Title
The Cube Problem for Linear Orders

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Publication Date
2017

UNIVERSITY OF CALIFORNIA,
IRVINE

The Cube Problem for Linear Orders

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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2017

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ACKNOWLEDGMENTS

Much of this thesis is adapted from my article “Every linear order isomorphic to its cube is isomorphic to its square,” appearing in *Advances in Mathematics*, Volume 313(2017), 237-281 (http://www.sciencedirect.com/science/article/pii/S0001870816313597). I am indebted to Justin Moore, whose close reading of my original draft not only helped significantly improve the final presentation, but was so encouraging to me.

I gratefully acknowledge the support of NSF grant DMS-1044150.

Thank you Martin, for letting me do my thing and listening, no matter how far afield I went;
and Matt, for your interest and support over the years;
and Lana, for bringing me to Irvine;
and Geoff, for being my companion and competitor in my mathematical life and outside of it;
and Micah, for being my intellectual light.

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ABSTRACT OF THE DISSERTATION

The Cube Problem for Linear Orders

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2017

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In 1958, Sierpiński asked whether there exists a linear order X that is isomorphic to its lexicographically ordered cube but is not isomorphic to its square. The corresponding question has been answered positively for many different classes of structures, including groups, rings, graphs, Boolean algebras, and topological spaces of various kinds. However, the main result of this thesis is that the answer to Sierpiński's question is negative: every linear order X that is isomorphic to its cube is already isomorphic to its square. More generally, if X is isomorphic to any one of its finite powers X^n, $n > 1$, it is isomorphic to all of them.

The proof relies on a general representation theorem that characterizes, for a fixed structure A from a class of structures $\mathfrak{C}$, those structures $X \in \mathfrak{C}$ that satisfy the isomorphism $A \times X \cong X$. This characterization is based on an analysis of an arbitrary bijection $f : A \times X \to X$, and is closely connected to the tail-equivalence relation on the Baire space A^ω.

In Chapter 1, we study the tail-equivalence relation as well as those continuous maps on A^ω that preserve tail-equivalence. In Chapter 2, we give our characterization of the isomorphism $A \times X \cong X$, and specify it for several particular classes of structures, including the class of linear orders in which we are primarily interested. In Chapter 3, we use this characterization to solve Sierpiński's problem, as well as several other problems concerning the multiplication

of linear orders. In Chapter 4, we solve a related problem, also due to Sierpiński, by showing there exist non-isomorphic orders X and Y that divide one another on both sides.

Introduction

> We wake and find ourselves on a stair;
> there are stairs below us, which we
> seem to have ascended; there are stairs
> above us, many a one, which go
> upward and out of sight...
>
> *Emerson*

Self-Similar Structures

Suppose that $(\mathfrak{C}, \times)$ is a class of structures equipped with an associative cartesian product. Given a fixed structure $A \in \mathfrak{C}$, a natural problem is to determine which structures $X \in \mathfrak{C}$ are invariant under left multiplication by A, that is, which X satisfy the isomorphism $A \times X \cong X$. It turns out that for many classes it is possible to characterize such structures. Roughly speaking, they can only be obtained by replacing points in the infinite product A^ω with structures from $\mathfrak{C}$, so that tail-equivalent points are replaced by isomorphic structures.

Before we state the result precisely, we recall what it means for two sequences to be tail-equivalent in A^ω.

Definition. Given two sequences $u, v \in A^\omega$, we say u and v are *tail-equivalent*, and write $u \sim v$, if there exist finite sequences $r, s \in A^{<\omega}$ and an infinite sequence $u' \in A^\omega$ such that $u = ru'$ and $v = su'$.

Tail-equivalent sequences have equal tails, but the initial sequences preceding those tails can be of differing lengths. It is easy to see that tail-equivalence is in fact an equivalence relation. We denote the equivalence class of a given $u \in A^\omega$ by $[u]$.

Tail-equivalence classes are formally the smallest subsets of A^ω that are invariant under left multiplication by A. That is, for an arbitrary subset $X \subseteq A^\omega$, if we define $A \times X$ as the set $\{au : a \in A, u \in X\}$, then $A \times X = X$ if and only if X is a union of tail-equivalence classes.

This fact can be used to produce many examples of structures invariant under left multiplication by A, as follows. Suppose that for every tail-equivalence class $[u]$ we fix a structure $I_{[u]} \in \mathfrak{C}$. Let $A^\omega(I_{[u]})$ denote the "structure" obtained by replacing every point $u \in A^\omega$ with the corresponding $I_{[u]}$. The underlying set of points in $A^\omega(I_{[u]})$ is $\{(u, x) : u \in A^\omega, x \in I_{[u]}\}$. If $u \sim v$, then both u and v are replaced by $I_{[u]} = I_{[v]}$, but inequivalent points may be replaced by different structures. Depending on context, certain extra restrictions may need to be placed on the $I_{[u]}$ in order to make $A^\omega(I_{[u]})$ a sensible structure. This "replacement" operation generalizes the usual product, since if there is a structure Y such that $I_{[u]} = Y$ for all u, then $A^\omega(I_{[u]})$ is simply $A^\omega \times Y$.

Structures of this form are naturally invariant under left multiplication by A: if $X = A^\omega(I_{[u]})$, then $A \times X \cong X$. The isomorphism is defined by the rule $(a, u, x) \mapsto (au, x)$. The fact that structures replacing tail-equivalent points are identical is necessary for this map to make sense.

It turns out that this is the only way to form such structures.

"Theorem". Fix a class of structures $\mathfrak{C}$ and a structure $A \in \mathfrak{C}$. For any structure $X \in \mathfrak{C}$, we have $A \times X \cong X$ if and only if X is isomorphic to a structure of the form $A^\omega(I_{[u]})$.

How to turn this "theorem" into a theorem depends on the class $\mathfrak{C}$. Here are some examples:

Representation Theorems.

a. Fix a set A. Then for any set X, there is a bijection between $A \times X$ and X if and only if $X \cong A^\omega(I_{[u]})$ for some collection of sets $I_{[u]}, u \in A^\omega$. This holds even in the absence of the axiom of choice.

b. Fix a group G. If H is a subgroup of G^ω that is closed under tail-equivalence, and $X = H \ltimes N$ is a semi-direct product of H defined with respect to a "$\sim$-invariant" homomorphism, then $G \times X \cong X$. In the other direction, if $G \times X \cong X$ then there is a normal subgroup N of G such that G/N is isomorphic to a subgroup H of G^ω that is closed under tail-equivalence.

c. Fix a topological space T. For any topological space X, we have $T \times X \cong X$ if and only if $X \cong T^\omega(I_{[u]})$, where the topology on T^ω can be the product topology, the box topology, or any intermediate topology that is "closed under multiplication by T."

d. Fix a linear order L and let $\times$ denote the lexicographical product of linear orders. Then for any order X, we have $L \times X \cong X$ if and only if $X \cong L^\omega(I_{[u]})$ for some collection of linear orders $I_{[u]}$.

An *iterated function system* (IFS) is a finite collection of contraction mappings $\{f_1, \ldots, f_n\}$ on a complete metric space. A fundamental result, due to Hutchinson [5], is that any such system has a unique attractor. That is, there is a unique compact set K such that $K = \bigcup f_i(K)$. Moreover, this attractor is naturally homeomorphic to a quotient of Cantor space (on n symbols), and under this homeomorphism each f_i becomes the shift map $u \mapsto iu$.

The representation theorems above can be viewed as analogues to Hutchinson's result. If A and X are structures such that $A \times X \cong X$, then X can be decomposed into "A-many copies of itself." Hence there is a collection of mappings $\{f_a : a \in A\}$ such that for each $a \in A$, the map f_a sends X onto the ath copy of itself within itself, and we have $X = \bigcup f_a(X)$. Moreover there is a natural isomorphism identifying X, not as a quotient of Cantor space, but as a replacement of A^ω. Under this isomorphism the f_a become shift maps on A^ω. Since there is no notion of metric, the f_a are not contractions. As a result, the iterated images of X under a sequence of these maps need not converge to a point, as they do in the case of an IFS. However, they do converge to a substructure (or, in certain instances, the "coset of a substructure"), and it is possible to show that substructures associated to tail-equivalent sequences are isomorphic.

