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Abstract

For decisions whose consequences accrue over time, several techniques are possible to compute total utility. One is to discount utilities of future consequences at some appropriate rate. The second is to discount per-period certainty equivalents. And the third is to compute net present value of various possible streams and then apply utility function to these net present values. When consequences are income streams, our main result shows that for a strict concave utility function, discounting utilities of incomes or discounting per-period certainty equivalents can result in a paradoxical preference for receiving more money later to more money now. For income streams, the correct approach is to …rst compute net present values of various possible income streams and then take the utility of such net present value. The discounted utility model is appropriate for consumption streams, provided that the time intervals between periods are su¢ciently large. Otherwise, we have the unrealistic situation where a short delay in consumption produces a discontinuous jump in the utility evaluation.

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1 Introduction

Consider the decision problem of an MBA student who is faced with the decision of selecting a job from a set of alternative job o¤ers that he has received. The student evaluates these jobs on the criteria of …rst year salary and future salary (salary three years from now), along with other criteria such as location of the job, functional area, travel requirements, etc. For simplicity, we assume that attributes other than the monetary attributes (…rst year salary and future salary) are …xed at a reference level and a pricing out procedure (see Kenney and Rai¤a 1976, Chapter 4) is used to reduce all jobs in terms of equivalent …rst year salary, x_1, and future salary, x_2. Thus, all jobs have identical values on non-monetary attributes and di¤er with one another only on equivalent …rst year and future salary. Further, suppose that x_1 and x_2 may be uncertain; for example, a part of the salary may be dependent on the performance of the company.

Keeney and Rai¤a (1976, Chapter 9, Page 477) note that several techniques are suggested in practice to evaluate alternatives when consequences accrue over time. The …rst approach is to take expected utilities at each point in time and discount these expected utilities. The second approach is to take certainty equivalents at each point in time and discount these certainty equivalents. The third approach is to discount the various possible certain streams, then assess a utility for such present values, and then weight these utilities by the respective probabilities of the streams. We will de…ne these approaches precisely in the ensuing sections. Our aim is to evaluate these three approaches from a prescriptive/normative standpoint.

For simplicity, we assume that there are only two time periods and $(x_1; x_2)$ denotes the stream where a consequence x_1 occurs in period 1 (now) and x_2 occurs in period 2 (later). We consider two cases. In the …rst case, consequences represent income streams (cash ‡ows, earnings or lottery winnings). In this case it is reasonable to assert that money today is preferred to money tomorrow. Sections 2 ¡ 4 deal with this case. In the second case, consequences represent consumption. In this case an incremental consumption today may not be preferred to an incremental consumption tomorrow as a decision maker may consider it desirable to smooth consumption over time. This case is considered in Section 5.

In Section 2 we examine the normative appropriateness of the discounted utility model to evaluate income streams. Section 3 examines an alternative model that discounts the per-period certainty equivalents. We show that both models are normatively inappropriate. These models result in a paradoxical preference for more money later to more money now. Instead of discounting utilities or certainty equivalents, one should first discount the cash flows and then take the utility of these discounted cash flows. In Section 4 we show that the model using utility of net present value has the desired normative properties. In Section 5 we discuss the discounted utility model for consumption streams. We argue that the model may yield undesirable results if the time interval between periods is small. Finally, conclusions are provided in Section 6.

2 Discounting Utility of Income

Consider the decision problem where the consequence accrue over time. For simplicity, assume that there are only two periods and let $(x_1; x_2)$ denote the stream where a consequence x_1 occurs in period 1 (now) and x_2 occurs in period 2 (later). Two consequences x_1 and x_2 could be uncertain. One approach for evaluating such time streams is the utility discount:

$$V(x_1; x_2) = u(x_1) + \beta u(x_2),\ 0 < \beta \le 1; \tag{1}$$

where V is the multi-period utility function, u is the single-period utility function, and β is the discount factor. In our formulation, it is assumed that the time at which x_2 is received, t, is a variable, and that β is continuously decreasing in t, tending to one as t tends to zero; and tending to zero as t tends to infinity. When x_1 and x_2 are uncertain, $u(x_1)$ and $u(x_2)$ represent expected utilities of these uncertain payoffs. Formula (1) is imposed on all β and $t \ge 0$. Typically, the relation between β and t will take the form $\beta = e^{-rt}$, where r is the relevant (continuous time) discount rate.

