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### Authors

Javandel, I.  
Witherspoon, P.A.

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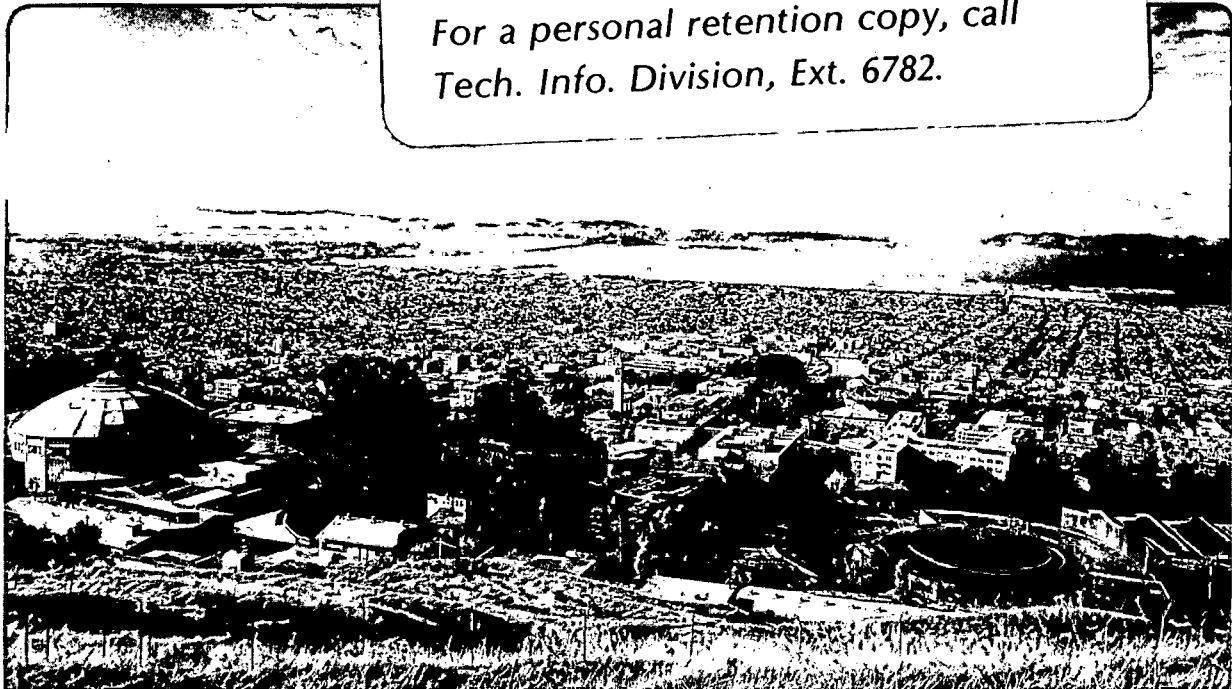
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Iraj Javandel and P.A. Witherspoon

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PRESSURE TRANSIENT TESTING WITH A PARTIALLY PENETRATING WELL  
IN A TWO-LAYER RESERVOIR

Iraj Javandel and P.A. Witherspoon  
Lawrence Berkeley Laboratory  
University of California  
and

Department of Materials Science & Mineral Engineering  
University of California  
Berkeley, CA 94720

ABSTRACT

Estimating flow properties of a heterogeneous reservoir consisting of two or more layers with different flow properties is often a problem of great interest to reservoir engineers. Analysis of pressure transient data from a partially penetrating well producing from either layer of a two-layer reservoir may be used to estimate the individual transmissivities.

An analytic solution has been derived to investigate the transient pressure response of two-layer reservoirs with cross-flow. Results of the analytic method have been verified using a finite-element model and they reveal important details of pressure transient behavior such as the limiting conditions for detecting a multilayer situation. A procedure has been developed to evaluate the permeability of the producing layer as well as that of the second layer. The method can be applied to both pressure buildup and interference well tests. As has been previously shown from earlier numerical techniques, the early-time response of such a two-layer reservoir producing from a fraction of the thickness of one of the layers closely follows the behavior of a single-layer case. An inflection point in the pressure response can be expected under certain circumstances, and this provides important data. At large times the system behaves like an equivalent homogeneous reservoir.

INTRODUCTION

The behavior of a layered reservoir under the influence of a pumping well is a problem of interest in the field of reservoir engineering. Numerous papers have been written on various aspects of this problem. Hantush and Jacob<sup>1</sup> have presented solutions for steady state flow to a well draining one of the layers of a two-layer bounded reservoir. Lefkovits et al.<sup>2</sup> studied the transient performance of a stratified bounded reservoir where the producing well is completely penetrating and there is no crossflow. Papadopoulos<sup>3</sup> has studied the same problem for only two layers of infinite areal extent.

References and illustrations at end of paper.

A similar problem, but with crossflow between adjacent layers, has also been investigated by Katz<sup>4</sup> and Russell and Prats<sup>5</sup> for the case of constant head at the wellbore, and by Jacquard<sup>6</sup> and Boulton and Streltsova<sup>7</sup> for constant flow rate.

In addition to the above works, which are all based on the analytical approach, many authors<sup>8-13</sup> have applied numerical as well as analog models to handle problems of flow in layered reservoirs. Recently, Javandel and Witherspoon<sup>14</sup> studied the problem of flow to a partially penetrating well in a two-layer aquifer where the well is open in the top layer and the lower layer is considered to be infinitely thick.

