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ABSTRACT

We formulate and solve an integral equation approach to spontaneous breakdown in a certain dual model. The model is not yet physical, but Goldstone and Higgs phenomena are visible in the explicit spectrum shift which removes the (tachyonic) vacuum instability.

I. INTRODUCTION

This paper is concerned with the problem of exhibiting spontaneous symmetry breakdown in dual models. In a series of earlier papers,[1,2,3] it was shown that in any dual model with at least one zero mass scalar, the vacuum may be unstable against the emission of these scalars (the spurion emission). A nontrivial minimum of the effective spurion potential, if it exists, corresponds to the spontaneous breakdown of some symmetry. Unfortunately, the computation of the effective potential in general turns out to be a very difficult task, except in the relatively uninteresting case of the zero-intercept dual model.[2] One of the things that makes the zero-intercept model uninteresting is the absence of a tachyonic state, which, if present, guarantees the instability of the vacuum. In contrast, the model we are going to discuss has a tachyon in its spectrum and is therefore expected to undergo spontaneous symmetry breaking. Our starting point is a particular type of "dual M-model", introduced in Ref. 3. We are unable to handle this model in its original form, so we consider the limit of the dimensions of the space going to infinity; the so-called large N limit.[4] This limiting procedure, in addition to some further simplifications, enables us to construct a manageable model. In this model, the problem of spurion summation turns out to be mathematically equivalent to a one-dimensional linear chain with only nearest and next-to-nearest neighbor correlations. This gives rise to a set of single variable linear integral equations for the spurion S matrix, which is the S matrix in the presence of a uniform (zero-momentum) external field [usually called the W function.[5]]. A similar set of integral equations are also derived for the propagator of the theory.

The rest of the paper is devoted to the examination of various solutions to these integral equation. One solution that is always present is the trivial perturbation solution, for which the propagator returns to its original value when the external field vanishes, or equivalently, when the spurion to vacuum coupling goes to zero. We are able to show that the equations also possess solutions different from the perturbation solution, and we explicitly exhibit the propagator and the W function for some of the solutions in the limit of vanishing external field. The existence of several solutions is equivalent to a many-sheeted analytic structure as a function of the external field, and the perturbation solution changes into the new solutions smoothly as one goes around the relevant Riemann sheet. In terms of the effective potential, the new solutions correspond to nontrivial local minima (or stationary points) of the potential. Their existence is no accident; it is made possible by the fact that the kernel of the integral equation is singular. This singularity in turn is caused by the existence of a tachyon in the spectrum; the chain of reasoning that relates vacuum instability to the existence of a tachyon is complete, and a satisfactory picture of spontaneous symmetry breaking emerges. Having the propagator, we show that, in one of the solutions, which we call the leading solution, both the tachyon and and the (zero mass) vector meson masses are raised by one unit. The old tachyon is therefore promoted into a zero mass particle and the final spectrum is tachyon free!

The emphasis throughout the paper is on the techniques used to derive the fundamental set of integral equations and on their solutions. The model is admittedly not physical; we expect, however, that similar methods will continue to work in the case of more physical (and more complicated) models. We also expect our results to hold in more physical models: There will in general be a number of vacua lower in energy than the intrinsically unstable tachyonic vacuum of present dual models. As we find here, we expect in general to see the entire spectrum shift up until the tachyon is no longer present.

The organization of the paper is as follows. In Section II, we briefly review the general approach to spontaneous symmetry breaking in dual models. In Section III, the model of interest is defined and its properties are discussed. The integral equations for the W function (spurion summation) and their solutions are discussed in Section IV. Section V deals with the propagator and the mass spectrum. Finally, Section VI summarizes our conclusions. There are also two appendices. Appendix A is the derivation of our model as a large N limit of a ghost-free dual M-model.[3] Appendix B presents another spontaneous breakdown solution, this time in which the M fields themselves pick up an explicit vacuum expectation value.

II. DUAL MODELS AND SPONTANEOUS SYMMETRY-BREAKING

We begin with a brief review of the standard generating function(al) approach to spontaneous breakdown.[5] As we are interested only in spurion emission at zero-four-momentum, we need take the standard source function $J(k^\mu) = J\,\delta^4(k^\mu)$. In this case, we work only with functions, not functionals. Our interest is then focused on $W(J)$, the generating function for connected S-matrix elements at zero four-momentum,

$$W(J) = \sum_n \frac{J^n}{n!} G_n \tag{2.1}$$

where

$$G_n = i^{n-1} \langle 0|T(\phi_1 \cdots \phi_n)|0\rangle \Big|_{\substack{\text{connected,} \\ k_i^\mu = 0}} . \tag{2.2}$$

In the case of a single species of particle of mass μ^2 (and no vacuum expectation value), we have

$$W(J) = \frac{J^2}{2\mu^2} - i \sum_{n=3}^{\infty} \frac{J^n}{n!} \frac{S_n}{(\mu^2)^n} \tag{2.3}$$

where S_n are connected S-matrix elements suitably extrapolated to all $k_i^\mu = 0$. In terms of $S_n \equiv i\,T_n$ (the T-matrix elements), all i's disappear, and we expect the T_n's to be simple coefficients times the beta-functions of dual theory. The effective potential V is defined as follows

$$\phi(J) \equiv \frac{\partial W}{\partial J}, \qquad V(\phi) \equiv J\phi - W, \qquad J = \frac{\partial V}{\partial \phi} . \tag{2.4}$$

Also, the vacuum expectation value of ϕ, $\langle\phi\rangle = \phi(0)$. In this paper we will never construct $V(\phi)$ explicitly,[6] staying directly with the more physical $W(J)$, but we will from time to time refer to properties of V and $\langle\phi\rangle$ easily deducible from W.

It is well known for example that $W(J)$ being multisheeted corresponds to spontaneous breakdown: We define the first sheet as in (2.1), being the power series of S-matrix elements before spontaneous breakdown. If indeed there is a branch point at some finite J, we can go around it and come back to $J = 0$ on the second sheet. Suppose on the second sheet, near $J = 0$, $W(J)$ looks like $W(J) \sim \omega_0 + J\omega_1 + \cdots$.

Then it is easy to see that we have found a new stationary point of the potential $(\partial V/\partial \phi = J = 0)$ at $\langle \phi \rangle = \omega_1$. ω_0 is the value of the potential at the new stationary point. The foregoing is quite standard; there are however a number of novel features introduced here because we are working in a dual model, with no known ghost-free off-mass-shell extrapolation.

In the Lagrangian field theory with many species of scalar fields, both W and V are functions of many variables, and one is free to work with them all. In dual models however, we cannot do this. As explained in Refs. (1, 2, 3), we have only a few zero mass scalars, (here zero mass M-particles) for which we can reach $k^{\mu} = 0$, because we really have no valid off-mass-shell extrapolation at all. We are forced then to work with only the zero mass scalars externally, while all other scalars appear internally. Nevertheless, <u>this procedure is entirely adequate for spontaneous breakdown</u>.

We use a simple argument to illustrate what is going on in this approach. Suppose $V(\phi, \sigma)$ is a function of two scalar fields ϕ and σ. Let J_ϕ and J_σ be the corresponding external fields. Using only J_ϕ as an external probe is equivalent to setting $J_\sigma = 0$. This is equivalent to eliminating σ from the definition of V through the equation of motion $\partial V/\partial \sigma = 0$, and one arrives at a reduced potential V_r as follows

$$V_r(\phi) \equiv V\big(\phi, \sigma(\phi)\big) \tag{2.5}$$

where $\sigma(\phi)$ is the solution of $\partial V/\partial \sigma = 0$. Since $\partial V/\partial \sigma = 0$ is a condition for a stationary point, all stationary points of V are also stationary ppoints of V_r. Therefore no information (about stationary points of V) is lost if only ϕ is coupled to its external field. In principle, any massive scalars in the theory can be eliminated in this way, in favor of the others, until one is left with only massless fields. This process of elimination fails for the massless fields however, since the equations of motion to be solved become singular in that case. This question will be treated at length in a separate publication[7]--here we need only the result that stationary points of the potential are adequately probed by the zero-mass scalars of the theory.

Once we begin using these reduced potentials, however, yet another novel feature needs discussion. Although V is usually a single valued field (at least in tree approximation), V_r is in general multi-valued, and sometimes spontaneous breakdown may show up by changing sheets of V_r (rather than a new minimum on the same sheet).

At this point, a simple example may prove illuminating. Consider the following effective potential of two fields ϕ and σ

$$V(\phi,\sigma) \equiv \frac{1}{4}\lambda_1\phi^4 + \lambda_2\phi^2\sigma + \frac{1}{3}\lambda_3\sigma^3 - \frac{1}{2}m^2\sigma^2 \qquad (2.6)$$

where the λ's are coupling constants. The condition $\partial V/\partial\sigma = 0$ can be solved for σ to yield the following,

$$\sigma^{(\pm)} = \frac{m^2}{2\lambda_3} \pm \frac{1}{2\lambda_3}(m^4 - 4\lambda_2\lambda_3\phi^2)^{1/2} ,$$

$$V_r^{(\pm)}(\phi) = \frac{1}{6}\frac{m^2\lambda_2}{\lambda_3}\phi^2 + \left(\frac{2}{3}\lambda_2\phi^2 - \frac{1}{6}\frac{m^4}{\lambda_3}\right) \qquad (2.7)$$

$$\times\left\{\frac{m^2}{2\lambda_3} \pm \frac{1}{2\lambda_3}(m^4 - 4\lambda_2\lambda_3\phi^2)^{1/2}\right\} .$$

The reduced potential V_r is a many-valued function of ϕ. Of the two solutions for σ, the one corresponding to the negative sign of the radical, $\sigma^{(-)}$, is the normal or the perturbation solution. This definition follows from the fact that $\sigma^{(-)} \to 0$ as $\phi \to 0$, and therefore the trajectory of $\sigma^{(-)}$ in the ϕ - σ plane passes through the normal stationary point $\phi = 0$, $\sigma = 0$ of the potential V. The other stationary points of V are also the stationary points of either $V_r^{(+)}$ or $V_r^{(-)}$. Of particular interest is the stationary point $\phi = 0$, $\sigma = m^2/\lambda_3$, which corresponds to the stationary point $\phi = 0$ on the nonperturbative branch $V_r^{(+)}$. Here we have an example of spontaneous breakdown occurring by slipping into a different branch of the reduced effective potential, where the field ϕ does not acquire a vacuum expectation value at all! There is, however, no paradox, since the field σ that has been eliminated from V_r is the one that acquired a vacuum expectation value. This is a common phenomenon when the potential is even in one of the fields (in this case in ϕ).

One can also easily construct the $W_{red} = J_\phi \phi - V_r$, being the W with only ϕ external. The upshot of these examples (the reader should continue such exercises to see for himself) is that: in our dual models <u>both</u> W_r and V_r are multisheeted. <u>Every</u> sheet of W_r corresponds to a spontaneous breakdown. Thus, even if on the second sheet $W_r \to 0$ like J^2, we interpret this as a spontaneous breakdown in the <u>hidden</u> scalar degrees of freedom (while $\langle\phi\rangle = 0$), by shifting to another sheet of V_r. In Section IV, we shall in fact find such a case, spontaneous breakdown in our dual model in which the probing field itself acquires no vacuum expectation value.

