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OPTIONS ON COMMODITY FORWARD CONTRACTS*

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We develop formulas for "European" options on commodity forward contracts. The assumptions and derivations are simple. The qualitative behavior of the formulas is developed for an intuitive overview. The put formula and related ideas were applied to successfully manage a quarter billion dollar hedge of GNMA futures versus standbys.

(OPTIONS; FORWARD CONTRACTS; COMMODITIES)

1. Introduction

We derive a formula for "European" options on commodity forward contracts. The approach is based on Black (1976). A commodity forward contract is an agreement to deliver a specified quantity and quality of the commodity at a designated time and price. Let c be the contract price, t^* the delivery date, r the riskless rate of return (see Thorp 1976, pp. 237–240 for a discussion of this concept), t a time less than t^* , and $\tilde{x} = \tilde{x}(t, t^*)$ the price at the time t for a forward contract for delivery at t^* . Then the current value $f = (f(t, t^*; c, t^*))$ of a forward contract at price c is given by Black (1976, p. 178, equation (19)) as

$$f = (\tilde{x} - c)\exp\{r(t - t^*)\}. \quad (1)$$

For an easy proof, make the usual assumptions of Black (1976) including (1) no transaction costs (either commissions or losses or market makers), (2) all riskless investments yield the (constant) riskless rate, (3) forward contracts may be freely traded at any time $t \leq t^*$, and exist for all choices of c and t^* . Now suppose at t you are party to a forward contract to buy described by c and t^* . Enter (by (3)) into a forward contract to sell described by $\tilde{x}$ and t^* . Now $\tilde{x}$, the contract price, is set at t to clear the market so the making of the contract involves no additional payment. At time t^* you will buy at c under the original contract and under the second contract also sell the same thing at $\tilde{x}$, gaining (or losing) $\tilde{x} - c$ (by (1)). Thus f is worth $\tilde{x} - c$ at t^* so by (2) it is worth $(\tilde{x} - c)\exp\{r(t - t^*)\}$ at t .

Recall next the formula for the value of a European call on a futures contract¹ (Black 1976),

$$w(x, t) = \exp\{r(t - t^*)\} [xN(d_1) - c^*N(d_2)] \quad (2)$$

where $x(t, t^*)$ is the price at t of a future contract for delivery at t^* , c^* is the option strike price, $N(t)$ is the (cumulative) normal distribution function, and $d_{1,2} = [\ln(x/c^*) \pm v^2(t^* - t)/2]/v\sqrt{t^* - t}$.

Note: As used in this paper, futures contracts and forward contracts have an essential difference. A forward contract at price c is settled by a cash payment of c at time t^* . There is no other encumbrance or flow of funds between buyer and seller. On the other hand, for a futures contract entered into at time $t_0 < t^*$ and settlement price

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¹The same formula was derived for the different situation of warrant hedging in Thorp (1973). For general background on option theory, and specifically for aspects related to this paper, see Cox and Rubenstein (to appear) and Margrabe (1978).

c^* , as the current price $x(t, t^*)$ of the contract changes, the accounts of buyer and seller are marked to the market daily using a closing settlement price determined by an exchange. Forwards are generally traded over the counter and futures are generally traded on an exchange.

Equation (2) is derived from the boundary condition

$$w(x, t^*) = \max(x - c, 0) \quad \text{for all } x \geq 0. \quad (\text{B fut CE})$$

Since $\lim_{x \rightarrow \infty} [w(x, t) - \exp\{r(t - t^*)\}[x - c^*]] = 0$ for $t < t^*$, $r > 0$, we do not have $w(x, t) \geq \max(x - c, 0)$ for all $x \geq 0$ and any $t < t^*$. Thus (2) does not satisfy the additional boundary condition for an American call option,

$$w(x, t) \geq \max(x - c, 0) \quad \text{for all } x \geq 0, \quad t \leq t^*. \quad (\text{B fut CA})$$

Hence (2) is not the solution for an American call option on a futures contract. That problem can be solved by iterative methods (see, e.g., Parkinson 1977).

