Title

Power and Energy Analysis for Robust Gaussian Joint Source-Channel Coding With a Distortion-Noise Profile

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Power and Energy Analysis for Robust Gaussian Joint Source-Channel Coding With a Distortion-Noise Profile

A Dissertation submitted in partial satisfaction
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The content of this thesis is a reprint of the materials that are appeared in the following publications:

1. **M. Baniasadi** and E. Tuncel, "Robust Gaussian Joint Source-Channel Coding Under the Near-Zero Bandwidth Regime," *IEEE International Symposium on Information Theory*, Los Angeles, CA, USA, Jun. 2020.

2. **M. Baniasadi** and E. Tuncel, "Minimum Energy Analysis for Robust Gaussian Joint Source-Channel Coding with a Square-Law Profile," *The International Symposium on Information Theory and Its Applications*, Kapolei, Hawaii, USA, Oct. 2020.

3. **M. Baniasadi**, E. Köken, and E. Tuncel, " Minimum Energy Analysis for Robust Gaussian Joint Source-Channel Coding with a Distortion-Noise Profile," submitted to *IEEE Transactions on Communications*.

This thesis also includes some materials which have not been published in peer-reviewed venues yet. We plan to publish the following paper in near future.

1. **M. Baniasadi** and E. Tuncel, "Robust Gaussian JSCC Under the Near-Infinity Bandwidth Regime with Side Information at the Receiver," to be published.

To my lovely parents.

ABSTRACT OF THE DISSERTATION

Power and Energy Analysis for Robust Gaussian Joint Source-Channel Coding With a Distortion-Noise Profile

by

Mohammadamin Baniasadi

Doctor of Philosophy, Graduate Program in Electrical Engineering
University of California, Riverside, December 2020
Dr. Ertem Tuncel, Chairperson

In this thesis, we investigate robust Gaussian joint source-channel coding with a distortion-noise profile. A distortion-noise profile is a function indicating the maximum allowed distortion value for each noise level. We analyze three different scenarios, and propose novel hybrid digital-analog based joint source-channel coding schemes which generalize or outperform existing schemes.

In the first scenario, we look at power-distortion trade-off for the case of bandwidth compression when the bandwidth ratio is near zero. We propose hybrid digital-analog schemes and prove a general lower bound for minimum power which is valid for any distortion-noise profile. We also find upper bounds for minimum power for specific profiles including rational profiles with order one and two.

In the second scenario, we consider energy-distortion trade-off for bandwidth expansion when the bandwidth ratio is near infinity. As in the bandwidth compression case, we propose hybrid digital-analog schemes and derive a general lower bound for

minimum achievable energy for any profile. We discuss certain profiles including inversely linear, exponential, square-law, and staircase in more detail and establish upper bounds for minimum energy for these profiles.

In the third scenario, we add side information available at the receiver onto the second scenario, and similarly calculate a general lower bound for minimum energy. We also propose coding schemes providing upper bounds on the minimum energy for linear and staircase profiles.

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Chapter 1

Introduction

In lossy transmission of a source over a noisy point to point bandwidth mismatched channel, it is shown in Shannon's paper [1] that the minimum distortion can be achieved asymptotically by using separate source and channel codes when both source and channel codewords goes to infinity. However separable schemes suffer from the threshold effect, i.e., the codewords cannot be decoded reliably when the channel quality is lower than expected and the performance degrades drastically. Also, they cannot benefit from an occasional increase in the channel quality either, which is called the leveling-off effect. Direct mapping from the source sequence to the channel input, which is called joint source-channel coding, is expected to perform at least as good as separate coding.

One of the prominent class of joint source-channel coding schemes is hybrid digital analog (HDA) coding. HDA schemes are robust in the sense that they mitigate the adverse effects of the ambiguity in the channel signal-to-noise-ratio (SNR). HDA coding increases the reconstruction quality of the source at the receivers in many scenarios such as point-to-point and broadcast communication systems. In this thesis, we propose novel robust HDA-based joint source channel coding schemes with a distortion noise profile for different scenarios. A distortion-noise profile is a function indicating the maximum allowed distortion value for each noise level.

In one scenario, we address minimum power analysis for bandwidth compression case and specifically near-zero bandwidth ratio. In another scenario, we address the case where the energy per source symbol is limited but the channel uses per source symbol is unlimited, which corresponds to unlimited bandwidth. We also analyze the near-infinity bandwidth case with having side information at the decoder in our last scenario.

In Chapter 2, we study the transmission of Gaussian sources over Gaussian channels under a regime of bandwidth approaching zero. An instrumental lower bound to the minimum required power for a given profile is presented. For an upper bound, a dirty-paper based coding scheme is proposed and its power-distortion tradeoff is analyzed. Finally, upper and lower bounds to the minimum power are analyzed and compared for specific distortion-noise profiles, namely rational profiles with order one and two.

In Chapter 3, the minimum energy required to achieve a distortion-noise profile is studied for robust transmission of Gaussian sources over Gaussian channels. We discuss the previous results for the inversely linear and exponential profiles. For square-law and staircase profiles, we propose coding schemes to upper bound the minimum energy needed. Conversely, utilizing a family of lower bounds originally derived for broadcast channels with power constraints, we lower bound the minimum required energy for both square-law and staircase profiles, and compare with the corresponding upper bounds.

In Chapter 4, minimum energy required to achieve a distortion-noise profile is studied for robust transmission of Gaussian sources over Gaussian channels when there is a side information about source at the receiver, where the quality of the side information is also unknown. In this case, the quality parameter would be two-dimensional. We propose coding schemes to upper bound the minimum energy needed for square-law and staircase profiles. Conversely, a general family of lower bounds is derived for the minimum required energy which works for any profiles including square-law and staircase.

In Chapter 5, we conclude our work and discuss the future work. One possible future work can be expanding our results to multiple access channels (MAC), where for each transmitter, there is a separate distortion-noise profile dictating maximum distortion levels as a function of the noise level of the MAC.

Chapter 2

Robust Gaussian JSCC Under the Near-Zero Bandwidth Regime

2.1 Introduction

In this chapter, minimum power required to achieve a distortion-noise profile, i.e., a function indicating the maximum allowed distortion value for each noise level, is studied for the transmission of Gaussian sources over Gaussian channels under a regime of bandwidth approaching zero. A simple but instrumental lower bound to the minimum required power for a given profile is presented. For an upper bound, a dirty-paper based coding scheme is proposed and its power-distortion tradeoff is analyzed. Finally, upper and lower bounds to the minimum power are analyzed and compared for specific distortion-noise profiles, namely rational profiles with order one

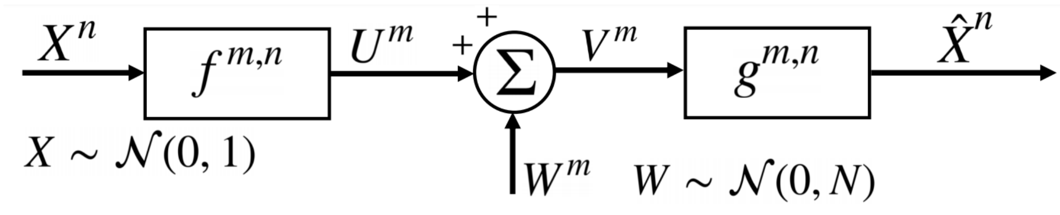

Figure 2.1: Lossy transmission of a Gaussian source over an additive white Gaussian noise (AWGN) channel

and two.

We consider the classical scenario of lossy transmission of a Gaussian source over an additive white Gaussian noise (AWGN) channel as shown in Figure 2.1, where the channel input constraint is on power. We also do not assume any feedback in our system model. When the channel noise variance N is fixed, it is well-known (thanks to the famous separation theorem) that the minimum distortion that can be achieved with input power P is given by

$$D_\kappa = \frac{1}{\left(1+\frac{P}{N}\right)^\kappa} \tag{2.1}$$

where κ is the bandwidth factor (with a unit of *channel uses per source symbol*).

In this work, however, we instead consider the scenario where N is not known at the transmitter (but known at the receiver as usual) and can assume any positive value $N > 0$. The system is to be designed to combat the unknown level of noise and comply with a *distortion-noise profile* $\mathcal{D}_\kappa(N)$, i.e., achieve

$$D_\kappa(N) \le \mathcal{D}_\kappa(N)$$

for all $N > 0$, while minimizing its power use, where $D_\kappa(N)$ denotes the achieved distortion at noise level N. This setting reflects a very adverse situation in which even though the channel may be originally of very high quality ($N \approx 0$), it could be suffering from occasional interferences of a wide spectrum of noise levels (including $N \gg 0$).

This scenario was previously tackled in [2] in the context of infinite bandwidth, i.e., $\kappa \to \infty$, where *energy* naturally replaces power as the currency (see [3], [4], [5], [6], and [7] for other work on energy-distortion tradeoff). Here, we address the other extreme, where the bandwidth is severely limited, i.e., $\kappa \approx 0$. This near-zero bandwidth condition might arise in cases where too many devices (e.g., in Internet-of-Things networks) share the same communication medium through multiplexing (e.g., TDMA, FDMA, etc.) Part of the theoretical and intellectual appeal, admittedly, is also the fact that performance of achievable schemes and converses simplify as $\kappa \to 0$.

Now, it should be clear that at $\kappa = 0$, there could be no communication, and as a result the squared error distortion is exactly 1 (assuming a unit variance source). Therefore, using a first order approximation with respect to κ, we expect the distortion to behave as

$$D_\kappa(N) \approx 1 + \kappa \left.\frac{\mathrm{d}D_\kappa(N)}{\mathrm{d}\kappa}\right|_{(\kappa=0)}$$

when κ is small but non-zero[1]. Thus, the quantity of interest throughout this chapter will be the *fidelity* the coding scheme achieves, defined as the negative slope of the

[1] The derivative is always negative as the distortion can only be improved with positive bandwidth.

distortion at $\kappa = 0$, i.e.,

$$F(N) = -\left.\frac{\mathrm{d}D_\kappa(N)}{\mathrm{d}\kappa}\right|_{(\kappa=0)} .$$

It will also prove more convenient to describe the fidelity as a function of *quality* level $Q = \frac{1}{N}$, i.e., as $F(Q)$. Our goal then is to analyze the minimum power needed to achieve a given *fidelity-quality profile* $\mathcal{F}(Q)$, i.e., to ensure

$$F(Q) \geq \mathcal{F}(Q) .$$

We derive a family of lower bounds to the minimum achievable power for a general profile $\mathcal{F}(Q)$, and discuss certain profiles in more detail. Specifically, we show that (i) the optimal scheme for rational profile with order one is simple uncoded transmission, and (ii) establish upper and lower bounds on the minimum energy for rational profile with order two[2].

One of the similar universal coding scenarios in the literature is given in [8], where a maximum regret approach for compound channels is proposed. The objective in their scenario is to minimize the maximum ratio of the capacity to the achieved rate at any noise level. Other related work in the literature includes [9], [10], [11], [12], [13], [14], and [15]. We will explain some of these in details in section 2.3.

The rest of this chapter is organized as follows. The next section is devoted to preliminaries and notation. In Section 2.3, we review the related work. In Section 2.4, a simple lower bounds on $P_{min}(\mathcal{F})$ is derived. Finally, in Section 2.5, we analyze

[2]We refer to $\mathcal{F}(Q) = \frac{\alpha Q}{1+\alpha Q}$ and $\mathcal{F}(Q) = \frac{\alpha Q^2}{1+\alpha Q^2}$ for some α as rational profiles with order one and two, respectively.

rational fidelity-quality profiles of order one and two and propose upper and lower bounds for them. The results of this chapter have been published in [16].

2.2 Preliminaries and Notation

Let X^n be an i.i.d. unit-variance Gaussian source to be transmitted over an AWGN channel $V^m = U^m + W^m$, where U^m is the channel input, $W^m \sim \mathcal{N}(\mathbf{0}, N\mathbf{I}_m)$ is the additive noise, and V^m is the observation at the receiver.

Definition 1. A pair of distortion-noise profile $\mathcal{D}_\kappa(N)$ and power level P is said to be *achievable* if for every $\epsilon > 0$, there exists (m, n), an encoder

$$f^{m,n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m ,$$

and decoders

$$g_N^{m,n} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

for every $0 < N < \infty$, such that

$$\frac{m}{n} \leq \kappa + \epsilon$$

together with

$$\frac{1}{m}\mathbb{E}\left\{||f^{m,n}(X^n)||^2\right\} \leq P + \epsilon$$

and

$$\frac{1}{n}\mathbb{E}\left\{||X^n - g_N^{m,n}(f^{m,n}(X^n) + W_N^m)||^2\right\} \leq \mathcal{D}_\kappa(N) + \epsilon$$

with W_N^m being the i.i.d. channel noise with variance N.

For a given function $\mathcal{D}_\kappa$, the main quantity of interest would be

$$P_{\min}(\mathcal{D}_\kappa) = \inf\{P : (\mathcal{D}_\kappa, P) \text{ achievable}\}$$

with the understanding that $P_{\min}(\mathcal{D}_\kappa) = \infty$ if there is no finite P for which $(\mathcal{D}_\kappa, P)$ is achievable.

As mentioned in the Introduction, we investigate this problem at the extreme of $\kappa \to 0$, in which case no distortion level less than 1 can be achieved for any N, and the problem as it is stated becomes trivialized. We instead look into what can be achieved for near-zero κ in terms of *how fast the distortion decreases* as a function of κ for all levels of noise $N > 0$, or equivalently, for all levels of quality $Q = \frac{1}{N}$.

Definition 2. A pair of fidelity-quality profile $\mathcal{F}(Q)$ and power level P is said to be *achievable* if for every $\epsilon > 0$, there exists an achievable $(\mathcal{D}_\kappa, P)$ for all $0 \leq \kappa < \epsilon$ such that

$$\mathcal{F}(Q) = -\left.\frac{\mathrm{d}\mathcal{D}_\kappa(\frac{1}{Q})}{\mathrm{d}\kappa}\right|_{(\kappa=0)} .$$

Note that $\mathcal{D}_\kappa(N)$ needs to be differentiable at $\kappa = 0$. Also, $P_{\min}(\mathcal{F})$ is similarly defined as

$$P_{\min}(\mathcal{F}) = \inf\{P : (\mathcal{F}, P) \text{ achievable}\} .$$

2.3 Related Work

In this section, we review the previous work.

In [9], the tradeoff between the distortion when the channel quality is good versus bad is investigated for transmission of memoryless Gaussian sources over channels with additive white Gaussian noise (AWGN). They propose novel schemes for $\frac{1}{2} \leq \kappa \leq 1$ and $1 \leq \kappa \leq 2$ achieving nontrivial tradeoffs outperforming all known schemes.

In [13], lossy transmission of a memoryless bivariate Gaussian source over a bandwidth mismatched AWGN channel with two receivers is studied. The authors show that their scheme for bandwidth compression outperforms the HDA coding scheme of [18] if their proposed conjecture (supported by numerical observations) is indeed true.

In [14], the problem of broadcasting a Gaussian source to two users over an AWGN channel is considered. A framework is developed which shows a duality between source-channel coding schemes for bandwidth expansion ($\kappa > 1$) and those for bandwidth compression ($\kappa < 1$). The authors then utilized the bandwitdh expansion scheme proposed by Reznic, Zamir, and Feder in [19] to develop achievable schemes for $\kappa < 1$. The authors also provide an analysis of performance of source-channel coding schemes in the presence of a signal-to-noise ratio (SNR) mismatch.

In [15], three hybrid digital-analog (HDA) systems for the transmission of Gaussian sources over AWGN channels under bandwidth compression are studied. Upper bounds on the asymptotically optimal mean squared error distortion are calculated for both matched and mismatched channel conditions.

As we discussed previously, we target robust communication with noise variance

N is unknown at the transmitter. At a first glance, it seems that we can employ the results in the aforementioned work to obtain achievability results simply by letting $\kappa \to 0$. However, they all are limited to those values of N for which at least one digital layer would be decoded (i.e., $N \leq N_1$ for some N_1), and sacrifice the distortion when $N > N_1$. We instead need a scheme which can operate at *all* noise levels $N > 0$. Towards that end, we develop our own achievability schemes.

2.4 A Family of Lower Bound on $P_{\min}(\mathcal{F})$

An immediate lower bound on $P_{\min}(\mathcal{F})$ follows from (2.1). Despite its simplicity, it will be instrumental in the sequel.

Lemma 1.

$$P_{\min}(\mathcal{F}) \geq \sup_{Q>0} \frac{\exp(\mathcal{F}(Q)) - 1}{Q} \ . \tag{2.2}$$

Proof. From (2.1), it follows that for any fixed N_0, the distortion $D_\kappa(N_0)$ achieved by any scheme with bandwidth κ has to satisfy

$$D_\kappa(N_0) \geq \frac{1}{\left(1 + \frac{P}{N_0}\right)^\kappa} \ . \tag{2.3}$$

Defining $Q_0 = \frac{1}{N_0}$ and

$$F(Q_0) = -\frac{\mathrm{d}D_\kappa\left(\frac{1}{Q_0}\right)}{\mathrm{d}\kappa}\Bigg|_{(\kappa=0)} ,$$

we can approximate (2.3) around $\kappa \approx 0$ as

$$1 - \kappa F(Q_0) \geq 1 - \kappa \ln\left(1 + PQ_0\right) \ .$$

Since $Q_0 > 0$ is arbitrary, this implies that $(\mathcal{F}, P)$ is achievable only if

$$\mathcal{F}(Q) \leq \ln(1 + PQ)$$

for all $Q > 0$. The result (2.2) then follows by rearranging.

2.5 Analysis for Specific Profiles

2.5.1 Rational Profile with Order One

Consider the fidelity-quality profile given as

$$\mathcal{F}(Q) = \frac{\alpha Q}{1 + \alpha Q} . \tag{2.4}$$

In what follows, we show that a simple uncoded transmission in fact achieves $P_{\min}(\mathcal{F})$, and therefore is optimal.

