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I. Introduction

In an earlier paper[1] an approximate theory was developed which contains the relationship between the frequency and propagation constant (or wave length) for axisymmetric motions of transversely isotropic rods. The rods are circular and solid, and the material of the rod is arranged so that axes of isotropy are parallel to the axis of the rod. It is a three-mode theory, and for those three modes the relationship between frequency and wave length corresponds accurately to the relationship given by the exact three dimensional theory of transversely isotropic rods. The approximate theory is given in terms of differential equations governing three "generalized displacements."

In an appendix of the same paper, trial solutions for the generalized displacements representing steady state vibrations were assumed, particularly as they apply to finite rods with homogeneous end boundary conditions. Space dependent functions were established using these trial solutions which, because they satisfy both the governing differential equations and homogeneous end boundary conditions represent mode shapes for finite rods vibrating freely. It was then established that these space functions form an orthogonal set and the orthogonality condition was derived. It is the orthogonality property of the normal or free-mode functions that permits finding the response of finite rods to an arbitrary forcing function.

In this paper the general forced vibration problem of finite, transversely isotropic rods is studied. The solution is given first in terms of an arbitrary input and following this a specific problem is described and solved. The specific problem is one of finding the response, expressed as the radial displacement, of an isotropic rod of specified length when a normal force at one end of the rod vibrates at two frequencies: one a resonant frequency of the rod, and one between resonant frequencies.

The paper begins with a very brief outline of the approximate theory which allows the paper to be self contained. In the third section the mode shapes for freely vibrating finite rods are developed in more detail than was possible in the previous treatment[1] and the orthogonality condition for these mode shapes is restated.

The nature of the mode shapes resembles closely those derived for finite, isotropic rods by McNiven and Perry[2]. For a rod of a particular geometry (length-to-radius ratio), the mode shapes for successive resonant frequencies alternate between "symmetric" and "antisymmetric" displacement distributions. As in reference 2, a symmetric mode is defined as one in which the radial displacement is symmetrical about the mid-length of the rod and the axial displacement is antisymmetrical. An antisymmetric mode has the opposite distributions. For any particular frequency there will theoretically be contributions from all modes, so it follows that the total motion must be described in terms of both symmetric and antisymmetric mode shapes.

In the fourth section an arbitrary forcing function is introduced and the system is subjected to arbitrary initial conditions. The solution to the problem involving inhomogeneous differential equations is established by taking the trial functions for the generalized displacements as infinite series of normal modes both symmetrical and antisymmetrical. The coefficients of the normal modes are unknown functions of time which are established by exploiting the orthogonality property of the normal modes.

In the fifth section a particular problem is chosen which, it is hoped, is realistic. A finite, isotropically elastic rod is subjected to a normal force on its left end which is applied with a specified frequency; the right end is free of traction. The solution, though complicated, is found and in the last section the results of the numerical analysis of the particular problem are disclosed and discussed.

II. The Approximate Theory

The approximate theory governing the axisymmetric motions in transversely isotropic rods was derived in a previous paper[1]. In this section, we review its content briefly.

The rod is circular and of radius "a". It is referred to a cylindrical coordinate system (r, θ, z) within which the "z" axis coincides with the axis of the rod. The material is arranged so that axes of isotropy are parallel to the axis of the rod.

In the development which follows, when it is appropriate, we use indicial notation and all the rules that apply to its use.

The theory is based on the constitutive equation

$$\tau_\alpha = c_{\alpha\beta}\,\varepsilon_\beta \qquad (\alpha, \beta = 1,\ldots,6) \tag{1}$$

where

$$(\tau_\alpha) = (\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{\theta z}, \tau_{zr}, \tau_{r\theta})$$

$$(\varepsilon_\alpha) = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, 2\varepsilon_{\theta z}, 2\varepsilon_{zr}, 2\varepsilon_{r\theta}),$$

and

$$(c_{\alpha\beta}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix}, \tag{2}$$

τ_{ij}, ε_{ij} being stress and strain tensors respectively, and $c_{\alpha\beta}$ the elastic coefficient matrix.

The theory is described in terms of generalized displacements u, w and ψ that are related to the radial and axial displacements u_r and u_z according to

$$u_r = \bar{r}u(x,\tau)$$
$$u_z = w(x,\tau) + (1-2\bar{r}^2)\,\psi(x,\tau), \qquad (3)$$

where

$\bar{r} = r/a$, a dimensionless radial distance

$x = \dfrac{\delta z}{a}$, a dimensionless axial distance

$\tau = \dfrac{\delta t}{a}\left(\dfrac{c_{44}}{\rho}\right)^{\frac{1}{2}}$, a dimensionless time

δ : a constant defined as the first nonzero root of $J_1(\delta_m) = 0$[3], J_1 being the Bessel function of the first kind

ρ : the mass density.

The constitutive equations relating these generalized displacements and generalized forces are

$$\left(\frac{a}{c_{44}}\right)P_r = K_1^2(2-\gamma_5)\frac{\gamma_3}{\gamma_4}u + \delta K_1(\gamma_3-1)\,w,_x$$
$$2\left(\frac{a}{c_{44}}\right)P_z = \delta\gamma_2^2\,w,_x + 2K_1(\gamma_3-1)u$$
$$4\left(\frac{a}{c_{44}}\right)P_{rz} = K_2^2(\delta\,u,_x - 4\psi) \qquad (4)$$
$$6\left(\frac{a}{c_{44}}\right)P_\psi = \delta\,\gamma_2^2\,\psi,_x\,,$$

where the generalized forces are defined by

$$P_r = \int_0^1 (\tau_{rr} + \tau_{\theta\theta})\, \bar{r} d\bar{r}$$

$$P_z = \int_0^1 (\tau_{zz}\, \bar{r} d\bar{r}$$

$$P_{rz} = \int_0^1 \tau_{rz}\, \bar{r}^2 d\bar{r} \tag{5}$$

$$P_\psi = \int_0^1 \tau_{zz}\, (1-2\bar{r}^2)\, \bar{r} d\bar{r}\ .$$

Using the notation of Reference 3,

$$\gamma_2 = \left(\frac{c_{33}}{c_{44}}\right)^{\frac{1}{2}} \quad ; \quad \gamma_4 = \frac{c_{13} + c_{44}}{c_{11}} \quad ;$$

$$\gamma_3 = \frac{c_{13} + c_{44}}{c_{44}} \ ; \ \gamma_5 = \frac{c_{11} - c_{12}}{c_{11}} \quad , \tag{6}$$

and the K_i's (i = 1-4) are adjustment factors introduced in the theory to make the three spectral lines of the theory match more closely the lowest three branches of the exact theory.

The three mode approximate theory is contained in the three equations

$$\delta K_2^2(\delta u,_{xx} - 4\psi,_x) - 4K_1^2(2-\gamma_5)\frac{\gamma_3}{\gamma_4} u - 4K_1(\gamma_3 - 1)\ \delta w,_x + \frac{4a}{c_{44}} R = K_3^2\, \delta^2\, u,_{\tau\tau}$$

$$\delta\gamma_2^2\ w,_{xx} + 2K_1(\gamma_3 - 1)u,_x + \frac{2a}{\delta c_{44}}\ Z \ = \delta w,_{\tau\tau} \tag{7}$$

$$\delta^2\gamma_2^2\ \psi,_{xx} + 6K_2^2(\delta u,_x - 4\psi) - \frac{6a}{c_{44}}\ Z \ = \delta^2 K_4^2\ \psi,_{\tau\tau}\ ,$$

where

$$R = \tau_{rr}(a,z,t)\ ;\ Z = \tau_{rz}(a,z,t).$$

III. Free Vibrations

In this section free vibrations of a finite rod, having zero stress end boundary conditions, are discussed and the mode shapes of the rod established. This knowledge, together with the information about the orthogonality conditions for mode shapes, which were derived in a previous paper[1], will be used in Section IV to study forced vibrations of a finite rod by means of a mode superposition technique.