In a multiattribute analysis, the discounted utility model (1) may be subsumed within a more general multiattribute utility model. For example, in the job selection problem described in Keeney and Raiffa (1976, Section 7:7:4), immediate and future compensation are treated as two attributes along with other attributes such as location, travel requirements, and nature of work. An additive value function or utility function over these multiple attributes that includes immediate and future compensation as two separate attributes with a higher weight on immediate compensation and a lower weight on future compensation is essentially a model where the total utility of a job is computed as sum of the discounted utility of monetary compensation and the utilities derived from other attributes.

We make the following assumptions about the preferences of a decision-maker.

A1 The single-period utility function, u, is monotonically increasing and strictly concave.

A2 A shift of payoff from period 2 (later) to period 1 (now) is preferred, i.e.,

$$(x_1 + \Delta; x_2 - \Delta) \succ (x_1; x_2), \text{ for all } x_1; x_2, \text{ and } \Delta > 0.$$

Assumption 2 is equivalent to $(x_1 + \Delta; x_2) \succ (x_1; x_2 + \Delta)$, $\Delta > 0$, and implies $(x; 0) \succ (0; x)$, for all $x > 0$. The appeal of this assumption comes from the observation that money today is preferred to money tomorrow. For income streams this assumption should be normatively acceptable, provided that the decision-maker has access to a competitive market for borrowing and lending money. In descriptive settings people who lack self-control may violate it (see Thaler 1992, Page 93). Assumption 2 requires, for example, that a lottery winner should opt to receive all of \$50;000 now rather than receiving \$25;000 now and \$25;000 a year from now. This assumption should not be confused with consumption where indeed a level of consumption may be more desirable than enormous immediate consumption now and subsistence level consumption later. In Section 5, we discuss the case where consequences are consumption streams rather than income streams. Throughout Sections $2-4$, consequences are assumed to be income streams.

We now state our first result, which shows that for any increasing, concave utility function u there is a discount factor $\beta < 1$ such that equation (1) and assumption A2 cannot be simultaneously satisfied.

Proposition 1 Model (1), A1, and A2 are incompatible.

Proof. Consider two payoff streams $(x; 0)$ and $(x/2; x/2)$. Set $u(0) = 0$ and $u(x) = 1$. For an increasing, strictly concave u, $u(x/2) = 1/2 + \delta$, for some $0 < \delta < 1/2$. As the time interval t between period 1 and 2 decreases, β increases approaching 1 as $t \to 0$. Now it is easy to see that for $\beta > (1/2 - \delta)/(1/2 + \delta)$, $(1 + \beta)u(x/2) > u(x)$ and $V(x/2; x/2) > V(x; 0)$ thus violating assumption 2. ■

Our observation that the discounting of utilities and risk aversion cannot coexist without violation of Assumption 2 is not an artifact of some extreme case analysis that obtains when $t \to 0$. For a mildly risk averse person, a violation of assumption A2 will appear for a reasonable time interval t. Consider a case where $0 \leq x \leq 1$ and $t = 1$ year. Further, assume that a decision-maker is endowed with the well known exponential utility function $u(x) = -e^{-x}$. In this case, $\beta > e^{-x/2}$ will lead to a preference of $(.5; .5)$ over $(1; 0)$. If $\beta = e^{-rt}$, then $t < \;/2r$ will produce such a reversal. Similarly for the utility function $u(x) = x^{\gamma}$, $0 < \gamma < 1$, $\beta > 2^{\gamma} - 1$ will lead to the preference of $(.5; .5)$ over $(1; 0)$. In the latter case, $\beta = .9$ (present value of \$1 received a year from now is 90 cents) and $\gamma < .9261$ (modest risk aversion) yields $V(.5; .5) > V(1; 0)$. A γ-value less than $.9261$ implies that the certainty equivalent of a $50 - 50$ lottery between \$0 and \$1000 is less than \$473. Since γ and β are chosen independently, the inconsistency noted above cannot be avoided. Consequently, a decision-maker that has a reasonable discount factor and a reasonable degree of risk aversion may be prescribed a plan that yields more money later compared to a plan that yields more money now. We note that if the additive form of model (1) is replaced with a multiplicative form (see Keeney and Raiffa 1976), the paradox remains.