Lemma 2. $P_{\min}(\mathcal{F}) = \alpha$ for the profile given in (2.4). Moreover, uncoded transmission with $m = 1$ and

$$U = \sqrt{\frac{\alpha}{n}} \sum_{t=1}^{n} X_t \tag{2.5}$$

achieves the minimum power.

Proof. Clearly, uncoded transmission as described in (2.5) uses a bandwidth factor of $\kappa = \frac{1}{n}$, and expends power α. It can easily be shown that the resultant expected distortion given by

$$D_\kappa(N) = 1 - \kappa \frac{\alpha}{\alpha + N}$$

for any $\kappa \geq 0$ and $N > 0$, translating into

$$F(Q) = -\frac{\mathrm{d}D_\kappa(\frac{1}{Q})}{\mathrm{d}\kappa}\bigg|_{(\kappa=0)} = \frac{\alpha Q}{1+\alpha Q}$$

for all $0 < Q < \infty$. Since this coincides (and hence complies) with $\mathcal{F}(Q)$, we conclude that $P_{\mathrm{min}}(\mathcal{F}) \leq \alpha$.

To show that $P_{\mathrm{min}}(\mathcal{F}) \geq \alpha$, it suffices to use the lower bound (2.2):

$$\begin{aligned}
P_{\mathrm{min}}(\mathcal{F}) &\geq \sup_{Q>0} \frac{\exp(\mathcal{F}(Q)) - 1}{Q} \\
&= \sup_{Q>0} \frac{\exp\left(\frac{\alpha Q}{1+\alpha Q}\right) - 1}{Q} \\
&\geq \lim_{Q\to 0} \frac{\exp\left(\frac{\alpha Q}{1+\alpha Q}\right) - 1}{Q} \\
&= \lim_{Q\to 0} \frac{\alpha \exp\left(\frac{\alpha Q}{1+\alpha Q}\right)}{(1+\alpha Q)^2} \\
&= \alpha \, .
\end{aligned}$$

Lemma 2 may not be surprising as the profile $\mathcal{F}(Q)$ in (2.4) is "tailored" to the performance of uncoded transmission. Nevertheless, just as in the energy-distortion context in [2], it is a somewhat surprising example where uncoded transmission is optimal in any context other than *matched bandwidth* scenarios.

2.5.2 Rational Profile with Order Two

In this section, we consider the fidelity-quality profile given as

$$\mathcal{F}(Q) = \frac{\alpha Q^2}{1+\alpha Q^2} \, . \tag{2.6}$$

We begin by lower bounding $P_{min}(\mathcal{F})$. According to (2.2), we can write

$$P_{\min}(\mathcal{F}) \geq \sup_{Q>0} \frac{\exp\left(\frac{\alpha Q^2}{1+\alpha Q^2}\right) - 1}{Q} \,. \tag{2.7}$$

Unfortunately, it is not easy to solve the optimization problem in (2.7) analytically. Therefore, we solve it numerically and plot it as a function of α in Figure 2.2.

Towards developing an achievable scheme for the profile $P_{\min}(\mathcal{F})$, and therefore obtaining an upper bound for it, we first show that it is not possible to achieve $P_{\min}(\mathcal{F})$ using purely uncoded transmission. That is because uncoded transmission with power P, as analyzed in Lemma 2, would achieve a fidelity of $\frac{PQ}{1+PQ}$. Thus, P needs to satisfy

$$\frac{PQ}{1+PQ} \geq \frac{\alpha Q^2}{1+\alpha Q^2} \tag{2.8}$$

for all $Q > 0$, which simplifies to $P \geq \alpha Q$. But this is not possible with a finite P.

To remedy this, we propose a hybrid scheme with one digital and one analog transmission layer. We describe it for any fixed $\kappa = \frac{m}{n}$, but then specialize its performance to $m = 1$ and $n \to \infty$, and thus to $\kappa \to 0$. We divide the available power as $P = P_a + P_1$, where P_a and P_1 are the power levels of the analog and digital layers, respectively. For any (m, n), we treat X^n and U^m as super symbols in our mapping of long source blocks of length nl onto channel words of length ml, where l is large enough to approach the Shannon limits. We quantize X^{nl} using to the super-letter distribution

$$X^n = S_1^n + E_1^n$$

with $S_1^n \perp E_1^n$. Note that this constrains the covariance matrices of E_1^n and X^n such that

$$\mathbf{0} \leq \mathbf{C}_{E_1^n} \leq \mathbf{C}_{X^n}.$$

We assume that $\mathbf{C}_{X^n} = \mathbf{I}$ which gives us the following constraint,

$$\mathbf{0} \leq \mathbf{C}_{E_1^n} \leq \mathbf{I}. \tag{2.9}$$

Now, we use an $m \times n$ matrix $\mathbf{K}$ to transmit X^n using

$$U_a^m = \mathbf{K}X^n$$

such that

$$\frac{1}{m}E[\|U_a^m\|^2] = \frac{1}{m}\mathrm{Tr}(\mathbf{K}\mathbf{K}^T) = P_a$$

Let U_1^{ml} denote the codeword for conveying S_1^{nl} such that

$$\frac{1}{m}E[\|U_1^m\|^2] = P_1.$$

This codeword is superimposed on U_a^{ml} using dirty-paper coding where U_a^{ml} is treated as channel state information (CSI) known at the encoder.

Let $V^{ml} = U_a^{ml} + U_1^{ml} + W_N^{ml}$ be the received vector. We designate a noise threshold N_1 such that if $N \leq N_1$, the digital information (i.e., the quantized block S_1^{nl}) is successfully decoded, and otherwise reconstruction should rely purely on analog information.

Thus, for $N > N_1$, the MMSE estimator is given by

$$\hat{X}^n = \mathbf{A}_1 V^m = \mathbf{A}_1(U_a^m + U_1^m + W_N^m),$$

where

$$
\begin{aligned}
&\mathbf{A}_1 = \mathbf{C}_{X^n V^m} \mathbf{C}_{V^m}^{-1} \\
&\mathbf{C}_{X^n V^m} = \mathbf{K}^T \\
&\mathbf{C}_{V^m}^{-1} = (\mathbf{K}\mathbf{K}^T + (P_1 + N)\mathbf{I})^{-1}.
\end{aligned} \tag{2.10}
$$

The corresponding distortion is given by

$$
\begin{aligned}
D(N) &= \frac{1}{n} \sum_{t=1}^{n} E[(X_t - \hat{X}_t)^2] \\
&= \frac{1}{n} [\mathrm{Tr}(\mathbf{C}_{X^n}) - \mathrm{Tr}(\mathbf{A}_1 \mathbf{K} \mathbf{C}_{X^n})] \\
&= \frac{1}{n} \Big[n - \mathrm{Tr}(\mathbf{K}^T (\mathbf{K}\mathbf{K}^T + (P_1 + N)\mathbf{I})^{-1} \mathbf{K}) \Big].
\end{aligned} \tag{2.11}
$$

For $N \leq N_1$, on the other hand, one can conclude using standard arguments in dirty-paper and Wyner-Ziv coding that to be able to transmit S_1^{nl} successfully to the receiver, we need

$$
\begin{aligned}
&\frac{m}{2n} \log\left(1 + \frac{P_1}{N_1}\right) \\
&\quad \geq \frac{1}{n} I(X^n; S_1^n | V^m) \\
&\quad = \frac{1}{n} [I(X^n; S_1^n) - I(V^m; S_1^n)] \\
&\quad = \frac{1}{n} [h(X^n) - h(E_1^n) - h(V^m) + h(V^m | S_1^n)] \\
&\quad = \frac{1}{2n} \log \frac{\det(\mathbf{C}_X^n) \det\big(\mathbf{K}\mathbf{C}_{E_1}^n \mathbf{K}^T + (P_1 + N_1)\mathbf{I}\big)}{\det\big(\mathbf{C}_{E_1}^n\big) \det(\mathbf{K}\mathbf{C}_X^n \mathbf{K}^T + (P_1 + N_1)\mathbf{I})} \\
&\quad = \frac{1}{2n} \log \frac{\det\big(\mathbf{K}\mathbf{C}_{E_1}^n \mathbf{K}^T + (P_1 + N_1)\mathbf{I}\big)}{\det\big(\mathbf{C}_{E_1}^n\big) \det(\mathbf{K}\mathbf{K}^T + (P_1 + N_1)\mathbf{I})}.
\end{aligned} \tag{2.12}
$$

The resultant MMSE estimator is given by

$$\hat{E}_1^n = \mathbf{A}_2 \tilde{V}^m = \mathbf{A}_2(U_1^m + \mathbf{K}E_1^n + W_N^m),$$

where

$$\begin{aligned} \mathbf{A}_2 &= \mathbf{C}_{E_1^n \tilde{V}^m} \mathbf{C}_{\tilde{V}^m}^{-1} \\ \mathbf{C}_{E_1^n \tilde{V}^m} &= \mathbf{C}_{E_1^n} \mathbf{K}^T \\ \mathbf{C}_{V^m}^{-1} &= (\mathbf{K}\mathbf{C}_{E_1^n}\mathbf{K}^T + (P_1 + N)\mathbf{I})^{-1}. \end{aligned} \tag{2.13}$$

The corresponding distortion is given by

$$D(N) \tag{2.14}$$

$$\begin{aligned} &= \frac{1}{n}\sum_{t=1}^{n} E[(E_{1t} - \hat{E}_{1t})^2] \\ &= \frac{1}{n}[\mathrm{Tr}(\mathbf{C}_{E_1^n}) - \mathrm{Tr}(\mathbf{A}_2\mathbf{K}\mathbf{C}_{E_1{}^n})] \\ &= \frac{1}{n}\Big[\mathrm{Tr}(\mathbf{C}_{E_1^n}) \\ &\quad -\mathrm{Tr}(\mathbf{C}_{E_1^n}\mathbf{K}^T(\mathbf{K}\mathbf{C}_{E_1^n}\mathbf{K}^T + (P_1 + N)\mathbf{I})^{-1}\mathbf{K}\mathbf{C}_{E_1^n})\Big]. \end{aligned} \tag{2.15}$$

The scheme described up until this point is general enough to be used for any bandwidth factor κ (in fact, even for bandwidth expansion). The next theorem states achievable fidelity levels as a function of quality Q when $\kappa \to 0$.

Theorem 1. The pair $(\mathcal{F}, P)$ is achievable if there exists a triplet (P_a, P_1, Q_1) such that $P = P_a + P_1$ and $F(Q) \geq \mathcal{F}(Q)$ where

$$F(Q) = \begin{cases} \frac{P_a Q}{1+PQ} & 0 < Q < Q_1 \\ \ln(1 + P_1 Q_1) + \frac{P_a Q}{1+PQ} & Q \geq Q_1 \end{cases} .$$

Proof. Let $m = 1$ and therefore $\kappa = \frac{1}{n}$ in the scheme described above. We also choose

$$\mathbf{K} = \begin{bmatrix} k & k & \dots & k \end{bmatrix}$$

and $\mathbf{C}_{E_1}^n = \sigma_1^2 \mathbf{I}_n$ with some $\sigma_1^2 \leq 1$ to be specified below.

The distortion in (2.11) then simplifies to

$$D(Q) = 1 - \kappa \frac{P_a Q}{1 + PQ}$$

for all $0 < Q < Q_1$, resulting in

$$F(Q) = -\frac{dD(Q)}{d\kappa}\bigg|_{\kappa=0} = \frac{P_a Q}{1 + PQ} . \tag{2.16}$$

On the other hand, for $Q \geq Q_1$, (2.15) and (2.12) reduce to

$$D(Q) = \sigma_1^2 \left(1 - \kappa \frac{\sigma_1^2 P_a Q}{1 + (\sigma_1^2 P_a + P_1) Q} \right), \tag{2.17}$$

and

$$1 + P_1 Q_1 \geq \frac{1 + (\sigma_1^2 P_a + P_1) Q_1}{(\sigma_1^2)^n (1 + PQ_1)} \tag{2.18}$$

respectively, where $n = \frac{1}{\kappa}$. It is straightforward to show that $D(Q)$ in (2.17) is increasing in σ_1^2. Thus, the minimum $D(Q)$ is achieved by minimum σ_1^2 satisfying (2.18). For convenience, let $\beta = \sigma_1^2$ and rewrite (2.18) as

$$f_n(\beta) = \beta^n (1 + PQ_1)(1 + P_1 Q_1) - \beta P_a Q_1 - (1 + P_1 Q_1) \geq 0.$$

Since $f_n(0) < 0$ and $f_n(1) > 0$, there has to be a $0 < \beta_n^* < 1$ such that $f_n(\beta_n^*) = 0$. Furthermore, it is easy to show that β_n^* is unique and $\beta_n^* \to 1$ as $n \to \infty$. Therefore, we can approximate β_n^* as the solution to

$$\hat{f}_n(\beta) = \beta^n(1 + PQ_1)(1 + P_1Q_1) - P_aQ_1 - (1 + P_1Q_1) = 0$$

instead. It can therefore be seen that $\beta_n^* \approx a^{\frac{1}{n}}$ for very large n, where

$$a = \frac{1}{1 + P_1Q_1} .$$

In fact, simply choosing $\beta_n = a^{\frac{1}{n}}$ readily satisfies $f_n(\beta_n) \geq 0$ for all n, so we do not need to make this approximation more precise mathematically.

We can then rewrite (2.17) as a function of κ as

$$D_\kappa(Q) = a^\kappa \left(1 - \kappa \frac{a^\kappa P_a Q}{1 + (a^\kappa P_a + P_1)Q}\right)$$

for small κ. Hence,

$$F(Q) = -\left.\frac{dD_\kappa(Q)}{d\kappa}\right|_{\kappa=0} = \ln(1 + P_1Q_1) + \frac{P_aQ}{1 + PQ}. \tag{2.19}$$

Bringing together (2.16) and (2.19) finishes the proof.

We would like to draw the parallel to the achievable scheme presented in [2] in the context of energy-distortion tradeoff. In the achievable scheme we presented, there is a persistent behavior of $\frac{P_aQ}{1+PQ}$, very much like the piecewise linear behavior in the energy-distortion case. This behavior is disrupted by a "jump" of magnitude $\ln(1 + P_1Q_1)$ after $Q = Q_1$, also as in the energy-distortion case.

Figure 2.3 depicts how the proposed achievable scheme can *potentially* comply with the rational profile $\mathcal{F}(Q)$ of order two, with properly chosen (P_a, P_1, Q_1). We next show this can indeed be done. For $0 \le Q < Q_1$, we need

$$\frac{P_a Q}{1 + PQ} \ge \frac{\alpha Q^2}{1 + \alpha Q^2}$$

which can be simplified to

$$P_a \ge \alpha Q + \alpha P_1 Q^2 \ .$$

Since the right-hand side is increasing in Q, it suffices to have

$$P_a \ge \alpha Q_1 + \alpha Q_1^2 P_1 \ . \tag{2.20}$$

On the other hand, for $Q \ge Q_1$, we must satisfy

$$\ln(1 + P_1 Q_1) + \frac{P_a Q}{1 + PQ} \ge \frac{\alpha Q^2}{1 + \alpha Q^2}$$

or rearranging, $P_a \ge g(Q)$ with

$$g(Q) = \frac{\left[\frac{\alpha Q^2}{1+\alpha Q^2} - \ln(1 + P_1 Q_1)\right](1 + P_1 Q)}{Q\left[\frac{1}{1+\alpha Q^2} + \ln(1 + P_1 Q_1)\right]}.$$

Since finding the maximum of $g(Q)$ over $Q \ge Q_1$ analytically seems difficult, we upper bound it as $z(Q) \ge g(Q)$ where

$$\begin{aligned} z(Q) &= \frac{\alpha Q^2 (1 + P_1 Q)}{\alpha Q^3 \ln(1 + P_1 Q_1)} \\ &= \frac{1}{\ln(1 + P_1 Q_1)} \left(\frac{1}{Q} + P_1\right). \end{aligned} \tag{2.21}$$

Since $z(Q)$ is decreasing for $Q \geq Q_1$, $P_a \geq z(Q)$ is the same as

$$P_a \geq \frac{1}{\ln(1+P_1Q_1)}\left(\frac{1}{Q_1}+P_1\right). \tag{2.22}$$

Combining (2.20) and (2.22), we obtain

$$P_a \geq \max\left[\frac{1}{\ln(1+P_1Q_1)}\left(\frac{1}{Q_1}+P_1\right), \alpha Q_1 + \alpha Q_1^2 P_1\right] \tag{2.23}$$

as the minimum possible P_a for any choice of (P_1, Q_1). Since this maximum is finite, it is indeed possible to achieve $\mathcal{F}(Q)$ using the proposed hybrid scheme.

By searching through the space of $P_1, Q_1 > 0$, we obtained the minimum possible $P = P_a + P_1$ as a function of α, which is depicted in Figure 2.2. The gap between the lower and upper bounds appear to saturate to a constant around 13dB.