We consider a finite rod of length 2L bounded by free surfaces at $r = a$ and at $z = \mp L$. For this problem the governing equations, Eqs. (7) with $R = Z = 0$, can be put into the following form

$$A_{ij}\, u_{j,xx} + B_{ij}\, u_{j,x} + C_{ij}\, u_j - D_{ij}\, u_{j,\tau\tau} = 0, \tag{8}$$

where

$$(A_{ij}) = \delta^2 \begin{bmatrix} 3K_2^2 & 0 & 0 \\ 0 & 6\gamma_2^2 & 0 \\ 0 & 0 & 2\gamma_2^2 \end{bmatrix}$$

$$(B_{ij}) = \delta \begin{bmatrix} 0 & -12K_1(\gamma_3 - 1) & -12K_2^2 \\ 12K_1(\gamma_3 - 1) & 0 & 0 \\ 12K_2^2 & 0 & 0 \end{bmatrix} \tag{9}$$

$$(C_{ij}) = \begin{bmatrix} -12K_1^2 \dfrac{\gamma_3}{\gamma_4}(2 - \gamma_5) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -48K_2^2 \end{bmatrix}$$

$$(D_{ij}) = \delta^2 \begin{bmatrix} 3K_3^2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2K_4^2 \end{bmatrix}$$ cont. (9)

and

$$(u_i) = (u, w, \psi) .$$

Using Eqs. (4,5), the free stress end boundary conditions can be written in terms of generalized displacements as

$$\begin{aligned} p\, w,_x(\mp\mathcal{L},\tau) + q u(\mp\mathcal{L},\tau) &= 0 \\ \delta\, u,_x(\mp\mathcal{L},\tau) - 4\psi(\mp\mathcal{L},\tau) &= 0 \\ \psi,_x(\mp\mathcal{L},\tau) &= 0 \end{aligned} \tag{10}$$

where

$$p = \tfrac{1}{2}\,\delta\,\gamma_2^2 \;;\quad q = K_1(\gamma_3 - 1) \;;\quad \mathcal{L} = \frac{\delta L}{a} . \tag{11}$$

For free vibrations we let the solution be in the form

$$u_i(x,\tau) = v_i(x)\, e^{i\Omega\tau} , \tag{12}$$

where the normalized angular frequency Ω is defined by

$$\Omega = \frac{\omega}{\omega_1^s} . \tag{13}$$

In Eq. (13) : ω is the angular frequency, $\omega_1^s = \frac{\delta G_{ns}}{a}$ is the first axial shear cut-off frequency and $G_{ns} = \left(\frac{c_{44}}{\rho}\right)^{\frac{1}{2}}$, the shear velocity[3].

Substitution of the trial solution, Eq. (12), into the governing equations of motion, Eq. (8) gives

$$A_{ij}\, v_{j,xx} + B_{ij}\, v_{j,x} + C_{ij}\, v_j + \Omega^2 D_{ij}\, v_j = 0 . \tag{14}$$

The boundary conditions, Eq. (10), in terms of v_i become

$$p\,\overline{w},_x(\mp\mathcal{L},\tau) + q\overline{u}(\mp\mathcal{L},\tau) = 0$$

$$\delta\,\overline{u},_x(\mp\mathcal{L},\tau) - 4\overline{\psi}(\mp\mathcal{L},\tau) = 0 \tag{15}$$

$$\overline{\psi},_x(\mp\mathcal{L},\tau) = 0$$

where

$$(v_i) = (\overline{u}\ ,\ \overline{w}\ ,\ \overline{\psi})\ .$$

We now seek the solution of Eq. (14), subject to the boundary conditions, Eq. (15).

The general solution of Eq. (14) is

$$\begin{bmatrix} \overline{u}(x) \\ \overline{w}(x) \\ \overline{\psi}(x) \end{bmatrix} = \sum_{n=1}^{3} \left\{ f_n \begin{bmatrix} \alpha_{un} \sin \zeta_n x \\ \alpha_{wn} \cos \zeta_n x \\ \alpha_{\psi n} \cos \zeta_n x \end{bmatrix} + h_n \begin{bmatrix} \beta_{un} \cos \zeta_n x \\ \beta_{wn} \sin \zeta_n x \\ \beta_{\psi n} \sin \zeta_n x \end{bmatrix} \right\}, \tag{16}$$

where the dimensionless wave propagation constant $\zeta_n = \frac{a\,k_n}{\delta}$ (n=1-3) (k_n is the wave propagation constant) is governed by the equation

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{vmatrix} = 0\ , \tag{17}$$

and

$$\begin{bmatrix} \alpha_{un} \\ \alpha_{wn} \\ \alpha_{\psi n} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{a_{12}}{a_{22}} \\ \frac{a_{13}}{a_{33}} \end{bmatrix} ; \begin{bmatrix} \beta_{un} \\ \beta_{wn} \\ \beta_{\psi n} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{a_{12}}{a_{22}} \\ -\frac{a_{13}}{a_{33}} \end{bmatrix} . \tag{18}$$

The f_n's and h_n's (n = 1-3) are six arbitrary constants to be determined from the boundary conditions.

In Eqs. (17-18):

$$
\begin{aligned}
a_{11} &= 3\{\delta^2 K_2^2\, \zeta^2 + 4K_1^2 \frac{\gamma_3}{\gamma_4} (2 - \gamma_5) - \delta^2 K_3^2\, \Omega^2\} \\
a_{22} &= 6\delta^2(\gamma_2^2\, \zeta^2 - \Omega^2) \\
a_{33} &= 2(\delta^2\gamma_2^2\, \zeta^2 + 24K_2^2 - \delta^2 K_4^2\, \Omega^2) \qquad (19) \\
a_{12} &= 12\delta K_1(\gamma_3 - 1)\zeta \\
a_{13} &= 12\delta K_2^2\, \zeta\ .
\end{aligned}
$$

For a given Ω, each a_{ij} has three values derived from the three values of ζ_n, the roots of Eq. (17).

We now proceed to apply boundary conditions, Eqs. (15), to the general solution, Eq. (16). We find

$$
\begin{aligned}
&\mp \sum_{n=1}^{3} \left[(q\alpha_{un} - p\alpha_{wn}\, \zeta_n) \sin \zeta_n \mathcal{L} \right] f_n + \sum_{n=1}^{3} \left[(q\beta_{un} + p\beta_{wn}\, \zeta_n) \cos\, \zeta_n \mathcal{L} \right] h_n = 0 \\
&\sum_{n=1}^{3} \left[(\delta\alpha_{un}\, \zeta_n - 4\alpha_{\psi n}) \cos\, \zeta_n \mathcal{L} \right] f_n \pm \sum_{n=1}^{3} \left[(\delta\beta_{un}\zeta_n + 4\beta_{\psi n}) \sin\, \zeta_n \mathcal{L} \right] h_n = 0 \qquad (20) \\
&\pm \sum_{n=1}^{3} (\alpha_{\psi n}\, \zeta_n \sin\, \zeta_n \mathcal{L})\, f_n + \sum_{n=1}^{3} (\beta_{\psi n}\, \zeta_n \cos\, \zeta_n \mathcal{L})\, h_n = 0.
\end{aligned}
$$

After manipulation Eqs. (20) can be put into two sets of equations in which f_n and h_n are uncoupled;

$$
\begin{aligned}
&\sum_{n=1}^{3} M_{mn}\, f_n = 0 \\
& \qquad\qquad\qquad (m = 1 - 3) \qquad\qquad (21) \\
&\sum_{n=1}^{3} N_{mn}\, h_n = 0\ ,
\end{aligned}
$$

where

$$M_{1n} = (q\alpha_{un} - p\alpha_{wn}\,\zeta_n)\,\sin\zeta_n\mathcal{L}$$

$$M_{2n} = (\delta\alpha_{un}\,\zeta_n - 4\alpha_{\psi n})\,\cos\zeta_n\mathcal{L} \qquad (n=1-3) \tag{22}$$

$$M_{3n} = \alpha_{\psi n}\,\zeta_n\,\sin\zeta_n\mathcal{L}$$

and

$$N_{1n} = (q\beta_{un} + p\beta_{wn}\,\zeta_n)\,\cos\zeta_n\mathcal{L}$$

$$N_{2n} = (\delta\beta_{un}\,\zeta_n + 4\beta_{\psi n})\,\sin\zeta_n\mathcal{L} \qquad (n=1-3) \tag{23}$$

$$N_{3n} = \beta_{\psi n}\,\zeta_n\,\cos\zeta_n\mathcal{L}\ .$$

There are two ways to satisfy Eqs. (21) nontrivially:

(i) For the first case we have

$$\det\,(M_{mn}) = 0\ . \tag{24}$$

Eq. (24) will give the relation between the length of the rod $\mathcal{L} = \frac{\delta L}{a}$ and the frequency Ω. Then, from Eqs. (21) we find

$$h_n = 0 \qquad (n = 1 - 3)$$
$$f_n = c\ f_n^*$$

$$(f_n^*) = \begin{bmatrix} 1 \\ \dfrac{M_{13}\,M_{21} - M_{11}\,M_{23}}{\Delta_M} \\ \dfrac{M_{11}\,M_{22} - M_{12}\,M_{21}}{\Delta_M} \end{bmatrix}, \tag{25}$$

where

$$\Delta_M = M_{12}\,M_{23} - M_{13}\,M_{22}\ , \tag{26}$$

and c is an arbitrary constant.

Substitution of Eqs. (25) into Eq. (16) gives

$$v_i = c\,\Phi_i \ , \qquad (27)$$

where

$$(\Phi_i) = \sum_{n=1}^{3} f_n^* \begin{bmatrix} \alpha_{un} \sin \zeta_n x \\ \alpha_{wn} \cos \zeta_n x \\ \alpha_{\psi n} \cos \zeta_n x \end{bmatrix} . \qquad (28)$$

From Eqs. (27) and (28) we see that the radial displacement (u) is antisymmetrical and the axial displacements (w, ψ) are symmetrical about the mid-length of the rod. These modes will be referred to as antisymmetric modes. Eq. (24) is the frequency equation for such modes.

