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A Parking Function Setting for Nabla Images of Schur Functions

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A Parking Function Setting for Nabla Images of Schur Functions

A dissertation submitted in partial satisfaction of the
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in

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by

Yeonkyung Kim

Committee in charge:

Professor Adriano M. Garsia, Chair
Professor James Buckwalter
Professor Walter Burkhard
Professor Ronald Evans
Professor Jeffrey Remmel

2015

The dissertation of Yeonkyung Kim is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To my lovely sons, Jayden and Ryan.

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PUBLICATIONS

(with A. Hicks), "An explicit formula for ndinv, a new statistic for two-shuffle parking functions", *Journal of Combinatorial Theory, Series A*, Vol 120, January 2013.

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ABSTRACT OF THE DISSERTATION

A Parking Function Setting for Nabla Images of Schur Functions

by

Yeonkyung Kim

Doctor of Philosophy in Mathematics

University of California, San Diego, 2015

Professor Adriano M. Garsia, Chair

Jim Haglund, Jennifer Morse, and Mike Zabrocki have published papers introducing symmetric function operators, **C** operator and **B** operator, and their combinatorial interpretations and identities. Their compositional refinement of the Shuffle Conjecture can be used to represent the image of a schur function under the Bergeron-Garsia nabla operator as a weighted sum of a suitable collection of parking functions. First, we express a schur function under nabla operator with **C** operator using its definition and fundamental identities about symmetric functions. Then, the compositional Shuffle conjecture can be used as the expression of a family of $\nabla \mathbf{C}_{p_1} \cdots \mathbf{C}_{p_k} \mathbf{1}$, with $p = (p_1, \cdots, p_k)$, composition of n, in terms of a sum of weight of a suitable set of parking functions. Therefore, in this paper, we express the image of a schur function under nable operator as the difference

between two sum of the weight of certain sets of parking functions. Then, we introduce injections between those sets of parking functions for a few cases of schur functions, so that an image of schur functions under nabla operator can be simply expressed using the complementary set of parking functions. The validity of these expressions is still conjectural until the compositional refinement of the Shuffle conjecture is proved.

Chapter 1

Introduction

First, we begin with basic definitions and their auxiliaries.

1.1 Parking Functions

There is a one-way street with n available parking spaces. We label the parking spaces, 1 to n, from the entrance to the exit of the street.

Cars 1 to n enter the street one by one in order. They have their own preferred parking space from 1 to n and some of them may have the same preference. We represent the preferences as a sequence, for example, a sequence $(2, 4, 1, 1, 3)$ means that there are 5 cars and the first car wants to be parked in space 2, and the second car wants to be parked in space 4. The third and fourth cars have the same preference, both of them want to be parked in space 1, and the last car prefers space 3.

The rule for parking these n cars is following:

1. Each car goes straight to its preferred parking space.
2. If the parking spot is empty, the car parks there. If not, the car goes further and parks in the first available spot.

For some preferences some cars cannot park. When all n cars can park, we call the sequence representing the preferences a *parking function*, and it was introduced by Konheim and Weiss in [KW66].

Example 1.1.1. Suppose 5 cars have the preferences $(2, 1, 4, 1, 4)$. Then the first car parks in space 2, the second car parks in space 1, and the third car parks in space 4. The fourth car's preferred space is 1, but since it is already filled, it goes further and parks in space 3, which is the first available spot. The last car's preference is space 4, but the third car was already parked there, so it should park in space 5. All 5 cars have parked, so the sequence $(2, 1, 4, 1, 4)$ is a parking function. However, suppose $(3, 4, 3, 2)$ is the sequence representing the preferences of the 4 cars. Then, the first car parks in space 3 and the second car parks in space 4. The third car goes straight to space 3, but the first car was already parked in space 3, so it can't be parked there, but since the space 4 is also filled with the second car, it cannot park anywhere. Therefore, the sequence $(3, 4, 3, 2)$ is *not* a parking function.

1.1.1 Parking Functions on a Lattice Square

We will represent a parking function by a labeled Dyck path on an n by n lattice square.

Let us call the *main diagonal* the straight line straight line from $(0, 0)$, to (n, n), of the n by n lattice square. A Dyck path on an n by n lattice square is a path from $(0, 0)$ to (n, n) that proceeds only by east or north steps always remaining weakly above the main diagonal. That is the path may hit the main diagonal, but never crosses it. To represent a parking function, we put the car numbers in the cells to the right of every north step in a column increasing manner.

Figure 1.1 shows an example of a parking function with 12 cars in a 12 by 12 lattice square. Notice that there are 4 cars, car 3, car 6, car 8 and car 11, are stacked in increasing order of their numbers.

Now we will define some statistics for a parking function on a lattice square.

Definition 1.1.1. We define the *area* of a parking function to be the number of complete cells between its Dyck path and the main diagonal.

Example 1.1.2. In Figure 1.1, there are 16 complete cells between the Dyck path and the main diagonal. Therefore, the *area* of this parking function is 16.

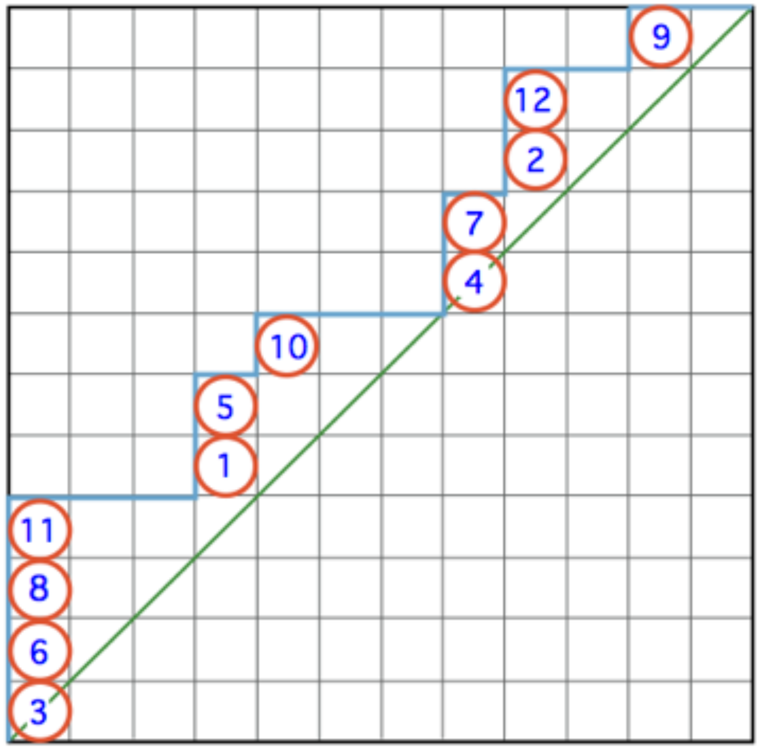


Figure 1.1: A parking function with 12 cars

Definition 1.1.2. We define a *diagonal inversion*(*dinv*) as the number of pairs of two cars, which are not in the same column, satisfying one of the following two conditions:

- They are in the same diagonal and the car number on the right is larger than the one on the left.
- The car on the left is on the diagonal one level above the diagonal of the car on the right. Also, it has a larger number than the one on the right.

We call the number of pairs satisfying the first condition the *primary dinv* and the number of pairs satisfying the second condition the *secondary dinv*.

Example 1.1.3. In Figure 1.1, the pair car 3 and 4 contributes to the primary *dinv* since they are in the same diagonal(the main diagonal) and the left car, car 3, is smaller than the right car, car 4. Similarly, the pairs $(6, 7), (6, 9), (1, 7), (1, 2), (1, 9)$,

$(7,9),(2,9),(8,10),(8,12),(5,10),(5,12)$ and $(10,12)$ are contributing to the primary *dinv*. Therefore, the primary *dinv* of this parking function is 13. On the other hand, the pair $(4,6)$ contributes to the secondary *dinv* since the car 6 is on the second diagonal, one level above the main diagonal, where the car 4 is, and the left car 6 is larger than the right car 4. The pairs $(1,8),(7,8),(2,8),(2,5),(7,10),(2,10),$ $(9,10),(5,11),(10,11)$ and $(9,12)$ also contribute to the secondary *dinv*, so the secondary *dinv* of PF is 11. Hence, the *dinv* of PF is 24.

Definition 1.1.3. Suppose a labeled Dyck path of a parking function hits the main diagonal at $(0,0)$, (f_1,f_1), $\ldots$,(f_k,f_k). Then we say the cars from the $(f_{i-1}+1)^{\text{th}}$ column to the f_i^{th} column are in i^{th} *part.* We also define the *composition of* PF, $(comp(PF))$, as the sequence $(p_1,p_2,\ldots,p_k)$, where p_i counts the number of cars in i^{th} part.

Example 1.1.4. In Figure 1.1, the cars $3,6,8,11,1,5,10$ are in the first part and the cars $4,7,2,12,9$ are in the second part. Therefore, the composition of PF is $(7,5)$ and we write $comp(PF)=(7,5)$.

Definition 1.1.4. We define the *diagonal word* of a parking function as the word obtained by reading cars from the top diagonal to the main diagonal. Where, within each diagonal line, we read cars from northeast to southwest.

Example 1.1.5. For the parking function in Figure 1.1, the diagonal word is $[11,12,10,5,8,9,2,7,1,6,4,3]$.

Definition 1.1.5. We finally define the *ides* as the set of i coming after $i+1$ in the diagonal word.

Example 1.1.6. The *ides* of the parking function in Figure 1.1 is $\{1,3,4,6,7,9,10\}$.

Finally, we can now define the *weight* of each parking function using these statistics.

Definition 1.1.6. We set for each parking function PF

$$w(PF)=t^{area(PF)}q^{dinv(PF)}Q_{ides(PF)}[X]$$

where for a subset $S \subset \{1, 2, \ldots, n-1\}$, $Q_S[X]$, introduced in [Ges84], denotes Gessel's fundamental quasi-symmetric function in the variables $X = (x_1, x_2, \ldots)$.

$$Q_S[X] = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ i \in S \to a_i < a_{i+1}}} x_{a_1} x_{a_2} \cdots x_{a_n}$$

Example 1.1.7. The weight of the parking function in Figure 1.1 is

$$w(PF) = t^{16}\ q^{24}\ Q_{\{1,3,4,6,7,9,10\}}[X].$$

1.1.2 Parking Functions as a Two-line Array

We also can represent a parking function by a two-line array. First, we let c_i be the car number in the i^{th} row from bottom to top. Next, we let d_i be the number of complete cells in the i^{th} row, between the Dyck path and the main diagonal. We can then represent a parkIng function as a two line array by setting

$$PF = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \\ d_1 & d_2 & \ldots & d_n \end{bmatrix}$$

Example 1.1.8. The parking function in Figure 1.1 can be represented by a two-line array

$$\begin{bmatrix} 3 & 6 & 8 & 11 & 1 & 5 & 10 & 4 & 7 & 2 & 12 & 9 \\ 0 & 1 & 2 & 3 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

To be a parking function representation, the two-line array has to satisfy three conditions:

1. $[c_1, c_2, \ldots, c_n]$ is a permutation of $1, 2, \ldots, n$ and d_i is a nonnegative integer for all i with $d_1 = 0$.
2. For all $1 \le i \le n-1$, $d_{i+1} \le d_i + 1$.
3. If $d_i = d_{i+1}$, then $c_{i+1} > c_i$.

We can easily see that the *area* and *dinv* can be expressed using c_i and d_i,

$$area(PF) = \sum_i d_i$$

$$dinv(PF) = \sum_{1 \le i < j \le n} \chi(d_i = d_j \text{ and } c_i < c_j) + \chi(d_i = d_j + 1 \text{ and } c_i > c_j)$$

1.2 Symmetric Functions, Macdonald Polynomials and the Nabla Operator

Now, we will introduce Macdonald symmetric functions and Nabla operator, first introduced by Macdonald in [Mac95] and Bergeron-Garsia in [BG99] respectively.

We denote the space of symmetric polynomials by Λ. The subspace of Λ consisting of the homogeneous symmetric polynomials of degree n is denoted by $\Lambda^{=n}$. We express symmetric polynomials in terms of the five following classical bases:

- the power basis $\{p_\lambda\}_{\lambda \vdash n}$
- the monomial basis $\{m_\lambda\}_{\lambda \vdash n}$
- the homogeneous basis $\{h_\lambda\}_{\lambda \vdash n}$
- the elementary basis $\{e_\lambda\}_{\lambda \vdash n}$
- the schur basis $\{s_\lambda\}_{\lambda \vdash n}$

The *Plethystic substitution* of an expression E is sometimes very useful when we work with symmetric function identities. For any expression E with the variables $t_1, t_2, t_3, \ldots$, $E = E[t_1, t_2, t_3, \ldots]$, we define

$$p_k[E] = E(t_1^k, t_2^k, t_3^k, \ldots).$$

For any symmetric function F, we can express it as $F = Q(p_1, p_2, p_3, \ldots)$, where the polynomial Q gives the expansion of F in terms of power basis. Then we can define

$$F[E] = Q(p_1[E], p_2[E], p_3[E], \ldots).$$

In particular, when we want to express F as a polynomial in the variables $x_1, x_2, \ldots, x_n$, using plethystic notation, we can write

$$f[X_n], \quad \text{where } X_n = x_1 + x_2 + \cdots + x_n.$$

The usual scalar product for symmetric functions, $\langle \cdot\,, \cdot \rangle$ is defined by

$$\langle s_\lambda, s_\mu \rangle = \chi(\lambda = \mu).$$

Also, using

$$z_\lambda = \prod_i i^{n_i} n_1!$$

where $n_i = n_i(\lambda)$ is the number of parts of λ equal to i, the original Macdonald scalar product, $\langle \cdot\,, \cdot \rangle_{q,t}$ is defined by

$$\langle \alpha_\lambda, \alpha_\mu \rangle_{q,t} = z_\lambda\, \chi(\lambda = \mu)\, \alpha_\lambda \left[\frac{1-t}{1-q} \right].$$

To work with Macdonald polynomials, we need to introduce further notation.

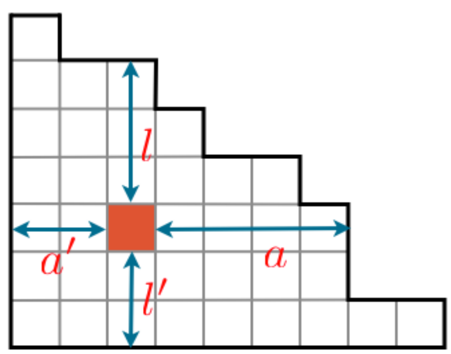


Figure 1.2: Standard Young tableau in French notation

Given a partition λ and a lattice cell $c \in \lambda$, we denote the *leg, arm, coleg* and *coarm* of c by $l_\lambda(c)$ (or just l sometimes), $a_\lambda(c)$ (or a), $l'_\lambda(c)$ (or l') and $a'_\lambda(c)$ (or a') respectively. The $l_\lambda(c)$, $l'_\lambda(c)$, $a_\lambda(c)$ and $a'_\lambda(c)$ respectively represents the number of cells in λ which are strictly north, south, east and west of the cell c in λ.

To proceed, we need to introduce some further constructs from Ferrers diagram.

$$n(\lambda) = \sum_{c\in\lambda} l(c) = \sum_{i=1}^{l(\lambda)} \lambda_i(i-1), \quad l(\lambda) \text{ is a length of a partition } \lambda.$$

$$n(\lambda') = \sum_{c\in\lambda} a'_\lambda(c) \qquad \tilde{h}_\lambda(q,t) = \prod_{c\in\lambda}\left(q^{a(c)} - t^{l(c)+1}\right)$$

$$B_\lambda(q,t) = \sum_{c\in\lambda} t^{l'(c)} q^{a'(c)} \qquad \tilde{h}'_\lambda(q,t) = \prod_{c\in\lambda}\left(t^{l(c)} - q^{a(c)+1}\right)$$

$$T_\lambda(q,t) = t^{n(\lambda)} q^{n(\lambda')} \qquad w_\lambda(q,t) = \tilde{h}_\lambda(q,t)\tilde{h}'_\lambda(q,t)$$

and

$$D_\lambda(q,t) = MB_\lambda(q,t) - 1 \text{ with } M = (1-t)(1-q)$$

Theorem 1.2.1. *For each partition λ, and generic q, t, there exists a unique family of polynomials $\{P_\lambda(x; q,t)\}_\lambda$ such that*

1. $P_\lambda = s_\lambda + \sum_{\mu<\lambda} s_\mu\, \xi_{\mu\lambda}(q,t)$
2. $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$ *if* $\lambda \neq \mu$.

Given this theorem, *the Macdonald integral forms* are defined by setting

$$J_\mu[X; q,t] = h_\mu(q,t)\, P_\mu(x; q,t)$$

with

$$h_\mu(q,t) = \prod_\mu (1 - q^a\, t^{l+1}).$$

The Macdonald q, t - Kotska, $K_{\lambda\mu}(q,t)$ is defined through the expansion

$$J_\mu[X; q,t] = \sum_\mu s_\lambda[X(1-t)] K_{\lambda\mu}(q,t).$$

Then we can set

$$H_\mu[X;q,t] = J_\mu\left[\frac{X}{1-t};q,t\right]$$

and now we can have

$$\widetilde{H}_\mu[X;q,t] = t^{n(\mu)}\, H_\mu[X;q,1/t].$$

The polynomials $\widetilde{H}_\mu[X;q,t]$ can be shown to satisfy the following fundamental identities

1. $T_\mu\,\omega\,\widetilde{H}_\mu[X;1/q,1/t] = \widetilde{H}_\mu[X;q,t]$,
2. $\widetilde{H}_{\mu'}[X;q,t] = \widetilde{H}_\mu[X;t,q]$.

Moreover we have

$$\widetilde{H}_\mu\Big|_{s_n} = 1 \quad \text{for all} \quad \mu.$$

Given this, we derive the following basic fact.

Theorem 1.2.2. *The polynomial $\widetilde{H}_\mu[X;q,t]$ are the unique symmetric function basis satisfying the two triangularity conditions*

1. $\widetilde{H}_\mu = \sum_{\lambda\le\mu} s_\lambda\left[\frac{X}{t-1}\right]\, c_{\lambda\mu}(q,t),$
2. $\widetilde{H}_\mu = \sum_{\lambda\ge\mu} s_\lambda\left[\frac{X}{1-q}\right]\, d_{\lambda\mu}(q,t).$

Proof. For some coefficients $\eta_{\lambda\mu}(q,t)$, we have

$$H_\mu[X;q,t] = \sum_{\lambda\le\mu} s_\lambda\left[\frac{X}{1-t}\right]\eta_{\lambda\mu}(q,t).$$

Thus

$$H_\mu[X;q,1/t] = \sum_{\lambda\le\mu} s_\lambda\left[\frac{X}{t-1}\right]\eta_{\lambda\mu}(q,1/t)t^{|\lambda|}$$

and then we have

$$\widetilde{H}_\mu[X;q,t] = \sum_{\lambda\le\mu} s_\lambda\left[\frac{X}{t-1}\right]\eta_{\lambda\mu}(q,1/t)t^{|\lambda|+n(\mu)}.$$

This proves the first equation with

$$c_{\lambda\mu}(q,t) = \eta_{\lambda\mu}(q,1/t)t^{|\lambda|+n(\mu)}.$$

Next, now we have

$$\widetilde{H}_{\mu'}[X;q,t] = \sum_{\lambda\leq\mu} s_\lambda \left[\frac{X}{q-1}\right] c_{\lambda\mu}(t,q) = \sum_{\lambda\leq\mu}(-1)^{|\lambda|} s_{\lambda'} \left[\frac{X}{1-q}\right] c_{\lambda\mu}(t,q)$$

from which we derive that

$$\widetilde{H}_{\mu}[X;q,t] = \sum_{\lambda'\geq\mu}(-1)^{|\lambda|} s_{\lambda'} \left[\frac{X}{1-q}\right] c_{\lambda\mu'}(t,q)$$

or better

$$\widetilde{H}_{\mu}[X;q,t] = \sum_{\lambda\geq\mu}(-1)^{|\lambda|} s_{\lambda} \left[\frac{X}{1-q}\right] c_{\lambda'\mu'}(t,q).$$

This proves the second equation with

$$d_{\lambda\mu}(q,t) = (-1)^{|\lambda|} c_{\lambda'\mu'}(t,q).$$

The *uniqueness* assertion then follows from an elementary Linear Algebra result that holds true under such triangularity properties up to a normalization.

□

Finally, the fundamental symmetric functions operator *Nabla*, ∇, which was first introduced in [BG99], is defined as

$$\nabla\ \widetilde{H}_\lambda[X;q,t] = T_\lambda\ \widetilde{H}_\lambda[X;q,t].$$

Chapter 2

Plethystic Operators and the Haglung-Morse Zabrocki Conjectures

We start this chapter by introducing two operators $\mathbf{C}_a$ and $\mathbf{B}_a$, which are first introduced in [HMZ11], for a symmetric polynomial $P[X]$ as following:

$$\mathbf{C}_a P[X] = \left(-\frac{1}{q}\right)^{a-1} P\left[X - \frac{1-1/q}{z}\right] \Omega[zX]\bigg|_{z^a},$$

where

$$\Omega[zX] = \sum_{m\geq 0} z^m h_m[X]$$

is the generating function of the homogeneous symmetric functions with the alphabet X, or

$$\mathbf{C}_a P[X] = \left(-\frac{1}{q}\right)^{a-1} \sum_{k\geq 0} P\left[X - \frac{1-1/q}{z}\right] z^k h_k[X]\bigg|_{z^a}$$

and

$$\mathbf{B}_a P[X] = \omega\, \mathbf{B}_a^* \,\omega\, P[X]$$

where

$$\mathbf{B}_a^* P[X] = P\left[X - \frac{1-q}{z}\right] \Omega[zX]\bigg|_{z^a}.$$

Since $\omega P[X] = P[-\epsilon X]$, we also have

$$\mathbf{B}_a P[X] = \omega\, \mathbf{B}_a^*\, P[-\epsilon X].$$

Then

$$\mathbf{B}_a^*\, \omega\, P[X] = P\left[-\epsilon X + \epsilon \frac{1-q}{z}\right] \Omega[zX]\bigg|_{z^a},$$

and we finally have

$$\mathbf{B}_a P[X] = P\left[X + \epsilon \frac{1-q}{z}\right] \Omega[-\epsilon z X]\bigg|_{z^a}.$$

Recall that a composition of a parking function PF, $comp(PF)$ is the sequence $(p_1, p_2, \ldots, p_k)$, where p_i counts the number of cars in i^{th} part. Also, we denote a set of parking fucntions with n cars by $\mathcal{PF}_n$ and for any composition $p = (p_1, p_2, \ldots, p_k)$ and any symmetric polynomial $F[X]$, we use

$$\mathbf{C}_p\, F[X] = \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k} F[X].$$

The original Shuffle conjecture was introduced in [HHL$^+$05]. Now, we introduce one of the compositional refinements of the Shuffle conjecture due to Haglund-Morse-Zabrocki in [HMZ11].

Conjecture 2.0.1. For any composition $p = (p_1, p_2, \ldots, p_k)$ of n, we have

$$\nabla \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k} \mathbf{1} = \sum_{\substack{PF \in \mathcal{PF}_n \\ comp(PF) = (p_1, p_2, \ldots, p_k)}} t^{area(PF)}\, q^{dinv(PF)}\, Q_{ides(PF)}.$$

For the convenience, we will use plethystic notation for identities of symmetric function. The detail of this notation is introduced in [GH96]. The operator *"row adder"* for schur function is defined as

$$\mathbf{S}_a P[X] = P\left[X - \frac{1}{z}\right]\ \Omega[zX]\bigg|_{z^a}$$

where

$$\Omega[zX] = \sum_{m \geq 0} z^m\, h_m[X]$$

is the generating function of the homogeneous symmetric functions with the alphabet X.

Proposition 2.0.3. *For any integral vector* $p = (p_1, p_2, \ldots, p_k)$*,*

$$s_{p_1,p_2,\ldots,p_k}[X] = \Omega[Z_k X] \prod_{1\le i<j\le k} \big(1 - z_j/z_i\big)\Big|_{z_1^{p_1} z_2^{p_2} \ldots z_k^{p_k}}$$

where $Z_k = z_1 + z_2 + \cdots + z_k$*.*

Proof. It is well known that

$$s_{p_1,p_2,\ldots,p_k}[X] = \mathbf{S}_{p_1}\mathbf{S}_{p_2}\cdots\mathbf{S}_{p_k}\mathbf{1}.$$

By the definition of the row adder operator $\mathbf{S}$,

$$\begin{aligned}
\mathbf{S}_{p_1}\mathbf{S}_{p_2}\mathbf{1} = \mathbf{S}_{p_1}\Omega[z_2X]\Big|_{z_2^{p_2}} &= \Omega\left[z_2\big(X - \frac{1}{z_1}\big)\right]\Omega[z_1X]\Big|_{z_1^{p_1} z_2^{p_2}} \\
&= \Omega[(-z_2/z_1)]\,\Omega[z_1X + z_2X]\Big|_{z_1^{p_1} z_2^{p_2}} \\
&= (1 - z_2/z_1)\Omega[z_1X + z_2X]\Big|_{z_1^{p_1} z_2^{p_2}}.
\end{aligned}$$

After iteration, we can obtain

$$\mathbf{S}_{p_1}\mathbf{S}_{p_2}\cdots\mathbf{S}_{p_k}\mathbf{1} = \Omega[Z_kX] \prod_{1\le i<j\le n} \big(1 - z_j/z_i\big)\Big|_{z_1^{p_1} z_2^{p_2} \ldots z_k^{p_k}}.$$

Thus, we have

$$s_{p_1,p_2,\ldots,p_k}[X] = \mathbf{S}_{p_1}\mathbf{S}_{p_2}\cdots\mathbf{S}_{p_k}\mathbf{1} = \Omega[Z_kX] \prod_{1\le i<j\le n} \big(1 - z_j/z_i\big)\Big|_{z_1^{p_1} z_2^{p_2} \ldots z_k^{p_k}}.$$

□

The device θ_i acts on the operator $\mathbf{C}_p = \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}$ as the formula

$$\theta_i\mathbf{C}_p = \mathbf{C}_{p-e_i}$$

where e_i is the coordinate vector with 1 in the i^{th} position. Using this device, we have the following theorem.

Theorem 2.0.4. *For any composition* $(p_1, p_2, \ldots, p_k)$*,*

$$s_{p_1,p_2,\ldots,p_k}[X] = (-q)^{p_1+\cdots+p_k-k} \prod_{1\le i<j\le n} \big(1-\theta_j/q\theta_i\big)\,\mathbf{C}_p\,\mathbf{1}$$

where $s_{p_1,p_2,\ldots,p_k}[X]$ *is the schur function indexed by the composition* $(p_1, p_2, \ldots, p_k)$*.*

Proof. By the definition of the operator $\mathbf{C}_a$, for any symmetric function $F[X]$,

$$\mathbf{C}_a F[X] = \left(-\frac{1}{q}\right)^{a-1} F\left[X - \frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^a}.$$

Then we have

$$\begin{aligned}
&(-q)^{p_1+p_2-2}\,\mathbf{C}_{p_1}\mathbf{C}_{p_2}F[X] \\
&\quad= (-q)^{p_1-1}\mathbf{C}_{p_1}F\left[X-\frac{1-1/q}{z_2}\right]\Omega[z_2X]\Big|_{z_2^{p_2}} \\
&\quad= F\left[X-\frac{1-1/q}{z_1}-\frac{1-1/q}{z_2}\right]\Omega\left[z_2\big(X-\frac{1-1/q}{z_1}\big)\right]\Omega[z_1X]\Big|_{z_1^{p_1}z_2^{p_2}} \\
&\quad= F\left[X-\frac{1-1/q}{z_1}-\frac{1-1/q}{z_2}\right]\Omega\left[-z_2\frac{1-1/q}{z_1}\right]\Omega[z_1X+z_2X]\Big|_{z_1^{p_1}z_2^{p_2}}.
\end{aligned}$$

Using $\Omega\left[-z_2\frac{1-1/q}{z_1}\right] = \frac{1-z_2/z_1}{1-z_2/qz_1}$,

$$\begin{aligned}
&(-q)^{p_1+p_2-2}\,\mathbf{C}_{p_1}\mathbf{C}_{p_2}F[X] \\
&\quad= F\left[X-\frac{1-1/q}{z_1}-\frac{1-1/q}{z_2}\right]\Omega[z_1X+z_2X]\frac{1-z_2/z_1}{1-z_2/qz_1}\Big|_{z_1^{p_1}z_2^{p_2}}.
\end{aligned}$$

By iteration, we have

$$\begin{aligned}
&(-q)^{p_1+\cdots+p_k-k}\,\mathbf{C}_{p_1}\cdots\mathbf{C}_{p_k}F[X] \\
&\quad= F\left[X-\sum_{i=1}^{k}\frac{1-1/q}{z_i}\right]\Omega[Z_kX]\prod_{1\le i<j\le k}\frac{1-z_j/z_i}{1-z_j/qz_i}\Big|_{z_1^{p_1}\cdots z_k^{p_k}}
\end{aligned}$$

where $Z_k = z_1 + z_2 + \cdots + z_k$.

