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A MIXED FORMULATION OF NONLINEAR-ELASTIC PROBLEMS

by

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ABSTRACT

The objective of this work is to present a systematic development of mixed finite element models within the context of nonlinear material models. The framework for this effort is the Hu-Washizu functional, due to the relative ease with which it incorporates nonlinear material laws. The functional is subjected to constraints on the spaces of admissible distributions of the assumed stress and strain fields, added to the classical form of the Hu-Washizu functional by the use of Lagrangian multipliers. As a result, the spaces of admissible assumed stress and strain fields are limited to those which satisfy internal constraints pointwise (e.g., incompressibility). In addition, the sensitivity to mesh distortion observed in classical finite element models is greatly reduced. Four-node plane strain elements are presented as an illustration of the proposed formulation. The excellent performance of these elements is illustrated in a number of numerical examples.

NOMENCLATURE

$\mathbb{R}^n$	Euclidian n-space
$\mathcal{B}$	reference configuration of the body
$\mathcal{B}^e$	reference configuration of the element domain
$\mathbf{d}$	nodal displacement vector
$\mathbf{e}$	independent strain parameters vector
$\mathbf{E}$	strain shape function
$\mathbf{N}$	isoparametric shape function
$\mathbf{s}$	independent stress parameters vector
$\mathbf{S}$	stress shape function
$\mathbf{u}$	displacement field
$W(\varepsilon)$	stored energy function
λ	Lagrange multiplier
ε	strain field
Π	total energy functional
σ	stress field
ξ,η	element's natural coordinates

1. INTRODUCTION

Since the early days, finite element research has pursued three main objectives: **i.** To reduce sensitivity to mesh distortion; **ii.** To improve performance in bending dominated problems; and **iii.** To avoid locking behavior in problems involving internal constraints (e.g., incompressibility). In addition, common to all these objectives has been the attempt to ensure that the proposed methods will be general in scope of applications, and carry over to nonlinear regimes.

The initial approach taken was to utilize either Reduced Integration (RI) or Selective Reduced Integration (SRI) (see e.g., Zienkiewicz and Taylor[1] and references therein). To avoid the appearance of spurious zero energy modes, while maintaining the reduced computational effort necessitated by these approaches, the RI scheme was used in conjunction with stabilization methods.[2,3] Later, in view of the equivalence theorem[4], more rigorous approaches capitalizing on the large body of literature regarding the convergence of mixed methods were proposed. Most notable among these approaches is the B-bar method.[5,6]

Working within the realm of linear elasticity, Weissman and Taylor[7] proposed a method to generate assumed stress and strain fields within the context of mixed finite element formulations. These fields, assumed discontinuous across element boundaries, are constrained to *a priori* satisfy the homogeneous part of the equilibrium equations in a weak sense (at the element level). As a result, the set of admissible stress and strain fields satisfies internal constraints pointwise. The proposed methodology was shown to yield excellent results when applied to model plane stress/strain problems[8] as well as to bending of thin plates.[9]

The objective of this work is two–fold: one, to extend the approach taken by Weissman and Taylor[7] to the more general realm of nonlinear material behavior, and two, to recast the proposed method into a more classical form. With respect to the first objective, the proposed method is formulated in the context of a three-field formulation of the Hu-Washizu[10] type, which makes it attractive to use in the area of nonlinear material behavior.[6] In particular, the case of nonlinear elasticity is considered. The second objective is achieved by appending the constraints to the functional by means of Lagrange multipliers, thus rephrasing it as an unconstrained minimization problem.

An outline of the paper follows. The formal statement of the problem is summarized in Section 2. In Section 3, the method proposed by Weissman and Taylor[7] is extended to the nonlinear regime, and recast in a classical form by the use of Lagrangian multipliers. Algorithmic stability is considered in Section 4. The assumed fields for a four-node plane strain element used to illustrate the proposed method are presented in Section 5. The performance of the proposed element is demonstrated in a set of examples in Section 6. Conclusions and closing remarks are contained in Section 7.

2. THE BOUNDARY VALUE PROBLEM: STRONG FORM

The strong (local) form of the boundary value problem considered in this work is summarized in this section. The framework for the present effort is that of hyperelastic bodies in an n_{dim} Euclidian space $\mathbb{R}^{n_{dim}}$ ($1 \le n_{dim} \le 3$ is the number of space dimensions), with $\{ e_i \}$ in $\mathbb{R}^{n_{dim}}$ the standard basis. Thus, a vector, α, and a rank two tensor, β, are given in component form by:

$$\alpha = \alpha_i\, e_i \quad \text{and} \quad \beta = \zeta_{ij}\, e_i \otimes e_j, \tag{2.1}$$

where repeated indicies imply the usual summation convention, and $\otimes$ denotes tensor product. Finally, the space of second rank symmetric tensors is introduced as

$$\mathcal{S} := \left\{ \beta : \mathbb{R}^{n_{dim}} \rightarrow \mathbb{R}^{n_{dim}} \mid \beta \text{ is linear, and } \beta = \beta^T \right\}. \tag{2.2}$$

Consider a bounded body, $\mathcal{B} \subset \mathbb{R}^{n_{dim}}$, with a smooth boundary $\partial\mathcal{B}$. The boundary, $\partial\mathcal{B}$, is assumed to have a continuous outward unit normal field, $\mathbf{n}$, and furthermore, possesses the following structure

$$\overline{\partial\mathcal{B}} = \overline{\partial_\sigma\mathcal{B} \cap \partial_u\mathcal{B}} \quad \text{and} \quad \partial_\sigma\mathcal{B} \cup \partial_u\mathcal{B} = \varnothing, \tag{2.3}$$

where $\partial_u\mathcal{B}$ is the part of $\partial\mathcal{B}$ where displacements are specified as

$$\mathbf{u}\,\big|_{\partial_u\mathcal{B}} = \overline{\mathbf{u}} \quad \text{(given)}, \tag{2.4}$$

and $\partial_\sigma\mathcal{B}$ is the part of $\partial\mathcal{B}$ where tractions are prescribed as

$$\mathbf{t} = \sigma \cdot \mathbf{n}\,\big|_{\partial_\sigma\mathcal{B}} \quad \text{(given)}. \tag{2.5}$$

In the above, $\mathbf{u}(\mathbf{x})$ is the displacement field associated with particles $\mathbf{x} \in \mathcal{B}$. The strain field is obtained from the displacement field through the following relation:

$$\varepsilon = \nabla^S\mathbf{u} := \tfrac{1}{2}\left[\; \nabla\mathbf{u} + (\nabla\mathbf{u})^T \;\right] = \tfrac{1}{2}\left[\; u_{i,j} + u_{j,i} \;\right] e_i \otimes e_j\,. \tag{2.6}$$

Let $\mathbf{b}: \mathcal{B} \rightarrow \mathbf{R}^{n_{dim}}$ and $\rho: \mathcal{B} \rightarrow \mathbf{R}$ be the (given) body force and mass density, respectively. The local form of the equilibrium equations is given by:

$$\text{in } \mathcal{B} \begin{cases} \operatorname{div} \sigma + \rho \mathbf{b} = 0 \\ \sigma^T = \sigma \end{cases} \tag{2.7}$$