(ii) For this case we let

$$\det\,(N_{mn}) = 0 \ . \qquad (29)$$

Then from Eq. (21) we find

$$\begin{aligned} f_n &= 0 \\ h_n &= d h_n^* \end{aligned} \qquad (n = 1 - 3)$$

$$(h_n^*) = \begin{bmatrix} 1 \\ \dfrac{N_{13}\,N_{21} - N_{11}\,N_{23}}{\Delta_N} \\ \dfrac{N_{11}\,N_{22} - N_{12}\,N_{21}}{\Delta_N} \end{bmatrix} , \qquad (30)$$

where

$$\Delta_N = N_{12}\,N_{23} - N_{13}\,N_{22} \ , \qquad (31)$$

and d is an arbitrary constant.

Substitution of Eqs. (30) into Eq. (16) gives

$$v_i = d\bar{\Phi}_i \qquad (32)$$

where

$$(\bar{\Phi}_i) = \sum_{n=1}^{3} h_n^* \begin{bmatrix} \beta_{un} \cos \zeta_n x \\ \beta_{wn} \sin \zeta_n x \\ \beta_{\psi n} \sin \zeta_n x \end{bmatrix} . \qquad (33)$$

From Eqs. (32) and (33) we see that the radial displacement (u) is symmetrical and the axial displacements (w, ψ) are antisymmetrical about the mid-length of the rod. These modes will be referred to as symmetrical modes. Equation (29) is the frequency equation for such modes, which governs the relation between the length of the rod $\mathcal{L} = \frac{\delta L}{a}$ and the frequency Ω.

Orthogonality Conditions

The orthogonality conditions of mode shapes were studied in a previous paper[1]. Since we will make use of these conditions when we study forced vibrations of a rod of a finite length in the next section, we restate the orthogonality conditions here. They are

$$\begin{aligned} \langle \Phi_i^m , D_{ij} \Phi_j^p \rangle &= 0 \quad \text{for } m \neq p \\ \langle \bar{\Phi}_i^m , D_{ij} \bar{\Phi}_j^p \rangle &= 0 \quad \text{for } m \neq p \\ \langle \bar{\Phi}_i^m , D_{ij} \Phi_j^p \rangle &= 0 \quad \text{for all m and p,} \end{aligned} \qquad (34)$$

where

$$\langle \Phi_i^m , D_{ij} \Phi_j^p \rangle = \int_{-\mathcal{L}}^{\mathcal{L}} D_{ij} \Phi_i^m \Phi_j^p \, dx \quad \ldots \text{etc.},$$

and Φ_i^m is the mode shape corresponding to the "m"th antisymmetric resonant frequency (Ω_m), and $\bar{\Phi}_i^m$ is the mode shape corresponding to the "m"th symmetric resonant frequency ($\bar{\Omega}_m$).

IV. Forced Vibrations

A general forced vibration problem, with non homogeneous stress end boundary conditions, can be reduced to finding the solution of the equation

$$A_{ij}\, u_{j,xx} + B_{ij}\, u_{j,x} + C_{ij}\, u_j - D_{ij}\, u_{j,\tau\tau} + G_i = 0 \tag{35}$$

subject to the initial conditions

$$\begin{aligned} u_i(x,o) &= \varphi_i(x) \\ u_{i,\tau}(x,o) &= \theta_i(x) \ , \end{aligned} \tag{36}$$

and zero stress end boundary conditions. In Eq. (35) the G_i are, in general, functions of x and τ.

In what follows we seek the solution for the above problem by means of a mode superposition technique. We start by letting the solution have the form

$$u_i = \sum_{p=1}^{\infty} c^p(\tau)\, \Phi_i^p(x) + \sum_{p=1}^{\infty} \bar{c}^p(\tau)\, \bar{\Phi}_i^p(x) \ , \tag{37}$$

where Φ_i^p and $\bar{\Phi}^p$ are the "p'th antisymmetric and symmetric mode shapes of free vibrations respectively, which satisfy

$$\begin{aligned} A_{ij}\, \Phi^p_{j,xx} + B_{ij}\, \Phi^p_{j,x} + C_{ij}\, \Phi^p_j + \Omega_p^2\, D_{ij}\, \Phi^p_j &= 0 \\ A_{ij}\, \bar{\Phi}^p_{j,xx} + B_{ij}\, \bar{\Phi}^p_{j,x} + C_{ij}\, \bar{\Phi}^p_j + \bar{\Omega}_p^2\, D_{ij}\, \bar{\Phi}^p_j &= 0 \ , \end{aligned} \tag{38}$$

respectively. $c^p(\tau)$, $\bar{c}^p(\tau)$ are functions of τ to be determined.

We now let

$$G_i(x,\tau) = \sum_{p=1}^{\infty} g^p(\tau)\, D_{ij}\, \Phi_j^p(x) + \sum_{p=1}^{\infty} \bar{g}^p(\tau)\, D_{ij}\, \bar{\Phi}_j^p(x)$$

$$\varphi_i(x) = \sum_{p=1}^{\infty} \varphi^p\, \Phi_i^p(x) + \sum_{p=1}^{\infty} \bar{\varphi}^p\, \bar{\Phi}_i^p(x) \tag{39}$$

$$\theta_i(x) = \sum_{p=1}^{\infty} \theta^p\, \Phi_i^p(x) + \sum_{p=1}^{\infty} \bar{\theta}^p\, \bar{\Phi}_i^p(x)\ .$$

Since $G_i(x,\tau)$, $\varphi_i(x)$ and $\theta_i(x)$ are specified functions, the coefficients in Eqs. (39) can be determined using the orthogonality conditions, Eqs. (34). The coefficients are

$$g^p(\tau) = \frac{\langle G_i,\ \Phi_i^p \rangle}{\langle \Phi_i^p,\ D_{ij}\Phi_j^p \rangle} \quad ; \quad \bar{g}^p(\tau) = \frac{\langle G_i,\ \bar{\Phi}_i^p \rangle}{\langle \bar{\Phi}_i^p,\ D_{ij}\bar{\Phi}_j^p \rangle}$$

$$\varphi^p = \frac{\langle \Phi_j^p,\ D_{ji}\,\varphi_i \rangle}{\langle \Phi_j^p,\ D_{ji}\,\Phi_i^p \rangle} \quad ; \quad \bar{\varphi}^p = \frac{\langle \bar{\Phi}_j^p,\ D_{ji}\varphi_i \rangle}{\langle \bar{\Phi}_j^p,\ D_{ji}\bar{\Phi}_i^p \rangle} \tag{40}$$

$$\theta^p = \frac{\langle \Phi_j^p,\ D_{ji}\theta_i \rangle}{\langle \Phi_j^p,\ D_{ji}\Phi_i^p \rangle} \quad ; \quad \bar{\theta}^p = \frac{\langle \bar{\Phi}_j^p,\ D_{ji}\theta_i \rangle}{\langle \bar{\Phi}_j^p,\ D_{ji}\bar{\Phi}_i^p \rangle}\ .$$

Substituting Eq. (37) and the first of Eqs. (39) into the governing equation, Eq. (35), and making use of Eqs. (38), we find

$$\sum_{p=1}^{\infty} (c^p_{,\tau\tau} + \Omega_p^2\, c^p - g^p)\, D_{ij}\, \Phi_j^p$$
$$+ \sum_{p=1}^{\infty} (\bar{c}^p_{,\tau\tau} + \bar{\Omega}_p^2\, \bar{c}^p - \bar{g}^p)\, D_{ij}\, \bar{\Phi}_j^p = 0\ . \tag{41}$$

By again using the orthogonality conditions one can show that Eq. (41) will be satisfied if and only if

$$c^p_{,\tau\tau} + \Omega_p^2 c^p = g^p$$
$$\bar{c}^p_{,\tau\tau} + \bar{\Omega}_p^2 \bar{c}^p = \bar{g}^p \quad . \tag{42}$$

The initial conditions for $c^p(\tau)$ and $\bar{c}^p(\tau)$ can be obtained from Eq. (37) and the second and third of Eqs. (39). They are

$$c^p(0) = \varphi^p \; ; \; \bar{c}^p(0) = \bar{\varphi}^p \; ;$$
$$c^p_{,\tau}(0) = \theta^p \; ; \; \bar{c}^p_{,\tau}(0) = \bar{\theta}^p \quad . \tag{43}$$

The solutions of Eqs. (42) subject to the initial conditions, Eqs. (43), are

$$c^p(\tau) = \varphi^p \cos \Omega_p \tau + \frac{\theta^p}{\Omega_p} \sin \Omega_p \tau + \frac{1}{\Omega_p} \int_0^\tau g^p(\tau') \sin \Omega_p (\tau-\tau') d\tau'$$
$$\bar{c}^p(\tau) = \bar{\varphi}^p \cos \bar{\Omega}_p \tau + \frac{\bar{\theta}^p}{\bar{\Omega}_p} \sin \bar{\Omega}_p \tau + \frac{1}{\bar{\Omega}_p} \int_0^\tau \bar{g}^p(\tau') \sin \bar{\Omega}_p (\tau-\tau') d\tau' \tag{44}$$

Thus, the solution given by Eq. (37) is complete.

V. A Particular Problem

In this section we study the following problem: We seek the response of a rod of finite length, initially at rest, subjected to a uniform normal stress on the left end of the rod that has a sinusoidal dependence in time, while its right end is free of traction.

