh(r)\sinh(r) \\
\Gamma^\theta_{r\theta} &= \frac{1}{r} \\
\Gamma^z_{rz} &= \frac{\sinh(r)}{\cosh(r)}.
\end{aligned}$$

Because of the symmetries of the Christoffel symbols, this information fully determines Γ. We then apply the formula for the Riemmann curvature in terms of the Christoffel symbols

$$\mathrm{Rm}^l_{ijk} = \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js} + \Gamma^l_{jk,i} - \Gamma^l_{ik,j}.$$

Observe first that if r occurs only once as an index, then every term in the expression above must vanish. If instead it occurs twice, with θ and z each occurring once, the same is true. Otherwise we have

$$\begin{aligned}
\mathrm{Rm}^r_{r\theta\theta} &= (-r)(0) - (\frac{1}{r})(-r) + (-1) - (0) = 0 \\
\mathrm{Rm}^r_{rzz} &= (-\cosh(r))(0) - (\frac{\sinh(r)}{\cosh(r)})(-\cosh(r)\sinh(r)) + (-\sinh^2(r) - \cosh^2(r)) - (0) = -\cosh^2(\\
\mathrm{Rm}^z_{z\theta\theta} &= (-r)(\frac{\sinh(r)}{\cosh(r)}) - (0)(0) + 0 - 0 = -\frac{r\sinh(r)}{\cosh(r)}.
\end{aligned}$$

Raising the last lower index in each of the above expressions then gives (1).

Observe also that for any $v \in \mathbb{R}^3$

$$\begin{aligned}
(\nabla_v \mathrm{Rm})_{r\theta\theta r} = D_v(&\mathrm{Rm}_{r\theta\theta r}) + 2v^r(\mathrm{Rm}_{r\theta\theta r}\Gamma^\theta_{r\theta}) \\
&+ 2v^\theta(\mathrm{Rm}_{r\theta\theta r}\Gamma^\theta_{\theta r}) + 2v^z(\mathrm{Rm}_{r\theta\theta r}\Gamma^r_{zr}) \\
= 0 &
\end{aligned} \tag{2}$$

Strictly speaking this calculation only holds away from the origin, but by continuity it in fact holds over all of $\mathbb{R}^3$. Re-expressing our metric in Euclidian coordinates as

$$\tilde{g} = dx^2 + dy^2 + \cosh^2(\sqrt{x^2 + y^2})dz^2,$$

it is immediate that in these coordinates the Christoffel symbols vanish at the origin. Therefore, combining with (2),

$$(\Delta \mathrm{Rm})_{xyyx}(0, 0, 0) = 0$$

This is equivalent to the corresponding statement in our original coordinates that

$$(\Delta \mathrm{Rm})_{r\theta\theta r}(0,0,0) = 0.$$

Now, we are primarily interested in what happens when we evolve the metrics g_s by the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric(g).$$

Under this evolution, if the Riemannian curvature at a point for a given metric is expressed in coordinates as

$$\mathrm{Rm} = \begin{bmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{bmatrix}$$

then whenever we also have

$$\Delta \mathrm{Rm} = \begin{bmatrix} 0 & \cdots & \\ \vdots & \ddots & \\ & & \end{bmatrix}$$

then we have the following

$$\frac{d}{dt}\kappa_1 = 2(\kappa_1^2 + \kappa_2\kappa_3)$$

(see [13]). In our case, if we let $\sec_t(\cdot,\cdot)$ denote the sectional curvature of the metric $\tilde{g}$ at time t this implies that at the origin we have

$$\left.\frac{d}{dt}\right|_{t=0} \sec_t(\partial r, \partial\theta) = 2(-1)(-1) = 2$$

and in particular we see that $\tilde{g}$ instantly develops a positive sectional curvature at a point despite being of non-positive sectional curvature on M at $t = 0$. It is this behavior in $\tilde{g}$ that we wish to preserve in our metric g_0.