So far, we have been careful to distinguish between the stationary points of the potential well and an absolute minimum. Unfortunately, our analysis is inadequate to decide whether a given stationary point is an absolute minimum. We know one thing for sure, however, since the normal dual model has a tachyon in its spectrum and our solution does not, our stationary point has lower energy than the normal one. When there are several stationary points, all of which are free of tachyons, we are presently unable to decide between them.

The last and most technical point of the dual model application is the off-mass-shell extrapolation, for the series $W(J)$ obviously has a branch point at $J = 0$ when $\mu^2 = 0$. Our use of $\mu^2 \neq 0$ must be thought of as a regulator procedure, with $\mu^2 \to 0$ at the end of the calculation. Because $\mu^2 = 0$ is the canonical (conformal) mass, we can in principle watch to make sure that all ghost-structure is vanishing as $\mu^2 \to 0$. We will return to this in Sections III, IV and our conclusions.

III. THE ABELIAN MODEL

Our starting point is a dual model with internal symmetry introduced in Ref. 3. This model has the internal symmetry group U(N), where N is taken to be arbitrary; and it is an "axiomatic" model, in the sense that, as far as we know, it is free of ghosts and possesses all the desirable features a dual model should possess. Its only drawback is that it has tachyonic states at $m^2 = -1$ and vector mesons at zero mass. The latter feature is common to all dual models before spontaneous breakdown.[8] We are unable to treat this model as it stands; so we use it as a stepping stone to a simpler model we are able to handle. The transition to the simpler model is achieved by starting with singlet external states in the $U(N)$ space, and then by taking the limit[4] $N \to \infty$. Since we shall not need the original model in what follows, the details of this limiting operation and of the original model are given in Appendix A. The rest of the section is devoted to the definition and a brief description of the limiting model, which we call the Abelian model.

The low lying spectrum of the Abelian model consists of a tachyon at $m^2 = -1$, a zero mass vector meson (photon), and a zero mass scalar which we call M. Taking the external particles to be M's, we have the following operator formula for the planar n point amplitude B_n,

$$B_n(k_1,\cdots,k_n) \equiv \langle 0|\pi_1^{(1)}\, V^{(2)}(k_2)\, \frac{1}{R - s_{12} - 1}$$

$$\times\, V^{(3)}(k_3)\, \frac{1}{R - s_{13} - 1} \cdots V^{(n-1)}(k_{n-1})(\pi_1^{(n)})^\dagger|0\rangle\,. \tag{3.1}$$

Here n has to be even, the k's are the external momenta, and $s_{ij} = (k_i + k_{i+1} + \cdots + k_j)^2$. The multiperipheral configuration described by Eq. (3.1) is depicted in Fig. 1. The definition of the vertices V and the mass operator R is similar to those of the standard models,[9] with, however, some important differences. In addition to the standard[9] orbital operators a_p^μ, we need another set of Bose operators $\pi_p^{(i)}$ with the following properties:

$$\left[\pi_p^{(i)},\, (\pi_{p'}^{(i)})^\dagger\right] = p\,\delta_{pp'}\,, \qquad (\pi, a^\dagger) = 0\,, \tag{3.2}$$

where p runs from 1 to ∞, and

$$\pi_p^{(2)} \equiv \pi_p^{(3)}, \qquad \pi_p^{(4)} \equiv \pi_p^{(5)}, \qquad \ldots, \qquad \pi_p^{(n)} \equiv \pi_p^{(1)} .$$

In general, $\pi_p^{(i)}$ and $(\pi_p^{(j)})^\dagger$ commute unless $i = j$ or unless they are identified pairwise as in Eq. (3.2). We then have the following equations for V and R,

$$R \equiv \sum_{p=1}^{\infty} \left\{ (a_p^{\mu})^\dagger (a_{p\mu}) + \sum_i p(\pi_p^{(i)})^\dagger \pi_p^{(i)} \right\} ,$$

$$V^{(i)}(k_i) \equiv \pi^{(i)} \exp\left\{ \sqrt{2}\, k_{i\mu} \sum_{p=1}^{\infty} \frac{a_p^{\mu} + (a_p^{\mu})^\dagger}{(p)^{\frac{1}{2}}} \right\} , \qquad (3.3)$$

$$\pi^{(i)} \equiv \sum_{p=1}^{\infty} \left\{ \pi_p^{(i)} + (\pi_p^{(i)})^\dagger \right\} .$$

From these definitions, it is clear that in evaluating Eq. (3.1), as far as the π operators are concerned, only pairwise contractions between neighboring vertices are allowed; and hence the name "nearest neighbor model." These contractions are indicated by arrows in Fig. 1. It is, of course, possible to define a different amplitude by choosing the contractions $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, etc., instead of $2 \leftrightarrow 3$, $4 \leftrightarrow 5$, etc. as we have done. This alternate possibility for the pattern of contractions is indicated in Fig. 2.

At first sight, it may appear that we have a spectrum that increases with the number of external legs, since we have to introduce a new operator for each extra pair of legs. This is not true, however, since an equivalence can be established between different pairs of

π's. These matters are discussed at length in an article by Neveu and Thorn,[10] where operators similar to our π's were first introduced in order to shift the intercept of certain dual models. The price paid for the introduction of the new set of operators is an enlarged spectrum which in general contains ghosts. Therefore, the model defined by Eq. (3.1) has ghosts, although its parent model (Appendix A) does not.

This price has to be paid in order to arrive at a nearest-neighbor type model in the operator space, which, as we shall see, enables us to write a simple linear integral equation for the amplitude.

For the purposes of computation, it is convenient to convert Eq. (3.1) into an integral representation in the standard way.[9] The answer is particularly simple and useful when written in the multi-peripheral form. Assigning variables $u_{1,i}$ to the channels $(1,i)$, with $2 \leqq i \leqq n-2$, we have the following:

$$
\begin{aligned}
B_n(k_1,\cdots,k_n) = & \int_0^1 \int_0^1 \cdots \int_0^1 du_{1,2}\, du_{1,3} \cdots du_{1,n-2} \\
& \times (1-u_{1,2})^{-2} (1-u_{1,4})^{-2} (1-u_{1,6})^{-2} \cdots (1-u_{1,n-2})^{-2} \\
& \times \prod_{\substack{i,j=2 \\ i<j}}^{n-2} (1 - u_{1,i}\, u_{1,i+1} \cdots u_{1,j-1})^{-2k_i \cdot k_j} \\
& \times u_{1,2}^{-s_{1,2}}\, u_{1,3}^{-1-s_{1,3}}\, u_{1,4}^{-s_{1,4}}\, u_{1,5}^{-s_{1,5}-1} \cdots u_{1,n-2}^{-s_{1,n-2}} .
\end{aligned}
\tag{3.4}
$$

An alternative representation, completely equivalent to Eq. (3.4), results from the choice of $u_{2,i}$ as independent variables:

$$B_n(k_1,\cdots,k_n) = \int_0^1\int_0^1\cdots\int_0^1 du_{2,3}\cdots du_{2,n-1}$$

$$\times (1-u_{2,4})^{-2}(1-u_{2,6})^{-2}\cdots(1-u_{2,n-2})^{-2}$$

$$\times u_{2,3}^{-2-s_{2,3}}\, u_{2,4}^{-1-s_{2,4}}\, u_{2,5}^{-2-s_{2,5}}\cdots u_{2,n-1}^{-2-s_{2,n-1}}$$

$$\times \prod_{\substack{i,j=3\\ i<j}}^{n-1}\left(1-u_{2,i}\cdots u_{2,j-1}\right)^{-2k_i\cdot k_j}\,. \qquad (3.5)$$

The lowest lying spectrum can easily be read off either from (3.4) and (3.5), or directly from (3.1), and for the purpose of studying the spectrum, it is convenient to classify the channels into three distinct groups, called the M-type channels, Abelian channels and heavy channels. Any channel containing an odd number of external lines is an M-type channel, and its lowest lying state is the zero mass scalar M discussed earlier. The Abelian channels are the channels of the type (2,3), (2,5), (2,7), (4,5), (4,7), (6,7) etc. The lowest lying states are a tachyon at $m^2 = -1$ and a vector meson at zero mass. The heavy channels are channels like (1,2), (1,4), (3,4) etc. The lowest lying state in these channels has $m^2 = 1$. Due to the presence of the tachyon, the vacuum is unstable and the model is a quite satisfactory laboratory for the investigation of spontaneous symmetry breakdown.

As mentioned earlier, we need an off-mass shell extrapolation of the model in order to be able to define W. This we do in the following way: All intercepts may be shifted by $+c^2$ (via say the method of Thorn and Neveu[10]--an additional nearest neighbor interaction). Then, the external (M) masses may be continued to zero[11] (via an additional next-nearest neighbor interaction, roughly analogous to Thorn and Neveu). The integral representations (3.4) and (3.5) are then modified to the forms given below

$$B_n(k_1,\cdots,k_n,c^2) = \int_0^1 \int_0^1 \cdots \int_0^1 du_{1,2} \cdots du_{1,n-2}$$

$$\times (1-u_{1,2})^{c^2-2}(1-u_{1,3})^{c^2}(1-u_{1,4})^{c^2-2}(1-u_{1,5})^{c^2}\cdots(1-u_{1,n-2})^{c^2-2}$$

$$\times (1-u_{1,2}\,u_{1,3})^{-c^2}(1-u_{1,3}\,u_{1,4})^{-c^2}\cdots(1-u_{1,n-3}\,u_{1,n-2})^{-c^2}$$

$$\times u_{1,2}^{c^2-s_{1,2}}\, u_{1,3}^{c^2-1-s_{1,3}}\, u_{1,4}^{c^2-s_{1,4}}\, u_{1,5}^{c^2-1-s_{1,5}} \cdots u_{1,n-2}^{c^2-s_{1,n-2}}$$

$$\times \prod_{\substack{i,j=2 \\ i<j}}^{n-2} (1-u_{1,i}\cdots u_{1,j-1})^{-2k_i\cdot k_j} \quad . \qquad (3.6)$$

$$B_n(k_1,\cdots,k_n,c^2) = \int_0^1\int_0^1\cdots\int_0^1 du_{2,3}\cdots du_{2,n-1}\,(1-u_{2,3})^{c^2}$$

$$\times\,(1-u_{2,4})^{c^2-2}(1-u_{2,5})^{c^2}(1-u_{2,6})^{c^2-2}\cdots(1-u_{2,n-2})^{c^2-2}$$

$$\times\,(1-u_{2,3}\,u_{2,4})^{-c^2}(1-u_{2,4}\,u_{2,5})^{-c^2}\cdots(1-u_{2,n-2}\,u_{2,n-1})^{-c^2}$$

$$\times\,u_{2,3}^{c^2-2-s_{2,3}}\,u_{2,4}^{c^2-1-s_{2,4}}\,u_{2,5}^{c^2-2-s_{2,5}}\cdots u_{2,n-1}^{c^2-2-s_{2,n-1}}$$

$$\times\prod_{\substack{i,j=3\\ i<j}}^{n-1}(1-u_{2,i}\cdots u_{2,j-1})^{-2k_i\cdot k_j}\,. \qquad (3.7)$$

After the sum given by Eq. (2.1) is performed, c^2 should be set equal to zero and thereby the original model is recovered. There is a complication in our case, however, which invalidates this conclusion. The shifted-intercept model has a larger spectrum than the original one. The extra states introduced by the shifting of the intercept couple to the rest of the states through coupling constants proportional to c, so that in the limit $c \to 0$, their contribution to W is expected to disappear. The above argument is correct for states whose mass remains finite in the limit $c \to 0$. However, it turns out that there are two extra scalar particles under the vector meson at mass $m^2 = c^2$. Although these states also have coupling constants proportional to c, they do not entirely decouple as $c \to 0$, since their propagators become singular in the same limit. It is therefore necessary to subtract their contribution explicitly before taking the limit $c \to 0$. The resulting set of equations are somewhat complicated, and will be dealt with in a future publication.[12] In the interests of simplicity, we shall ignore this complication in this paper, and use the Eqs. (3.6) and (3.7) as they stand. There is no obvious difficulty if we then keep c^2 finite and avoid the limit $c \to 0$. This simplified version of the model then serves as a good training ground for the more complicated situation. Apart from this, the simplified model is a perfectly satisfactory one for the study of vacuum instability in its own right, since for $c^2 < 1$, there is still a tachyon in the theory.