Mathematically, we seek the solution of Eq. (8) subject to the boundary conditions

$$\begin{aligned} &\tau_{zz}(-\mathcal{L},\tau) = P_o \sin \Omega\tau \ ; \ \tau_{zz}(\mathcal{L},\tau) \equiv 0 \\ &\tau_{zr}(\pm \mathcal{L},\tau) \equiv 0 \quad , \end{aligned} \tag{45}$$

and the initial conditions

$$\begin{aligned} u(x,o) &= u,_\tau(x,o) \equiv 0 \\ w(x,o) &= w,_\tau(x,o) \equiv 0 \\ \psi(x,o) &= \psi,_\tau(x,o) \equiv 0 \quad . \end{aligned} \tag{46}$$

In Eqs. (45), Ω is the frequency of the input, which is specified, and P_o is a known constant.

Using Eqs. (4,5), the boundary conditions can be written in terms of the kinematic variables. They are

$$\begin{aligned} &pw,_x(-\mathcal{L},\tau) + qu(-\mathcal{L},\tau) = A_o \sin \Omega\tau \\ &pw,_x(\mathcal{L},\tau) + qu(\mathcal{L},\tau) \equiv 0 \\ &\delta u,_x(\pm \mathcal{L},\tau) - 4\psi(\pm \mathcal{L},\tau) \equiv 0 \\ &\psi,_x(\pm \mathcal{L},\tau) \equiv 0 \quad , \end{aligned} \tag{47}$$

where

$$A_o = \frac{aP_o}{2c_{44}} \quad . \tag{48}$$

First, we wish to make the boundary conditions homogeneous. To this end we let the solution be

$$u_i = u_i^* + m_i \ , \tag{49}$$

where

$$(m_i) = \left(0,\ A\left(\mathcal{L}x - \frac{x^2}{2}\right) \sin \Omega\tau,\ 0\right), \tag{50}$$

and

$$(u_i^*) = (u^*,\ w^*,\ \psi^*) \ . \tag{51}$$

In Eq. (50):

$$A = \frac{A_o}{2\mathcal{L}p} \ . \tag{52}$$

Then the problem reduces to the following one: We seek the solution of the equation

$$A_{ij}u^*_{j,xx} + B_{ij}u^*_{j,x} + C_{ij}u^*_j - D_{ij}u^*_{j,\tau\tau} + G_i = 0 \ , \tag{53}$$

subject to the boundary conditions

$$\begin{aligned} &pw^*_{,x}(\pm \mathcal{L},\tau) + qu^*(\pm \mathcal{L},\tau) \equiv 0 \\ &\delta u^*_{,x}(\pm \mathcal{L},\tau) - 4\psi^*(\pm \mathcal{L},\tau) \equiv 0 \\ &\psi^*_{,x}(\pm \mathcal{L},\tau) \equiv 0 \ , \end{aligned} \tag{54}$$

and the initial conditions

$$\begin{aligned} &u^*(x,o) = u^*_{,\tau}(x,o) \equiv 0 \\ &w^*(x,o) = 0 \ ; \ w^*_{,\tau}(x,o) = A(s_1x + s_2x^2) \\ &\psi^*(x,o) = \psi^*_{,\tau}(x,o) \equiv 0 \ , \end{aligned} \tag{55}$$

where

$$(G_i) = A(G_{10} + G_{11}x,\ G_{20} + G_{21}x + G_{22}x^2,\ 0)\sin \Omega\tau \tag{56}$$

$$\begin{aligned} G_{10} &= B_{12}\mathcal{L} \\ G_{11} &= -\ B_{12} \\ G_{20} &= -\ A_{22} \\ G_{21} &= D_{22}\Omega^2\mathcal{L} \\ G_{22} &= -\ D_{22}\Omega^2/2 \end{aligned} \tag{57}$$

$$\begin{aligned} s_1 &= -\ \mathcal{L}\Omega \\ s_2 &= \frac{\Omega}{2}\ . \end{aligned} \tag{58}$$

As outlined in section IV, the solution of this particular problem can be obtained explicitly. It is

$$\begin{aligned} u &= \sum_{p=1}^{\infty} c^p(\tau)\Phi_1^p(x) + \sum_{p=1}^{\infty} \bar{c}^p(\tau)\bar{\Phi}_1^p(x) \\ w &= \sum_{p=1}^{\infty} c^p(\tau)\Phi_2^p(x) + \sum_{p=1}^{\infty} \bar{c}^p(\tau)\bar{\Phi}_2^p(x) + A\left(\mathcal{L}x - \frac{x^2}{2}\right)\sin \Omega\tau \\ \psi &= \sum_{p=1}^{\infty} c^p(\tau)\Phi_3^p(x) + \sum_{p=1}^{\infty} \bar{c}^p(\tau)\bar{\Phi}_3^p(x)\ . \end{aligned} \tag{59}$$

In Eqs. (59):

$$c^p(\tau) = \frac{A}{d_p \Omega_p}\left\{\theta_o^p \sin \Omega_p \tau + \frac{g_o^p}{2}\left[\frac{1}{\Omega + \Omega_p}(\sin \Omega\tau + \sin \Omega_p \tau) - \frac{1}{\Omega - \Omega_p}(\sin \Omega\tau - \sin \Omega_p \tau)\right]\right\} \quad (\text{for } \Omega \neq \Omega_p)$$

$$c^p(\tau) = \frac{A}{d_p \Omega_p}\left\{\theta_o^p \sin \Omega_p \tau + \frac{g_o^p}{2}\left(\frac{1}{\Omega_p}\sin \Omega_p \tau - \tau \cos \Omega_p \tau\right)\right\} \quad (\text{for } \Omega = \Omega_p)$$

$$\bar{c}^p(\tau) = \frac{A}{\bar{d}_p \bar{\Omega}_p}\left\{\bar{\theta}_o^p \sin \bar{\Omega}_p \tau + \frac{\bar{g}_o^p}{2}\left[\frac{1}{\Omega + \bar{\Omega}_p}(\sin \Omega\tau + \sin \bar{\Omega}_p \tau) - \frac{1}{\Omega - \bar{\Omega}_p}(\sin \Omega\tau - \sin \bar{\Omega}_p \tau)\right]\right\} \quad (\text{for } \Omega \neq \bar{\Omega}_p) \tag{60}$$

$$\bar{c}^p(\tau) = \frac{A}{\bar{d}_p \bar{\Omega}_p}\left\{\bar{\theta}_o^p \sin \bar{\Omega}_p \tau + \frac{\bar{g}_o^p}{2}\left(\frac{1}{\bar{\Omega}_p}\sin \bar{\Omega}_p \tau - \tau \cos \bar{\Omega}_p \tau\right)\right\} \quad (\text{for } \Omega = \bar{\Omega}_p)$$

where

$$d_p = D_{11}\sum_{n=1}^{3}\sum_{m=1}^{3} f_n^{*p} f_m^{*p} \alpha_{un}^p \alpha_{um}^p S_{nm}^p + D_{22}\sum_{n=1}^{3}\sum_{m=1}^{3} f_n^{*p} f_m^{*p} \alpha_{wn}^p \alpha_{wm}^p C_{nm}^p + D_{33}\sum_{n=1}^{3}\sum_{m=1}^{3} f_n^{p*} f_m^{*p} \alpha_{\psi n}^p \alpha_{\psi m}^p C_{nm}^p \tag{61}$$

$$\bar{d}_p = D_{11} \sum_{n=1}^{3} \sum_{m=1}^{3} h_n^{*p} h_m^{*p} \beta_{un}^p \beta_{um}^p C_{nm}^p$$

$$+ D_{22} \sum_{n=1}^{3} \sum_{m=1}^{3} h_n^{*p} h_m^{*p} \beta_{wn}^p \beta_{wm}^p S_{nm}^p \qquad \text{cont. (61)}$$

$$+ D_{33} \sum_{n=1}^{3} \sum_{m=1}^{3} h_n^{*p} h_m^{*p} \beta_{\psi n}^p \beta_{\psi m}^p S_{nm}^p$$

$$S_{nm}^p = \begin{cases} \left[\dfrac{\sin(\zeta_n^p - \zeta_m^p)\mathcal{L}}{(\zeta_n^p - \zeta_m^p)} - \dfrac{\sin(\zeta_n^p + \zeta_m^p)\mathcal{L}}{(\zeta_n^p + \zeta_m^p)} \right] & \text{for } m \neq n \\ \left[\mathcal{L} - \dfrac{\sin(2\zeta_n^p \mathcal{L})}{2\zeta_n^p} \right] & \text{for } m = n \end{cases} \qquad (62)$$

$$C_{nm}^p = \begin{cases} \left[\dfrac{\sin(\zeta_n^p - \zeta_m^p)\mathcal{L}}{(\zeta_n^p - \zeta_m^p)} + \dfrac{\sin(\zeta_n^p + \zeta_m^p)\mathcal{L}}{(\zeta_n^p + \zeta_m^p)} \right] & \text{for } m \neq n \\ \left[\mathcal{L} + \dfrac{\sin(2\zeta_n^p \mathcal{L})}{2\zeta_n^p} \right] & \text{for } m = n \end{cases} \qquad (63)$$