2.5.3 Extension of the Achievable Scheme to K Layers

The proposed hybrid scheme can be extended into multiple layers of digital information by simply quantizing the quantization error from the previous round and building a coding hierarchy where the kth layer "sees" the channel words of the layers below it as noise and above it as interference that can be canceled by virtue of dirty paper coding. Each layer k will bring about a similar jump in the fidelity at some quality level Q_k. We provide the corresponding $F(Q)$ as follows. For $0 < Q < Q_1$,

$$F(Q) = d_0 \triangleq \frac{P_a Q}{1+PQ}.$$

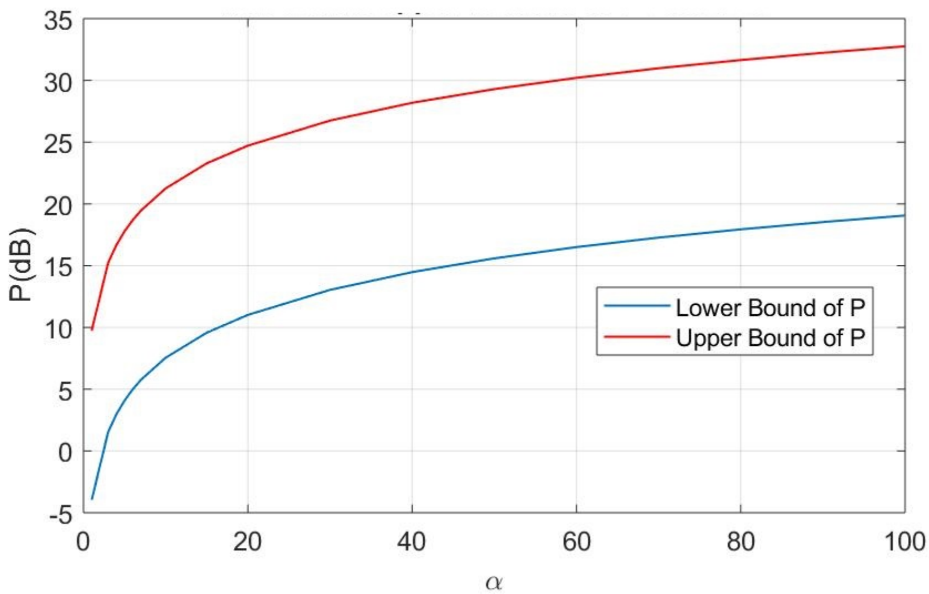

Figure 2.2: Lower and upper bounds of P (in dB) as a function of α.

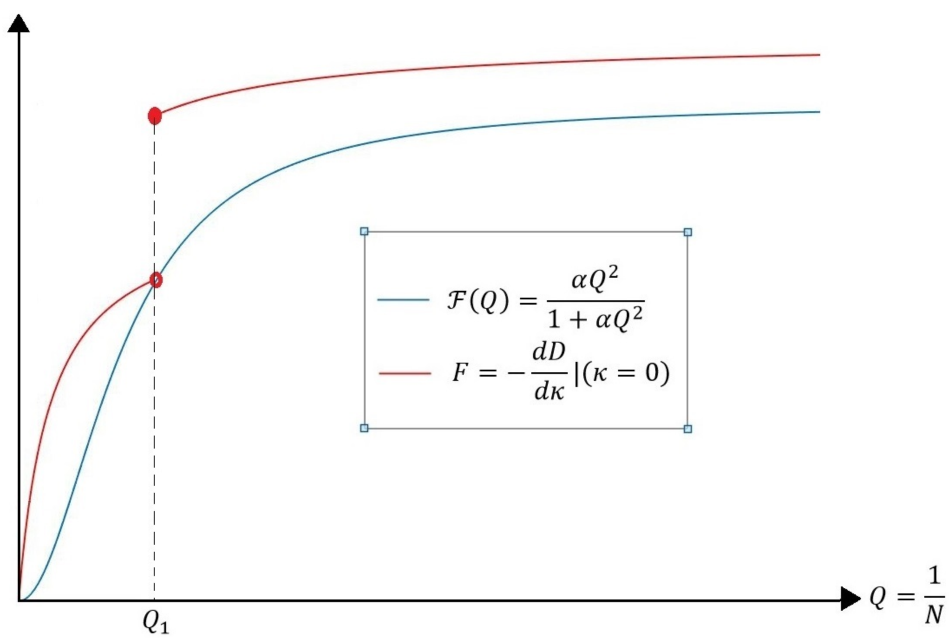

Figure 2.3: The achieved fidelity $F(Q)$ versus $\mathcal{F}(Q) = \frac{\alpha Q^2}{1+\alpha Q^2}$.

For $Q_k \leq Q < Q_{k+1}$, and $1 \leq k < K$,

$$F(Q) = d_k \triangleq d_{k-1} + \ln\left(\frac{1 + (\sum_{i=k}^{K} P_i)Q_k}{1 + (\sum_{i=k+1}^{K} P_i)Q_k}\right).$$

Finally, for $Q \geq Q_K$,

$$F(Q) = d_K \triangleq d_{K-1} + \ln\left(1 + P_K Q_K\right). \tag{2.24}$$

Chapter 3

Robust Gaussian JSCC Under the Near-Infinity Bandwidth Regime

3.1 Introduction

In this chapter, minimum energy required to achieve a distortion-noise profile, i.e., a function indicating the maximum allowed distortion value for each channel noise level, is studied for robust transmission of Gaussian sources over Gaussian channels. In [2], it is shown that for the inversely linear profile, uncoded transmission is optimal. As a negative result, it is also shown that exponential profiles are not achievable with finite energy. For square-law and staircase profiles, we propose coding schemes to upper bound the minimum energy needed. Conversely, utilizing a family of lower bounds originally derived for broadcast channels with power constraints, the minimum required energy is lower bounded for both square-law and staircase profiles, and compared with the corresponding upper bounds.

We consider lossy transmission of a Gaussian source over an additive white Gaussian noise (AWGN) channel, where the channel input constraint is not on power and bandwidth, but on *energy per source symbol*. This regime has drawn much attention recently, [3, 4, 5, 6] to name a few references. In our opinion, part of the appeal is the simplifications to both achievable schemes and converses as the bandwidth expansion factor approaches infinity [5].

When the channel noise variance N is fixed, it is well-known (for example, see [3]) that the minimum distortion that can be achieved with a energy quota of E is given by[1]

$$D = \exp\left(-\frac{E}{N}\right) . \tag{3.1}$$

In this chapter, we instead consider the robust setting where N is unknown at the transmitter (but known at the receiver as usual) and can take on any value in the interval $(0, \infty)$. The system is to be designed to fulfill with a distortion-noise profile $\mathcal{D}(N)$ so that it achieves

$$D \leq \mathcal{D}(N)$$

for all $0 < N < \infty$, while minimizing its energy use. We consider this wide spectrum of noise variances to account for the scenarios that we may know *absolutely nothing* about the noise level. For example, even though the channel may be originally of very

[1] All exponentials and logarithms in this chapter are natural.

high quality ($N \approx 0$), it could be suffering occasional interferences of unknown (and conceivably very high) power ($N \gg 0$). It is also worth noting that even when the range of unknown noise variance is limited to a finite interval $N_{\min} \leq N \leq N_{\max}$, we can still tackle the problem within our framework by setting $\mathcal{D}(N) = 1$ for $N > N_{\max}$ and $\mathcal{D}(N) = \mathcal{D}(N_{\min})$ for $N < N_{\min}$.

In this chapter, we discuss a family of lower bounds to the minimum achievable energy for a general profile, and analyze certain profiles in more detail: inversely linear, exponential, square-law and staircase. Specifically, authors in [2] show that the optimal scheme in the inversely linear case is simple uncoded transmission and exponential profiles are not achievable with finite energy. We establish upper and lower bounds on the minimum energy for the square-law and staircase profiles.

One of the similar universal coding scenarios in the literature is given in [8], where a maximum regret approach for compound channels is proposed. The objective in their scenario is to minimize the maximum ratio of the capacity to the achieved rate at any noise level. Other related works include [10], [11], and [12].

The rest of the chapter is organized as follows. The next section is devoted to preliminaries and notation. In section 3.3, we present the general lower bound for any noise profile. In Section 3.4, we present the results for inversely linear, exponential, square-law and staircase profiles, respectively. The results of this chapter have been published in [7] and [17]

3.2 Preliminaries and Notation

Suppose that X^n is an i.i.d unit-variance Gaussian source which is transmitted over an AWGN channel $V^m = U^m + W^m$, where U^m is the channel input, $W^m \sim (\mathbf{0}, N\mathbf{I_m})$ is the noise, and V^m is the observation at the receiver. We define the bandwidth expansion factor as $\kappa = \frac{m}{n}$, which can be arbitrarily large. The energy per source symbol and the achieved distortion is measured by $\frac{1}{n}\mathbb{E}\{||U^m||^2\}$ and $\frac{1}{n}\mathbb{E}\{||X^n - \hat{X}^n||^2\}$, respectively, where $\hat{X}^n$ is the reconstruction at the receiver.

Definition 3. A pair of distortion-noise profile $\mathcal{D}(N)$ and energy level E is said to be *achievable* if for every $\epsilon > 0$, there exists large enough (m, n), an encoder

$$f^{m,n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m \ ,$$

and decoders

$$g_N^{m,n} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

for every $0 < N\infty$, such that

$$\frac{1}{n}\mathbb{E}\left\{||f^{m,n}(X^n)||^2\right\} \leq E \tag{3.2}$$

and

$$\frac{1}{n}\mathbb{E}\left\{||X^n - g_N^{m,n}(f^{m,n}(X^n) + W_N^m)||^2\right\} \leq \mathcal{D}(N) + \epsilon \tag{3.3}$$

for all N, with W_N^m being the i.i.d. channel noise with variance N.

For given $\mathcal{D}$, the main quantity of interest would be

$$E_{\min}(\mathcal{D}) = \inf\{E : (\mathcal{D}, E) \text{ achievable}\}$$

with the understanding that $E_{\min}(\mathcal{D}) = \infty$ if there is no finite E for which $(\mathcal{D}, E)$ is achievable.

For two distinct profiles $\mathcal{D}_1$ and $\mathcal{D}_2$, we use the notation $\mathcal{D}_1 \preceq \mathcal{D}_2$ if for all $0 < N < \infty$, $\mathcal{D}_1(N) \leq \mathcal{D}_2(N)$.

Definition 4. A distortion-noise profile $\mathcal{D}$ is called *degenerate*, if there exists another profile $\mathcal{D}^* \preceq \mathcal{D}$ such that

$$E_{\min}(\mathcal{D}^*) = E_{\min}(\mathcal{D}) \ .$$

Clearly, any profile $\mathcal{D}$ which is not monotonically non-decreasing is degenerate, as decreasing noise levels can only improve the distortion performance. Another simple example for degeneracy is when $\lim_{N\to\infty} \mathcal{D}(N) > 1$, as one can always ignore the channel output and estimate $\hat{X}^n = \mathbf{0}$, resulting in $D = 1$.

For our purposes, it will be much more convenient to use the notation $F = \frac{1}{D}$ and $Q = \frac{1}{N}$, F and Q standing for signal *fidelity* and channel *quality*[2], respectively. For any $\mathcal{D}(N)$, we define the corresponding *fidelity-quality profile* as

$$\mathcal{F}(Q) = \frac{1}{\mathcal{D}(\frac{1}{Q})}$$

and state that $(\mathcal{F}, E)$ is achievable if and only if $(\mathcal{D}, E)$ is achievable according to Definition 3. $E_{\min}(\mathcal{F})$ and the concept of degeneracy is similarly defined.

[2]We cannot use the usual channel SNR as a quality measure since for any finite energy E, the expended power per channel symbol $\frac{E}{\kappa}$ approaches 0 as $\kappa \to \infty$.

3.3 A Family of Lower Bounds on $E_{\min}(\mathcal{D})$

In this section, we review a family of lower bounds on $E_{\min}(\mathcal{D})$ from [2]. An immediate lower bound on $E_{\min}(\mathcal{D})$ follows from (3.1). Since for any fixed N_0 and D_0 the expended energy cannot be lower than $N_0 \log \frac{1}{D_0}$, we obtain a first-order lower bound given by

$$E_{\min}(\mathcal{D}) \geq \sup_{N>0} N \log \frac{1}{\mathcal{D}(N)} \tag{3.4}$$

or equivalently by

$$E_{\min}(\mathcal{F}) \geq \sup_{Q>0} \frac{\log \mathcal{F}(Q)}{Q} . \tag{3.5}$$

Authors in [2] utilize the connection between this problem and lossy transmission of Gaussian sources over Gaussian broadcast channels where the bandwidth expansion factor κ is fixed and the power per channel symbol is limited. More specifically, they employ the converse result by Tian *et al.* [20], which is a generalization of the 2-receiver outer bound shown by Reznic *et al.* [19] to K receivers. The result is a Kth-order lower bound presented in the lemma below.

Lemma 3. For any $K \geq 1$, $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_{K-1} \geq \tau_K = 0$, and $N_1 \geq N_2 \geq \ldots \geq N_K \geq N_{K+1} = 0$,

$$E_{\min}(\mathcal{D}) \geq \sum_{k=1}^{K} \Delta N_k \log \frac{(1+\tau_k) \prod_{j=2}^{k} (\mathcal{D}(N_j) + \tau_{j-1})}{\prod_{j=1}^{k} (\mathcal{D}(N_j) + \tau_j)} \tag{3.6}$$

where $\Delta N_k = N_k - N_{k+1}$. [3]

[3] It should be understood that $\prod_{j=2}^{k} a_j = 1$ for $k = 1$, for any $\{a_j\}$.

Proof. For finite κ and power constraint

$$\frac{1}{m}\mathbb{E}\left\{||U^m||^2\right\} \leq P \;, \tag{3.7}$$

Tian *et al.* [20, Theorem 2] showed that for fixed $N_1 \geq N_2 \geq \ldots \geq N_K \geq N_{K+1} = 0$, achievable $(D_1, D_2, \ldots, D_K)$ in the aforementioned broadcast scenario must satisfy

$$\sum_{k=1}^{K} \Delta N_k \left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right]^{\frac{1}{\kappa}} \leq P + N_1$$

for any $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_{K-1} \geq \tau_K = 0$. Since (3.7) is the same as (3.2) with $P = \frac{E}{\kappa}$, this implies

$$\sum_{k=1}^{K} \Delta N_k \left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right]^{\frac{1}{\kappa}} \leq \frac{E}{\kappa} + N_1$$

for any $\kappa > 0$.

Noting that we are only interested in the case $\kappa \to \infty$, we observe[4]

$$\begin{aligned}
&E_{\min} \\
&\geq \lim_{\kappa\to\infty} \kappa\left(\sum_{k=1}^{K} \Delta N_k \left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right]^{\frac{1}{\kappa}} - N_1\right) \\
&\overset{(a)}{=} \lim_{\rho\to 0} \frac{1}{\rho}\left(\sum_{k=1}^{K} \Delta N_k \left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right]^{\rho} - N_1\right) \\
&\overset{(b)}{=} \lim_{\rho\to 0} \sum_{k=1}^{K} \Delta N_k \left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right]^{\rho} \cdot \log\left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right] \\
&= \sum_{k=1}^{K} \Delta N_k \log\left[\frac{(1+\tau_k)\prod_{j=2}^{k}(D_j+\tau_{j-1})}{\prod_{j=1}^{k}(D_j+\tau_j)}\right]
\end{aligned}$$

where $\rho = \frac{1}{\kappa}$ in (a) and (b) follows from L'Hôpital's rule. Setting $D_k = \mathcal{D}(N_k)$ then completes the proof.

[4] One can always choose to ignore some of the available bandwidth, so the minimum energy is a non-increasing function of κ.

Remark 1. The first-order lower bound (3.4) can easily be obtained from (3.6) by setting $K = 1$, $\tau_1 = 0$, and $N_2 = 0$.

Remark 2. Equation (3.6) can alternatively be stated as

$$E_{\min}(\mathcal{D}) \quad \geq \quad N_1 \log \frac{1+\tau_1}{\mathcal{D}(N_1)+\tau_1} + \sum_{k=2}^{K} N_k \log \frac{(1+\tau_k)\,(\mathcal{D}(N_k)+\tau_{k-1})}{(1+\tau_{k-1})\,(\mathcal{D}(N_k)+\tau_k)} \, . \tag{3.8}$$

3.4 Analysis for Specific Profiles

In this section, we discuss four different noise profiles, namely linear, exponential, square-law, and stair-case. We find upper and lower bounds for them.

3.4.1 Linear Fidelity-Quality Profiles

Consider the fidelity-quality profile given as

$$\mathcal{F}(Q) = 1 + \alpha Q \, . \tag{3.9}$$

In what follows, authors in [2] show that simple uncoded transmission in fact achieves $E_{\min}(\mathcal{F})$, and therefore is optimal.

Lemma 4. $E_{\min}(\mathcal{F}) = \alpha$ for the linear profile given in (3.9). Moreover, uncoded transmission

$$U_t = \begin{cases} \sqrt{\alpha} X_t & 1 \leq t \leq n \\ 0 & t > n \end{cases}$$

achieves the minimum energy.

Proof. Clearly, uncoded transmission expends energy α, and is well-known to achieve the expected distortion given by

$$D = \frac{1}{1 + \frac{\alpha}{N}}$$

for all $0 < N < \infty$, translating into

$$F = 1 + \alpha Q$$

for all $0 < Q < \infty$. Since this coincides (and hence complies) with $\mathcal{F}(Q)$, it is concluded that $E_{\min}(\mathcal{F}) \leq \alpha$.

To show that $E_{\min}(\mathcal{F}) \geq \alpha$, it suffices to use the first-order lower bound (3.5):

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \sup_{Q>0} \frac{\log(1+\alpha Q)}{Q} \\ &= \lim_{Q\to 0} \frac{\log(1+\alpha Q)}{Q} \\ &= \lim_{Q\to 0} \frac{\alpha}{1+\alpha Q} \\ &= \alpha \, . \end{aligned}$$

Lemma 4 may not be surprising as the profile $\mathcal{F}(Q)$ in (3.9) is "tailored" to the performance of uncoded transmission. Nevertheless, it constitutes the first example where uncoded transmission is optimal in any context other than *matched bandwidth* scenarios, to the best of our knowledge. However, the most powerful use of Lemma 4 is actually in the following result, which shows that *any* concave fidelity-quality profile $\mathcal{F}$ other than $\mathcal{F}(Q) = 1 + \alpha Q$ is *degenerate* and uncoded transmission always achieves $E_{\min}(\mathcal{F})$.