4.3 Conditions on the metrics g_s

Referring again to the Christoffel symbols, we calculate the curvature for the metrics g_s. Similarly to $\tilde{g}$ we have the nonzero terms

$$\begin{aligned}
\Gamma^r_{\theta\theta} &= -\phi\phi_r \\
\Gamma^r_{zz} &= -\cosh(r)\sinh(r) \\
\Gamma^\theta_{r\theta} &= \frac{\phi_r}{\phi} \\
\Gamma^z_{rz} &= \frac{\sinh(r)}{\cosh(r)}
\end{aligned}$$

and because ϕ may depend on z we also have

$$\Gamma^z_{\theta\theta} = -\frac{\phi\phi_z}{\cosh^2(r)}$$
$$\Gamma^\theta_{z\theta} = \frac{\phi_z}{\phi}.$$

All terms that are not determined by the symmetries of the Christoffel symbols from the values above are equal to zero. For the curvatures, firstly we have

$$\mathrm{Rm}^r_{r\theta\theta} = (\phi\phi_r)(0) - (\frac{\phi_r}{\phi})(-\phi\phi_r) + (-\phi_r^2 - \phi\phi_{rr}) - (0) = -\phi\phi_{rr}$$
$$\mathrm{Rm}^r_{rzz} = (-\cosh(r))(0) - (\frac{\sinh(r)}{\cosh(r)})(-\cosh(r)\sinh(r)) + (-\sinh^2(r) - \cosh^2(r)) - (0) = -\cosh^2($$
$$\mathrm{Rm}^z_{z\theta\theta} = (-\phi\phi_r)(\frac{\sinh(r)}{\cosh(r)}) - (\frac{\phi_z}{\phi})(-\frac{\phi\phi_z}{\cosh^2(r)}) + (\frac{\phi_z^2 - \phi\phi_{zz}}{\cosh^2(r)}) - 0 = -\frac{\phi\phi_r\sinh(r)}{\cosh(r)} - \frac{\phi\phi_{zz}}{\cosh^2(r)}$$

and for the remaining curvature terms, we have

$$\begin{aligned}
\mathrm{Rm}^r_{r\theta z} &= 0 \\
\mathrm{Rm}^z_{r\theta\theta} &= (-\frac{\phi\phi_z}{\cosh^2(r)})(\frac{\sinh(r)}{\cosh(r)}) - (\frac{\phi_r}{\phi})(-\frac{\phi\phi_z}{\cosh^2(r)}) \\
&\qquad + (\frac{\phi_r\phi_z + \phi\phi_{rz}}{\cosh^2(r)} + \frac{2\sinh(r)\phi\phi_z}{\cosh^3(r)}) - 0 \\
&= \frac{\phi\phi_z\sinh(r) + \phi\phi_{rz}\cosh(r)}{\cosh^3(r)} \\
\mathrm{Rm}^z_{rz\theta} &= (\frac{\phi_z}{\phi})(0) - (\frac{\phi_r}{\phi})(0) + 0 - 0 = 0.
\end{aligned}$$

We again express Rm with respect to an orthonormal basis by normalizing each term to give

$$\mathrm{Rm}_{g_s} = \begin{bmatrix} -\frac{\phi_{rr}}{\phi} & 0 & \frac{\phi_z\sinh(r)+\phi_{rz}\cosh(r)}{\phi\cosh^2(r)} \\ 0 & -1 & 0 \\ \frac{-\phi_z\sinh(r)+\phi_{rz}\cosh(r)}{\phi\cosh^2(r)} & 0 & -\frac{\phi_r\sinh(r)}{\phi\cosh(r)} - \frac{\phi_{zz}}{\phi\cosh^2(r)} \end{bmatrix},$$

which we put into the more digestible form

$$\mathrm{Rm}_{g_s} = \begin{bmatrix} f_1 & 0 & \epsilon \\ 0 & -1 & 0 \\ \epsilon & 0 & f_2 \end{bmatrix},$$

and we note, if f_1 and f_2 are negative functions, that the eigenvalues of this matrix are nonpositive when

$$f_1 f_2 - \epsilon^2 \geq 0.$$

Thus, ϕ must determine f_1, f_2, and ϵ such that, on $C(R_{,0}, Z_0)$, we have that $f_1, f_2 \leq 0$ and $f_1 f_2 - \epsilon^2 \geq 0$ for g_s to have nonpositive sectional curvature, and to force negative sectional curvature these inequalities must be made strict.