$$g_o^p = G_{11}a_{11}^p + G_{20}a_{20}^p + G_{22}a_{22}^p$$

$$\bar{g}_o^p = G_{10}b_{10}^p + G_{21} b_{21}^p \qquad (64)$$

$$\theta_o^p = D_{22}s_2a_{22}^p$$

$$\bar{\theta}_o^p = D_{22}s_1b_{21}^p \qquad (65)$$

$$a_{11}^{p} = 2 \sum_{n=1}^{3} f_n^{*p} \frac{\alpha_{un}^{p}}{\zeta_n^{p}} \left(\frac{\sin \zeta_n^{p} \mathfrak{L}}{\zeta_n^{p}} - \mathfrak{L} \cos \zeta_n^{p} \mathfrak{L} \right)$$

$$a_{20}^{p} = 2 \sum_{n=1}^{3} f_n^{*p} \frac{\alpha_{wn}^{p}}{\zeta_n^{p}} \sin \zeta_n^{p} \mathfrak{L}$$

$$a_{22}^{p} = 2 \sum_{n=1}^{3} f_n^{*p} \frac{\alpha_{wn}^{p}}{\zeta_n^{p}} \left(\mathfrak{L}^2 \sin \zeta_n^{p} \mathfrak{L} + 2\mathfrak{L} \frac{\cos \zeta_n^{p} \mathfrak{L}}{\zeta_n^{p}} - 2 \frac{\sin \zeta_n^{p} \mathfrak{L}}{(\zeta_n^{p})^2} \right) \qquad (66)$$

$$b_{10}^{p} = 2 \sum_{n=1}^{3} h_n^{*p} \beta_{un}^{p} \frac{\sin \zeta_n^{p} \mathfrak{L}}{\zeta_n^{p}}$$

$$b_{21}^{p} = 2 \sum_{n=1}^{3} \frac{h_n^{*p} \beta_{wn}^{p}}{\zeta_n^{p}} \left(\frac{\sin \zeta_n^{p} \mathfrak{L}}{\zeta_n^{p}} - \mathfrak{L} \cos \zeta_n^{p} \mathfrak{L} \right) .$$

VI. Numerical Analysis

Our choice is to calculate and exhibit the radial displacement. Even though the theory is developed for transversely isotropic rods, numerical computations are carried out for an isotropic rod having a Poisson's ratio of $\nu = 0.29$. This is possible because transversely isotropic materials contain isotropic materials as a special case. For the ratio of the length of the rod to its radius we choose $2 \frac{L}{a} = 26$.

Computations are carried out for two different input frequencies:

(i) First the input frequency is chosen equal to the frequency of the second antisymmetric mode, i.e., $\Omega = \Omega_2 = 0.20132$. The variations of the radial displacement (along the rod at fixed times, and in time at a fixed station) are shown in Figs. (1-6).

The significant point about the response of the rod to a forcing frequency Ω_p which is a particular resonant frequency (in this case Ω_2), is that the second term of the first infinite series solutions for "u" ($c^2(\tau)$) contains the term $\tau \cos \Omega_2 \tau$ (see the second Eqs. (60)). The influence of these terms can be seen by examining Figs. (2-6). First by comparing Figs. (2, 3, 4) the influence of τ in the $\cos \Omega_2 \tau$ part of the term can be seen. When $\tau = 311.80$ $\cos \Omega_2 \tau$ is close to one allowing the second mode to predominate, whereas when $\tau = 307.80$ and 305.80, even though τ is large, the $\cos \Omega_2 \tau$ is small reducing the influence of the term and therefore the influence of the second mode. This influence can also be seen in Fig. 1. By examining Figs. (2, 5, 6), the influence of the τ preceding $\cos \Omega_2 \tau$ is recognized. For $\tau = 311.80$ the displacement amplitude is large and the second mode accounts for practically all of the motion whereas for the short times $\tau = 44$ and 79 the amplitudes are much smaller and the second mode no longer has such a predominant influence. We also note that at $\tau = 44$ the wave has not as yet reached the right end of the rod, thus a portion of the rod at the right end remains undisturbed (Fig. 5). At the later time $\tau = 79$ the disturbance has reached the end of the rod and the distribution shown in Fig. (6) contains the influence of the wave reflection from the right end.

(ii) The second input frequency is $\Omega = 0.2250$ which is between the frequency of the second antisymmetric mode, $\Omega_2 = 0.20132$, and that of the third symmetric mode, $\bar{\Omega}_3 = 0.25067$. The radial displacement distribution for the time $\tau = 300$ is shown in Fig. (7). Study of the figure shows that the distribution is neither symmetric nor antisymmetric and that the values of "u" are small compared to those of Fig. (2). It is important to note from Fig. (7) which modes are contributing to the motion. If only the modes below the forcing frequency are accounted for (i.e., four modes), the actual distribution is poorly described. However, when the contribution of the mode whose resonant frequency is immediately above the forcing frequency (the fifth mode) is added to the four lower order contributions, the distribution is quite accurate. This conclusion can be reached when it is noted that adding the influences of the seven modes above this fifth mode alters the five-mode distribution very little.

Acknowledgement

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References

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2. H. D. McNiven and D. C. Perry, "Axially Symmetric Waves in Finite, Elastic Rods," The Journal of the Acoustical Society of America, vol. 34, No. 4, April 1962, pp. 433-437.

3. H. D. McNiven and Y. Mengi, "Dispersion of Waves in Transversely Isotropic Rods," to appear in The Journal of the Acoustical Society of America, September, 1970.

Captions for Figures

Fig. 1 Radial displacement for the frequency $\Omega = 0.20132$ at the station $x = -14.943$.

Fig. 2 Radial displacement distribution for the frequency $\Omega = 0.20132$ at the time $\tau = 311.80$.

Fig. 3 Radial displacement distribution for the frequency $\Omega = 0.20132$ at the time $\tau = 307.80$.

Fig. 4 Radial displacement distribution for the frequency $\Omega = 0.20132$ at the time $\tau = 305.80$.

Fig. 5 Radial displacement distribution for the frequency $\Omega = 0.20132$ at the time $\tau = 44$.

Fig. 6 Radial displacement distribution for the frequency $\Omega = 0.20132$ at the time $\tau = 79$.

Fig. 7 Radial displacement distribution for the frequency $\Omega = 0.2250$ at the time $\tau = 300$.