With $F[X]=1$,

$$(-q)^{(p_1-a_1)+\cdots+(p_k-a_k)-k}\,\mathbf{C}_{p_1-a_1}\cdots\mathbf{C}_{p_k-a_k}\mathbf{1}$$
$$=\prod_{1\le i<j\le k}\frac{1-z_j/z_i}{1-z_j/qz_i}\,\Omega[Z_kX]\Big|_{z_1^{p_1-a_1}z_2^{p_2-a_2}\cdots z_k^{p_k-a_k}}$$
$$=\prod_{1\le i<j\le k}\frac{1-z_j/z_i}{1-z_j/qz_i}\,\Omega[Z_kX]z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}\Big|_{z_1^{p_1}z_2^{p_2}\cdots z_k^{p_k}}.$$

By the definition of the device θ_i acting on the operator $\mathbf{C}_p=\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}$, we have

$$\theta_i^{a_i}\,\mathbf{C}_p=\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_i-a_i}\cdots\mathbf{C}_{p_k}$$

and

$$\theta_1^{a_1}\theta_2^{a_2}\cdots\theta_k^{a_k}\,\mathbf{C}_p=\mathbf{C}_{p_1-a_1}\mathbf{C}_{p_2-a_2}\cdots\mathbf{C}_{p_k-a_k}.$$

Thus, we obtain

$$(-q)^{p_1+\cdots+p_k-k}\,(-\theta_1/q)^{a_1}(-\theta_2/q)^{a_2}\cdots(-\theta_k/q)^{a_k}\mathbf{C}_{p_1}\cdots\mathbf{C}_{p_k}\mathbf{1}$$
$$=(-q)^{p_1+\cdots+p_k-k}\left(-\frac{1}{q}\right)^{a_1+\cdots+a_k}\mathbf{C}_{p_1-a_1}\cdots\mathbf{C}_{p_k-a_k}\mathbf{1}$$
$$=(-q)^{(p_1-a_1)+\cdots+(p_k-a_k)-k}\,\mathbf{C}_{p_1-a_1}\cdots\mathbf{C}_{p_k-a_k}\mathbf{1}$$
$$=\prod_{1\le i<j\le k}\frac{1-z_j/z_i}{1-z_j/qz_i}\,\Omega[Z_kX]z_1^{a_1}z_2^{a_2}\cdots z_k^{a_k}\Big|_{z_1^{p_1}z_2^{p_2}\cdots z_k^{p_k}}.$$

Therefore

$$(-q)^{p_1+\cdots+p_k-k}\prod_{1\le i<j\le k}(1-\theta_j/q\theta_i)\mathbf{C}_p\mathbf{1}=\prod_{1\le i<j\le k}(1-z_j/z_i)\Omega[Z_kX]\Big|_{z_1^{p_1}z_2^{p_2}\cdots z_k^{p_k}}.$$

Then, by Proposition 2.0.3, we have

$$(-q)^{p_1+\cdots+p_k-k}\prod_{1\le i<j\le k}(1-\theta_j/q\theta_i)\mathbf{C}_p\mathbf{1}=s_{p_1,p_2,\ldots,p_k}[X].$$

□

Example 2.0.1. Suppose $k = 2$ and $a \geq b \geq 1$. Then, by Theorem 2.0.4, we have

$$\begin{aligned} s_{a,b}[X] &= (-q)^{a+b-2}(1 - \theta_2/q\theta_1)\, \mathbf{C}_a \mathbf{C}_b\, \mathbf{1} \\ &= (-q)^{a+b-2}\big(\mathbf{C}_a \mathbf{C}_b \mathbf{1} - \mathbf{C}_{a+1} \mathbf{C}_{b-1} \mathbf{1}/q\big) \\ &= (-q)^{a+b-2}\big(\mathbf{C}_a \mathbf{C}_b \mathbf{1} - \mathbf{C}_{a+1} \mathbf{C}_{b-1} \mathbf{1}/q\big) \\ &= (-q)^{a+b-3}\big(\mathbf{C}_{a+1} \mathbf{C}_{b-1} \mathbf{1} - q\, \mathbf{C}_a \mathbf{C}_b \mathbf{1}\big). \end{aligned}$$

Example 2.0.2. Suppose $k = 3$ and $a \geq b \geq c \geq 1$. Again, by Theorem 2.0.4, we obtain

$$\begin{aligned} s_{a,b,c}[X] &= (-q)^{a+b+c-3}(1 - \theta_2/q\theta_1)(1 - \theta_3/q\theta_1)(1 - \theta_3/q\theta_2)\, \mathbf{C}_a \mathbf{C}_b \mathbf{C}_c\, \mathbf{1} \\ &= (-q)^{a+b+c-3}\big(\mathbf{C}_a \mathbf{C}_b \mathbf{C}_c \mathbf{1} - \mathbf{C}_{a+1} \mathbf{C}_{b-1} \mathbf{C}_c \mathbf{1}/q - \mathbf{C}_a \mathbf{C}_{b+1} \mathbf{C}_{c-1} \mathbf{1}/q \\ &\qquad - \mathbf{C}_{a+1} \mathbf{C}_b \mathbf{C}_{c-1} \mathbf{1}/q + \mathbf{C}_{a+1} \mathbf{C}_b \mathbf{C}_{c-1} \mathbf{1}/q^2 \\ &\qquad + \mathbf{C}_{a+2} \mathbf{C}_{b-1} \mathbf{C}_{c-1} \mathbf{1}/q^2 + \mathbf{C}_{a+1} \mathbf{C}_{b+1} \mathbf{C}_{c-2} \mathbf{1}/q^2 \\ &\qquad - \mathbf{C}_{a+2} \mathbf{C}_b \mathbf{C}_{c-2} \mathbf{1}/q^3\,\big). \end{aligned}$$

We can get an idea that by applying ∇ to both sides, we may be able to get a paking function interpretation of schur functions under ∇ operator assuming that the compositional refinement of the Shuffle conjecture 2.0.1 holds.

For example, for $a = 4$ and $b = 3$, we have

$$\begin{aligned} s_{4,3}[X] = (-q)^5\big(\mathbf{C}_4 \mathbf{C}_3 \mathbf{1} - \mathbf{C}_5 \mathbf{C}_2 \mathbf{1}/q\big) &= (-q)^5\big(\mathbf{C}_4 \mathbf{C}_3 \mathbf{1} - \mathbf{C}_5 \mathbf{C}_2 \mathbf{1}/q\big) \\ &= q^4\big(\mathbf{C}_5 \mathbf{C}_2 \mathbf{1} - q\, \mathbf{C}_4 \mathbf{C}_3 \mathbf{1}\big). \end{aligned}$$

Applying ∇ to both sides gives us

$$\nabla s_{4,3}[X] = q^4\big(\nabla \mathbf{C}_5 \mathbf{C}_2 \mathbf{1} - q \nabla \mathbf{C}_4 \mathbf{C}_3 \mathbf{1}\big).$$

Then, if we assume that the compositional refinement of the Shuffle conjecture 2.0.1 holds, we can get

$$\begin{aligned} \nabla s_{4,3}[X] = q^4\Bigg(&\sum_{PF \in \Pi[5,2]} t^{area(PF)}\, q^{dinv(PF)}\, Q_{ides(PF)} \\ &\qquad - q \sum_{PF \in \Pi[4,3]} t^{area(PF)}\, q^{dinv(PF)}\, Q_{ides(PF)} \Bigg) \end{aligned}$$

where $\Pi[5,2]$ is the set of parking functions whose diagonal composition is $[5,2]$ and $\Pi[4,3]$ is the set of parking functions whose diagonal composition is $[4,3]$. Hence, to get the combinatorial image of $s_{4,3}[X]$ under ∇ operator, we need to construct an injection ϕ from $\Pi[4,3]$ to $\Pi[5,2]$ which preserves the *area* and *ides*, but increases the *dinv* exactly one, and then identify the complementary collection $\Pi[5,2]\backslash\phi(\Pi[4,3])$, so that we can have the identity

$$\nabla s_{4,3}[X] = q^4 \sum_{PF \in \Pi[5,2]\backslash\phi(\Pi[4,3])} t^{area(PF)}\, q^{dinv(PF)}\, Q_{ides(PF)}.$$

Also, Example 2.0.2 suggests that the combinatorial image of $s_{a,b,c}[X]$ under nabla operator can be obtained by carrying our an “inclusion-exclusion” process on the collection of parking functions whose diagonal compositions are the indices of the operators occured in the equation.

In this work, we will introduce the injections yielding parking function settings for the nabla image of some cases of schur functions indexed by two-row or two-column partitions.

Chapter 3

Plethystic Operators Identities and Their Combinatorial Interpretation

In this chapter, we will introduce identities between symmetric function operators and their combinatorial interpretations.

First, we introduce the several different expressions for the definitions of **B** operator and **C** operator. For **B** operator, we defined as

$$\mathbf{B}_a P[X] = P\left[X + \epsilon\frac{1-q}{z}\right]\Omega[-\epsilon z X]\Big|_{z^a}.$$

Also, it may be expressed as

$$\mathbf{B}_a P[X] = \sum_{r\geq 0}(-1)^r e_{a+r}[X]\, h_r[X(1-q)]^{\perp} P[X].$$

Also, for **C** operator, we defined as

$$\begin{aligned}\mathbf{C}_a P[X] &= \left(-\frac{1}{q}\right)^{a-1} P\left[X - \frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^a} \\ &= \left(-\frac{1}{q}\right)^{a-1}\sum_{k\geq 0} P\left[X - \frac{1-1/q}{z}\right] z^k h_k[X]\Big|_{z^a},\end{aligned}$$

but we may also define as

$$\mathbf{C}_a P[X] = -q\, P\left[X + \epsilon\frac{1-q}{z}\right]\Omega\left[\epsilon\left(\frac{z}{q}\right)X\right]\Bigg|_{z^a}$$
$$= \left(-\frac{1}{q}\right)^{a-1}\sum_{r\geq 0} q^{-r} h_{a+r}[X]\, h_r[X(1-q)]^{\perp} P[X].$$

Using these definitions, we have various identities of operators.

Theorem 3.0.5. *For any integers $a, b \geq 0$, letting $n = a+b$, we have*

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \sum_{r=0}^{a} q^{a-r}(e_r[X]e_{n-r}[X] - e_{r-1}[X]e_{n-r+1}[X]).$$

Proof. We will use

$$\mathbf{B}^*_a P[X] = P\left[X - \frac{1-q}{z}\right]\Omega[zX]\Bigg|_{z^a}.$$

We have

$$\mathbf{B}^*_a\mathbf{1} = 1\Omega[zX]\Big|_{z^a} = h_a[X],$$

and since $\omega 1 = 1$, we have

$$\mathbf{B}_a\mathbf{1} = \omega h_a[X] = e_a[X].$$

Now, by the definition of $\mathbf{B}$ operator, we have

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \mathbf{B}_b e_a[X] = e_a\left[X + \epsilon\frac{1-q}{z}\right]\Omega[-\epsilon z X]\Bigg|_{z^b}$$
$$= \sum_{r=0}^{a} e_r[X]e_{a-r}\left[\epsilon\frac{1-q}{z}\right]\Omega[-\epsilon z X]\Bigg|_{z^b}$$
$$= \sum_{r=0}^{a} e_r[X]\frac{e_{a-r}[\epsilon(1-q)]}{z^{a-r}}\Omega[-\epsilon z X]\Bigg|_{z^b}$$
$$= \sum_{r=0}^{a} e_r[X](-1)^{a-r}e_{a-r}[(1-q)]e_{a+b-r}[X].$$

Since

$$e_m[1-q] = \begin{cases} 1 & m=0 \\ (-q)^{m-1}h_m[1-q] = (-q)^{m-1}(1-q) & m>0 \end{cases},$$

we have

$$e_{a-r}[1-q] = \begin{cases} 1 & a=r \\ (-q)^{a-r-1}(1-q) & a>r \end{cases}.$$

Thus

$$\begin{aligned}
&\mathbf{B}_a\mathbf{B}_b\mathbf{1} \\
&= \sum_{r=0}^{a-1} e_r[X]q^{a-r-1}(q-1)e_{n-r}[X] + e_a[X]e_b[X] \\
&= \sum_{r=0}^{a-1} e_r[X]q^{a-r}e_{n-r}[X] - \sum_{r=0}^{a-1} e_r[X]q^{a-r-1}e_{n-r}[X] + e_a[X]e_b[X] \\
&= \sum_{r=0}^{a-1} e_r[X]q^{a-r}e_{n-r}[X] - \sum_{r=1}^{a} e_{r-1}[X]q^{a-r}e_{n-r+1}[X] + e_a[X]e_b[X] \\
&= \sum_{r=0}^{a-1} q^{a-r}(e_r[X]e_{n-r}[X] - e_{r-1}[X]e_{n-r+1}[X]) + e_a[X]e_b[X] - e_{a-1}[X]e_{b+1}[X] \\
&= \sum_{r=0}^{a} q^{a-r}(e_r[X]e_{n-r}[X] - e_{r-1}[X]e_{n-r+1}[X])
\end{aligned}$$

as desired. □

Theorem 3.0.6. *For $n = a+b$ and $a \leq b$, we have*

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \sum_{r=0}^{a} q^{a-r} s_{2^r,1^{n-2r}}[X]$$

and if $a > b$, then we have

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \sum_{r=0}^{b} q^{a-r} s_{2^r,1^{n-2r}}[X] - \sum_{r=b+1}^{\lfloor \frac{n}{2} \rfloor} (q^{a-r} - q^{r-b-1}) s_{2^r,1^{n-2r}}[X].$$

Proof. From Theorem 3.0.5, the expression $e_r e_{n-r} - e_{r-1}e_{n-r+1}$ may be written as a Jacobi-Trudi determinant. In other words, if $n - r \geq r$, we have

$$e_r e_{n-r} - e_{r-1}e_{n-r+1} = \det \begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r \end{bmatrix}.$$

If $n-r=r-1$,

$$e_r e_{n-r} - e_{r-1}e_{n-r+1} = 0.$$

If $n-r<r-1$, then we have

$$e_r e_{n-r} - e_{r-1}e_{n-r+1} = -\det\begin{bmatrix} e_{r-1} & e_r \\ e_{n-r} & e_{n-r+1}\end{bmatrix}.$$

Also, note that we have the classical formula

$$\det\begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r\end{bmatrix} = s_{2^r,1^{n-2r}}[X].$$

Given this formula, if $a \leq b$, then we have

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \sum_{r=0}^{a} q^{a-r} s_{2^r,1^{n-2r}}[X].$$

Suppose $a>b$. We will divide into two subcases, one is when n is even and the other is when n is odd. If n is even, $n=2d$, then since we suppose $a>b$, $a>d$. Now we can split Theorem 3.0.5 into the two sums

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \sum_{r=0}^{d} q^{a-r}(e_r[X]e_{n-r}[X] - e_{r-1}[X]e_{n-r+1}[X]) + \sum_{s=d+1}^{a} q^{a-s}(e_s[X]e_{n-s}[X] - e_{s-1}[X]e_{n-s+1}[X]).$$

In the first sum, since $r \leq d$, $n-r \geq r$, so we have

$$e_r[X]e_{n-r}[X] - e_{r-1}[X]e_{n-r+1}[X] = \det\begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r\end{bmatrix}.$$

Also, in the second sum, since $s \geq d+1$, we have $s-1 \geq d = n-d \geq n-(s-1) = n-s+1$. Thus we have

$$e_s[X]e_{n-s}[X] - e_{s-1}[X]e_{n-s+1}[X] = -\det\begin{bmatrix} e_{s-1} & e_s \\ e_{n-s} & e_{n-s+1}\end{bmatrix}.$$

Now we have

$$\mathbf{B}_b\mathbf{B}_a\mathbf{1} = \sum_{r=0}^{d} q^{a-r}\det\begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r \end{bmatrix} - \sum_{s=d+1}^{a} q^{a-s}\det\begin{bmatrix} e_{s-1} & e_s \\ e_{n-s} & e_{n-s+1} \end{bmatrix}.$$

If we substitute s by $1+n-r$, the determinant in the second sum becomes the one in the first sum. Also, since $d+1 \le s \le a$, we have $d+1 \le 1+n-r \le a$, and it means $b+1 \le r \le d$. Therefore, we now have

$$\begin{aligned}\mathbf{B}_b\mathbf{B}_a\mathbf{1} &= \sum_{r=0}^{d} q^{a-r} s_{2^r,1^{n-2r}}[X] - \sum_{r=b+1}^{d} q^{a-(1+n-r)} s_{2^r,1^{n-2r}}[X] \\ &= \sum_{r=0}^{b} q^{a-r} s_{2^r,1^{n-2r}}[X] + \sum_{r=b+1}^{d} (q^{a-r} - q^{r-b-1}) s_{2^r,1^{n-2r}}[X].\end{aligned}$$

Next, suppose n is odd, say $n = 2d+1$. For all $r \le d$,

$$n - r \ge n - d = d + 1 \ge r + 1 > r.$$

If $s = d+1$, then

$$e_s[X]e_{n-s}[X] - e_{s-1}[X]e_{n-s+1}[X] = e_{d+1}[X]e_d[X] - e_d[X]e_{d+1}[X] = 0.$$

For all s such that $d+2 \le s \le a$, we have

$$s - 1 \ge d + 1 = n - d - 1 + 1 = n - d \ge n - (s-2) = n - s + 2 > n - s + 1.$$

Thus,

$$\begin{aligned}\mathbf{B}_b\mathbf{B}_a\mathbf{1} &= \sum_{r=0}^{d} q^{a-r}(e_r[X]e_{n-r}[X] - e_{r-1}[X]e_{n-r+1}[X]) \\ &\qquad + \sum_{s=d+1}^{a} q^{a-s}(e_s[X]e_{n-s}[X] - e_{s-1}[X]e_{n-s+1}[X]) \\ &= \sum_{r=0}^{d} q^{a-r}\det\begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r \end{bmatrix} - \sum_{s=d+2}^{a} q^{a-s}\det\begin{bmatrix} e_{s-1} & e_s \\ e_{n-s} & e_{n-s+1} \end{bmatrix}.\end{aligned}$$

Again, substituting s by $1+n-r$, $d+2 \le s \le a$ becomes

$$1+n-a = 1+b \le r = 1+n-s \le 1+n-(d+2) = 1+2d+1-d-2 = d,$$

i.e,

$$1+b \le r \le d.$$

Therefore, we have

$$\begin{aligned}
\mathbf{B}_b\mathbf{B}_a\mathbf{1} &= \sum_{r=0}^{d} q^{a-r} \det \begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r \end{bmatrix} - \sum_{r=b+1}^{d} q^{r-b-1} \det \begin{bmatrix} e_{n-r} & e_{n-r+1} \\ e_{r-1} & e_r \end{bmatrix} \\
&= \sum_{r=0}^{b} q^{a-r} s_{2^r,1^{n-2r}}[X] - \sum_{r=b+1}^{d} q^{r-b-1} s_{2^r,1^{n-2r}}[X] \\
&= \sum_{r=0}^{b} q^{a-r} s_{2^r,1^{n-2r}}[X] + \sum_{r=b+1}^{d} (q^{a-r} - q^{r-b-1}) s_{2^r,1^{n-2r}}[X]
\end{aligned}$$

as desired. □

Now, we will introduce an identity about $\mathbf{C}$ operators.

Theorem 3.0.7. *For all $a \le b$, where $n = a+b$,*

$$\mathbf{C}_b\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{n-2} \sum_{r=0}^{a} \left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X].$$

Proof. For any symmetric polynomial $P[X]$,

$$\mathbf{C}_a P[X] = \left(-\frac{1}{q}\right)^{a-1} P\left[X - \frac{1-1/q}{z}\right] \Omega[zX] \Big|_{z^a}.$$

Thus, we have

$$\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{a-1} h_a[X].$$

Then,

$$\begin{aligned}
\mathbf{C}_b\mathbf{C}_a\mathbf{1} &= \left(-\frac{1}{q}\right)^{a+b-2} h_a\left[X-\frac{1-1/q}{z}\right]\Omega[zX]\bigg|_{z^b}\\
&= \left(-\frac{1}{q}\right)^{a+b-2} \sum_{r=0}^{a} h_r[X]h_{a-r}\left[-\frac{1-1/q}{z}\right]\Omega[zX]\bigg|_{z^b}\\
&= \left(-\frac{1}{q}\right)^{a+b-2} \sum_{r=0}^{a} h_r[X]h_{a-r}[1-q]h_{a+b-r}[X]/q^{a-r}.
\end{aligned}$$

Using

$$h_m[1-q] = \begin{cases} 1-q & m>0 \\ 1 & m=0 \end{cases},$$

we have

$$
\begin{aligned}
&\mathbf{C}_b\mathbf{C}_a\mathbf{1}\\
&= \left(-\frac{1}{q}\right)^{n-2} h_a[X]h_b[X] + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a-1} h_r[X](1-q)h_{a+b-r}[X]/q^{a-r}\\
&= \left(-\frac{1}{q}\right)^{n-2} h_a[X]h_b[X] + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a-1} h_r[X]\left(\frac{1}{q}\right)^{a-r} h_{n-r}[X]\\
&\qquad\qquad - \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a-1} h_r[X]h_{n-r}[X]\left(\frac{1}{q}\right)^{a-r-1}\\
&= \left(-\frac{1}{q}\right)^{n-2} h_a[X]h_b[X] + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a-1} h_r[X]\left(\frac{1}{q}\right)^{a-r} h_{n-r}[X]\\
&\qquad\qquad - \left(-\frac{1}{q}\right)^{n-2}\sum_{r=1}^{a} h_{r-1}[X]h_{n-r+1}[X]\left(\frac{1}{q}\right)^{a-r}\\
&= \left(-\frac{1}{q}\right)^{n-2} h_a[X]h_b[X] + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=1}^{a-1} h_r[X]\left(\frac{1}{q}\right)^{a-r} h_{n-r}[X]\\
&\qquad\qquad + \left(-\frac{1}{q}\right)^{n-2}\left(\frac{1}{q}\right)^{a} h_n[X] - \left(-\frac{1}{q}\right)^{n-2} h_{a-1}[X]h_{n-a+1}[X]\\
&\qquad\qquad\qquad - \left(-\frac{1}{q}\right)^{n-2}\sum_{r=1}^{a-1} h_{r-1}[X]h_{n-r+1}[X]\left(\frac{1}{q}\right)^{a-r}\\
&= \left(-\frac{1}{q}\right)^{n-2}\left(\frac{1}{q}\right)^{a} h_n[X] + \left(-\frac{1}{q}\right)^{n-2}\big[h_a[X]h_b[X] - h_{a-1}[X]h_{n-a+1}[X]\big]\\
&\qquad\qquad + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=1}^{a-1}\left(\frac{1}{q}\right)^{a-r}\big[h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big]\\
&= \left(-\frac{1}{q}\right)^{n-2}\left(\frac{1}{q}\right)^{a} h_n[X]\\
&\qquad\qquad + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=1}^{a}\left(\frac{1}{q}\right)^{a-r}\big[h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big]\\
&= \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a}\left(\frac{1}{q}\right)^{a-r}\big[h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big].
\end{aligned}
$$

Now, recall that

$$s_\lambda = \det_{i,j} |h_{\lambda_i+j-i}|$$

and since $a \le b$ and $n = a + b$, for all $r \le a$, $r \le n - r$ and we have

$$h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X] = s_{n-r,r}[X],$$

thus

$$\mathbf{C}_b\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X]$$

as desired. □

We can also derive the analogue of Theorem 3.0.7 for the case $a > b$.

Theorem 3.0.8. *For all $a > b$, where $n = a + b$,*

$$\mathbf{C}_b\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{b}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X] + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=b+1}^{\lfloor \frac{n}{2}\rfloor}\left(\left(\frac{1}{q}\right)^{a-r} - \left(\frac{1}{q}\right)^{r-1-b}\right) s_{n-r,r}[X].$$

This is valid when $a = b$ and it becomes Theorem 3.0.7.

Proof. We will start from

$$\mathbf{C}_b\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a}\left(\frac{1}{q}\right)^{a-r}\big(h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big).$$

Suppose n is even, say $n = 2d$. Since $a > b$, we also have $a > d$. Now we have

$$\begin{aligned}\mathbf{C}_b\mathbf{C}_a\mathbf{1} &= \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{d}\left(\frac{1}{q}\right)^{a-r}\big(h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big) \\ &\quad + \left(-\frac{1}{q}\right)^{n-2}\sum_{s=d+1}^{a}\left(\frac{1}{q}\right)^{a-s}\big(h_s[X]h_{n-s}[X] - h_{s-1}[X]h_{n-s+1}[X]\big) \\ &= \left(-\frac{1}{q}\right)^{n-2}\left[\sum_{r=0}^{d}\left(\frac{1}{q}\right)^{a-r} S_{n-r,r}[X] - \sum_{s=d+1}^{a}\left(\frac{1}{q}\right)^{a-s} S_{s-1,n-s+1}[X]\right].\end{aligned}$$

Substituting s by $1+n-r$, we have

$$\mathbf{C}_b\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{n-2}\left[\sum_{r=0}^{d}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X] - \sum_{r=b+1}^{d}\left(\frac{1}{q}\right)^{a-(n+1-r)} s_{n-r,r}[X]\right]$$
$$= \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{b}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X]$$
$$+ \left(-\frac{1}{q}\right)^{n-2}\sum_{r=b+1}^{d}\left[\left(\frac{1}{q}\right)^{a-r} - \left(\frac{1}{q}\right)^{r-1-b}\right] s_{n-r,r}[X]$$

as desired.

Now, suppose n is odd, say $n = 2d+1$. Note that we still have $a > d$. If $r = d+1$, then $n-r = 2d+1-(d+1) = d$, and in this case,

$$h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X] = h_{d+1}[X]h_d[X] - h_d[X]h_{d+1}[X] = 0.$$

Thus,

$$\mathbf{C}_b\mathbf{C}_a\mathbf{1} = \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{a}\left(\frac{1}{q}\right)^{a-r}\big(h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big)$$
$$= \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{d}\left(\frac{1}{q}\right)^{a-r}\big(h_r[X]h_{n-r}[X] - h_{r-1}[X]h_{n-r+1}[X]\big)$$
$$+ \left(-\frac{1}{q}\right)^{n-2}\sum_{s=d+2}^{a}\left(\frac{1}{q}\right)^{a-s}\big(h_s[X]h_{n-s}[X] - h_{s-1}[X]h_{n-s+1}[X]\big)$$
$$= \left(-\frac{1}{q}\right)^{n-2}\left[\sum_{r=0}^{d}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X] - \sum_{s=d+2}^{a}\left(\frac{1}{q}\right)^{a-s} s_{s-1,n-s+1}[X]\right].$$

Again, substituting s by $n+1-r$,

$$\begin{aligned}\mathbf{C}_b\mathbf{C}_a\mathbf{1} &= \left(-\frac{1}{q}\right)^{n-2}\left[\sum_{r=0}^{d}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X] - \sum_{r=b+1}^{d}\left(\frac{1}{q}\right)^{r-1-b} s_{n-r,r}[X]\right] \\ &= \left(-\frac{1}{q}\right)^{n-2}\sum_{r=0}^{b}\left(\frac{1}{q}\right)^{a-r} s_{n-r,r}[X] \\ &\quad + \left(-\frac{1}{q}\right)^{n-2}\sum_{r=b+1}^{d}\left[\left(\frac{1}{q}\right)^{a-r} - \left(\frac{1}{q}\right)^{r-1-b}\right] s_{n-r,r}[X]\end{aligned}$$

Therefore, Theorem 3.0.8 is valid in this case.

Suppose $a=b$. Then, the summation in the second term doesn't have any terms and the first term is same as Theorem 3.0.7, so our proof is completed.