IV. COMPUTATION OF W IN THE PRESENCE OF SPONTANEOUS BREAKDOWN

In this section, we wish to compute W given by Eq. (2.1), using the expression given by (3.6) for B_n. For $c^2 > 1$, we can directly set $k_i = 0$ in (3.6), since the integral representation is convergent, which, parenthetically, is another technical simplification gained by the shift of the intercept. The interesting interval $0 < c^2 < 1$ will be reached later by analytic continuation in c^2. When the external momenta vanish, all correlations between different vertices except the ones between nearest and next-nearest neighbors disappear, and it is possible to write a recursion relation for B_n. It is this feature of the model that makes it manageable. In order to derive the promised recursion relations, we define the following pair of auxiliary functions:

$$f_{2n}(u) \equiv \int_0^1 \int_0^1 \cdots \int_0^1 du_{1,2} \cdots du_{1,2n-2}$$

$$\times (1 - u_{1,2})^{c^2-2}(1 - u_{1,3})^{c^2}(1 - u_{1,4})^{c^2-2} \cdots (1 - u_{1,2n-2})^{c^2-2}$$

$$\times (1 - u_{1,2}\, u_{1,3})^{-c^2}(1 - u_{1,3}\, u_{1,4})^{-c^2} \cdots (1 - u_{1,2n-2}\, u)^{-c^2}$$

$$\times u_{1,2}^{c^2}\, u_{1,3}^{c^2-1}\, u_{1,4}^{c^2} \cdots u_{1,2n-2}^{c^2} ,$$

$$f_{2n+1}(u) \equiv \int_0^1 \int_0^1 \cdots \int_0^1 du_{1,2} \cdots du_{1,2n-1}$$

$$\times (1 - u_{1,2})^{c^2-2} (1 - u_{1,3})^{c^2} \cdots (1 - u_{1,2n-1})^{c^2}$$

$$\times (1 - u_{1,2}\, u_{1,3})^{-c^2}(1 - u_{1,3}\, u_{1,4})^{-c^2} \cdots (1 - u_{1,2n-1}\, u)^{-c^2}$$

$$\times u_{1,2}^{c^2}\, u_{1,3}^{c^2-1}\, u_{1,4}^{c^2} \cdots u_{1,2n-1}^{c^2-1}, \tag{4.1}$$

where

$$f_{2n}(0) = B_{2n}(k_i = 0, c^2) ,$$

as can readily be verified by comparing it with Eq. (3.6). The following set of recursion relations immediately follow from the defining relations of (4.1),

$$f_{2n+1}(u) = \int_0^1 du'\,(u')^{c^2-1}\,(1-u')^{c^2}(1-uu')^{-c^2}\,f_{2n}(u')\,,$$

$$f_{2n}(u) = \int_0^1 du'(u')^{c^2}(1-u')^{c^2-2}\,(1-uu')^{-c^2}\,f_{2n-1}(u')\,. \tag{4.2}$$

We wish to use the above recursion relations to carry out the sum of Eq. (2.1) at zero external momentum. We have, however, first to establish a relation between the S-matrix elements S_n of Eq. (2.3) and B_n of Eqs. (3.4) or (3.6). By definition, S_n is the complete S matrix; it includes all the Feynman graphs in the tree approximation. On the other hand, B_n is defined to include only the planar Feynman graphs.[9] It then follows that we have to symmetrize B_n with respect to the external lines. When all the lines carry zero momentum, this merely introduces a factor of $n!$ On the other hand, since B_n is already cyclically symmetric, the planar graphs are overcounted by a factor of n in this process. Therefore, the precise relationship is the following:

$$T_n = (n-1)!\;\; B_n(k_i = 0)\,. \tag{4.3}$$

It is now convenient to define the following auxiliary function $\overline{W}(J)$,

$$\overline{W}(J) \equiv \sum_{n=4}^{\infty} \left(\frac{J}{c^2}\right)^n B_n(k_i = 0, c^2)$$

$$= J\,\frac{d}{dJ}\Big(W(J)\Big) - \frac{J^2}{c^2}\,. \tag{4.4}$$

$\overline{W}$ contains the same amount of information as W, given $\overline{W}$ -- apart from a trivial constant, one can solve for W. The advantage of using $\overline{W}$ is the fact that it satisfies a simple integral equation. To derive this equation, we further define the following,

$$\overline{W}_1(J,u) \equiv \sum_{n=2}^{\infty} \left(\frac{J}{c^2}\right)^{2n} f_{2n}(u) ,$$

$$\overline{W}_2(J,u) \equiv \left(\frac{J}{c^2}\right)^3 + \sum_{n=2}^{\infty} \left(\frac{J}{c^2}\right)^{2n+1} f_{2n+1}(u) , \tag{4.5}$$

where

$$\overline{W}(J) = \overline{W}_1(J, u=0) .$$

The recursion relations of Eq. (4.2) can then be used to arrive at the following set of equations:

$$\overline{W}_1(\lambda,u) = \lambda \int_0^1 du'(u')^{c^2} (1-u')^{c^2-2} (1-uu')^{-c^2} \overline{W}_2(\lambda,u') , \tag{4.6a}$$

$$\overline{W}_2(\lambda,u) = \lambda^3 + \lambda \int_0^1 du'(u')^{c^2-1} (1-u')^{c^2} (1-uu')^{-c^2} \overline{W}_1(\lambda,u') , \tag{4.6b}$$

where

$$\lambda = \frac{J}{c^2} .$$

The above equations are the fundamental result of this section, and the rest of the section will be devoted to solving them. We already know one solution, namely the perturbation solution mentioned earlier. This solution is generated by iteration from the starting point $\overline{W}_2^{(0)}(\lambda,u) = \lambda^3$, and corresponds to the original sum of

Eq. (4.4). In what follows, we will find other solutions to the fundamental set of equations. These solutions also admit a power series expansion in λ; however, their starting point is different from the perturbation solution. The existence of the extra solutions depend on the fact that the kernel of the integral equations is singular. The nature of this singularity depends on the value of c^2. Initially, we have to start with $c^2 > 1$, since the integrals of Eq. (4.1) only exist in this range. The kernels of Eqs. (4.6a) and (4.6b) are then at most marginally (integrably) singular at either end point $u' = 0$ and $u' = 1$. However, when c^2 is continued to the critical interval $0 < c^2 < 1$, Eq. (4.6a) develops a non-integrable singularity at $u' = 1$. It is this singularity that makes the existence of nonperturbative solutions possible when $c^2 < 1$. Since a tachyon is introduced in the spectrum under the same condition, the new solutions and the instability of the vacuum are related in a very satisfactory manner.

It is clear that the behavior of the kernel of Eq. (4.6a) near $u' = 1$ is of crucial importance, whereas the point $u' = 0$ at most corresponds to a mild singularity in either equation, so long as $c^2 > 0$. It is then convenient to change variables by $u = 1 - x$, $u' = 1 - y$, in order to arrive at the following form of the equations:

$$\overline{W}_1(\lambda,x) = \lambda \int_0^1 dy\, y^{c^2-2}(1-y)^{c^2}(x+y-xy)^{-c^2}\, \overline{W}_2(\lambda,y) , \tag{4.7a}$$

$$\overline{W}_2(\lambda,x) = \lambda^3 + \lambda \int_0^1 dy\, y^{c^2}(1-y)^{c^2-1}(x+y-xy)^{-c^2}\, \overline{W}_1(\lambda,y) . \tag{4.7b}$$

We make the ansatz that the solution can be written as an infinite superposition of different powers of x as follows:

$$\bar{W}_{1,2} = \sum_{p} \sum_{n=0}^{\infty} d_{p,n}^{(1,2)} x^{\alpha_p + n} , \qquad (4.8)$$

where the d's are x-independent. The α_p are in general expected to depend on λ, as well as the d's. In fact, we argue that all α_p have to be λ dependent, and fixed integer values for α_p are not permissible: Any fixed power, upon iteration, is not stable; it eventually generates powers of $\log(x)$, which, when summed, convert the fixed power to a moving (λ dependent) power. Moving powers, on the other hand, are stable under the iteration procedure, since, unlike the fixed power case, there is no mechanism for the generation of polynomials in $\log(x)$. The situation is similar to the absence of fixed singularities in the complex angular momentum plane in the solution to Schroedinger or Bethe-Salpeter type equations.

Notice that, for convenience, the values of α differing by integer units are lumped together in one "family" in Eq. (4.8), again in analogy to the "daughters" of Regge theory.

We are going to break up our approach to solving these equations into two steps. (1) Consistency conditions: The homogeneous form of the equations relate the coefficients of all daughters in a moving family, leaving one normalization parameter (family undetermined). (2) Cancellation (or normalization) conditions which fix the remaining parameter of each family--by requiring cancellation of all fixed poles (say in the inhomogeneous term).

Consistency Conditions

If a member of the family belonging to a given power α_p is substituted into the right-hand side of Eq. (4.7), a member of the same family is again obtained as the output. It then becomes possible to obtain a consistency equation involving only one family at a time. Defining

$$\bar{W}_{1,2}^{(p)} = \sum_{n=0}^{\infty} d_{p,n}^{(1,2)} x^{\alpha_p + n} , \tag{4.9}$$

and also making the following change of variable for convenience,

$$x = \frac{\omega}{1+\omega} , \qquad y = \frac{\omega t}{1+\omega t} , \tag{4.10}$$

we obtain the following consistency equations for $\bar{W}_p$,

$$W_1^{(p)}(\omega) = \frac{\lambda}{\omega} (1+\omega)^{c^2} \int_0^\infty dt\, t^{c^2-2} (1+\omega t)^{-c^2} (1+t+\omega t)^{-c^2} \bar{W}_2^{(p)}(\omega t), \tag{4.11a}$$

$$\bar{W}_2^{(p)}(\omega) = \lambda\omega (1+\omega)^{c^2} \int_0^\infty dt\, t^{c^2} (1+\omega t)^{-1-c^2} (1+t+\omega t)^{-c^2} \bar{W}_1^{(p)}(\omega t). \tag{4.11b}$$

Notice that we use here only the homogeneous form of the equations. This is adequate for matching moving powers. One can now substitute Eq. (4.6) in Eqs. (4.11) and match powers of ω on both sides. The resulting equations enable one to determine both α_p and all d's in terms of one arbitrary normalization parameter. We exhibit below the equations for two leading powers of ω ,

$$d_{p,0}^{(2)} = 0 ,$$

$$d_{p,0}^{(1)} = \lambda\, d_{p,1}^{(2)} B(c^2 + \alpha_p, \quad -\alpha_p) ,$$

$$d_{p,1}^{(2)} = \lambda\, d_{p,0}^{(1)} B(c^2 + \alpha_p + 1, \quad -\alpha_p - 1) ,$$

$$d_{p,1}^{(1)} = \lambda\, d_{p,2}^{(2)} B(c^2 + \alpha_p + 1, \quad -\alpha_p - 1) - c^2\lambda\, d_{p,1}^{(2)} B(c^2 + \alpha_p + 1, \quad -\alpha_p) + (\alpha_p + c^2) d_{p,0}^{(1)} ,$$

$$d_{p,2}^{(2)} = \lambda(d_{p,1}^{(1)} - d_{p,0}^{(1)}) B(c^2 + 2 + \alpha_p, \quad -2 - \alpha_p) - c^2\lambda\, d_{p,0}^{(1)} B(c^2 + \alpha_p + 2, \quad -1 - \alpha_p) + (\alpha_p + c^2 + 1) d_{p,1}^{(2)}, \qquad (4.12)$$

where B is the Euler beta-function.