The "theorem" above can be generalized to characterize the isomorphism $A^n \times X \cong X$ for a fixed positive integer n. For this isomorphism the relevant equivalence relation is not the tail-equivalence relation $\sim$, but rather the finer n-tail-equivalence relation $\sim_n$. Because they are central to all subsequent work in the thesis, the relations $\sim_n$ are studied in detail in Chapter 1. There, we will not work over some particular class of structures, but simply take A to be an arbitrary set with the discrete topology and A^ω to be its infinite product with the product topology. We will also consider those continuous maps on A^ω that preserve tail-equivalence. In Chapter 2, we prove the "theorem" above and derive the representation theorems $(a) - (d)$.

Cube Problems

In many classes of structures $(\mathfrak{C}, \times)$, it is possible to find an infinite structure X that is isomorphic to its own square. If $X^2 \cong X$, then by multiplying again we have $X^3 \cong X$. Determining whether the converse holds for a given class $\mathfrak{C}$, that is, whether $X^3 \cong X$ implies

$X^2 \cong X$ for all $X \in \mathfrak{C}$, is called the *cube problem* for $\mathfrak{C}$. If the cube problem for $\mathfrak{C}$ has a positive answer, then $\mathfrak{C}$ is said to have the *cube property.*

The cube problem is related to three other basic problems concerning the multiplication of structures in a given class $(\mathfrak{C}, \times)$.

1. Does $A \times Y \cong X$ and $B \times X \cong Y$ imply $X \cong Y$ for all $A, B, X, Y \in \mathfrak{C}$? Equivalently, does $A \times B \times X \cong X$ imply $B \times X \cong X$ for all $A, B, X \in \mathfrak{C}$?

2. Does $X^2 \cong Y^2$ imply $X \cong Y$ for all $X, Y \in \mathfrak{C}$?

3. Does $A \times Y \cong X$ and $A \times X \cong Y$ imply $X \cong Y$ for all $A, X, Y \in \mathfrak{C}$? Equivalently, does $A^2 \times X \cong X$ imply $A \times X \cong X$ for all $A, X \in \mathfrak{C}$?

The first question is sometimes called the *Schroeder-Bernstein problem* for $\mathfrak{C}$, and if it has a positive answer, then $\mathfrak{C}$ is said to have the *Schroeder-Bernstein property.* The second question is called the *unique square root problem* for $\mathfrak{C}$, and if its answer is positive, then $\mathfrak{C}$ has the *unique square root property.* Taken together, the first two questions are sometimes called the *Kaplansky test problems*, after Irving Kaplansky who posed them in [9] as a heuristic test for whether a given class of abelian groups (under the direct product) has a satisfactory structure theory ("I believe their defeat is convincing evidence that no reasonable invariants exist"). Tarski [17] had posed them previously for the class of Boolean algebras. All three questions are listed in Hanf's seminal paper [4] on products of Boolean algebras. We will refer to Question 3 as the *weak Schroeder-Bernstein problem* for $\mathfrak{C}$, and the corresponding property as the *weak Schroeder-Bernstein property.*

A negative solution to Question 3 obviously gives a negative solution to Question 1. If the product for $\mathfrak{C}$ is commutative, it gives a negative solution to Question 2 as well. If the cube problem for $\mathfrak{C}$ has a negative solution, that is, if there is an $X \in \mathfrak{C}$ that is isomorphic to its cube but not to its square, then all three questions have a negative solution, without assuming commutativity of the product. In practice, it is often by constructing such an X that these three problems are solved.

If the class $\mathfrak{C}$ does not contain any infinite structure isomorphic to its cube, then the cube property holds trivially. When the cube property does not hold trivially, it usually fails. The first result in this direction is due to Hanf, who constructed in [4] a Boolean algebra that is isomorphic to its cube but not its square. Tarski [18] and Jónsson [7] immediately adapted Hanf's result to show the failure of the cube property for the class of semigroups, the class of groups, the class of rings, and various other classes of algebraic structures. Hanf's example, and consequently many of those produced by Tarski and Jónsson, is of size continuum, and for some time it was open whether there were countable examples witnessing the failure of the cube property for these various classes.

In 1965, Corner showed in [1] that indeed there exists a countable (torsion-free, abelian) group G isomorphic to G^3 but not G^2. Later, Jones [6] showed that it is even possible to construct a finitely generated (necessarily non-abelian) group isomorphic to its cube but not its square. In 1979, Ketonen [10] solved the so-called Tarski cube problem by producing a countable Boolean algebra isomorphic to its cube but not its square.

Throughout the 1970s and 1980s, Trnková solved the cube problem negatively for many different classes of topological spaces and relational structures, including the class of graphs under several different notions of graph product [22]. Her topological results are summarized in [21]. Answering a question of Trnková, Orsatti and Rodino showed in [13] that there is even a *connected* topological space homeomorphic to its cube but not its square. Koubek, Nešetřil, and Rödl [11] showed that the cube property fails for the class of partial orders, as well as for other classes of relational structures. More recently, Eklof and Shelah [2] constructed an $\aleph_1$-separable group isomorphic to its cube but not its square, and Gowers [3] constructed a Banach space linearly homeomorphic to its cube but not its square.

On the other hand, there are rare instances when the cube property holds nontrivially. It holds for the class of sets under the cartesian product, since any set in bijective correspondence with its cube is either infinite, empty, or a singleton, and hence in bijection with its square. This is immediate if one assumes the axiom of choice, but it can be proved without the axiom of choice using the Schroeder-Bernstein theorem. Similarly easily, the cube property holds for the class of vector spaces (over a fixed field) under the direct product. Less trivially, the cube property holds for the class of countably complete Boolean algebras. This follows from the Schroeder-Bernstein theorem for such algebras. Trnková [19] showed that the cube property also holds for the class of countable metric spaces (where isomorphism means homeomorphism), as well as for closed subspaces of Cantor space [20]. Koubek, Nešetřil, and Rödl showed in [11] that the cube property holds for the class of equivalence relations. It is worth noting that for all of these classes, it is actually possible to establish the stronger Schroeder-Bernstein property.

In his 1958 book *Cardinal and Ordinal Numbers* [15], Sierpiński posed the cube problem (although he does not use the term) for the class $(LO, \times_{lex})$ of linear orders under the lexicographical product. On page 232, he writes,

"We do not know so far ... any type α such that $\alpha = \alpha^3 \neq \alpha^2$."

Here, "type" means linear order type, and the ordering on the cartesian powers α^2 and α^3 is the lexicographical ordering[1]. Although the cube problem has been solved for many other classes of structures, Sierpiński's question has remained open. One major difference in this version of the cube problem is that, unlike the products for the other classes so far discussed, the lexicographical product of linear orders is not commutative. Though he does not make a conjecture in his book, his language suggests that Sierpiński expected that such an α exists, that is, that the cube property fails for $(LO, \times_{lex})$. He was already aware of examples of

[1]Sierpiński actually ordered these powers anti-lexicographically, though in this thesis we will use the lexicographical ordering. This does not change the problem.

linear orders witnessing the failure of the unique square root property and (the right-sided and left-sided versions of) the Schroeder-Bernstein property.

The main result of this thesis is that in fact the cube property holds for $(LO, \times_{lex})$. This is proved in Chapter 3, appearing as Theorem 3.3.15 below.

Main Theorem. If X is a linear order and $X^3 \cong X$, then $X^2 \cong X$. More generally, if $X^n \cong X$ for some $n > 1$, then $X^2 \cong X$.

Thus the cube property holds for the class of linear orders despite the fact that the Schroeder-Bernstein property and unique square root property fail. We will show in Chapter 3 that even the weak Schroeder-Bernstein property fails for $(LO, \times_{lex})$. In this sense, the cube property is closer to failing for $(LO, \times_{lex})$ than it is for the other classes for which it is known to hold.