□

For $E_1, E_2, \cdots, E_k$ given expressions and $P[X]$ a symmetric polynomial, we set

$$P^{(r_1,r_2,\cdots,r_k)}[X] \;=\; P[X+E_1u_1+E_2u_2+\cdots+E_ku_k]\Big|_{u_1^{r_1}u_2^{r_2}\cdots u_k^{r_k}}.$$

The important property is that if

$$Q^{(r_1)}[X] \;=\; P[X+E_1u_1]\Big|_{u_1^{r_1}},$$

then

$$Q^{(r_1)}[X+E_2u_2]\Big|_{u_2^{r_2}} \;=\; P[X+E_1u_1+E_2u_2]\Big|_{u_1^{r_1}u_2^{r_2}} \;=\; P^{(r_1,r_2)}[X].$$

Using this expression, now we introduce an important identity about the combination of $\mathbf{C}$ operators.

Theorem 3.0.9. *For $b \le a-1$, we have*

$$q(\mathbf{C}_b\mathbf{C}_a + \mathbf{C}_{a-1}\mathbf{C}_{b+1}) = \mathbf{C}_a\mathbf{C}_b + \mathbf{C}_{b+1}\mathbf{C}_{a-1}.$$

Proof. For any homogeneous symmetric function $P[X]$, we have

$$\begin{aligned}
\mathbf{C}_a P[X] &= \left(-\frac{1}{q}\right)^{a-1} P\left[X - \frac{1-1/q}{z_1}\right] \Omega[z_1 X]\Big|_{z_1^a} \\
&= \left(-\frac{1}{q}\right)^{a-1} \sum_{r_1 \geq 0} P^{r_1}[X] \frac{1}{z_1^{r_1}} \Omega[z_1 X]\Big|_{z_1^a} \\
&= \left(-\frac{1}{q}\right)^{a-1} \sum_{r_1 \geq 0} P^{r_1}[X] h_{r_1+a}[X].
\end{aligned}$$

Thus, using the definition again,

$$\begin{aligned}
&\mathbf{C}_b \mathbf{C}_a P[X] \\
&\quad = \left(-\frac{1}{q}\right)^{a+b-2} \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] \frac{1}{z_2^{r_2}} h_{r_1+a}[X + \frac{1-q}{q z_2}] \Omega[z_2 X]\Big|_{z_2^b} \\
&\quad = \left(-\frac{1}{q}\right)^{a+b-2} \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] \frac{1}{z_2^{r_2}} \sum_{s=0} h_{r_1+a-s}[X] h_s[1-q] \frac{1}{q^s z_2^s} \Omega[z_2 X]\Big|_{z_2^b} \\
&\quad = \left(-\frac{1}{q}\right)^{a+b-2} \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] \sum_{s=0}^{r_1+a} h_{r_1+a-s}[X] h_s[1-q] \frac{1}{q^s} h_{r_2+s+b}[X].
\end{aligned}$$

Using the fact that

$$h_s[1-q] = \begin{cases} 1-q & s > 0 \\ 1 & s = 0 \end{cases},$$

we can also write

$$\begin{aligned}
&\mathbf{C}_b\mathbf{C}_a P[X] \\
&\quad = \left(-\frac{1}{q}\right)^{a+b-2}(1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]\sum_{s=1}^{r_1+a} h_{r_1+a-s}[X]\frac{1}{q^s}h_{r_2+s+b}[X] \\
&\qquad + \left(-\frac{1}{q}\right)^{a+b-2}\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]h_{r_1+a}[X]h_{r_2+b}[X] \\
&\quad = \left(-\frac{1}{q}\right)^{a+b-2}(1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]\sum_{u=1+r_2+b}^{r_1+r_2+a+b} h_{r_1+r_2+a+b-u}[X]\frac{1}{q^{u-r_2-b}}h_u[X] \\
&\qquad + \left(-\frac{1}{q}\right)^{a+b-2}\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]h_{r_1+a}[X]h_{r_2+b}[X] \\
&\quad = (-1)^b\left(-\frac{1}{q}\right)^{a-2}(1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]q^{r_2}\sum_{u=1+r_2+b}^{r_1+r_2+a+b} h_{r_1+r_2+a+b-u}[X]h_u\left[\frac{X}{q}\right] \\
&\qquad + \left(-\frac{1}{q}\right)^{a+b-2}\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]h_{r_1+a}[X]h_{r_2+b}[X].
\end{aligned}$$

Notice that we have used the substitution $u = r_2 + s + b$ in the range $1 + r_2 + b \leq u \leq r_1 + r_2 + a + b$ and set $s = u - r_2 - b$, giving $r_1 + a - s = r_1 + a - (u - r_2 - b) = r_1 + r_2 + a + b - u$.

Then we can write

$$\begin{aligned}
&\sum_{u=1+r_2+b}^{r_1+r_2+a+b} h_{r_1+r_2+a+b-u}[X]h_u\left[\frac{X}{q}\right] \\
&\qquad = h_{r_1+r_2+a+b}\left[X\frac{1+q}{q}\right] - \sum_{u=0}^{r_2+b} h_{r_1+r_2+a+b-u}[X]h_u\left[\frac{X}{q}\right].
\end{aligned}$$

so we can decompose $\mathbf{C}_b\mathbf{C}_a P[X]$ into three terms

$$\mathbf{C}_b\mathbf{C}_a P[X] = \left(-\frac{1}{q}\right)^{a+b-2} A[P;b,a] + q^{2-a}B[P;a+b] - \left(-\frac{1}{q}\right)^{a+b-2} C[P;b,a]$$

with

$$A[P;b,a] = \sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]h_{r_1+a}[X]h_{r_2+b}[X],$$

$$B[P;a+b] = (1-q)(-1)^{a+b-2} \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] q^{r_2} h_{r_1+r_2+a+b}\left[X\frac{1+q}{q}\right],$$

and

$$C[P;b,a] = q^b(1-q) \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] q^{r_2} \sum_{u=0}^{r_2+b} h_{r_1+r_2+a+b-u}[X] h_u\left[\frac{X}{q}\right],$$

since

$$(-1)^b \left(-\frac{1}{q}\right)^{a-2} = q^{2-a}(-1)^{a+b-2}.$$

Next, note that we have the identity

$$q(q^{2-a}+q^{2-b-1}) - (q^{2-b}+q^{2-a+1}) = q(q^{2-a}+q^{2-b-1}) - q(q^{2-b-1}+q^{2-a}) = 0,$$

so

$$q^{2-a}B[P;a+b] = 0.$$

Thus, omitting the common factor $\left(-\frac{1}{q}\right)^{a+b-2}$, to prove the theorem, we need to show that

$$\begin{aligned} q(A[P;b,a] - C[P;b,a] + A[P;a-1,b+1] - C[P;a-1,b+1]) \\ = A[P;a,b] - C[P;a,b] + A[P;b+1,a-1] - C[P;b+1,a-1]. \end{aligned} \tag{3.1}$$

We can write

$$\begin{aligned} qA[P;b,a] &= q \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] h_{r_1+a}[X] h_{r_2+b}[X] \\ &= q \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] \frac{1}{z_1^{r_1} z_2^{r_2}} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}, \end{aligned}$$

$$\begin{aligned} qA[P;a-1,b+1] &= q \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] h_{r_2+b+1}[X] h_{r_1+a-1}[X] \\ &= q \sum_{r_1 \geq 0} \sum_{r_2 \geq 0} P^{r_1,r_2}[X] \frac{1}{z_1^{r_1-1} z_2^{r_2+1}} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}, \end{aligned}$$

$$A[P;a,b] = \sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X] h_{r_2+b}[X] h_{r_1+a}[X]$$
$$= \sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X] \frac{1}{z_1^{r_1} z_2^{r_2}} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b},$$

and

$$A[P;b+1,a-1] = \sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X] h_{r_1+a-1}[X] h_{r_2+b+1}[X]$$
$$= \sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X] \frac{1}{z_1^{r_1-1} z_2^{r_2+1}} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}.$$

Thus

$$qA[P;b,a] - A[P;a,b] = (q-1)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X] \frac{1}{z_1^{r_1} z_2^{r_2}} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}$$
$$= (q-1) P\left[X + \frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right] \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}$$

and

$$qA[P;a-1,b+1] - A[P;b+1,a-1]$$
$$= (q-1)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X] \frac{1}{z_1^{r_1} z_2^{r_2}} \frac{z_1}{z_2} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}$$
$$= (q-1) P\left[X + \frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right] \frac{z_1}{z_2} \Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}.$$

Likewise, we can get

$$\begin{aligned} qC[P;b,a] &- C[P;b+1,a-1] \\ &= q^{b+1}(1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]q^{r_2}\sum_{u=0}^{r_2+b} h_{r_1+r_2+a+b-u}[X]h_u\left[\frac{X}{q}\right] \\ &\quad - q^{b+1}(1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]q^{r_2}\sum_{u=0}^{r_2+b+1} h_{r_1+r_2+a+b-u}[X]h_u\left[\frac{X}{q}\right] \\ &= -q^{b+1}(1-q)\left(\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]q^{r_2}h_{r_1+r_2+a+b-r_2-b-1}[X]h_{r_2+b+1}\left[\frac{X}{q}\right]\right). \end{aligned}$$

Thus

$$\begin{aligned} qC[P;b,a] &- C[P;b+1,a-1] \\ &= -(1-q)\left(\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]h_{r_1+a-1}[X]h_{r_2+b+1}[X]\right) \\ &= (q-1)P\left[X+\frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right]\frac{z_1}{z_2}\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}. \end{aligned}$$

Finally

$$\begin{aligned} C[P;a,b] &- qC[P;a-1,b+1] \\ &= q^a(1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]q^{r_2}h_{r_1+b}[X]h_{r_2+a}\left[\frac{X}{q}\right] \\ &= (1-q)\sum_{r_1\geq 0}\sum_{r_2\geq 0} P^{r_1,r_2}[X]\frac{1}{z_1^{r_1}z_2^{r_2}}\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b} \\ &= (1-q)P\left[X+\frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right]\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}. \end{aligned}$$

Recapitulating we have proved that

$$\begin{aligned} qA[P;b,a] &- A[P;a,b] \\ &= (q-1)P\left[X+\frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right]\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} qA[P;a-1,b+1] &- A[P;b+1,a-1] \\ &= (q-1)P\left[X+\frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right]\frac{z_1}{z_2}\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} qC[P;b,a] &- C[P;b+1,a-1] \\ &= (q-1)P\left[X+\frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right]\frac{z_1}{z_2}\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} qC[P;a-1,b+1] &- C[P;a,b] \\ &= (q-1)P\left[X+\frac{1-q}{q}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\right]\Omega[(z_1+z_2)X]\Big|_{z_1^a z_2^b}. \end{aligned} \tag{3.5}$$

Thus subtracting 3.5 from 3.2 gives

$$\big(qA[P;b,a]-A[P;a,b]\big)-\big(qC[P;a-1,b+1]-C[P;a,b]\big)=0$$

and subtracting 3.4 from 3.3 gives

$$\big(qA[P;a-1,b+1]-A[P;b+1,a-1]\big)-\big(qC[P;b,a]-C[P;b+1,a-1]\big)=0.$$

Now we have the desired conclusion 3.1 which was

$$\begin{aligned} q\big(A[P;b,a]-C[P;b,a]+A[P;a-1,b+1]-C[P;a-1,b+1]\big) \\ = A[P;a,b]-C[P;a,b]+A[P;b+1,a-1]-C[P;b+1,a-1]. \end{aligned}$$

□

Applying ω to the theorem above and replacing q by $1/q$ gives us the following identity on $\mathbf{B}$ operator.

Corollary 1. *For $a,b\in\mathbb{Z}$ with $b\le a-1$,*

$$\mathbf{B}_a\mathbf{B}_b - q\mathbf{B}_{a+1}\mathbf{B}_{b-1} = q\mathbf{B}_b\mathbf{B}_a - q\mathbf{B}_{b-1}\mathbf{B}_{a+1}.$$

In case $b=a+1$,

$$q\mathbf{B}_{a+1}\mathbf{B}_a = \mathbf{B}_a\mathbf{B}_{a+1}.$$

Now, we will introduce identities about the combination of **C** and **B** operators.

Theorem 3.0.10. *For* $a, b \in \mathbb{Z}$*,*

$$\mathbf{B}_a \mathbf{C}_b = q\mathbf{C}_b \mathbf{B}_a.$$

Proof. To prove this, we will use the definitions of **B** and **C** operators as

$$\mathbf{B}_a P[X] = P\left[X + \epsilon \frac{1-q}{z}\right] \Omega[-\epsilon z X]\Big|_{z^a}$$

and

$$\mathbf{C}_b P[X] = \left(-\frac{1}{q}\right)^{b-1} P\left[X - \frac{1-1/q}{z}\right] \Omega[zX]\Big|_{z^b}.$$

Given this, we have

$$\begin{aligned}
&\mathbf{C}_b \mathbf{B}_a P[X] \\
&= \mathbf{C}_b P\left[X + \epsilon \frac{1-q}{z_1}\right] \Omega[-\epsilon z_1 X]\Big|_{z_1^a} \\
&= \left(-\frac{1}{q}\right)^{b-1} P\left[X - \frac{1-1/q}{z_2} + \epsilon \frac{1-q}{z_1}\right] \Omega\left[-\epsilon z_1 \left(X - \frac{1-1/q}{z_2}\right)\right] \Omega[z_2 X]\Big|_{z_1^a z_2^b} \\
&= \left(-\frac{1}{q}\right)^{b-1} P\left[X - \frac{1-1/q}{z_2} + \epsilon \frac{1-q}{z_1}\right] \Omega\left[\epsilon z_1 \frac{1-1/q}{z_2}\right] \Omega[-\epsilon z_1 X] \Omega[z_2 X]\Big|_{z_1^a z_2^b}.
\end{aligned}$$

Since

$$\Omega\left[\epsilon z_1 \frac{1-1/q}{z_2}\right] = \frac{1-\epsilon z_1/z_2 q}{1-\epsilon z_1/z_2} = \frac{1}{q}\frac{z_1 + z_2 q}{z_1 + z_2},$$

we get

$$\begin{aligned}
&\mathbf{C}_b \mathbf{B}_a P[X] \\
&\quad = \left(-\frac{1}{q}\right)^{b-1} \frac{1}{q}\frac{z_1 + z_2 q}{z_1 + z_2} P\left[X - \frac{1-1/q}{z_2} + \epsilon \frac{1-q}{z_1}\right] \Omega[-\epsilon z_1 X] \Omega[z_2 X]\Big|_{z_1^a z_2^b}.
\end{aligned}$$

On the other hand, we also get

$$\begin{aligned}
&\mathbf{B}_a\mathbf{C}_b P[X] \\
&= \left(-\frac{1}{q}\right)^{b-1} \mathbf{B}_a P\left[X - \frac{1-1/q}{z_2}\right] \Omega[z_2X]\bigg|_{z_2^b} \\
&= \left(-\frac{1}{q}\right)^{b-1} P\left[X + \epsilon\frac{1-q}{z_1} - \frac{1-1/q}{z_2}\right] \Omega\left[z_2\left(X + \epsilon\frac{1-q}{z_1}\right)\right] \Omega[-\epsilon z_1 X]\bigg|_{z_1^a z_2^b} \\
&= \left(-\frac{1}{q}\right)^{b-1} P\left[X + \epsilon\frac{1-q}{z_1} - \frac{1-1/q}{z_2}\right] \Omega\left[z_2\epsilon\frac{1-q}{z_1}\right] \Omega[-\epsilon z_1 X]\Omega[z_2X]\bigg|_{z_1^a z_2^b}.
\end{aligned}$$

Since

$$\Omega\left[z_2\epsilon\frac{1-q}{z_1}\right] = \frac{1-\epsilon q z_2/z_1}{1-\epsilon z_2/z_1} = \frac{z_1+qz_2}{z_1+z_2},$$

we can derive that

$$\mathbf{B}_a\mathbf{C}_b P[X] = \left(-\frac{1}{q}\right)^{b-1} \frac{z_1+qz_2}{z_1+z_2} P\left[X + \epsilon\frac{1-q}{z_1} - \frac{1-1/q}{z_2}\right] \Omega[-\epsilon z_1 X]\Omega[z_2X]\bigg|_{z_1^a z_2^b}.$$

Therefore, our proof is completed. □

In Chapter 2, we defined the operator *"row adder"* for schur function as

$$\mathbf{S}_a P[X] = P\left[X - \frac{1}{z}\right] \Omega[zX]\bigg|_{z^a}$$

where

$$\Omega[zX] = \sum_{m\geq 0} z^m h_m[X]$$

is the generating function of the homogeneous symmetric functions with the alphabet X.

First, we will introduce the other expressions for **S** operator.

Proposition 3.0.11.

$$\begin{aligned}\mathbf{S}_a P[X] &= \sum_{k\geq 0}(-1)^k h_{a+k}[X] e_k^\perp P[X] \\ &= (-q)^{a-1}\sum_{k\geq 0}\mathbf{C}_{a+k} e_k^\perp P[X] \\ &= \sum_{k\geq 0}(-q)^k \tilde{\mathbf{B}}_{a+k} e_k^\perp P[X] \\ &= \sum_{k\geq 0}(-q)^k \omega \mathbf{B}_{a+k}\omega e_k^\perp P[X].\end{aligned}$$

Proof. We will start from the definition of $\mathbf{S}_a$ operator.

$$\begin{aligned}\mathbf{S}_a P[X] &= P\left[X - \frac{1-1/q}{z} - \frac{1}{qz}\right]\Omega[zX]\bigg|_{z^a} \\ &= \sum_{k\geq 0}(e_k^\perp P)\left[X - \frac{1-1/q}{z}\right](-1)^k \frac{1}{q^k z^k}\Omega[zX]\bigg|_{z^a} \\ &= \sum_{k\geq 0}(e_k^\perp P)\left[X - \frac{1-1/q}{z}\right](-1)^k \frac{1}{q^k}\Omega[zX]\bigg|_{z^{a+k}} \\ &= (-q)^{a-1}\sum_{k\geq 0}\left(-\frac{1}{q}\right)^{a+k-1}(e_k^\perp P)\left[X - \frac{1-1/q}{z}\right]\Omega[zX]\bigg|_{z^{a+k}} \\ &= (-q)^{a-1}\sum_{k\geq 0}\mathbf{C}_{a+k} e_k^\perp P[X].\end{aligned}$$

Note that we can also write

$$
\begin{aligned}
\mathbf{S}_a P[X] &= P\left[X - \frac{1-q}{z} - \frac{q}{z}\right]\Omega[zX]\bigg|_{z^a} \\
&= \sum_{k\geq 0}(e_k^{\perp}P)\left[X - \frac{1-q}{z}\right](-1)^k\,\frac{q^k}{z^k}\,\Omega[zX]\bigg|_{z^a} \\
&= \sum_{k\geq 0}(e_k^{\perp}P)\left[X - \frac{1-q}{z}\right](-q)^k\,\Omega[zX]\bigg|_{z^{a+k}} \\
&= \sum_{k\geq 0}(-q)^k(e_k^{\perp}P)\left[X - \frac{1-q}{z}\right]\Omega[zX]\bigg|_{z^{a+k}} \\
&= \sum_{k\geq 0}(-q)^k\,\widetilde{\mathbf{B}}_{a+k}\,e_k^{\perp}P[X].
\end{aligned}
$$

Using the fact that

$$\mathbf{B}_a = \omega\widetilde{\mathbf{B}}_a\omega,$$

we finally obtain that

$$\mathbf{S}_a P[X] = \sum_{k\geq 0}(-q)^k\,\omega\mathbf{B}_{a+k}\omega\,e_k^{\perp}P[X].$$

□

Before we introduce identities about **S** operator related to the other operators, we will introduce an identity of **C** operator and use it for identities about **S** operator.

Proposition 3.0.12. *For all k and $a \geq 1$, we have*

$$e_k^{\perp}\mathbf{C}_a = \mathbf{C}_a e_k^{\perp} - \frac{1}{q}\mathbf{C}_{a-1}e_{k-1}^{\perp},$$

provided we set $e_k^{\perp} = 0$ for $k < 0$.

Proof. First, we will prove that

$$e_k^{\perp}H_a^q = H_a^q e_k^{\perp} + H_{a-1}^q e_{k-1}^{\perp} \tag{3.6}$$

with the Hall Littlewood operator H_a defined as

$$H_a^q P[X] = P\left[X - \frac{1-q}{z}\right]\Omega[zX]\bigg|_{z^a}.$$

Set

$$\Gamma(z) = \sum_{r\geq 0} (-z)^r e_r^{\perp}$$

and note that we have

$$\Gamma(z)P[X] = P[X - z].$$

Given this, we have

$$\begin{aligned}
(-1)^k e_k^{\perp} H_a^q P[X] &= \Gamma(z_2) H_a^q P[X]\Big|_{z_2^k} \\
&= P\left[X - \frac{1}{z_2} - \frac{1-q}{z_1}\right] \Omega[z_1(X - z_2)]\Big|_{z_1^a z_2^k} \\
&= P\left[X - \frac{1}{z_2} - \frac{1-q}{z_1}\right] \Omega[z_1 X](1 - z_1 z_2)\Big|_{z_1^a z_2^k} \\
&= P\left[X - \frac{1}{z_2} - \frac{1-q}{z_1}\right] \Omega[z_1 X]\Big|_{z_1^a z_2^k} \\
&\qquad - P\left[X - \frac{1}{z_2} - \frac{1-q}{z_1}\right] \Omega[z_1 X]\Big|_{z_1^{a-1} z_2^{k-1}} \\
&= (-1)^k H_a^q e_k^{\perp} P[X] - (-1)^{k-1} H_{a-1}^q e_{k-1}^{\perp} P[X].
\end{aligned}$$

This proves 3.6 and by the replacement $q \to 1/q$, we obtain

$$e_k^{\perp} H_a^{\frac{1}{q}} = H_a^{\frac{1}{q}} e_k^{\perp} + H_{a-1}^{\frac{1}{q}} e_{k-1}^{\perp}, \tag{3.7}$$

and since

$$\mathbf{C}_a = (-1/q)^{a-1} H_a^{\frac{1}{q}},$$

by multiplying both sides of 3.7 by $(-1/q)^{a-1}$, we get

$$e_k^{\perp} \left(-\frac{1}{q}\right)^{a-1} H_a^{\frac{1}{q}} = \left(-\frac{1}{q}\right)^{a-1} H_a^{\frac{1}{q}} e_k^{\perp} + \left(-\frac{1}{q}\right)^{a-1} H_{a-1}^{\frac{1}{q}} e_{k-1}^{\perp}$$

or better

$$e_k^{\perp} \left(-\frac{1}{q}\right)^{a-1} H_a^{\frac{1}{q}} = \left(-\frac{1}{q}\right)^{a-1} H_a^{\frac{1}{q}} e_k^{\perp} + \left(-\frac{1}{q}\right) \left(-\frac{1}{q}\right)^{a-2} H_{a-1}^{\frac{1}{q}} e_{k-1}^{\perp}$$

and our proof is completed. □

We introduced the expansion of schur function in terms of $\mathbf{C}$ operator in Theorem 2.0.4. We will prove the expansion using Proposition 3.0.12.

Theorem 3.0.13. *For all $a > b+1$, we have*

$$\left(-\frac{1}{q}\right)^{a+b-3} s_{a-1,b+1}[X] = \mathbf{C}_a\mathbf{C}_b\mathbf{1} - q\mathbf{C}_{a-1}\mathbf{C}_{b+1}\mathbf{1}.$$

Proof. We will start from

$$s_{a-1,b+1}[X] = \mathbf{S}_{a-1}\mathbf{S}_{b+1}\mathbf{1}.$$

Proposition 3.0.11 gives

$$\mathbf{S}_{b+1}\mathbf{1} = (-q)^b\mathbf{C}_{b+1}\mathbf{1}$$

and using Proposition 3.0.11 and Proposition 3.0.12, we have

$$\begin{aligned} s_{a-1,b+1}[X] &= (-q)^{a+b-2}\sum_{k\geq 0}\mathbf{C}_{a-1+k}e_k^{\perp}\mathbf{C}_{b+1}\mathbf{1} \\ &= (-q)^{a+b-2}\sum_{k\geq 0}\mathbf{C}_{a-1+k}\left(\mathbf{C}_{b+1}e_k^{\perp}\mathbf{1} - \frac{1}{q}\mathbf{C}_b e_{k-1}^{\perp}\mathbf{1}\right) \\ &= (-q)^{a+b-2}\left(\mathbf{C}_{a-1}\mathbf{C}_{b+1}\mathbf{1} - \frac{1}{q}\mathbf{C}_a\mathbf{C}_b\mathbf{1}\right). \end{aligned}$$

Dividing both sides by $(-q)^{a+b-3}$ completes our proof. □

Example 2.0.2 shows the expansion of schur function with three parts in terms of $\mathbf{C}$ operators. Now, we will see one example of the expansion of schur function with three parts in terms of $\mathbf{C}$ operators using Proposition 3.0.12.