Multiplying the third and the second equations, we obtain the following eigenvalue condition for α_p ,

$$\lambda^2 B(c^2 + \alpha_p, \quad -\alpha_p) B(c^2 + \alpha_p + 1, \quad -1 - \alpha_p) = 1, \qquad (4.13)$$

This is a transcendental equation for α_p, which is in general difficult to solve. However, a power series solution in λ is easy to obtain. As $\lambda \to 0$, one or both of the beta-functions must develop a pole (s) to satisfy Eq. (4.13). This means that α_p must approach either an integer greater than or equal to minus one, or it must tend to the points $-c^2 + n$. It is then convenient to classify α_p according to the value it reaches at $\lambda = 0$ as follows:

$$\alpha_p \to -1 + p, \qquad \text{for} \quad p \geqq 0 \,,$$
$$\alpha_p \to -c^2 + p + 1, \qquad \text{for} \quad p \leqq -1 \,, \tag{4.14}$$

as $\lambda \to 0$. The function α_p can be evaluated to lowest order in λ by replacing B's by the relevant poles and then solving Eq. (4.13). The results are given below,

$$\alpha_0 = -1 - \frac{\lambda^2}{c^2 - 1} + \cdots , \qquad \alpha_{-1} = -c^2 + \frac{\lambda^2}{c^2 - 1} + \cdots ,$$

$$\alpha_p = -1 + p \pm \frac{i\lambda}{\Gamma(c^2)} \left[\Gamma(c^2 - 1 + p)\,\Gamma(c^2 + p)\right]^{1/2} + \cdots , \quad (p > 0) ,$$

$$\alpha_p = -c^2 + p + 1 \pm \frac{i\lambda}{\Gamma(c^2)} \left[\Gamma(c^2 - 1 - p)\,\Gamma(c^2 - p - 2)\right]^{1/2} + \cdots , \quad (p < -1) . \tag{4.15}$$

Notice that there are two trajectories of α_p and hence two families starting at each point except for the points $p = 0$ and $p = -1$. One can now substitute α_p back into the equations (4.12), and determine all the d_p's in a given family in terms of one arbitrarily chosen d_p. For example, for the $p = -1$ family, everything can be determined in terms of $d^{(1)}_{-1,0}$, and some of these relations in lowest order λ are given below:

$$d^{(2)}_{-1,1} \cong \frac{\lambda}{c^2 - 1} \, d^{(1)}_{-1,0} ,$$
$$d^{(2)}_{-1,2} \cong \frac{\lambda}{(c^2 - 1)(2 - c^2)} \, d^{(1)}_{-1,0} ,$$
$$d^{(1)}_{-1,1} \cong \frac{\lambda^2}{(c^2 - 1)(2 - c^2)} \, d^{(1)}_{-1,0} , \quad \text{etc.} \tag{4.16}$$

From these expressions, it is clear that in the family $p = -1$, all d's are at least one power of λ down compared to $d^{(1)}_{-1,0}$, the leading term in $\overline{W}_1$. Since this point will be of importance in what follows, it is worthwhile to understand how it comes about. A glance at Eqs. (4.12) shows that there is always an extra factor of λ on the right-hand side. However, this λ can easily be canceled by a singularity of the beta-function. This is what happens in the second equation; the beta-function is near a pole as $\alpha_{-1} \cong -c^2$ and develops a singularity of the form $1/\lambda^2$. It then follows that $d^{(1)}_{p,0} \sim \frac{1}{\lambda} d^{(2)}_{p,1}$, contrary to the initial impression. In all the other equations, however, the first argument of the beta-function is sufficiently shifted, and there is no singularity as $\alpha_{-1} \to -c^2$. Hence, the extra power of λ for all the d's except for $d^{(1)}_{-1,0}$.

Cancellation Conditions

The problem of solving the Eqs. (4.7) then reduces to determining a set of an infinite number of normalization parameters, one for each family. These parameters are to be determined by what we call the cancellation conditions: We have already postulated that there are no fixed integer powers in the expansion of the $\overline{W}$'s; however, the right-hand side of Eq. (4.7) can readily develop such powers. We have to require the coefficients of these fixed powers to vanish. This is achieved by expanding the integrals on the right-hand side of Eq. (4.7) directly into power series in x and setting the coefficients equal to zero. The result is the following set of cancellation conditions:

$$\int_0^1 dy\, y^{-2-n} (1 - y)^{c^2+n}\, \overline{W}_2(y) = 0 , \qquad (4.17a)$$

$$\lambda^2 \delta_{n,0} + \int_0^1 dy\, y^{-n} (1 - y)^{c^2-1+n}\, \bar{W}_1(y) = 0 , \qquad (4.17b)$$

where n ranges over all positive integers, from 0 to ∞.

Needless to say, we are not going to attempt solving Eqs. (4.17) exactly. Instead, we will resort to an expansion in powers of λ, which simplifies matters considerably. It is of interest to reexamine the perturbation solution, obtained by the straightforward iteration of the inhomogeneous term, in the light of the present approach. The perturbation solution uses only the families $p \geqq 0$ in the expansion of Eq. (4.9). This becomes clear when one observes that, in the iteration of the inhomogeneous term, no power of the form x^{-c^2+p+1}, where p is an integer, can ever appear. Therefore, the families belonging to $p \leqq -1$, as defined by (4.14), are absent. The expansion of (4.9) can then be substituted in Eqs. (4.17), yielding a set of condition which determine d's. We have checked that these equations yield the normal iteration solution, and we shall not pursue this topic any further.

Our main interest lies in nonperturbation solutions. These solutions clearly have to make use of families with $p \leqq -1$ in order to be different from the perturbation solution. The simplest possibility is to replace the family at $p = 0$ by the family at $p = -1$, which leads to the following ansatz,

$$\bar{W}_{1,2} = \sum_{n=0}^{\infty} d^{(1,2)}_{-1,n}\, x^{\alpha_{-1}+n} + \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} d^{(1,2)}_{p,n}\, x^{\alpha_p+n} . \qquad (4.18)$$

In Appendix B, we shall generalize this ansatz by allowing several positive p families to be replaced by the negative p families. In what follows immediately, we shall show that the ansatz of Eq. (4.18) indeed produces a unique solution to the fundamental set of equations (4.7). The significance of this ansatz is further discussed at the end of the section.

Substituting Eq. (4.18) into (4.17), we obtain the following set of conditions on the unknown coefficients:

$$\sum_{p} \sum_{n=1}^{\infty} d^{(2)}_{p,n} \; B(\alpha_p + n - 1 - m, \;\; c^2 + m + 1) \;=\; 0 \,, \qquad (4.19a)$$

$$\lambda^2 \, \delta_{m,0} \;+\; \sum_{p} \sum_{n=0}^{\infty} d^{(1)}_{p,n} \; B(\alpha_p + n - m + 1, \;\; c^2 + m) \;=\; 0 \,, \qquad (4.19b)$$

where $0 \leqq m \leqq \infty$, and the summation over p starts at $p = -1$, skips the value $p = 0$, and then runs over all positive integers starting with one.

We now assert that the $d^{(1)}$'s are down by a factor of λ compared to $d^{(1)}_{-1,0}$;

$$d^{(1)}_{-1,n} \sim \lambda \, d^{(1)}_{-1,0} \qquad \text{for} \quad n \geqq 1 \,, \qquad (4.20a)$$

$$d^{(1)}_{p,n} \sim \lambda \, d^{(1)}_{-1,0} \qquad \text{for} \quad p \geqq 1 \,. \qquad (4.20b)$$

Via our previous discussion including Eq. (4.16), we have already proven (4.20a). To prove (4.20b), consider Eqs. (4.19b). Since for $p \geqq 1$, α_p is close to an integer, certain terms in the sum in

Eq. (4.19b) are near the poles of the beta function. This leads to singularities of the form $1/\lambda$ as follows:

$$B(\alpha_p + n - m + 1, \quad c^2 + m) \sim \frac{1}{\lambda} \tag{4.21}$$

for $p \geqq 1$, and $p + n - m = 0$. This result immediately follows from Eq. (4.15). However, for $p = -1$, one is not near a pole, and there are no $1/\lambda$ type singularities. Therefore, the coefficient of $d^{(1)}_{-1,0}$ has an extra factor of λ compared to the other d's. If, now, one imagines solving the Eq. (4.19b), $d^{(1)}_{-1,0}$ will clearly be the leading term by one power of λ. This enables us to write the following result, valid to the leading order,

$$\overline{W}_1(x) \cong d^{(1)}_{-1,0}\, x^{-c^2 + \lambda^2/(c^2-1)} . \tag{4.22}$$

Substituting this in Eq. (4.19b) with $m = 0$, we obtain the following final formula, again valid to the leading order in λ,

$$d^{(1)}_{-1,0} \cong \frac{\lambda^2}{B(1 - c^2, \quad c^2)} ,$$

$$\overline{W}_1(x) \cong - \frac{\lambda^2}{B(1 - c^2, \quad c^2)}\, x^{-c^2 + \lambda^2/(c^2-1)} . \tag{4.23}$$

This equation is the starting point of a power series expansion in λ for the new solution. It can be substituted in Eqs. (4.19) and (4.11) to obtain the terms higher order in λ. For example, Eq. (4.16) yields the result below,

$$d^{(2)}_{-1,1} \cong \frac{\lambda}{c^2 - 1}\, d^{(1)}_{-1,0} = \frac{\lambda^3}{(1 - c^2)\, B(1 - c^2, \quad c^2)} . \tag{4.24}$$

This approach has the merit of keeping the powers of x intact. An alternative approach is to set $\lambda = 0$ in Eq. (4.22), and use that as a starting point for a straightforward iteration of the original Eqs. (4.7). In this case, one encounters powers of $\log(x)$ arising from the expansion of powers of x in λ. In either approach, it is clear that $\overline{W}_2$, as well as corrections to $\overline{W}_1$ given by Eq. (4.22), are at least of the order of λ^3.