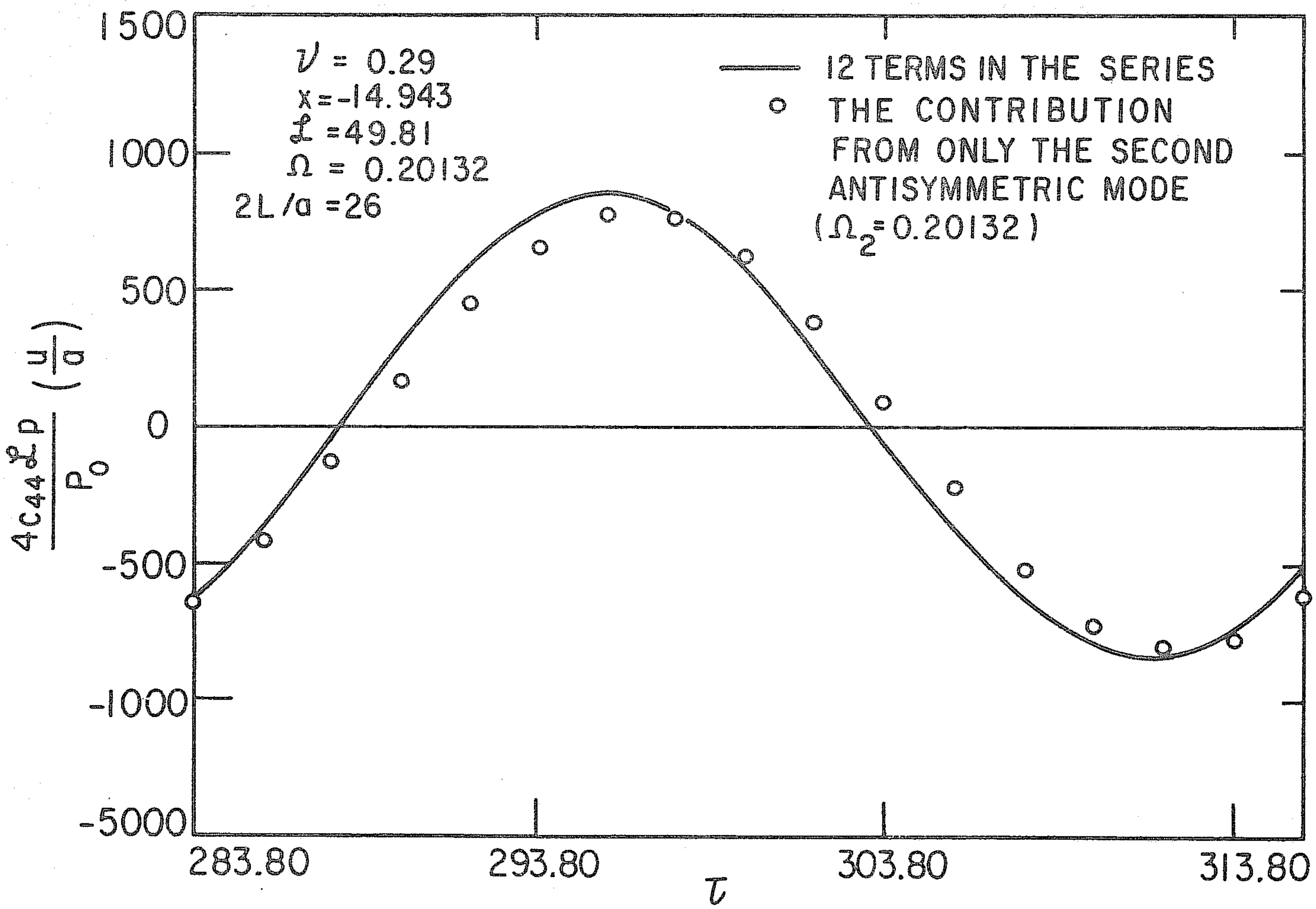

1500
1000
500
0
-500
-1000
-5000
ν = 0.29
x=-14.943
ℒ =49.81
Ω = 0.20132
2L/a =26
12 TERMS IN THE SERIES
THE CONTRIBUTION
FROM ONLY THE SECOND
ANTISYMMETRIC MODE
(Ω2=0.20132)
283.80
293.80
303.80
313.80
τ

FIG. 1

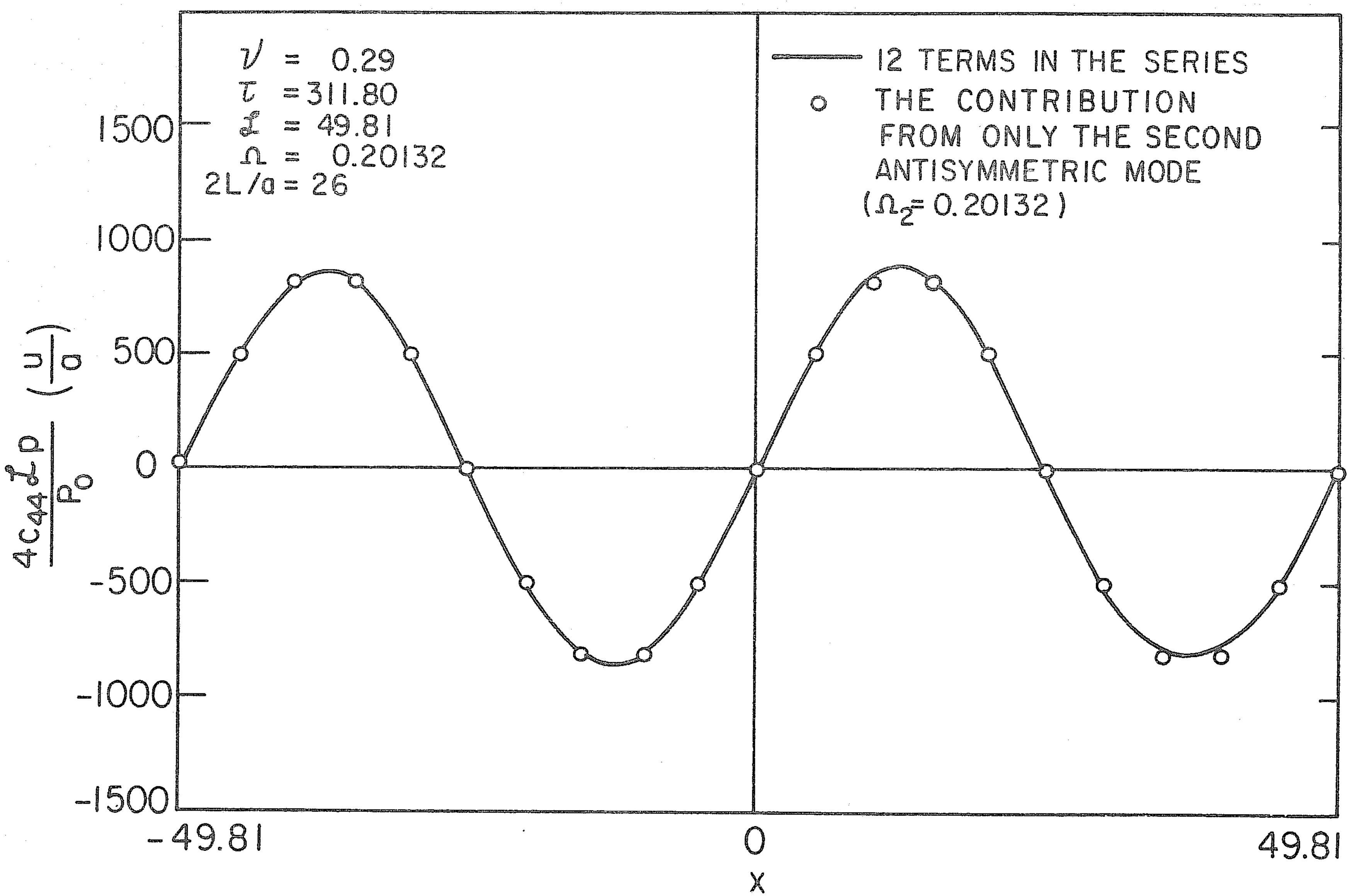
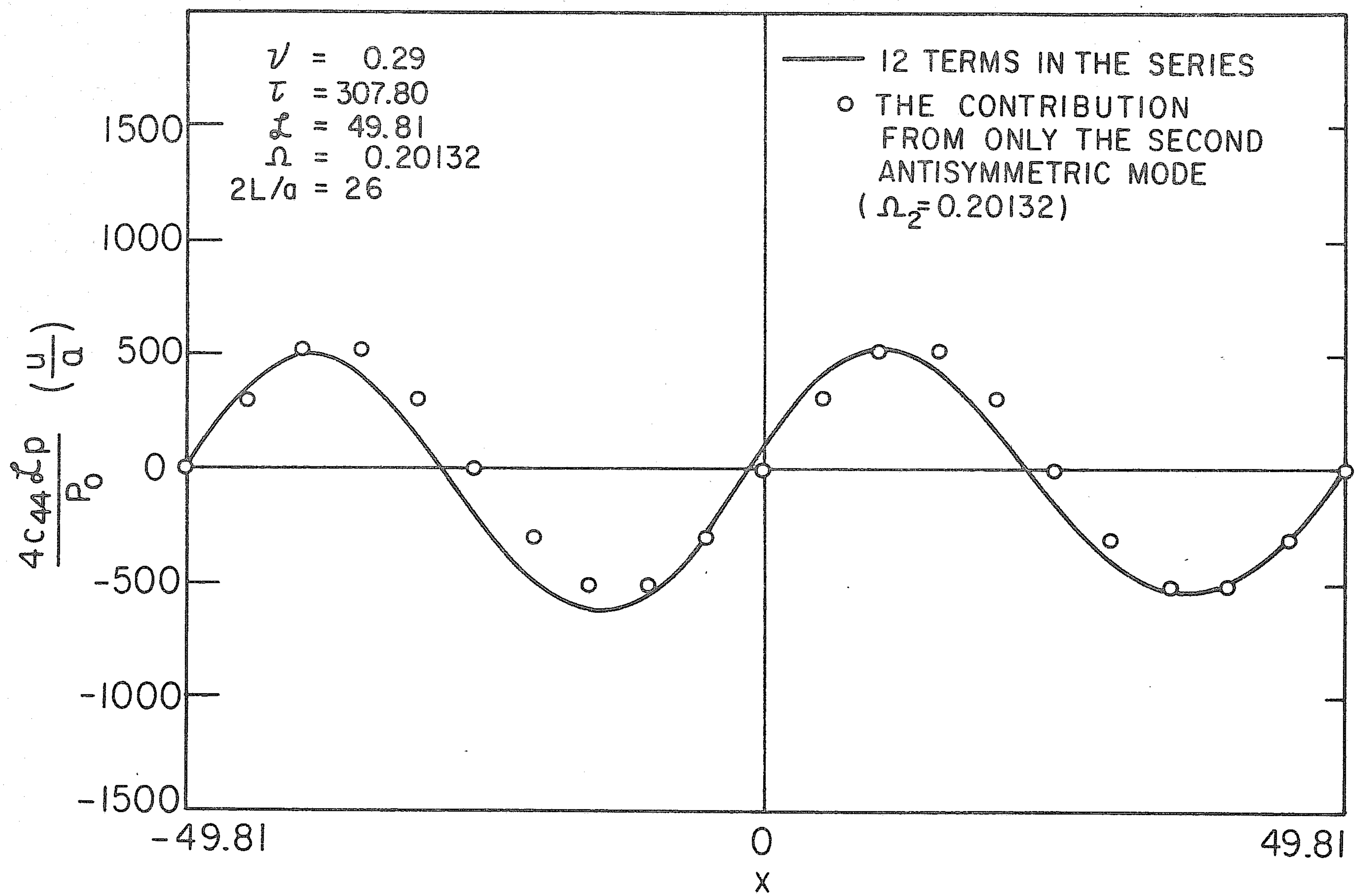
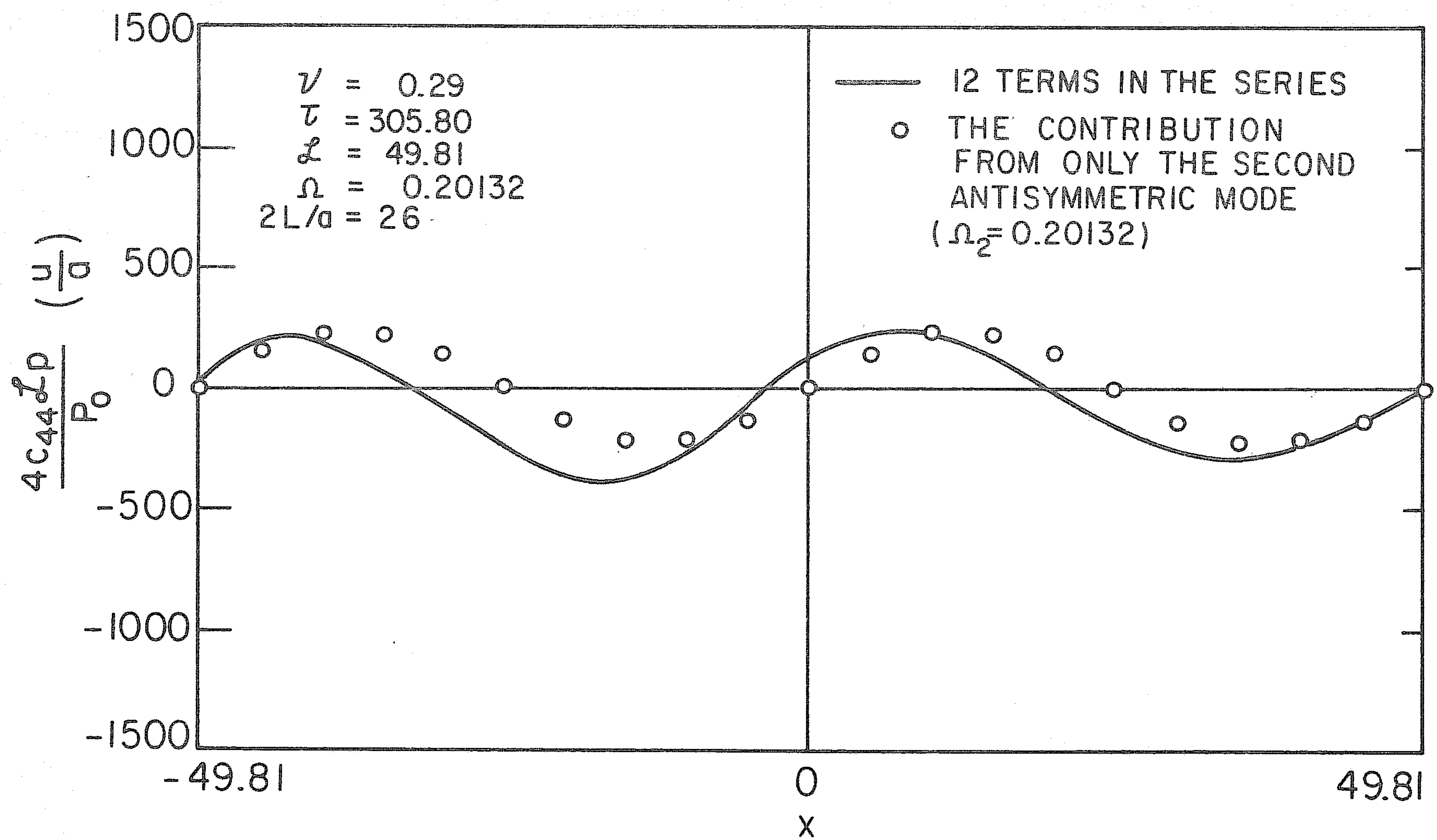