If the function u is differentiable, then we can compute the intertemporal marginal rate of substitution. We recall that an intertemporal marginal rate of substitution bigger than one implies that the decision-maker prefers to reduce a bit today's income in order to increase a bit tomorrow's income, in violation of A2. In Model (1),

$$IT_{MRS} = \frac{\partial V/\partial x_2}{\partial V/\partial x_1} = \frac{\beta u'(x_2)}{u'(x_1)}.$$

The concavity of u readily implies that for $x_1 > x_2$, $u'(x_2)/u'(x_1) > 1$. Thus, we can always obtain $IT_{MRS} > 1$ by letting β approach 1.

3 Discounting Certainty Equivalents

We now examine the following model. Begin by computing the expected utility for each period, then calculate the certainty equivalent for each period, and, finally, discount the certainty equivalents. We recall that if X denotes a random payoff, then the certainty equivalent $e(X)$ is the given by

$$e(X) = u^{-1}(E[u(X)]).$$

The appeal of this model comes from thinking that the per-period certainty equivalent is the amount of money for which the decision-maker is willing to sell the uncertainty of that given period. The model is then

$$V(X_1; X_2) = e(X_1) + \beta e(X_2),\ 0 < \beta \leq 1; \qquad (2)$$

where $V(X_1; X_2)$ is the discounted certainty equivalent. We have that

Proposition 2 Model (2), A1, and A2 are incompatible.

Proof. Consider alternatives A and B. In alternative A the decision-maker receives a deterministic payoff stream of $(x/2; x/2)$. In alternative B, on the contrary, the payoffs are uncertain: with probability $1/2$ the decision-maker receives a payoff stream of $(x; 0)$, and with probability $1/2$ he receives a payoff stream of $(x/2; x/2)$. Clearly, B dominates A in that with probability $1/2$ the decision maker receives the entirety of x in period 1, as opposed to receiving $x/2$ now and $x/2$ later otherwise. By A2, B should always be preferred to A. The evaluation of alternative A is immediate:

$$V^A(X_1; X_2) = \frac{x}{2} + \beta\frac{x}{2} = x(1+\beta)/2:$$

For the evaluation of B, note that the strict concavity of u given by A1 implies

$$\begin{aligned} Eu(X_1) &= \tfrac{1}{2}u(x) + \tfrac{1}{2}u(x/2) < u(\tfrac{1}{2}x + \tfrac{1}{2}x/2) = u(3x/4); \text{ and} \\ Eu(X_2) &= \tfrac{1}{2}u(0) + \tfrac{1}{2}u(x/2) < u(\tfrac{1}{2}0 + \tfrac{1}{2}x/2) = u(x/4); \end{aligned}$$

and because $e(x)$ is an increasing function we conclude that $e(X_1) = u^{-1}[Eu(X_1)] < 3x/4$ and $e(X_2) = u^{-1}[Eu(X_2)] < x/4$ (see Figure 1).