Lemma 5. Let $\mathcal{F}(Q)$ be an arbitrary concave function of Q such that

$$\lim_{Q \to 0} \mathcal{F}(Q) = 1$$

and

$$\lim_{Q \to 0} \mathcal{F}'(Q) = \alpha$$

for some $\alpha > 0$. Then $E_{\min}(\mathcal{F}) = \alpha$, achieved by uncoded transmission.

Proof. Since $\mathcal{F}$ is concave,

$$\mathcal{F}(Q) \leq \mathcal{F}^*(Q) = 1 + \alpha Q$$

implying using Lemma 4 that

$$E_{\min}(\mathcal{F}) \leq E_{\min}(\mathcal{F}^*) = \alpha \ .$$

The fact that $E_{\min}(\mathcal{F}) \geq \alpha$ follows exactly as in the proof of Lemma 4:

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \lim_{Q \to 0} \frac{\log \mathcal{F}(Q)}{Q} \\ &= \lim_{Q \to 0} \frac{\mathcal{F}'(Q)}{\mathcal{F}(Q)} \\ &= \alpha \ . \end{aligned}$$

Remark 3. Because $\mathcal{F} \preceq \mathcal{F}^*$ while $E_{\min}(\mathcal{F}) = E_{\min}(\mathcal{F}^*)$, $\mathcal{F}$ becomes a degenerate profile.

3.4.2 Exponential Fidelity-Quality Profiles

Consider the fidelity-quality profile

$$\mathcal{F}(Q) = \exp(\alpha Q)$$

for some $\alpha > 0$. It is clear using the first-order lower bound (3.5) that

$$E_{\min}(\mathcal{F}) \geq \alpha \ .$$

In fact, α coincides with the minimum possible energy when the channel quality Q is known beforehand, given in (3.1). It would then be naive to expect $E_{\min}(\mathcal{F}) = \alpha$. What one might hope for instead is perhaps some finite function of α as the minimum energy. The following theorem from [2] shows that it is in fact impossible to achieve $\mathcal{F}(Q)$ with finite energy.

Theorem 2. For the exponential profile $\mathcal{F}(Q) = \exp(\alpha Q)$,

$$E_{\min}(\mathcal{F}) = \infty \ .$$

Proof. First, observe that the corresponding distortion-noise profile is given as

$$\mathcal{D}(N) \ = \ \frac{1}{\mathcal{F}(\frac{1}{N})} = \exp\left(-\frac{\alpha}{N}\right) \ .$$

Then, invoking Lemma 3 with the parameters

$$N_k = \frac{1}{k}$$

for $k = 1, \ldots, K$ and

$$\tau_k = \mathcal{D}(N_k) = \exp(-\alpha k)$$

for $k = 1, \ldots, K-1$, and $\tau_K = 0$, we obtain

$$\begin{aligned} E_{\min}(\mathcal{D}) \ &\geq \ N_1 \log \frac{1+\tau_1}{\mathcal{D}(N_1)+\tau_1} + \sum_{k=2}^{K} N_k \log \frac{(1+\tau_k)\,(\mathcal{D}(N_k)+\tau_{k-1})}{(1+\tau_{k-1})\,(\mathcal{D}(N_k)+\tau_k)} \\ &\geq \ \log \frac{1+\exp(-\alpha)}{2\exp(-\alpha)} + \sum_{k=2}^{K} \frac{1}{k} \log \frac{[1+\exp(-\alpha k)]\,[1+\exp(\alpha)]}{2\,[1+\exp(-\alpha(k-1))]} \ . \end{aligned} \tag{3.10}$$

Now, using inequalities

$$1 + \exp(-\alpha) > 2\exp(-\alpha) \tag{3.11}$$

and

$$\frac{1 + \exp(-\alpha k)}{1 + \exp(-\alpha(k-1))} \geq \frac{1 + \exp(-\alpha)}{2}\ , \tag{3.12}$$

where (3.12) follows from the fact that its left-hand side is increasing in k, (3.10) can be further lower bounded as

$$E_{\min}(\mathcal{D}) \geq \left(\log \frac{[1 + \exp(-\alpha)]\,[1 + \exp(\alpha)]}{4}\right) \sum_{k=2}^{K} \frac{1}{k} \tag{3.13}$$

for any $K \geq 2$. But since

$$[1 + \exp(-\alpha)]\,[1 + \exp(\alpha)] > 4$$

for all $\alpha > 0$, the right-hand side of (3.13) diverges as $K \to \infty$.

3.4.3 Square-Law Fidelity-Quality Profiles

So far, we completely discussed the minimum energy for a linear profile, and showed that it is not possible to achieve an exponential profile with finite energy. What about polynomially increasing $\mathcal{F}(Q)$? While it would be interesting to analyze any profile $\mathcal{F}(Q)$ which is a polynomial of Q, we focus on

$$\mathcal{F}(Q) = 1 + \alpha Q^2$$

for arbitrary $\alpha > 0$, as the first step.

Lower Bound for Minimum Energy

We begin with lower bounding $E_{\min}(\mathcal{F})$ using the first-order lower bound (3.5):

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \sup_{Q>0} \frac{\log \mathcal{F}(Q)}{Q} \\ &= \sup_{Q>0} \frac{\log(1+\alpha Q^2)}{Q} \\ &= \left(\sup_{q>0} \frac{\log(1+q^2)}{q} \right) \sqrt{\alpha} \end{aligned} \tag{3.14}$$

where $q = \sqrt{\alpha}Q$. Solving (3.14) numerically, it can be found that the supremum achieved at $q^* \approx 1.9803$ and thus

$$E_{\min}(\mathcal{F}) \geq 0.8047\sqrt{\alpha}. \tag{3.15}$$

Now, to improve this lower bound, we utilize Lemma 3 for $K = 2$. Specifically, by setting $K = 2$, $\tau_1 = \tau \geq \tau_2 = 0$, and $Q_i = \frac{1}{N_i}$ in Lemma 3, we obtain

$$\begin{aligned} &E_{\min}(\mathcal{F}) \\ &\geq \sup_{Q_2>Q_1>0,\tau>0} \left[\frac{\log\left(1+\frac{\alpha Q_1^2}{1+\tau(1+\alpha Q_1^2)}\right)}{Q_1} + \frac{\log\left(1+\frac{\alpha\tau Q_2^2}{1+\tau}\right)}{Q_2} \right] \\ &\geq \left(\sup_{q_2>q_1>0,\tau>0} \left[\frac{\log\left(1+\frac{q_1^2}{1+\tau(1+q_1^2)}\right)}{q_1} + \frac{\log\left(1+\frac{\tau q_2^2}{1+\tau}\right)}{q_2} \right] \right) \sqrt{\alpha} \end{aligned} \tag{3.16}$$

where $q_1 = \sqrt{\alpha}Q_1$ and $q_2 = \sqrt{\alpha}Q_2$, respectively.

In order to solve (3.16), we use the gradient ascent algorithm. As the initial point, we set $q_1 = 1.9803$ and $\tau = 0$, which together give us the same lower bound (3.15) for any arbitrary choice of q_2. Starting from this initial point (together with the arbitrary choice $q_2 = 3$), the algorithm converged to $q_1^* \approx 1.5721$, $q_2^* \approx 6.1638$ and $\tau^* \approx 0.1151$, and the corresponding lower bound was achieved as

$$E_{\min}(\mathcal{F}) \geq 0.9057\sqrt{\alpha}. \tag{3.17}$$

Upper Bound for Minimum Energy

To upper bound $E_{\min}(\mathcal{F})$, we introduce an infinite-layer coding scheme, in which the source is quantized successively, and the quantization index in each layer is Wyner-Ziv coded and transmitted digitally (using infinite bandwidth at each round). Both the source and each quantization error are also transmitted in an uncoded fashion (using matched bandwidth), serving as the side information the digital coding relies on.[5]

The available energy is divided into (i) A_k, used for uncoded transmission rounds $k = 0, 1, 2, \ldots$, and (ii) B_k, used for digital transmission rounds, $k = 1, 2, 3, \ldots$. The kth quantization index is to be decoded whenever $N \leq N_k$ for some predetermined sequence of noise levels $N_1 \geq N_2 \geq N_3 \geq \ldots$.

The source X^n is successively quantized into source codewords $\hat{S}_k^n$ for $k = 1, 2, 3, \ldots$, where the underlying single-letter characterization satisfies

$$S_k = \hat{S}_{k+1} + S_{k+1} \tag{3.18}$$

with $S_0 = X$ and $\hat{S}_{k+1} \perp S_{k+1}$. Each S_k^n is then sent in an uncoded fashion, i.e., as $\sqrt{A_k} S_k^n$. For any noise variance $0 < N < \infty$, the received signals will then be given

[5] Note that none of these infinitely many rounds of transmission are superposed on or interfere with each other because we are utilizing an infinite available bandwidth to send them separately.

Table 3.1: Utilization of information in the proposed coding scheme

Noise interval	$N > N_1$	$N_1 \geq N > N_2$	$N_2 \geq N > N_3$	$\cdots$
Decoded digital information	$-$	$\hat{S}_1^n$	$\hat{S}_1^n$ $\hat{S}_2^n$	$\cdots$
Effective side information	$\sqrt{A_0}S_0^n + W_{0,N}^n$	$\sqrt{A_0}S_0^n + W_{0,N}^n$ $\sqrt{A_1}S_1^n + W_{1,N}^n$	$\sqrt{A_0}S_0^n + W_{0,N}^n$ $\sqrt{A_1}S_1^n + W_{1,N}^n$ $\sqrt{A_2}S_2^n + W_{2,N}^n$	$\cdots$

by

$$Y_{i,N}^n = \sqrt{A_i}S_i^n + W_{i,N}^n, \tag{3.19}$$

for $i = 0, 1, 2, \ldots$. When $N > N_1$, the X^n will be estimated only by utilizing $Y_{0,N}^n$. On the other hand, when $N_{k+1} < N \leq N_k$, since the first k layers of quantization indices will already be decoded, the estimation can rely on all

$$\tilde{Y}_{i,N}^n = \sqrt{A_i}S_k^n + W_{i,N}^n \tag{3.20}$$

as *effective* side information, as all $\hat{S}_i^n$ for $i = 1, ..., k$ can be subtracted from X^n. The utilization of information in our coding scheme is summarized in Table 3.1.

Now, to be able to decode $\hat{S}_k$ whenever $N \leq N_k$, it suffices to use a binning rate of

$$\begin{aligned}
R_k &= I(S_{k-1}; \hat{S}_k | \tilde{Y}_{0,N_k}, \tilde{Y}_{1,N_k}, ..., \tilde{Y}_{k-1,N_k}) \\
&= I(S_{k-1}; \hat{S}_k) - I(\tilde{Y}_{0,N_k}, \tilde{Y}_{1,N_k}, ..., \tilde{Y}_{k-1,N_k}; \hat{S}_k) \\
&= h(S_{k-1}) - h(S_k) - h(\tilde{Y}_{0,N_k}, \tilde{Y}_{1,N_k}, ..., \tilde{Y}_{k-1,N_k}) \\
&\quad + h(\tilde{Y}_{0,N_k}, \tilde{Y}_{1,N_k}, ..., \tilde{Y}_{k-1,N_k} | \hat{S}_k) \\
&= \frac{1}{2} \log \frac{\sigma^2_{S_{k-1}}}{\sigma^2_{S_k}} - \frac{1}{2} \log \frac{\det \mathbf{\Sigma}_{\mathbf{Y}_k}}{\det \mathbf{\Sigma}_{\mathbf{Y}_k | \hat{S}_k}} \qquad (3.21)
\end{aligned}$$

where $\mathbf{\Sigma}$ denotes covariance matrix, and

$$\mathbf{\Sigma}_{\mathbf{Y}_k} = \mathbf{A}_k \mathbf{\Sigma}_{\mathbf{Z}_k} \mathbf{A}_k^T$$

$$\mathbf{\Sigma}_{\mathbf{Y}_k | \hat{S}_k} = \mathbf{A}_k \mathbf{\Sigma}_{\tilde{\mathbf{Z}}_k} \mathbf{A}_k^T$$

with

$$\mathbf{Z}_k = \begin{bmatrix} S_{k-1} & W_{0,N_k} & W_{1,N_k} & \cdots & W_{k-1,N_k} \end{bmatrix}^T$$

$$\tilde{\mathbf{Z}}_k = \begin{bmatrix} S_k & W_{0,N_k} & W_{1,N_k} & \cdots & W_{k-1,N_k} \end{bmatrix}^T$$

and

$$\mathbf{A}_k = \begin{bmatrix} \mathbf{a}_k & \mathbf{I}_k \end{bmatrix}$$

with

$$\mathbf{a}_k = \begin{bmatrix} \sqrt{A_0} & \sqrt{A_1} & \sqrt{A_2} & \cdots & \sqrt{A_{k-1}} \end{bmatrix}^T.$$

Since the source and channel noise are independent, both $\mathbf{\Sigma}_{\mathbf{Z}_k}$ and $\mathbf{\Sigma}_{\tilde{\mathbf{Z}}_k}$ are diagonal, and that makes the computation of $\mathbf{\Sigma}_{\mathbf{Y}_k}$ and $\mathbf{\Sigma}_{\mathbf{Y}_k|\hat{S}_k}$ easy. Specifically, defining the $k \times k$ matrix

$$\mathbf{G}_k = \mathrm{diag}(1, 0, 0, \ldots, 0) \ ,$$

one can write

$$\mathbf{\Sigma}_{\mathbf{Z}_k} = N_k \mathbf{I}_k + (\sigma^2_{S_{k-1}} - N_k)\mathbf{G}_k \ .$$

and

$$\mathbf{\Sigma}_{\tilde{\mathbf{Z}}_k} = N_k \mathbf{I}_k + (\sigma^2_{S_k} - N_k)\mathbf{G}_k \ .$$

We then have

$$\begin{aligned} \mathbf{\Sigma}_{\mathbf{Y}_k} &= \mathbf{A}_k \Big(N_k \mathbf{I}_k + (\sigma^2_{S_{k-1}} - N_k)\mathbf{G}_k \Big) \mathbf{A}_k^T \\ &= N_k \mathbf{A}_k \mathbf{A}_k^T + (\sigma^2_{S_{k-1}} - N_k) \mathbf{A}_k \mathbf{G}_k \mathbf{A}_k^T \\ &= N_k (\mathbf{a}_k \mathbf{a}_k^T + \mathbf{I}_k) + (\sigma^2_{S_{k-1}} - N_k) \mathbf{a}_k \mathbf{a}_k^T \\ &= N_k \mathbf{I}_k + \sigma^2_{S_{k-1}} \mathbf{a}_k \mathbf{a}_k^T \ . \end{aligned} \tag{3.22}$$

Similarly,

$$\mathbf{\Sigma}_{\mathbf{Y}_k|\hat{S}_k} = N_k \mathbf{I}_k + \sigma^2_{S_k} \mathbf{a}_k \mathbf{a}_k^T. \tag{3.23}$$

By substituting (3.22) and (3.23) in (3.21), we then get

$$R_k = \frac{1}{2} \log \frac{\sigma^2_{S_{k-1}}}{\sigma^2_{S_k}} - \frac{1}{2} \log \frac{\det\Big(N_k \mathbf{I}_k + \sigma^2_{S_{k-1}} \mathbf{a}_k \mathbf{a}_k^T \Big)}{\det\big(N_k \mathbf{I}_k + \sigma^2_{S_k} \mathbf{a}_k \mathbf{a}_k^T \big)} \ . \tag{3.24}$$

Using the Matrix Determinant Lemma [21], which states for arbitrary invertible $\mathbf{M}$ and column vectors $\mathbf{u}$ and $\mathbf{v}$ that

$$\det\left(\mathbf{M} + \mathbf{u}\mathbf{v}^T\right) = \det(\mathbf{M}) \cdot (1 + \mathbf{v}^T \mathbf{M}^{-1} \mathbf{u}).$$

We can write

$$\begin{aligned} R_k &= \frac{1}{2} \log \frac{\sigma^2_{S_{k-1}}}{\sigma^2_{S_k}} - \frac{1}{2} \log \frac{(1 + \frac{\sigma^2_{S_{k-1}}}{N_k} \mathbf{a}_k^T \mathbf{a}_k)}{(1 + \frac{\sigma^2_{S_k}}{N_k} \mathbf{a}_k^T \mathbf{a}_k)} \\ &= \frac{1}{2} \log \frac{\sigma^2_{S_{k-1}} (1 + \frac{\sigma^2_{S_k}}{N_k} \mathbf{a}_k^T \mathbf{a}_k)}{\sigma^2_{S_k} (1 + \frac{\sigma^2_{S_{k-1}}}{N_k} \mathbf{a}_k^T \mathbf{a}_k)} \\ &= \frac{1}{2} \log \frac{\beta_k + Q_k A_{k,\text{total}}}{\beta_{k-1} + Q_k A_{k,\text{total}}} \end{aligned} \tag{3.25}$$

where $\beta_k = \frac{1}{\sigma^2_{S_k}}$, $Q_k = \frac{1}{N_k}$, and

$$A_{k,\text{total}} \triangleq A_0 + A_1 + ... + A_{k-1} \ .$$

For this digital message, we use the channel with infinite bandwidth and energy B_k. Therefore, the rate must not exceed the channel capacity under the noise level N_k, i.e.,

$$\frac{1}{2} \log \frac{\beta_k + Q_k A_{k,\text{total}}}{\beta_{k-1} + Q_k A_{k,\text{total}}} \leq \frac{B_k Q_k}{2} \tag{3.26}$$

or equivalently,

$$\frac{\beta_k + Q_k A_{k,\text{total}}}{\beta_{k-1} + Q_k A_{k,\text{total}}} \leq \exp(B_k Q_k) \ . \tag{3.27}$$

When $N_{k+1} < N \leq N_k$, or equivalently $Q_k \leq Q < Q_{k+1}$, the MMSE estimation boils down to estimating S_k^n using all the available effective side information, that is

$$\tilde{S}_k^n = \sum_{i=0}^{k} C_i \tilde{Y}_{i,N}$$

with appropriate C_i for $i = 0, 1, \ldots, k$. This is standard, and the resultant distortion can be calculated with the help of the Sherman-Morrison-Woodbury identity [21] as

$$\begin{aligned} D &= \sigma_{S_k}^2 - (\sigma_{S_k}^2)^2 \mathbf{a}_{k+1}^T \left(N\mathbf{I}_{k+1} + \sigma_{S_k}^2 \mathbf{a}_{k+1}\mathbf{a}_{k+1}^T \right)^{-1} \mathbf{a}_{k+1} \\ &= \sigma_{S_k}^2 - (\sigma_{S_k}^2)^2 \mathbf{a}_{k+1}^T \left[Q\mathbf{I}_{k+1} - \frac{Q^2 \mathbf{a}_{k+1}\mathbf{a}_{k+1}^T}{\beta_k + Q\mathbf{a}_{k+1}^T \mathbf{a}_{k+1}} \right] \mathbf{a}_{k+1} \\ &= \sigma_{S_k}^2 - (\sigma_{S_k}^2)^2 Q A_{k+1,\text{total}} \left[1 - \frac{Q A_{k+1,\text{total}}}{\beta_k + Q A_{k+1,\text{total}}} \right] \\ &= \sigma_{S_k}^2 - \sigma_{S_k}^2 Q A_{k+1,\text{total}} \left[\frac{1}{\beta_k + Q A_{k+1,\text{total}}} \right] \\ &= \frac{1}{\beta_k + Q A_{k+1,\text{total}}}. \end{aligned} \tag{3.28}$$

Equivalently, the fidelity can be written as

$$F(Q) = \beta_k + Q A_{k+1,\text{total}} \,. \tag{3.29}$$

Therefore, $F(Q)$ is an "inclined" staircase function with changing slope $A_{k+1,\text{total}}$ within each $Q_k \leq Q < Q_{k+1}$. Figure 3.1 depicts this behavior. We are now ready to prove an upper bound on $E_{min}(\mathcal{F})$.