ν = 0.29
τ = 311.80
ℒ = 49.81
Ω = 0.20132
2L/a = 26
12 TERMS IN THE SERIES
THE CONTRIBUTION FROM ONLY THE SECOND ANTISYMMETRIC MODE
(Ω2= 0.20132)
1500
1000
500
0
-500
-1000
-1500
-49.81
0
49.81
X

FIG. 2



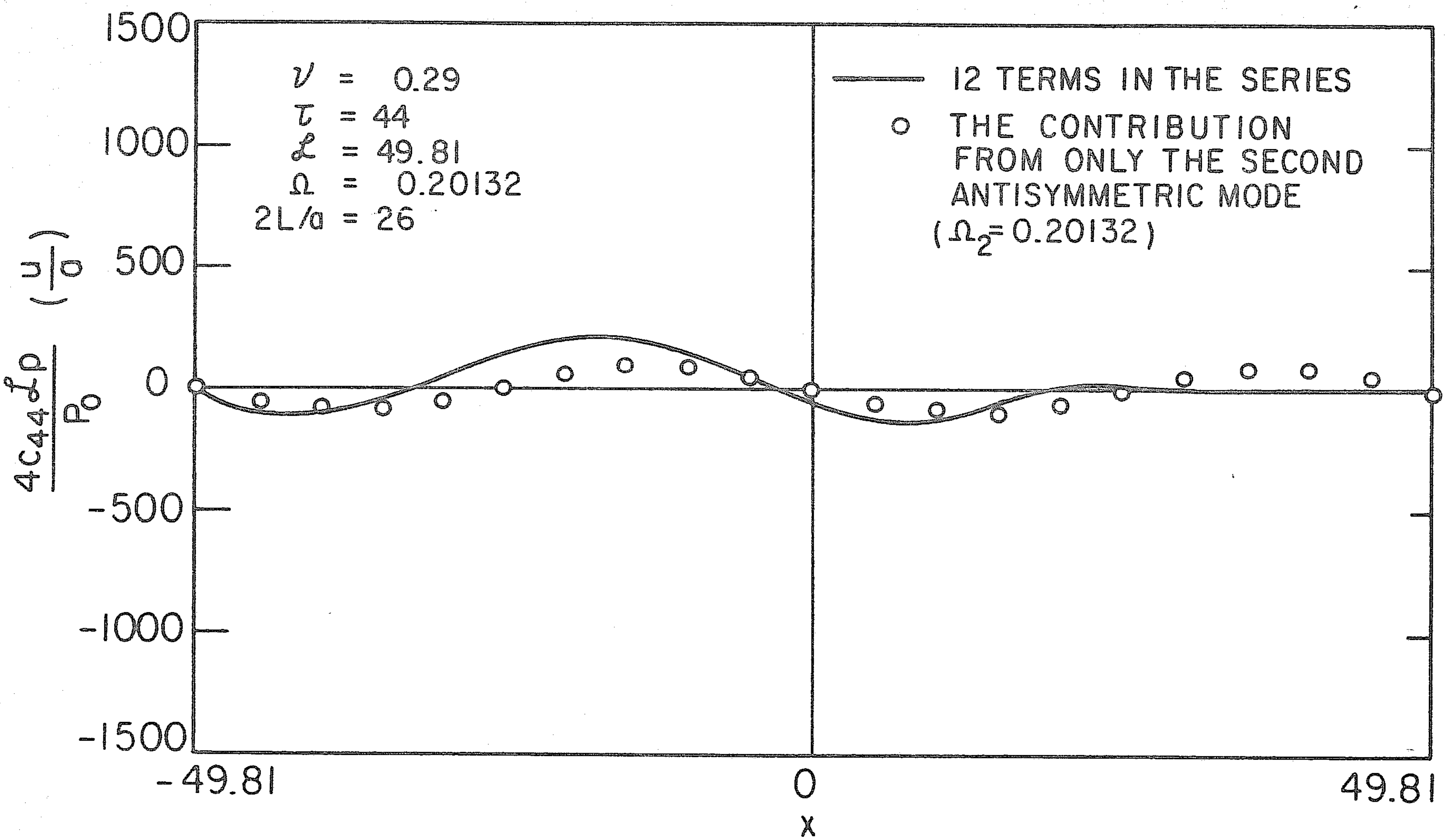
ν = 0.29
τ = 307.80
ℒ = 49.81
Ω = 0.20132
2L/a = 26
12 TERMS IN THE SERIES
THE CONTRIBUTION FROM ONLY THE SECOND ANTISYMMETRIC MODE
(Ω₂= 0.20132)
1500
1000
500
0
-500
-1000
-1500
4c44ℒp/Po (u/a)
-49.81
0
49.81
X

FIG. 3



1500
1000
500
0
-500
-1000
-1500
$\frac{4c_{44}\mathcal{L}p}{P_o}\left(\frac{u}{a}\right)$
-49.81
0
49.81
X
ν = 0.29
τ = 305.80
$\mathcal{L}$ = 49.81
Ω = 0.20132
2L/a = 26
12 TERMS IN THE SERIES
THE CONTRIBUTION FROM ONLY THE SECOND ANTISYMMETRIC MODE
(Ω_2= 0.20132)