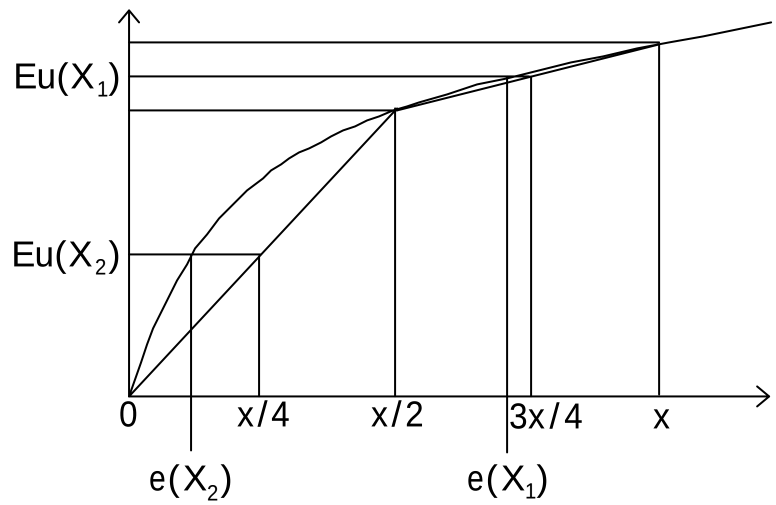

Figure 1: Certainty equivalents.

Thus, for some $\delta > 0$,[1] $V^B(X_1; X_2) = e(X_1) + \beta e(X_2) = (3+\beta)x/4 - \delta$. As $\beta \to 1$, $V^A(X_1; X_2) \to x$ and $V^B(X_1; X_2) \to x - \delta$. Because $\delta > 0$, $V^A(X_1; X_2) > V^B(X_1; X_2)$ for β close to 1 (small t), a contradiction. ■

[1] δ is the discounted risk premium: if $RP(X_i) = E(X_i) - e(X_i)$ is the per-period risk premium, then $\delta = RP(X_1) + \alpha RP(X_2)$.

4 Utility of Discounted Payoffs

We now propose a modification of equation (1)

$$V(x_1; x_2) = u(x_1 + \beta x_2),\ 0 < \beta < 1. \quad (3)$$

In model (3) above, first payoffs are discounted and then the single period utility function is applied. The expectation of V is used to evaluate lotteries over payoff streams. Such an approach is used by Smith (1998).

Proposition 3 Model (3), A1, and A2 are compatible.

Proof. Consider two streams:

$$(x_1 + \Delta; x_2 - \Delta) \text{ and } (x_1; x_2), \text{ where } \Delta > 0.$$

Since u is monotonically increasing, by model (3) we obtain

$$u(x_1 + \beta x_2 + \Delta(1 - \beta)) > u(x_1 + \beta x_2).$$

Thus, $(x_1 + \Delta; x_2 - \Delta)$ is always preferred to $(x_1; x_2)$ for any $x_1; x_2$, and $\Delta > 0$: assumption A2 is satisfied. ■

The intertemporal marginal rate of substitution in Model (3) is given by

$$IT_{MRS} = \frac{\partial V / \partial x_2}{\partial V / \partial x_1} = \beta,$$

which is clearly less than 1 by assumption, i.e., the decision maker always prefers the reception of an incremental income in period 1 than in period 2.

5 Evaluation of Consumption Streams

In Sections 2 − 4, we assumed that the consequence are income streams. For income streams, the assumption A2 is desirable because money today can be used to make money tomorrow and therefore an earlier receipt of money is always preferred. Now we consider the case when the consequences are consumption bundles.

For consumption, the assumption A2 loses its normative appeal. An additional consumption Δ is not necessarily sweeter today than it would be tomorrow; for example, if one already has high consumption today an faces a low consumption tomorrow. Most people prefer one pizza in each period to two pizzas in period 1 and none in period 2.

Discounted utility model (1) is appropriate for consumption streams provided that the time intervals between periods are large. Koopmans (1960) and Koopmans et al. (1964) have axiomatized the discounted utility model for countable infinite streams. The discounted utility model rests on the assumption of separability over time.

We note, however, that the assumption of separability over time is suspect when time intervals are small.[2] To illustrate the difficulty, consider the example below. We assume that the utility function is concave representing diminishing marginal utility of additional consumption.

Example 4 Consumption stream A is given by $(2c, 0)$, which rewards the decision-maker with a utility of $u(2c) + \beta u(0)$. The decision-maker is allowed to postpone his consumption plan and receive consumption stream B given by (c, c), which results in a utility evaluation of $u(c) + \beta u(c)$.