By relating the stress field, σ, to the displacement field, the boundary value problem for the displacements $\mathbf{u}(\mathbf{x})$ subjected to the boundary conditions [equations (2.4) and (2.5)] is obtained. In the present work a hyperelastic material is assumed. Consequently, the stress response is formulated in terms of a stored energy function

$$W(\mathbf{x},\varepsilon(\mathbf{x})) : \mathcal{B} \times \mathcal{S} \rightarrow \mathbb{R}, \tag{2.8}$$

such that the second rank stress tensor ($\sigma \in \mathcal{S}$) is given by:

$$\sigma(\mathbf{x}) := \partial_\varepsilon W(\mathbf{x},\varepsilon(\mathbf{x})). \tag{2.9}$$

The elasticity tensor, $\mathbf{C}$, (a rank four tensor) is defined as

$$\mathbf{C} := \partial_{\varepsilon\varepsilon} W(\mathbf{x},\varepsilon(\mathbf{x})). \tag{2.10}$$

Remarks 2:

1. By the definition of $\mathbf{C}$ it possesses the following symmetries:

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jilk} \,. \tag{2.11}$$

2. $\mathbf{C}$ is assumed to be positive definite relative to $\mathcal{S}$, i.e.,

$$\zeta^T \mathbf{C}\, \zeta \geq \beta\, \zeta^T \zeta \tag{2.12}$$

for some $\beta > 0$ and for all $\zeta \in \mathcal{S}$. This condition is known as "pointwise stability" (e.g., Marsden and Hughes,[11] Chapter 4) and is equivalent to the convexity of W. ■

3. PROPOSED FORMULATION

The method proposed by Weissman and Taylor[7] is recast in a classical form by the use of Lagrange multipliers. As stated above, the formulation is presented within the context of nonlinear hyperelasticity.

The total free energy is introduced via the notion of a three-field functional, commonly referred to as the Hu-Washizu[10] functional:

$$\boxed{\Pi(\sigma,\varepsilon,\mathbf{u}) := \int_{\mathcal{B}} \left[W(\varepsilon) + \sigma^T(\nabla^s\mathbf{u} - \varepsilon) \right] dV - \Pi_{EXT}(\mathbf{u}).} \tag{3.1}$$

Here, $W(\varepsilon)$ is the stored energy function (where for convenience the explicit dependence on $\mathbf{x}$ has been omitted), and $\Pi_{EXT}(\mathbf{u})$ is the potential energy of the external loading, which under the assumption of dead loading is given by

$$\boxed{\Pi_{EXT}(\mathbf{u}) := -\int_{\mathcal{B}} \rho \mathbf{b}\cdot\mathbf{u}\, dV - \int_{\partial_\sigma \mathcal{B}} \mathbf{t}\cdot\mathbf{u}\, d\Gamma.} \tag{3.2}$$

In equation (3.1), $\{\sigma, \varepsilon, \mathbf{u}\}$ are taken as independent variables. Thus, σ may be viewed as a Lagrange multiplier. This form is obtained from the standard total energy functional ($\varepsilon = \nabla^s\mathbf{u}$ pointwise) by the introduction of the two following Legendre transformations:

$$\sigma = \partial_\varepsilon W(\varepsilon) \quad \text{and} \quad \varepsilon = \partial_\sigma \mathcal{K}(\sigma), \tag{3.3}$$

where $\mathcal{K}(\sigma)$ is the complementary energy, given by:

$$\mathcal{K}(\sigma) := \sigma^T \varepsilon \; - \; W(\varepsilon). \tag{3.4}$$

The finite element presented in this work is based on discontinuous interpolations of the stress and strain fields over the boundary of a typical element $\mathcal{B}^e \subset \mathcal{B}$ of a discretization $\mathcal{B} \approx \bigcup_{n=1}^{nel}$ (where nel is the number of elements in the discretization), while using the standard isoparametric interpolation for the displacement field. Consequently, the total energy is approximated by

$$\Pi(\sigma,\varepsilon,\mathbf{u}) \approx \sum_{i=1}^{nel} \Pi_i^e(\sigma^e,\varepsilon^e,\mathbf{u}^e), \tag{3.5}$$

where

$$\Pi^e(\sigma^e,\varepsilon^e,\mathbf{u}^e) = \int_{\mathcal{B}^e}\left[W(\varepsilon^e) + \sigma^{eT}(\nabla^s\mathbf{u}^e - \varepsilon^e)\right]dV - \Pi^e_{EXT}(\mathbf{u}^e). \tag{3.6}$$

The remainder of this section is carried out at the element level. Thus, it is possible to omit the superscript "e" denoting the element level without any confusion.

Unfortunately, the use of mixed finite elements does not guarantee improved results over those obtained by the displacement formulation. Indeed, if the stress and strain fields derived from the displacement field form a subset of the corresponding assumed fields, the best result will be that produced by the displacement formulation (Fraeis de Veubeke[12]). Moreover, the constrained optimization problem of minimizing the energy functional, equation (3.6), subjected to internal constraints (e.g., incompressibility), may result in locking behavior (i.e., poor results for coarse meshes, and very slow convergence). One way to overcome this setback is to choose the assumed stress and strain fields from a subset of $\mathcal{S}$, denoted $\mathcal{S}'$; where

$$\mathcal{S}' := \left\{\, \mathcal{S}' \subset \mathcal{S} \mid \zeta \in \mathcal{S}' \Leftrightarrow \zeta \in \mathcal{S} \text{ and } \zeta \text{ satisfies the internal constraints pointwise.} \right\} \tag{3.7}$$

How to construct the assumed fields to be elements of $\mathcal{S}'$ constitutes the reminder of this Section.

Let the assumed strain, stress and displacement fields (at the element level) be approximated by

$$\varepsilon := \varepsilon_1 + \varepsilon_2 = \mathbf{E}_1\,\mathbf{e}_1 + \mathbf{E}_2\,\mathbf{e}_2\,, \tag{3.8a}$$

$$\sigma := \sigma_1 + \sigma_2 = \mathbf{S}_1\,\mathbf{s}_1 + \mathbf{S}_2\,\mathbf{s}_2, \tag{3.8b}$$

and

$$\mathbf{u} := \mathbf{N}\ \mathbf{d}, \tag{3.8c}$$

where the additive split is such that the distributions obey

$$\varepsilon_1 \cap \varepsilon_2 = \varnothing \quad \text{and} \quad \sigma_1 \cap \sigma_2 = \varnothing, \tag{3.9}$$

and **d** is the vector of nodal displacements.