4.4 a construction for ϕ

We first take ϕ of the form

$$\phi(r,z) = \psi(r) + a(z)(\sinh(r) - \psi(r))$$

for smooth one variable functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ and $a : \mathbb{R} \to [0,1]$. We assume $a(z) = 0$ for $|z| < z_0$ and $a(z) = 1$ for $|z| > Z_0$, for some constants $0 < z_0 < Z_0$. The function ψ will model how we interpolate between r and $\sinh(r)$, so we impose the following:

$$\begin{gathered} \psi(r) \leq \sinh(r), \quad \psi''(r) \geq 0 \\ \psi(r_0) = r \text{ for } r < r_0 \\ \psi(r) = \sinh(r) \text{ for } r > R_0. \end{gathered} \tag{3}$$

We deconstruct the functions

$$f_1(r,z) = -\frac{\phi_{rr}}{\phi}$$

$$f_2(r,z) = -\frac{\phi_r(r,z)\sinh(r)}{\phi(r,z)\cosh(r)} - \frac{\phi_{zz}}{\phi(r,z)\cosh^2(r)}$$

$$\epsilon(r,z) = \frac{\phi_z(r,z)\sinh(r) + \phi_{rz}(r,z)\cosh(r)}{\phi(r,z)\cosh^2(r)}$$

by first noting that, on $C(R_0, Z_0)$, we have

$$0 < c_1 < \frac{\sinh(r)}{\psi(r)} < C_1$$

$$0 \leq \sinh(r) - \psi(r) < C_2 r^2$$

$$0 \leq \cosh(r) - \psi'(r) < C_3 r^2$$

$$0 < c_4 < \frac{\sinh(r) - \psi(r)}{\cosh(r) - \psi'(r)} < C_4$$

for suitable constants $c_1, C_1, C_2, C_3, c_4, C_4$ depending on ψ. Thus, because the derivatives of ϕ take the form

$$\phi_r(r,z) = \psi'(r) + a(z)(\cosh(r) - \psi'(r))$$

$$\phi_z(r,z) = a'(z)(\sinh(r) - \psi(r))$$

$$\phi_{rz}(r,z) = \psi'(r) + a'(z)(\cosh(r) - \psi'(r))$$

$$\phi_{rr}(r,z) = \psi''(r) + a(z)(\sinh(r) - \psi''(r))$$

$$\phi_{zz}(r,z) = a''(z)(\sinh(r) - \psi(r))$$

we have

$$0 \leq k_1 a(z) < |f_1(r,z)|$$

$$0 < k_2 - k_3 a''(z) < |f_2(r,z)|$$

$$0 \leq |\epsilon(r,z)| < K_4 a'(z) r^2$$

for suitable constants k_1, k_2, k_3, K_4. Now, by rescaling ϕ so that

$$\phi(r,z) = \psi(r) + a(\xi z)(\sinh(r) - \psi(r))$$

for some small $\xi > 0$, we can make the terms $|a'(z)|$ and $|a''(z)|$ arbitrarily small, and we can therefore achieve the inequalities

$$0 \leq k_1 a(z) < |f_1(r,z)|$$

$$0 < \frac{k_2}{2} < |f_2(r,z)|$$

$$0 \leq |\epsilon(r,z)| < \sqrt{\frac{k_1 k_2}{2}} |a'(z)|.$$

Thus, if for all z we have $|a'(z)|^2 < a(z)$, we arrive at the inequality

$$f_1(r,z) f_2(r,z) - \epsilon^2(r,z) \geq 0.$$

Because the only conditions on ψ and a are those given in (3) and the added condition $|a'(z)|^2 < a(z)$, all of which are easily achieved, we see that there exist ϕ such that the corresponding metric g_0 has nonpositive sectional curvature. We finally observe that, had we imposed the strict condition $\psi''(r) > 0$, then we could have similarly achieved a negative sectional curvature condition, and so we can set

$$\phi(r,z,s) = \psi(r,s) + a(z)(\sinh(r) - \psi(r,s))$$

where $\psi(\cdot, s)$ are a smooth family of functions satisfying $\psi(r,0) = \psi(r)$ and otherwise satisfying $\psi_{rr}(r,s) > 0$, and the resulting family of metrics g_s have negative sectional curvature away from $s = 0$.