Proposition 3.0.14. *We have*

$$\left(\frac{1}{q}\right)^2 s_{3,2,1}[X] = -q\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1\mathbf{1} + (1-q)\mathbf{C}_4\mathbf{C}_2\mathbf{1} + \mathbf{C}_5\mathbf{C}_1\mathbf{1} - q\mathbf{C}_3\mathbf{C}_3\mathbf{1} + \mathbf{C}_4\mathbf{C}_1\mathbf{C}_1\mathbf{1}.$$

Proof. From Theorem 3.0.13, we derive for $a = 0$ and $b = 0$

$$s_{2,1}[X] = \mathbf{C}_3\mathbf{C}_0\mathbf{1} - q\mathbf{C}_2\mathbf{C}_1\mathbf{1}$$

and $\mathbf{C}_0\mathbf{1} = -q\mathbf{1}$ gives

$$\left(-\frac{1}{q}\right) s_{2,1}[X] = \mathbf{C}_3\mathbf{1} + \mathbf{C}_2\mathbf{C}_1\mathbf{1}.$$

Given this, we have

$$\frac{1}{q^2}s_{3,2,1}[X] = \left(-\frac{1}{q}\right)\mathbf{S}_3\big(\mathbf{C}_3\mathbf{1} + \mathbf{C}_2\mathbf{C}_1\mathbf{1}\big).$$

Since Proposition 3.0.11 for $a = 3$ reduces to

$$\mathbf{S}_3 = (-q)^2 \sum_{k\geq 0} \mathbf{C}_{3+k} e_k^{\perp},$$

we have

$$\begin{aligned}\frac{1}{q^2}s_{3,2,1}[X] &= \left(-\frac{1}{q}\right)(-q)^2 \sum_{k\geq 0} \mathbf{C}_{3+k} e_k^{\perp}\big(\mathbf{C}_3\mathbf{1} + \mathbf{C}_2\mathbf{C}_1\mathbf{1}\big) \\ &= (-q)\sum_{k\geq 0}\big(\mathbf{C}_{3+k}e_k^{\perp}\mathbf{C}_3\mathbf{1} + \mathbf{C}_{3+k}e_k^{\perp}\mathbf{C}_2\mathbf{C}_1\mathbf{1}\big).\end{aligned} \tag{3.8}$$

From Proposition 3.0.12, we can derive that

$$e_k^{\perp}\mathbf{C}_3\mathbf{1} = \mathbf{C}_3 e_k^{\perp}\mathbf{1} - \frac{1}{q}\mathbf{C}_2 e_{k-1}^{\perp}\mathbf{1} = \begin{cases} \mathbf{C}_3\mathbf{1} & k = 0 \\ -\frac{1}{q}\mathbf{C}_2\mathbf{1} & k = 1 \\ 0 & k > 1 \end{cases}$$

and

$$e_k^{\perp}\mathbf{C}_2\mathbf{C}_1\mathbf{1} = \mathbf{C}_2 e_k^{\perp}\mathbf{C}_1\mathbf{1} - \frac{1}{q}\mathbf{C}_1 e_{k-1}^{\perp}\mathbf{C}_1\mathbf{1} = \begin{cases} \mathbf{C}_2\mathbf{C}_1\mathbf{1} & k = 0 \\ \mathbf{C}_2 e_1^{\perp}\mathbf{C}_1\mathbf{1} - \frac{1}{q}\mathbf{C}_1\mathbf{C}_1\mathbf{1} & k = 1 \\ -\frac{1}{q}\mathbf{C}_1 e_1^{\perp}\mathbf{C}_1\mathbf{1} & k = 2 \\ 0 & k > 2 \end{cases}.$$

Note that we have

$$e_1^{\perp}\mathbf{C}_1\mathbf{1} = e_1^{\perp}h_1 = \mathbf{1}$$

and thus we now can derive

$$e_k^{\perp}\mathbf{C}_2\mathbf{C}_1\mathbf{1} = \mathbf{C}_2 e_k^{\perp}\mathbf{C}_1\mathbf{1} - \frac{1}{q}\mathbf{C}_1 e_{k-1}^{\perp}\mathbf{C}_1\mathbf{1} = \begin{cases} \mathbf{C}_2\mathbf{C}_1\mathbf{1} & k = 0 \\ \mathbf{C}_2\mathbf{1} - \frac{1}{q}\mathbf{C}_1\mathbf{C}_1\mathbf{1} & k = 1 \\ -\frac{1}{q}\mathbf{C}_1\mathbf{1} & k = 2 \\ 0 & k > 2 \end{cases}.$$

Using this in 3.8 finally gives

$$\begin{aligned}\frac{1}{q^2}s_{3,2,1}[X] &= (-q)\sum_{k\geq 0}\left(\mathbf{C}_{3+k}e_k^{\perp}\mathbf{C}_3\mathbf{1} + \mathbf{C}_{3+k}e_k^{\perp}\mathbf{C}_2\mathbf{C}_1\mathbf{1}\right)\\
&= (-q)\Big[\mathbf{C}_3\mathbf{C}_3\mathbf{1} + \mathbf{C}_3\mathbf{C}_2\mathbf{C}_2\mathbf{1} - \frac{1}{q}\mathbf{C}_4\mathbf{C}_2\mathbf{1} + \mathbf{C}_4\left(\mathbf{C}_2\mathbf{1} - \frac{1}{q}\mathbf{C}_1\mathbf{C}_1\mathbf{1}\right)\\
&\qquad - \frac{1}{q}\mathbf{C}_5\mathbf{C}_1\mathbf{1}\Big]\\
&= -q\mathbf{C}_3\mathbf{C}_3\mathbf{1} - q\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1\mathbf{1} + (1-q)\mathbf{C}_4\mathbf{C}_2\mathbf{1} + \mathbf{C}_4\mathbf{C}_1\mathbf{C}_1\mathbf{1} + \mathbf{C}_5\mathbf{C}_1\mathbf{1}\end{aligned}$$

as desired. □

This is also agreed with Example 2.0.2. Since

$$\mathbf{C}_{-1}\mathbf{1} = 0$$

and

$$\mathbf{C}_0\mathbf{1} = -q\mathbf{1},$$

substituting $a = 3, b = 2$, and $c = 1$ gives

$$\begin{aligned}s_{3,2,1}[X] &= (-q)^{3+2+1-3}(1-\theta_2/q\theta_1)(1-\theta_3/q\theta_1)(1-\theta_3/q\theta_2)\,\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1\,\mathbf{1}\\
&= (-q)^{3+2+1-3}[\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1\mathbf{1} - \mathbf{C}_{3+1}\mathbf{C}_{2-1}\mathbf{C}_1\mathbf{1}/q - \mathbf{C}_3\mathbf{C}_{2+1}\mathbf{C}_{1-1}\mathbf{1}/q\\
&\qquad - \mathbf{C}_{3+1}\mathbf{C}_2\mathbf{C}_{1-1}\mathbf{1}/q + \mathbf{C}_{3+1}\mathbf{C}_2\mathbf{C}_{1-1}\mathbf{1}/q^2\\
&\qquad + \mathbf{C}_{3+2}\mathbf{C}_{2-1}\mathbf{C}_{1-1}\mathbf{1}/q^2 + \mathbf{C}_{3+1}\mathbf{C}_{2+1}\mathbf{C}_{1-2}\mathbf{1}/q^2\\
&\qquad - \mathbf{C}_{3+2}\mathbf{C}_2\mathbf{C}_{1-2}\mathbf{1}/q^3]\\
&= (-q)^3[\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1\mathbf{1} - \mathbf{C}_4\mathbf{C}_1\mathbf{C}_1\mathbf{1}/q - \mathbf{C}_3\mathbf{C}_3\mathbf{C}_0\mathbf{1}/q - \mathbf{C}_4\mathbf{C}_2\mathbf{C}_0\mathbf{1}/q\\
&\qquad + \mathbf{C}_4\mathbf{C}_2\mathbf{C}_0\mathbf{1}/q^2 + \mathbf{C}_5\mathbf{C}_1\mathbf{C}_0\mathbf{1}/q^2 + \mathbf{C}_4\mathbf{C}_3\mathbf{C}_{-1}\mathbf{1}/q^2\\
&\qquad - \mathbf{C}_5\mathbf{C}_2\mathbf{C}_{-1}\mathbf{1}/q^3]\\
&= (-q)^3[\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1\mathbf{1} - \mathbf{C}_4\mathbf{C}_1\mathbf{C}_1\mathbf{1}/q + \mathbf{C}_3\mathbf{C}_3\mathbf{1} + \mathbf{C}_4\mathbf{C}_2\mathbf{1} - \mathbf{C}_4\mathbf{C}_2\mathbf{1}/q\\
&\qquad - \mathbf{C}_5\mathbf{C}_1\mathbf{1}/q].\end{aligned}$$

Now, we will see identities about relations between **S** operator and **C** operator.

Theorem 3.0.15. *For all $a, b \in \mathbb{Z}$, we have*

$$q\,\mathbf{S}_a\mathbf{C}_b = \mathbf{C}_{b-1}\mathbf{S}_{a+1} + \mathbf{C}_b\,\mathbf{S}_a.$$

Proof. We will use the definition of $\mathbf{C}$ operator

$$\mathbf{C}_a P[X] = \left(-\frac{1}{q}\right)^{a-1} P\left[X - \frac{1-1/q}{z}\right]\Omega[zX]\bigg|_{z^a}$$

and the definition of $\mathbf{S}$ operator

$$\mathbf{S}_a P[X] = P\left[X - \frac{1}{z}\right]\ \Omega[zX]\bigg|_{z^a}.$$

Therefore, we have

$$\begin{aligned}(-q)^{b-1}\mathbf{C}_b P[X] &= \sum_{r_1=0}^{d} P^{(r_1)}[X]\frac{1}{z^{r_1}}\sum_{m\geq 0} z^m h_m[X]\bigg|_{z^b} \\ &= \sum_{r_1=0}^{d} P^{(r_1)}[X] h_{b+r_1}[X].\end{aligned}$$

Then,

$$
\begin{aligned}
&(-q)^{b-1}\mathbf{S}_a\mathbf{C}_b P[X] \\
&\quad= \sum_{r_1=0}^{d} P^{(r_1)}\left[X-\frac{1}{z_2}\right] h_{b+r_1}\left[X-\frac{1}{z_2}\right]\Omega[z_2X]\Big|_{z_2^a} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\frac{1}{z_2^{r_2}}h_{b+r_1}\left[X-\frac{1}{z_2}\right]\Omega[z_2X]\bigg|_{z_2^a} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\frac{1}{z_2^{r_2}}\sum_{s=0}^{b+r_1} h_{b+r_1-s}[X]h_s\left[-\frac{1}{z_2}\right]\Omega[z_2X]\bigg|_{z_2^a} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\sum_{s=0}^{b+r_1} h_{b+r_1-s}[X]h_s\left[-\frac{1}{z_2}\right]\Omega[z_2X]\bigg|_{z_2^{a+r_2}} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\sum_{s=0}^{b+r_1} h_{b+r_1-s}[X](-1)^s e_s\left[\frac{1}{z_2}\right]\Omega[z_2X]\bigg|_{z_2^{a+r_2}} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\sum_{s=0}^{1} h_{b+r_1-s}[X](-1)^s e_s[1]\Omega[z_2X]\bigg|_{z_2^{a+r_2+s}} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\sum_{s=0}^{1} h_{b+r_1-s}[X](-1)^s e_s[1]h_{a+r_2+s}[X] \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\big[h_{b+r_1}[X]h_{a+r_2}[X]-h_{b+r_1-1}[X]h_{a+r_2+1}[X]\big].
\end{aligned}
$$

Now, we will start from $\mathbf{S}$ operator

$$
\begin{aligned}
\mathbf{S}_a P[X] &= \sum_{r_2=0}^{d} P^{(r_2)}[X]\frac{1}{z_2^{r_2}}\sum_{m\geq 0} z_2^m h_m[X]\bigg|_{z_2^a} \\
&= \sum_{r_2=0}^{d} P^{(r_2)}[X]h_{a+r_2}[X].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&(-q)^{b-1}\mathbf{C}_b\mathbf{S}_a P[X] \\
&\quad= \sum_{r_2=0}^{d} P^{(r_2)}\left[X-\frac{1-1/q}{z}\right] h_{a+r_2}\left[X-\frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^b} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\frac{1}{z^{r_1}} h_{a+r_2}\left[X-\frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^b} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\frac{1}{z^{r_1}} \sum_{m=0}^{a+r_2} h_{a+r_2-m}[X] h_m\left[-\frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^b} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] \sum_{m=0}^{a+r_2} h_{a+r_2-m}[X] h_m\left[-\frac{1-1/q}{z}\right]\Omega[zX]\Big|_{z^{b+r_1}} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] \sum_{m=0}^{a+r_2} h_{a+r_2-m}[X] \left(\frac{1}{q}\right)^m \left(\frac{1}{z}\right)^m h_m[1-q]\Omega[zX]\Big|_{z^{b+r_1}} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] \sum_{m=0}^{a+r_2} h_{a+r_2-m}[X] \left(\frac{1}{q}\right)^m h_m[1-q]\Omega[zX]\Big|_{z^{b+r_1+m}} \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] \sum_{m=0}^{a+r_2} h_{a+r_2-m}[X] \left(\frac{1}{q}\right)^m h_m[1-q] h_{b+r_1+m}[X] \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] h_{a+r_2}[X] h_{b+r_1}[X] \\
&\qquad\qquad + \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] \sum_{m=1}^{a+r_2} h_{a+r_2-m}[X] \left(\frac{1}{q}\right)^m (1-q) h_{b+r_1+m}[X].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&(-q)^{(b-1)-1}\mathbf{C}_{b-1}\mathbf{S}_{a+1} P[X] \\
&\quad= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] h_{a+1+r_2}[X] h_{b-1+r_1}[X] \\
&\qquad\qquad + \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X] \sum_{m=1}^{a+1+r_2} h_{a+1+r_2-m}[X] \left(\frac{1}{q}\right)^m (1-q) h_{b-1+r_1+m}[X]
\end{aligned}
$$

and

$$
\begin{aligned}
&(-q)^{b-1}\mathbf{C}_{b-1}\mathbf{S}_{a+1}P[X] \\
&= (-q)\sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]h_{a+1+r_2}[X]h_{b-1+r_1}[X] \\
&\quad + (-q)\sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\sum_{m=1}^{a+1+r_2} h_{a+1+r_2-m}[X]\left(\frac{1}{q}\right)^{m}(1-q)h_{b-1+r_1+m}[X] \\
&= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X](-q)h_{a+1+r_2}[X]h_{b-1+r_1}[X] \\
&\quad - \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\sum_{m=1}^{a+1+r_2} h_{a+1+r_2-m}[X]\left(\frac{1}{q}\right)^{m-1}(1-q)h_{b-1+r_1+m}[X].
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&(-q)^{b-1}\mathbf{C}_b\mathbf{S}_aP[X]+(-q)^{b-1}\mathbf{C}_{b-1}\mathbf{S}_{a+1}P[X]\\
&\quad=\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]h_{a+r_2}[X]h_{b+r_1}[X]\\
&\qquad+\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]\sum_{m=1}^{a+r_2}h_{a+r_2-m}[X]\left(\frac{1}{q}\right)^m(1-q)h_{b+r_1+m}[X]\\
&\qquad+\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X](-q)h_{a+1+r_2}[X]h_{b-1+r_1}[X]\\
&\qquad-\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]\sum_{m=1}^{a+1+r_2}h_{a+1+r_2-m}[X]\left(\frac{1}{q}\right)^{m-1}(1-q)h_{b-1+r_1+m}[X]\\
&\quad=\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]h_{a+r_2}[X]h_{b+r_1}[X]\\
&\qquad+\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]\sum_{m=1}^{a+r_2}h_{a+r_2-m}[X]\left(\frac{1}{q}\right)^m(1-q)h_{b+r_1+m}[X]\\
&\qquad+\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X](-q)h_{a+1+r_2}[X]h_{b-1+r_1}[X]\\
&\qquad-\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]\sum_{m=0}^{a+r_2}h_{a+r_2-m}[X]\left(\frac{1}{q}\right)^m(1-q)h_{b+r_1+m}[X]\\
&\quad=\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]h_{a+r_2}[X]h_{b+r_1}[X]\\
&\qquad+\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X](-q)h_{a+1+r_2}[X]h_{b-1+r_1}[X]\\
&\qquad-\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X]h_{a+r_2}[X](1-q)h_{b+r_1}[X]\\
&\quad=\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X](-q)h_{a+1+r_2}[X]h_{b-1+r_1}[X]\\
&\qquad-\sum_{r_1=0}^{d}\sum_{r_2=0}^{d}P^{(r_1,r_2)}[X](-q)h_{a+r_2}[X]h_{b+r_1}[X]
\end{aligned}
$$

Therefore, we finally have

$$= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X](-q)\big[h_{a+1+r_2}[X]h_{b-1+r_1}[X] - h_{a+r_2}[X]h_{b+r_1}[X]\big]$$
$$= \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\big[qh_{a+r_2}[X]h_{b+r_1}[X] - qh_{a+1+r_2}[X]h_{b-1+r_1}[X]\big].$$

Since

$$(-q)^{b-1}\mathbf{S}_a\mathbf{C}_bP[X] = \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\big[h_{b+r_1}[X]h_{a+r_2}[X] - h_{b+r_1-1}[X]h_{a+r_2+1}[X]\big],$$

$$q(-q)^{b-1}\mathbf{S}_a\mathbf{C}_bP[X] = \sum_{r_1=0}^{d}\sum_{r_2=0}^{d} P^{(r_1,r_2)}[X]\big[qh_{b+r_1}[X]h_{a+r_2}[X] - qh_{b+r_1-1}[X]h_{a+r_2+1}[X]\big].$$

Now we have

$$(-q)^{b-1}\big(\mathbf{C}_b\mathbf{S}_aP[X] + \mathbf{C}_{b-1}\mathbf{S}_{a+1}P[X]\big) = q(-q)^{b-1}\mathbf{S}_a\mathbf{C}_bP[X],$$

which means

$$\mathbf{C}_b\mathbf{S}_aP[X] + \mathbf{C}_{b-1}\mathbf{S}_{a+1}P[X] = q\mathbf{S}_a\mathbf{C}_bP[X].$$

□

Corollary 2. *For all $a, b \in \mathbb{Z}$, we have*

$$\mathbf{B}_b\mathbf{S}_a = \mathbf{S}_{a+1}\mathbf{B}_{b-1} + q\,\mathbf{S}_a\,\mathbf{B}_b.$$

Chapter 4

Applications and Examples

In this chapter, we will introduce a parking function setting to the Nabla image of a two-row schur function case. Throughout this chapter, figures show the rightmost part of parking functions unless indicated otherwise.

4.1 The Case of $\nabla s_{n-3,3}$

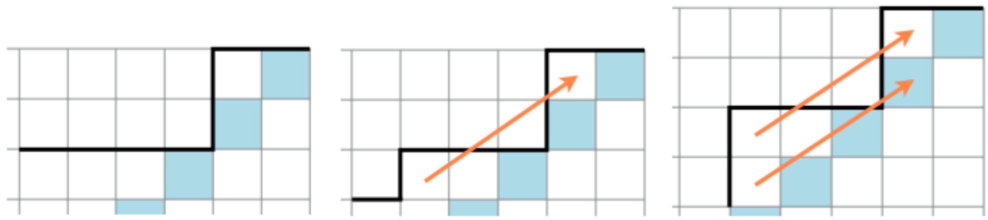
Let $n > 5$. Let $\mathcal{NS}_3$ be a set of parking functions with diagonal composition $[n-2, 2]$ whose Dyck path terminates according to one of the following three patterns in Figure 4.1 and the cars adjacent to the north steps satisfy the conditions indicated by the arrows, which means that the car in the cell where the arrow starts should be smaller than the one in the cell where the arrow ends. For example, in the second pattern in Figure 4.1, we should have $v_{n-2} < v_n$ and in the third pattern in Figure 4.1, we should have $v_{n-2} < v_n$ and $v_{n-3} < v_{n-1}$.

Theorem 4.1.1. *Assume that the compositional refinement of the Shuffle conjecture holds. Then,*

$$\nabla s_{n-3,3}[X] = (-q)^{n-3} \sum_{PF \in \mathcal{NS}_3} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X]$$

Proof. From the Haglund-Morse-Zabrocki conjecture, we have

$$\nabla s_{n-3,3}[X] = (-q)^{n-3} \Big(\nabla \mathbf{C}_{n-2} \mathbf{C}_2 \mathbf{1} - q \nabla \mathbf{C}_{n-3} \mathbf{C}_3 \mathbf{1} \Big).$$



(a) (b) $v_{n-2} < v_n$ (c) $v_{n-2} < v_n$, $v_{n-3} < v_{n-1}$

Figure 4.1: A set of parking functions $\mathcal{NS}_3$.

To get the images of parking functions under Nabla operator for $s_{n-3,3}$, we will construct an injection ϕ_3 from $\Pi[n-3,3]$, the set of parking functions whose diagonal composition is $[n-3,3]$, to $\Pi[n-2,2]$, the set of parking functions with diagonal composition $[n-2,2]$. This injection ϕ_3 should preserve the *area* and the *ides* of the parking functions in $\Pi[n-3,3]$ and $\Pi[n-2,2]$, but the *dinv* of the image parking functions by this injection in $\Pi[n-2,2]$ should be increased by 1.

Let PF be a parking function in $\Pi[n-3,3]$. There are two possible shapes for the rightmost three columns of the Dyck path of PF as shown in Figure 4.2.

Figure 4.2: Two possible rightmost three columns.

Note that since $n > 5$ and Dyck path can touch the main diagonal only after the first $n-3$ columns, two steps preceding the rightmost three columns should be going east. We will use a red dot to indicate the point that Dyck path must pass through in each diagram and denote v_{n-2}, v_{n-1}, v_n and v_{n-3} by a, b, c, and d, respectively.

For the first possible shape in Figure 4.2, we divide into three cases as in

Figure 4.3.

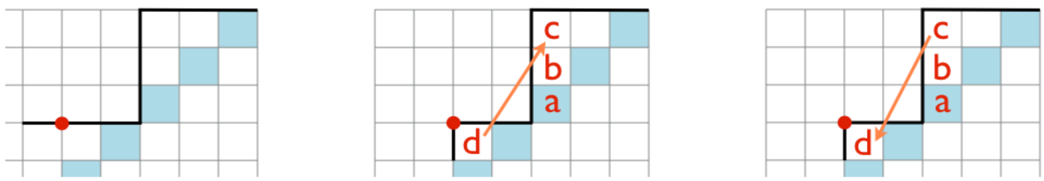


Figure 4.3: Three cases for the first possible shape in Figure 4.2. $d < c$ in the second case and $c < d$ in the third case.

Suppose that the last five columns in PF are as the first case in Figure 4.3. Then, we replace the last five columns of PF as we can see in Figure 4.4.

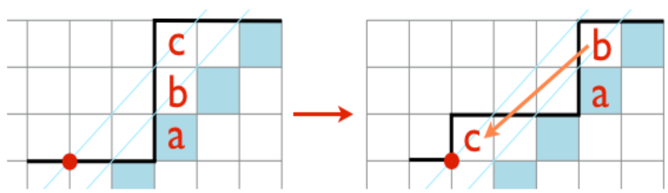


Figure 4.4: For any a, b and c

Note that the right side of this figure gives a legal parking function. We will denote this parking function by PF'. Then, $area(PF) = area(PF')$ and $ides(PF) = ides(PF')$ since PF and PF' have the same diagonal word. Also, since cars move within the same diagonal lines, if they contribute to the $dinv$ in PF, then the pairs also contribute to the $dinv$ in PF' only except the pair (b, c). The pair (b, c) does not contribute to the $dinv$ in PF, but since $b < c$, it contributes to the secondary $dinv$ in PF'. Hence, we have $dinv(PF') = dinv(PF) + 1$, and we can have $\phi_3(PF) = PF'$.

For the second case in Figure 4.3, we can obtain a parking function PF' by replacing the last five columns of PF with the columns of the right side of the Figure 4.5.

Note that PF and PF' have the same $area$ and $ides$ as in the first case. For the $dinv$, the pair (b, c) does not contibute to the $dinv$ in PF, but it contriutes to the secondary $dinv$ in PF'. Therefore, $\phi_3(PF) = PF'$.

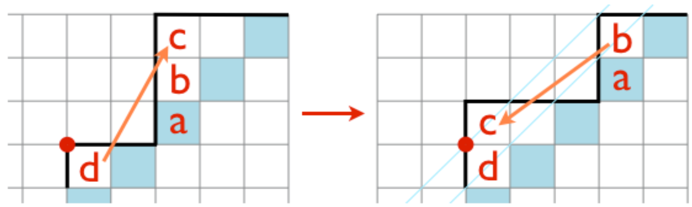


Figure 4.5: $d < c$

For the last case in Figure 4.3, suppose that we have the last four cars in the last five columns in PF as in the left side of Figure 4.6. Then, we move these four cars as the right side of Figure 4.6 and get a new parking function PF'.

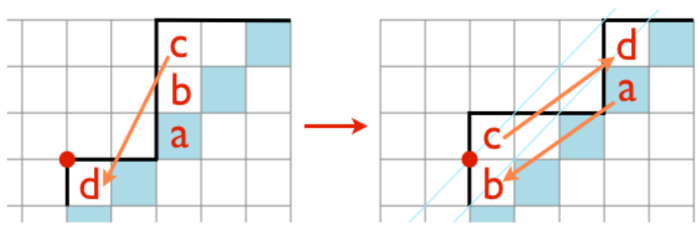


Figure 4.6: $c < d$

First, PF' is a legal parking function since $a < c < d$. Note that $area(PF) = area(PF')$. The diagonal word is changed since b and d have switched the places. Recall that the descent set of the inverse of a permutation is the set of all i such that $i + 1$ occurs before i in the permutation. Therefore, the $ides$ only can be changed when two consecutive cars interchange their orders in the diagonal word. However, since $b < c < d$ in this case, b and d are not consecutive, so the $ides$ is not changed. We need to show $dinv(PF') = dinv(PF) + 1$. The pair (b, d) is not the primary $dinv$ since $b < c < d$, but the pair (a, d) is the secondary $dinv$ in PF. In PF', the pair (a, d) does not contribute to the $dinv$, but the pair (a, b) becomes the secondary $dinv$ and the pair (b, d) becomes the primary $dinv$. Hence, we have $dinv(PF') = dinv(PF) + 1$ and $\phi_3(PF) = PF'$.

Now, we will consider the second possible shape in Figure 4.2.

Suppose $v_{n-2} < v_n$, i.e, $a < c$ as in the left side of Figure 4.7. Replacing the last four columns in PF with the columns of the right side of Figure 4.7 gives a legal parking function and we will denote this parking function by PF'.

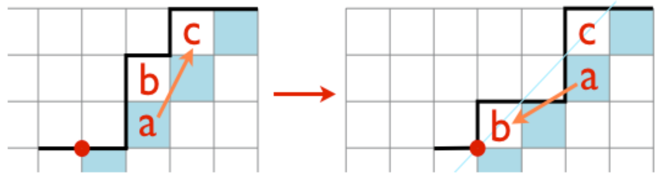


Figure 4.7: $a < c$

Notice that $area(PF) = area(PF')$ and $ides(PF) = ides(PF')$ since PF and PF' have the same diagonal word. The pair (a, b) now becomes the additional secondary $dinv$ in PF' while all other pairs contibuting to the primary or the secondary $dinv$ have not been changed. Therefore, $dinv(PF') = dinv(PF) + 1$ and then we let $\phi_3(PF) = PF'$.

For the last case, suppose $c < a$ as in the left side of the Figure 4.8. Again, we move the last three cars a, b and c in the last three columns in PF as the right side of the Figure 4.8 and let this parking function be PF'.

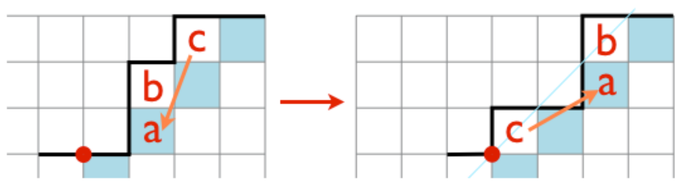


Figure 4.8: $c < a$

Note that the $area(PF) = area(PF')$, but the diagonal word has been changed since the car b and c have changed their order in the diagonal word. However, as in the third case, the car b and c are not consecutive since $c < a < b$, so the $ides$ has not been changed. Now, we will show $dinv(PF') = dinv(PF) + 1$. The pair (b, c) is not the primary $dinv$ in PF, but it contributes to the secondary $dinv$ in PF'. Also note that the pair (a, c) is not the secondary $dinv$ since $c < a$. Hence again, we have $dinv(PF') = dinv(PF) + 1$ as desired and we can have $\phi_3(PF) = PF'$.

The sets of the image parking functions from each case are disjoint in $\Pi[n-2,2]$. Therefore, ϕ_3 is an injection from $\Pi[n-3,3]$ to $\Pi[n-2,2]$. Also, we can easily see that the complementary set of $\phi_3(\Pi[n-3,3])$ in $\Pi[n-2,2]$ is $\mathcal{NS}_3$.

Therefore, by the Haglund-Morse-Zabrocki conjecture, we now have that

$$\nabla s_{n-3,3}[X] = (-q)^{n-3}\Big(\sum_{PF\in\Pi[n-2,2]} w(PF) - q\sum_{PF\in\Pi[n-3,3]} w(PF)\Big)$$
$$= (-q)^{n-3}\Big(\sum_{PF\in\Pi[n-2,2]} w(PF) - \sum_{PF\in\Pi[n-3,3]} w(\phi_3(PF))\Big)$$
$$= (-q)^{n-3}\sum_{PF\in\mathcal{NS}_3} t^{area(PF)}q^{dinv(PF)}Q_{ides(PF)}[X]$$

.

□

Suppose $n=5$. First, we can not have the first case if $n=5$. For the second, fourth and fifth case, we can apply the same injection as above. We should consider the third case carefully. Let $v_1 = e$ as the left side of the Figure 4.9.

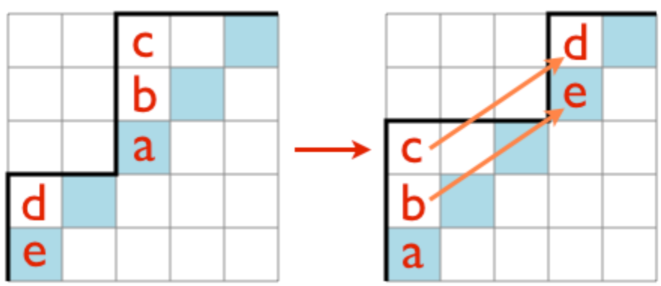


Figure 4.9: $c<d$ and $b<e$ when $n=5$

If $e<b$, then we can use the same injection as the third case above. However, if $b<e$ in PF, we can't use ϕ_3 since it doesn't give a legal parking function. Instead, in this case, we move all five cars as the right side of the Figure 4.9 and let this parking function be PF'. Note that $area(PF) = area(PF')$. The diagonal word has been changed from $[c,b,d,a,e]$ to $[c,d,b,e,a]$. However, since $b<c<d$ and $a<b<e$, we have the same $ides$ for PF and PF'. The $dinv(PF)=1$ since

the pair (a, d) in PF is the secondary *dinv*, while the $dinv(PF') = 2$ since the pairs (a, e) and (b, e) contribute to the primary *dinv*, but there is no secondary *dinv* in PF'. Hence $dinv(PF') = dinv(PF) + 1$ and we let $\phi_3(PF) = PF'$.

In this case $n = 5$, $\mathcal{NS}_3$ becomes an empty set after removing the image parking functions in $\Pi[3, 2]$ because we may not have the parking function with the first and second shapes in Figure 4.1 and the parking functions with the third case should be the image of the parking functions with $v_5 < v_2$ and $v_4 < v_1$ under ϕ_3, as we see above. This is also verified as

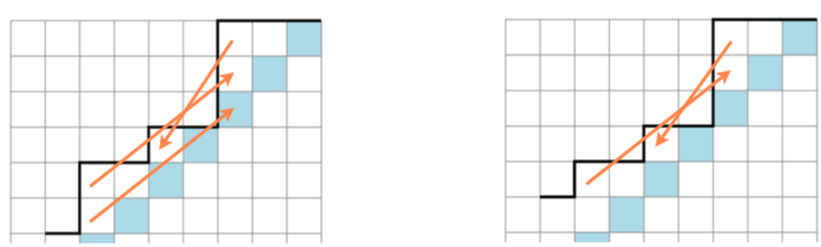
$$\nabla s_{2,3}[X] = (-q)^2 \Big(\nabla \mathbf{C}_3 \mathbf{C}_2 \mathbf{1} - q \nabla \mathbf{C}_2 \mathbf{C}_3 \mathbf{1} \Big) = 0.$$

by Theorem 3.0.9.