The objective of constraining σ and ε to be elements of $\mathcal{S}'$ can be obtained by constraining these fields to *a priori* satisfy the homogeneous part of the equilibrium equations in a weak sense.[7,13,14] Thus, by using the method of Lagrange multipliers, the constrained optimization problem may be rephrased as an unconstrained one. Accordingly, the problem is to minimize the following functional:

$$\Pi_M(\sigma,\varepsilon,\mathbf{u},\lambda_1,\lambda_2) = \int_{\mathcal{B}^e} [W(\varepsilon) + \sigma^T(\nabla^s\mathbf{u} - \varepsilon) + \lambda_1^T \sigma + \lambda_2^T \partial_\varepsilon W(\varepsilon)]dV - \Pi_{EXT}(\mathbf{u}).) \tag{3.10}$$

where λ_1 and λ_2 are Lagrange multipliers, interpreted as strains so that the expression has the dimensionality of energy. In particular, λ_1 and λ_2 possess the following structure:

$$\boxed{\lambda_1 := \mathbf{E}^i \mathbf{z}_1 \quad \text{and} \quad \lambda_2 := \mathbf{E}^i \mathbf{z}_2.} \tag{3.11}$$

The Euler-Lagrange equations associated with the modified Lagrangian are:

$$D\Pi_M \cdot \delta\varepsilon_1 = \int_{\mathcal{B}^e} \delta\varepsilon_1^T [\partial_\varepsilon W(\varepsilon) - \sigma + \partial_{\varepsilon\varepsilon} W(\varepsilon)\lambda_2]dV = 0, \tag{3.12a}$$

$$D\Pi_M \cdot \delta\varepsilon_2 = \int_{\mathcal{B}^e} \delta\varepsilon_2^T [\partial_\varepsilon W(\varepsilon) - \sigma + \partial_{\varepsilon\varepsilon} W(\varepsilon)\lambda_2]dV = 0, \tag{3.12b}$$

$$D\Pi_M \cdot \delta\sigma_1 = \int_{\mathcal{B}^e} \delta\sigma_1^T [\nabla^s\mathbf{u} - \varepsilon + \lambda_1]dV = 0, \tag{3.13a}$$

$$D\Pi_M \cdot \delta\sigma_2 = \int_{\mathcal{B}^e} \delta\sigma_2^T [\nabla^s\mathbf{u} - \varepsilon + \lambda_1]dV = 0, \tag{3.13b}$$

$$D\Pi_M \cdot \delta\mathbf{u} = \int_{\mathcal{B}^e} (\nabla^s\delta\mathbf{u})^T \sigma \, dV + \Pi_{EXT}(\delta\mathbf{u}) = 0, \tag{3.14}$$

$$D\Pi_M \cdot \delta\lambda_1 = \int_{\mathcal{B}^e} \delta\lambda_1^T \sigma \, dV = 0, \tag{3.15}$$

$$D\Pi_M \cdot \delta\lambda_2 = \int_{\mathcal{B}^e} \delta\lambda_2^T \partial_\varepsilon W(\varepsilon) \, dV = 0, \tag{3.16}$$

where equations (3.12) are the weak form of the constitutive equations; equations (3.13) are the weak form of the compatability equations; equation (3.14) is the weak form of the momentum equation; and equations (3.15) and (3.16) are the constraint equations on the assumed stress and strain fields, respectively.

Equation (3.15) is now used to relate s_1 to s_2:

$$\boxed{s_2 = -\left(\int_{\mathcal{B}^e} \mathbf{E}^{iT} \mathbf{S}_2 \, dV\right)^{-1} \int_{\mathcal{B}^e} \mathbf{E}^{iT} \mathbf{S}_1 \, dV \, s_1} \tag{3.16}$$

and equation (3.16) may be used to relate $\mathbf{e}_1$ to $\mathbf{e}_2$. However, as $W(\varepsilon)$ is in general a nonlinear function, equation (3.16) must be linearized, and only $\Delta\mathbf{e}_1$ can be related to $\Delta\mathbf{e}_2$. The linearized form is given by:

$$D\Pi_M \cdot \delta\lambda_2 \approx \int_{\mathcal{B}^e} \delta\lambda_2^T [\partial_\varepsilon W(\varepsilon_n) + \mathbf{C}_n(\mathbf{E}_1 \Delta\mathbf{e}_{1n} + \mathbf{E}_2 \Delta\mathbf{e}_{2n})]dV = 0 \tag{3.17}$$

where the subscript n denotes the iteration in a Newton scheme. Δe_2 can be expressed as:

$$\Delta e_{2n} = -\left(\int_{\mathcal{B}^e} E^{iT} C_n E_2 \, dV\right)^{-1} \left[\int_{\mathcal{B}^e} E^{iT} \partial_\varepsilon W(\varepsilon_n) \, dV + \int_{\mathcal{B}^e} E^{iT} C_n E_1 \, dV \, \Delta e_{1n}\right] \tag{3.18}$$

Substituting the above relations into the Euler-Lagrange equations yields a reduced system of equations, given by:

$$D\Pi \cdot \delta e_1 = \delta e_1^T \int_{\mathcal{B}^e} E^T [\partial_\varepsilon W(\varepsilon_n) + C_n E \, \Delta e_{1n} - S \, (s_{1n} + \Delta s_{1n})] dV = 0, \tag{3.19}$$

$$D\Pi \cdot \delta s_1 = \delta s_1^T \int_{\mathcal{B}^e} S^T [B \, \Delta d_n - E_n \, \Delta e_{1n} - E_2 \, V_n] dV = 0, \tag{3.20}$$

$$D\Pi \cdot \delta u = \int_{\mathcal{B}^e} (\nabla^s \delta u)^T \sigma \, dV + \Pi_{EXT}(\delta u) = 0, \tag{3.21}$$

where the following definitions have been introduced:

$$S := S_1 - S_2 \left(\int_{\mathcal{B}^e} E^{iT} S_2 \, dV\right)^{-1} \int_{\mathcal{B}^e} E^{iT} S_1 \, dV \tag{3.22}$$

$$E_n := E_1 - E_2 \left(\int_{\mathcal{B}^e} E^{iT} C_n E_2 \, dV\right)^{-1} \int_{\mathcal{B}^e} E^{iT} C_n E_1 \, dV \tag{3.23}$$

$$V_n := -\left(\int_{\mathcal{B}^e} E^{iT} C_n E_2 \, dV\right)^{-1} \int_{\mathcal{B}^e} E^{iT} \partial_\varepsilon W(\varepsilon_n) \, dV \tag{3.24}$$

and taking notice that

$$\boxed{\int_{\mathcal{B}^e} \lambda_1^T S \, dV = 0 \quad \text{and} \quad \int_{\mathcal{B}^e} \lambda_2^T C_n E \, dV = 0.} \tag{3.25}$$

The following definitions are now introduced:

$$G := \int_{\mathcal{B}^e} S^T B \, dV \qquad A_n := \int_{\mathcal{B}^e} S^T E_n \, dV \tag{3.26a}$$

$$H_n := \int_{\mathcal{B}^e} E_n^T C_n E \, dV \qquad R_n := \int_{\mathcal{B}^e} E_n^T \partial_\varepsilon W(\varepsilon_n) \, dV \tag{3.26b}$$

$$r_{1n} := \int_{\mathcal{B}^e} E_n^T S \, dV \, s_{1n} - \int_{\mathcal{B}^e} E_n^T \partial_\varepsilon W(\varepsilon_n) \, dV \tag{3.26c}$$

and

$$r_{2n} := \int_{\mathcal{B}^e} S^T[\, E_n\, e_{1n} + E_2\, e_{2n} + E_2 V_n - B\, d_n \,]dV \tag{3.26d}$$

where

$$B := \nabla^s N \tag{3.27}$$

is the finite element strain displacement operator. The incremental discrete problem can now be written into a matrix form as:

$$\begin{bmatrix} H_n & -A_n^T & 0 \\ -A_n & 0 & G \\ 0 & G^T & 0 \end{bmatrix} \begin{bmatrix} \Delta e_{1n} \\ \Delta s_{1n} \\ \Delta d_n \end{bmatrix} = \begin{bmatrix} r_{1n} \\ r_{2n} \\ f_n \end{bmatrix} \tag{3.28}$$