Now let us examine the solution we have obtained from the point of view of spontaneous symmetry breaking. We set $x = 1$ $(u = 0)$ in Eq. (4.23), as we are instructed to do by Eq. (4.5), remind ourselves that $\lambda = J/c^2$, and solve for W from Eq. (4.4). The result, to the lowest order in J, is the following:

$$W(J) \cong -\frac{1}{2}\,\frac{J^2}{c^2}\left(\frac{1}{c^2\,B(1 - c^2,\ c^2)} - 1\right). \tag{4.25}$$

The fact that the quadratic term in J is different from the standard perturbation value $\left(\frac{1}{2}\,\frac{J^2}{c^2}\right)$ shows that some kind of spontaneous breakdown must have taken place. However, the absence of a linear term in J in Eq. (4.25) tells us that the probing field has not acquired any vacuum expectation value; this follows from the well-known relation

$$\langle\, \phi\, \rangle = \left(\frac{\partial W}{\partial J}\right)_{J=0}. \tag{4.26}$$

The question is, what has happened? As discussed in Section II, our answer is that the effective potential at hand is many-valued, and the spontaneous breakdown occurred by going to a different branch of the potential. We refer the reader to Eqs. (2.6) and (2.7) and to the discussion that follows these equations. It was shown there, that, in a simple Lagrangian example, spontaneous breakdown takes place in exactly the same way as we are suggesting in the present case. Although the probing field acquires no vacuum expectation value, the hidden fields corresponding to internal scalar particles do acquire nonzero expectation values, and this reflects itself in the many-valuedness of the effective potential. Notice also that the probing field (the M field) must possess some kind of G parity for this to happen, and in the present case, this requirement is satisfied. We see no plausible alternative to the above explanation.

Are there any solutions to our integral equations where the probing field acquires an expectation value? As we shall show in Appendix B, such solutions emerge when we allow a more complicated reshuffling of the trajectories α_p . However, as far as we can tell, the solution given above is a perfectly satisfactory and nontrivial example of spontaneous symmetry breaking. Our form of the propagator for this solution (next section) will fully support this conclusion.

At this point, we have to remind the reader that all we have accomplished so far is find new solutions to Eqs. (4.6) and (4.7). It is gratifying to note that in the critical interval $0 < c^2 < 1$, the perturbation solution, where $\overline{W}_1$ behaves like x^{-1} , is too singular to satisfy Eq. (4.7b), whereas the new solution, where $\overline{W}_1$ behaves like x^{-c^2} , is less singular and satisfies Eq. (4.7b).

Sheet-Structure in J

What remains to be shown is that starting with the perturbation solution and going around some branch point in the J or λ plane, one ends up with the new solution. This is not a trivial point since one could end up with a linear combination of various solutions to the integral equations! Since we are able to solve the integral equations only near $\lambda = 0$ as a power series in λ, the Riemann sheet structure of the solution is lost, and it may seem that we are forced to go beyond the power series expansion. Luckily, things are not that complicated; all we need to know is the Riemann sheet structure of α_p in Eq. (4.8) as a function of λ. Since the α_p are defined to be the solutions of Eq. (4.13), the analyticity properties of this equation in λ is all that needs be studied. Equation (4.13) is however a complicated transcendental equation, and we have not studied its solutions in full generality. What we have done, instead, is to study the hypothesis of interchange of trajectories embodied in the ansatz of Eq. (4.18), in a somewhat simplified version of Eq. (4.13). Since we are dealing only with trajectories at $p = 0$ and $p = -1$ in this ansatz, it seems like a sensible approximation to replace the beta-functions in Eq. (4.13) by the poles that generate the above trajectories. The effect of the other poles is represented by a constant background term. This amounts to the following approximations:

$$\begin{aligned} B(c^2 + \alpha_p, \ -\alpha_p) &\cong \frac{1}{c^2 + \alpha_p} + c_1 , \\ B(c^2 + \alpha_p + 1, \ -1 - \alpha_p) &\cong -\frac{1}{1 + \alpha_p} + c_2 , \end{aligned} \qquad (4.27)$$

where c_1 and c_2 are two constants representing the background, whose precise values are not of interest to us. With these approximations, Eq. (4.13) becomes the following quadratic equation

$$\alpha_p^2 (1 - \lambda^2 c_1 c_2) + \alpha_p \left[c^2 + 1 - \lambda^2 (c_1 c_2 - c_1 + c_2 + c_1 c_2 c^2) \right] + c^2 - \lambda^2 (c^2 c_1 + 1)(c^2 - 1) = 0 . \qquad (4.28)$$

The solutions to this equation look messy; however, one can easily verify the following simple statements. Let $\alpha^{(\pm)}$ be the solutions corresponding to the positive and negative branches of the square root respectively. It then follows that,

for $c^2 > 1$,

$$\alpha^{(+)} \to -1, \qquad \alpha^{(-)} \to -c^2, \qquad \text{as} \qquad \lambda \to 0 ;$$

for $0 < c^2 < 1$,

$$\alpha^{(+)} \to -c^2, \qquad \alpha^{(-)} \to -1, \qquad \text{as} \qquad \lambda \to 0. \tag{4.29}$$

In this approximation, our ansatz about the interchange of the trajectories with $p = 0$ and $p = -1$ [when we continue from one branch of the solution to another branch] is then completely justified.

V. THE PROPAGATOR

In this section, we shall examine the propagator of the non-perturbation (sic) solution derived in the last section. Our main result is that the new propagator has its intercept raised by one unit, so that the M and the photon move to the point $m^2 = 1 + c^2$, and the tachyon is lifted to $m^2 = c^2$. To compute the new propagator, we imagine an arbitrary number of M's emitted at zero momentum from an internal line of the dual model, as shown in Fig. 3. The contribution of all such M's is summed, and then the external field J to which they couple is set equal to zero. One possible final answer is the original propagator one started from. However, if the propagator is a many valued function of J, in analogy to W, then a nontrivial final answer is possible. This approach is similar to Lee's treatment of the sigma model.[13]

The demonstration that emission of M-spurions (only) is adequate to obtain the propagator follows the same lines of reasoning as in Section II. The reader is invited to follow Lee's reasoning through, e.g. in the simple-model of Section II--now emitting only J_ϕ spurions. The point is that one obtains exactly the expected equations for both propagators and for $\langle \sigma \rangle$ and $\langle \phi \rangle$, now as a function of J_ϕ (and $J_\sigma = 0$). As one intends solving these (nonlinear) relations at both $J_\phi = J_\sigma = 0$, we miss no solutions at all.[14]

The basic idea in computing the propagator is to derive integral equations similar to Eqs. (4.6). There is, however, an additional complication in the case of the propagator. The external lines can be emitted both in the "up" and the "down" direction symmetrically. This makes it difficult to set up a multiperipheral integral similar to Eq. (3.4). Using arguments based on duality, however, we can move, say, the lines emitted downward to the right-hand side, so that there is no intermixing between the up and the down lines, as shown in Fig. 4. We are then able to sum the up and down multiperipheral chains separately, and then combine them at the end as in Fig. 4.

Let us now establish some notation. For the sake of being definite, let D denote the subpropagator where all the spurions are emitted in the up direction. The subpropagator where the spurions are emitted downward is simply related to the above and need not be calculated separately. By definition, the sum for D starts with the emission of at least one line, and after the summation is performed, the external field J is set equal to zero.

It is convenient to label D as follows. First of all, D can depend on the channels it connects. Let the letters A, M, and H stand for the Abelian, M and heavy channels respectively. This then leads to the labeling D_{AA}, D_{AM}, D_{MH}, etc. Furthermore, the subpropagator is a function of the variables u and v, associated with the initial and final lines, and of the variable s, the square of the momentum it carries. In addition, there is some operator dependence that can be read off from Eq. (3.1). We imagine having performed the nearest-neighbor contractions between the π's, so they will not appear in the final integral representation. In contrast, the a's are capable of long-range contractions that cannot be carried out solely in the propagator, and so they have to remain in the definition of D. These arguments show that the subpropagator must be of the following form:

$$D \equiv D(u, v, \tilde{R} - s) , \tag{5.1}$$

where

$$\tilde{R} = \sum_{p=1}^{\infty} (a_p{}^{\mu})^+ a_{p\mu} .$$

Notice that D depends on the orbital operators only through the combination $\tilde{R} - s$, which means that the trajectories always stay linear, even after the spontaneous breakdown.[2]

There is one further complication involved in the definition of D_{MM} --that there are two different patterns of contraction for the emitted spurion lines. (The term "contraction" is used in the sense of the section following Eq. (3.3).) One pattern of contraction leads to the sequence of channels M, H, M, H, etc. as we follow the propagator line, and the other results in the sequence M, A, M, A, etc., as shown in Fig. 5. We denote the second kind of D by $\tilde{D}_{MM}$, to distinguish it from the first possibility, which has no wiggly line.

We are now ready to proceed with our derivation in a manner similar to that of Section IV. In fact, the steps leading to Eq. (4.6) can be taken over with only minor modifications. In parallel to Eq. (4.5), it is convenient to define the following series:

$$F_{MM}(\lambda,\ u,\ v,\ \tilde{R} - s) \equiv \sum_{n=1}^{\infty} \lambda^{2n} F_{MM}^{(2n)} ,$$
$$F_{MH}(\lambda,\ u,\ v,\ \tilde{R} - s) \equiv \sum_{n=0}^{\infty} \lambda^{2n+1} F_{MH}^{(2n+1)} , \tag{5.2}$$

with similar definitions for the other combinations of channels. Here $\lambda = J/c^2$ as before, and the connection between D and F is

$$D(u,\ v,\ \tilde{R} - s) = F(\lambda = 0,\ u,\ v,\ \tilde{R} - s) . \tag{5.3}$$

The functions $F^{(2n)}$ and $F^{(2n+1)}$ can be written in form of integrals very similar to Eq. (4.1), i.e.,

$$F_{MM}^{(2n)}(u,\ v,\ \tilde{R} - s) = \int_0^1 \int_0^1 \cdots \int_0^1 du_1 \cdots du_{2n-1}$$
$$\times (1 - u_1)^{c^2-2} (1 - u_2)^{c^2} (1 - u_3)^{c^2-2} \cdots (1 - u_{2n-1})^{c^2-2}$$
$$\times (1 - uu_1)^{-c^2} (1 - u_1u_2)^{-c^2} (1 - u_2u_3)^{-c^2} \cdots (1 - u_{2n-1}v)^{-c^2}$$
$$\times u_1^{c^2+\tilde{R}-s}\ u_2^{c^2-1+\tilde{R}-s} \cdots u_{2n-1}^{c^2+\tilde{R}-s} ,$$

(Eq. 5.4 continued)

$$F_{MH}^{(2n+1)}(u,\ v,\ \tilde{R} - s) = \int_0^1 \int_0^1 \cdots \int_0^1 du_1 \cdots du_{2n}$$

$$\times (1 - u_1)^{c^2-2} (1 - u_2)^{c^2} (1 - u_3)^{c^2-2} \cdots (1 - u_{2n})^{c^2}$$

$$\times (1 - uu_1)^{-c^2} (1 - u_1 u_2)^{-c^2} \cdots (1 - u_{2n}\ v)^{-c^2}$$

$$\times u_1^{c^2+\tilde{R}-s}\ u_2^{c^2-1+\tilde{R}-s} \cdots u_{2n}^{c^2-1+\tilde{R}-s} . \tag{5.4}$$

A similar set of equations hold for the $\tilde{F}$'s ,

$$\tilde{F}_{MM}^{(2n)}(u,\ v,\ \tilde{R} - s) = \int_0^1 \cdots \int_0^1 du_1 \cdots du_{2n-1}$$

$$\times (1 - u_1)^{c^2} (1 - u_2)^{c^2-2} (1 - u_3)^{c^2} \cdots (1 - u_{2n-1})^{c^2}$$

$$\times (1 - uu_1)^{-c^2} (1 - u_1 u_2)^{-c^2} \cdots (1 - u_{2n-1}\ v)^{-c^2}$$

$$\times u_1^{c^2-2+\tilde{R}-s}\ u_2^{c^2-1+\tilde{R}-s} \cdots u_{2n-1}^{c^2-2+\tilde{R}-s} ,$$

$$\tilde{F}_{MA}^{(2n+1)}(u,\ v,\ \tilde{R} - s) = \int_0^1 \cdots \int_0^1 du_1 \cdots du_{2n}$$

$$\times (1 - u_1)^{c^2} (1 - u_2)^{c^2-2} \cdots (1 - u_{2n})^{c^2-2}$$

$$\times (1 - uu_1)^{-c^2} (1 - u_1 u_2)^{-c^2} \cdots (1 - u_{2n}\ v)^{-c^2}$$

$$\times u_1^{c^2-2+\tilde{R}-s}\ u_2^{c^2-1+\tilde{R}-s} \cdots u_{2n}^{c^2-1+\tilde{R}-s} . \tag{5.5}$$

The reader can easily supply the expressions for the other F's, by factorizing, for example, Eqs. (3.6) and (3.7).