In this paper we shall present an analytic solution to the problem of transient flow to a partially penetrating well that is open in either layer of a two-layer system where both layers are finite in thickness. Crossflow is permitted at the interface between the two layers. These solutions can readily be evaluated numerically. Asymptotic forms of the solution for small and large values of time are developed from the general solution. The approach here is to start with the problem when the pumping well is open only in the top layer. A solution is then developed for the case when the well is partially penetrating only in the lower layer. The analytical solutions are evaluated numerically and results are presented in dimensionless form on semilogarithmic plots for a few different parameters. Based on the application of these results, a method is proposed for interpretation of the pump-test data in two-layer reservoirs.

WELL OPEN IN TOP LAYER

Let us consider a reservoir consisting of two layers that are confined above and below by impervious layers as illustrated on Figure 1. Each layer has its own flow properties, is finite in thickness, and extends radially to infinity. The interface between the two layers is an open boundary, meaning that no discontinuity of potential or its gradient is allowed across this surface. The top layer of the system is partially penetrated by a well of infinitesimal radius for a length  $l$  from the top of

the aquifer. If the well is pumped at a constant rate,  $q$ , we are interested in determining the potential distribution  $\phi(r, z, t)$  within the reservoir after pumping starts. The differential equations and initial and boundary conditions to describe this problem can be written as:

$$\frac{\partial^2 \phi_i}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_i}{\partial r} + \frac{\partial^2 \phi_i}{\partial z^2} = \frac{1}{\alpha_i} \frac{\partial \phi_i}{\partial t}, \quad i = 1, 2 \quad (1)$$

$$\phi_i(r, z, 0) = \phi_0 \quad (2)$$

$$\frac{\partial \phi_1}{\partial z}(r, h_1, t) = 0 \quad (3)$$

$$\frac{\partial \phi_2}{\partial z}(r, -h_2, t) = 0 \quad (4)$$

$$\lim_{r \rightarrow \infty} \phi_i(r, z, t) = \phi_0 \quad (5)$$

$$\phi_1(r, 0, t) = \phi_2(r, 0, t) \quad (6)$$

$$k_1 \frac{\partial \phi_1}{\partial z}(r, 0, t) = k_2 \frac{\partial \phi_2}{\partial z}(r, 0, t) \quad (7)$$

$$\lim_{r \rightarrow 0} \left( r \frac{\partial \phi_1}{\partial r} \right) = -\frac{q\mu}{2\pi k_1 \rho}, \quad \text{for } (h_1 - l) < z < h_1 \quad (8)$$

$$\lim_{r \rightarrow 0} \left( r \frac{\partial \phi_1}{\partial r} \right) = 0, \quad \text{for } -h_2 < z < (h_1 - l) \quad (9)$$

In order to handle the nonuniform boundary condition along the axis of the well, one can arbitrarily divide the top layer of the reservoir into two separate layers by considering an imaginary interface at the elevation of  $z = h_1 - l$ . The system is then made of three layers, two of them having the same flow properties. Let us then designate three different symbols for potential  $\phi_1$  for the top layer in the zone between the top of the reservoir and the imaginary horizontal plane passing through the bottom of the well;  $\phi_2$  for the bottom layer; and  $\phi_3$  for the zone between the elevation of the bottom of the well and the top of the lower layer.

#### Solution for the Two-layer Case

The general solution of the problem can be obtained by successive application of Laplace and Hankel transformations over  $t$  and  $r$ , respectively. The details of the analytic solution are given elsewhere<sup>15</sup>. By introducing the following dimensionless parameters:

$$P_D = \frac{2\pi k \rho h_1 (\phi - \phi_0)}{q\mu}, \quad t_D = \frac{\alpha_1 t}{r_w^2}, \quad r_D = \frac{r}{r_w}$$

$$h_{D2} = \frac{h_2}{r_w}, \quad l_D = \frac{l}{h_1}, \quad D = \frac{\alpha_2}{\alpha_1}$$

the expressions for dimensionless pressure in the different layers may be given by the following set of equations.

$$P_{D1} = \frac{1}{k_D} \int_0^\infty x J_0(xr_D) \left[ \frac{1}{x^2} - f_1(x) + \sum_{n=1}^\infty \frac{B'_1}{A'_1} \exp \left\{ - \left( \frac{\gamma_n^2}{h_{D2}^2} + x^2 \right) Dt_D \right\} \right] dx \quad (10)$$

$$P_{D2} = \frac{1}{k_D} \int_0^\infty x J_0(xr_D) \left[ f_2(x) + \sum_{n=1}^\infty \frac{B'_2}{A'_1} \exp \left\{ - \left( \frac{\gamma_n^2}{h_{D2}^2} + x^2 \right) Dt_D \right\} \right] dx \quad (11)$$

$$P_{D3} = \frac{1}{k_D} \int_0^\infty x J_0(xr_D) \left[ f_3(x) + \sum_{n=1}^\infty \frac{B'_3}{A'_1} \exp \left\{ - \left( \frac{\gamma_n^2}{h_{D2}^2} + x^2 \right) Dt_D \right\} \right] dx \quad (12)$$

Definitions for  $f_1(x)$ ,  $f_2(x)$ ,  $f_3(x)$ ,  $A'_1$ ,  $B'_1$ ,  $B'_2$ , and  $B'_3$  are given in Appendix A.