The proof of the cube property relies crucially on the characterizations given in Section 2.2.1 of the isomorphisms $A \times X \cong X$ and $A^2 \times X \cong X$ for linear orders. Using these characterizations, it is possible to write down a sufficient condition, namely the existence of a *parity-reversing automorphism* of A^ω, for the implication $A^2 \times X \cong X \implies A \times X \cong X$ to hold for every linear order X. This is done in Section 3.2.2.

The combinatorial heart of the proof is contained in Section 3.3, where parity-reversing automorphisms of A^ω are constructed for various orders A, including all countable orders. These maps are built using generalized versions of the classical Schroeder-Bernstein bijection.

In this context, the isomorphism $X^3 \cong X$ can be rewritten $X^2 \times X \cong X$. The results of Section 3.2.2 then give that if X^ω has a parity-reversing automorphism, then we must also have $X \times X \cong X$ (i.e. $X^2 \cong X$). We will show at the end of Section 3.3 that if $X^3 \cong X$, then indeed X^ω has a parity-reversing automorphism, completing the proof of the cube property.

The proof is then generalized to give $X^n \cong X \implies X^2 \cong X$ for any linear order X and $n > 1$.

In Section 3.4, we show that the main theorem is not vacuous by illustrating a general way of constructing orders X such that $X^n \cong X$ for a fixed $n > 1$. Such orders can be arranged to be of any cardinality. In Section 3.5 we show that there exists an order A such that A^ω does not have a parity-reversing automorphism, and as a consequence we will be able to construct a counterexample to the weak Schroeder-Bernstein property for $(LO, \times_{lex})$.

In *Cardinal and Ordinal Numbers*, Sierpiński posed several other questions related to the cube problem concerning the multiplication of linear orders.

1. (Sierpiński) Do there exist non-isomorphic countable orders X and Y that are right-hand divisors of one another? That is, do there exist countable orders $X \ncong Y$ such that $X \cong A \times Y$ and $Y \cong B \times X$ for some orders A, B?

In other words, Sierpiński is asking for countable witnesses to the failure of the (left-sided) Schroeder-Bernstein property. He was aware of uncountable witnesses. We will show that uncountability is in fact necessary.

Theorem. If X and Y are countable orders such that divide one another on the right, then $X \cong Y$.

This is proved in Section 3.6. A more delicate question is the following:

2. (Sierpiński) Do there exist non-isomorphic orders X and Y that are both right-handed *and* left-handed divisors of one another? That is, are there orders $X \ncong Y$ such that for some A_0, B_0, A_1, B_1 we have $X \cong A_0 \times Y \cong Y \times B_0$ and $Y \cong A_1 \times X \cong X \times B_1$?

As already indicated, Sierpiński was aware of examples of non-isomorphic orders X_0, Y_0 that divide each other on the right. Separately he knew of non-isomorphic orders X_1, Y_1 that divide each other on the left. (In other words, he was aware of examples witnessing the failure of the left-sided Schroeder-Bernstein property, and separately, the right-sided Schroeder-Bernstein property.) It is natural to ask if there are distinct orders that divide each other on both sides. We will refer to this question as the *two-sided Schroeder-Bernstein problem*. If there were an order X isomorphic to X^3 but not X^2, then the pair X, X^2 would give a positive answer. By our main theorem there are no such orders, but it turns out the answer is still positive.

Theorem. There exist non-isomorphic orders X, Y of size $2^{\aleph_0}$ that divide one another on both the left and right.

This is proved in Chapter 4 using an adaptation of the argument from Section 3.5. While such orders are necessarily uncountable, it is unknown if they can consistently have cardinality smaller than $2^{\aleph_0}$. The theorem gives further evidence that the cube property for $(LO, \times_{lex})$ is "close" to being false.

Chapter 1

Tail-equivalence Relations

1.1 The tail-equivalence relation

Let A be a nonempty set, and let A^ω denote the set of infinite sequences of elements of A. We take ω to include 0, so elements u of A^ω are written $u = (u_0, u_1, \ldots)$. Later on, A will often be a structure of some kind, such as a group or linear order, and A^ω will be its infinite direct product. Let $A^{<\omega}$ denote the set of finite sequences of elements of A, including the empty sequence. If r is a finite sequence, and u is either a finite or infinite sequence, we let ru denote the sequence beginning with r and ending with u.

Definition 1.1.1. Given two sequences $u, v \in A^\omega$, we say u and v are *tail-equivalent*, and write $u \sim v$, if there exist finite sequences $r, s \in A^{<\omega}$ and an infinite sequence $u' \in A^\omega$ such that $u = ru'$ and $v = su'$.

In dealing with sequences of elements of A, the letters $a, b, \ldots$ will usually denote single elements of A, whereas $r, s, \ldots$ will denote finite sequences, and $u, v, \ldots$ will denote infinite sequences. If $u = ru'$ for some finite sequence r, then u' is called a *tail-sequence* of u and r is called an *initial sequence* of u. In an abuse, no distinction is made between elements of A

and sequences of length 1, so that au refers to the sequence with first entry $a \in A$ followed by the tail-sequence $u \in A^\omega$. The length of a finite sequence r is denoted $|r|$. Given $u \in A^\omega$, u_i refers to the ith entry of u, and $u \upharpoonright n = (u_0, u_1, \ldots, u_{n-1})$ denotes the initial sequence of the first n entries of u. If $a \in A$ is an element of A, then $a^k \in A^{<\omega}$ denotes the finite sequence $aa \ldots a$ of a repeated k times. If $k = 0$, then a^k denotes the empty sequence, but we will usually tacitly assume that $k > 0$ in an expression of this form.

If $u \sim v$, so that $u = ru'$ and $v = su'$ for some r, s, u', the pair ru' and su' is called a *meeting representation* of u and v. Alternatively, we will sometimes think of a meeting representation as a triple (r, s, u'), where r, s are finite sequences and u' is an infinite sequence. If we say (r, s, u') is a meeting representation of u and v, we mean $u = ru'$ and $v = su'$.

Meeting representations are not unique: if $u = ru'$, $v = su'$, and a is the first entry of u', so that $u' = au''$, then letting $r' = ra$, $s' = sa$ we have that $u = r'u''$ and $v = s'u''$, giving a different representation. Usually, unzipping along a tail-sequence like this is the only way of generating distinct representations. However, if u' is an eventually periodic sequence there are other ways. We will return to this issue later.

It is easy to see that tail-equivalence is an equivalence relation on A^ω. The equivalence class of u is denoted $[u]$. It consists exactly of those elements in A^ω of the form ru', where $r \in A^{<\omega}$ and u' is a tail-sequence of u.

Let $\times$ denote the cartesian product of sets. We will often suppress this symbol, writing for example XY instead of $X \times Y$. In later chapters $\times$ will usually be an extension of the cartesian product to a particular class of structures, like the direct product of groups or the lexicographical product of linear orders. Given a fixed structure A from our class, we will be interested in determining which structures X have the property that $A \times X \cong X$. It turns out that such structures can essentially be represented as unions of tail-equivalence classes in the infinite product A^ω. We give the precise result in Chapter 2.

For now, note that there is a natural bijection between $A \times A^\omega$ and A^ω defined by $(a, u) \mapsto au$. Since this "flattening map" will play an important role in what follows, we denote it by fl, so that $fl(a, (u_0, u_1, \ldots)) = (a, u_0, u_1, \ldots)$ for every $a \in A$ and sequence $(u_0, u_1, \ldots) \in A^\omega$.

The tail-equivalence classes are the smallest subsets of A^ω that are "invariant under left multiplication by A" in the following sense. If $[u]$ is a tail-equivalence class, then viewing it as a subset of A^ω, we may form the product $A[u] = \{(a, v) : a \in A, v \in [u]\}$. Then $fl[A[u]] = [u]$. To check this, note that if $v \in [u]$, then $av \sim v$ and hence $av \in [u]$ for any $a \in A$. Hence $fl[A[u]] \subseteq [u]$. And if $v \in [u]$, say $v = (v_0, v_1, \ldots)$, then $v' = (v_1, v_2, \ldots)$ is tail-equivalent to v and hence also in $[u]$. But then $v_0 v' = v$ is in the image of $A[u]$, giving the reverse containment.