4.2 The Case of $\nabla s_{n-4,4}$

Now, we will consider the image of the parking function under Nabla operator, $\nabla s_{n-4,4}$, by introducing an injection from a set of parking functions whose diagonal composition is $[n-4, 4]$, $\Pi[n-4, 4]$, to the one whose diagonal composition is $[n-3, 3]$, $\Pi[n-3, 3]$.

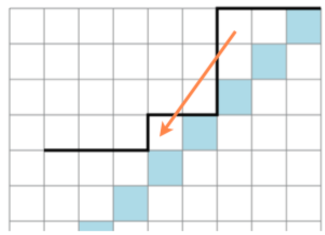
First, let $n > 7$. Also, let $\mathcal{NS}_4$ be a set of parking functions with diagonal composition $[n-3, 3]$ whose the rightmost columns on the diagram are one of the following sixteen patterns with conditions indicated by arrows in each diagram.



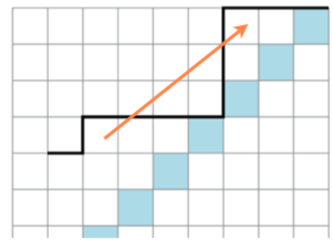
(a) $v_n < v_{n-3}$, $v_{n-4} < v_{n-1}$, $v_{n-5} < v_{n-2}$

(b) $v_n < v_{n-3}$, $v_{n-4} < v_{n-1}$

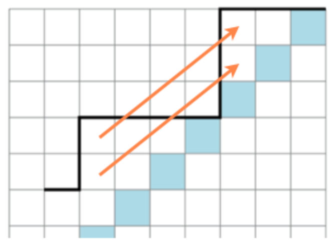
Figure 4.10: A set of parking functions $\mathcal{NS}_4$.



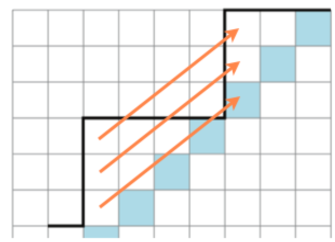
(a) $v_n < v_{n-3}$



(b) $v_{n-3} < v_n$

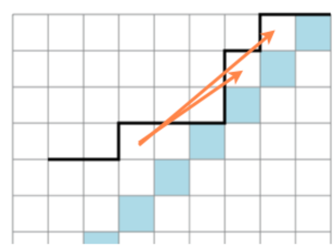


(c) $v_{n-3} < v_n$, $v_{n-4} < v_{n-1}$

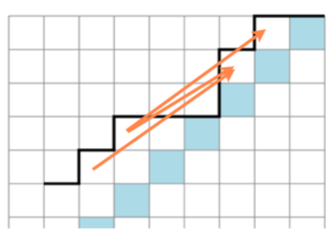


(d) $v_{n-3} < v_n$, $v_{n-4} < v_{n-1}$, $v_{n-5} < v_{n-2}$

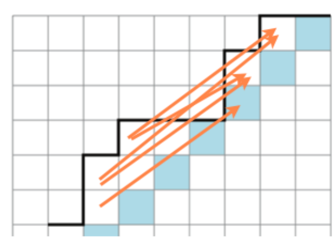
(e)



(f) $v_{n-3} < v_n$, $v_{n-3} < v_{n-1}$

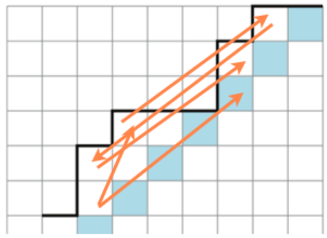


(g) $v_{n-3} < v_n$, $v_{n-3} < v_{n-1}$, $v_{n-4} < v_{n-1}$

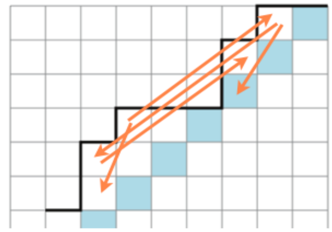


(h) $v_{n-3} < v_{n-1}$, $v_{n-3} < v_n$, $v_{n-4} < v_{n-1}$, $v_{n-4} < v_n$, $v_{n-5} < v_{n-2}$

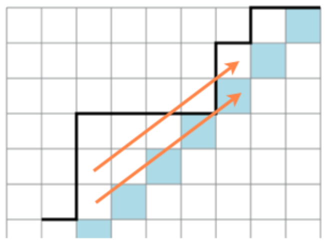
Figure 4.11: A set of parking functions $\mathcal{NS}_4$, continued



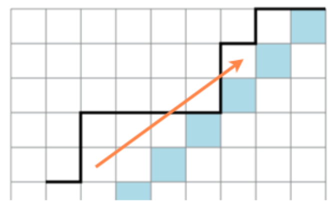
(a) $v_{n-3} < v_n$, $v_{n-4} < v_{n-1}$, $v_n < v_{n-4}$, $v_{n-5} < v_{n-3}$, $v_{n-5} < v_{n-2}$



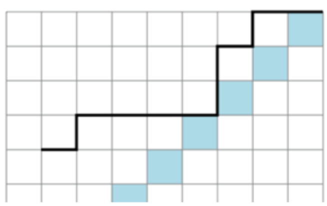
(b) $v_{n-3} < v_n$, $v_{n-4} < v_{n-1}$, $v_n < v_{n-4}$, $v_{n-3} < v_{n-5}$, $v_n < v_{n-2}$



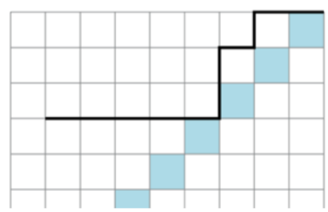
(c) $v_{n-4} < v_{n-1}$, $v_{n-5} < v_{n-2}$



(d) $v_{n-4} < v_{n-1}$



(e)



(f)

Figure 4.12: A set of parking functions $\mathcal{NS}_4$, continued.

Theorem 4.2.1. *Assume that the compositional refinement of the Shuffle conjecture holds. Then,*

$$\nabla s_{n-4,4} = (-q)^{n-3} \sum_{PF \in \mathcal{NS}_4} t^{area(PF)} q^{dinv(PF)} Q_{ides(PF)}[X].$$

Proof. Again, we can obtain

$$\nabla s_{n-4,4}[X] = (-q)^{n-3} \Big(\nabla \mathbf{C}_{n-3} \mathbf{C}_3 \mathbf{1} - q \nabla \mathbf{C}_{n-4} \mathbf{C}_4 \mathbf{1} \Big).$$

We need to find an injection ϕ_4 from $\Pi[n-4,4]$ to $\Pi[n-3,3]$. As we did in $\nabla s_{n-3,3}$ case, this injection ϕ_4 should preserve the *area* and the *ides* of the parking functions in $\Pi[n-4,4]$ and $\Pi[n-3,3]$, but the image parking function in $\Pi[n-3,3]$ by this injection should have one more *dinv* than the preimage parking function in $\Pi[n-4,4]$.

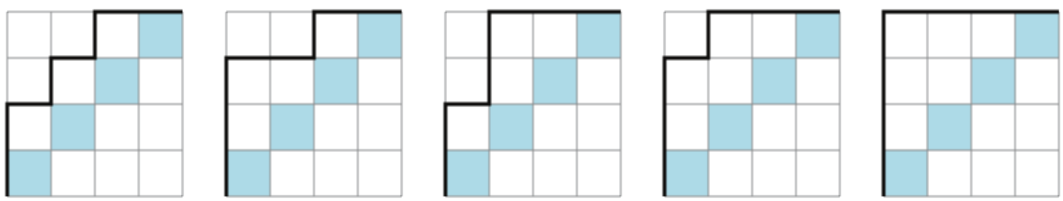


Figure 4.13: Five possible shapes of the rightmost four columns.

Let PF be a parking function in $\Pi[n-4,4]$. Then there are five possible patterns for the rightmost four columns of the Dyck path of PF as in Figure 4.13. For the convenience, we denote v_{n-3}, v_{n-2}, v_{n-1}, v_n, v_{n-4} and v_{n-5} by a, b, c, d, e and f, respectively.

Case A

To consider the first case in Figure 4.13, note that the two steps preceding the last four columns must be east steps since PF cannot hit the main diagonal twice.

Suppose that $a < c$. Then, we replace the last five columns of PF as the right side of the Figure 4.14 and denote this parking function by PF'.

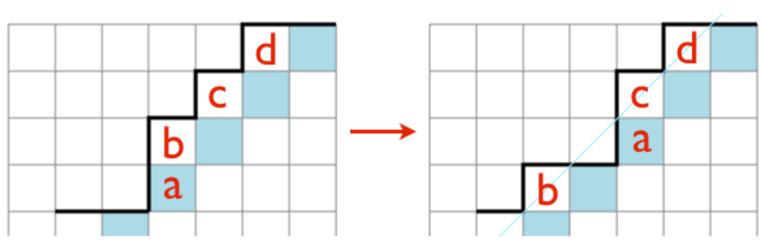


Figure 4.14: $a < c$

PF' is a legal parking function since $a < c$ and has the same *area* and *ides* as PF. Since all cars have been moved within the same diagonal lines, if they contribute the $dinv$ with any car not in these last five columns, they still contribute the $dinv$ after we replace the columns. Hence, the only difference on $dinv$ can be occured due to pairs of a, b, c, and d. In this case, PF' has one more $dinv$ than PF since the pair (a, b) in PF' contributes to the secondary $dinv$. Therefore we have $dinv(PF') = dinv(PF) + 1$ and let $\phi_4(PF) = PF'$.

Now, suppose $c < a$. Let PF' be the parkging function obtained by replacing the last five columns of PF as the right side of the Figure 4.15.

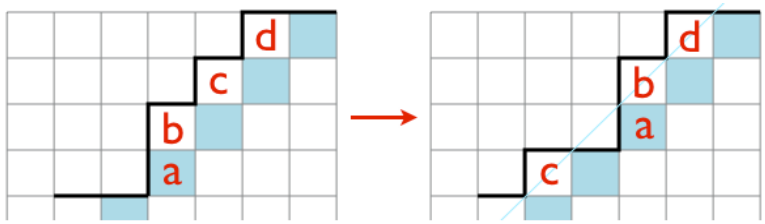


Figure 4.15: $c < a$

The $area$ is unchanged, but the diagonal word has been changed. Again, as we have seen in the case $\nabla S_{[n-3,3]}$, $ides$ is unchanged unless two consecutive cars interchange the order in the diagonal word. In this case, the order of b and c has been changed, but since $c < a < b$, b and c are not consecutive, so the $ides$ is unchanged. The pair (b, c) is not the primary $dinv$ in PF, but it becomes the primary $dinv$ in PF' while the pair (a, c) does not contribute to the secondary $dinv$ in PF'. Hence, we can have $\phi_4(PF) = PF'$.

Case B

For the second possible pattern in Figure 4.13, we divide into the cases again as Figure 4.16. Again, note that we only have these two possible patterns since there must be two east steps before the Dyck path hit the main diagonal.

For the first pattern in Figure 4.16, we move the last four cars, a, b, c, and d, as the right side of Figure 4.17 and let this parking function be PF'.

The $area$ and the $ides$ are unchanged. The pair (b, c) does not contribute to the $dinv$ in PF, but it does to the secondary $dinv$ in PF'. Hence $dinv(PF') = dinv(PF) + 1$ as desired and we have $\phi_4(PF) = PF'$.

For the second pattern in Figure 4.16, we should divide into two cases again.

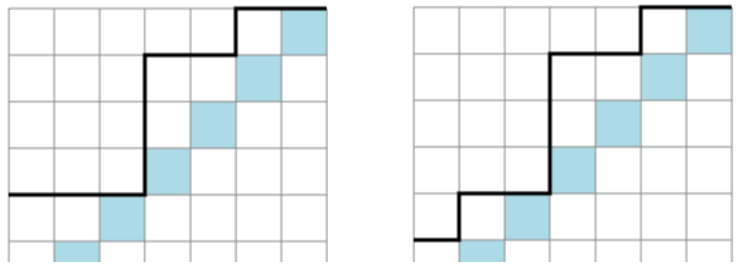


Figure 4.16: Two possible cases for *Case B*

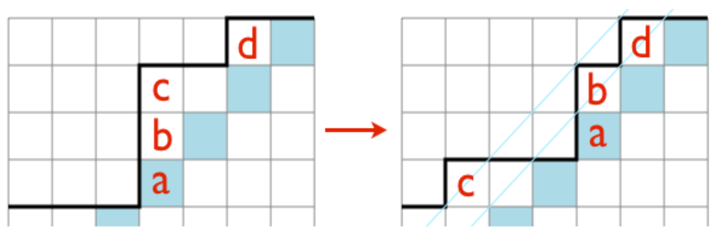


Figure 4.17: Any a, b, c and d

Suppose $e < c$ in the left side of Figure 4.18. Then we modify the last six columns as the right side of Figure 4.18 and let this parking function be PF'.

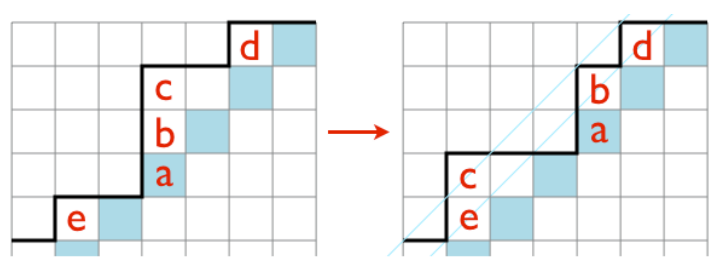


Figure 4.18: $e < c$

Note that PF' is a legal parking function. Also, $area(PF) = area(PF')$ and $ides(PF) = ides(PF')$. The pair (b, c) is not the *dinv* in PF, but it contributes to the secondary *dinv* in PF'. Since PF' has one more *dinv* than PF with the same *area* and *ides*, we have $\phi_4(PF) = PF'$.

Now, suppose $c < e$ as in the left side of Figure 4.19. Replacing the last six columns as the right side of Figure 4.19 gives a legal parking function since

$a < c < e$.

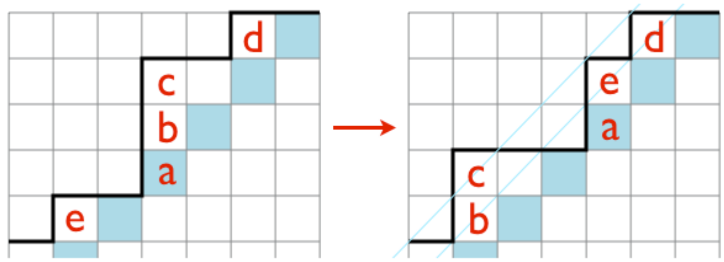


Figure 4.19: $c < e$

The *area* is unchaged. The diagonal word has been changed from $[c, d, b, e, a]$ to $[c, d, e, b, a]$, but the *ides* is unchanged since $b < c < e$. In PF, the pair (a, e) is the secondary *dinv* and the pair (b, e) is not the primary *dinv*. In PF', the pair (c, e) is not the secondary *dinv*, but the pair (b, e) is the primary *dinv* and the pair (a, b) is the secondary *dinv*. Note that the pair (a, e) doesn't contribute to the *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$ as desired, so $\phi_4(PF) = PF'$.

Case C

For the third possible pattern in Figure 4.13, we again divide into two cases as Figure 4.20.

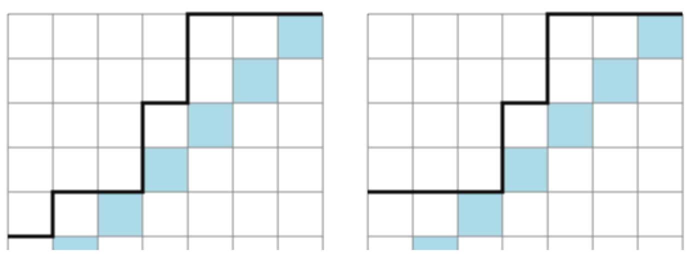


Figure 4.20: Two possible shapes for *Case C*

For the first possible pattern in Figure 4.20, suppose $a < c$ as the left side of 4.21. Then, we move the last five cars as the right side of Figure 4.21 and get a legal parking function PF'.

Note that the *area* and the *ides* are not changed. In PF', the pair (a, b) contributes to the secondary *dinv*, so $dinv(PF') = dinv(PF) + 1$ and we can let

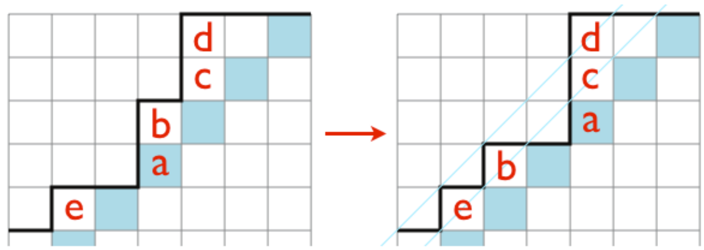


Figure 4.21: $a < c$

$\phi_4(PF) = PF'$.

Now, suppose $c < a$ in the first possible pattern in Figure 4.20. Also, suppose $b < d$. Then, we replace the last six columns as the right side of Figure 4.22.

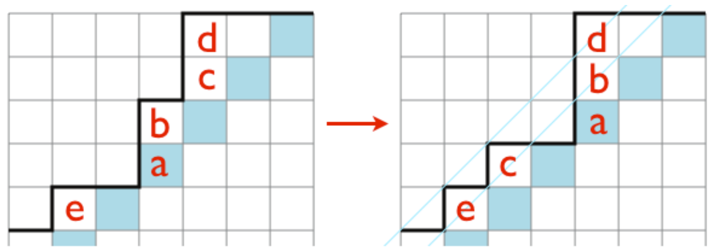


Figure 4.22: $c < a$ and $b < d$

Note that this new parking function, PF', is legal and has the same *area* as PF. The diagonal word of PF' is different from the one of PF, but the *ides* has not been changed since $c < a < b$, c and b are not consecutive. The pair (b, c) is not the primary *dinv* in PF, but it is the primary *dinv* in PF' while the pair (a, c) is not the secondary *dinv* in PF'. Hence, we can let $\phi_4(PF) = PF'$.

Suppose $c < a$ and $d < b$. Then, we divide into four subcases, either 1) $e < a$ and $e < d$, or 2) $a < e$ and $d < e$, or 3) $e < a$ and $d < e$, or 4) $a < e$ and $e < d$.

Suppose $e < a$ and $e < d$. Replacing the last six columns as the right side of Figure 4.23 gives a legal parking function, PF', and it has the same *area* and *ides* as PF.

In PF', the pair (c, d) contributes to the secondary *dinv* while the pair

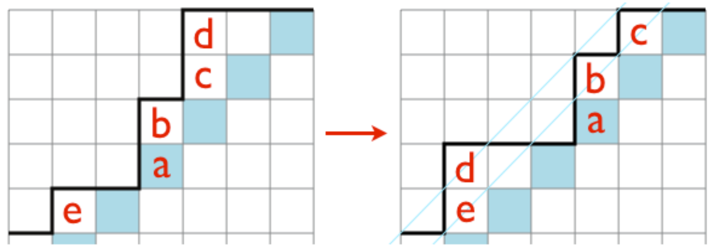


Figure 4.23: $c < a$, $d < b$, $e < a$ and $e < d$

(b, d) does not conribute to the *dinv* since $d < b$. Therefore, we have $dinv(PF') = dinv(PF) + 1$ and let $\phi_4(PF) = PF'$.

Suppose $a < e$ and $d < e$. We move the last five cars as the right side of Figure 4.24 and get a legal parking function, PF'.

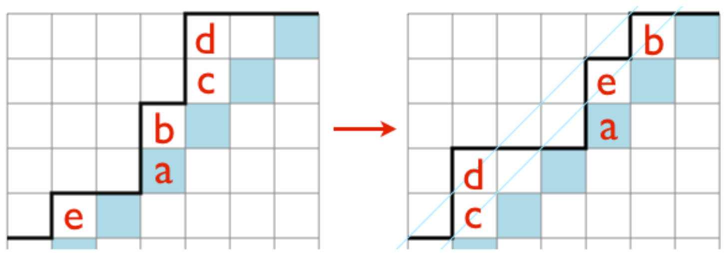


Figure 4.24: $c < a$, $d < b$, $a < e$ and $d < e$

Again, PF' has the same *area* as PF. The diagonal word has been changed from $[d, c, b, e, a]$ to $[d, b, e, c, a]$. However, since $c < a < b$, c and b can interchange their order and since $c < a < e$, c and e can interchange their order again in the diagonal word without changing the *ides*. In PF, the pair (a, e) is the secondary *dinv* while the pair (c, e) and (b, c) are not the primary *dinv*. In PF', the pair (d, e), (b, d) and (a, c) are not the secondary *dinv* while the pair (c, e) and (b, c) are the primary *dinv*. Note that the pair (a, e) doesn't contribute to the secondary *dinv* in PF'. Then $dinv(PF') = dinv(PF) + 1$ and we let $\phi_4(PF) = PF'$.

Now, suppose $e < a$ and $d < e$. We replace the last six columns as the right side of Figure 4.25 and get PF' whose the *area* is the same as PF.

The diagonal word of PF is $[d, c, b, e, a]$ and the one of PF' is $[d, e, b, c, a]$. However, their *ides* are same, since $c < d < e < a < b$, without changing the *ides*,

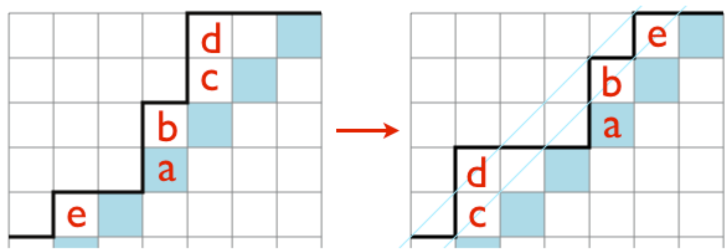


Figure 4.25: $c < a$, $d < b$, and $\Big[[e < a \text{ and } d < e] \text{ or } [a < e \text{ and } e < d]\Big]$

the diagonal word can be changed from $[d, c, b, e, a]$ to $[d, e, b, c, a]$. For the *dinv* in PF, the pair (a, e), (c, e) and (b, c) do not contribute to the *dinv* while the pair (b, e) is the primary *dinv*. In PF', the pair (a, c),(b, d), (d, e), (a, c) and (b, e) do not contribute to the *dinv* while the pair (b, c) and (c, e) are the primary *dinv*. Then $dinv(PF') = dinv(PF) + 1$ and we let $\phi_4(PF) = PF'$.

We have the same injection when $a < e$ and $e < d$ as Figure 4.25. In fact, in this case, we have $c < a < e < d < b$. Then, the diagonal word of PF can be changed from $[d, c, b, e, a]$ to $[d, e, b, c, a]$, which is the diagonal wokd of PF' without changing the *ides*. For the *dinv*, the pair (a, e) is the secondary *dinv* and (b, e) is the primary *dinv* while the pair (b, c) and (c, e) are not the primary *dinv* in PF. In PF', the pair (b, c) and (c, e) contribute to the primay *dinv* and the pair (d, e) does to the secondary *dinv* while the pair (b, e), (a, e), (a, c) and (b, d) are not the *dinv*. Hence, we have one more *dinv* in PF' than in PF, so we can let $\phi_4(PF) = PF'$.

Now, suppose the rightmost seven columns of PF have the shape as the second possible case in Figure 4.20. In this case, we have three subcases.

First, suppose $d < b$. Then we replace the last six columns as the right side of Figure 4.26 to get a parking function PF'.

The *area* and the *ides* are unchanged. The pair (c, d) contributes to the secondary *dinv* while the pair (b, d) does not to the secondary *dinv* in PF'. Therefore, $dinv(PF') = dinv(PF) + 1$ and we can have $\phi_4(PF) = PF'$.

Now, suppose $b < d$ and $a < c$. We move the last four cars as the right side of Figure 4.27.

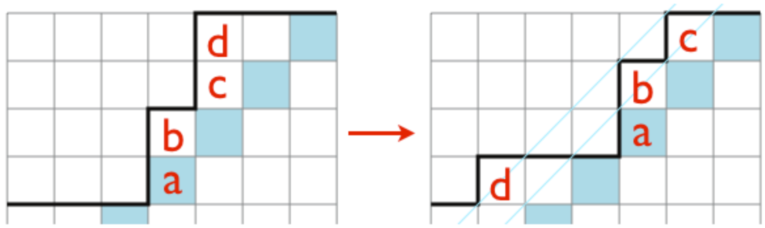


Figure 4.26: $d < b$

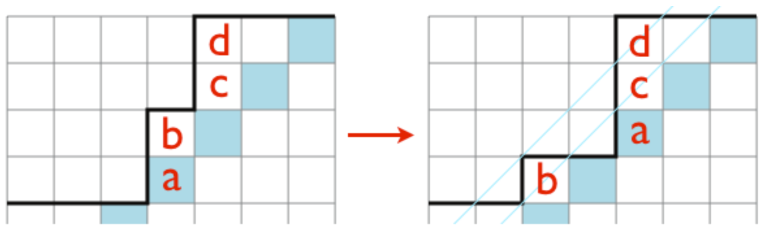


Figure 4.27: $b < d$ and $a < c$

Note that this new parking function is legal since $a < c$ and we denote this parking function by PF'. Also, note that the *area* and the *ides* are unchanged. We have one additional *dinv* in PF' as the pair (a, b) becomes the secondary *dinv*. We let $\phi_4(PF) = PF'$.

For the last case, suppose $b < d$ and $c < a$. We replace the last five columns as Figure 4.28 and get a legal parking function PF'.

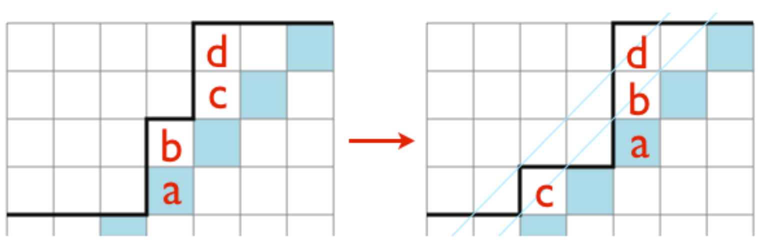


Figure 4.28: $b < d$ and $c < a$

PF and PF' have the same *area*. The diagonal word has been changed, but the *ides* is unchanged, since $c < a < b$, b and c are not consecutive. The pair b, c does not contribute to the primary *dinv* in PF, but it does in PF', while the pair (a, c) does not contribute to the secondary *dinv* in PF'. Hence, we have exactly one additional *dinv* in PF' and $\phi_4(PF) = PF'$.

Case D

Now, we consider the fourth possible case in Figure 4.13, which is the most complicated case in $\nabla_{n-4,4}$.

First, we divide into two cases as Figure 4.29.

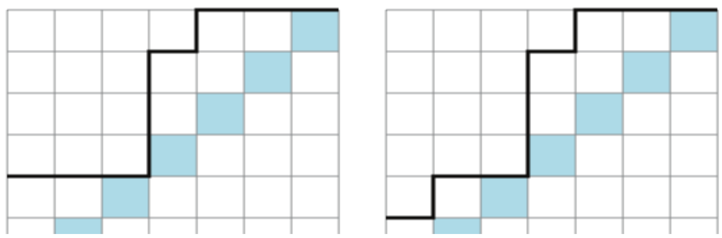


Figure 4.29: Two possible patterns for *Case D*

The case with the first shape in Figure 4.29 has only two subcases. First, suppose $b < d$ as the left side of Figure 4.30.

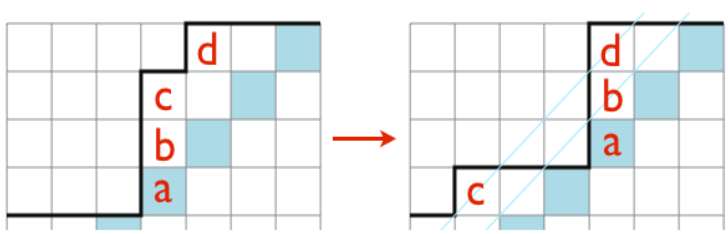


Figure 4.30: $b < d$

Replacing the last six columns as Figure 4.30 gives us a legal parking function since we suppose $b < d$. Notice that the *area* and the *ides* are unchanged. The pair (b, c) contributes to the secondary *dinv* in PF', so $dinv(PF') = dinv(PF) + 1$ and we have $\phi_4(PF) = PF'$.