Theorem 3. The minimum required energy for profile $\mathcal{F}(Q) = 1 + \alpha Q^2$ is upper bounded as

$$E_{\min}(\mathcal{F}) \leq \sqrt{\alpha} \sum_{k=1}^{\infty} \left[\frac{c d^{k-1}}{1 + c^2 k^2 - k c^2 \left(\frac{1-d^k}{1-d} \right)} + \frac{1}{kc} \log \left(1 + \frac{c^2 \left(2k + 1 - k d^k - \frac{1-d^{k+1}}{1-d} \right)}{1 + k^2 c^2} \right) \right] \tag{3.30}$$

for any $0 < d < 1$ and $c > 0$.

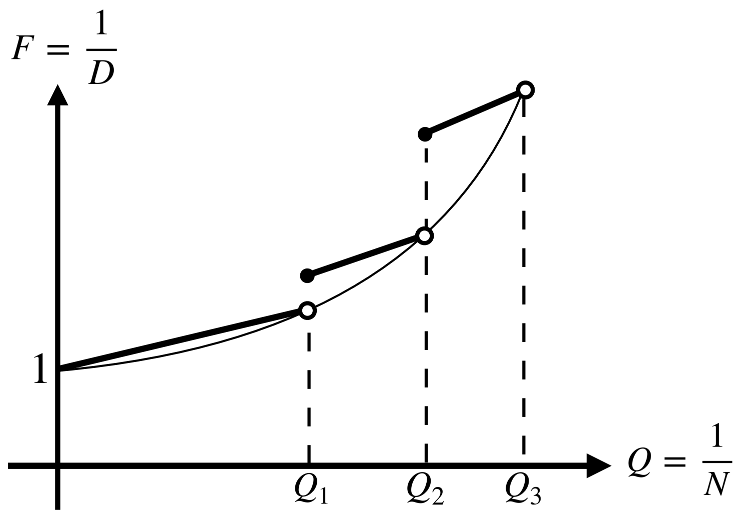

Figure 3.1: The fidelity-quality tradeoff of the proposed scheme and $F(Q) = 1 + \alpha Q^2$ coinciding only at Q_k.

Remark 4. In order to minimize this upper bound, we evaluated (3.30) numerically and perform an exhaustive search over $0 < d < 1$ and $c > 0$ with very high precision. The resultant best values are given by $d^* = 0.99913$ and $c^* = 0.00125$, for which the upper bound yields

$$E_{\min}(\mathcal{F}) \leq 2.3201\sqrt{\alpha}\ .$$

Proof. We will use the scheme described above such that for any $0 = Q_0 < Q_1 < Q_2 < ...$, the energy A_k and the source coding parameters $1 = \beta_0 < \beta_1 < \beta_2 < ...$ will be chosen such that the fidelity-quality tradeoff in (3.29) is always above the profile $F(Q)$, coinciding with it at the jump points Q_k, as shown in Figure 3.1. In other words,

$$A_{k,\text{total}} Q_k + \beta_{k-1} = 1 + \alpha Q_k^2 \tag{3.31}$$

for all $k = 1, 2, \ldots$.

Thus, we obtain

$$\begin{aligned}\beta_{k-1} &= 1 + \alpha Q_k^2 - A_{k,\text{total}} Q_k \\ &= 1 + \alpha Q_k^2 - (A_0 + A_1 + \ldots + A_{k-1}) Q_k. \end{aligned} \tag{3.32}$$

The requirement that β_k is increasing in k leads to the following constraint:

$$\begin{aligned} A_0 &= \alpha Q_1 \\ A_k &< \frac{\alpha(Q_{k+1}^2 - Q_k^2) - A_{k,\text{total}}(Q_{k+1} - Q_k)}{Q_{k+1}}, \end{aligned} \tag{3.33}$$

for all $k \geq 1$.

Lemma 6. For fixed α, the choice $Q_k = k\Delta$, $A_0 = \alpha\Delta$ and $A_k = d^k \alpha \Delta$ for $k \geq 1$ satisfies (3.33) for any $0 < d < 1$ and $\Delta > 0$.

Proof. Substituting A_k and Q_k in (3.33) yields:

$$d^k < \frac{(2k+1) - (\frac{1-d^k}{1-d})}{k+1}. \tag{3.34}$$

We use induction to prove (3.34). For $k = 1$, (3.34) reduces to $d < 1$ which is true. Substituting $k = l$, we get to the following:

$$d^l(l+1) < (2l+1) - (d^{l-1} + \ldots + d + 1) \tag{3.35}$$

We assume (3.35) is true. Now we substitute $k = l+1$ in (3.34) and have the following:

$$d^{l+1}(l+2) < (2l+3) - (d^l + d^{l-1} + \ldots + d + 1). \tag{3.36}$$

In order to complete the proof, we show (3.36) is true as follows. First, we multiply both sides of (3.35) with d and then add d^{l+1} to both sides, yielding

$$d^{l+1}(l+2) < (2l+1)d + d^{l+1} - (d^l + ... + d). \tag{3.37}$$

Now, it suffices to show the right hand side of (3.37) is less than or equal to the right hand side of (3.36), which is the same as

$$d^{l+1} + (2l+1)d \leq 2l+2. \tag{3.38}$$

Since $0 < d < 1$, we have $(2l+1)d < (2l+1)$ and $d^{l+1} < 1$. Thus, (3.38) is valid and the proof is complete.

By substituting $A_k = d^k \alpha \Delta$ and $Q_k = k\Delta$ in (3.32), we get

$$\beta_{k-1} = 1 + \alpha k^2 \Delta^2 - k\alpha\Delta^2 \left(\frac{1-d^k}{1-d} \right)$$

for $k = 1, 2, \ldots$. The total energy expended for transmitting the uncoded information then becomes

$$\begin{aligned} E_{\text{unc}} &= \sum_{k=0}^{\infty} \frac{A_k}{\beta_k} \\ &= \sum_{k=0}^{\infty} \frac{d^k \alpha \Delta}{1 + \alpha(k+1)^2\Delta^2 - (k+1)\alpha\Delta^2 \left(\frac{1-d^{k+1}}{1-d} \right)} \\ &= \sum_{k=1}^{\infty} \frac{d^{k-1} \alpha \Delta}{1 + \alpha k^2 \Delta^2 - k\alpha\Delta^2 \left(\frac{1-d^k}{1-d} \right)}. \end{aligned} \tag{3.39}$$

On the other hand, choosing B_k to satisfy equality in (3.26), the total expended

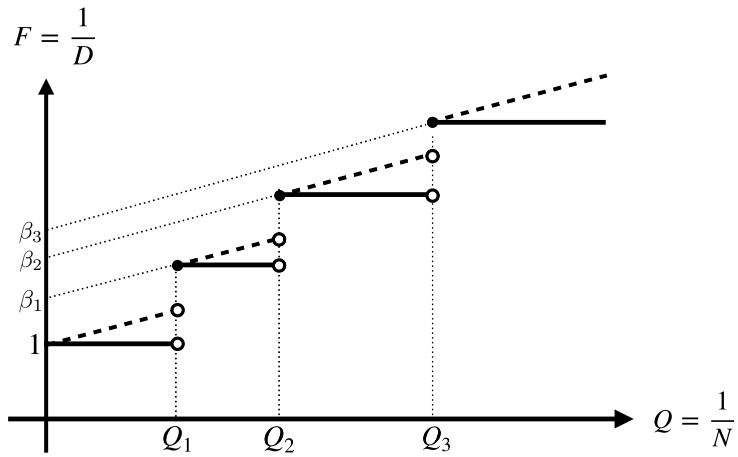

Figure 3.2: The Staircase fidelity-quality profile with $K = 3$ steps (solid lines), and the achieved performance with one layer of uncoded information expending energy A_0 (dashed lines).

digital energy can be written as

$$\begin{aligned} E_{\text{dig}} &= \sum_{k=1}^{\infty} B_k \\ &= \sum_{k=1}^{\infty} \frac{1}{k\Delta} \log\left(1 + \frac{\alpha\Delta^2\left(2k+1-kd^k-\frac{1-d^{k+1}}{1-d}\right)}{1+\alpha k^2\Delta^2}\right). \end{aligned} \tag{3.40}$$

Letting $\Delta = \frac{c}{\sqrt{\alpha}}$ then yields the desired result.

3.4.4 Staircase Fidelity-Quality Profiles

In this section, we analyze the minimum energy needed for K-step staircase profiles characterized by

$$\mathcal{F}(Q) = a_k \qquad Q_k \le Q < Q_{k+1} \tag{3.41}$$

for $0 \le k \le K$, with the understanding that $a_0 = 1$, $Q_0 = 0$, and $Q_{K+1} = \infty$. An example profile is depicted in solid lines in Figure 3.2 for $K = 3$. Note that this scenario can be reduced to the pursuit of minimum energy in a broadcast channel with K receivers, where the kth receiver observes a noise level of $\frac{1}{Q_k}$ and aims for distortion $\frac{1}{a_k}$.

A General Upper bound for $E_{\min}(\mathcal{F})$

The first approach that comes to mind to obtain an upper bound for the minimum energy needed to comply with this profile is using a multi-layered digital coding scheme designed to decode one more layer each time the channel quality Q exceeds a discontinuity point Q_k. As a matter of fact, this digital coding scheme would be a finite-layer special case of the scheme described in Section 3.4.3, whereby all uncoded energy levels are set to zero, i.e., $A_0 = A_1 = \cdots = A_K = 0$. In this case, (3.26) and (3.29) will respectively simplify to

$$\frac{1}{Q_k} \log \frac{\beta_k}{\beta_{k-1}} \leq B_k$$

and

$$F(Q) = \beta_k$$

whenever $Q_k \leq Q < Q_{k+1}$. Clearly, the choice $\beta_k = a_k$ satisfies $F(Q) = \mathcal{F}(Q)$ for all Q, thereby providing an upper bound for the minimum energy needed as

$$E_{\min}(\mathcal{F}) \leq \sum_{k=1}^{K} \frac{1}{Q_k} \log \frac{a_k}{a_{k-1}} \ . \tag{3.42}$$

Now, despite the fact that we meet the profile $\mathcal{F}(Q)$ *exactly* with this scheme, we know from the study of broadcast channels (see [5] for example) that the minimum energy can be further reduced by employing analog information (unlike in the case of linear profiles discussed in Section 3.4.1, where uncoded transmission achieved the profile exactly and was provably optimal).

We therefore employ another special case of the scheme in Section 3.4.3, where $A_0 > 0$ while all the other analog energy levels are still set to zero, i.e, $A_k = 0$ for $k = 1, \ldots, K$. In this case, $A_{k,\text{total}} = A_0$ for $k = 1, \ldots, K$, and therefore (3.26) and (3.29) become

$$\frac{1}{Q_k} \log \frac{\beta_k + Q_k A_0}{\beta_{k-1} + Q_k A_0} \leq B_k \tag{3.43}$$

and

$$F(Q) = \beta_k + A_0 Q$$

for $Q_k \leq Q < Q_{k+1}$. See Figure 3.2 which depicts $F(Q)$ in dashed lines.

To ensure $F(Q) \geq \mathcal{F}(Q)$, it then suffices to set A_0 and β_k such that

$$\beta_k \geq a_k - A_0 Q_k \tag{3.44}$$

for $k = 1, \ldots, K$, while also satisfying

$$1 = \beta_0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_K \, . \tag{3.45}$$

We are now ready to state the main theorem of this section.

Theorem 4. For a staircase profile described in (3.41), the minimum required energy is upper-bounded as

$$\begin{aligned} E_{\min}(\mathcal{F}) &\le E^*_{\mathrm{UB}}(A_0) \\ &\triangleq A_0 + \sum_{k=1}^{K} \frac{1}{Q_k}\left(\log \frac{a_k}{A_0(Q_k - Q_{k-1}) + a_{k-1}}\right)^+ \end{aligned} \tag{3.46}$$

where $x^+ = \max\{x, 0\}$.

Proof. For any fixed A_0 in the allowed range, and β_k statisfying (3.44) and (3.45), it follows from (3.43) that

$$\begin{aligned} E_{\min}(\mathcal{F}) &\le E_{\mathrm{UB}}(\beta_1, \beta_2, \ldots, \beta_K | A_0) \\ &\triangleq A_0 + \sum_{k=1}^{K} \frac{1}{Q_k} \log \frac{\beta_k + Q_k A_0}{\beta_{k-1} + Q_k A_0}\ . \end{aligned} \tag{3.47}$$

Now, differentiating $E_{\mathrm{UB}}(\beta_1, \beta_2, \ldots, \beta_K | A_0)$ with respect to β_k yields

$$\begin{aligned} \frac{\partial E_{\mathrm{UB}}}{\partial \beta_k} &= \frac{1}{Q_k(\beta_k + Q_k A_0)} - \frac{1}{Q_{k+1}(\beta_k + Q_{k+1} A_0)} \\ &= \frac{\beta_k(Q_{k+1} - Q_k) + A_0(Q_{k+1}^2 - Q_k^2)}{Q_k Q_{k+1}(\beta_k + Q_k A_0)(\beta_k + Q_{k+1} A_0)} \\ &> 0 \end{aligned}$$

implying that $E_{\mathrm{UB}}(\beta_1, \beta_2, \ldots, \beta_K | A_0)$ is increasing in β_k. But this implies that it is minimized by the choice

$$\beta_k = \max\{\beta_{k-1}, a_k - A_0 Q_k\} \tag{3.48}$$

for $k = 1, \ldots, K$, due to (3.44) and (3.45). Substituting (3.48) in (3.47) then yields the desired result.

Remark 5. The upper bound $E^*_{\mathrm{UB}}(A_0)$ is a convex function of A_0 and hence is not difficult to optimize for given $Q_1, \ldots, Q_K$ and $a_1, \ldots, a_K$. Essentially, it is continuous and has continuous derivatives everywhere with the exception of K discontinuity points at $\frac{a_k - a_{k-1}}{Q_k - Q_{k-1}}$ for $1 \leq k \leq K$. It is also easy to show that $\left.\frac{dE^*_{\mathrm{UB}}(A_0)}{dA_0}\right|_{A_0=0} < 0$ and therefore it is *always* beneficial to set $A_0 > 0$, i.e., transmit analog information.

In the rest of this discussion, we define

$$E^*_{\mathrm{UB}} \triangleq \min_{A_0>0} E_{\mathrm{UB}}(A_0) \ .$$

A General Lower bound for $E_{\min}(\mathcal{F})$

We will invoke Lemma 3 to obain the following Theorem.

Theorem 5. For a staircase profile described in (3.41), the minimum required energy is lower-bounded as

$$E_{\min}(\mathcal{F}) \geq E^*_{\mathrm{LB}} \triangleq \sum_{k=1}^{K} \frac{1}{Q_k} \log \frac{a_k}{4a_{k-1}} \ . \tag{3.49}$$

Proof. In Lemma 3, we choose $\frac{1}{N_k}$ to coincide with the discontinuity points Q_k in the staircase profile and set $\tau_k = \mathcal{D}(N_k) = \frac{1}{\mathcal{F}(Q_k)} = \frac{1}{a_k}$ for $1 \leq k \leq K-1$ and $\tau_K = 0$

in (3.8) to obtain

$$
\begin{aligned}
E_{\min}(\mathcal{F}) &\geq N_1 \log \frac{1+\mathcal{D}(N_1)}{2\mathcal{D}(N_1)} + \sum_{k=2}^{K-1} N_k \log \frac{[1+\mathcal{D}(N_k)]\,[\mathcal{D}(N_k)+\mathcal{D}(N_{k-1})]}{2[1+\mathcal{D}(N_{k-1})]\mathcal{D}(N_k)} \\
&\quad + N_K \log \frac{\mathcal{D}(N_k)+\mathcal{D}(N_{k-1})}{[1+\mathcal{D}(N_{k-1})]\mathcal{D}(N_k)} \\
&\overset{(a)}{>} N_1 \log \frac{1}{2\mathcal{D}(N_1)} + \sum_{k=2}^{K} N_k \log \frac{\mathcal{D}(N_{k-1})}{4\mathcal{D}(N_k)} \\
&\overset{(b)}{>} \sum_{k=1}^{K} N_k \log \left(\frac{\mathcal{D}(N_{k-1})}{4\mathcal{D}(N_k)} \right) \\
&= \sum_{k=1}^{K} \frac{1}{Q_k} \log \frac{a_k}{4a_{k-1}}
\end{aligned}
$$

where (a) follows from $0 < \mathcal{D}(N_k) < 1$ for all k, and (b) is written with the understanding that $\mathcal{D}(N_0) = \mathcal{D}(\infty) = 1$.