Notice that, when the external legs of the propagator are on the mass shell, the difference between Eqs. (5.4) and (5.5) disappear. The two expressions are then related by a cyclic transformation, just like (3.4) and (3.5). When the legs are off the mass shell, however, we have to distinguish between them.

Equations (5.5) and (5.4) lead to recursion relations of the type given by Eq. (4.2), from which we finally obtain the fundamental set of integral equations, analogous to (4.2). In terms of the convenient set of variables $z = 1 - u$, $x = 1 - v$, $y = 1 - v'$, $s - \tilde{R} = \bar{s}$, these equations are given below:

$$F_{MM}(\lambda, x, z, \bar{s}) = \lambda \int_0^1 dy\, y^{c^2-2} (1 - y)^{c^2-\bar{s}} (x + y - xy)^{-c^2} F_{MH}(\lambda, y, z, \bar{s}),$$

$$F_{MH}(\lambda, x, z, \bar{s}) = \lambda(z + x - xz)^{-c^2} + \lambda \int_0^1 dy\, y^{c^2} (1 - y)^{c^2-1-\bar{s}} \times (x + y - xy)^{-c^2} F_{MM}(\lambda, x, z, \bar{s}) .$$

(5.6a)

$$F_{AA}(\lambda, x, z, \bar{s}) = \lambda \int_0^1 dy\, y^{c^2-2} (1 - y)^{c^2-\bar{s}-1} (x + y - xy)^{-c^2} F_{AM}(\lambda, y, z, \bar{s}),$$

$$F_{AM}(\lambda, x, z, \bar{s}) = \lambda(x + z - xz)^{-c^2} + \lambda \int_0^1 dy\, y^{c^2} (1 - y)^{c^2-\bar{s}-2} \times (x + y - xy)^{-c^2} F_{AA}(\lambda, y, z, \bar{s}) .$$

(5.6b)

Once the above integral equations are solved, the other F's can be determined directly through the recursion relations. For example, $\tilde{F}_{MM}$ and F_{HH} can be computed as follows:

$$\tilde{F}_{MM}(\lambda, x, z, \bar{s}) = \lambda \int_0^1 dz'(z')^{c^2} (1-z')^{c^2-2-\bar{s}} (z+z'-zz')^{-c^2} \times F_{AM}(\lambda, x, z', \bar{s}) ,$$

$$F_{HH}(\lambda, x, z, \bar{s}) = \lambda \int_0^1 dz'(z')^{c^2} (1-z')^{c^2-1-\bar{s}} (z+z'-zz')^{-c^2} \times F_{MH}(\lambda, x, z', \bar{s}) . \tag{5.7}$$

In addition, the following symmetry relations are useful:

$$F_{MA}(\lambda, x, z, \bar{s}) = F_{AM}(\lambda, z, x, \bar{s}) ,$$

$$F_{MH}(\lambda, z, x, \bar{s}) = F_{HM}(\lambda, x, z, \bar{s}) , \text{ etc.} \tag{5.8}$$

It is also a useful check on the results to notice that the diagonal F's of the form F_{AA}, F_{MM}, etc. have to be symmetric in the variables x and z .

The method of solving Eqs. (5.6) completely parallels the approach we used in solving Eqs. (4.7). We again make the ansatz (4.8), where the d's can now be functions of z and $\bar{s}$. The analogue of Eqs. (4.11) in the case of F_{MM} and F_{MH} are the following:

$$F_{MM}^{(p)}(\omega) = \frac{\lambda}{\omega}(1+\omega)^{c^2}\int_0^\infty dt\, t^{c^2-2}(1+\omega t)^{-c^2-\bar{s}}(1+t+\omega t)^{-c^2}F_{MH}^{(p)}(t\omega),$$

$$(5.9)$$

$$F_{MH}^{(p)}(\omega) = \lambda\omega(1+\omega)^{c^2}\int_0^\infty dt\, t^{c^2}(1+\omega t)^{-1-c^2-\bar{s}}(1+t+\omega t)^{-c^2}F_{MM}^{(p)}(t\omega).$$

It is now easy to verify that the equations for the coefficients of the leading powers are again the first three equations in (4.12), with no modification. In particular, Eq. (4.13) is unchanged, so we have the same trajectories as before. The equations for nonleading d's are in general modified by terms proportional to $\bar{s}$. In this section, however, we shall not need the nonleading terms.

The cancellation conditions of Eqs. (4.17) still remain valid, with some minor modifications. For example, F_{MM} and F_{MH} satisfy the following:

$$\int_0^1 dy\, y^{-2-n}(1-y)^{c^2+n-\bar{s}}\, F_{MH}(y,\, z,\, \bar{s}) = 0\,, \qquad (5.10a)$$

$$\int_0^1 dy\, y^{-n}\,(1-y)^{c^2-1+n-\bar{s}}\, F_{MM}(y,\, z,\, \bar{s})$$

$$+ \frac{\Gamma(1-c^2)}{\Gamma(1-c^2-n)}\, z^{-c^2-n}(1-z)^n = 0,$$

$$(5.10b)$$

where n again ranges from 0 to ∞.

To obtain a solution different from the perturbation solution, we again adopt the ansatz of Eq. (4.18), and in analogy to Eq. (4.19), we arrive at the following set of equations:

$$\sum_p \sum_{n=1}^{\infty} d_{p,n}^{(MH)} B(\alpha_p + n - 1 - m, \quad c^2 + m + 1 - \bar{s}) = 0 , \tag{5.11a}$$

$$\sum_p \sum_{n=0}^{\infty} d_{p,n}^{(MM)} B(\alpha_p + n - m + 1, \quad c^2 + m - \bar{s}) + \frac{\Gamma(1 - c^2)}{\Gamma(1 - c^2 - n)} z^{-c^2 - n}(1 - z)^n = 0 , \tag{5.11b}$$

where, again, $0 \leqq m \leqq \infty$. The arguments about the order of various terms in λ [which follow Eq. (4.16)] remain valid in the present case also. It then follows that the only d which is zeroth order in λ is $d_{-1,0}^{(MM)}$; all other d's are proportional to at least one power of λ. Solving Eq. (5.11b) with $m = 0$ for $d_{-1,0}^{(MM)}$, we obtain the following,

$$F_{MM}(\lambda = 0, x, z, \bar{s}) = - \frac{(xz)^{-c^2}}{B(1 - c^2, c^2 - \bar{s})} \tag{5.12}$$

$$F_{MH}(\lambda = 0, x, z, \bar{s}) = 0 .$$

Equations (5.6b) can be treated along the same lines with only trivial modifications. The results are given below:

$$F_{AA}(\lambda = 0, x, z, \bar{s}) = - \frac{(xz)^{-c^2}}{B(1 - c^2, c^2 - \bar{s} - 1)} , \tag{5.13}$$

$$F_{AM}(\lambda = 0, x, z, \bar{s}) = 0 .$$

Further, the use of Eqs. (5.7) and (5.8) shows that all the other F's are zero:

$$F_{MH}(\lambda = 0,\ x,\ z,\ \bar{s}) = 0\ ,$$

$$F_{HH}(\lambda = 0,\ x,\ z,\ \bar{s}) = 0\ ,$$

$$F_{MM}(\lambda = 0,\ x,\ z,\ \bar{s}) = 0\ , \text{ etc.} \tag{5.14}$$

Since F at $\lambda = 0$ is the sub-propagator D, we have to collect our results in order to construct the full propagator. Consider the MM propagator, given in Fig. 5. As argued earlier, if the spurions are emitted in the up direction, we get the sequence of channels M, H, M, H, etc.; whereas if they are emitted in the down direction, the sequence is M, A, M, A, etc. The contribution of the second sequence is zero, however, since $\tilde{D}_{MM}$ is zero. Hence, we obtain the full propagator Δ_{MM} by multiplying D_{MM} by the propagators of the two external lines and the "cross channel" factor $(1 - u)^{c^2}(1 - v)^{c^2}$, which were omitted in the definition of (5.5), and then adding it to the bare propagator, which again was omitted from the definition:

$$\Delta_{MM} = u^{\tilde{R}-s-1+c^2}(1 - u)^{c^2}\,\delta(u - v) - \frac{1}{B(1 - c^2,\ c^2 - s + \tilde{R})}(uv)^{c^2-1-s+\tilde{R}}, \tag{5.15}$$

Here we have also switched back to the original variables $u = 1 - z$, $v = 1 - x$.

The expression analogous to Eq. (5.15) for the Abelian propagator is the following:

$$\Delta_{AA} = u^{\tilde{R}-s-2+c^2}(1 - u)^{c^2}\,\delta(u - v) - \frac{1}{B(1 - c^2,\ c^2 - s + \tilde{R} - 1)}(uv)^{c^2-2-s+\tilde{R}}. \tag{5.16}$$

Actually, Eq. (5.16) has to be symmetrized with respect to "up" and "down" emissions; however, since this does not alter any of our results, we shall not enter into this further complication.

The propagator of Eq. (5.15) is to be sandwiched between vertices that depend on u and v, and the result should be integrated over these variable. One may first suspect that there are double poles coming from the second term on the right because of the double integration; however, these cancel against the poles of the beta-function at the same positions.

It is also of interest to determine the location of the lowest pole. We imagine having expanded the vertices in powers of u and v, and notice that the lowest pole comes from the zeroth order terms. This pole located at $c^2 + \tilde{R} - s = 0$ cancels between the two terms in Eq. (5.15). We interpret this as the M trajectory having moved up a unit. A similar analysis of Eq. (5.16) leads to the same conclusion, that the photon-tachyon trajectory $c^2 + \tilde{R} - s - 1 = 0$ is canceled, and these particles move up a unit.

In more detail, trajectories appear at $s = c^2 + \tilde{R} + (n + 1)$ and $s = \tilde{R} + n + 1$ in the M-channels, and $s = c^2 + \tilde{R} + n$, $s = \tilde{R} + n$ in the Abelian channels ($n = 0, 1, \cdots$). For very small c^2, residues of particles near the same mass tend to cancel, revealing a bad ghost-structure in this spectrum. Such diseases cannot be present when the ghost scalars are properly subtracted at $c^2 = 0$.[12]

One remaining question is, how do we know that the above solution for the propagator goes together with the solution for W obtained in Section IV?

The comparison of the two results rests on the fact that the M propagator must reduce to $\overline{W}$ when the two external legs are on the mass shell; i.e., when $\overline{s} = 0$ and $u = v = 0$. It is easy to verify that both the equations and the solutions of this Section go over to those of Section IV in this limit.