If the time interval between now and later is short, then β is effectively 1 so that stream A produces $u(2c) + u(0)$, whereas stream B produces $2u(c)$. The strict concavity of u implies that $2u(c) > u(2c) + u(0)$, and the consumer experiences a positive jump in the utility evaluation by instantly delaying the reception of a part of $2c$, regardless of how short is the time separation between now and later. Exploiting this opportunity, the decision maker could consume c/n over n consecutive instants and obtain $nu(c/n)$. In the limit, his utility approaches $u'(0)$.[3]

The previous example indicates that a desirable property for a multiperiod utility function $V(x_1, x_2)$ is that as $t \to 0$, $V(x_1, x_2) \to u(x_1 + x_2)$. By continuity at $t = 0$, this implies $u(x_1) + u(x_2) = u(x_1 + x_2)$. It is well known (see Aczél 1966) that such a functional equation is satisfied only if $u(x) = \alpha x$, for some positive α. Thus, the paradox in the example arises because the additive separability in (1) and the strict concavity of u are incompatible with a linear behavior of V near $t = 0$.

A simple example makes our argument vivid. Suppose you consume a pizza now and another pizza a month later. Then it seems appropriate to compute total utility by adding the utilities derived from consuming one pizza now and one pizza a month

[2] For example, the preference for a menu for dinner may depend on what one had for lunch, but may be largely independent of the consumption a month ago. For a model that relaxes separability see Gilboa (1989).

[3] For some functional forms accepted in the literartue, such as $u(x) = x^\alpha$, $u'(0) = \infty$.

from now with some suitable discounting. If, however, you consume one pizza now and another soon after consuming the first one, then you do not receive a total utility that is twice the utility of one pizza as the model (1) will imply. Instead, you get the utility of consuming two pizzas, which is likely to be less than twice the utility derived from consuming one pizza because of diminishing marginal utility. The additive model (1) assumes that the utility in each period is computed afresh. This is reasonable if there are no lingering effects of past consumption. The time interval within which the satiation due to past consumption disappears completely would vary a great deal based on the nature of consumption. For the pizza example, it may be a week or less but for a vacation the effects of past vacations may last a long time. In general, the separability assumption is a good approximation if time intervals between consumptions are large, but some accounting of satiation due to past consumption is needed if the time intervals between consumptions are small.

We now propose a realistic model that accommodates such short time intervals. The model introduces an element of intertemporal satiation so that previous consumption levels affect the utility evaluation derived from the current consumption. An example of such a model would be (see Figure 2)

$$V'(x_1; x_2) = u(x_1) + \beta[u(x_2 + y_2) - u(y_2)]; \tag{4}$$

where y_2 is some satiation level produced by previous consumption. To permit a decay in satiation over time, one may use $y_2 = x_1 e^{-\sigma t}$, where σ is a parameter that captures the decay in satiation due to the passage of time.[4] Thus, to evaluate the consumption stream B in Example 4 given by $(x_1; x_2) = (c; c)$, we have

$$V'(c; c) = u(c) + \beta[u(c + ce^{-\sigma t}) - u(ce^{-\sigma t})]:$$

Clearly, as $t \to 0$, $V'(c; c) \to u(2c)$, producing the desired property that an instant delay in consumption should not create a discontinuity in the utility evaluation.

The above example illustrates that the discounted utility model (1) assumes a restrictive behavioral assumption: in each period, the consumer evaluates the utility of the current consumption afresh. This may be reasonable when time intervals between periods are large and therefore satiation from the previous consumption can

[4] Thus, if $\sigma = 0$, then the satiation level is equal to the cumulative consumption; and if σ is large, then there is virtually no satiation. For a more general case with n periods, we have $V'(x_1; \ldots x_n) = \sum_{n=1}^{N} [u_n(x_n + y_n) - u(y_n)] e^{-rn\Delta t}$, $y_n = \sum_{s=1}^{s=n-1} x_s e^{-\sigma \Delta t (n-s)}$, and $y_1 = 0$. For a fixed σ, if Δt is large, then the model particularizes to the usual discounted utility model without satiation.