Eliminating the stress and strain coefficients yields:

$$\boxed{K_n\, \Delta d_n = R_n}, \tag{3.29}$$

where K_n and R_n are the tangent stiffness matrix and residual, given by:

$$\boxed{K_n := G^T (\, A_n\, H_n^{-1} A_n^T\,)^{-1} G} \tag{3.30}$$

and

$$\boxed{R_n := f_n + G^T (\, A_n\, H_n^{-1} A_n^T\,)^{-1}\,(\, H_n\, A^{-1} r_{1n} + r_{2n}\,)}. \tag{3.31}$$

The algebraic condition to perform this elimination is:

$$\boxed{n_e + n_d \geq n_s \quad \text{and} \quad n_s \geq n_d} \tag{3.32}$$

("Mixed patch test," Zienkiewicz, *et al.* [15]); where n_e, n_s, and n_d are the number of strain, stress, and displacement parameters, respectively (n_d is equal to the number of nodal degrees-of-freedom minus the number of rigid body modes).

To save in the computational effort, A_n can be constructed to be invertible. Henceforth, A_n is assumed to be invertible. It follows that $n_e = n_s$, and consequently, inequality $(3.32)_1$ is automatically satisfied. As a result, the minimal number of independent stress and strain parameters that satisfies inequalities (3.32) is $n_s = n_e = n_d$. Furthermore, in view of the desire to mimimize the computational effort, this is the optimal number. The tangent stiffness matrix is now given by:

$$\boxed{K_n := G^T A_n^{-T} H_n\, A_n^{-1} G}, \tag{3.33}$$

and the residual is given by:

$$\boxed{\mathbf{R}_n := \mathbf{f}_n + \mathbf{G}^T\mathbf{A}^{-T}(\mathbf{r}_{1n} + \mathbf{H}\mathbf{A}^{-1}\mathbf{r}_{2n}).} \tag{3.34}$$

Finally, for completeness, the admissible spaces are now stated. The space of trial displacement solutions is as follows:

$$U := \{ \mathbf{u} \mid \mathbf{u} \in H^1(\mathcal{B}^e),\ \mathbf{u} = \overline{\mathbf{u}} \text{ on } \partial_u\mathcal{B} \} \tag{3.35}$$

Space of displacement weight functions:

$$V := \{ \mathbf{u} \mid \mathbf{u} \in H^1(\mathcal{B}^e),\ \mathbf{u} = \mathbf{0} \text{ on } \partial_u\mathcal{B} \} \tag{3.36}$$

Space of stress solution and weight functions:

$$\Sigma := \{ \sigma \mid \sigma \in H^0(\mathcal{B}^e) \} \tag{3.37}$$

Space of strain solution and weight functions:

$$\psi := \{ \varepsilon \mid \varepsilon \in H^0(\mathcal{B}^e) \} \tag{3.35}$$

Remarks 3:

1. Note that the constraint equations make sense only in a finite dimensional space obtained by the projection of the continuum problem (infinite dimensional) on a finite dimensional subspace via the finite element discretization.

2. The constraint equation on the strain field, equation (3.16), does not involve the strain field directly. Rather, the constraint is imposed on the stress field which is related, pointwise, to the strain field via the constitutive equations. Thus, in an abstract sense, equations (3.15) and (3.16) impose the same constraint on different fields.

3. Pian and Chen[16] constrained the stress equilibrium equations (the homogeneous part of equation $(2.1)_1$) using incompatible displacements as Lagrange multipliers. However, while their stated objective was the same, the implementation involved the stress equilibrium equation explicitly. As a result, when applied by Pian and Sumihara,[13] the procedure did not produce enough independent equations and further assumptions were required, something avoided by the current procedure.

4. Equation (3.16) is a basic requirement in the free formulation procedure.[17] However, it is used to select the higher order displacements, not to impose constraints on the assumed strain field.

5. The completeness requirement represented by the constant stress/strain patch test[18] implies that the initial assumed stress and strain fields must be constructed hierarchically in the Pascal triangle. As to the number of parameters, the minimum is dictated by the mixed patch test. By choosing the first complete polynomial expansion that contains this minimum number of parameters, the number of independent parameters in σ_2 and ε_2 is obtained. Note that $\dim(\sigma_2) = \dim(\varepsilon_2)$.

6. It follows from equations (3.16) and (3.18) that the dimension of λ_1 equals that of σ_2, and that of λ_2 equals that of σ_2. Furthermore, in view of the the assumption on the invertibility of $\mathbf{A}_n$ and the previous remarks, the same interpolation can be used for both λ_1 and λ_2, a fact used in equation (3.11). ■

4. ALGORITHMIC STABILITY

The algorithmic stability of the proposed formulation is examined in this section, and a number of constraints on the admissible fields that can be used as the Lagrange multipliers are added to those already established in Section 3. The discussion in this section is restricted to linear elasticity with $\mathbf{C} = \partial_{\varepsilon\varepsilon} W$ = constant.

First, the constraints already established in Section 3 on ε and σ are summarized.

1. Same interpolation is used for ε and σ.

2. $\varepsilon = \varepsilon_1 + \varepsilon_2$ and $\sigma = \sigma_1 + \sigma_2$, such that $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ and $\sigma_1 \cap \sigma_2 = \emptyset$; these fields are obtained by a hierarchical expansion in the Pascal triangle (completeness), with the lower order distributions in σ_1 and ε_1; and the number of independent parameters in ε_1 and σ_1 is determined by the mixed patch test.

3. $\dim(\lambda_1) = \dim(\lambda_2) = \dim(\sigma_2) = \dim(\varepsilon_2)$.

4. The matrix of the shape function for σ and ε has full column rank in the distributional sense (i.e., each column is a different distribution, see, e.g., equation (5.1) below).

5. The distributions in ε_1 (σ_1) are chosen from the basis of $\mathbf{B}$.

6. $\partial_{\varepsilon\varepsilon}W$ is positive definite.

With the above assumptions at hand, it follows from the definitions of $\mathbf{H}_n$ and $\mathbf{A}_n$, equation (3.26), that they are fully ranked (note that this crucially depends on assumption 5).

Let Ket[**B**] and Ker[**G**] denote the null spaces of **B** and **G** respectively, and recall that

$$\mathrm{Ker}[\mathbf{B}] = \{\ \mathbf{d} \mid \mathbf{Bd} = \mathbf{0}\ \}. \tag{4.1}$$

It is clear from the definition of **G** that Ker[B] $\subset$ Ker[**G**]. However, the condition for stability is

$$\boxed{\mathrm{Ker}[\mathbf{B}] = \mathrm{Ker}[\mathbf{G}]}. \tag{4.2}$$

Proposition: If assumptions 1 through 6 hold, condition (4.2) holds if and only if:

$$\boxed{\lambda_1 \cap \sigma_1 = \varnothing}, \tag{4.3}$$

or equivalently, $\lambda_2 \cap \varepsilon_1 = \varnothing$.