#### Solution for Single-layer Case

Javandel and Witherspoon<sup>15</sup> have shown that if the permeability of the lower layer vanishes, Eq. 10 will take the following form:

$$P_{D1} = \frac{1}{2} \int_u^\infty \frac{e^{-y}}{y} dy + \frac{1}{\pi k_D} \sum_{n=1}^\infty \frac{1}{n} \sin(n\pi l_D) \cdot \cos[n\pi(z_D - 1)] \int_u^\infty \exp \left\{ -y - \frac{(n\pi)^2 r_D^2}{4yh_{D1}^2} \right\} \frac{dy}{y} \quad (13)$$

which is the Hantush<sup>16</sup> solution for single-layer partial penetration. Equation 13, thus, provides an independent check on the two-layer solution.

#### Solution for Small Values of Time

It can be shown<sup>15</sup> that at early time, Eq. 10 reduces to Eq. 13, provided

$$t_D < \frac{h_{D1}^2 (1 - l_D)^2}{10}$$

This means that for sufficiently small values of time, the two-layer reservoir behaves as if the lower layer were absent.

### Solution for Large Values of Time

At large values of time and at radial distances that exceed the thickness of the reservoir, the pressure transient behavior of the producing layer may be given by<sup>15</sup>:

$$P_{D_1} = \frac{1}{2[1 + (k_2 h_2 / k_1 h_1)]} \int_v^\infty \frac{e^{-y}}{y} dy \quad (14)$$

where

$$v = \frac{r^2(\psi_1 \mu c_1 h_1 + \psi_2 \mu c_2 h_2)}{4t(k_1 h_1 + k_2 h_2)} \int_v^\infty \frac{e^{-y}}{y} dy \quad (15)$$

Equation 14 may be approximated by:

$$P_{D_1} = \frac{1.15}{1 + (k_2 h_2 / k_1 h_1)} \cdot \left[ \log(t_D / r_D^2) + \log \frac{2.25(1 + k_2 h_2 / k_1 h_1)}{1 + (\phi \mu ch)_2 / (\phi \mu ch)_1} \right] \quad (16)$$

This is an interesting result because it indicates that a semilog plot of dimensionless pressure versus dimensionless time at the wellbore or at any shut-in observation well will yield a straight line when the pumping time becomes sufficiently large. From Eq. 16, it is apparent that the slope of this line will be:

$$m = \frac{1.15}{1 + k_2 h_2 / k_1 h_1} \quad (17)$$

When  $r > (h_1 + h_2)$  the value of  $t_{D_0}$  corresponding to  $P_{D_1} = 0$  will be given by

$$t_{D_0} = \frac{r_D^2(1 + (\psi ch)_2 / (\psi ch)_1)}{2.25[1 + k_2 h_2 / k_1 h_1]} \quad (18)$$

Although Eq. 18 holds for  $r > (h_1 + h_2)$ , Eq. 17 is true for all radial distances.

### WELL OPEN IN LOWER LAYER

When the pumping well is open along length  $l$  in the upper part of the lower layer (Fig. 2), the general solution for this particular case is given by the following set of equations<sup>15</sup>:

$$\hat{P}_{D_1} = \frac{1}{\hat{l}_D} \int_0^\infty x J_0(xr_D) \left\{ R_1(x) + \sum_{n=1}^\infty \frac{E_1}{\delta} \exp\left[-(\gamma_n^2 + x^2) \hat{t}_D / h_{D_2}^2\right] \right\} dx \quad (19)$$

$$\hat{P}_{D_2} = \frac{1}{\hat{l}_D} \int_0^\infty x J_0(xr_D) \left\{ \frac{1}{x^2} - R_2(x) + \sum_{n=1}^\infty \frac{E_2}{\delta} \exp\left[-(\gamma_n^2 + x^2) \hat{t}_D / h_{D_2}^2\right] \right\} dx \quad (20)$$

$$\hat{P}_{D_3} = \frac{1}{\hat{l}_D} \int_0^\infty x J_0(xr_D) \left\{ R_3(x) + \sum_{n=1}^\infty \frac{E_3}{\delta} \exp\left[-(\gamma_n^2 + x^2) \hat{t}_D / h_{D_2}^2\right] \right\} dx \quad (21)$$

Definitions for  $R_1(x)$ ,  $R_2(x)$ ,  $R_3(x)$ ,  $E_1$ ,  $E_2$ ,  $E_3$  and  $\delta$  are given in Appendix A.

### DISCUSSION OF RESULTS

Figure 3 shows a semilog plot of dimensionless pressure versus dimensionless time to illustrate how the effects of partial penetration at different radial distances in a two-layer reservoir differ from those for the single layer. These results were determined for the particular location  $z = h_1$ , which means the top of the producing well, but as will be shown below, the average pressure drop over the open wellbore does not differ significantly from these values. This has been reported earlier by Kazemi and Seth<sup>13</sup>. One notes that at early time the solution for the two-layer reservoir coincides with that of the single-layer case. This was shown to be the case from the properties of the two-layer solution. At large values of time, the slopes of the curves are in agreement with the values obtained from Eq. 17. As is apparent on this figure, the slopes of the curves must converge to  $m = 1.15$  when  $k_2 h_2 = 0$ , which of course corresponds to the single-layer case for large values of time.