Now, suppose $d < b$ as the left side of Figure 4.31. Then, we move the last four cars as the right side of Figure 4.31 and get a parking function PF', which has the same *area* as PF.

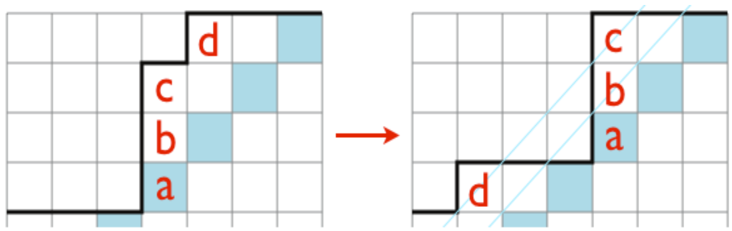


Figure 4.31: $d < b$

The diagonal word has been changed, but since $d < b < c$, PF' has the same *ides* as PF. In PF, the pair (c, d) is not the primary *dinv*. However, it becomes the primary *dinv*, while the pair (b, d) is not the secondary *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$ and we have $\phi_4(PF) = PF'$.

Suppose PF has the second possible shape in Figure 4.29. Also, suppose $b < d$ and $e < c$. Then, we replace the last five cars as the right side of Figure 4.32 and get a legal parking function.

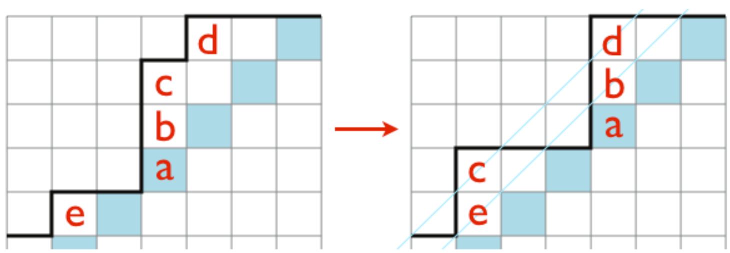


Figure 4.32: $b < d$ and $e < c$

Notice that the *area* and the *ides* have not been changed. There is one additional *dinv* in PF' since the pair (b, c) becomes the secondary *dinv*. Then, we can have $\phi_4(PF) = PF'$.

Next, suppose $d < b$ and $e < d$ with the second possible shape in Figure 4.29. We move the last five cars as Figure 4.33.

Note that the new parking function, PF', is legal since we assume $e < d$ and PF and PF' have the same *area*. The car c and d interchaged their order in the diagonal word, but since $d < b < c$, the *ides* is unchanged. The pair (c, d) is not the primary *dinv* in PF while it is in PF' and the pair (b, d) is not the

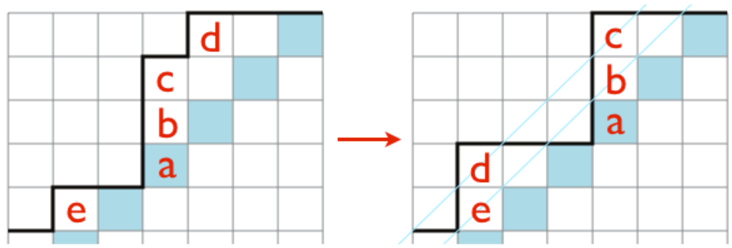


Figure 4.33: $d < b$ and $e < d$

secondary *dinv* in PF'. Hence, we may have $\phi_4(PF) = PF'$.

Now, the case when $b < d$ and $c < e$ and the case when $d < b$ and $d < e$ are left. Unfortunately, in these two cases, we should consider the last seven columns as Figure 4.34 to divide into cases.

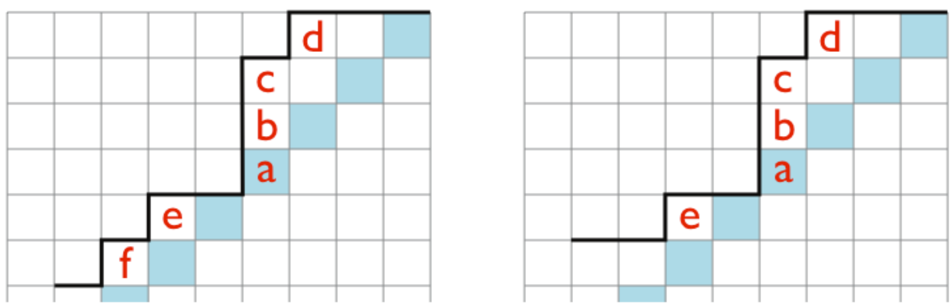


Figure 4.34: $[b < d$ and $c < e]$ or $[d < b$ and $d < e]$

First, suppose PF has the first shape in Figure 4.34 with the conditions $b < d$ and $c < e$. For the first subcase, suppose also that $e < d$ and $f < d$. Then, we replace the last seven columns as Figure 4.35 and get PF'.

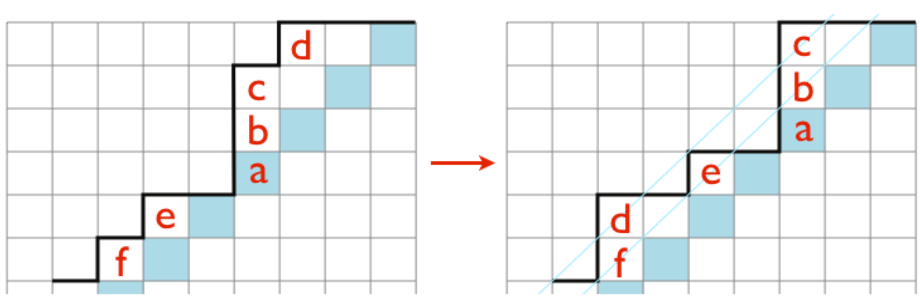


Figure 4.35: $b < d$, $c < e$, $e < d$ and $f < d$

Note that PF' is a legal parking function since we assume $f < d$ and it has the same *area* as PF. The car c and d interchanged their order in the diagonal word, but the *ides* has not changed since $c < e < d$ and they are not consecutive. In PF, the pair (c, d) is the primary *dinv*. In PF', it is not the primary *dinv*, but the pair (b, d) and the pair (d, e) are the secondary *dinv*. Hence, $dinv(PF') = dinv(PF) + 1$ as desired and we have $\phi_4(PF) = PF'$.

Now, suppose PF has the first shape in Figure 4.34 with the conditions $b < d$, $c < e$, $e < d$ and $d < f$. In fact, in this case, we have $a < b < c < e < d < f$. Replacing the last seven columns as Figure 4.36 gives us a legal parking function PF'.

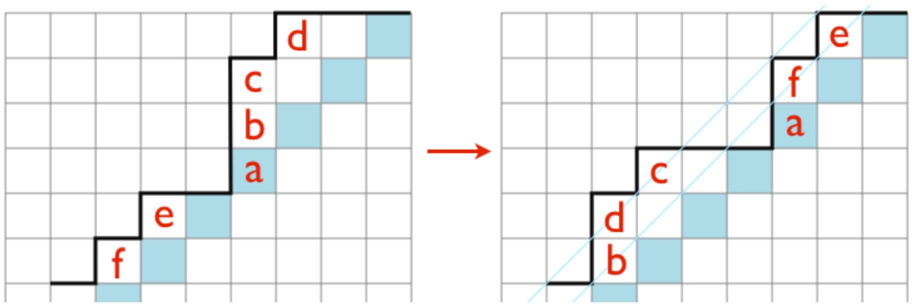


Figure 4.36: $b < d$, $c < e$, $e < d$ and $d < f$

The *area* is unchanged. The diagonal word is changed from $[d, c, b, e, f, a]$ to $[c, d, e, f, b, a]$. However, the *ides* has not been changed since the diagonal word of PF', $[c, d, e, f, b, a]$, can be obtained from $[d, c, b, e, f, a]$ by interchanging the order of two cars which are not consecutive when the condition $a < b < c < e < d < f$ is given. For the *dinv*, the pair (c, d) is the primary *dinv*, and the pair (a, e) and the pair (a, f) are the secondary *dinv* in PF. In PF', the pairs (b, e) and (b, f) are the primary *dinv*, and the pair (a, b) and the pair (d, e) are the secondary *dinv*, while the pair (c, d), the pair (d, f), the pair (c, f), and the pair (c, e) do not contribute to the *dinv*. Hence, $dinv(PF') = dinv(PF) + 1$ as desired and we have $\phi_4(PF) = PF'$.

Suppose that PF has the first shape in Figure 4.34 with the conditions $b < d$, $c < e$, $d < e$ and $f < c$. Then, we replace the last seven columns as Figure

4.37 and get a legal parking function PF'.

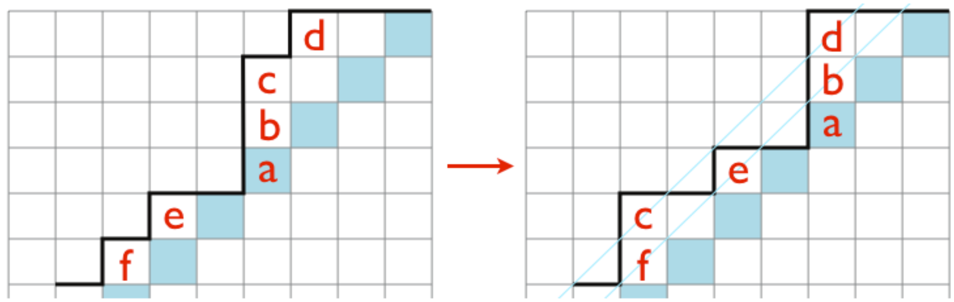


Figure 4.37: $b < d$, $c < e$, $d < e$ and $f < c$

Note that PF and PF' have the same *area* and the same *ides*. The pair (b, c) becomes the secondary *dinv* in PF', so $dinv(PF') = dinv(PF)+1$ as desired. We let $\phi_4(PF) = PF'$.

Suppose now that PF has the first shape in Figure 4.34 with the conditions $b < d$, $c < e$, $d < e$, $c < f$ and $d < f$. Then, we move the last six cars as Figure 4.38 and get a legal parking function PF' whose *area* is the same as the one of PF.

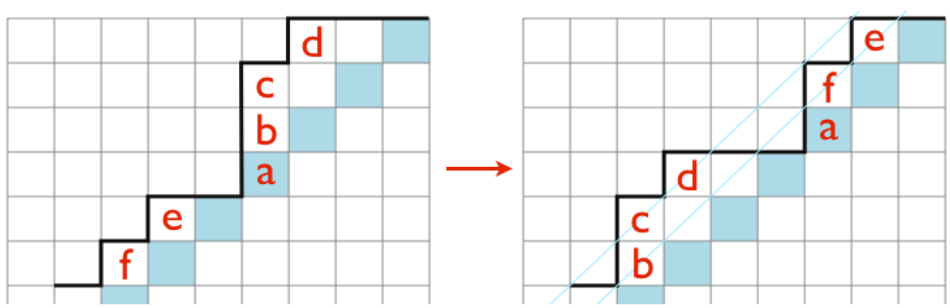


Figure 4.38: $b < d$, $c < e$, $d < e$, $c < f$ and $d < f$

Without changing the *ides*, the diagonal word is changed from $[d, c, b, e, f, a]$, which is in PF, to $[d, c, e, b, f, a]$ since $b < c < e$ and to $[d, c, e, f, b, a]$, which is the diagonal word in PF', since $b < c < f$. In PF, the pair (b, f) and the pair (b, e) are not the primary *dinv*, while the pair (a, e) and the pair (a, f) are the secondary *dinv*. On the other hand, in PF', the pair (b, f) and the pair (b, e) are the primary *dinv* and the pair (a, b) is the secondary *dinv*. The pair (c, f), (c, e), (d, f) and (d, e) do not the secondary *dinv* in PF'. Hence, we have exactly one more *dinv* in PF', so we can let $\phi_4(PF) = PF'$.

For the last case with the first shape in Figure 4.34, we suppose that $b < d$, $c < e$, $d < e$, $c < f$ and $f < d$. In fact, we have $a < b < c < f < d < e$. Then, we move the last six cars as Figure 4.39 and get a legal parking function PF'.

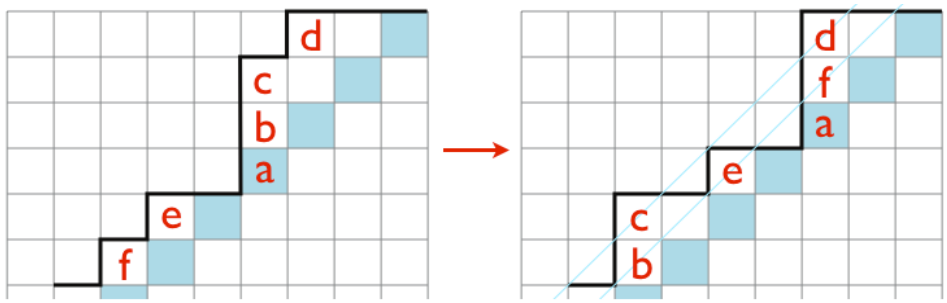


Figure 4.39: $b < d$, $c < e$, $d < e$, $c < f$ and $f < d$

Note that the *area* is unchanged. The dianogal word has been changed, but since we have $a < b < c < f < d < e$, we can obtain the diagonal word of PF' by interchanging the order of two nonconsecutive cars in the diagonal word of PF without changing the *ides*. For the *dinv*, the pair (e, f) is the primary *dinv* and the pair (a, f) is the secondary *dinv*, while the pair (b, e) and the pair (b, f) are not the primary *dinv* in PF. In PF', the pair (b, e) and the pair (b, f) contribute to the primary *dinv* and the pair (a, b) does to the secondary *dinv*, while the pair (e, f) is not the primary *dinv*, and the pair (c, f) and the pair (c, e) are not the secondary *dinv*. Hence, $dinv(PF') = dinv(PF) + 1$ as desired, and we let $\phi_4(PF) = PF'$.

Now, suppose PF has the second possible shape in Figure 4.34. Remember that we also have conditions $b < d$ and $c < e$ now. We divide into two cases. First, suppose also that $e < d$. Then, we move the last five cars as Figure 4.40.

The *area* has not been changed. The diagonal word is changed from $[d, c, b, e, a]$ to $[c, d, b, e, a]$, but since $c < e < d$, the car c and d are not consecutive, so the *ides* is unchanged. The pair (c, d) contributes to the primary *dinv*

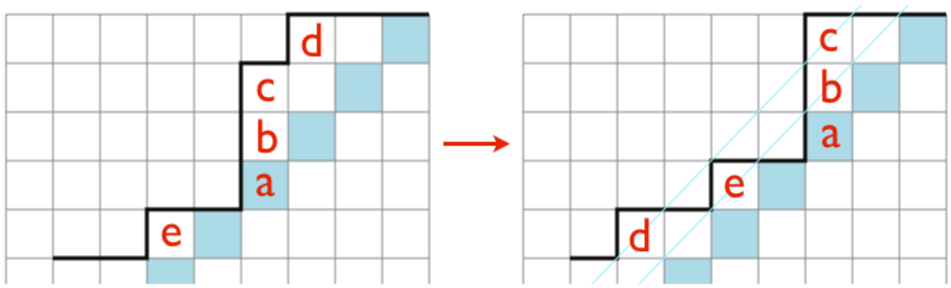


Figure 4.40: $b < d$, $c < e$ and $e < d$

in PF, but it does not in PF'. Instead, the pair (b, d) and the pair (d, e) contribute to the secondary *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$ as desired, and we have $\phi_4(PF) = PF'$.

Suppose $d < e$ with the second possible shape in Figure 4.34. Then, moving the last five cars as Figure 4.41 gives us a legal parking function and we denote this by PF'.

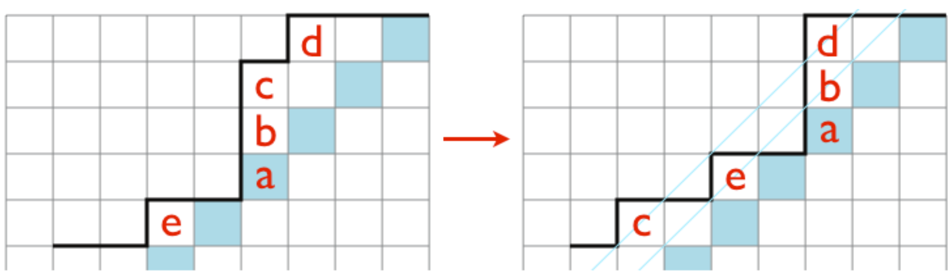


Figure 4.41: $b < d$, $c < e$ and $d < e$

The *area* and the *ides* are unchanged. There is one addition secondary *dinv* in PF' because of the pair (b, c). Again, we have $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Now, we need to consider the case $d < b$ and $d < e$ with the second possible shape in Figure 4.29. As the case $b < d$ and $c < e$, we should consider the last six cars as Figure 4.34 to divide the cases.

First, suppose PF has the first possible pattern in Figure 4.34 with the conditions $d < b$ and $d < e$. Then, we divide into five cases, 1) $f < c$ and $e < a$, 2) $f < c$ and $c < e$, 3) $f < c$ and $a < e < c$, 4) $c < f$ and $c < e$, and 5) $c < f$ and

$e < c$.

Suppose $f < c$ and $e < a$. Remember that we also suppose $d < b$ and $d < e$. Moving the last six cars as Figure 4.42 gives us a legal parking function PF'.

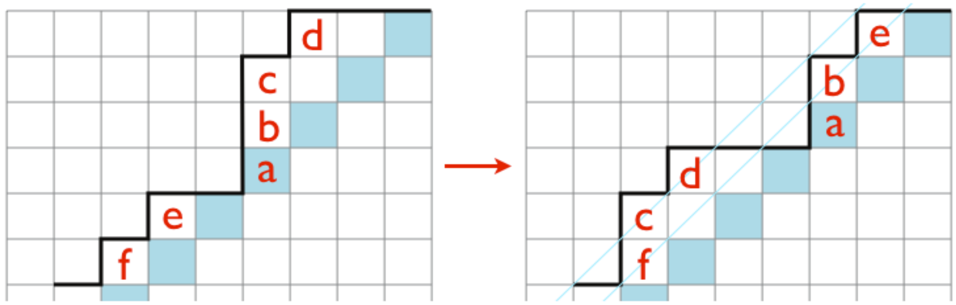


Figure 4.42: $d < b$, $d < e$, $f < c$ and $[e < a$ or $c < e]$

Notice that the *area* is unchanged. The diagonal word has been changed from $[d, c, b, e, f, a]$ to $[d, c, e, b, f, a]$, but since $e < a < b$, the car b and e are not consecutive, so the *ides* of PF is the same as the one of PF'. For *dinv*, the pair (b, e) is the primary *dinv* and the pair (a, e) is not the secondary *dinv* in PF. In PF', the pair (b, c) and the pair (c, e) contribute to the secondary *dinv* while the pair (b, e), the pair (b, d) and the pair (d, e) do not contribute to the primary or the secondary *dinv*. Hence, $dinv(PF') = dinv(PF)+1$ as desired and $\phi_4(PF) = PF'$.

Next, suppose $f < c$ and $c < e$. In fact, we use the same injection as above, so we move the last six cars as Figure 4.42 again. Note that the *area* is still unchaged and the *ides* is also unchaged, since $b < c < e$ in this case. In PF, the pair (a, e) is the secondary *dinv* while the pair (b, e) is not the primary *dinv*. On the other hand, in (PF'), the pair (b, e) is the primary *dinv* and the pair (b, c) is the secondary *dinv*, while the pair (c, e), the pair (b, d) and the pair (d, e) are not the secondary *dinv*. Again, we have $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Now, suppose $f < c$ and $a < e < c$. Note that we assume that PF has the first possible pattern in Figure 4.34 with the conditions $d < b$ and $d < e$. Replacing the last seven columns as Figure 4.43 gives a legal parking function.

The *area* and *ides* are unchanged. The pair (a, e) contributes to the sec-

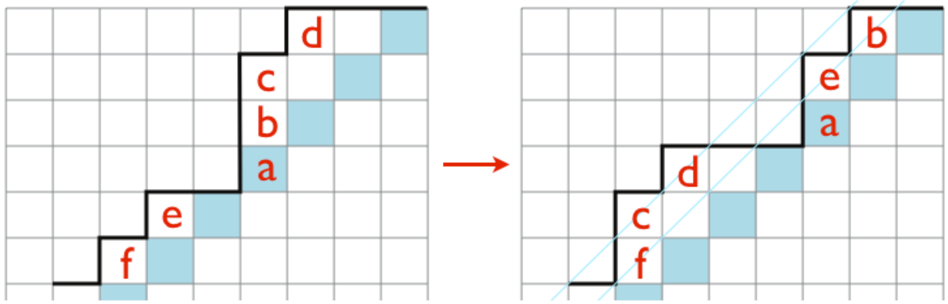


Figure 4.43: $d < b$, $d < e$, $f < c$ and $a < e < c$

ondary *dinv* in PF, while the pair (c, e) and the pair (b, c) contribute to the secondary *dinv* in PF', but the pair (b, d) and the pair (d, e) are not the secondary *dinv* in PF'. We have $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Next, suppose $c < f$ and $c < e$. Moving the last six cars as Figure 4.44 gives us a legal parking function PF' with the same *area* as the one of PF.

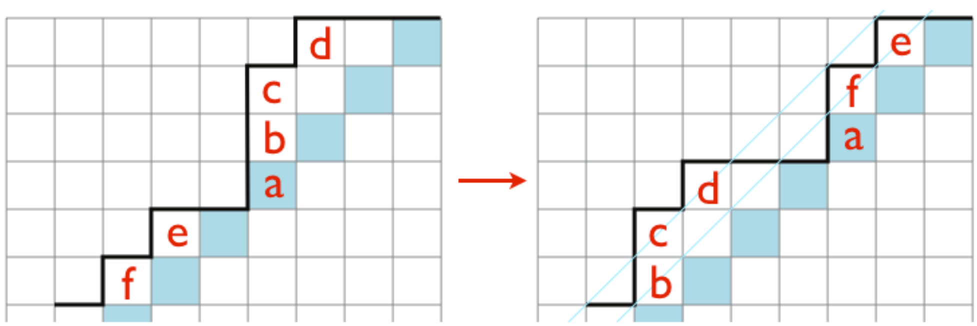


Figure 4.44: $d < b$, $d < e$, $c < f$ and $c < e$

The diagonal word is changed without changing the *ides* from $[d, c, b, e, f, a]$ to $[d, c, e, b, f, a]$ since $b < c < e$, and to $[d, c, e, f, b, a]$ again, which is the diagonal word of PF', since $b < c < f$. Hence PF and PF' have the same *ides*. For *dinv*, the pair (b, e) and the pair (b, f) are not the primary *dinv*, but the pair (a, e) and the pair (a, f) are the secondary *dinv* in PF. In PF', the pair (b, e) and the pair (b, f) are the primary *dinv* and the pair (a, b) is the secondary *dinv*. The pair (c, f), (c, e), (d, f) and (d, e) are not the secondary *dinv* in PF'. Therefore, we have exactly one additional *dinv* in PF', and we let $\phi_4(PF) = PF'$.

Finally, we suppose that $c < f$ and $e < c$. Then, we replace the last seven columns as Figure 4.45 and get a legal parking function since $a < c < f$.

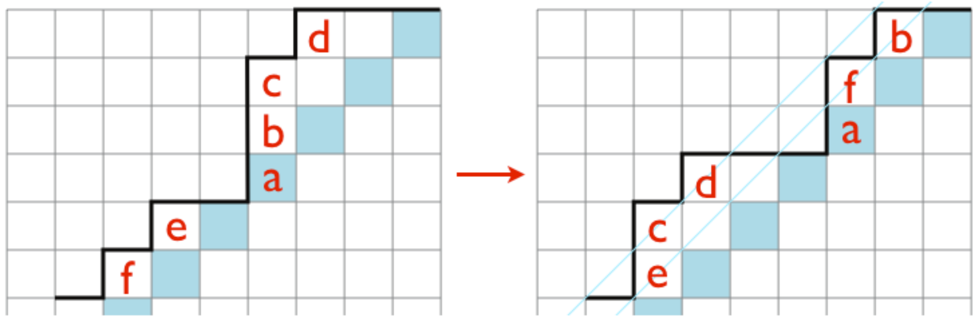


Figure 4.45: $d < b$, $d < e$, $c < f$ and $e < c$

The *area* is unchanged. The diagonal word is changed from $[d, c, b, e, f, a]$ to $[d, c, b, f, e, a]$, but the *ides* has not been changed since $e < c < f$ and the car e and f are not consecutive. In PF, the pair (a, f) is the secondary *dinv* and the pair (e, f) is not the primary *dinv*. On the other hand, in PF', the pair (e, f) becomes the primary *dinv* and the pair (b, c) is the secondary *dinv*. The pair (c, f), (b, d) and (d, f) are not the secondary *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Now, for the last possible case in *Case D*, we suppose that PF has the second possible shape in Figure 4.34 with the conditions $d < b$ and $d < e$. For the first subcase, we also suppose that $e < a$. We get a new parking function PF' by replacing the last seven columns as Figure 4.46.

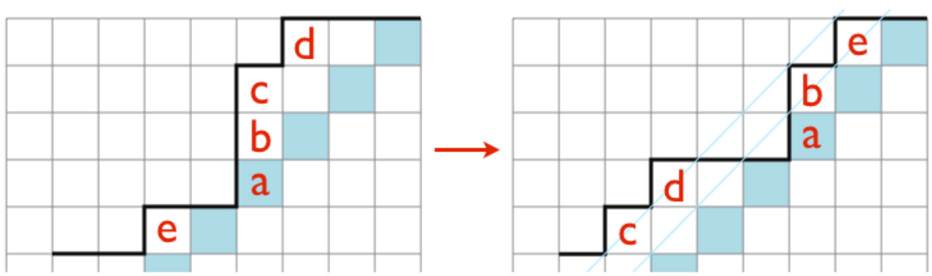


Figure 4.46: $d < b$, $d < e$ and $[e < a$ or $c < e]$

The *area* is unchaged. The diagonal word has been changed from $[d, c, b, e, a]$

to $[d, c, e, b, a]$, but since $e < a < b$, the *ides* of PF and PF' are the same. For the *dinv*, the pair (b, e) contributes to the primary *dinv*, while the pair (a, e) does to the secondary *dinv* in PF. In PF', the pair (b, c) and the pair (c, e) are the secondary *dinv*, while the pair (b, e), the pair (b, d) and the pair (d, e) do not contribute to the *dinv*. Therefore, there is exactly one more *dinv* in PF', so we can let $\phi_4(PF) = PF'$.

If $c < e$, then we use the same injection as above. Again, the *area* and the *ides* have not been changed, since $b < c < e$. In PF, the pair (a, e) is the secondary *dinv*, but the pair (b, e) is not the primary *dinv*. In PF', the pair (b, e) contributes to the primary *dinv* and the pair (b, c) does to the secondary *dinv*, while the pair (c, e), the pair (b, d) and the pair (d, e) are not the secondary *dinv*. We have $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Suppose $a < e < c$. Moving the last five cars as Figure 4.47 gives us a legal parking function PF'.

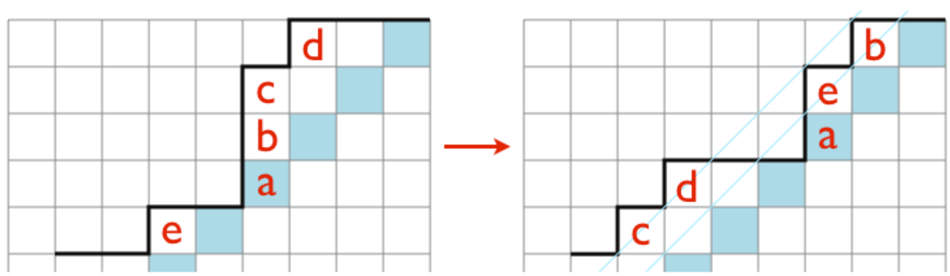


Figure 4.47: $d < b$, $d < e$ and $a < e < c$

Note that the *area* and the *ides* are not changed. The pair (a, e) is the secondary *dinv* in PF. The pair (b, c) and the pair (c, e) are the secondary *dinv*, while the pair (b, d) and the pair (d, e) are not the secondary *dinv* in PF'. Therefore, we have $dinv(PF') = dinv(PF) + 1$ and $\phi_4(PF) = PF'$.