Proof: First note that if $\lambda_1 \cap \sigma_1 \neq \varnothing$, then by construction there is a vector **d** (nodal displacements) such that $\mathbf{G}\ \mathbf{d} = \mathbf{0}$, and $\mathbf{d} \notin$ Ker[**B**] and consequently, Ker[**B**] $\neq$ Ker[**G**].

Conversely, suppose there is a vector $\mathbf{d} \notin$ Ker[**B**] but $\mathbf{d} \in$ Ker[**G**] (i.e., Ker[**B**] $\neq$ Ker[**G**]). Then, by the definition of S, S_1 can be obtained as a linear combination of S_2. This, however, contradicts assumption 2. ∎

Remarks 4:

1. Note that assumption 5 is crucial to proof as it guarantees that

$$\mathrm{Ker}\Big[\int_{\mathcal{B}^e} S_1^T\, \mathbf{B}\ dV\Big] = \mathrm{Ker}[\mathbf{B}]. \tag{4.4}$$

2. Note further, that when remark 6 of Section 3 is added, the space of admissible distributions for λ_1 (λ_2) equals that of σ_2 (ε_2).

3. In some problems such as plate bending it may prove beneficial to derive the distribution for λ_1 (λ_2) from a displacement field, usually termed "incompatible displacements" (Weissman and Taylor[9]). This displacement field, however, is such that the resulting strain field is a linear combination of σ_2 (ε_2). ∎

5. EXAMPLE: FOUR-NODE PLANE STRAIN ELEMENTS

The method proposed in Section 3 is used to formulate four-node quadrilateral plane strain elements. The assumed stress, strain and displacement fields used in the finite element approximation are presented in Section 5.1, Section 5.2, and Section 5.3, respectively. Two options for the incompatible displacements used to generate the Lagrange multipliers are summarized in Section 5.4. The proposed elements are presented in Section 5.5.

5.1 Assumed Stress Field

First note that the number of nodal degrees-of-freedom is eight. Secondly, note that the number of rigid body modes is three. Thus, in view of the mixed patch test the number of independent stress parameters in σ_1 is taken to be five. The first complete order polynomial is linear, and so the number of independent parameters is nine (symmetry of the stress tensor). Following Pian and Sumihara[13], the assumed stress field is expressed in the element's natural coordinates (ξ,η) as:

$$\sigma^* = \begin{bmatrix} \sigma^*_{\xi\xi} \\ \sigma^*_{\eta\eta} \\ \sigma^*_{\xi\eta} \end{bmatrix} = \begin{bmatrix} 1 & \xi & \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \xi & \eta \end{bmatrix} \begin{bmatrix} s^*_1 \\ s^*_2 \\ \vdots \\ s^*_9 \end{bmatrix}. \tag{5.1}$$

The procedure presented in Section 3, however, would require the selection of a different set of parameters in each element for σ_1 (i.e., the procedure would depend upon the orientation of the element's natural coordinate with respect to the special coordinates). This difficulty can be avoided if the stress field is transformed from the (ξ,η) space into the (x_1,x_2) space by means of the following transformation:

$$\boxed{\sigma_{ij} = \frac{1}{J} F_{iI} F_{jJ} \, \sigma^*_{IJ}}, \tag{5.2}$$

where both subindicies i and j take the values x_1 and x_2; both subindicies I and J take the values ξ and η; J is the Jacobian of the transformation; F is the "deformation gradient," given in component form by:

$$\boxed{F_{iI} = \frac{\partial x_i}{\partial \xi_I}}; \tag{5.3}$$

and in order to maintain the ability to pass the constant stress/strain patch test the transformation is based on values of J and F at the element's center. After redefining the independent coefficients, the assumed stress field is given by:

$$\sigma = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a_1\eta & a_2\xi & a_1\xi & a_2\eta & a_3\xi & a_3\eta \\ 0 & 1 & 0 & b_1\eta & b_2\xi & b_1\xi & b_2\eta & b_3\xi & b_3\eta \\ 0 & 0 & 1 & c_1\eta & c_2\xi & c_1\xi & c_2\eta & c_3\xi & c_3\eta \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_9 \end{bmatrix} \tag{5.4}$$

where,

$$a_1 = xs^2 \qquad a_2 = xt^2 \qquad a_3 = 2\ xs\ xt \tag{5.5a}$$

$$b_1 = ys^2 \qquad b_2 = yt^2 \qquad b_3 = 2\ ys\ yt \tag{5.5b}$$

$$c_1 = xs\ ys \qquad c_2 = xt\ yt \qquad c_3 = xs\ yt + xt\ ys \tag{5.5c}$$

$$xs = \frac{1}{4}\xi_I\, x_{1I} \qquad xt = \frac{1}{4}\eta_I\, x_{1I} \qquad ys = \frac{1}{4}\xi_I\, x_{2I} \qquad yt = \frac{1}{4}\eta_I\, x_{2I} \tag{5.6}$$

$$s_1 = \frac{1}{J_0}(xs^2 s_1^* + xs^2 s_4^* + a_3 s_7^*) \qquad s_2 = \frac{1}{J_0}(ys^2 s_1^* + ys^2 s_4^* + b_3 s_7^*) \tag{5.7a}$$

$$s_3 = \frac{1}{J_0}(c_1 s_1^* + c_2 s_4^* + c_3 s_7^*) \qquad s_i = \frac{1}{J_0} s_j^* \tag{5.7b}$$

with

$$(i,j) \in \{\ (4,3),\ (5,5),\ (6,2),\ (7.6),\ (8,8),\ (9,9)\ \}. \tag{5.8}$$

With these definitions in hand, σ_1 can be chosen to contain the first five parameters (s_1– s_5), and σ_2 to contain the last four parameters (s_6– s_9).

5.2 Assumed Strain Field

A similar approach to that described for the stress field is taken. However, two approaches can be taken in the coordinate transformation:

- Same transformation as that used for the stress field.

- The inverse transformation:

$$\boxed{\varepsilon_{ij} = J\, F^{-1}_{Ii}\, F^{-1}_{Jj}\, \varepsilon^{*}_{IJ}}\,. \tag{5.9}$$

The first approach is motivated by the use of the same shape functions for the stress and strain fields for the modified functional. The second approach is motivated by the invariance of the total internal energy ($\sigma^T\varepsilon$) under coordinate transformation. In this case the strain field in the (x_1,x_2) space is given by:

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & b_2\eta & a_2\xi & b_2\xi & a_2\eta & ab_1\xi & ab_1\eta \\ 0 & 1 & 0 & b_1\eta & a_1\xi & b_1\xi & a_1\eta & ab_2\xi & ab_2\eta \\ 0 & 0 & 1 & b_4\eta & a_4\xi & b_4\xi & a_4\eta & c_3\xi & c_3\eta \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_9 \end{bmatrix} \tag{5.10}$$

where

$$ab_1 = -2\ xt\ yt \qquad ab_2 = -2\ xs\ ys \qquad a_4 = -\ xs\ xt \qquad b_4 = -\ ys\ yt. \tag{5.11}$$

ε_1 can now be chosen to contain the first five parameters ($e_1 - e_5$), and ε_2 to contain the last four parameters ($e_6 - e_9$).