Figure 4 illustrates the effects at the well bore of partial penetration in the upper layer of a two-layer reservoir as the penetration increases from 10% to 100%. As can be anticipated from Eq. 17, the slopes of the curves at large values of time are independent of the depth of penetration.

To illustrate the effect of transmissibility contrast ( $k_2 h_2 / k_1 h_1$ ) on pressure drawdowns in a shut-in observation well, Figure 5 has been prepared for the particular case of a radial distance corresponding to  $r_D = 40$ . Note that the effect of the second layer becomes significant when  $t_D > 10$ .

Figure 6 shows a type curve for the pressure effects at the wellbore of a partially penetrating well for the particular case of 10% penetration in the upper layer. The Theis, or line source, solution is also included to illustrate that drawdowns

in this particular case are approximately ten times greater than would be observed with full penetration in a single layer equal in thickness to that of the top layer.

The partial penetration results on Figure 6 are for pressure drawdowns calculated for the top of the upper layer ( $z = h_1$ ). To examine the variation of pressures along the wellbore, an average value was determined and is shown on the figure. At early times ( $t_D < 1$ ) the averaged result is essentially the same as at  $z = h_1$ , and at large values of time the averaged results are about 10% low. It is important to note that the slopes of the semilog straight lines for either result are still in accord with Eq. 17.

A finite-element model was also used to provide a numerical approach to this same two-layer problem. Figure 7 shows a comparison of dimensionless results for  $r_D = 10, 20,$  and  $100$ . Two different meshes were used to duplicate the conditions of the analytical model. The first mesh only included 323 nodal points and it is evident that computed drawdowns at any given time were too low when  $r_D = 10$  and  $20$  and too high when  $r_D = 100$ . A second mesh using 681 nodal points gave much better agreement with the analytical results. This suggests that some care must be exercised when approaching this kind of complex problem from the numerical standpoint.

#### APPLICATION TO RESERVOIR PUMP TESTS

If a well is completed through the total thickness of a two-layer reservoir and is pumped at constant rate, the analysis of the results can only yield the properties of the equivalent system. However, if the well is completed in only one part of either layer of the system, the following procedure can be used to investigate the properties of the individual layers. The properties of the layer being pumped can be determined from the early time response in the pumping well or an appropriately located observation well. The late time response can then be used to determine the properties of the unpumped layer.

#### Properties of Pumped Layer

Two different methods can be used to obtain results for the pumped layer: (a) the inflection method, and (b) the type-curve method. We shall present both below and then discuss a method of determining the properties of the unpumped layer.

The inflection method has been introduced by Hantush<sup>17</sup> for a single-layer aquifer with partial penetration and will be reviewed briefly. One should construct a semilog plot of pressure drawdown data from the pumping well or a nearby shut-in observation well versus time. The data may reveal an inflection as is illustrated by the dimensionless plot on Figure 8. If an inflection point is clearly indicated, the slope of the tangent at the point of inflection,  $m_i$ , can be easily determined. Hantush<sup>17</sup> has shown that:

$$\beta^2/\sqrt{\pi} = x e^{x^2} \operatorname{erf}(x) \quad (22)$$

At the pumping well, or when the observation well is open only at the top of the reservoir, or when  $r > l$  and the depth of the observation well is less than that of the pumping well, then  $\beta = l/r$ . On the other hand, when the observation well is open at approximately the elevation of the bottom of the pumping well,  $\beta = 2l/r$ . Knowing  $\beta$ , one can evaluate  $x$  from Eq. 22 and then compute

$$u_i = (x/\beta)^2 \quad (23)$$

The permeability  $k_1$  can then be determined from:

$$k_1 = \frac{2.3 q\mu}{4\pi m_1 \rho g l} e^{-u_i} \operatorname{erf}(x_i), \quad \text{if } \beta = l/r$$

or

$$k_1 = \frac{2.3 q\mu}{8\pi m_1 \rho g l} e^{-u_i} \operatorname{erf}(x_i), \quad \text{if } \beta = 2l/r$$

The next step is to evaluate a function that Hantush<sup>17</sup> defined as:

$$M(u_i, \nu) = \int_{u_i}^{\infty} \frac{e^{-y}}{y} \operatorname{erf}(\beta\sqrt{y}) dy \quad (24)$$

and that has been tabulated<sup>18,19</sup>. One then calculates pressure drawdown at the inflection point from

$$\Delta P_i = \left( \frac{q\mu}{B\pi k_1 l} \right) M(u_i, \beta) \quad (25)$$

where  $B = 4$  when  $\beta = l/r$  and  $B = 8$  when  $\beta = 2l/r$ .

From the semilog plot of pressure drawdown data, one reads the time,  $t_i$ , corresponding to the value of  $\Delta P_i$ . Finally, the product of  $(\psi c)_1$  can be computed from:

$$(\psi c)_1 = 4k_1 t_i u_i / r^2 \mu \quad (26)$$

The type-curve method is essentially the same as the standard log-log, type-curve method except that one must prepare a special plot of  $P_D$  versus  $t_D$  for the appropriate parameters corresponding to Hantush's<sup>16</sup> solution for a single layer with partial penetration. In other words, the effects of the second layer are ignored. This method will again yield  $k_1$  and  $(\psi c)_1$ .