European puts and calls on futures contracts are connected by a formula like that for European puts and calls on stocks:

$$w(x, t) - p(x, t) = (x - c)\exp\{r(t - t^*)\}. \quad (2a)$$

The proof follows from that for stock options given in Merton (1973, p. 157, Theorem 12). From (2) and (2a) we obtain the formula for European puts on futures:

$$p(x, t) = \exp\{r(t - t^*)\}[-xN(-d_1) + cN(-d_2)]. \quad (2b)$$

This satisfies the boundary condition

$$p(x, t^*) = \max(c - x, 0) \quad \text{for all } x \geq 0. \quad (\text{B fut PE})$$

However it does not satisfy the additional boundary condition for an American put option,

$$p(x, t) \geq \max(c - x, 0) \quad \text{for all } x \geq 0, \quad \text{all } t \leq t^*, \quad (\text{B fut PA})$$

because $\lim_{x \rightarrow 0} p(x, t) = c\exp\{r(t - t^*)\} < c$ if $t < t^*$. Thus (2b) is not the solution for an American put option on a futures contract. Again, iterative methods yield a solution.

2. Calls on Forward Contracts

Next consider the valuation of a European call on a forward contract. The boundary condition is

$$\tilde{w}(\tilde{x}^*, t^*) = \max(\tilde{x}^* - c, 0). \quad (\text{B fwd CE})$$

As explained in Black (1976), the prices $x(t, t^*)$ of a futures contract and $\tilde{x}(t, t^*)$ of a forward contract are equal for each $t = t^*$. In particular if $t \leq t^*$, $\tilde{x}^* \equiv \tilde{x}(t^*) = x(t^*)$ so (B fut CE) and (B fwd CE) are identical; hence the solutions are identical. Therefore (2) is also the solution for a European call on a forward contract.

The remarkable and useful fact is that this is also the solution for an American call on a forward contract. To see this note first the additional boundary condition

$$\tilde{w}(f, t) \geq \max(f, 0) \quad \text{for all } x \geq 0, \quad t \leq t^*. \quad (\text{B fwd CA})$$

By (1), $\max(f, 0) = \max[(\tilde{x} - c)\exp\{r(t - t^*)\}, 0]$ and the calculation below shows that $w(\tilde{x}, t) \geq \max[(\tilde{x} - c)\exp\{r(t - t^*)\}, 0]$. Thus $w(\tilde{x}, t)$ satisfies both (B fwd CE) and (B fwd CA), the boundary conditions for an American option. Hence it is a solution and by the uniqueness of such (class C') solutions, it is the solution. The calculation

follows. It also develops a number of formulas and relationships that are useful both here and in the management of portfolios.

PROOF that $w(\tilde{x}, t) \geq \max[(\tilde{x} - c)e^{r(t-t^*)}, 0]$ where $w(\tilde{x}, t) = \exp\{r(t - t^*)\}[\tilde{x}N(d_1) - cN(d_2)]$. (We write x for $\tilde{x}$ from here on.)

$$1^\circ \lim_{x \downarrow 0} w(x, t) = 0.$$

2°

$$\lim_{x \uparrow \infty} \{w(x, t) - (x - c)e^{r(t-t^*)}\} = e^{r(t-t^*)} \lim_{x \uparrow \infty} \{x(N(d_1) - 1) + c(1 - N(d_2))\} = 0.$$

$$3^\circ \partial w / \partial x = N'(d_1)e^{r(t-t^*)} \text{ because}$$

$$e^{r(t^*-t)} \frac{\partial w}{\partial x} = N(d_1) + xN'(d_1) \frac{\partial d_1}{\partial x} - cN'(d_2) \frac{\partial d_2}{\partial x} \quad \text{and}$$

$$\frac{\partial d_1}{\partial x} = \frac{1}{v\sqrt{t^* - t}} \cdot \frac{1}{x}$$

whence

$$e^{r(t^*-t)} \frac{\partial w}{\partial x} = N(d_1) + \frac{1}{xv\sqrt{t^* - t}} \{xN'(d_1) - cN'(d_2)\}.$$