5.3 Assumed Displacement Field

The isoparametric interpolation is used for the displacement field. Accordingly, the displacement field is approximated by:

$$\mathbf{u} = N_I\, \mathbf{d}_I \tag{5.11}$$

where, $\mathbf{d}_I$ is the vector of nodal displacements, and N_I is the shape function associated with node I (I = 1,2,3, or 4), which for the four-node element is given in the element's natural coordinates (ξ,η) by:

$$N_I(\xi,\eta) = \frac{1}{4}\,(1 + \xi_I\xi)(1 + \eta_I\eta). \tag{5.12}$$

5.4 Assumed Incompatible Displacements

In view of the restriction placed on λ_1 and λ_2 in Sections 3 and 4, and the fact that both σ_2 and ε_2 contain four parameters each, the assumed incompatible displacements are given by:

$$U_1^i = N_1^i z_1 + N_2^i z_2 \qquad \text{and} \qquad u_2^i = N_1^i z_3 + N_2^i z_4 \,. \tag{5.13}$$

where z_1, z_2, z_3, and z_4 are the independent incompatible displacements parameters, and N_1^i and N_2^i are the incompatible shape functions. Thus, the issue is how to select these two shape functions. Two options, selected from the literature, that satisfy the requirements set in Section 4 are considered:

- Wu, *et al.* [14]:

$$N_1^i = \xi^2 - \frac{2}{3}\frac{J_1}{J_0}\xi + \frac{2}{3}\frac{J_2}{J_0}\eta \qquad \text{and} \qquad N_2^i = \eta^2 + \frac{2}{3}\frac{J_1}{J_0}\xi - \frac{2}{3}\frac{J_2}{J_0}\eta \tag{5.14}$$

- Taylor, *et al.* [15]:

$$N_1^i = (1 - \frac{J_2}{J_0}\eta)(1-\xi^2) + \frac{J_1}{J_0}\xi(1-\eta^2) \tag{5.15a}$$

and

$$N_2^i = (1 - \frac{J_1}{J_0}\xi)(1-\eta^2) + \frac{J_2}{J_0}\eta(1-\xi^2). \tag{5.15b}$$

5.5 Proposed Elements

The fields proposed above are used to generate a number of four-node quadrilateral plane strain elements, termed PS_n (plane strain) (n = 1,2,3, or 4). These elements differ by the incompatible shape functions used and by the transformation used for the assumed strain. Elements PS_1 and PS_3 use the incompatible shape functions (5.14), while elements PS_2 and PS_4 are based on the shape functions (5.15); in elements PS_1 and PS_2 the same transformation is used for the stress and strain fields, while the other two elements use transformation (5.9) for the strain field.

Remarks 5:

1. Note that the transformation (5.3) is identical to the transformation used to push forward the second Piloa-Kirchhoff stress tensor to the Cauchy stress tensor.

2. The numerical results obtained in the linear case (Weissman and Taylor[8]) were independent of the type of transformation used for the strain field. This characteristic also carries over to the nonlinear case presented in this work. ■

6. NUMERICAL EXAMPLES

The performance of the elements proposed in Section 5 is evaluated with three problems available in the literature. First considered is the constant stress/strain patch test in Section 6.1. Secondly, the sensitivity to mesh distortion is examined in Section 6.2. Finally, Cook's membrane problem is presented in Section 6.3.

Throughout this section the following strain energy function is used:

$$W(\varepsilon) = \frac{1}{2} K I_1^2 + 2G I_2' + \beta I_1^2 \, I_2' \tag{6.1}$$

where $I_1 = \mathrm{trace}(\varepsilon)$ is the first invariant of the strain tensor; $I_2' = \frac{1}{2} e_{ij} e_{ij}$ is the second invariant of the strain deviator strain (**e**); K is the bulk modulus; G is the shear modulus; and β is a material constant.

In the examples given below, all four proposed elements yield identical results, and are designated by the term "Present." The 5-parameter element recently proposed by Simo and Rifai[19] (considered to be one of the best available elements) is used for comparison.

6.1 Constant Stress/Strain Patch Test

A rectangular mesh is modeled by one element in Figure 6.1a and by a skewed mesh in Figure 6.1b. The following material properties are used: K = 10, G = 3.75, β = 1000. The mesh is subjected to a constant state of tension/compression. All elements presented pass this test.

6.2 Sensitivity to Mesh Distortion

In this example, a beam is modeled by two elements, as shown in Figure 6.2. The top edge displacement, v (normalized with the exact beam theory solution, w = 56.2565), is shown as Δ is increased from 0 to 5. The material properties used are: K = 1666.67, G = 0.3333, and β = 5000 (these properties imply a nearly incompressible material).

The proposed elements are shown to avoid locking at the nearly incompressible limit, and to yield excellent results in a bending dominated problem; furthermore, they exhibit less sensitivity to mesh distortion than the 5-parameter element.

6.3 Cook's Membrane Problem

A tapered panel is clamped at one end and loaded by a uniformly distributed in-plane bending load on the other end, as shown in Figure 6.4. The material properties used are: $K = 10$, $G = 3.75$, and $\beta = 1000$. The transverse displacement at point C, normalized with a result obtained by a mesh consisting of 1024 elements ($w = 2.11885$), is reported in Figure 6.5.

The results are practically identical to the results obtained by the 5-parameter element (with the exception of the one element mesh where a better result is obtained).

7. CONCLUSION

The method proposed by Weissman and Taylor[7] was generalized to the nonlinear elasticity regime. Furthermore, using the method of Lagrange multipliers, the method was rephrased as the minimization of an unconstrained problem.

The elements presented yield excellent results. Moreover, no sensitivity to the transformation used for the strain field or to the incompatible displacements used is observed. It must be noted, however, that the latter property holds only if at least two opposite sides are parallel (Weissman and Taylor[8]).

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REFERENCES

1. O.C. Zienkiewicz and R.L. Taylor, [1989], *The Finite Element Method*, 4th edition, London: MacGraw-Hill Book Co.

2. D.P. Flanagan and T. Belytschko, [1981], "A Uniform Strain Hexahedron and Quadrilateral With Orthogonal Hourglass Control," *International Journal for Numerical Methods in Engineering*, Vol. 17, pp. 679-706.

3. T. Belytschko and W.E. Bachrach, [1986], "Efficient Implementation of Quadrilaterals With High Coarse-Mesh Accuracy," *Computer Methods in Applied Mechanics and Engineering*, Vol. 54, pp. 279-301.

4. T.J.R. Hughes, [1980], "Generalization of Selective Integration Procedures to Anisotropic and Nonlinear Media," *International Journal for Numerical Methods in Engineering,* Vol. 15, pp. 1413-1418.