The application of the inflection and type-curve methods is based on the assumption that the early time response of the two-layer system is essentially controlled by the properties of the pumped layer. This is generally the case when the observation well is located at a radial distance from the pumping well that is less than half the thickness of the pumped layer.

Properties of Unpumped Layer

To obtain the properties of the unpumped layer, the semilog plot mentioned above should reveal a straight line if the pumping test has been run for a sufficiently large period of time. The slope,  $m$ , of this straight line can be used to obtain  $(k_2 h_2 + k_1 h_1)$  from:

$$m = \frac{1.5 q\mu}{2\pi(k_1 h_1 + k_2 h_2)} \quad (27)$$

Since  $k_1$  has been evaluated and  $h_1$  is known,  $k_2 h_2$  is readily calculated. If  $h_2$  is also known,  $k_2$  is easily determined.

In order to determine the storage term,  $(\psi c)_2$ , of the unpumped layer, it is necessary to have interference drawdown data from a shut-in observation well at a distance large enough to satisfy  $r > h_1 + h_2$ . As indicated by Eq. 16, an extrapolation of the straight-line portion of a semilog plot of  $\Delta p_i$  versus  $\log t$  back to the axis for zero drawdown yields the time intercept,  $t_0$ . From Eq. 18, one can then derive:

$$(\psi c)_2 = \left[ \frac{2.25 t_0 (k_1 h_1 + k_2 h_2)}{\mu r^2} - (\psi ch)_1 \right] \frac{1}{h_2} \quad (28)$$

CONCLUSIONS

An analytic solution for the problem of transient flow toward a partially penetrating well in a two-layer reservoir has been presented. Solutions have been developed for the effects of a pumping well that is open in either layer. The solutions have been evaluated numerically, and graphical results for some typical cases are presented. The results have also been checked by comparison with finite-element calculations. It was shown that the solutions reduce to the case for a single layer with partial penetration. Asymptotic solutions for small and large values of time have been developed to show that: (1) at early times with partial penetration, the behavior of the pumped layer is exactly the same as that of a single layer, and (2) at large values of time, a semilog plot of drawdown versus time yields a straight line whose slope is only a function of the ratio  $k_2 h_2 / k_1 h_1$ . Finally, a method of analyzing field data to determine the hydraulic properties of both the pumped and unpumped layers is proposed.

NOMENCLATURE

A	$k_2 h_1 / k_1 h_2$	--
c	compressibility	$LT^2/M$
D	$a_2/a_1$	--
erf(x)	error function	--
$h_1$	thickness of the top layer	L
$h_2$	thickness of the lower layer	L
$h_{D1}$	$= h_1/r_w$	--
$h_{D2}$	$= h_2/r_w$	--
H	$h_2/h_1$	--
$\hat{H}$	$h_1/h_2$	--
$J_0(x)$	Bessel's function of the first kind and zero order	--

$k_1, k_2$	hydraulic conductivity of upper and lower layers, respectively	$L^2$
$l$	depth of penetration	L
$l_D$	$l/h_1$	--
$\hat{l}_D$	$l/h_2$	--
$m_i$	slope of the tangent at the inflection point	--
q	rate of discharge	$L^3/T$
r	radial distance	L
$r_D$	$r/h_w$	--
$r_w$	radius of the pumping well	L
$P_{Di}$	$[2\pi k_1 h_1 \rho (\psi_i - \psi_0)]/q\mu$ , dimensionless pressure	--
$\hat{P}_{Di}$	$[2\pi k_2 h_2 \rho (\psi_i - \psi_0)]/q\mu$ , dimensionless pressure	--
$\Delta P_i$	pressure drawdown at the inflection point	$M/LT^2$
t	time	T
$t_D$	$\alpha_1 t/r_w^2$ , dimensionless time	--
$\hat{t}_D$	$\alpha_2 t/r^2$ , dimensionless time	--
$t_i$	time at the inflection point	T
u	$= r^2/4\alpha_1 t$	L
z	vertical coordinate	L
$z_D$	$z/h_1$	--
$\hat{z}_D$	$z/h_2$	--
$\alpha_1, \alpha_2$	diffusivity of layer 1 and 2, respectively	$L^2/T$
$\gamma_n$	roots of characteristic equations	--
$\mu$	viscosity of fluid	$M/TL$
$\rho$	density of fluid	$M/L^3$
$\phi$	porosity	--
$\phi_1, \phi_2$	potential in upper and lower layer, respectively	$L^2/T^2$

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## APPENDIX A

Following are definitions of some of the functions used in Eq. 10 through 12 and 19 through 21 in the text:

$$f_1(x) = \frac{\cosh[xh_{D_1}(1 - z_D)]}{x^2}$$

$$\frac{AH \tanh(xh_{D_2}) \cosh[xh_{D_1}(1 - k_D)] + \sinh[xh_{D_1}(1 - k_D)]}{AH \tanh(h_{D_2} x) \cosh(xh_{D_1}) + \sinh(xh_{D_1})} \quad (A-1)$$

$$f_2(x) = \frac{1}{x^2}$$

$$\frac{\cosh[xh_{D_1}(z_D + H)] \sinh(xh_{D_1} k_D)}{AH \sinh(xh_{D_2}) \cosh(xh_{D_1}) + \cosh(xh_{D_2}) \sinh(xh_{D_1})} \quad (A-2)$$