On the other hand, if $X \subseteq A^\omega$ and $fl[AX] = X$, then X is a union of tail-equivalence classes. For, if $u = (u_0, u_1, \ldots)$ is in X, then $fl^{-1}(u) = (u_0, (u_1, \ldots))$ is in AX and therefore the tail-sequence $(u_1, u_2, \ldots)$ is in X. By induction, any tail-sequence u' of u is in X. Then for any $a \in A$, we must have $(a, u') \in AX$ and hence $fl((a, u')) = au' \in X$. By induction, for any finite sequence r we have $ru' \in X$. Thus $[u] \subseteq X$, giving the claim. We have proved the following proposition.

Proposition 1.1.2. Fix a subset $X \subseteq A^\omega$. Then $fl[AX] = X$ if and only if X is a union of tail-equivalence classes.

In Chapter 2, when A is not simply a set but a structure of some kind, we will see how this proposition allows us to produce examples of structures X such that $A \times X \cong X$.

1.2 The n-tail-equivalence relations

We now turn our attention to the n-tail-equivalence relations $\sim_n$, $n > 1$, which refine the tail-equivalence relation. Just as the tail-equivalence relation partitions A^ω into subsets invariant under left multiplication by A, the n-tail-equivalence relation partitions A^ω into subsets invariant under left multiplication by A^n. In the rest of the chapter, after a proving a proposition or theorem about $\sim_n$ for an arbitrary n, we will often specify it for the case when $n = 2$, since this is the case in which we are primarily interested.

Definition 1.2.1. Fix a positive integer $n > 1$. Two sequences $u, v \in A^\omega$ are called *n-tail-equivalent* if there exist $r, s \in A^{<\omega}$ with $|r| \equiv |s| \pmod{n}$ and a sequence $u' \in A^\omega$ such that $u = ru'$ and $v = su'$. If u and v are n-tail-equivalent, we write $u \sim_n v$.

The definition of n-tail-equivalence is identical to that of tail-equivalence, except that in the meeting representation $u = ru'$, $v = su'$ witnessing $u \sim_n v$, the lengths of r and s are required to be the same modulo n. Reflexivity and symmetry of the relation $\sim_n$ are immediate, and transitivity can be checked. The $\sim_n$-equivalence class of u is denoted $[u]_n$. It consists exactly of those sequences of the form ru', where $u' = (u_k, u_{k+1}, \ldots)$ is a tail-sequence of u, and r is a finite sequence such that $|r| \equiv k \pmod{n}$.

We will need to see precisely how tail-equivalence classes are related to n-tail-equivalence classes. If $u \sim_n v$, then certainly $u \sim v$ as well. Hence the n-tail-equivalence relation refines the tail-equivalence relation, and every $\sim$-class $[u]$ splits into some number of $\sim_n$-classes. Clearly in fact, it splits into at most n such classes: if u and v are tail-equivalent, and $u = ru', v = su'$ is a meeting representation of u and v, then for some k, $0 \leq k < n$, we have $|s| \equiv |r| + k \pmod{n}$. Hence for any fixed $a \in A$ we have $v \sim_n a^k u$, as witnessed by the meeting representation $u = a^k ru', v = su'$. (Here, a^k can be replaced by any k-sequence of elements of A.) This shows that for any $\sim$-class $[u]$ and fixed $a \in A$ we have

$$[u] = \bigcup_{0 \leq k < n} [a^k u]_n.$$

For distinct non-negative integers $l < l' < n$, the classes $[a^l u]_n$ and $[a^{l'} u]_n$ are either equal or disjoint, depending on whether $a^l u \sim_n a^{l'} u$ or not. It will be useful to know precisely when we have $a^l u \sim_n a^{l'} u$. Notice that this relation holds if and only if $u \sim_n a^{l'-l} u$. Hence to determine when we have $a^l u \sim_n a^{l'} u$ for distinct integers $l, l' < n$, it is sufficient to determine when we have $u \sim_n a^k u$ for a given k, $0 < k < n$.

Usually, $a^k u \not\sim_n u$. Certainly the obvious representation $u = ru'$, $a^k u = su'$, where $r = \emptyset$, $s = a^k$, and $u' = u$ does not witness $a^k u \sim_n u$. Most of the time this obvious representation is the only one, up to unzipping along the tail-sequence. However, if u is an eventually periodic sequence then it is possible to get truly distinct representations, and in some of these cases we have $a^k u \sim_n u$. Proposition 1.2.2 below tells us exactly when this happens.

For any finite sequence $s \in A^{<\omega}$, let $\overline{s}$ denote the sequence $sss\ldots \in A^\omega$. Define a sequence $u \in A^\omega$ to be *eventually periodic* if there exist $r, s \in A^{<\omega}$ such that $u = rsss\ldots = r\overline{s}$. Such a sequence is said to be *repeating in s*. If $r = \emptyset$ we say u is simply *periodic*. If p is the shortest possible length of a sequence in which u repeats, we say u has *period p*.

Proposition 1.2.2. Fix a sequence $u \in A^\omega$ and element $a \in A$. Let n be an integer greater than 1, and fix k, $0 < k < n$. Then $a^k u \sim_n u$ if and only if u is eventually periodic, and if p is the period of u, then there is a nonzero integer $l \in \mathbb{Z}$ such that $k \equiv lp \pmod{n}$.

Proof. Note that the l in the statement of the proposition is possibly a negative integer: the last clause says that k is either a positive *or* negative multiple of p (modulo n).

Before proving the proposition, we establish the following claim: if u is eventually periodic of period p, and we have a "periodic representation" of u as $u = r\overline{s}$, then p divides $|s|$. If $p = |s|$ there is nothing to show, so suppose $p < |s|$. Then there are finite sequences r', t such that $u = r'\overline{t}$ and $|t| = p$. It is safe to assume that $r' = r$, since if say r' is shorter than r, then by extending r' to r at the beginning of the second representation for u, we see that $u = r\overline{t'}$, where t' is of the same length as t (and similarly, if r' is longer than r we may extend

r to r' in the first representation). Hence $u = r\overline{s} = r\overline{t}$. Thus the tail-sequences $\overline{s}$ and $\overline{t}$ are equal. In particular, t is an initial sequence of s.

Let $d = \gcd(|s|, |t|)$. We wish to show $d = |t|$. Suppose to the contrary $d < |t|$. Write t as $x_1 x_2 \ldots x_m$, where the x_i are sequences all of length d, and $m = \frac{|t|}{d}$ is greater than 1. Then since t is an initial sequence of s, we have $s = x_1 \ldots x_m s'$ for some finite sequence s'.

By Bezout's identity there exist integers a and b such that $a|s| + b|t| = d$, where we may assume $a < 0$ and $b > 0$. This means that if we consider the words $x = ss\ldots s$ consisting of a-many copies of s, and $y = tt\ldots t$ consisting of b-many copies of t, then y is d entries longer than x. But these words must be equal up to the end of x, since they begin the sequences $\overline{s}$ and $\overline{t}$, which are equal. Since the last d entries of y is exactly the sequence x_m, we have $y = xx_m$. Thus we may rewrite $\overline{s}$ as $xs\overline{s}$ and $\overline{t}$ as $xx_m t\overline{t}$. Expanding the solitary s and t in these representations gives $\overline{s} = xx_1x_2\ldots x_m s'\overline{s}$ and $\overline{t} = xx_m x_1 \ldots x_{m-1}x_m\overline{t}$. Since these sequences are equal we must have the identities $x_1 = x_m$, $x_2 = x_1$, $\ldots$, $x_m = x_{m-1}$. Hence all of the x_i are equal to x_1. But then $t = x_1x_1\ldots x_1$ so that $\overline{t} = \overline{x_1}$ and hence $u = r\overline{x_1}$. This is a contradiction, since we assumed that the period of u is $p = |t|$, and $|x_1|$ is strictly less than $|t|$. We have the claim.