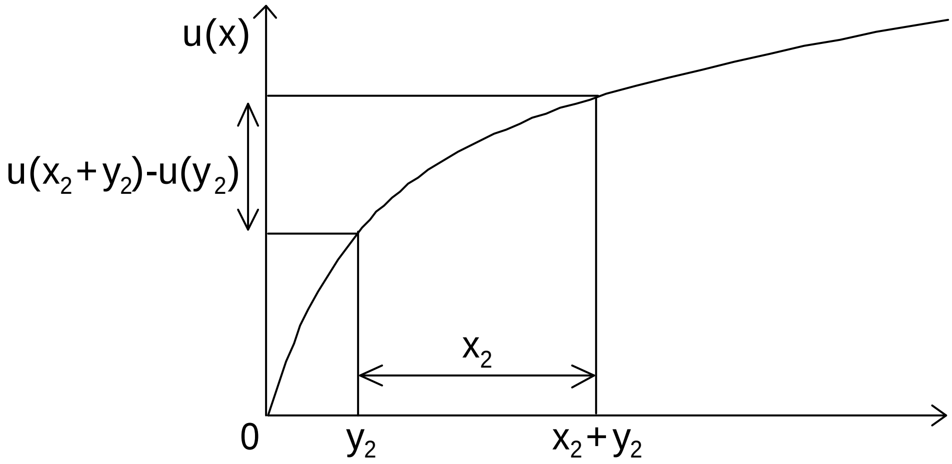

Figure 2: The evaluation of period 2 utility when satiation level due to period 1 consumption is y_2.

be ignored. Indeed, for large t, $e^{-\sigma t} \approx 0$ and $V'(x_1; x_2) \approx V(x_1; x_2)$. In the continuous time model,[5] the difficulty is more serious because one does not have the freedom to choose time intervals. The continuous time model assumes that the consumer evaluates the utility of x(t) afresh at each instant of time t, without experiencing satiation due to past consumption.

6 Conclusions

In this paper we consider the problem where the consequences of a decision accrue over time and are uncertain. We examine two cases: first, where the consequences are income streams, and second, where the consequences are consumption streams.

[5] In the continuous time case, we have

$$V(x) = \int_0^T u(x(t))e^{-rt}dt.$$

Satiation is easily introduced by considering

$$V'(x) = \int_0^\infty [u(x(t) + y(t)) - u(y(t))]e^{-rt}dt;$$

where $y(t) = \int_0^t x(s)e^{-\sigma(t-s)}ds$. Alternatively, $y'(t) = x(t) - \sigma y(t)$ and $y(0) = 0$: y(t) accumulates current consumption but depreciates at rate σ. Becker (1996) used a similar model to explain habit formation and addiction. See also Chakravarty and Manne (1968) for a model where instant utility depends on the rate of change of consumption.

The ...rst case arises in decision analysis or multiattribute utility analysis problems such as job selection where immediate and future compensation serve as a proxy attributes for economic well-being. The second case is common in economics where maximization of the utility of consumption subject to a budget constraint is assumed both in modeling and theoretical analysis.

Our main result is that risk aversion and discounting of utilities of income cannot coexist without violating the principle that money now is preferable to money later. Speci...cally, we show that the discounted utility model when applied to income steams or cash ‡ows leads to an undesirable result: for an individual with concave utility, postponing income increases utility. The paradox remains if, instead of using discounted utility, one discounts the certainty equivalents in each period. We show that discounting cash ‡ows ...rst and then applying utility to net present values leads to desirable results.

In dealing with consumption streams, it is appropriate to use discounted utility model, provided that the time periods are separated enough. For short time periods, the model loses realism in that an arbitrarily short delay of part of a unit consumption leads to a jump in the utility evaluation. This is because utility is evaluated afresh in each period. Thus, $(c; c)$ gives a utility of approximately equal to $2u(c)$ if the time interval between two periods is very small (discount factor ¼ 1), whereas $(2c; 0)$ gives a lower utility of $u(2c) + u(0)$. The continuous time model, in particular, hides the restrictive behavioral assumption that utility of consumption is evaluated afresh at each instant in time. Exploring the e¤ect of satiation in a time preference model is an agenda for future research.

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