$$f_3(x) = \frac{\sinh(xh_{D_1} k_D)}{x^2}$$

$$\frac{AH \sinh(xh_{D_2}) \sinh(xh_{D_1} z_D) + \cosh(xh_{D_2}) \cosh(xh_{D_1} z_D)}{AH \sinh(xh_{D_2}) \cosh(xh_{D_1}) + \cosh(xh_{D_2}) \sinh(xh_{D_1})} \quad (A-3)$$

$$A' = \left(\frac{1}{2}\right) \left( \frac{Y_n^2}{h_{D_2}^2} + x^2 \right) \left[ \beta_n \left\{ D - H^2 \left( \frac{\beta_n}{Y_n} \right)^2 \right\} \right. \\ \left. + \cos \gamma_n \sin \beta_n + \beta_n^2 (AH^2 + D) \cos \gamma_n \cos \beta_n \right. \\ \left. - \left( H^2 \frac{\rho_n}{Y_n} + AD \frac{Y_n}{\beta_n} \right) \beta_n^2 \sin \gamma_n \sin \rho_n \right] \quad (A-4)$$

$$B'_1 = \cos \left[ \rho_n (z_D - 1) \right] \left\{ A \gamma_n \sin \gamma_n \cos \left[ \rho_n (1 - k_D) \right] \right. \\ \left. + \beta_n \cos \gamma_n \sin \left[ \beta_n (1 - k_D) \right] \right\} \quad (A-5)$$

$$B'_2 = -\rho_n \sin(\beta_n k_D) \cos \left[ \gamma_n \left( 1 + \frac{z_D}{H} \right) \right] \quad (A-6)$$

$$B'_3 = \sin(\beta_n k_D) \left\{ A \gamma_n \sin \gamma_n \sin(\beta_n z_D) \right. \\ \left. - \rho_n \cos \gamma_n \cos(\beta_n z_D) \right\} \quad (A-7)$$

$\gamma_n$  and  $\beta_n$  are roots of appropriate characteristic equations<sup>15</sup>.

$$R_1(x) = \frac{\cosh(xh_{D_2}(\hat{z}_D - \hat{H}))}{x^2} \cdot$$

$$\frac{\sinh(xh_{D_2}) - \sinh(xh_{D_2}(1 - \hat{k}_D))}{\cosh(xh_{D_1})\sinh(xh_{D_2}) + (\hat{H}/A)\sinh(xh_{D_1})\cosh(xh_{D_2})}$$

(A-8)

$$R_2(x) =$$

$$\frac{1}{x^2[\cosh(xh_{D_1})\sinh(xh_{D_2}) + (\hat{H}/A)\sinh(xh_{D_1})\cosh(xh_{D_2})$$

$$\cdot \{(\hat{H}/A)\sinh(xh_{D_1})\cosh(xh_{D_2}(\hat{z}_D + 1))$$

$$+ \sinh(xh_{D_2}(1 - \hat{k}_D))[\cosh(xh_{D_1})\cosh(xh_{D_2}\hat{z}_D)$$

$$- (\hat{H}/A)\sinh(xh_{D_2}\hat{z}_D)\sinh(xh_{D_1})\}]} \quad (A-9)$$

$$R_3(x) =$$

$$\frac{\cosh(xh_{D_2}(\hat{z}_D + 1))}{x^2[\cosh(xh_{D_1})\sinh(xh_{D_2}) + (\hat{H}/A)\sinh(xh_{D_1})\cosh(xh_{D_2})$$

$$[-(\hat{H}/A)\sinh(xh_{D_1}) + (\hat{H}/A)\sinh(xh_{D_1})\cosh(xh_{D_2}\hat{k}_D)$$

$$+ \cosh(xh_{D_1})\sinh(xh_{D_2}\hat{k}_D)]} \quad (A-10)$$

$$\delta = \frac{1}{2} \left( \frac{\gamma_n^2}{h_{D_2}^2} + x^2 \right) \left\{ \beta_n \sin \beta_n \cos \gamma_n \left( \frac{DH^2 \gamma_n^2}{A \beta_n^2} - \frac{1}{A} \right) \right.$$

$$- \gamma_n^2 \sin \beta_n \sin \gamma_n \left( \frac{DH^2 \gamma_n}{\beta_n} + \frac{\beta_n}{A \gamma_n} \right)$$

$$\left. + \gamma_n^2 \cos \beta_n \cos \gamma_n \left( 1 + \frac{DH^2}{A} \right) \right\} \quad (A-11)$$

$$E_2 = \sin[\gamma_n(1 - \hat{k}_D)] \left\{ \gamma_n \cos \beta_n \cos(\gamma_n \hat{z}_D) \right.$$

$$+ (\beta_n/A) \sin(\gamma_n \hat{z}_D) \sin \beta_n \left. \right\}$$

$$+ (\beta_n/A) \sin \beta_n \cos[\gamma_n(\hat{z}_D + 1)] \quad (A-13)$$

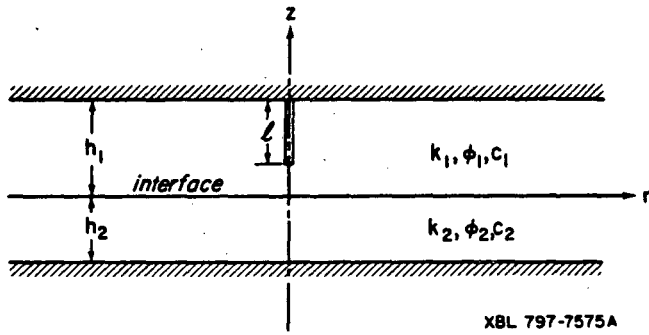
$$E_1 = -\gamma_n \cos \left[ \beta_n \left( \frac{\hat{z}_D}{H} - 1 \right) \right] \left\{ \sin \gamma_n - \sin[\gamma_n(1 - \hat{k}_D)] \right\}$$