4.5 Behavior of g_s under the Ricci flow

If $g_s(t)$ are solutions to the Ricci flow with initial condition g_s, then wherever these solutions are defined we have that

$$\lim_{s \to 0} g_s(t) = g_0(t)$$

smoothly and thus, if $\mathrm{Rm}(s,t)$ denotes the curvature tensor for $g_s(t)$, that

$$\lim_{s \to 0} \mathrm{Rm}(s,t) = \mathrm{Rm}(0,t).$$

This implies that for small s, g_s must also develop positive curvature under the Ricci flow. We now give the proof of proposition 0.1.

Proof. Our chosen metrics g_s all have the property that the identity map restricted to $C(R_0, Z_0)^c$ gives an isometry to the complement of a cylinder in $\mathbb{H}^3$. Furthermore, for any $R > 0$ there exists a compact hyperbolic Riemannian manifold (N, h), a point $p \in N$ and a local isometry

$$\pi : \mathbb{H}^3 \to N$$

such that the restriction of π to $B = B(\pi^{-1}(p), R) \subset \mathbb{H}^3$ is an isometry (see for instance [22], chapter 11). We can also assume that $\pi|_B^{-1}(p) = \vec{0}$ in our chosen coordinates above. Noting that the metrics g_s and $g_{\mathbb{H}^3}$ are defined over the same underlying manifold, the map

$$\pi|_B : M \to N$$

pushes forward the metrics g_s to metrics

$$g'_s := (\pi|_B)_* g_s$$

defined on the image of $\pi|_B$. If R has been chosen sufficiently large, these metrics are all identical to h in a neighborhood of the boundary of $\pi(B)$ and so the metrics g'_s immediately extend to smooth metrics defined on all of N by the equality $g'_s = h$ outside of the image of $\pi|B$. □

4.6 General Setting

In this section we assume we are only given a non-positively curved Riemannian manifold (M^n, g) that is locally isometric to $\mathbb{H}^n$ outside of a compact set but for which we are not explicitly given an embedding into $\mathbb{H}^n$ from this region, as we have in the example above (the inclusion $\mathbb{I}$). We show that we can replicate the above construction by finding such an embedding.

Proposition 4.6.1. *For any complete simply connected n-dimensional Riemannian manifold (M, g) with $n \geq 3$ satisfying sec ≤ 0 and for which there exists $K \subset\subset M$ with such that $M \backslash K$ is locally isometric to $\mathbb{H}^n$, there exists a (noncompact) Riemannian manifold (N, h) and a compact set $B \subset N$ such that B is isometric to a ball in (M, g) containing K and B^c is isometric to the complement of a compact set in $\mathbb{H}^n$.*

Proof. Fix $x \in M$. Because (M, g) satisfies $\sec(\cdot, \cdot) \equiv -1$ outside of a compact set, there exists $l > 0$ for which

$$(B(x, l)^c, g)$$

is hyperbolic, Were $B(x, l)$ denotes the ball of radius l centered at x. Furthermore, the condition sec ≤ 0 implies that for any $p \in M$

$$\exp_P : T_pM \to M$$

is a diffeomorphism and therefore that any metric ball entirely contained in $B(x, l)^c$ is isometric to a ball in $\mathbb{H}^n$. Thus, by increasing if necessary we can choose l such that there exists $r > 0$ such that for all $p \in B(x, l)^c$, $B(p, r)$ is isometric to a ball in $\mathbb{H}^n$. Because $B(x, l)^c$ is also simply connected $(n \geq 3)$, this implies that there exists a local isometry

$$\phi : B(x,l)^c \to \mathbb{H}^n$$

We fix a point $p \in B(x,l)^c$.

claim 1. *For all sufficiently large $R_1 > 0$ there exists $\overline{R}_1 > 0$ such that the image of ϕ restricted to $B(p,R_1)^c$ contains $B(\phi(p),\overline{R_1})^c$.*

Proof. Let $R_1 > 0$ such that $B(p,R_1)^c \subset B(x,l)^c$. Let d' be the induced distance defined as the infimum of lengths of curves lying entirely in $B(p,l)^c$. Choose $\overline{R_1} := \max\{d'(q,p) \mid q \in \partial B(p,R_1)^c)\}$. Let x in $B(\phi(p),\overline{R_1})^c$ be given. choose $p' \in B(p,R_1)^c$ satisfying $d_{\mathbb{H}^n}(\phi(p),\phi(p')) = d_H(p,p') > \overline{R}_1$. This can be done by considering the image of a geodesic ray based at p and contained in $B(p,l)^c$, since local isometries map geodesics to geodesics and any geodesic in $\mathbb{H}^n$ is minimizing.