Now we can prove the proposition. Suppose first that we have $u \sim_n a^k u$, as witnessed by the meeting representation $u = ru'$ and $a^k u = tu'$. We may assume t is at least as long as a^k, that is $t = a^k r'$ for some $r' \in A^{<\omega}$, since if not we can unzip along the tail-sequence until we have a meeting representation where this holds. Since both r, r' are initial sequences of u, it must be that one is an initial sequence of the other. Let us assume first that r' is an initial sequence of r, that is, $r = r's$ for some finite sequence s. We are going to show that u is repeating in s.

We know that $|r| \equiv |a^k r'| \pmod{n}$. Hence $|r| \equiv k + |r'| \pmod{n}$. But $|r| = |r'| + |s|$, so we have $|r'| + |s| \equiv |r'| + k \pmod{n}$, which gives $|s| \equiv k \pmod{n}$.

Now, on one hand we have $u = ru' = r'su'$, and on the other we have $u = r'u'$. Hence $u' = su'$. The only way this is possible is if $u' = \overline{s}$. Therefore $u = r\overline{s}$ is eventually periodic. It may be that the period p of u is shorter than $|s|$, but by above we have $|s| = lp$ for some (positive) integer l. Combining this with the final clause in the previous paragraph gives $lp \equiv k \pmod{n}$.

The other case is when r is an initial sequence of r', that is $r' = rs$ for some finite sequence s. Then as before we can show that $u = r\overline{s}$, and by assumption we have $|r| \equiv |a^k r'| \pmod{n}$. The difference is that now this congruence becomes $|r| \equiv k+|r|+s \pmod{n}$, so that $k+s \equiv 0 \pmod{n}$. Since $|s|$ must be a positive multiple m of the period p, this congruence becomes $k + mp \equiv 0 \pmod{n}$, that is, $k \equiv lp \pmod{n}$, where $l = -m$, as desired.

For the backwards direction, assume that u is eventually periodic of period p and $k \equiv lp \pmod{n}$ for some $l \in \mathbb{Z}$, $l \neq 0$. Write u as $u = r\overline{s}$ for some s of length p. Suppose first that $l > 0$. Then we have $u = r\overline{s} = rs^l\overline{s}$, and $a^k u = a^k r\overline{s}$. The meeting representation $(rs^l, a^k r, \overline{s})$ witnesses $u \sim_n a^k u$. If $l < 0$, so that $l = -m$ for some positive m, then we have $k + mp \equiv 0 \pmod{n}$. Writing u as $r\overline{s}$ and $a^k u$ as $a^k rs^m\overline{s}$, we see that the meeting representation $(r, ra^k s^m, \overline{s})$ witnesses $u \sim_n a^k u$. □

An equivalent statement of the proposition is that $[u]_n = [a^k u]_n$ if and only if u is eventually periodic and k is congruent (modulo n) to integer multiple of the period p. Since they play an especially important role in what follows, we emphasize that for sequences u of period 1, we have $[u]_n = [a^k u]_n$ for *any* k. In this case, all of the n-tail-equivalence classes $[a^k u]_n$ are the same, and hence they all are simply equal to the tail-equivalence class $[u]$. On the other hand, if u is not periodic in any period, all of the n-tail-equivalence classes $[a^k u]_n$ are distinct, so that that the union

$$[u] = \bigcup_{0 \leq k < n} [a^k u]_n.$$

is in fact a disjoint union.

Since we will need to refer to this particular case later, let us state Proposition 1.2.2 for the case $n = 2$.

Corollary 1.2.3. Fix elements $u \in A^\omega$ and $a \in A$. Then $au \sim_2 u$ if and only if u is eventually periodic and the period of u is odd.

Just as the $\sim$-classes $[u]$ were the smallest suborders of A^ω invariant under left multiplication by A, the $\sim_n$-classes $[u]_n$ are the smallest suborders of A^ω invariant under left multiplication by A^n. This is stated formally as Proposition 1.2.4 below. Let $fl_n : A^n \times A^\omega \to A^\omega$ denote the "n-flattening bijection" defined by $fl_n((a_0, a_1, \dots, a^{n-1}), u) = a_0 a_1 \dots a_{n-1} u$.

Proposition 1.2.4. Fix a subset $X \subseteq A^\omega$ and $n \geq 1$. Then $fl_n[A^n X] = X$ if and only if X is a union of $\sim_n$-classes.

Proof. The proof is essentially identical to the proof of Proposition 1.1.2. There, the relevant fact was that $au \sim u$ for all a, u. Here, it is that $a_0 a_1 \dots a_{n-1} u \sim_n u$ for any n-sequence $a_0 \dots a_{n-1} \in A^n$ and $u \in A^\omega$. $\square$

Thus, just as $A[u]$ is naturally in bijection (via fl) with $[u]$ for every $u \in A^\omega$, we have that $A^n[u]_n$ is naturally bijective with $[u]_n$ (via fl_n). We can express this informally by writing $A[u] = [u]$ and $A^n[u]_n = [u]_n$. A natural question is what happens to a given n-tail-equivalence class $[u]_n$ when it is multiplied by a single factor of A.

Proposition 1.2.5. Fix $n \geq 1$. For all $u \in A^\omega$ and $a \in A$ we have $fl[A[u]_n] = [au]_n$.

Proof. If $x \in fl[A[u]_n]$, then $x = bv$ for some $b \in A$ and $v \sim_n u$. But then $bv \sim_n au$, i.e. $x \sim_n au$. Conversely, if $x \sim_n au$ then writing $x = x_0 x'$ we have that $x' \sim_n u$ and hence $x \in fl[A[u]_n]$. $\square$

It follows that $fl_k[A^k[u]_n] = [a^k u]_n$ for any $a \in A$ and $k > 0$. We can express this informally by saying $A^k[u]_n = [a^k u]_n$.

In later chapters, when A is a structure of some kind, Proposition 1.2.4 will allow us to produce examples of structures X such that $A^n X \cong X$. Proposition 1.2.5 will then give us a way of describing the "intermediate" structures $A^k X$, $0 < k < n$, in terms of the original structure X.

1.3 Tail-equivalence preserving maps on A^ω

In this section we will view A^ω as a topological space and prove a theorem about continuous maps $f : A^\omega \to A^\omega$ that are invariant on the tail-equivalence classes of A^ω. Maps of this kind will figure prominently in our proof of the cube property for linear orders.

As before, assume A is an arbitrary nonempty set. We may view A as a topological space equipped with the discrete topology. Let A^ω be equipped with the product topology. For a given finite sequence $r \in A^{<\omega}$, let $\mathcal{O}_r$ denote the set of sequences beginning with r, that is, $\mathcal{O}_r = \{u : u = ru' \text{ for some } u' \in A^\omega\}$. The collection $\{\mathcal{O}_r : r \in A^{<\omega}\}$ is the standard basis for the product topology on A^ω. It is a well-known fact that A^ω equipped with this topology is a Baire space.

We are interested in understanding continuous maps $f : A^\omega \to A^\omega$ that preserve tail-equivalence.

Definition 1.3.1. Given a function $f : A^\omega \to A^\omega$, we say that f *preserves tail-equivalence*, or f is *$\sim$-preserving*, if for every $u \in A^\omega$ we have $f(u) \sim u$. If $\mathcal{O}$ is a subset of A^ω, we say f is *$\sim$-preserving on $\mathcal{O}$* if for every $u \in \mathcal{O}$ we have $f(u) \sim u$.

Equivalently, f is $\sim$-preserving if for every tail-equivalence class $[u]$ we have that the image of $[u]$ under f is contained in $[u]$.

The requirement that a given function f be $\sim$-preserving is quite a strong one, even by itself: it implies, for one, that f is far from being a constant function.

In this section, we are interested in understanding maps that are both $\sim$-preserving *and* continuous. The identity function gives one example of such a map. In a moment we will see that there are non-identity examples.