<u>*Case E*</u>

Finally, suppose PF has the last possible shape in Figure 4.13. For this shape, we divide into four cases according to the shape of the Dyck path on four columns preceding the point where the Dyck path hit the main diagonal as Figure 4.48.

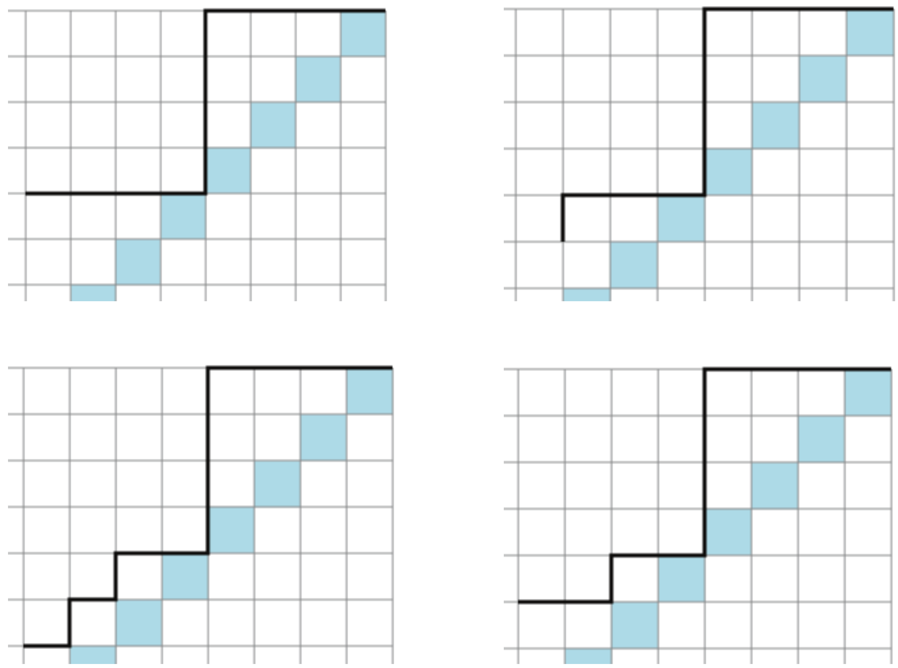


Figure 4.48: Four possible shapes for *Case E*

For the first possible shape in Figure 4.48, we move the car d as Figure 4.49 and get a new parking function PF'.

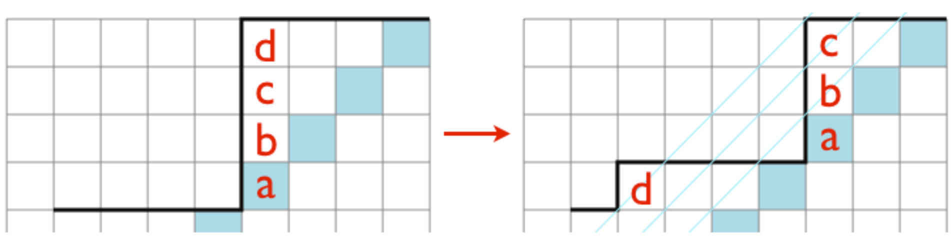


Figure 4.49: Any a, b, c and d

Notice that the *area* and the *ides* are unchanged. We have exactly one additional *dinv* in PF' since the pair (c, d) becomes the secondary *dinv* in PF'. Hence, we can have $\phi_4(PF) = PF'$.

For the second possible shape in Figure 4.48, suppose $e < d$. Then, replacing the last seven columns as Figure 4.50 gives us a new legal parking function PF'.

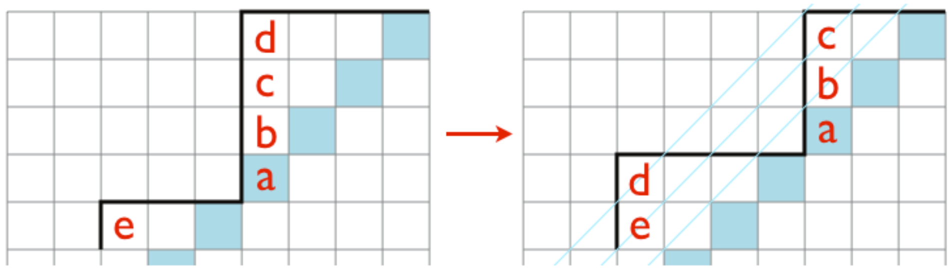


Figure 4.50: $e < d$

Again, PF and PF' have the same *area* and *ides*. There is one additional secondary *dinv* in PF' because of the pair (c, d). Therefore, we have $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Suppose $d < e$ with the second possible shape in Figure 4.48. Then, we have to divide into two subcases as Figure 4.51.

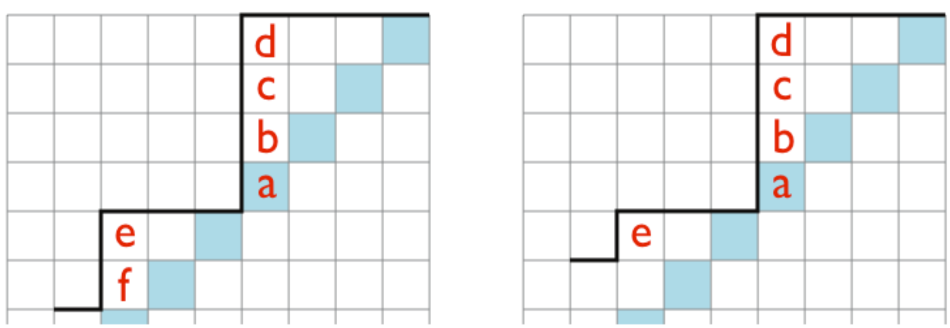


Figure 4.51: Two possible shapes when $d < e$ with the second shape in *Case E*

Suppose PF has the first possible shape in Figure 4.51. Also, suppose $f < c$. Then, we move the last six cars as Figure 4.52 and get a legal parking function PF'.

The *area* is unchaged. Also, even though the diagonal word has been changed from $[d, c, e, b, f, a]$ to $[d, e, c, b, f, a]$, since $c < d < e$, the car c and e are not consecutive, so the *ides* is unchanged. For *dinv*, the pair (b, e) is the secondary *dinv* while the pair (c, e) is not the primary *dinv* in PF. The pair (c, e) contributes to the primary *dinv* and the pair (b, c) contributes to the secondary *dinv*, while the pair (d, e) is not the secondary *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$

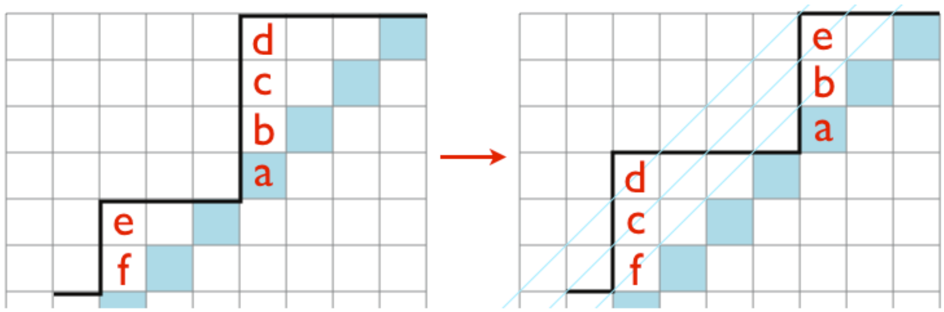


Figure 4.52: $d < e$ and $f < c$

as desired and $\phi_4(PF) = PF'$.

Suppose now that $c < f$. Replacing the last seven columns as Figure 4.53 gives us a legal parking function PF' whose *area* is the same as the one of PF.

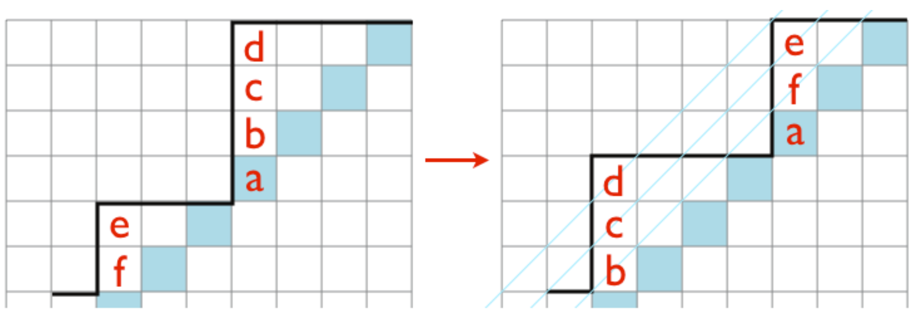


Figure 4.53: $d < e$ and $c < f$

The diagonal word has been changed from $[d, c, e, b, f, a]$ to $[d, e, c, f, b, a]$, but since $c < d < e$ and $b < c < f$, the car c and e can interchange their order in the diagonal word without changing the *ides* and the car b and f do the same thing again. For *dinv*, the pair (c, e) and the pair (b, f) are not the primary *dinv*, but the pair (a, f) and the pair (b, e) are the secondary *dinv* in PF. On the other hand, the pair (c, e) and the pair (b, f) contribute to the primary *dinv* and the pair (a, b) does to the secondary *dinv*, while the pair (c, f) and the pair (d, e) are not the secondary *dinv* in PF'. Hence, we have exactly one additional *dinv* in PF', so we have $\phi_4(PF) = PF'$.

Now, suppose PF has the second possible shape in Figure 4.51 with the condition $d < e$. In fact, in this case, we have $a < b < c < d < e$. Moving the last five cars as Figure 4.54 gives us a legal parking function PF'.

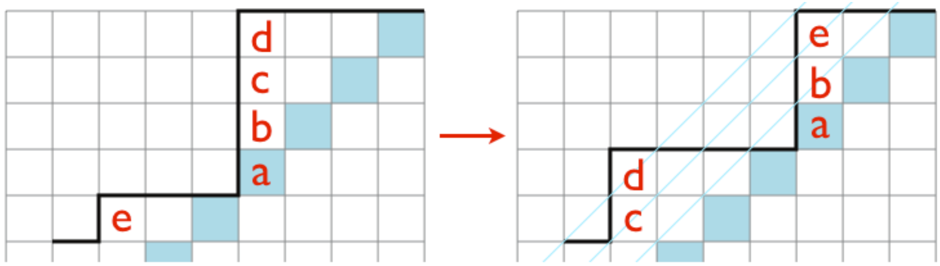


Figure 4.54: $d < e$

Note that the *area* is not changed. The diagonal word has been changed from $[d, c, e, b, a]$ to $[d, e, c, b, a]$, but since $c < d < e$, the car c and the car e can interchange their order in the diagonal word keeping the *ides* same. The pair (c, e) is not the primary *dinv*, but the pair (b, e) is the secondary *dinv* in PF. In PF', the pair (c, e) is the primary *dinv* and the pair (b, c) is the secondary *dinv*, while the pair (d, e) is not the secondary *dinv*. Hence, $dinv(PF') = dinv(PF) + 1$, so $\phi_4(PF) = PF'$.

Next, suppose PF has the third possible shape in Figure 4.48.

First, suppose $f < c$ and $e < a$. Moving the last six cars as Figure 4.55 gives us a legal parking function PF' whose *area* is the same as the one of PF.

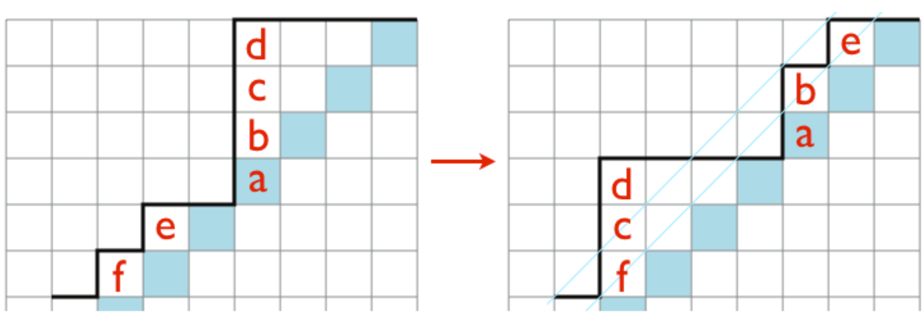


Figure 4.55: $f < c$ and $[e < a$ or $c < e]$

The diagonal word has been changed from $[d, c, b, e, f, a]$ to $[d, c, e, b, f, a]$, but since $e < a < b$, the *ides* is unchanged. The pair (b, e) is the primary *dinv*, but the pair a, e is not the secondary *dinv* in PF. In PF', the pair (b, e) is not the primary *dinv*, but the pair (b, c) and the pair (c, e) are the secondary *dinv*. Therefore, we have $dinv(PF') = dinv(PF) + 1$, and $\phi_4(PF) = PF'$.

If $f < c$ and $c < e$, then we use the same injection as above. In this case, we still have the same *ides*, since $b < c < e$, the car b and e still can interchange their order in the diagonal word without changing the *ides*. In PF, the pair (b, e) is not the primary *dinv*, but the pair (a, e) is the secondary *dinv*. In PF', the pair (b, e) is the primary *dinv* and the pair (b, c) is the secondary *dinv*, but the pair (c, e) is not the secondary *dinv*. Therefore, we have one more *dinv* in PF', so we can let $\phi_4(PF) = PF'$.

Now, suppose $f < c$ and $a < e < c$. Then, we move the last six cars as Figure 4.56 and get a legal parking function PF'.

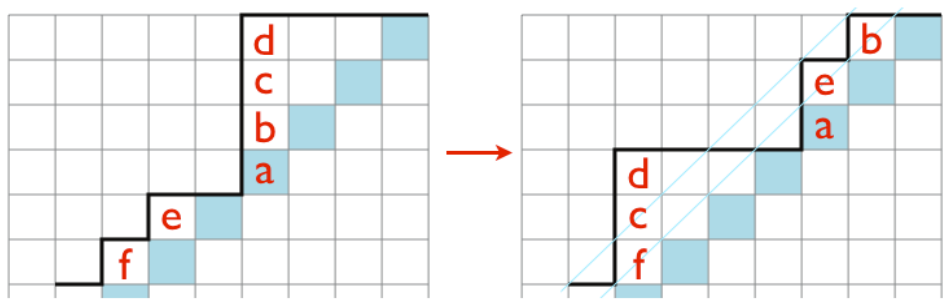


Figure 4.56: $f < c$ and $a < e < c$

Note that the *area* and the diagonal word are unchanged, so *ides* is also unchanged. The pair (a, e) is the secondary *dinv* in PF, while the pair (b, c) and the pair (c, e) are the secondary *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$ as desired, and $\phi_4(PF) = PF'$.

Next, suppose $c < f$ and $e < a$ with the third possible shape in Figure 4.48. Then, we replace the last seven columns as Figure 4.57 and get a legal parking function PF' with the same *area* as PF.

Since $c < a < b$, the car b and c are not consecutive, and since $b < c < f$, the car b and f are not consecutive, either. Therefore, even though the diagonal word has been changed from $[d, c, b, e, f, a]$ to $[d, c, e, f, b, a]$, the *ides* has not been

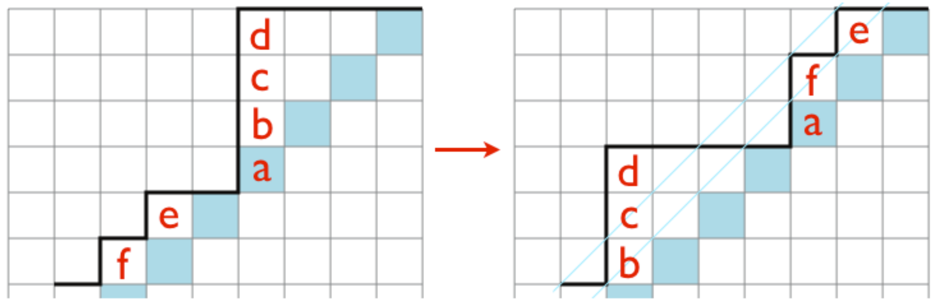


Figure 4.57: $c < f$ and $[e < a$ or $c < e]$

changed. For *dinv*, the pair (b, e) is the primary *dinv* and the pair (a, f) is the secondary *dinv*, while the pair (b, f) is not the primary *dinv* and the pair (a, e) is not the secondary *dinv* in PF. For PF', the pair (b, f) is the primary *dinv*, and the pair (a, b) and the pair (c, e) are the secondary *dinv*, while the pair (b, e) is not the primary *dinv* and the pair (c, f) is not the secondary *dinv*. Hence, $dinv(PF') = dinv(PF) + 1$ as desired, and $\phi_4(PF) = PF'$.

When $c < f$ and $c < e$, we apply the same injection as above. Again, since $b < c < e$ and $b < c < f$, we still have the same *ides* in PF and in PF'. For *dinv*, the pair (b, e) and the pair (b, f) are not the primary *dinv*, while the pair (a, e) and the pair (a, f) are the secondary *dinv* in PF. In PF', the pair (b, e) and the pair (b, f) are the primary *dinv* and the pair (a, b) is the secondary *dinv*, but the pair (c, f) and the pair (c, e) are not the secondary *dinv*. Hence, there is exactly one additional *dinv* in PF', so we can have $\phi_4(PF) = PF'$.

Now, suppose $c < f$ and $a < e < c$. Replacing the last seven columns as Figure 4.58 gives us a legal parking function PF' with the same *area* as PF.

The diagonal word has been changed from $[d, c, b, e, f, a]$ to $[d, c, b, f, e, a]$, but since $e < c < f$, the *ides* is unchanged. The pair (e, f) is not the primary *dinv*, but the pair (a, f) is the secondary *dinv* in PF. The pair (e, f) becomes a primary *dinv* and the pair (b, c) is the secondary *dinv*, but the pair (c, f) is not the secondary *dinv* in PF'. Therefore, PF' has exactly one more *dinv* than PF, so we can have $\phi_4(PF) = PF'$.

For the last possible case in Figure 4.48, suppose $e < a$ first. Moving the last five cars as Figure 4.59 gives us a new parking function PF' whose *area* is the

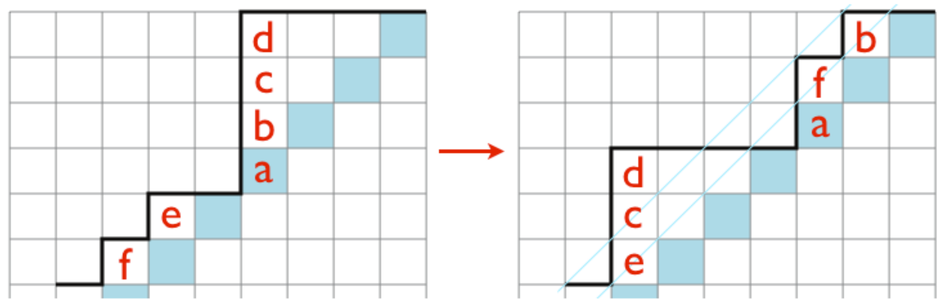


Figure 4.58: $c < f$ and $a < e < c$

same as the one of PF.

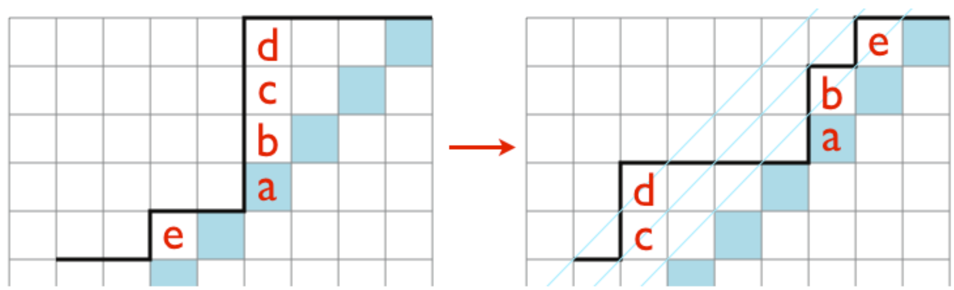


Figure 4.59: $e < a$ or $c < e$

Since $e < a < b$, the car b and the car e are not consecutive, so the *ides* of PF and the one of PF' are the same. The pair (b, e) is the primary *dinv*, but the pair (a, e) is not the secondary *dinv* in PF. Meanwhile, the pair (b, e) is not the primary *dinv*, but the pair (b, c) and the pair (c, e) are the secondary *dinv* in PF'. Hence, we have $dinv(PF') = dinv(PF) + 1$ and $\phi_4(PF) = PF'$.

If $c < e$, we can apply the same injection. The *ides* is still unchanged, since $b < c < e$. The pair (b, e) is not the primary *dinv*, but the pair (a, e) is not the secondary *dinv* in PF. In PF', the pair (b, e) is the primary *dinv* and the pair (b, c) is the secondary *dinv*, but the pair (c, e) is not the secondary *dinv*. Again, we have $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Finally, suppose $a < e < c$ with the fourth possible shape in Figure 4.48. Then, we replace the last seven columns as Figure 4.60 gives us a new legal parking function PF' with the same *area* and the *ides* as PF.

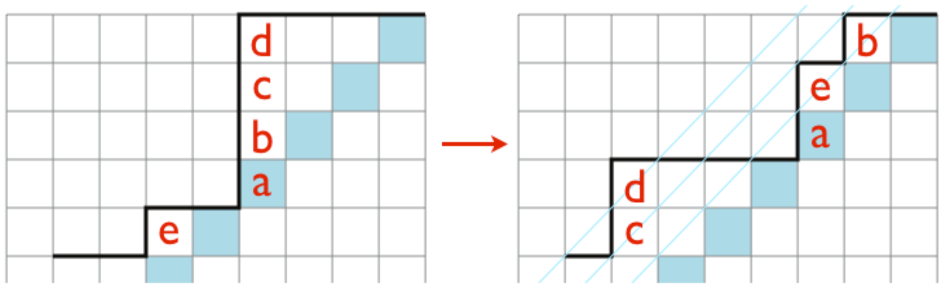


Figure 4.60: $a < e < c$

The pair (a, e) is the secondary *dinv* in PF, while the pair (c, e) and the pair (b, c) are the secondary *dinv* in PF'. Therefore, $dinv(PF') = dinv(PF) + 1$ as desired and $\phi_4(PF) = PF'$.

Note that the sets of the image parking functions from each case are disjoint in $\Pi[n-3,3]$. Therefore, ϕ_4 is an injection from $\Pi[n-4,4]$ to $\Pi[n-3,3]$. Also, the complementary set of $\phi_4(\Pi[n-4,4])$ in $\Pi[n-3,3]$ is $\mathcal{NS}_4$.

Therefore, by the Haglund-Morse-Zabrocki conjecture, we now have that

$$\nabla s_{[n-4,4]}[X] = (-q)^{n-3}\Big(\sum_{PF\in\Pi[n-3,3]} w(PF) - q\sum_{PF\in\Pi[n-4,4]} w(PF)\Big)$$

$$= (-q)^{n-3}\Big(\sum_{PF\in\Pi[n-3,3]} w(PF) - \sum_{PF\in\Pi[n-4,4]} w(\phi_4(PF))\Big)$$

$$= (-q)^{n-3}\sum_{PF\in\mathcal{NS}_4} t^{area(PF)}q^{dinv(PF)}Q_{ides(PF)}[X]$$

.

□

Now, suppose $n = 7$. We will denote v_{n-6} by g for convenience. For *Case A*, *Case B*, and *Case C*, we can apply the same injection with the case $n > 7$. For *Case D* with the first possible pattern in Figure 4.29, we also use the same injection as above. With the second possible pattern in Figure 4.29, if we have conditions $[e < c$ and $b < d]$ or $[d < b$ and $e < d]$, we can use the same injection as above.

We need to think carefully when [$c < e$ and $b < d$] or [$d < b$ and $d < e$] with the second possible pattern in Figure 4.29. Suppose $c < e$ and $b < d$. We can't have the shape in the right side of Figure 4.34 when $n = 7$. With the shape in the left side of Figure 4.34, if [$e < d$ and $f < d$] or [$d < e$ and $f < c$], then we can use the same injection as the case when $n > 7$. Also, if [$c < f, d < f, d < e$ and $g < b$], [$e < d, d < f$ and $g < b$], and [$d < e, c < f, f < d$ and $g < b$], we also use the same injection as above. The cases we need to make a new injection are when we have additional conditions [$c < f, d < f, d < e$ and $b < g$], [$e < d, d < f$ and $b < g$], and [$d < e, c < f, f < d$ and $b < g$].

Suppose $c < e$, $b < d$, $c < f$, $d < f$, $d < e$ and $b < g$. Replacing these seven cars as Figure 4.61 gives a legal parking function PF'.

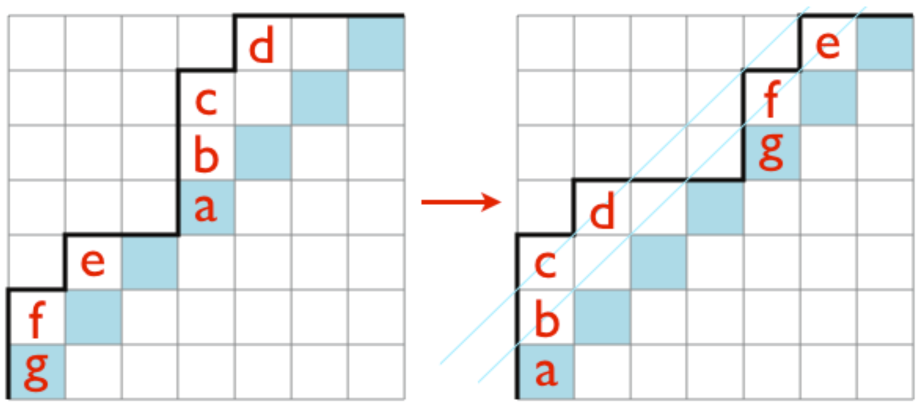


Figure 4.61: $c < e$, $b < d$, $c < f$, $d < f$, $d < e$ and $b < g$

Note that PF' has the same *area* as PF. The diagonal word has been changed without changing the *ides* from $[d, c, b, e, f, a, g]$ to $[d, c, e, b, f, a, g]$ since $b < c < e$, to $[d, c, e, f, b, a, g]$ agian since $b < d < f$, and to $[d, c, e, f, b, g, a]$ finally since $a < b < g$. For *dinv* in PF, the pairs (b, f), (b, e), and (a, g) are not the primary *dinv*, while the pairs (a, f) and (a, e) are the secondary *dinv*. In PF', the pairs (b, f), (b, e) and (a, g) are the primary *dinv*, while the pairs (b, g), (c, f), (c, e), (d, f) and (d, e) are not the secondary *dinv*. Hence, PF' has one more *dinv* than PF, so we can have $\phi_4(PF) = PF'$.

Suppose $c < e$, $b < d$, $e < d$, $d < f$ and $b < g$ in PF. We move seven cars as the right side of Figure 4.62 and get a legal parking function PF' with the same *area* as PF.

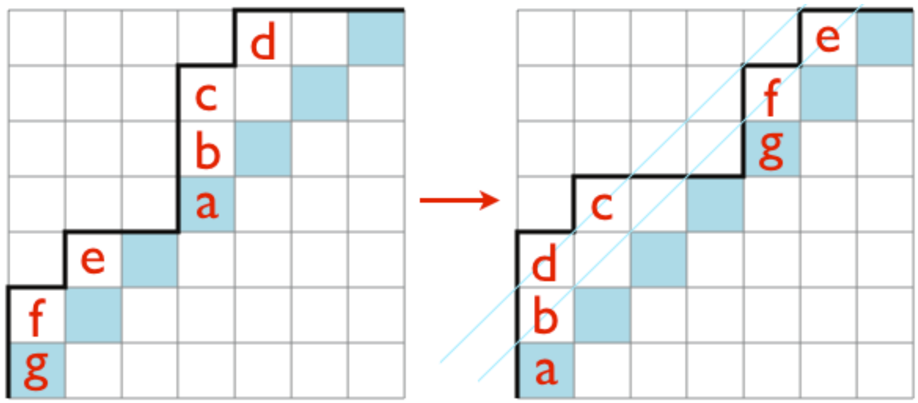


Figure 4.62: $c < e$, $b < d$, $e < d$, $d < f$ and $b < g$

The diagonal word of PF, $[d, c, b, e, f, a, g]$ can be changed, not changing the *ides*, to $[c, d, b, e, f, a, g]$ since $c < e < d$, to $[c, d, e, b, f, a, g]$ since $b < c < e$, to $[c, d, e, f, b, a, g]$ since $b < g < f$, and to $[c, d, e, f, b, g, a]$, which is the diagonal word of PF', since $a < b < g$. In PF, the pair (c, d) is the primary *dinv*, but the pairs (b, f), (b, e), and (a, g) are not the primary *dinv*. Also, the pairs (a, f) and (a, e) are the secondary *dinv* in PF. In PF', the pair (c, d) is not the primary *dinv*, but the pairs (b, f), (b, e), and (a, g) are the primary *dinv*. The pair (d, e) contributes to the secondary *dinv*, while the pairs (b, g), (d, f), (c, f) and (c, e) don't. Therefore, PF' has one additional *dinv* and we let $\phi_4(PF) = PF'$.