FIG. 4



1500
1000
500
0
-500
-1000
-1500
$\frac{4c_{44}\mathcal{L}\rho}{P_0}\left(\frac{u}{a}\right)$
-49.81
0
49.81
X
ν = 0.29
τ = 44
$\mathcal{L}$ = 49.81
Ω = 0.20132
2L/a = 26
12 TERMS IN THE SERIES
THE CONTRIBUTION FROM ONLY THE SECOND ANTISYMMETRIC MODE (Ω_2= 0.20132)

FIG. 5

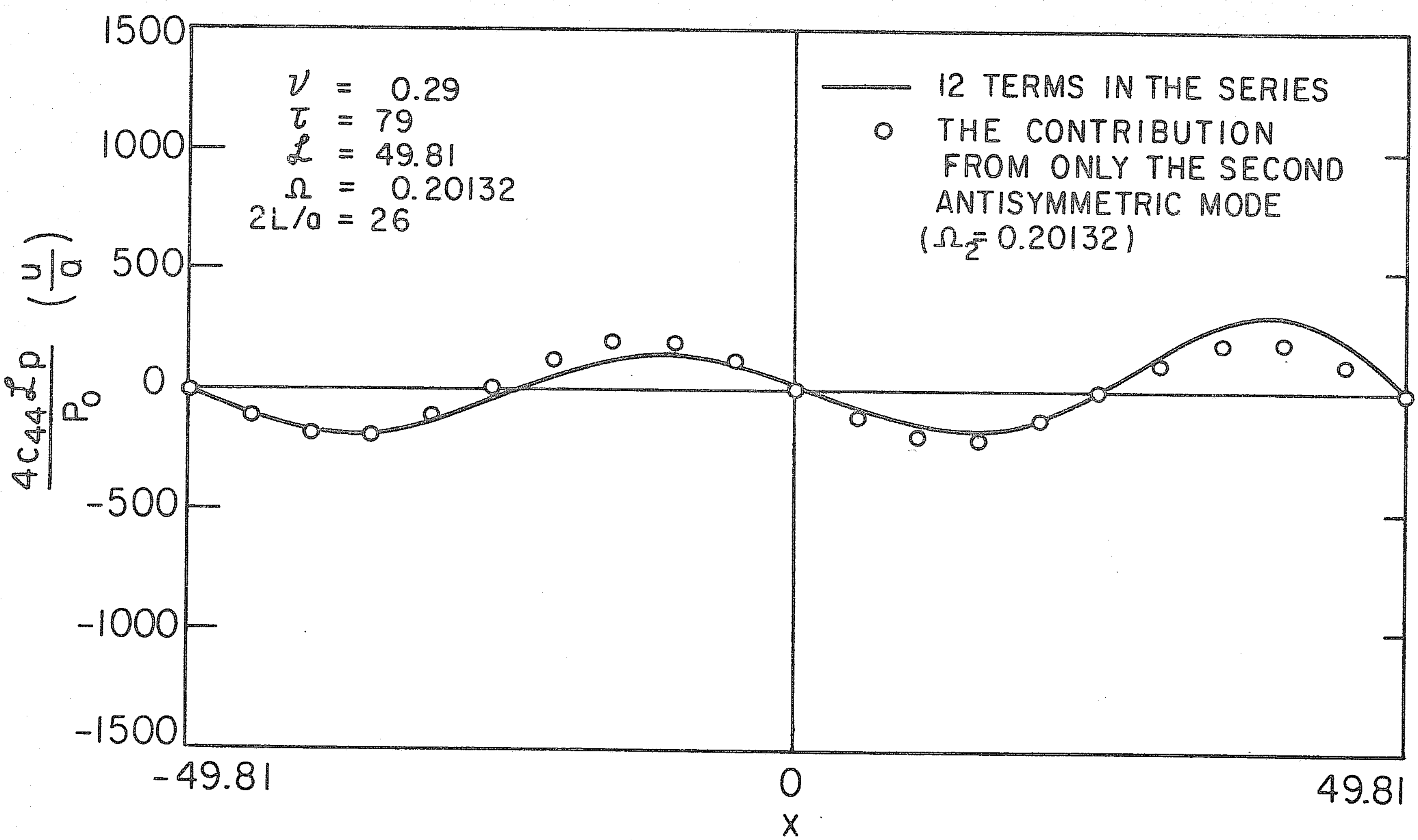

1500
1000
500
0
-500
-1000
-1500
$\frac{4c_{44}\mathcal{L}p}{P_o}\left(\frac{u}{a}\right)$
ν = 0.29
τ = 79
$\mathcal{L}$ = 49.81
Ω = 0.20132
2L/a = 26
12 TERMS IN THE SERIES
THE CONTRIBUTION FROM ONLY THE SECOND ANTISYMMETRIC MODE (Ω_2= 0.20132)
-49.81
0
49.81
X

FIG. 6

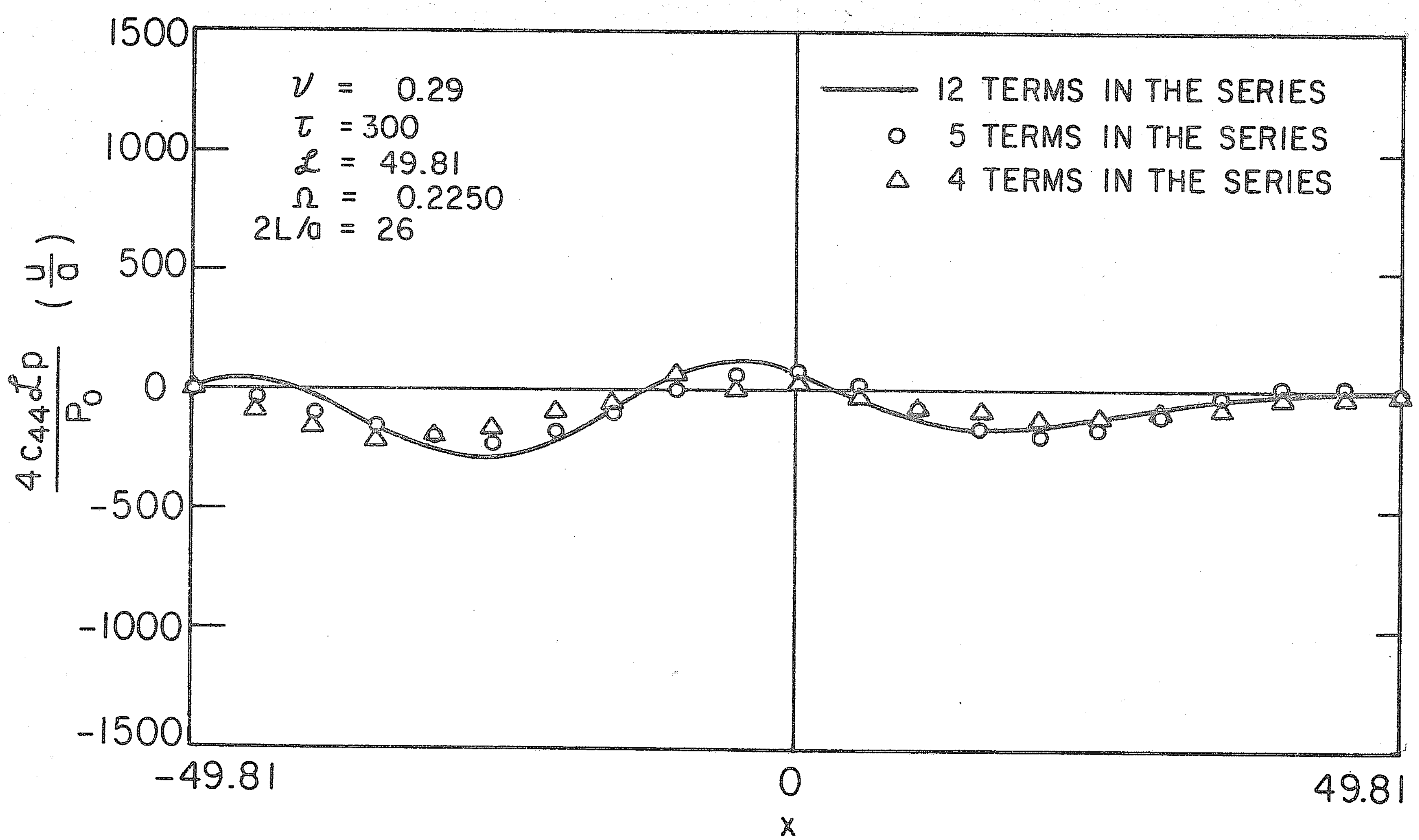

1500
1000
500
0
-500
-1000
-1500
$\frac{4 c_{44} \mathcal{L} p}{P_o} \left(\frac{u}{a}\right)$
ν = 0.29
τ = 300
$\mathcal{L}$ = 49.81
Ω = 0.2250
2L/a = 26
12 TERMS IN THE SERIES
5 TERMS IN THE SERIES
4 TERMS IN THE SERIES
-49.81
0
49.81
x

FIG. 7