But the bracketed expression is zero because:

$$xN'(d_1) - cN'(d_2) = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} (xe^{-(d_1^2 - d_2^2)/2} - c),$$

$$d_1 - d_2 = v^2(t^* - t)/v\sqrt{t^* - t}, \quad d_1 + d_2 = \left(2 \ln \frac{x}{c}\right)/v\sqrt{t^* - t},$$

$$d_1^2 - d_2^2 = 2 \ln \frac{x}{c} e^{-(d_1^2 - d_2^2)/2} = e^{-\ln(x/c)} = \frac{c}{x};$$

therefore $xN'(d_1) - cN'(d_2) = 0$. This yields

$$\frac{\partial w}{\partial x} = e^{r(t-t^*)}N(d_1) \quad (3)$$

which determines the extremely important neutral hedge ratio for portfolios. It follows that $0 < \partial w / \partial x < e^{r(t-t^*)}$. But then from 1° w lies above $y = 0$ and from 2° w lies above $y = (x - c^*)e^{r(t-t^*)}$. Therefore (B fwd CA) holds. Note further that:

4°

$$\frac{\partial^2 w}{\partial x^2} = e^{r(t-t^*)}N'(d_1) \frac{\partial(d_1)}{\partial x} = \frac{e^{r(t-t^*)}}{xv\sqrt{t^* - t} \sqrt{2\pi}} \exp(-d_1^2/2) > 0$$

so w is concave upward.

In the important market in options on GNMA (Government National Mortgage Association) pass through certificates, the options expire at t^* ("notification date") and the underlying forward contract matures at $t_m > t^*$. The appropriate formula then is

$$w(x, t) = \exp\{r(t - t_m)\} [xN(d_1) - cN(d_2)] \quad (4)$$

where $d_{1,2} = [\ln(x/c) \pm v^2(t^* - t)/2]/v\sqrt{t^* - t}$.

This follows from (2) and the fact that the payoff at t^* is deferred until t_m but can be hedged risklessly by a sale of a forward at t^* at price $x(t^*)$. Therefore we discount (2) by $e^{r(t^*-t_m)}$ to get (4).

3. Puts on Forward Contracts

European puts and calls on forward contracts are connected by a formula like that for European puts and calls on stocks:

$$w_e - p_e = (x - e)\exp\{r(t - t^*)\}. \quad (5e)$$

The proof follows from that for stock options given in Merton (1973). The subscript "e" indicates the option is European. We already have shown that $w_e = w$, the American call price. We now show that $p_e = p$, the American put price. The boundary condition for the European put is

$$p(f, t^*) = \max(-f, 0) \quad \text{for all } x \geq 0 \quad (\text{B fwd PE})$$

and the American put satisfies in addition the boundary condition

$$p(f, t) \geq \max(-f, 0) \quad \text{for all } x \geq 0, \quad t \leq t^*. \quad (\text{B fwd PA})$$

It is evident that p_e satisfies (B fwd PE) so it suffices to show that

$$p(f, t) \geq \max(-f, 0) = \max[(c - x)\exp\{r(t - t^*)\}, 0] \equiv p_{\min} \quad \text{for all } x \geq 0, \quad t \leq t^*.$$

This is established by a calculation similar to that in the call case, but (5e), with $w_e = w$, reduces the work.

$$1^\circ \lim_{x \uparrow \infty} p_e = 0 = p_{\min},$$

$$2^\circ \lim_{x \downarrow 0} p_e = ce^{r(t-t^*)} = p_{\min}.$$

It therefore suffices to show:

(a) if $\phi = p_e - (c - x)e^{r(t-t^*)}$ then $\partial\phi/\partial x > 0$ so p_e lies above $y = (c - x)e^{r(t-t^*)}$, and

(b) $\partial p_e/\partial x < 0$ whence p_e lies above $y = 0$.

PROOF OF (b). From (5e), $\partial p_e/\partial x = \partial w/\partial x - \exp\{r(t - t^*)\}$ which is

$$\exp\{r(t - t^*)\} [N(d_1) - 1] \quad (6)$$

the hedge ratio for puts. Hence $-e^{r(t-t^*)} < \partial p_e/\partial x < 0$.