Suppose $c < e$, $b < d$, $d < e$, $c < f$, $f < d$ and $b < g$. Moving seven cars as the right side of Figure 4.63 gives a legal parking function PF' whose *area* is the same as PF.

The diagonal word of PF, $[d, c, b, e, f, a, g]$ can be changed, not changing the *ides*, to $[d, c, e, b, f, a, g]$ since $b < c < e$, to $[d, c, e, f, b, a, g]$ since $b < c < f$, to $[d, c, f, e, b, a, g]$ since $f < d < e$, and to $[d, c, f, e, b, g, a]$, which is the diagonal word of PF', since $a < b < g$. For *dinv* in PF, the pair (f, e) is the primary *dinv*, but the pairs (b, f), (b, e), and (a, g) are not the primary *dinv*. Also, the pairs (a, f) and (a, e) are the secondary *dinv* in PF. In PF', the pair (f, e) is not the primary *dinv*, but the pairs (b, f), (b, e), and (a, g) are the primary *dinv*. The

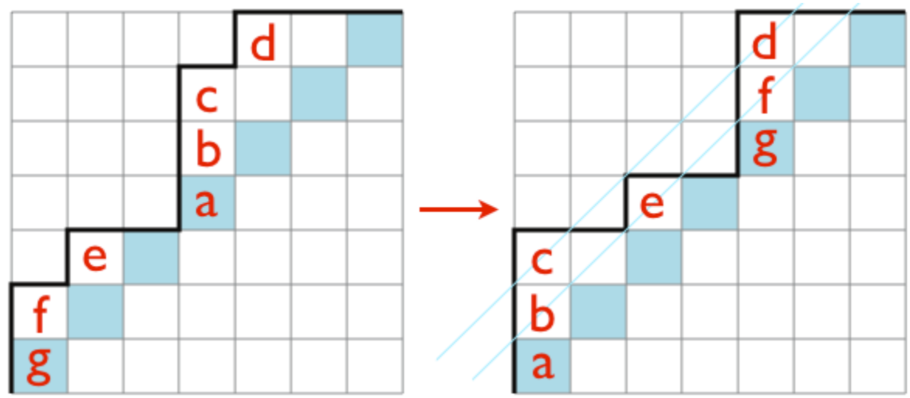


Figure 4.63: $c < e$, $b < d$, $d < e$, $c < f$, $f < d$ and $b < g$

pair (e, g) contributes to the secondary *dinv*, while the pairs (b, g), (c, f) and (c, e) don't. Therefore, we have $dinv(PF') = dinv(PF) + 1$ and we let $\phi_4(PF) = PF'$.

Now, suppose $d < b$ and $d < e$ with the second possible pattern in Figure 4.29. Again, we can't have the shape in the right side of Figure 4.34 when $n = 7$. With the shape in the left side of Figure 4.34, if $f < c$, then we can use the same injection as the case when $n > 7$. When $c < f$, if we have additional conditions $[c < e$ and $g < b]$ or $[e < c$ and $g < e]$, we also use the same injection as above. The cases we need to consider as new cases are when we have additional conditions $[c < e$ and $b < g]$ and $[e < c$ and $e < g]$.

First, suppose $c < e$ and $b < g$. Remember that we already have conditions, $d < b$, $d < e$ and $c < f$. In this case, we move seven cars as the right side of Figure 4.64 and get a new parking function PF' in $\Pi[4, 3]$.

PF and PF' have the same *area*. The diagonal word of PF, $[d, c, b, e, f, a, g]$, is changed without changing the *ides*, to $[d, c, e, b, f, a, g]$ since $b < c < e$, to $[d, c, e, f, b, a, g]$ since $b < g < f$, and to $[d, c, e, f, b, g, a]$ finally, the diagonal word of PF', since $a < b < g$. In PF, the pairs (a, g), (b, f) and (b, e) are not the primary *dinv* while the pairs (a, f) and (a, e) are the secondary *dinv*. The pairs (a, g), (b, f) and (b, e) contribute to the primary *dinv* and there is no secondary *dinv* in PF'. Hence, $dinv(PF') = dinv(PF) + 1$ and we let $\phi_4(PF) = PF'$.

Next, suppose $d < b$, $d < e$, $c < f$, $e < c$ and $e < g$. In this case, we need to divide into four subcases again. Suppose $e < a$ and $g < b$ additionally (Indeed,

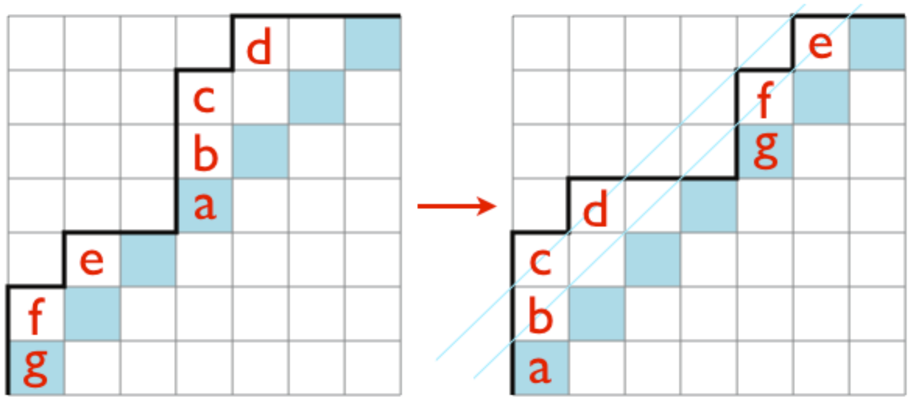


Figure 4.64: $d < b$, $d < e$, $c < f$, $c < e$ and $b < g$

in this case, we don't need a condition $e < c$ if we have $e < a$ since $a < b < c$). We rearrange seven cars as the right side of Figure 4.65 and get a new legal parking function PF'.

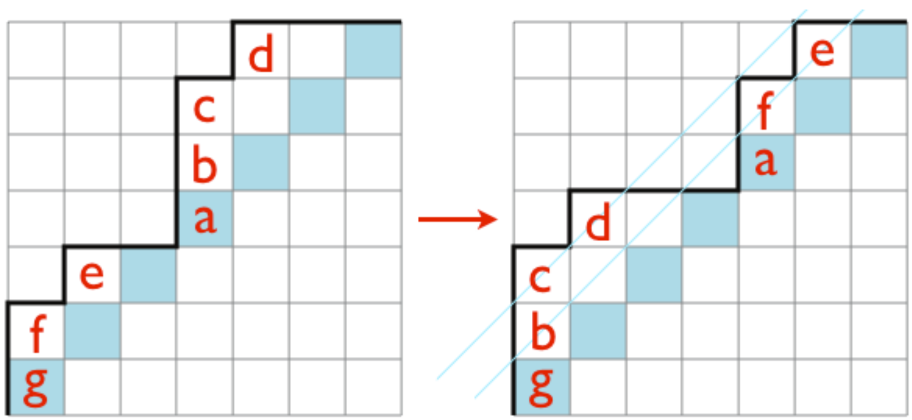


Figure 4.65: $d < b$, $d < e$, $c < f$, $e < c$, $e < g$, $e < a$ and $g < b$

Notice that PF and PF' have the same *area*. The diagonal word of PF has been changed from $[d, c, b, e, f, a, g]$ to $[d, c, e, b, f, a, g]$ and again to $[d, c, e, f, b, a, g]$, but the *ides* has not been changed since $e < a < b$ and $b < c < f$. For *dinv* of PF, the pair (b, e) is the primary *dinv*, but the pair (b, f) is not. Also, the pair (a, f) is the secondary *dinv*, but the pair (a, e) is not. For the *dinv* of PF', the pair (b, f) is the primary *dinv*, but the pair (b, e) is not. The pairs (a, b) and (c, e) are the secondary *dinv*, but the pairs (c, f), (d, f), and (d, e) are not the secondary *dinv*. Therefore, $dinv(PF') = dinv(PF) + 1$ as desired and we can let $\phi_4(PF) = PF'$.

Next, suppose $d < b$,$d < e$, $c < f$, $e < c$, $e < g$, $e < a$ and $b < g$. We replace these seven cars as the right side of Figure 4.66 and get a new parking fucntion PF' with the same *area* as PF.

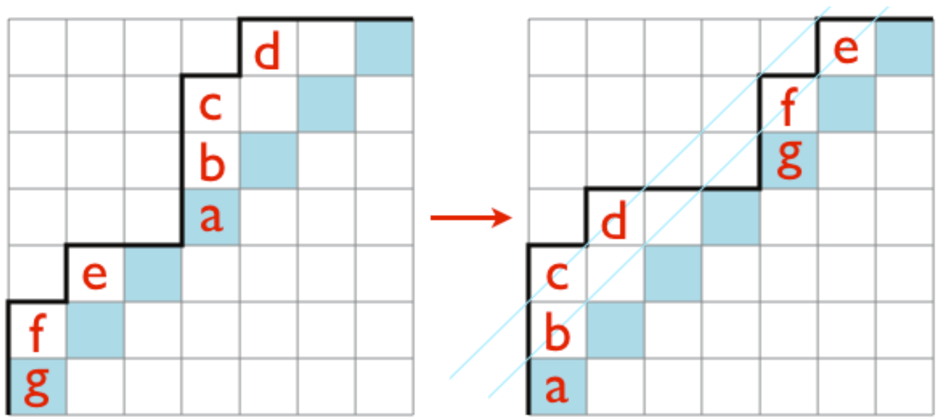


Figure 4.66: $[d < b, d < e, c < f, e < c, e < g, e < a$ and $b < g]$ or $[d < b, d < e, c < f, e < c, e < g, a < e$ and $g < b]$

The diagonal word of PF is $[d, c, b, e, f, a, g]$, but it can be changed, keeping the *ides* same, to $[d, c, e, b, f, a, g]$ since $e < a < b$, to $[d, c, e, f, b, a, g]$ since $b < c < f$, and to $[d, c, e, f, b, g, a]$ since $a < b < g$. For *dinv* in PF, the pair (b, e) is the primary *dinv* and the pair (a, f) is the secondary *dinv*, but the pairs (b, f), (a, g) and (a, e) are not the primary or secondary *dinv*. In PF', the pairs (b, f) and (a, g) are the primary *dinv* and the pair (c, e) is the secondary *dinv* while the pairs (b, e), (b, g), (c, f), (d, f) and (d, e) are not the primary or secodnary *dinv*. Hence, we have $dinv(PF') = dinv(PF) + 1$ as desired and we let $\phi_4(PF) = PF'$.

Now, suppose $d < b$, $d < e$, $c < f$, $e < c$, $e < g$, $a < e$ and $g < b$. In this case, we use the same injection as Figure 4.66. Now, the diagonal word of PF, $[d, c, b, e, f, a, g]$, can be changed with the same *ides* to $[d, c, e, b, f, a, g]$ since $e < g < b$, to $[d, c, e, f, b, a, g]$ since $b < c < f$, and to $[d, c, e, f, b, g, a]$, the diagonal word of PF', since $a < e < g$. For *dinv* in PF, the pair (b, e) is the primary *dinv* and the pairs (a, f) and (a, e) are the secondary *dinv*, but the pairs (b, f)

and (a, g) are not the primary *dinv*. In PF', the pairs (b, f) and (a, g) are the primary *dinv* and the pairs (b, g) and (c, e) are the secondary *dinv* while the pairs (b, e), (c, f), (d, f) and (d, e) are not the primary or secodnary *dinv*. Again, we have $dinv(PF') = dinv(PF) + 1$ and we let $\phi_4(PF) = PF'$.

Suppose $d < b$, $d < e$, $c < f$, $e < c$, $e < g$, $a < e$ and $b < g$. Replacing seven cars as Figure 4.67 gives a legal parking fucntion PF' with the same *area* as the one of PF.

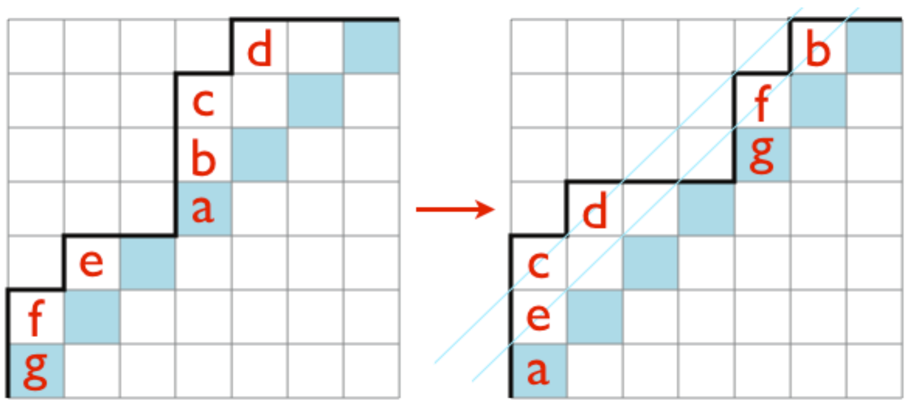


Figure 4.67: $d < b$, $d < e$, $c < f$, $e < c$, $e < g$, $a < e$ and $b < g$

The diagonal word has been changed from $[d, c, b, e, f, a, g]$ to $[d, c, b, f, e, a, g]$ and to $[d, c, b, f, e, g, a]$ again, but the *ides* is unchanged since $e < g < f$ and $a < b < g$. The pairs (e, f) and (a, g) are not the primary *dinv*, while the pairs (a, f) and (a, e) are the secondary *dinv* in PF. In PF', the pairs (e, f) and (a, g) are the primary *dinv* and the pair (b, c) is the secondary *dinv*, while the pairs (e, g), (c, f), (d, f), and (b, d) are not the secondary *dinv*. PF' has exactly one more *dinv* than PF, so we can let $\phi_4(PF) = PF'$.

Finally, we need to think about *Case E* when $n = 7$. Any parking function in $\Pi[3, 4]$ can't have the first possible shape in Figure 4.48. If the last eight columns of a parking function are the same as the second possible shape in Figure 4.48, and if $e < d$, we can use the same injection as $n > 7$ case. If $d < e$ with the second possible shape in Figure 4.48, we divide into two subcases as Figure 4.51. If a parking fucntion has the first shape in Figure 4.51 with $d < e$, and if $f < c$ or $[c < f$ and $g < b]$, we also can use the same injection as above. If a parking fucntion has the first shape in Figure 4.51 with $d < e$, $c < f$ and $b < g$, we replace the seven cars as the right side of Figure 4.68 and get a new parking function PF'.

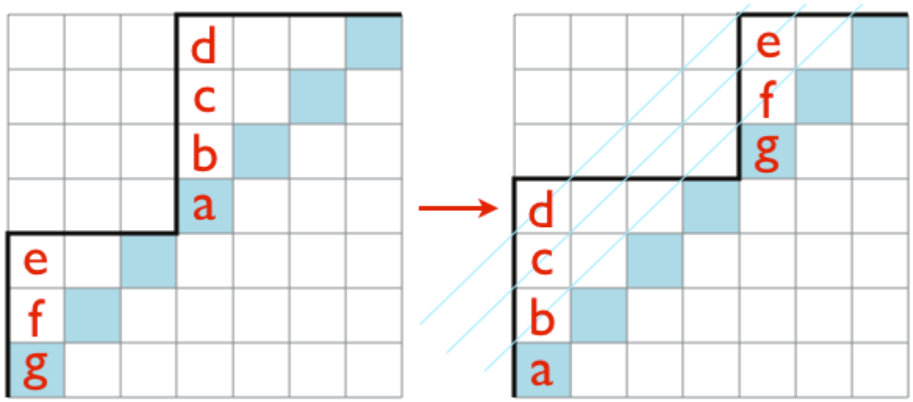


Figure 4.68: $d < e$, $c < f$ and $b < g$

Note that PF and PF' have the same *area*. The diagonal word of PF is $[d, c, e, b, f, a, g]$, and it has been changed to $[d, e, c, b, f, a, g]$, to $[d, e, c, f, b, a, g]$, and to $[d, e, c, f, b, g, a]$, but the *ides* of PF' is the same the one of PF since $c < d < e$, $b < c < f$ and $a < b < g$. In PF, the pairs (c, e), (b, f), and (a, g) are not the primary *dinv*, but the pairs (a, f) and (b, e) are the secondary *dinv*. In PF', the pairs (c, e), (b, f) and (a, g) are the primary *dinv*, while the pairs (d, e), (c, f) and (b, g) don't contribute to the secondary *dinv*. Therefore, PF' has exactly one more *dinv* than PF, so we can let $\phi_4(PF) = PF'$.

Any parking fucntion in $\Pi[3, 4]$ can not have the second shape in Figure 4.51. Also, it can't have the third or the fourth shape in Figure 4.48. Instead, we need to think about the case that the *area* of PF is $[0, 1, 1, 0, 1, 2, 3]$. In this case, if $f < c$, we have the injection as Figure 4.69 or Figure 4.70.

Note that these cases have the same injection as the case when $n > 7$. Since $f < c$, we can put the car c a top of the car f and this injection is the same as the one with the third possible shape in Figure 4.48.

Suppose $c < f$ with the *area* $[0, 1, 1, 0, 1, 2, 3]$. If $g < b$ and $[c < e$ or $e < a]$, we can use the same injection as the case $n > 7$. Also, if $g < e$ and $a < e < c$, the same injection can be used. Therefore, only cases we need to make new injection are when $\big[b < g$ and $[c < e$ or $e < a]\big]$ and when $[e < g$ and $a < e < c]$.

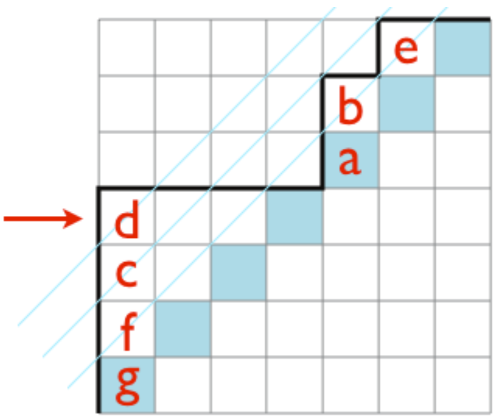


Figure 4.69: $f < c$ and $[e < a \text{ or } c < e]$

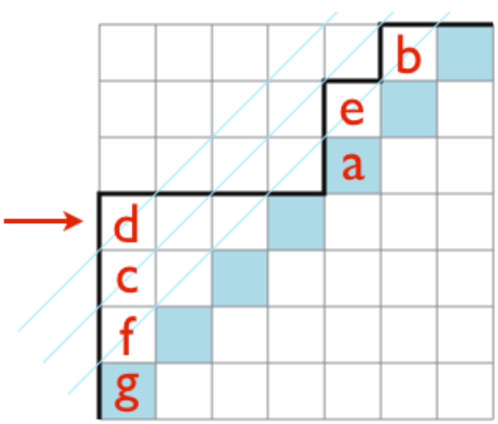


Figure 4.70: $f < c$ and $a < e < c$

Now, suppose $c < f$, $b < g$ and $e < a$ with the *area* $[0, 1, 1, 0, 1, 2, 3]$ as the left side of Figure 4.71. Replacing seven cars as the right side of Figure 4.71 gives us a new parking function PF' in $\Pi[4, 3]$.

PF' has the same *area* as the one of PF. The diagonal word has been changed from $[d, c, b, e, f, a, g]$ to $[d, c, e, b, f, a, g]$, to $[d, c, e, f, b, a, g]$, and finally to $[d, c, e, f, b, g, a]$ without any change of the *ides* since $e < a < b$, $b < g < f$, and $a < b < g$. The pair (b, e) is the primary *dinv* and the pair (a, f) is the secondary *dinv*, while the pairs (b, f), (a, g), and (a, e) are not the primary or secondary *dinv* in PF. For *dinv* of PF', the pairs (b, f) and (a, g) are the primary *dinv* and the

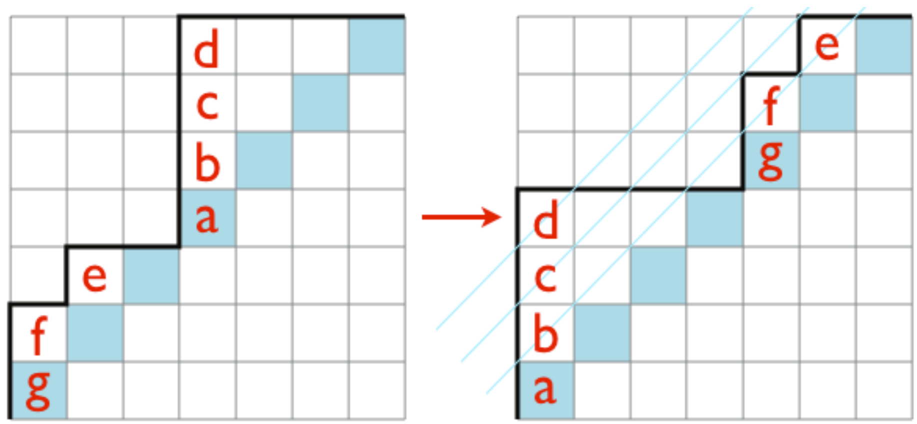


Figure 4.71: $c < f$, $b < g$ and $[e < a \text{ or } c < e]$

pair (c, e) is the secondary *dinv*, while the pairs (b, e) is not the primary *dinv* and the pairs (b, g) and (c, f) are not secondary *dinv*. Hence, PF' has exactly one more *dinv* than PF, so we can let $\phi_4(PF) = PF'$.

When $c < f$, $b < g$ and $c < e$ with the *area* $[0, 1, 1, 0, 1, 2, 3]$, we can use the same injection as Figure 4.71. The diagonal word has been changed with the same steps, but when it changed from $[d, c, b, e, f, a, g]$ to $[d, c, e, b, f, a, g]$, the *ides* is still unchanged since $b < c < e$. In PF, the pairs (b, e), (b, f) and (a, g) are not the primary *dinv*, but the pairs (a, f) and (a, e) are the secondary *dinv*. For PF', the pairs (b, e), (b, f), and (a, g) are the primary *dinv*, but the pairs (b, g), (c, f), and (c, e) are not the secondary *dinv*. Again, we have $dinv(PF') = dinv(PF) + 1$ and we let $\phi_4(PF) = PF'$.

Finally, suppose $c < f$, $e < g$ and $a < e < c$ with the *area* $[0, 1, 1, 0, 1, 2, 3]$. In this case, we move seven cars as the right side of Figure 4.72 and get a legal parking function PF' with the same *area*.

Since $e < c < f$ and $a < e < g$, the diagonal word can be changed from $[d, c, b, e, f, a, g]$ to $[d, c, b, f, e, a, g]$ and to $[d, c, b, f, e, g, a]$ again and the *ides* is unchanged. In PF, the pairs (e, f) and (a, g) are not the primary *dinv*, but the pairs (a, f) and (a, e) are the secondary *dinv*. Meanwhile, the pairs (e, f) and (a, g) contribute to the primary *dinv* and the pair (b, c) contributes to the secondary *dinv*, but the pairs (e, g) and (c, f) are not the secondary *dinv* in PF'. Hence,

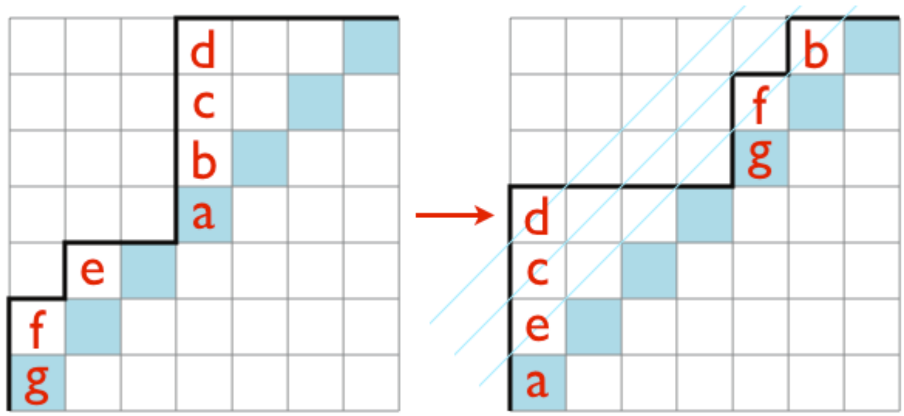


Figure 4.72: $c < f$, $e < g$ and $a < e < c$

$dinv(PF') = dinv(PF) + 1$ as desired and we can have $\phi_4(PF) = PF'$.

As the injection from $\Pi[n-3,3]$ to $\Pi[n-2,2]$, $\mathcal{NS}_4$ should be an empty set when $n = 7$ if we remove the image parking functions under ϕ_4. Now, we compare $\mathcal{NS}_4$ to the set of image parking functions which made by the injection only when $n = 7$. Any parking functions in $\Pi[4,3]$ can't have a Dick path with the shape of (b), (c), (d), (e), (g), (h) in Figure 4.11 and (a), (f), (g), or (h)Figure 4.12. The other cases in Figure 4.11 or Figure 4.12 are possible shapes when $n = 7$, if the first step is not going east, but going north, ending at the same point. The disjoint union of the set of parking functions with Figure 4.61 and Figure 4.64 is the same as the set of parking function with the shape of (b) in Figure 4.12. The set of parking functions with Figure 4.62 is the same as the case with the shape of (c) in Figure 4.12. Also, Figure 4.63 case is the same as the (a) case in Figure 4.11. The disjoint union of the set of parking functions with Figure 4.65, Figure 4.66, and Figure 4.67 is the same as the set of parking function with the shape of (d) in Figure 4.12. The set of parking functions with Figure 4.68 is the same as the case with the shape of (f) in Figure 4.11 and the disjoint union of the set of parking functions with Figure 4.71 and Figure 4.72 is the same as the set of parking function with the shape of (e) in Figure 4.12. Therefore, after removing the image parking function by ϕ_4 when $n = 7$, $\mathcal{NS}_4$ becomes an empty set and it can be also verified by

$$\nabla s_{3,4}[X] = (-q)^4\Big(\nabla\mathbf{C}_4\mathbf{C}_3\mathbf{1} - q\nabla\mathbf{C}_3\mathbf{C}_4\mathbf{1}\Big) = 0.$$

4.3 The Final Conclusion

We now can make the final conclusion that, to find an injection from $\Pi[n-m,m]$ to $\Pi[n-m+1,m-1]$ for any $n \geq 2m-1$, we only need to find a bijection between $\Pi[m-1,m]$ to $\Pi[m,m-1]$, which is the smallest case $n = 2m-1$.

Since

$$s_{a,b}[X] = (-q)^{a+b-3}\big(\mathbf{C}_{a+1}\mathbf{C}_{b-1}\mathbf{1} - q\mathbf{C}_a\mathbf{C}_b\mathbf{1}\big)$$

and

$$q(\mathbf{C}_b\mathbf{C}_a + \mathbf{C}_{a-1}\mathbf{C}_{b+1}) = \mathbf{C}_a\mathbf{C}_b + \mathbf{C}_{b+1}\mathbf{C}_{a-1},$$

we have

$$\nabla s_{m-1,m}[X] = (-q)^{a+b-3}\big(\nabla\mathbf{C}_m\mathbf{C}_{m-1}\mathbf{1} - q\nabla\mathbf{C}_{m-1}\mathbf{C}_m\mathbf{1}\big) = 0.$$

It means there must be a bijection between $\Pi[m-1,m]$ and $\Pi[m,m-1]$. Once we find the bijection, we can use the same mapping to make an injection from $\Pi[n-m,m]$ to $\Pi[n-m+1,m-1]$ for any $n > 2m-1$. It is because as we have seen in $\nabla s_{n-3,3}$ case and $\nabla s_{n-4,4}$ case, the few cars we have in the $m-1$ columns preceding the last m columns, the simpler conditions we have when we make an mapping, because we have fewer pairs of cars conflicting the condition of parking functions on a lattice square; the cars in one column should be in increasing order.

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