Now, comparing (3.49) and (3.42), we have

$$
E^*_{\text{UB}}(A_0) - E^*_{\text{LB}} \leq \log 4 \sum_{k=1}^{K} \frac{1}{Q_k} \ ,
$$

that is, the gap between the upper and lower bounds is itself upper-bounded by a number that is easy to compute.

Special Scenarios

We finish our analysis of the minimum energy under the staircase profile by looking into some special scenarios.

Logarithmic steps with $K = \infty$ If it is desirable that the reproduction quality to improve by a certain amount (in dB) every time the channel quality also improves by another certain amount (also in dB) indefinitely, we have a scenario where $K = \infty$, $Q_k = \gamma^k$, and $a_k = \lambda^k$ for all $k \geq 1$ with some appropriately chosen $\gamma, \lambda > 1$.

Under this scenario, (3.46) and (3.49) respectively become

$$E^*_{\text{UB}}(A_0) = A_0 + \frac{1}{\gamma}\left(\log\frac{\lambda}{A_0\gamma + 1}\right)^+ + \sum_{k=2}^{\infty}\frac{1}{\gamma^k}\left(\log\frac{\lambda}{A_0(\gamma-1)(\frac{\gamma}{\lambda})^{k-1}+1}\right)^+ \tag{3.50}$$

and

$$E^*_{\text{LB}} = \sum_{k=1}^{\infty}\frac{1}{\gamma^k}\log\frac{\lambda^k}{4\lambda^{k-1}} = \frac{1}{\gamma-1}\log\frac{\lambda}{4}\ . \tag{3.51}$$

An immediate observation is that $E^*_{\text{LB}} < 0$ for $\lambda < 4$, and hence is useless as a lower bound. We can actually derive a tighter lower bound for some (λ, γ) pairs by starting as in the first step in the proof of Theorem 5, i.e., $\tau_k = \mathcal{D}(N_k) = \frac{1}{a_k} = \lambda^{-k}$, but then proceeding differently afterwards:

$$\begin{aligned}
&E_{\min}(\mathcal{F}) \\
&\geq \frac{1}{\gamma}\log\frac{1+\lambda^{-1}}{2\lambda^{-1}} + \sum_{k=2}^{\infty}\frac{1}{\gamma^k}\log\frac{(1+\lambda^{-k})(\lambda^{-k}+\lambda^{-k+1})}{2(1+\lambda^{-k+1})\lambda^{-k}} \\
&= \frac{1}{\gamma}\log\frac{1+\lambda}{2} + \sum_{k=2}^{\infty}\frac{1}{\gamma^k}\log\frac{(1+\lambda^k)(1+\lambda)}{2(\lambda+\lambda^k)} \\
&= \frac{1}{\gamma-1}\log\frac{1+\lambda}{2} + \sum_{k=2}^{\infty}\frac{1}{\gamma^k}\log\frac{1+\lambda^k}{\lambda+\lambda^k} \\
&\geq \frac{1}{\gamma-1}\log\frac{1+\lambda}{2} - \sum_{k=2}^{\infty}\frac{1}{\gamma^k}\log\lambda \\
&= \frac{1}{\gamma-1}\log\frac{1+\lambda}{2\lambda^{\frac{1}{\gamma}}} \triangleq \bar{E}_{\text{LB}}\ .
\end{aligned} \tag{3.52}$$

It is not difficult to see that for a fixed γ, $\bar{E}_{\text{LB}} > E^*_{\text{LB}}$ when λ is close to 1, and $\bar{E}_{\text{LB}} < E^*_{\text{LB}}$ when λ is large. Figure 3.3 shows a comparison of E^*_{UB}, E^*_{LB}, and $\bar{E}_{\text{LB}}$

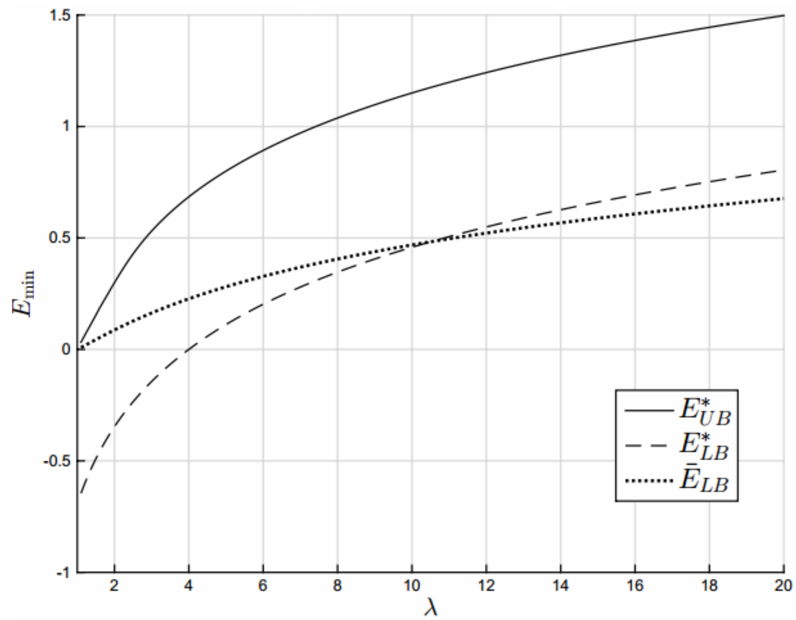

Figure 3.3: Comparison of bounds as a function of λ when $\gamma = 3$ is fixed.

as a function of λ for fixed γ.

Staircase profile with $K = 2$ When there are only two steps in the profile, we can find analytically compute (3.46) and find a tighter bound than (3.49).

Starting first with the upper bound, it can actually be shown that it suffices to limit ourselves to

$$A_0 \leq A_{0,\max} \triangleq \min\left\{\frac{a_1 - 1}{Q_1}, \frac{a_2 - a_1}{Q_2 - Q_1}\right\}$$

in which case (3.46) becomes

$$E_{\mathrm{UB}}(A_0) \quad = \quad A_0 + \frac{1}{Q_1}\log\frac{a_1}{A_0Q_1 + 1} + \frac{1}{Q_2}\log\frac{a_2}{A_0(Q_2 - Q_1) + a_1}\ .$$

Since $E_{\mathrm{UB}}(A_0)$ is convex everywhere, and decreasing at $A_0 = 0$, it will assume its minimum either at A_0 satisfying $\frac{dE_{\text{UB}}}{dA_0} = 0$ or at $A_0 = \min\left\{\frac{a_1-1}{Q_1}, \frac{a_2-a_1}{Q_2-Q_1}\right\}$, whichever is smaller. It is not difficult to show that $\frac{dE_{\text{UB}}}{dA_0} = 0$ is the same as the quadratic equation

$$A_0^2 + A_0\left[\frac{a_1}{Q_2-Q_1} - \frac{1}{Q_2}\right] - \frac{1}{Q_1Q_2} = 0 \,.$$

We therefore have

$$E_{\text{UB}}^* = E_{\text{UB}}(\min\{A_{0,\max}, A_0^*\}) \tag{3.53}$$

where

$$A_0^* = \frac{-\left[\frac{a_1}{Q_2-Q_1} - \frac{1}{Q_2}\right] + \sqrt{\left[\frac{a_1}{Q_2-Q_1} - \frac{1}{Q_2}\right]^2 + \frac{4}{Q_1Q_2}}}{2} \,.$$

As for the lower bound, (3.49) becomes

$$E_{\text{LB}}^* = \frac{1}{Q_1}\log\frac{a_1}{4} + \frac{1}{Q_2}\log\frac{a_2}{4a_1} \,. \tag{3.54}$$

We improve this lower bound by optimizing the only free parameter τ_1 in Lemma 3:

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \hat{E}_{\text{LB}} \\ &\triangleq \max_{\tau_1>0}\left[\frac{1}{Q_1}\log\frac{1+\tau_1}{\frac{1}{a_1}+\tau_1} + \frac{1}{Q_2}\log\frac{(\frac{1}{a_2}+\tau_1)}{(1+\tau_1)(\frac{1}{a_2})}\right] \,. \end{aligned} \tag{3.55}$$

Now, it is not difficult to show that the objective function in the above maximization problem is monotonically decreasing if $\frac{Q_1}{Q_2} \leq \frac{a_1-1}{a_2-1}$, monotonically increasing if $\frac{Q_1}{Q_2} \geq \frac{a_2(a_1-1)}{a_1(a_2-1)}$, and is unimodal otherwise. After some algebra, we then obtain the optimal

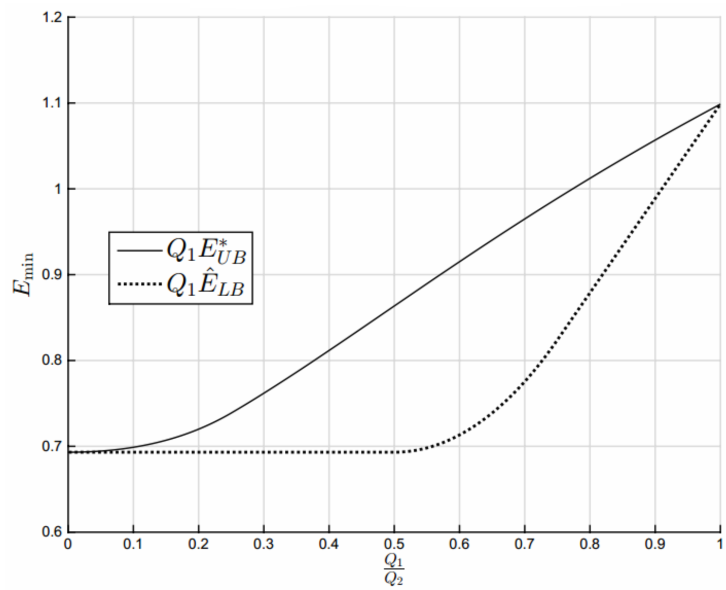

Figure 3.4: Comparison of bounds as a function of $\frac{Q_1}{Q_2}$ when $a_1 = 2$ and $a_2 = 3$ are fixed.

value $\tau_1 = \tau_1^*$ as

$$\tau_1^* = \begin{cases} \frac{\frac{Q_1}{Q_2}(a_2-1)-(a_1-1)}{a_2(a_1-1)-\frac{Q_1}{Q_2}a_1(a_2-1)} & \text{if } \frac{a_1-1}{a_2-1} < \frac{Q_1}{Q_2} < \frac{a_2(a_1-1)}{a_1(a_2-1)} \\ \infty & \text{if } \frac{Q_1}{Q_2} \geq \frac{a_2(a_1-1)}{a_1(a_2-1)} \\ 0 & \text{if } \frac{Q_1}{Q_2} \leq \frac{a_1-1}{a_2-1} \end{cases} \tag{3.56}$$

Substituting (3.56) in (3.55) then yields

$$\hat{E}_{\text{LB}} = \frac{1}{Q_1} \cdot \begin{cases} \log \frac{a_1(a_2-1)\left(1-\frac{Q_1}{Q_2}\right)}{a_2-a_1} - \frac{Q_1}{Q_2} \log \frac{(a_1-1)\left(\frac{Q_2}{Q_1}-1\right)}{a_2-a_1} & \text{if } \frac{a_1-1}{a_2-1} < \frac{Q_1}{Q_2} < \frac{a_2(a_1-1)}{a_1(a_2-1)} \\ \frac{Q_1}{Q_2} \log a_2 & \text{if } \frac{Q_1}{Q_2} \geq \frac{a_2(a_1-1)}{a_1(a_2-1)} \\ \log a_1 & \text{if } \frac{Q_1}{Q_2} \leq \frac{a_1-1}{a_2-1} \end{cases} . \tag{3.57}$$

It can be seen from (3.53) and (3.57) that both bounds, when multiplied by Q_1, can be expressed as a function of $\frac{Q_1}{Q_2}$. In Figure 3.4, we compare $Q_1 E^*_{\text{UB}}$ and $Q_1 \hat{E}_{\text{LB}}$ when $a_1 = 2$ and $a_2 = 3$ as $\frac{Q_1}{Q_2}$ varies between 0 and 1.

Chapter 4

Robust Gaussian JSCC Under the Near-Infinity Bandwidth Regime with Side Information at the Receiver

4.1 Introduction

In this chapter, minimum energy required to achieve a distortion-noise profile, i.e., a function indicating the maximum allowed distortion value for each channel noise level, is studied for robust transmission of Gaussian sources over Gaussian channels when there is a side information about the source at the decoder, where the quality of the side information is also unknown. In this case, the quality parameter would be two-dimensional. For square-law and staircase profiles, we proposed coding schemes

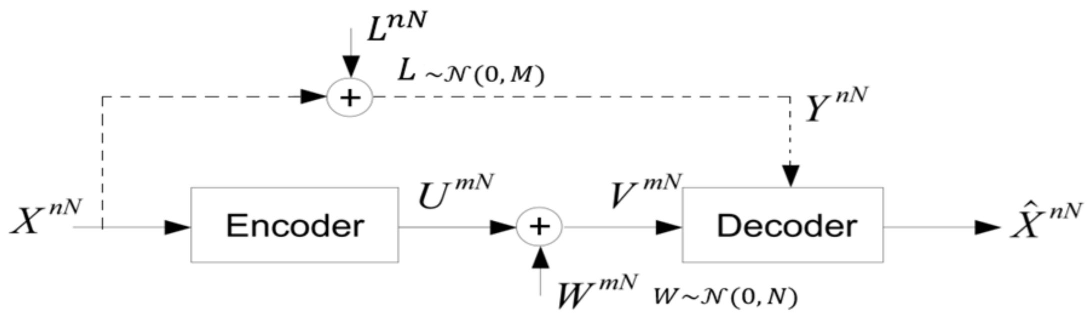

Figure 4.1: Lossy transmission of a Gaussian source over an additive white Gaussian noise (AWGN) channel with side information at the decoder

to upper bound the minimum energy needed. Conversely, a general family of lower bounds is derived for the minimum required energy which works for any profiles including square-law and staircase .

We consider lossy transmission of a Gaussian source over an additive white Gaussian noise (AWGN) channel with side information at the desoder as shown in Figure 4.1, where the channel input constraint is on *energy per source symbol*. Similarly to chapter 3, part of the appeal is the simplifications to both achievable schemes and converses as the bandwidth expansion factor approaches infinity.

When the channel noise variance N and the side information noise variance M are fixed, it is well-known (for example, see [3]) that the minimum distortion that can be achieved with a energy quota of E is given by

$$D = \frac{M}{M+1} \exp\left(-\frac{E}{N}\right) . \tag{4.1}$$

In this chapter, we instead consider the robust setting where N and M are unknown at the transmitter (but known at the receiver as usual) and can take on any value in the interval $(0, \infty)$. Please note that the fundamental difference between this chapter and previous chapters is that here since both N and M are varying in the interval $(0, \infty)$, we deal with two dimensional problem. If we set $M \to \infty$, then the problem could be similar to chapter 3. The system is to be designed to fulfill with a distortion-noise profile $\mathcal{D}(N, M)$ so that it achieves

$$D \leq \mathcal{D}(N, M)$$

for all $0 < N < \infty$ and $0 < M < \infty$, while minimizing its energy use. We consider this wide spectrum of variances to account for the scenarios that we may know *absolutely nothing* about the noise level.

In this chapter, we discuss a family of lower bounds to the minimum achievable energy for a general profile, and analyze certain profiles in more detail: inversely linear and staircase. We establish upper and lower bounds on the minimum energy for inversely linear and staircase profiles.

Some related works include [8], [10], [11], and [12]. However, none of the previous works consider the minimum energy analysis with side information at the decoder.

The rest of the chapter is organized as follows. The next section is devoted to preliminaries and notation. In section 4.3, we present the general lower bound for any noise profile. In Section 4.4, we present the results for inversely linear and staircase profiles, respectively.

4.2 Preliminaries and Notation

Suppose that X^n is an i.i.d unit-variance Gaussian source which is transmitted over an AWGN channel $V^m = U^m + W^m$, where U^m is the channel input, $W^m \sim (\mathbf{0}, N\mathbf{I_m})$ is the noise, and V^m is the observation at the receiver. Assume that there is side information Y^n at the decoder where $Y^{nN} = X^{nN} + L^{nN}$ and $L^n \sim (\mathbf{0}, M\mathbf{I_n})$.

We define the bandwidth expansion factor as $\kappa = \frac{m}{n}$, which can be arbitrarily large. The energy per source symbol and the achieved distortion is measured by $\frac{1}{n}\mathbb{E}\{||U^m||^2\}$ and $\frac{1}{n}\mathbb{E}\{||X^n - \hat{X}^n||^2\}$, respectively, where $\hat{X}^n$ is the reconstruction at the receiver.

Definition 5. A pair of distortion-noise profile $\mathcal{D}(N, M)$ and energy level E is said to be *achievable* if for every $\epsilon > 0$, there exists large enough (m, n), an encoder

$$f^{m,n} : \mathbb{R}^n \longrightarrow \mathbb{R}^m ,$$

and decoders

$$g^{m,n}_{N,M} : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

for every $0 < N < \infty$, such that

$$\frac{1}{n}\mathbb{E}\left\{||f^{m,n}(X^n)||^2\right\} \leq E \tag{4.2}$$

and

$$\frac{1}{n}\mathbb{E}\left\{||X^n - g^{m,n}_{N,M}(f^{m,n}(X^n) + W^m_N)||^2\right\} \leq \mathcal{D}(N, M) + \epsilon \tag{4.3}$$

for all N, with W^m_N being the i.i.d. channel noise with variance N.