(A-12)

$$E_3 = \cos[\gamma_n(\hat{z}_D + 1)] \left\{ (\beta_n/A) \sin \beta_n \right.$$

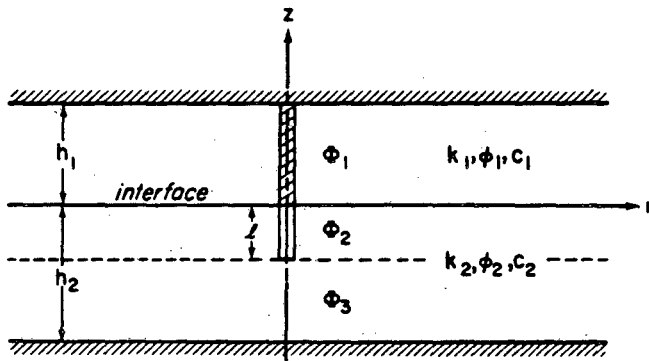
$$\left. - (\beta_n/A) \sin \beta_n \cos(\gamma_n \hat{k}_D) - \gamma_n \cos \beta_n \sin(\gamma_n \hat{k}_D) \right\}$$

(A-14)



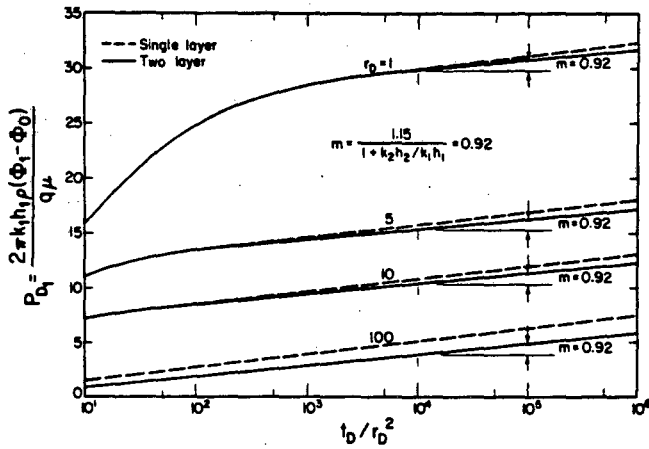
XBL 797-7575A

Figure 1. Two-layer reservoir with a partially penetrating well in the upper layer.



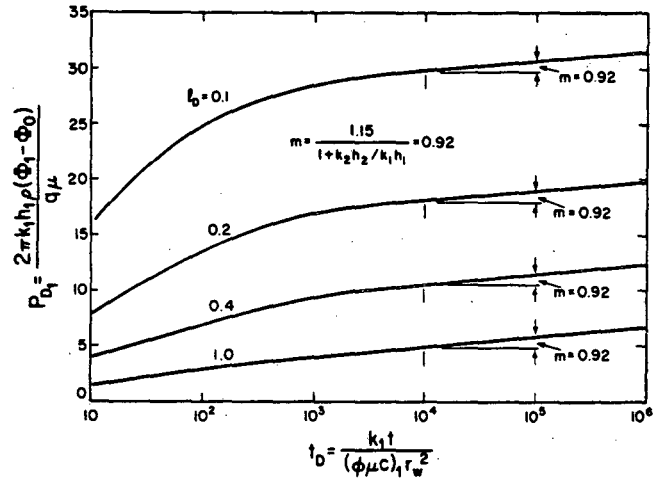
XBL 797-7574B

Figure 2. Two-layer reservoir with a partially penetrating well in the lower layer.



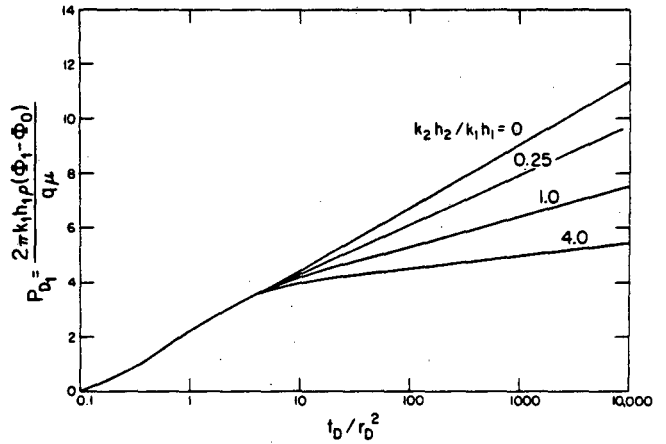
XBL 828-1004

Figure 3. Dimensionless pressure versus dimensionless time for partial penetration in the upper layer of a two-layer reservoir for the case  $l_D = 0.1$ ,  $k_2/k_1 = 0.5$ ,  $h_2/h_1 = 0.5$ ,  $\phi_2 c_2 = \phi_1 c_1$ , and  $r_w/h_1 = 0.01$ .