It is instructive to put our method for evaluating the propagator in perspective with a more standard operator approach. As discussed in Ref. 1, the whole spurion summation can be put in the form of the geometrical sum of the zero-four momentum spurion operator on the bare propagator, a form which we might schematically indicate as $(L_0 - 1)^{-1} \rightarrow (L_0 - 1 + J^2\Delta)^{-1}$ where Δ would be quadratic in the M spurion. It is quite clear that our integral equation formulation here is simply a way of giving meaning to this formal sum--in the limit as $J \rightarrow 0$. It may also be possible to develop other procedures (for diagonalizing at $J = 0$) that stay closer to the operators.

Finally, how do we construct the new dual amplitude? Having the propagator, we lack only the vertex. Since the only coupling in the dual model is a three point vertex, it is unchanged under spontaneous breakdown. (We have put all spurion corrections on the propagator.) The states, however, do change--the new ones constructible by diagonalizing the propagator. The new vertex is then the old vertex, sandwiched between the new states. This construction will not be attempted in this paper.

VI. CONCLUSIONS

Starting with a ghost-free dual M-model, we have made certain controllable approximations (a large N-limit, etc.) to cast the problem of spontaneous breakdown in the form of (singular) integral equations. Our conclusions are that spontaneous breakdown does occur, and we have given explicit constructions for the new connected vacuum functional and the propagator. Goldstone phenomena and the Higgs effect are all observed explicitly, and the mass spectrum cures itself of vacuum instability by shifting uniformly up until there are no more tachyons. We conjecture that such features will be true in more physical models: For example in a dual M-model with spin (Neveu-Schwarz type), one expects the tachyon at $m^2 = -1/2$ to dictate the magnitude of the shift. It would move to zero mass, carrying the entire spectrum by 1/2 unit. The resulting spectrum would be indeed exciting.

In our approach, we worked directly with the physical S matrix at zero 4-momentum $W(J)$, eschewing the effective potential itself. Nevertheless, we constantly regard the dual model as a complex Lagrangian involving an infinite number of scalar species. Many of our arguments in calculating and interpreting the behavior of the solutions are based on the dictum that "whatever is true in any Lagrangian is true." In particular, our contention that "probing with zero mass particles is adequate" is true in any Lagrangian. Our expectation that no new ghosts will be introduced in the spectrum by going from one sheet of the theory to another is also based on Lagrangians: Spontaneous breakdown of ghost-free Lagrangians lead to ghost-free broken theories.

As a laboratory in its own right, we find our model highly interesting. For contact with physics, it has probably one most outstanding problem--that we have not yet gone back to $c^2 = 0$. This involves subtracting certain "regulator" scalars (s) under the ρ explicitly, and will proceed by emitting these along with M's and solving with the constraint the $\partial W/\partial J_s = 0$, which is equivalent to eliminating these fields from the theory. Work is proceeding along this line, with the optimism that, after all, the whole problem of spontaneous breakdown in dual models now appears conceptually clear and essentially tractable.

APPENDIX A

The Model as Large N-Limit of Ghost-Free Model

In our paper "Dual M-Models"[3] we sketched a large class of dual models containing the correct M-scalars[15] for hadronic spontaneous breakdown. In that study, we concentrated on models "with spin," indicating the analogous procedures for similar orbital models. Here, for simplicity, we have use for the very simplest orbital model of this class.

From that reference, we recall that M's are the spin-zero mesons which, before spontaneous breakdown, transform as fundamental representations under both the hadronic (gauge) group (Chan-Paton factors), and the weak (exotic) group. Thus the simplest orbital M-vertex can be taken as

$$V_M^{i\alpha} \equiv V_0(1)\, \chi^i\, \pi^{\dagger\alpha}(1) , \qquad V_{M^\dagger}^{\alpha i} \equiv V_0(1)\, \pi^{\alpha}(1)\, \chi^{i\dagger} . \tag{A.1}$$

Here, as in Ref. 3, V_0 is the usual orbital vertex, the numerical SU(N) "spinors" χ^i, $\chi^{j\dagger}$ generate the usual Chan-Paton factors, and the complex projective vectors π, $\pi^\dagger$ carry the "weak" SU(N). For our purposes here, we are taking the M's as forming an $N \times N$ square matrix, that is with weak and strong groups the same size.

Following Ref. 3, we construct the desired sectors by alternating M's and $M^\dagger$'s in forming n-point functions. We are concerned with spontaneous breakdown via the emission of an arbitrary number of M-spurions. Drawing on our Lagrangian experience, the appropriate spurions are the trace of M (or $M^\dagger$), because a vacuum expectation value of these will leave the product group SU(N) intact:

$$SU(N)_{Final} = SU(N)_{Strong} + SU(N)_{Weak} . \qquad (A.2)$$

Thus we concern ourselves with vertices for the trace

$$\sum_{\alpha=1}^{N} V_M^{\alpha\alpha} = \frac{1}{N^{\frac{1}{2}}} V_0(1) \sum_{\alpha=1}^{N} \chi^{\alpha} \pi^{\dagger\alpha}(1) .$$

An amplitude for the scattering of n of these particles appears to have a multiplicative factor $(N^{\frac{1}{2}})^{-n}$. This is illusory, however, because the contractions among the π^{α}, $\pi^{\dagger\beta}$'s tend to give positive powers of N. For example, using the identity $\chi^{\dagger\alpha} \chi^{\beta} = \delta^{\alpha\beta}$

$$\left(\frac{1}{N^{\frac{1}{2}}}\right)^2 \langle 0 | \sum_{\alpha=1}^{N} \pi^{\alpha}(z) \chi^{\dagger\alpha} \sum_{\beta=1}^{N} \chi^{\beta} \pi^{\beta\dagger}(1) | 0 \rangle$$

$$= \frac{1}{N} \sum_{\alpha=1}^{N} \langle 0 | \pi^{\alpha}(z) \pi^{\alpha}(1) | 0 \rangle = \frac{z}{(1 - z)^2} . \qquad (A.3)$$

This positive power of N depends on having the π's contract in sympathy with the χ's; other π contractions, failing this, will indeed be down in size by one or more factors of N. This is illustrated graphically in Figs. 6a and 6b. In particular Fig. 6a shows the leading (order N^0) contribution to each S-matrix element. Figure 6b shows a particular (neglected) configuration of lower order. In these figures, each external M carries both a Chan-Paton index (dotted line) and a "weak" index (the solid line). In the leading approximation, the weak indices follow the strong indices in their contraction pattern. Each closed loop is a factor of N.

The leading (order N^{O}) configuration is evidently a nearest-neighbor contraction scheme (with respect to π, $\pi^{\dagger}$), and is precisely the model considered in the text. In this order, only the Abelian sector ("ninth") of both weak and strong groups survive to couple to the trace of M. This is apparent from Figs. (6a, b).

There are a number of remarks pertinent to our application of the large N limit. In order of increasing interest they are: (1) There are also order N^{O} contributions to the S matrix in the non-Abelian sector (non-trace-M's externally). In our spurion summation on the propagators, however, these do not matter. (2) It is not entirely clear that the non-Abelian vector mesons acquire a mass in the same order. They do however appear to acquire an N-independent mass, along with the non-Abelian member; if we sum the analogous set of propagator graphs of order $(\frac{1}{N})^{n/2} J^{n} = J^{n}_{eff}$ (n-spurions, <u>no</u> closed loops). This raises the interesting question of just how formal is our large N-approach: To what extent precisely does the leading approximation to the S matrix actually dominate a spontaneous breakdown? (3) Our large N connotes a large number of (effective) degrees of space-time, and we cannot think of N arbitrarily large. For this orbital model, remembering that each π^{α} is complex, we must imagine the weak and strong groups as SU(N), $N \geqslant 11$. For the corresponding models with spin, as discussed in Ref. 3, we would have to maintain $N \leqslant 3$, (SU(3)) . As long as N is bounded in this way, we can expect that ghost residues (introduced in the leading approximation) will be of order 1/N, and correspondingly smaller as the approximation is improved.

APPENDIX B

Solution with $\langle M \rangle \neq 0$

We wish to find a solution to Eqs. (4.7) which results in a W with a linear term in λ. This requires an ansatz more complicated than that of Eq. (4.18), and in particular, the ansatz must allow terms odd in λ to appear in the solution. There are a large number of possibilities, and we shall only present what we think is a particularly simple solution.

Let us distinguish between the two branches of the double trajectories at the points $p > 0$ and $p < -1$ by labeling them as $\alpha_p^{(\pm)}$, where the plus label goes with the positive sign on the right-hand side of Eq. (4.15), and similarly for the minus label. We now imagine that in addition to α_p at $p = 0$ moving to $p = -1$, one of the trajectories at $p = 1$ moves over to $p = -2$. There are four possibilities to choose from, and we assume that $\alpha_1^{(+)}$ goes over to $\alpha_{-2}^{(+)}$. The choice $\alpha_1^{(-)} \to \alpha_{-2}^{(-)}$ turns out to be physically equivalent, and the other two possibilities $\alpha_1^{(-)} \to \alpha_{-2}^{(+)}$ or $\alpha_1^{(+)} \to \alpha_{-2}^{(-)}$ are incapable of satisfying the fundamental integral equation, as will be discussed later. Notice that by splitting $\alpha_p^{(+)}$ from its accompanying $\alpha_p^{(-)}$, we are able to generate solutions odd in λ.

We have now to examine Eqs. (4.12) and (4.17) in order to segregate terms lowest order in λ. A careful analysis, which will not be reproduced here, indicates that, to the lowest order, $\bar{W}_1$ and $\bar{W}_2$ are of the following form:

$$\overline{W}_1(x) \cong \lambda\,\beta_1\, x^{-1-c^2+i\lambda c}\,\{1 + \frac{i\lambda x}{c}\} + \lambda^2\,\beta_2\, x^{-c^2+\lambda^2/(c^2-1)} + \lambda^3\beta_3\, x^{i\lambda c},$$

(B.1)

$$\overline{W}_2(x) \cong -\frac{i\lambda\beta_1}{c}\, x^{-c^2+i\lambda c} - i\,\lambda^3\, c\,\beta_3\, x^{1+i\lambda c},$$

where β's are the unknown constants. Notice that we have already solved the homogeneous system of Eqs. (4.12) in writing (B.1). Only Eqs. (4.17), or equivalently, (4.19) remain to be satisfied. To the lowest order in λ, it can be shown that only (4.17a) with $m = 0$ and (4.17b) with $m = 0$ and $m = 1$ need be considered. These lead to the following set of equations:

$$\frac{i\beta_1}{c} B(-1 - c^2 + i\lambda c,\ \ c^2 + 1) + \beta_3\lambda = 0,$$

$$\lambda + \beta_1 B(-c^2 + i\lambda c,\ \ c^2) + \frac{i\lambda}{c}\beta_1 B(1 - c^2,\ \ c^2) + \beta_2\lambda\, B(1 - c^2,\ \ c^2) = 0,$$

$$\beta_1 B(-c^2 - 1 + i\lambda c,\ \ c^2 + 1) + \frac{i\lambda}{c}\beta_1 B(-c^2,\ \ c^2 + 1) + \beta_2\lambda\, B(-c^2,\ \ c^2 + 1) - \frac{i\lambda}{c}\beta_3 = 0.$$

(B.2)

Some of the beta-functions in the above equations are of the order of λ, for example,

$$B(-c^2 + i\lambda c,\ \ c^2) \cong i\lambda c\,\Gamma(c^2)\,\Gamma(-c^2),$$

(B.3)

$$B(-1 - c^2 + i\lambda c,\ \ c^2 + 1) \cong i\lambda c\,\Gamma(1 + c^2)\,\Gamma(-1 - c^2).$$

Upon this observation, all the λ dependence from Eq. (B.2) cancels.