For given $\mathcal{D}$, the main quantity of interest would be

$$E_{\min}(\mathcal{D}) = \inf\{E : (\mathcal{D}, E) \text{ achievable}\}$$

with the understanding that $E_{\min}(\mathcal{D}) = \infty$ if there is no finite E for which $(\mathcal{D}, E)$ is achievable.

Definition 6. A distortion-noise profile $\mathcal{D}$ is called *degenerate*, if there exists another profile $\mathcal{D}^* \preceq \mathcal{D}$ such that

$$E_{\min}(\mathcal{D}^*) = E_{\min}(\mathcal{D}) \ .$$

For our purposes, it will be much more convenient to use the notation $F = \frac{1}{D}$, $Q = \frac{1}{N}$, and $s = \frac{1}{M}$ where F, Q and s standing for signal *fidelity*, channel *quality* and side information *quality* , respectively. For any $\mathcal{D}(N, M)$, we define the corresponding *fidelity-quality profile* as

$$\mathcal{F}(Q, s) = \frac{1}{\mathcal{D}(N, M)}$$

and state that $(\mathcal{F}, E)$ is achievable if and only if $(\mathcal{D}, E)$ is achievable according to Definition 5. $E_{\min}(\mathcal{F})$ and the concept of degeneracy is similarly defined.

4.3 A Family of Lower Bounds on $E_{\min}(\mathcal{D})$

In this section, we find a family of lower bounds on $E_{\min}(\mathcal{D})$. An immediate lower bound on $E_{\min}(\mathcal{D})$ follows from (4.1). Since for any fixed N_0, M_0 and D_0 the expended energy cannot be lower than $N_0 \log \frac{M_0}{(M_0+1)D_0}$, we obtain a first-order lower bound given by

$$E_{\min}(\mathcal{D}) \geq \sup_{N>0,M>0} N \log \frac{M}{(M+1)\mathcal{D}(N,M)} \tag{4.4}$$

or equivalently by

$$E_{\min}(\mathcal{F}) \geq \sup_{Q>0,s>0} \frac{\log \frac{\mathcal{F}(Q,s)}{1+s}}{Q} . \tag{4.5}$$

4.4 Analysis for Specific Profiles

In this section, we discuss linear and staircase profiles in detail. We find upper and lower bounds for them.

4.4.1 Linear Fidelity-Quality Profiles

Consider the fidelity-quality profile given as

$$\mathcal{F}(Q,s) = 1 + \alpha Q + \beta s . \tag{4.6}$$

In what follows, we show that simple uncoded transmission in fact achieves $E_{\min}(\mathcal{F})$, and therefore is optimal.

Lemma 7. $E_{\min}(\mathcal{F}) = \alpha$ for the linear profile given in (4.6). Moreover, uncoded transmission

$$U_t = \begin{cases} \sqrt{\alpha} X_t & 1 \leq t \leq n \\ 0 & t > n \end{cases}$$

achieves the minimum energy.

Proof.

To show that $E_{\min}(\mathcal{F}) \geq \alpha$, it suffices to use the lower bound (4.5):

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \sup_{Q>0,s>0} \frac{\log\left(\frac{1+\alpha Q+\beta s}{1+s}\right)}{Q} \\ &\geq \sup_{Q>0} \left[\sup_{s>0} \frac{\log\left(\frac{1+\alpha Q+\beta s}{1+s}\right)}{Q} \right] \\ &\geq \sup_{Q>0} \frac{1}{Q} \log\left(\frac{1+\alpha Q+\beta s^*(Q)}{1+s^*(Q)} \right) \end{aligned}$$

where

$$s^*(Q) = \begin{cases} \infty, & \text{if } 0 < Q < \frac{\beta-1}{\alpha} \\ 0, & \text{if } Q > \frac{\beta-1}{\alpha}. \end{cases}$$

Thus,

$$E_{\min}(\mathcal{F}) \geq \sup_{Q>0} f(Q) \tag{4.7}$$

where

$$f(Q) = \begin{cases} \frac{1}{Q}\log\beta, & \text{if } 0 < Q < \frac{\beta-1}{\alpha} \\ \frac{1}{Q}\log(1+\alpha Q), & \text{if } Q > \frac{\beta-1}{\alpha}. \end{cases}$$

Now, for $\beta > 1$, we have $\frac{\beta-1}{\alpha} > 0$, and thus $E_{\min}(\mathcal{F}) = \infty$. For $\beta \leq 1$, $\frac{\beta-1}{\alpha} < 0$ is guaranteed and thus we can conclude

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \sup_{Q>0} \frac{1}{Q}\log(1+\alpha Q) \\ &= \alpha \end{aligned} \tag{4.8}$$

To prove that $E_{\min}(\mathcal{F}) \leq \alpha$, we first assume that $\beta = 1$. It is easy to show that the distortion formula is as follows,

$$D = \frac{1}{1 + \frac{\alpha}{N} + \frac{1}{M}} \tag{4.9}$$

for all $0 < N < \infty$ and $0 < M < \infty$, translating into

$$F = 1 + \alpha Q + s \tag{4.10}$$

for all $0 < Q < \infty$ and $0 < s < \infty$.

Clearly, uncoded transmission expends energy α is well-known to achieve the expected distortion given in (4.9) or equivalently (4.10). We showed that profile $1 + \alpha Q + s$ is achievable with finite energy α. Since any other profiles like $\mathcal{F}(Q, s) = 1 + \alpha Q + \beta s$ are degenerate, i.e.,$1 + \alpha Q + s \geq 1 + \alpha Q + \beta s$, we can conclude that $E_{\min}(\mathcal{F}) \leq \alpha$.

Lemma 7 may not be surprising as the profile $\mathcal{F}(Q, s)$ in (4.6) is "tailored" to the performance of uncoded transmission. Nevertheless, it is an important example where uncoded transmission is optimal in any context other than *matched bandwidth* scenarios.

4.4.2 Staircase Fidelity-Quality Profiles

So far, we completely discussed the minimum energy for a linear profile. In this section, we analyze the minimum energy needed for K-step staircase profiles characterized by

$$\mathcal{F}(Q, s) = a_k \qquad (Q_k, s_k) \in T_k \tag{4.11}$$

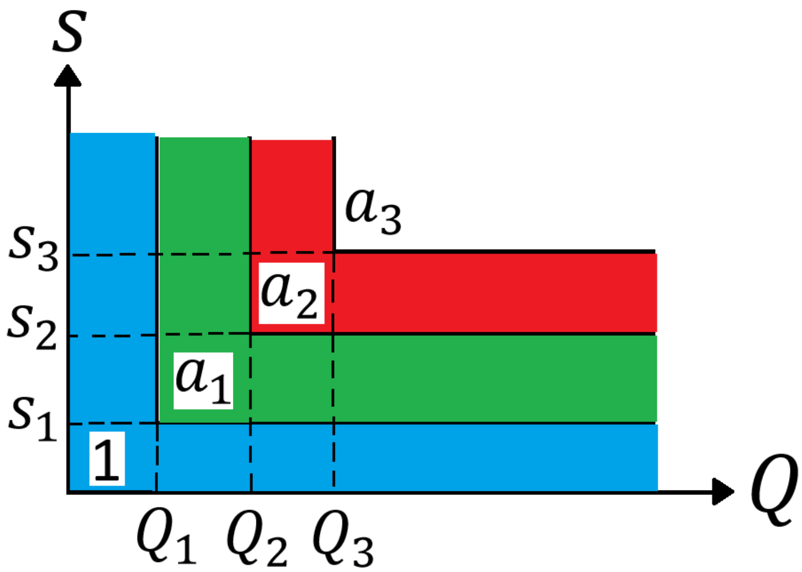

Figure 4.2: The Staircase fidelity-quality profile with $K = 3$ steps.

for $0 \leq k \leq K$, with the understanding that $a_0 = 1$, $Q_0 = 0$, $s_0 = 0$, and $Q_{K+1} = \infty$. An example profile is depicted in Figure 4.2 for $K = 3$ in which T_0, T_1, T_2 and T_3 regions are shown with blue, green, red and white colors respectively.

A General Upper bound for $E_{\min}(\mathcal{F})$

To upper bound $E_{\min}(\mathcal{F})$, we introduce an infinite-layer coding scheme, in which the source is quantized successively, and the quantization index in each layer is Wyner-Ziv coded and transmitted digitally (using infinite bandwidth at each round). The source is also transmitted in an uncoded fashion (using matched bandwidth), serving as the side information the digital coding relies on.[1]

[1]Note that none of these infinitely many rounds of transmission are superposed on or interfere with each other because we are utilizing an infinite available bandwidth to send them separately.

The available energy is divided into (i) E_0, used for uncoded transmission , and (ii) E_k, used for digital transmission rounds, $k = 1, 2, 3, \ldots$. The kth quantization index is to be decoded whenever $N \leq N_k$ for some predetermined sequence of noise levels $N_1 \geq N_2 \geq N_3 \geq \ldots$. For any noise variance $0 < N < \infty$, the received signal from the first subband is then given by

$$Z_N^n = \sqrt{E_0} X^n + W_N^n. \tag{4.12}$$

The source X^n is successively quantized into source codewords $\hat{S}_k^n$ for $k = 1, 2, 3, \ldots$, where the underlying single-letter characterization satisfies

$$S_k = \hat{S}_{k+1} + S_{k+1} \tag{4.13}$$

with $S_0 = X$ and $\hat{S}_{k+1} \perp S_{k+1}$. Thus, the quantization error S_k^n from layer k is quantized into $\hat{S}_{k+1}^n$ and transmitted from the $(k+1)$th infinite subband using Wyner-Ziv coding where the estimation can rely on all

$$\tilde{Z}_{k,N}^n = \sqrt{E_0} S_k^n + W_N^n$$

$$\tilde{Y}_{k,M}^n = S_k^n + L_M^n$$

as *effective* side information, as all $\hat{S}_i^n$ for $i = 1, ..., k$ can be subtracted from X^n.

Now, to be able to decode $\hat{S}_k$ whenever $N \leq N_k$, it suffices to use a binning rate of

$$
\begin{aligned}
R_k &= I(S_{k-1}; \hat{S}_k | \tilde{Y}_{k-1,M_k}, \tilde{Z}_{k-1,N_k}) \\
&= I(S_{k-1}; \hat{S}_k) - I(\tilde{Y}_{k-1,M_k}, \tilde{Z}_{k-1,N_k}; \hat{S}_k) \\
&= h(S_{k-1}) - h(S_k) - h(\tilde{Y}_{k-1,M_k}, \tilde{Z}_{k-1,N_k}) + h(\tilde{Y}_{k-1,M_k}, \tilde{Z}_{k-1,N_k} | \hat{S}_k) \\
&= \frac{1}{2}\log\frac{\sigma^2_{S_{k-1}}}{\sigma^2_{S_k}} - \frac{1}{2}\log\frac{\det \mathbf{\Sigma}_{\tilde{\mathbf{U}}_k}}{\det \mathbf{\Sigma}_{\tilde{\mathbf{U}}_k|\hat{S}_k}} \qquad (4.14)
\end{aligned}
$$

where

$$
\mathbf{\Sigma}_{\tilde{\mathbf{U}}_k} = \begin{bmatrix} E_0\sigma^2_{S_{k-1}} + N_k & \sqrt{E_0}\sigma^2_{S_{k-1}} \\ \sqrt{E_0}\sigma^2_{S_{k-1}} & \sigma^2_{S_{k-1}} + M_k \end{bmatrix},
$$

and

$$
\mathbf{\Sigma}_{\tilde{\mathbf{U}}_k|\hat{S}_k} = \begin{bmatrix} E_0\sigma^2_{S_k} + N_k & \sqrt{E_0}\sigma^2_{S_k} \\ \sqrt{E_0}\sigma^2_{S_k} & \sigma^2_{S_k} + M_k \end{bmatrix}
$$

respectively.

By substituting $\det \mathbf{\Sigma}_{\tilde{\mathbf{U}}_k}$ and $\det \mathbf{\Sigma}_{\tilde{\mathbf{U}}_k|\hat{S}_k}$ in (4.14), we then get

$$
\begin{aligned}
R_k &= \frac{1}{2}\log\frac{\sigma^2_{S_{k-1}}}{\sigma^2_{S_k}} - \frac{1}{2}\log\frac{(E_0M_k + N_k)\sigma^2_{S_{k-1}} + N_kM_k}{(E_0M_k + N_k)\sigma^2_{S_k} + N_kM_k} \\
&= \frac{1}{2}\log\frac{(E_0M_k + N_k) + \frac{N_kM_k}{\sigma^2_{S_k}}}{(E_0M_k + N_k) + \frac{N_kM_k}{\sigma^2_{S_{k-1}}}} \\
&= \frac{1}{2}\log\frac{(E_0Q_k + s_k) + \gamma_k}{(E_0Q_k + s_k) + \gamma_{k-1}} \qquad (4.15)
\end{aligned}
$$

where $\gamma_k = \frac{1}{\sigma^2_{S_k}}$, $Q_k = \frac{1}{N_k}$, and $s_k = \frac{1}{M_k}$.

Remark 6. Please note that when $M_k \to \infty$, we have

$$R_k = \frac{1}{2} \log \frac{E_0 Q_k + \gamma_k}{E_0 Q_k + \gamma_{k-1}}$$

which exactly matches rate formula we calculated in chapter 3.

For this digital message, we use the channel with infinite bandwidth and energy E_k. Therefore, the rate must not exceed the channel capacity under the noise level N_k, i.e.,

$$R_k \le C_k = \frac{E_k Q_k}{2}$$

or

$$\frac{1}{2} \log \frac{E_0 Q_k + s_k + \gamma_k}{E_0 Q_k + s_k + \gamma_{k-1}} \le \frac{E_k Q_k}{2} \tag{4.16}$$

or equivalently,

$$\frac{E_0 Q_k + s_k + \gamma_k}{E_0 Q_k + s_k + \gamma_{k-1}} \le \exp(E_k Q_k)\ . \tag{4.17}$$

When $N_{k+1} < N \le N_k$, or equivalently $Q_k \le Q < Q_{k+1}$, the MMSE estimation boils down to estimating S_k^n using all the available effective side information, that is

$$\tilde{S}_k^n = \sum_{i=0}^{k} G_i \tilde{Y}_{i,M} + \sum_{i=0}^{k} H_i \tilde{Z}_{i,N}$$

with appropriate G_i and H_i for $i = 0, 1, \ldots, k$. This is standard, and the resultant distortion can be calculated as

$$D = \sigma_{S_k}^2 - (\sigma_{S_k}^2)^2 \mathbf{a}_k \mathbf{\Sigma}_{\tilde{\mathbf{U}}}^{-1} \mathbf{a}_k^T$$

where

$$\mathbf{\Sigma}_{\tilde{\mathbf{U}}} = \begin{bmatrix} E_0\sigma_{S_k}^2 + N & \sqrt{E_0}\sigma_{S_k}^2 \\ \sqrt{E_0}\sigma_{S_k}^2 & \sigma_{S_k}^2 + M \end{bmatrix} ,$$

and

$$\mathbf{a}_k = \begin{bmatrix} \sqrt{E_0} & 1 \end{bmatrix} ,$$

respectively. Thus,

$$\begin{aligned} D &= \sigma_{S_k}^2 - \frac{(\sigma_{S_k}^2)^2(E_0M + N)}{E_0M\sigma_{S_k}^2 + N\sigma_{S_k}^2 + NM} \\ &= \frac{1}{\frac{E_0}{N} + \frac{1}{M} + \frac{1}{\sigma_{S_k}^2}} \\ &= \frac{1}{E_0Q + s + \gamma_k} \end{aligned} \tag{4.18}$$

Equivalently, the fidelity can be written as

$$F(Q, s) = E_0Q + s + \gamma_k . \tag{4.19}$$

To ensure $F(Q) \geq \mathcal{F}(Q)$, it then suffices to set E_0 and γ_k such that

$$\gamma_k \geq a_k - E_0Q_k - s_k \tag{4.20}$$

for $k = 1, \ldots, K$, while also satisfying

$$1 = \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_K . \tag{4.21}$$

We are now ready to prove an upper bound on $E_{min}(\mathcal{F})$.

Theorem 6. For a staircase profile described in (4.11), the minimum required energy is upper-bounded as

$$\begin{aligned} E_{min}(\mathcal{F}) &\le E_{\mathrm{UB}}(E_0) \\ &\triangleq E_0 + \sum_{k=1}^{K} \frac{1}{Q_k} \left(\log \frac{a_k}{E_0(Q_k - Q_{k-1}) + (s_k - s_{k-1}) + a_{k-1}} \right)^{+} \end{aligned} \tag{4.22}$$

where $x^+ = \max\{x, 0\}$.