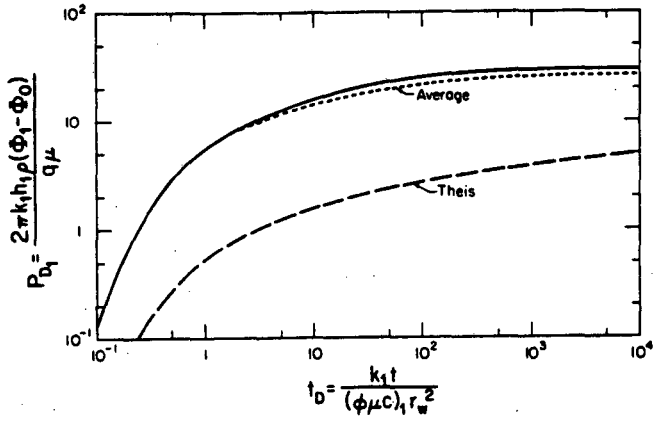
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Figure 4. Dimensionless pressure versus dimensionless time at the wellbore for partial penetration in the upper layer of a two-layer reservoir for the case  $k_2/k_1 = 0.5$ ,  $h_2/h_1 = 0.5$ ,  $\phi_2 c_2 = \phi_1 c_1$ ,  $z = h_1$ , and  $r_w/h_1 = 0.01$ .



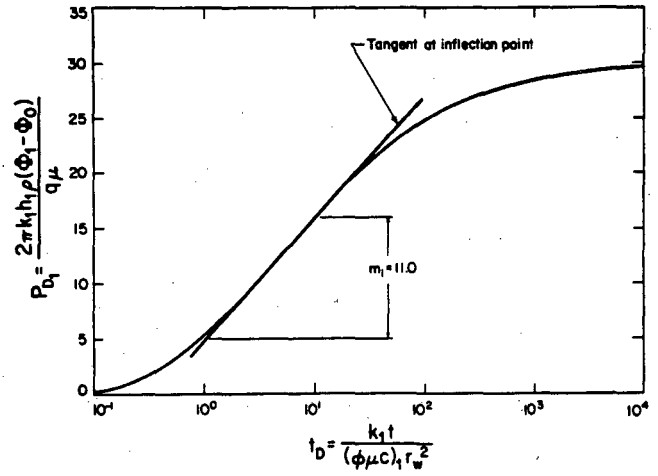
XBL 828-1004

Figure 5. Effect of transmissibility contrast on pressure drawdowns in a shut-in observation well with partial penetration in the upper layer of a two-layer reservoir for the case  $r_D = 40$ ,  $l_D = 0.2$ ,  $\phi_2 c_2 = \phi_1 c_1$ , and  $r_w/h_1 = 0.01$ .



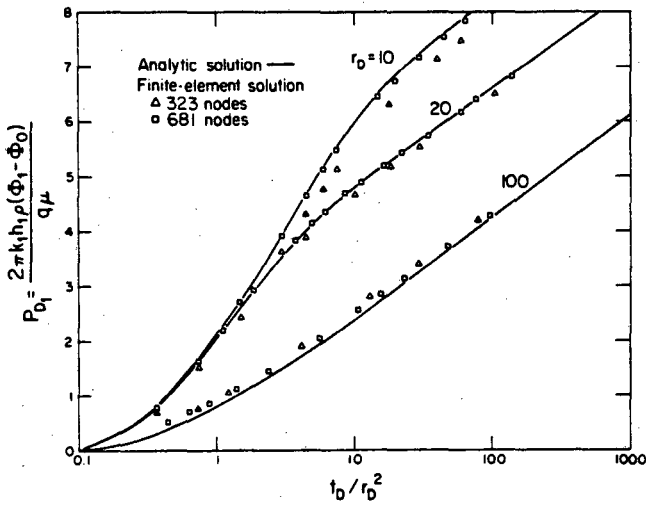
XBL 828-1005

Figure 6. Comparison of pressure drawdowns at the wellbore from the Theis solution with those of partial penetration in the upper layer of a two-layer reservoir for the case  $k_D = 0.1$ ,  $k_2/k_1 = 0.5$ ,  $h_2/h_1 = 0.5$ ,  $\psi_2 c_2 = \phi_1 c_1$ ,  $z = h_1$ , and  $r_w/h_1 = 0.01$ .



XBL 828-1007

Figure 8. Example of occurrence of an inflection in the dimensionless results for pressure versus time at the wellbore for the particular case  $k_D = 0.1$ ,  $k_2/k_1 = 0.5$ ,  $h_2/h_1 = 0.5$ ,  $\psi_2 c_2 = \phi_1 c_1$ ,  $z = h_1$ , and  $r_w/h_1 = 0.01$ .



XBL 828-1004

Figure 7. Comparison of analytic solution with finite-element calculation for the case  $k_D = 0.5$ ,  $k_2/k_1 = 0.5$ ,  $h_2/h_1 = 0.5$ ,  $\psi_2 c_2 = \phi_1 c_1$ ,  $z = h_1$ , and  $r_w/h_1 = 0.01$ .

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UNIVERSITY OF CALIFORNIA  
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