For now, let us observe that there is a natural way for a given map $f : A^\omega \to A^\omega$ to be $\sim$-preserving and continuous on a basic open subset of A^ω. Suppose $r, s \in A^{<\omega}$ are finite sequences, and for every $u \in \mathcal{O}_r$, if $u = ru'$ then $f(u) = su'$. Then f maps $\mathcal{O}_r$ bijectively onto $\mathcal{O}_s$. It does this by chopping off the initial sequence r from every sequence in $\mathcal{O}_r$ and replacing it with the sequence s. Clearly f is $\sim$-preserving on $\mathcal{O}_r$, and it is clearly continuous on $\mathcal{O}_r$ as well.

It turns out that a map on A^ω that is globally $\sim$-preserving and continuous looks locally like the chop-and-paste map described in the previous paragraph, at least on a open dense set. The precise result is stated as Theorem 1.3.3 below.

Definition 1.3.2. Suppose $f : A^\omega \to A^\omega$ is a function. If $r, s \in A^{<\omega}$ are finite sequences, and for every sequence u of the form $u = ru'$ we have $f(u) = su'$, then we say that f is *standard* on $\mathcal{O}_r$ and write $\mathcal{O}_r \xrightarrow{f} \mathcal{O}_s$. Given a fixed $u \in A^\omega$, we say that f is *standard at* u if there exists a neighborhood $\mathcal{O}_r$ containing u on which f is standard.

Thus if f is standard at u, then in particular f is continuous and $\sim$-preserving in a neighborhood of u.

Now, suppose that R is a subset of $A^{<\omega}$ with the property that for every $u \in A^\omega$ there is exactly one $r \in R$ that is an initial sequence of u. (For example, R could be the set of sequences of length 1.) Let S be another subset of $A^{<\omega}$ with this property of the same cardinality as R. Then if $F : R \to S$ is any bijection, we can define a map $f : A^\omega \to A^\omega$ by the following rule: given $u \in A^\omega$, if $u = ru'$ for $r \in R$, then $f(u) = su'$ where $s = F(r)$. Equivalently, for every $r \in R$, we have $\mathcal{O}_r \xrightarrow{f} \mathcal{O}_{F(r)}$. So defined, f is a bijection of A^ω with itself. Since it is standard everywhere, f is continuous and $\sim$-preserving (in fact, f is a $\sim$-preserving homeomorphism of A^ω). In particular, we can arrange that a (globally) continuous and $\sim$-preserving map not be the identity.

A natural question is whether every $\sim$-preserving and continuous map has the same property as the f defined above, namely, that it is standard everywhere. While this is not quite true, the following theorem says that any continuous and $\sim$-preserving map on A^ω is standard on a generic subset of A^ω.

Theorem 1.3.3. If a function $f : A^\omega \to A^\omega$ is continuous and preserves tail-equivalence, then the set of points at which f is standard contains an open dense set.

Proof. Let f be a continuous and $\sim$-preserving function on A^ω, and let $\mathcal{O} \subseteq A^\omega$ be an arbitrary open subset of A^ω. We will show that there exist finite sequences t, t' such that $\mathcal{O}_t \subseteq \mathcal{O}$ and $\mathcal{O}_t \xrightarrow{f} \mathcal{O}_{t'}$. That is, we will find an open subset of $\mathcal{O}$ on which f is standard. Since $\mathcal{O}$ was picked arbitrarily, this suffices to prove the theorem.

For every $u \in A^\omega$, we have that $u \sim f(u)$. Let us say that u is (m, n) if there is a meeting representation $u = ru'$ and $f(u) = su'$ such that $|r| = m$ and $|s| = n$. Let $K(m, n)$ denote the set of (m, n) points. Notice that if u is (m, n), then by unzipping along the tail-sequence in the meeting representation for u and $f(u)$, we can show that u is $(m + k, n + k)$ for any $k \in \omega$. Hence $K(m, n) \subseteq K(m + k, n + k)$ for any $k \in \omega$.

Since f is $\sim$-preserving, every u is (m,n) for some integers m and n, that is,

$$A^\omega = \bigcup_{(m,n)} K(m,n).$$

Similarly we have

$$\mathcal{O} = \bigcup_{(m,n)} [K(m,n) \cap \mathcal{O}].$$

Since $\mathcal{O}$ is an open subset of a Baire space, it is also a Baire space in the inherited topology. Since the union above is countable, it cannot be that all of the sets $K(m,n)$ are nowhere dense in $\mathcal{O}$. Hence there exists a pair (m,n) and a finite sequence $r \in A^{<\omega}$ such that $\mathcal{O}_r \subseteq \mathcal{O}$ and $K(m,n)$ is dense in $\mathcal{O}_r$.

Now, if $|r| \geq m$, say $|r| = m+k$ for some $k \in \omega$, then since $K(m,n) \subseteq K(m+k, n+k)$, we have that $K(m+k, n+k)$ is also dense in $\mathcal{O}_r$. On the other hand, if $|r| < m$, then if we let q be any extension of r of length m, we have that $\mathcal{O}_q$ is an open subset of $\mathcal{O}_r$, and hence $K(m,n)$ is dense in $\mathcal{O}_q$ as well. Thus in either case we can match the "rank" of our open set to the "rank" of our dense set, and so without loss of generality we may simply assume that $|r| = m$ to begin with.

Fix $u \in K(m,n) \cap \mathcal{O}_r$. Then $u = ru'$ for some tail-sequence u'. Since u is (m,n) we know that $f(u) = su'$ for some s of length n. By definition, then, we have $f(u) \in \mathcal{O}_s$.

Since f is continuous, for all v sufficiently near u we have that $f(v) \in \mathcal{O}_s$. That is, there is a t extending r such that t is also an initial sequence of u and for all $v \in \mathcal{O}_t$ we have $f(v) \in \mathcal{O}_s$. Suppose that $t = rr'$, so that $u = tu'' = rr'u''$, where $r'u'' = u'$.

Now, $\mathcal{O}_t$ is an open subset of $\mathcal{O}_r$, and so $K(m,n)$ is dense in $\mathcal{O}_t$ as well. Suppose $v \in K(m,n) \cap \mathcal{O}_t$, so that $v = tv' = rr'v'$ for some tail-sequence v'. Then since v is (m,n) and $f(v) \in \mathcal{O}_s$, it must be that $f(v) = sr'v'$.

Let $t' = sr'$. Then the above paragraph may be rephrased by saying, for densely many $v \in \mathcal{O}_t$, if $v = tv'$ then $f(v) = t'v'$. By continuity then, this actually must hold for every $v \in \mathcal{O}_t$. But this means precisely that $\mathcal{O}_t \xrightarrow{f} \mathcal{O}_{t'}$, as desired. □

Chapter 2

The Isomorphism $A \times X \cong X$

2.1 Analyzing bijections $f : A \times X \to X$

2.1.1 Introduction and preliminaries

Suppose that $(\mathfrak{C}, \times)$ is a class of structures with a cartesian product, and A is a fixed structure from $\mathfrak{C}$. In this chapter we will describe a general method for characterizing those structures $X \in \mathfrak{C}$ such that $A \times X \cong X$, and illustrate it by giving precise characterizations for several particular classes of structures.

The method amounts to an analysis of the case when A and X are simply sets, and we are given an arbitrary bijection $f : A \times X \to X$. We will show that such a bijection can be used to construct, without any non-constructive principle like the axiom of choice, another bijection F from X onto a set that looks something like a union of tail-equivalence classes in A^ω. This F can be thought of as a representation of X in terms of A. Conversely, given such a representation of X, it will be easy to see that there is a natural bijection between $A \times X$ and X.

Since any isomorphism is in particular a bijection, our analysis will apply even when the sets A and X carry extra structure, like that of a ring or topological space. In these cases, we will be able to adapt our analysis to prove theorems of the form "$A \times X$ is isomorphic to X if and only if X can be represented in this particular way." The only challenge in specifying such a theorem for a given class will be to determine how the representation of X is influenced by the fact that the isomorphism in play is not only a bijection, but a structure-preserving one.

The method can be easily generalized to characterize those X satisfying the isomorphism $A^n \times X \cong X$ for a given $n > 1$. These characterizations are essentially the same as in the case when $n = 1$, the only difference being that the relevant representation of X is in terms of the n-tail-equivalence classes of A^ω, instead of the tail-equivalence classes.