Now choose a curve $\gamma : [0,1] \to \mathbb{H}^n$ with $\gamma(0) = \phi(p')$, $\gamma(1) = x$ and $d_{\mathbb{H}^n}(\gamma(t),\phi(p)) > \overline{R_1}$ for all $t \in [0,1]$. Because ϕ is invertible on $B(p',r)$, for some $\epsilon_1 > 0$ we can lift γ uniquely to a curve

$$\phi^{-1} \cdot \gamma : [0,\epsilon_1) \to B(p',r)$$

Because ϕ is distance non-increasing with respect to d', if $\phi^{-1} \circ \gamma(t) \in \partial B(p,r_1)$ for some t, then $\overline{R}_1 < d_{\mathbb{H}^n}(\phi(p),\gamma(t)) \leq d'(p,\phi^{-1} \circ \gamma(t)) \leq \overline{R}_1$. Thus every point on this curve must lie within $B(p,R_1)^c$. Thus for some $\epsilon_2 > \epsilon_1$ we can extend this lift

$$\phi^{-1} \circ \gamma : [0,\epsilon_2) \to B(p,R_1)^c$$

and because we are able to choose the radius r of the ball within which we invert ϕ independently of its center we can ultimately extend to a curve

$$\phi^{-1} \circ \gamma : [0,1] \to B(p,R_1)^c$$

terminating at x. Thus ϕ is surjective. □

claim 2. *For any R_1 as above, there exists $R_2 > R_1$ such that ϕ restricted to $B(p,R_2)^c$ is mapped to $B(\phi(p),\overline{R}_1)^c$.*

Proof. Choose $R_2 > 2\overline{R}_1 + R_1$. For any $q \in B(p,R_2)^c$ there exists a geodesic $\gamma : [0,1] \to B(p,R_1)^c$ of length $2\overline{R}_1$ with $\gamma(0) = q$ and $\gamma(1) = q' \in \partial B(p,R_1)$. Because geodesics are mapped to geodesics, all geodesics in $\mathbb{H}^n$ are minimizing and we have $d_{\mathbb{H}^n}(\phi(p),\phi(q')) \leq \overline{R}_1$, it folows that

$$d_{\mathbb{H}^n}(\phi(p),\phi(q)) \geq \overline{R}_1.$$

□

claim 3. *ϕ restricted to $B(p,R_2)^c$ injects into $B(\phi(p),\overline{R_2})^c$.*

Proof. Let $p_1,p_2 \in B(p,R_2)^c$ satisfying $\phi(p_1) = \phi(p_2) \in B(\phi(p),\overline{R}_2)^c$ be given. Connect p_1 and p_2 with a curve $\gamma : [0,1] \to B(p,R_2)^c$. The projection $\phi \circ \gamma$ is a closed loop and by claim 2 lies entirely within $B(\phi(p),\overline{R}_1)^c$. It can therefore be continuously contracted to a point. Furthermore, following the argument in claim 1, We can lift this to a contraction of γ, and therefore $p_1 = p_2$. □

If we set

$$H = \phi^{-1}(B(\phi(p), \overline{R}_2)^c)$$

it now follows that

$$\phi : H \to B(\phi(p), \overline{R}_2)^c$$

is a diffeomorphism and therefore an isometry. Furthermore it follows from claim 2 above that for some $R_3 > R_2$ H^c is compactly contained in $B(p, R_3)$. Therefore, if we let

$$N = B(p, R_3) \cup_\phi \phi(B(p, R_3)^c)$$

(i.e. the gluing of these two sets along their boundaries by the map ϕ) and

$$h = \left\{ \begin{array}{cc} g & \text{on } B(p, R_3) \\ g_{\mathbb{H}^n} & \text{otherwise} \end{array} \right.$$

then (N, h) is a smooth Riemannian manifold with the desired properties. $\square$

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