5. D.S. Malkus and T.J.R. Hughes, [1978], "Mixed Finite Element Methods - Reduced and Selective Integration Techniques: A Unification of Concepts," *Computer Methods in Applied Mechanics and Engineering,* Vol. 15, pp. 63-81.

6. J.C. Simo, R.L. Taylor, and K.S. Pister, [1985], "Variational and Projection Methods for the Volume Constraint in Finite Deformations Elasto-Plasticity," *Computer Methods in Applied Mechanics and Engineering,* Vol. 51, pp. 177-208.

7. S.L. Weissman and R.L. Taylor, [1990], "Treatment of Internal Constraints by Mixed Finite Element Methods: Unification of Concepts," *International Journal for Numerical Methods in Engineering,* to appear.

8. S.L. Weissman. and R.L. Taylor, [1990], "A Unified Approach to Mixed Finite Element Methods: Application to In-Plane Problems," Report No. UCB/SEMM 90/12.

9. S.L. Weissman and R.L. Taylor, [1990], "Mixed Formulations for Plate Bending Elements," *Computer Methods in Applied Mechanics and Engineering,* to appear.

10. K. Washizu, [1982], *Variational Methods in Elasticity and Plasticity*, 3rd edition, Elmsford, New York: Pergamon Press, Inc.

11. J.E. Marsden and T.J.R. Hughes, [1983], *Mathematical Foundations of Elasticity,* Englewood Cliffs, New Jersey: Prentice-Hall, Inc.

12. B. Fraeijs de Veubeke, [1965], "Displacement and Equilibrium Models in the Finite Element Method," in *Stress Analysis*, editors O.C. Zienkiewicz and G.S. Holister. London: John Wiley.

13. T.H.H. Pian and K. Sumihara, [1984], "Rational Approach for Assumed Stress Finite Elements," *International Journal for Numerical Methods in Engineering,* Vol. 20, pp. 1685-1695.

14. C.-C. Wu, M.-G. Huang and T.H.H. Pian, [1987], "Consistency Condition and Convergence Criteria of Incompatible Elements: General Formulation of Incompatible Functions and its Application," *Computers & Structures*, Vol. 27, No. 5, pp. 639-644.

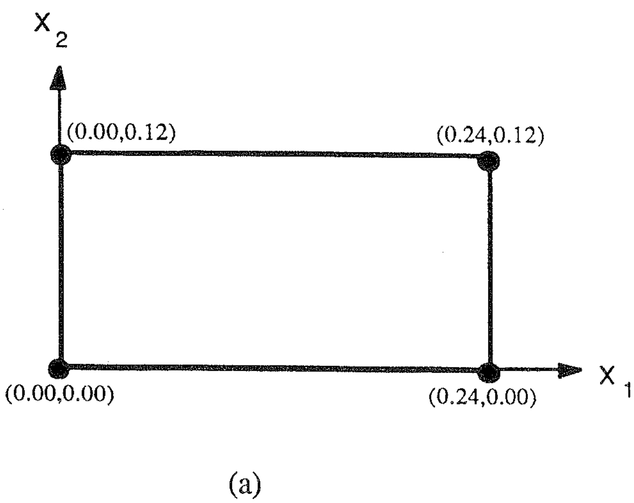
15. O.C. Zienkiewicz, S. Qu, R.L. Taylor and S. Nakazawa, [1986], "The Patch Test for Mixed Formulations," *International Journal for Numerical Methods in Engineering,* Vol. 23, pp. 1873-1883.

16. T.H.H. Pian and D.P. Chen, [1982], "Alternative Ways for Formulation of Hybrid Stress Elements," *International Journal for Numerical Methods in Engineering,* Vol. 18, pp. 1679-1684.

17. P.G. Bergan and M.K. Nygard, [1984], "Finite Element With Increased Freedom in Choosing Shape Functions," *International Journal for Numerical Methods in Engineering,* Vol. 20, pp. 643-663.

18. G.P. Bazeley, Y.K. Cheung, B.M. Irons and O.C. Zienkiewicz, [1965], "Triangular Elements in Bending-Conforming and Non-Conforming Solutions," *Proc. Conf. Matrix Methods in Structural Mechanics*, Air Force Inst. Tech. Wright-Patterson AF Base, Ohio.

19. J.C. Simo and M.S. Rifai, [1990], "A Class of Mixed Assumed Strain Methods and the Method of Incompatible Modes," *International Journal for Numerical Methods in Engineering,* Vol. 29, pp. 1595-1638.



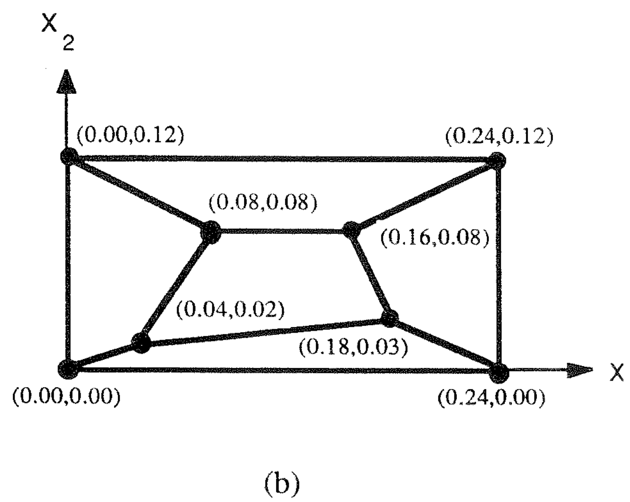


Figure 6.1: Patch test - (a) One-element mesh, (b) Skewed mesh.

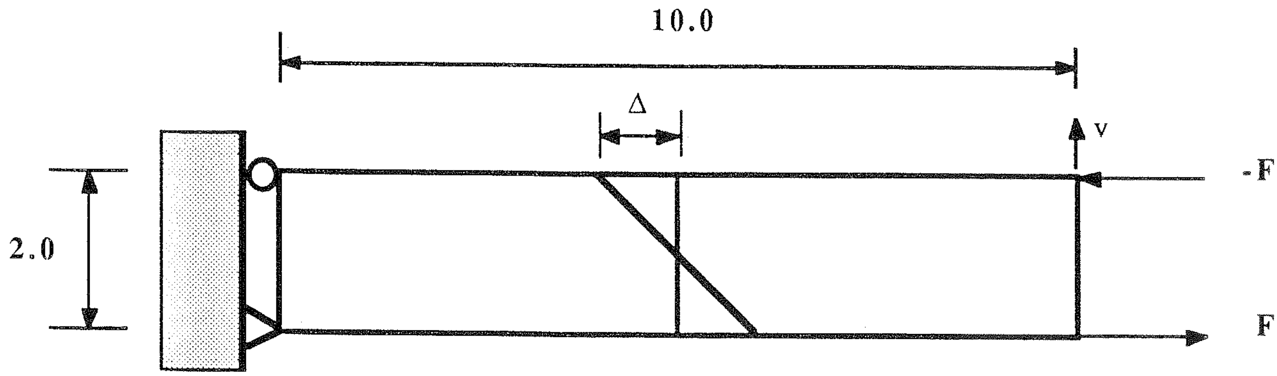


Figure 6.2: Beam bending problem, sensitivity to mesh distortion.