We begin by defining the necessary terminology, and giving some examples. For this section, as in the previous chapter, capital letters like $A, X, Y, \ldots$ denote sets, and $\times$ is the cartesian product. Also as before, we will freely omit $\times$ in expressions when there is no danger of confusion: XY means $X \times Y$. Later, capital letters will refer to structures from some fixed class, and $\times$ will be a product on that class extending the cartesian product. As such, it will be helpful to think of the sets that we deal with in this section not as sets purely, but rather as structures of some kind about which we have no information. In this vein, we can think of bijections not only as bijections, but as isomorphisms.

The cartesian product is naturally associative, in the sense that for any sets X, Y, Z there is a natural bijection from $(X \times Y) \times Z$ onto $X \times (Y \times Z)$, namely the map defined by $((x, y), z) \mapsto (x, (y, z))$. We will always assume that the products we deal with are associative, that is, that this map defines an isomorphism when X, Y, Z are structures from a given class. As such we will usually omit parentheses and write expressions of the form XYZ for threefold products, and similar expressions for longer products. If $X_0 X_1 \ldots X_{n-1}$ is an n-fold product of sets, then we write the underlying points as n-tuples $(x_0, \ldots, x_{n-1})$. The notation X^n is shorthand for the n-fold product $XX \ldots X$.

The cartesian product is of course also commutative: the map defined by $(x, y) \mapsto (y, x)$ is a natural bijection of XY with YX. We will not, however, assume that our products are commutative in general. Though many products of structures are commutative, the product in which we are primarily interested—the lexicographical product of linear orders—is not.

With this case in mind, we will think of the cartesian product in something a non-commutative way. In any product XY, each $x \in X$ indexes a "copy" of Y in XY, namely the set of points $(x, \cdot)$ with first coordinate x. We may think of XY as the set obtained by replacing every point $x \in X$ with a copy of Y, and say XY is X-many copies of Y.

Generalizing this idea, given a set X, and for every $x \in X$ a set I_x, we define the *replacement* of X by the sets I_x to be the set $\{(x, i) : x \in X, i \in I_x\}$. We denote this set by $X(I_x)$ and think of it as the set obtained by replacing every point $x \in X$ with the corresponding I_x. Importantly, we allow that for a given $x_0 \in X$ we have $I_{x_0} = \emptyset$. In this case there are no pairs with first coordinate x_0 in $X(I_x)$. The notion of a replacement generalizes the notion of a product, since if there is a Y such that $I_x = Y$ for every $x \in X$, then $X(I_x) = XY$.

A replacement of X is nothing more than a collection of sets indexed by X. However, when X is a structure of some kind the idea of replacing the points in X by other structures of the same kind will usually be less trivial. For example, suppose that X is a linear order, and for every point $x \in X$ we fix a linear order I_x. The replacement order $X(I_x)$ is formally defined to be the set of pairs $(x, i) : x \in X, i \in I_x$, ordered lexicographically. But visually, $X(I_x)$ is the linear order obtained by replacing every point $x \in X$ with the corresponding I_x.

It will be useful to see in what sense the operation of replacement is "associative" on the left in its relation to the cartesian product. Given a replacement $X(I_x)$ and a set A, we can multiply to form the product $A \times X(I_x)$. Points in this set are tuples of the form $(a, (x, i))$. Since this set is "A-many copies of $X(I_x)$," each I_x appears in it A-many times: the collection of points of the form $(a, (x, \cdot))$ will be a copy of I_x, regardless of a. Alternatively, we might have begun with A, formed $A \times X$, and then replaced each point (a, x) with I_x to form $(A \times X)(I_x)$. Points in this set are of the form $((a, x), i)$. The set of points of the form $((a, x), \cdot)$ is a copy to I_x regardless of a, and hence this set is naturally bijective with $A \times X(I_x)$ under the map $((a, x), i) \mapsto (a, (x, i))$.

However, to be consistent with our previous notation, the subscripts in the replacement $(A \times X)(I_x)$ should not (as they are written) range over X, but rather over $A \times X$. In forming the set the second way, by replacing after taking the product, we should have labeled the set replacing the point (a, x) as $J_{(a,x)}$ and used the notation $(A \times X)(J_{(a,x)})$. If, as in our example, $J_{(a,x)} = I_x$ for every a, then $(A \times X)(J_{(a,x)})$ is naturally bijective with $A \times X(I_x)$, as noted.

We will use the following convention. If we first form the replacement $X(I_x)$ and then multiply on the left by A, we will write $AX(I_x)$. If we first form the product AX and then replace each point (a, x) with some $J_{(a,x)}$, we will use the notation $AX(J_{(a,x)})$. It will sometimes be convenient to think of $AX(I_x)$ as being formed in the second way, with the left multiplication taking place first, in which case we will switch the notation to $AX(J_{(a,x)})$ and make clear that $J_{(a,x)} = I_x$ for all $a \in A$.

Finally, let us note how bijections factor through products and replacements. If X and Y are sets of the same cardinality, as witnessed by a bijection $f : X \to Y$, then for any set Z, the products XZ and YZ are naturally in bijection, as witnessed by the map $(x, z) \mapsto (f(x), z)$. Similarly, if we have collections of sets I_x, $x \in X$, and J_y, $y \in Y$ so that $I_x = J_{f(x)}$ for all $x \in X$, we will also have that the replacements $X(I_x)$ and $Y(J_y)$ are naturally in bijection, as witnessed by the map $(x, i) \mapsto (f(x), i)$.

2.1.2 Natural bijections $f : A \times X \to X$

Let A be a fixed, nonempty set. For which sets X can one find a bijection $f : AX \to X$? If X is infinite and at least as large as A, then the axiom of choice guarantees that such a bijection always exists. However, if A and X are structures, there is no reason for the bijection yielded by the axiom of choice to be an *isomorphism* of AX with X. Since we wish to generalize our results to cases when we are looking for an isomorphism instead of simply a bijection, it is more useful to ask for which sets X can we find a *natural* bijection between AX and X.

Proposition 1.1.2 gives that if X is a subset of A^ω that is closed under tail-equivalence, then the flattening map $(a, u) \mapsto au$ defines a bijection of AX with X. In particular, there is a natural bijection between AX and X when $X = A^\omega$. Extending this, if $X \subseteq A^\omega$ is closed under tail-equivalence and Y is any set, then if we let $X' = XY$ we can also get a bijection $f : AX' \to X'$ by defining $f((a, u, y)) = (au, y)$. This is just the flattening map on the first two coordinates of AXY, and the identity on the last. We write $f = (fl, \mathrm{id})$.

The proposition actually yields substantially more general examples. Suppose that for every $u \in A^\omega$ we fix a set I_u, with the restriction that if $u \sim v$, then $I_u = I_v$. Let $X = A^\omega(I_u)$. Then there is a natural bijection between AX and X. Indeed, this bijection is just the flattening map on the first two coordinates. To see this, rewrite $AX = A \times A^\omega(I_u)$ as $(A \times A^\omega)(J_{(a,u)})$, where $J_{(a,u)} = I_u$ for all $a \in A$. Then $J_{(a,u)} = I_{au}$ as well, since $au \sim u$. Thus the map $(a, u, x) \mapsto (au, x)$ makes sense, and defines a bijection of $(A \times A^\omega)(J_{(a,u)})$ with $A^\omega(I_u)$, that is, of AX with X.

Letting $I_{[u]}$ denote the single set I_v for all $v \in [u]$, we may denote X by $A^\omega(I_{[u]})$. Our previous examples were actually of this form: if $X \subseteq A^\omega$ is a union of tail-equivalence classes then X may be written as $A^\omega(I_{[u]})$, where $I_{[u]} = 1$ if $u \in X$, and otherwise $I_{[u]} = \emptyset$. Of course, since the $I_{[u]}$ may be chosen arbitrarily, there are many other examples besides these.