We therefore see that, although terms proportional to β_2 and β_3 appear with higher powers of λ compared to the term proportional to β_1 in Eq. (B.1), they all appear to the same order in λ in Eqs. (B.2).

Upon substitution of (B.3) in (B.2), and after some simplification, we have the following results:

$$\beta_1 = -\frac{ic}{2}\,\frac{1}{\Gamma(c^2+1)\,\Gamma(-c^2-1)}\,,$$

$$\beta_2 = -\frac{1}{\Gamma(c^2)\,\Gamma(1-c^2)}\,,$$

$$\beta_3 = -\frac{ic}{2}\,, \tag{B.4}$$

and hence, the linear term in W is given by,

$$W \cong -\frac{iJ}{2}\,\frac{c}{\Gamma(c^2+1)\,\Gamma(1-c^2)}\,. \tag{B.5}$$

It then follows that the classical field ϕ acquires a vacuum expectation value equal to the right-hand side of (B.5) divided by J. The imaginary value for this quantity is unphysical and presumably has to do with the existence of ghosts in the regulator procedure ($c^2 \neq 0$). If one started with the ansatz $\alpha_1^{(-)} \to \alpha_{-2}^{(-1)}$, this would lead to a mere change of sign in Eq. (B.5). The choices $\alpha_1^{(-)} \to \alpha_{-2}^{(+)}$ or $\alpha_1^{(+)} \to \alpha_{-2}^{(-)}$, on the other hand, would lead to an inconsistent set of equations for the β's.

The propagator for this solution is obtainable via the methods of Section V. It is more complicated than that of the leading solution, and will not be presented here. A qualitative remark is in order however: Since W is essentially λ^2 times the M-M propagator at $s = 0$, and the propagator is $O(\lambda^0)$, how can W have an order λ terms? The answer, born out explicitly in the M-M propagator, is that there is a Goldstone pole at $s = 0$, which is like $\frac{1}{s} \to \frac{1}{i\lambda}$ as we go from propagator to W : the Goldstone theorem that $\langle M \rangle \neq 0$ implies Goldstone pole is, as it should be, quite inescapable. The actual pole is expected to decouple in the diagonalization with the Abelian channels (Higgs phenomenon).

Finally we mention that this solution is really best defined in the range $c^2 < 0$ (or complex). A careful study of convergence of integrals used in the computation bears this out. This makes good physical sense, in that, by starting thus with tachyonic M's , we force $\langle M \rangle \neq 0$. On the other hand, this makes it even more clear that the solution is intimately involved with the regulator ghosts.

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$$W \sim \sum_n J^{2n} \alpha^{n-2} B_{2n} , \qquad V = \sum_{n=2}^{\infty} c_n (\phi^2)^n \alpha^{n-2} ,$$

where the c's are dimensionless numbers and α is the slope. Thus, the part of the B_n's which is contributing is order α^0 for all n. This is, unfortunately, deeper and deeper inside B_n as n grows.

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FIGURE CAPTIONS

Fig. 1. The standard multiperipheral configuration.

Fig. 2. An alternative channel assignment.

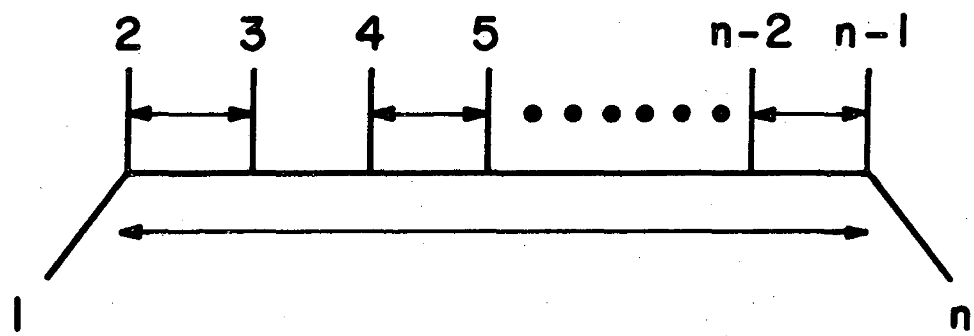
Fig. 3. Spurion emission from an internal line.

Fig. 4. Separation of up and down spurions.

Fig. 5. The two different M subpropagators.

Fig. 6a. The leading terms in the large N expansion.

Fig. 6b. An omitted contraction of order $(N)^{-(n/2)+1}$.



XBL748-3871

Fig. 1

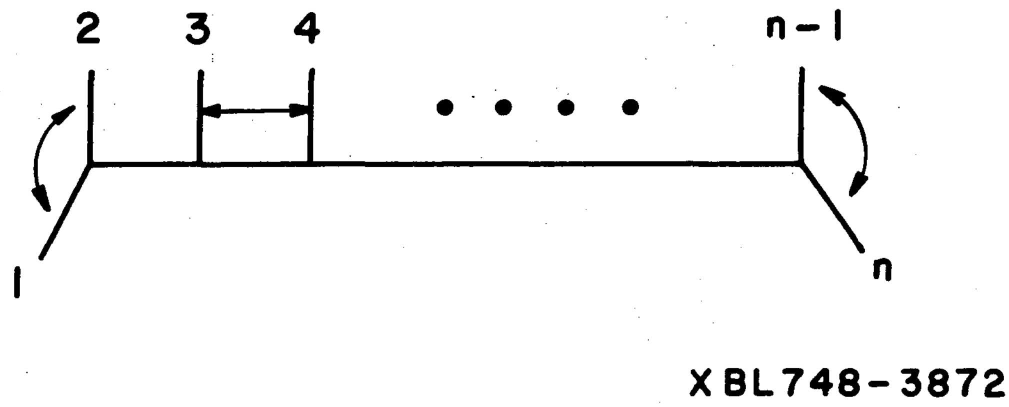
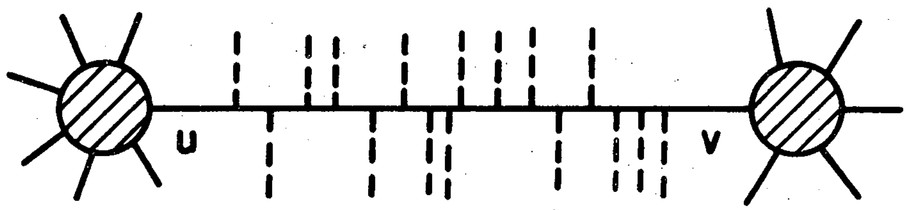
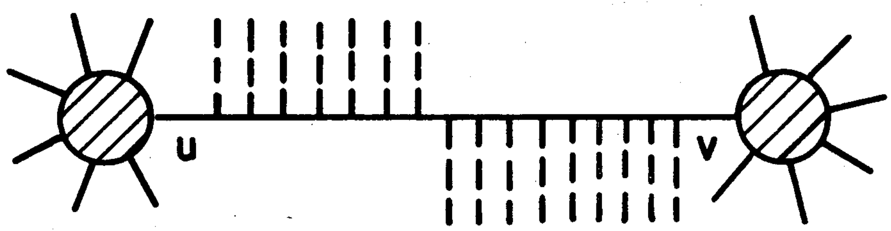


Fig. 2



XBL748-3870

Fig. 3



XBL748-3869

Fig. 4

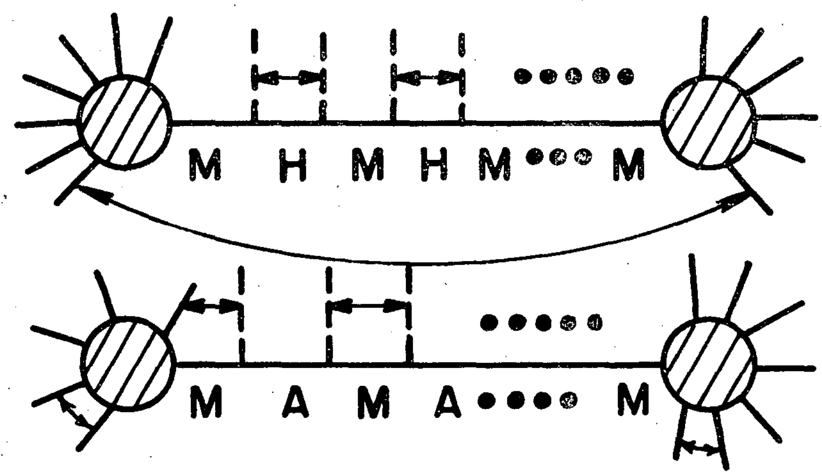


Fig. 5

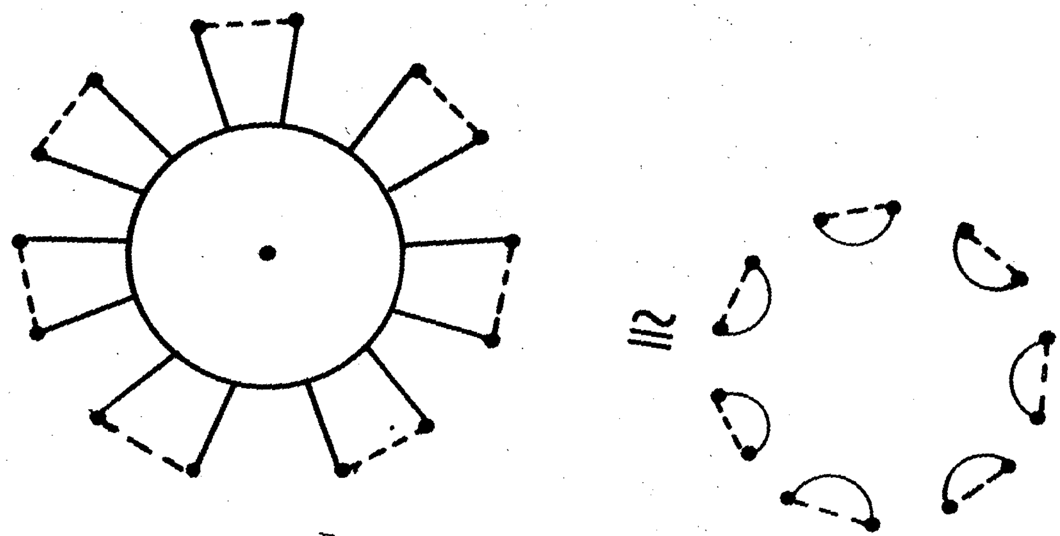


Fig. 6a

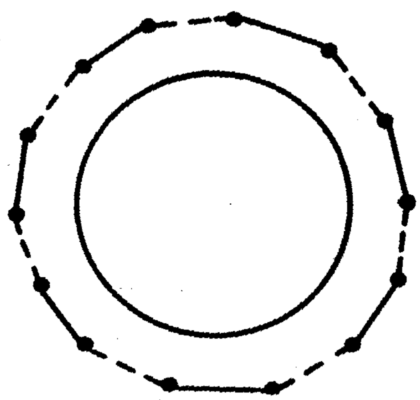


Fig. 6b

XBL748-3867

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