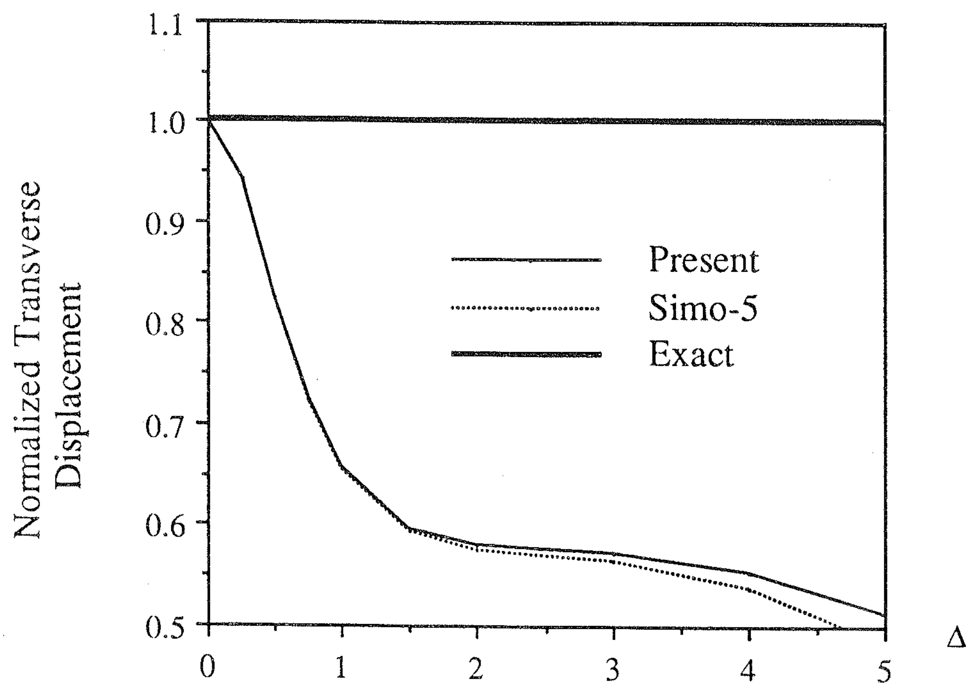


Figure 6.3: Sensitivity to mesh distortion; normalized top edge tip displacement, v.

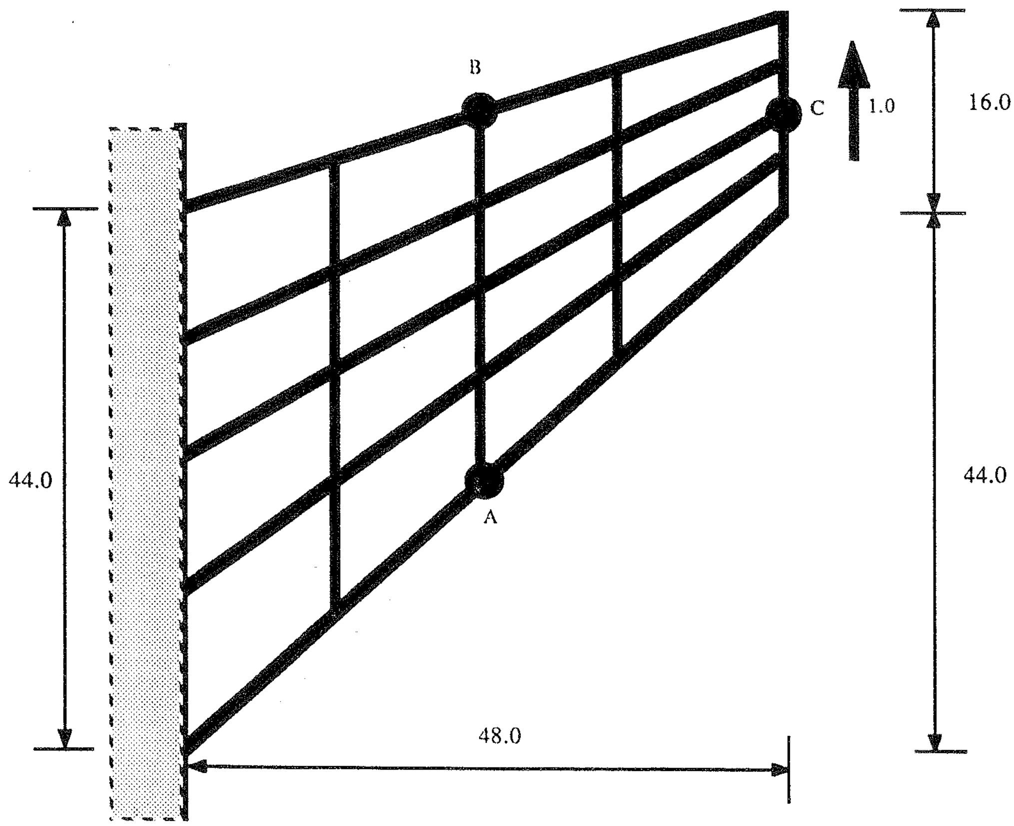


Figure 6.4: Cook's membrane problem.

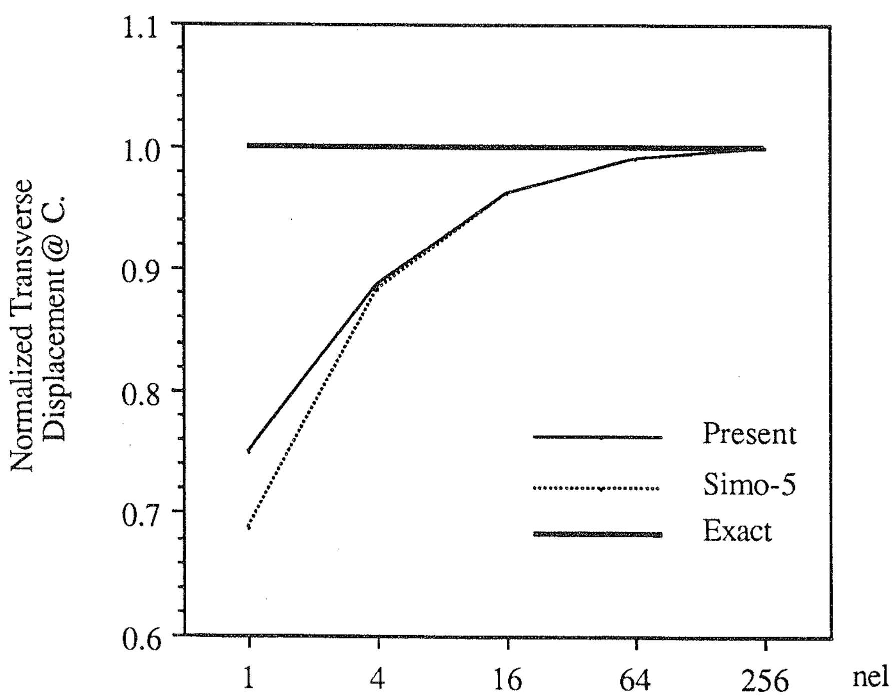


Figure 6.5: Cook's membrane problem; vertical displacement at point C.