PROOF OF (a). $\partial\phi/\partial x = \partial p_e/\partial x + e^{r(t-t^*)} > 0$. Therefore $p = p_e$.

Notes. 1° The neutral hedge ratio for a straddle s is

$$\partial s/\partial x = \partial w/\partial x + \partial p/\partial x = \exp\{r(t - t^*)\} [2N(d_1) - 1].$$

In particular a straddle is itself neutrally hedged if and only if $d_1 = 0$, i.e., $x = c \exp\{-v^2(t^* - t)/2\}$.

2° If $x = c$ then $w = p$, which also is a standard rule used by traders in options on futures.

3° $\partial^2 w/\partial x^2 = \partial^2 p/\partial x^2$, i.e., w and p have the same curvature.

As before we generalize to the case where the option expires at $t^* < t_m$, the maturity date of the underlying forward contract. The formula is:

$$p = \exp\{r(t, t_m)(t - t_m)\} [-xN(-d_1) + cN(-d_2)] \quad (7)$$

where $d_{1,2} = [\ln(x/c) \pm v^2(t^* - t)/2]/v\sqrt{t^* - t}$ and $r(t, t_m)$ is the riskless rate for the time period from t to t_m . This is the rate such that if $\exp\{r(t, t_m)(t - t_m)\}$ is risklessly invested at time t it becomes 1 at time t_m .

4. Application to the GNMA Standby Market

GNMA standbys are OTC European puts on forward contracts on GNMA pass through certificates. The certificates themselves trade OTC both for cash (immediate delivery) and as forward contracts. The options are often sold by savings and loan associations to broker-dealers who then mark them up and resell them. In case of default by the savings and loan, the dealer loses. In case of default by the dealer, the put buyer loses. To limit risk, one should therefore buy standbys only from well capitalized broker-dealers.

The purchaser of the put pays a fee F at time t . In a risk neutral universe² the expected return when the position is closed out at t_m is $\exp\{r(t_m - t)\}p = cN(-d_2) - xN(-d_1) \equiv p_m$. Thus F grows to p_m in time $t_m - t$ so the (market) rate of return on investment $r(t, t^*) = r_m$ is given by $F \exp\{r_m(t_m - t)\} = p_m$ or $r_m = (t_m - t)^{-1} \log_e(p_m/F)$ and the annual compound rate a_m is e^{r_m} or $a_m = (p_m/F)^{1/(t_m - t)}$.

A more precise analysis includes the effects of various costs, such as hedging in the CBT GNMA futures market, and bid asked spread costs if the GNMA is actually purchased and "put".

In practice $x(t, t_m)$ is only available for $t_m - t \leq 6$ months as a direct quote from the OTC forward market. However the CBT GNMA futures market is closely related and $x(t, t_m)$ can generally be inferred from it for $t^* - t \leq 30$ months.

A formula like (7), but a little more general, appears in Thorp (1976, p. 185). To derive (7) from this, let $v_p = 0$, whence $v_x \equiv v$, and assume $X/P = x(t, t_m)$, the current price of a forward contract. To make this consistent with observed market prices, X , the spot price in Thorp (1976), should be the market price at t minus the present value of the dividends until t^* . This makes sense too: the buyer of the spot GNMA keeps the dividends from t to $t_m \equiv t^*$. These can be regarded as a reduction in purchase price X by an amount equal to the value at t of this dividend stream. Note too that formula (7) should theoretically be generalized to incorporate v_p , the volatility of the riskless bond with term $[t^*, t_m]$, and the correlation ρ_{px} between this bond and the standby. However the correction is small since $[t^*, t_m]$ in practice is a month or less.

Formula (7) and the ideas of this paper were implemented during the heyday of the GNMA standby market in the late 1970s. During a two-year period a hedged portfolio long standbys and long about \$250 million of GNMA futures contracts was successfully and profitably managed.³

²The assumption of risk neutrality here is a convenience. It may be removed without affecting the validity of the conclusions, as Cox and Ross (1976) and others have observed.

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