Proof. For any fixed E_0 in the allowed range, and γ_k statisfying (4.20) and (4.21), it follows from (4.16) that

$$\begin{aligned} E_{\min}(\mathcal{F}) &\le E_{\mathrm{UB}}(\gamma_1, \gamma_2, \ldots, \gamma_K | E_0) \\ &\triangleq E_0 + \sum_{k=1}^{K} \frac{1}{Q_k} \log \frac{\gamma_k + Q_k E_0 + s_k}{\gamma_{k-1} + Q_k E_0 + s_k} \ . \end{aligned} \tag{4.23}$$

Now, differentiating $E_{\mathrm{UB}}(\gamma_1, \gamma_2, \ldots, \gamma_K | E_0)$ with respect to γ_k yields

$$\begin{aligned} \frac{\partial E_{\mathrm{UB}}}{\partial \gamma_k} &= \frac{1}{Q_k(\gamma_k + Q_k E_0 + s_k)} - \frac{1}{Q_{k+1}(\gamma_k + Q_{k+1} E_0 + s_k)} \\ &= \frac{(\gamma_k + s_k)(Q_{k+1} - Q_k) + E_0(Q_{k+1}^2 - Q_k^2)}{Q_k Q_{k+1}(\gamma_k + Q_k E_0 + s_k)(\gamma_k + Q_{k+1} E_0 + s_k)} \\ &> 0 \end{aligned}$$

implying that $E_{\mathrm{UB}}(\gamma_1, \gamma_2, \ldots, \gamma_K | E_0)$ is increasing in γ_k. But this implies that it is minimized by the choice

$$\gamma_k = \max\{\gamma_{k-1}, a_k - E_0 Q_k - s_k\} \tag{4.24}$$

for $k = 1, \ldots, K$, due to (4.20) and (4.21). Substituting (4.24) in (4.23) then yields the desired result.

Remark 7. The upper bound $E_{\text{UB}}(E_0)$ is a convex function of E_0 and hence is not difficult to optimize for given $Q_1, \ldots, Q_K$, $s_1, \ldots, s_K$ and $a_1, \ldots, a_K$. Essentially, it is continuous and has continuous derivatives everywhere with the exception of K discontinuity points at $\frac{(a_k - a_{k-1}) - (s_k - s_{k-1})}{Q_k - Q_{k-1}}$ for $1 \le k \le K$. It is also easy to show that $\left.\frac{dE_{\text{UB}}(E_0)}{dE_0}\right|_{E_0=0} < 0$ and therefore it is *always* beneficial to set $E_0 > 0$, i.e., transmit analog information.

In the rest of this discussion, we define

$$E_{\text{UB}}^* \triangleq \min_{E_0 > 0} E_{\text{UB}}(E_0) \ .$$

A General Lower bound for $E_{\min}(\mathcal{F})$

By using (4.5), we can write

$$\begin{aligned} E_{\min}(\mathcal{F}) &\geq \sup_{Q>0, s>0} \frac{\log \frac{\mathcal{F}(Q,s)}{1+s}}{Q} \\ &= \sup_{Q>0, s>0} \frac{\log \frac{a_k}{1+s}}{Q} \\ &= \max_{k=1,\ldots,K} \frac{1}{Q_k} \log\left(\frac{a_k}{1+s_k}\right). \end{aligned} \tag{4.25}$$

Special Scenarios

We finish our analysis of the minimum energy under the staircase profile by looking into two special scenarios.

Logarithmic steps with $K = \infty$ If it is desirable that the reproduction quality to improve by a certain amount (in dB) every time the channel quality also improves by

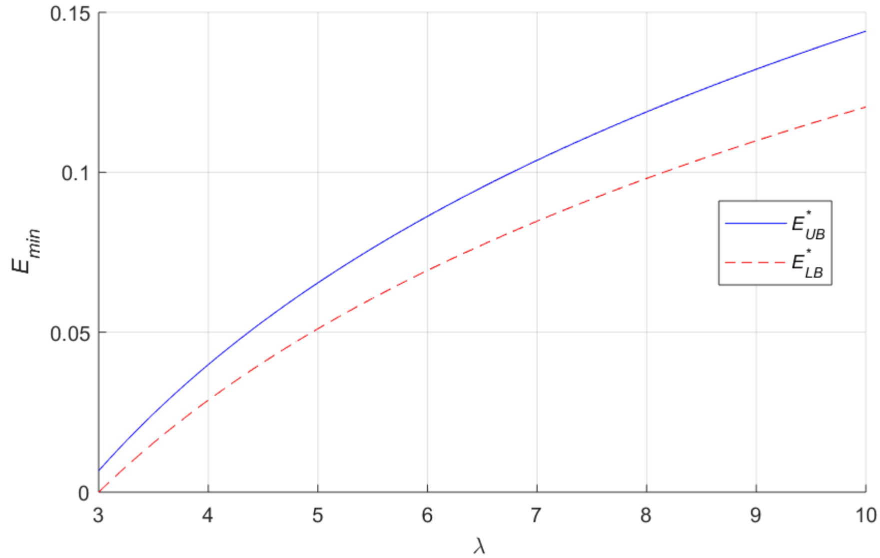

Figure 4.3: Comparison of bounds as a function of λ when $\gamma = 10$ and $\eta = 2$ are fixed.

another certain amount (also in dB) indefinitely, we have a scenario where $K = \infty$, $Q_k = \gamma^k$, $s_k = \eta^k$ and $a_k = \lambda^k$ for all $k \geq 1$ with some appropriately chosen $\gamma, \eta, \lambda > 1$.

Under this scenario, (4.22) and (4.25) respectively become

$$E_{\text{UB}}(E_0) = E_0 + \frac{1}{\gamma}\left(\log \frac{\lambda}{E_0\gamma + \eta + 1}\right)^+ + \sum_{k=2}^{\infty} \frac{1}{\gamma^k}\left(\log \frac{\lambda}{(\frac{\eta}{\lambda})^{k-1}(\eta - 1) + E_0(\frac{\gamma}{\lambda})^{k-1}(\gamma - 1) + 1}\right)^+ \tag{4.26}$$

and

$$E^*_{\text{LB}} = \max_{k=1,\ldots,K} \frac{1}{\gamma^k} \log\left(\frac{\lambda^k}{1 + \eta^k}\right). \tag{4.27}$$

Figure 4.3 shows a comparison of E^*_{UB} and E^*_{LB} as a function of λ for fixed γ and η as an example.

Profile with $K = 2$ In this section, we analyze our problem when there are only two steps in the profile.

Starting first with the upper bound, it can actually be shown that it suffices to limit ourselves to

$$E_0 \leq E_{0,\max} \triangleq \min\left\{\frac{a_1 - 1 - s_1}{Q_1}, \frac{(a_2 - a_1) - (s_2 - s_1)}{Q_2 - Q_1}\right\}$$

in which case (4.22) becomes

$$E_{\mathrm{UB}}(E_0) \quad = \quad E_0 + \frac{1}{Q_1}\log\frac{a_1}{E_0 Q_1 + s_1 + 1} + \frac{1}{Q_2}\log\frac{a_2}{E_0(Q_2 - Q_1) + (s_2 - s_1) + a_1} \; .$$

Since $E_{\mathrm{UB}}(E_0)$ is convex everywhere, and decreasing at $E_0 = 0$, it will assume its minimum either at E_0 satisfying $\frac{dE_{\mathrm{UB}}}{dE_0} = 0$ or at $E_0 = \min\left\{\frac{a_1 - 1 - s_1}{Q_1}, \frac{(a_2 - a_1) - (s_2 - s_1)}{Q_2 - Q_1}\right\}$, whichever is smaller.

The first approach that comes to mind to obtain an upper bound for the minimum energy needed to comply with this profile is using a two-layered digital coding scheme designed to decode one layer each time the channel quality Q exceeds a discontinuity point Q_k for $k = 1, 2$. As a matter of fact, this digital coding scheme would be a finite-layer special case of the scheme in which the uncoded energy level is set to zero, i.e., $E_0 = 0$. Thus, the upper bound is as follows,

$$\hat{E}_{\mathrm{UB}} = \frac{1}{Q_1}\log\frac{a_1}{s_1 + 1} + \frac{1}{Q_2}\log\frac{a_2}{(s_2 - s_1) + a_1}. \tag{4.28}$$

As for the lower bound, (4.25) becomes

$$E^*_{\mathrm{LB}} = \max\left(\frac{1}{Q_1}\log\frac{a_1}{s_1 + 1}, \frac{1}{Q_2}\log\frac{a_2}{s_2 + 1}\right). \tag{4.29}$$

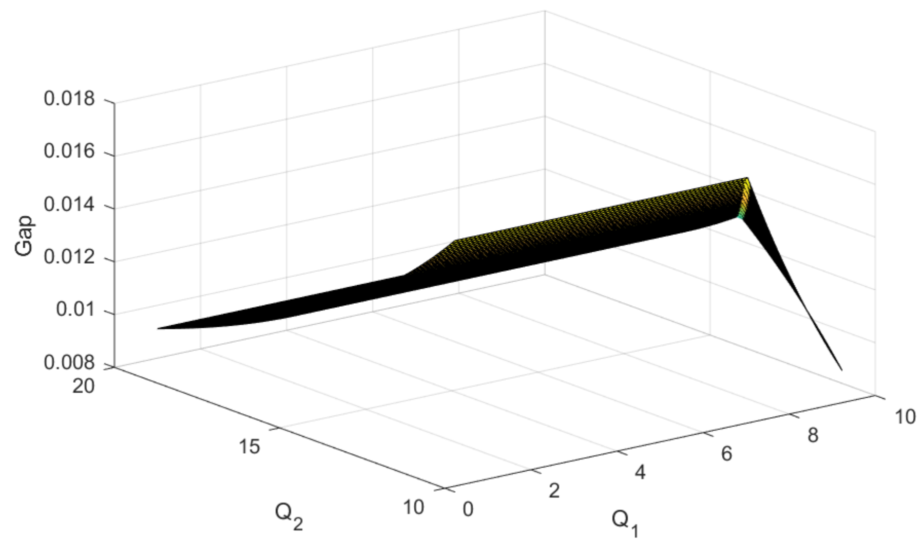

Figure 4.4: Gap between upper and lower bounds as a function of Q_1 and Q_2.

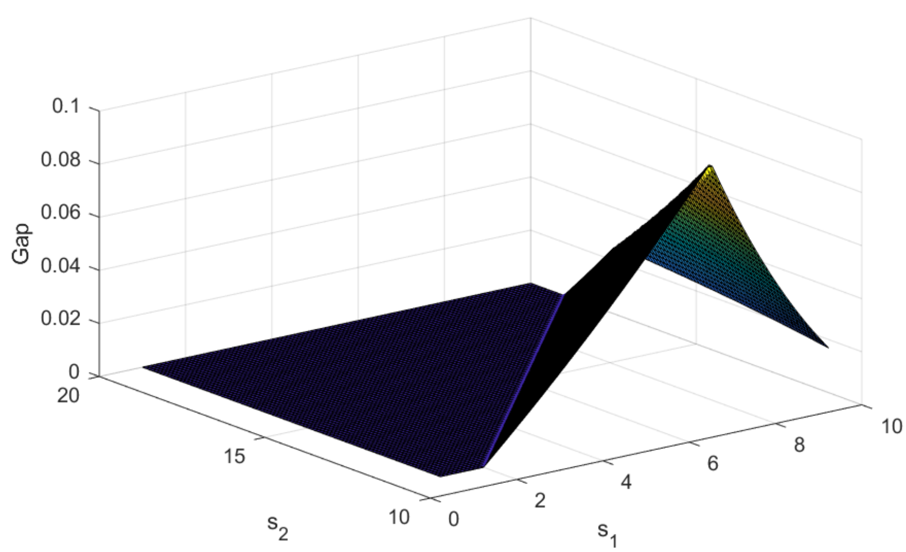

Figure 4.5: Gap between upper and lower bounds as a function of s_1 and s_2.

Figure 4.4 shows a gap between $\hat{E}_{\mathrm{UB}}$ and E^*_{LB} as a function of Q_1 and Q_2 for fixed $a_1 = 4$, $a_2 = 6$, $s_1 = 2$ and $s_2 = 3$ as an example. Figure 4.5 shows a gap between $\hat{E}_{\mathrm{UB}}$ and E^*_{LB} as a function of s_1 and s_2 for fixed $a_1 = 12$, $a_2 = 21$, $Q_1 = 2$ and $Q_2 = 3$ as another example.

Chapter 5

Conclusions and Future Work

5.1 Conclusions

In this thesis, the minimum energy required to achieve a distortion-noise profile, i.e., a function indicating the maximum allowed distortion value for each noise level, was studied for robust transmission of Gaussian sources over Gaussian channels.

In one scenario, we did minimum power analysis for bandwidth compression case and specifically near-zero bandwidth ratio. A general lower bound to the minimum required power for a given profile was presented. For an upper bound, a dirty-paper based coding scheme was proposed and its power-distortion trade-off is analyzed. Finally, upper and lower bounds to the minimum power were compared for specific distortion-noise profiles, namely rational profiles with order one and two.

In another scenario, we addressed the case where the energy per source symbol is limited but the channel uses per source symbol is unlimited, which corresponds to infinite bandwidth. The previous results for the inversely linear and exponential profiles were discussed. For square-law and staircase profiles, we proposed new coding schemes to upper bound the minimum energy needed. Conversely, finding a family of lower bounds, we lower bounded the minimum required energy for both square-law and staircase profiles, and compared with the corresponding upper bounds.

In our last scenario, we also analyzed the near-infinity bandwidth case where the receiver observes side information about the source. The quality of the side information is also unknown and the quality parameter would be two-dimensional. The coding schemes were proposed to upper bound the minimum energy needed for square-law and staircase profiles. Furthermore, a general family of lower bounds was derived for the minimum required energy which works for any profiles such as square-law and staircase.

5.2 Future Work

As future work, we are interested in expanding our results to multiple access channels (MAC), where for each transmitter, there would be a separate distortion-noise profile dictating maximum distortion levels as a function of the noise level of the MAC.

Bibliography

[1] C. Shannon, "A mathematical theory of communication," *Bell Sys. Tech. Journal*, vol. 27, pp. 379–423, 1948.

[2] E. Koken and E. Tuncel, "On minimum energy for robust Gaussian joint source-channel coding with a distortion-noise profile," *IEEE International Symposium on Information Theory*, Aachen, Germany, June 2017.

[3] A. Jain, D. Gündüz, S. R. Kulkarni, H. V. Poor, and S. Verdú, "Energy-distortion tradeoffs in Gaussian joint source-channel coding problems," *IEEE Transactions on Information Theory*, vol. 58, no. 5, pp. 3153–3168, May 2012.

[4] N. Jiang, Y. Yang, A. Host-Madsen, and Z. Xiong, "On the minimum energy of sending correlated sources over the Gaussian MAC," *IEEE Transactions on Information Theory*, vol. 60, no. 10, pp. 6254–6275, Aug. 2014.

[5] E. Köken and E. Tuncel, "On the energy-distortion tradeoff for the Gaussian broadcast problem," *IEEE International Symposium on Information Theory*, Barcelona, Spain, Jul. 2016.

[6] E. Köken, D. Gündüz, and E. Tuncel, "Energy-distortion exponents in lossy transmission of Gaussian sources over Gaussian channels," *IEEE Transactions on Information Theory*, vol. 63, no. 2, pp. 1227–1236, Feb. 2017.

[7] M. Baniasadi and E. Tuncel, "Minimum Energy Analysis for Robust Gaussian Joint Source-Channel Coding with a Square-Law Profile," *The International Symposium on Information Theory and Its Applications*, Kapolei, Hawaii, USA, Oct. 2020.

[8] K. Woyach, K. Harrison, G. Ranade, and A. Sahai, "Comments on unknown channels," in *Information Theory Workshop (ITW).*, IEEE, Sep. 2012, pp. 172–176

[9] E. Köken, and E. Tuncel, "On Robustness of Hybrid Digital/Analog Source-Channel Coding With Bandwidth Mismatch," *IEEE Transactions on Information Theory*, vol. 61, no. 9, pp. 4968 – 4983, Sep. 2015.

[10] K. Eswaran, A. D. Sarwate, A. Sahai, and M. Gastpar, "Using zero-rate feedback on binary additive channels with individual noise sequences," *IEEE International Symposium on Information Theory*, Nice, France, Jun. 2007.

[11] V. Misra and T. Weissman, "The porosity of additive noise sequences," *IEEE International Symposium on Information Theory*, Istanbul, Turkey, Jul. 2012.

[12] Y. Lomnitz and M. Feder, "Communication over individual channels," *IEEE Transactions on Information Theory*, vol. 57, no. 11, pp. 7333–7358, Nov. 2011.

[13] E. Köken, and E. Tuncel, "Joint Source-Channel Coding for Broadcasting Correlated Sources," *IEEE Transactions on Communications*, vol. 65, no. 7, pp. 3012 – 3022, July. 2017.

[14] K. Narayanan, G. Caire, and M. Wilson, "Duality between Broadcasting with Bandwidth Expansion and Bandwidth Compression," *IEEE International Symposium on Information Theory*, Nice, France, June. 2006.

[15] Y. Wang, F. Alajaji, and T. Linder, "Hybrid Digital-Analog Coding with Bandwidth Compression for Gaussian Source-channel Pairs," *IEEE Transactions on Communications*, vol. 57, no. 4, pp. 997 – 1012, April. 2009.

[16] M. Baniasadi and E. Tuncel, "Robust Gaussian Joint Source-Channel Coding Under the Near-Zero Bandwidth Regime," *IEEE International Symposium on Information Theory*, Los Angeles, CA, USA, Jun. 2020.

[17] M. Baniasadi, E. Köken, and E. Tuncel, " Minimum Energy Analysis for Robust Gaussian Joint Source-Channel Coding with a Distortion-Noise Profile," submitted to *IEEE Transactions on Communications*.

[18] H. Behroozi, F. Alajaji, and T. Linder, "On the performance of hybrid digital-analog coding for broadcasting correlated Gaussian sources," *IEEE Transactions on Communications*, vol. 59, no. 12, pp. 3335 – 3342, Dec. 2011.

[19] Z. Reznic, M. Feder, and R. Zamir, "Distortion bounds for broadcasting with bandwidth expansion," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3778–3788, Aug. 2006.

[20] C. Tian, S. Diggavi, S. Shamai, "Approximate characterization for the Gaussian source broadcast distortion region," *IEEE Transactions on Information Theory*, vol. 57, no. 8, pp. 124–136, Jan. 2011.

[21] J. Ding and A. Zhou, “Eigenvalues of rank-one updated matrices with some applications," *Applied Mathematics Letters*, vol. 20, no. 12, pp. 1223-1226, 2007.