Here is a concrete one. Let $\mathbb{Z}$ denote the set of integers, and $\mathbb{Z}^\omega$ the set of infinite sequences of integers. Each tail-equivalence class $[u] \subseteq \mathbb{Z}^\omega$ is a countable subset of $\mathbb{Z}^\omega$, hence the number of classes is 2^ω. Enumerate them as $\{C_\alpha : \alpha < 2^\omega\}$. Let X be the set obtained by replacing every point in the αth class with the ordinal α (not as a singleton but as a set). That is, if $[u] = C_\alpha$ let $I_{[u]} = \alpha$, and let $X = \mathbb{Z}^\omega(I_{[u]})$. By our observations above, there is a natural bijection between $\mathbb{Z}X$ and X given by the flattening map $(z, u, x) \mapsto (zu, x)$. In Chapter 3 we will revisit this example, but there we will consider $\mathbb{Z}^\omega$ and the ordinals α as linear orders.

It is worth noting that if we did not insist $I_u = I_v$ whenever $u \sim v$, the "map" between $A \times A^\omega(I_u)$ and $A^\omega(I_u)$ defined by $(a, u, x) \mapsto (au, x)$ may be meaningless. For example, if for some $u \in A^\omega$ and $a \in A$ we had that $I_u = \{x\}$ and $I_{au} = \emptyset$, then while (a, u, x) is a point in $A \times A^\omega(I_u)$, there is no point (au, x) in $A^\omega(I_u)$. Indeed, there are no points whatsoever with first coordinate au in $A^\omega(I_u)$.

We have just seen that for every set X of the form $X = A^\omega(I_{[u]})$, there is a natural bijection between AX and X, namely the flattening map (fl, id). Even if we do not have identity, but only an bijection $F : X \to A^\omega(I_{[u]})$, there is still a natural bijection $f : AX \to X$, namely $f = F^{-1} \circ (fl, \mathrm{id}) \circ (\mathrm{id}, F)$. This is the same flattening map, up to the relabeling F.

A converse to this statement is very nearly true, even in the absence of the axiom of choice. In order to prove a precise converse, we need to loosen the meaning of the notation $A^\omega(I_{[u]})$. Suppose that we have a collection of sets $I_u, u \in A^\omega$ such that for every pair of tail-equivalent sequences u, v there is a bijection $f_{u \to v} : I_u \to I_v$. Then while the sets I_u and I_v may not be identical, they are of the same "type" (i.e., cardinality).

To produce such a system of bijections $\{f_{u\to v} : u, v \in A^\omega, u \sim v\}$, it is actually sufficient that for every sequence u and element $a \in A$, we have a bijection $f_{u\to au} : I_u \to I_{au}$. Then, for every finite sequence $r = ab$ of length two, we can define the bijection $f_{u\to abu} : I_u \to I_{abu}$

to be the composition $f_{bu\to abu} \circ f_{u\to bu}$. By taking longer compositions, we can get a bijection $f_{u\to ru} : I_u \to I_{ru}$ for any finite sequence r and $u \in A^\omega$. By taking inverses, we obtain bijections $f_{ru\to u} : I_{ru} \to I_u$ for any r, u. Finally, if u and v are tail-equivalent sequences, as witnessed by a meeting representation $u = ru', v = su'$, we can define a bijection $f_{u\to v}$: $I_u \to I_v$ as $f_{u'\to su'} \circ f_{ru'\to u'}$. So we get a bijection $f_{u\to v}$ for every pair $u \sim v$, as claimed.

In what follows, we will take the notation $A^\omega(I_{[u]})$ to mean a replacement $A^\omega(I_u)$ for which we have a system of bijections $\{f_{u\to au} : u \in A^\omega, a \in A\}$. We record this in the following definition, so that we can refer back to it later.

Definition 2.1.1. Let A be a set. Then $A^\omega(I_{[u]})$ is a pair $\langle A^\omega(I_u), \{f_{u\to au} : a \in A, u \in A^\omega\}\rangle$, where $A^\omega(I_u)$ is a replacement of A^ω, and for all $u \in A^\omega$ and $a \in A$, $f_{u\to au}$ is a bijection of I_u and I_{au}. We refer to such a pair as a *replacement of* A^ω *up to tail-equivalence.*

We will often take $A^\omega(I_{[u]})$ as simply referring to the underlying replacement $A^\omega(I_u)$. For example, if we say "X is in bijection with $A^\omega(I_{[u]})$," we mean X is in bijection with a replacement $A^\omega(I_u)$ for which we have a system of bijections $\{f_{u\to au}\}$. If all of the bijections $f_{u\to au}$ are the identity, then $I_u = I_v$ for every tail-equivalent pair u, v, and this reduces to the case above. This will usually be the situation we are in, but not always: we will consider replacements $A^\omega(I_{[u]})$ for which the bijections $f_{u\to au}$ are not the identity. Given such a replacement $A^\omega(I_{[u]})$ and a sequence $u \in A^\omega$, one may think of $I_{[u]}$ as referring, not to a specific set, but rather the "isomorphism type" of every set I_v for $v \in [u]$.

Even with this loosened meaning, we still have that if X is a replacement of the form $A^\omega(I_{[u]})$, then there is still a natural bijection between AX and X, namely the map defined by $(a, u, x) \mapsto (au, f_{u\to au}(x))$. Also as before, even if we do not have identity, but only a bijection $F : X \to A^\omega(I_{[u]})$, we may obtain a bijection between AX and X by conjugating the above map with F.

We can now prove our converse: there is a bijection between AX and X only if X can be put into bijective correspondence with a set of the form $A^\omega(I_{[u]})$. In the presence of the axiom of choice, this is a trivial theorem. But the proof below does not use choice, and can be adapted to cases when A and X are structures, and we wish to characterize those X for which there is an *isomorphism* between AX and X.

Theorem 2.1.2. Let A and X be nonempty sets. Then there is a bijection $f : AX \to X$ if and only if there is a bijection F from X onto a replacement of the form $A^\omega(I_{[u]})$.

Proof. We have already seen that if there is a bijection F between X and a set of the form $A^\omega(I_{[u]})$, then there is a natural bijection $f : AX \to X$.

So assume that there is a bijection $f : AX \to X$. For every $a \in A$, let aX denote the set of pairs in AX with first coordinate a, and let X_a denote $f[aX]$. The sets X_a are pairwise disjoint and cover X. We think of our bijection $f : AX \to X$ as giving us a way of splitting X into A-many copies of itself, and X_a as being the ath copy of X within X. Let $f_a : X \to X_a$ be the bijection defined by $f_a(x) = f(a, x)$.

Given one of the sets X_b and some $a \in A$, we may think of the image $f_a[X_b]$ as the bth copy of X within X_a. Denote this set by $X_{(a,b)}$. Then $f_a \circ f_b$ is a bijection of X with $X_{(a,b)}$. Extending this, given a finite sequence $r = (a_0, a_1, \ldots, a_n) \in A^{<\omega}$, define X_r as $f_{a_0} \circ f_{a_1} \circ \ldots \circ f_{a_n}[X]$. Denote the bijection $f_{a_0} \circ f_{a_1} \circ \ldots \circ f_{a_n} : X \to X_r$ by f_r.

It is immediate that for any $r, s \in A^{<\omega}$, we have $f_r[X_s] = X_{rs}$, and conversely, $f_r^{-1}[X_{rs}] = X_s$. If t extends r, then X_t is a subset of X_r. If neither one of r, t extends the other, then X_r and X_t are disjoint. In particular, if $|r| = |t|$ but $r \neq t$, then $X_r \cap X_t = \emptyset$.

Now, given an infinite sequence $u \in A^\omega$, define I_u to be the set obtained by taking the natural nested intersection:

$$I_u = \bigcap_{n\in\omega} X_{u\restriction n}.$$

For a given u, the set I_u need not be in bijection with X. Indeed, I_u may be empty. However, every $x \in X$ is in some I_u. Furthermore, if u, v are distinct sequences in A^ω, then for some n we have $u \restriction n \neq v \restriction n$. Hence $X_{u\restriction n} \cap X_{v\restriction n} = \emptyset$, and so $I_u \cap I_v = \emptyset$. Thus we have that the I_u in fact partition X.

Because we have partitioned X into sets I_u indexed by sequences $u \in A^\omega$, there is a natural bijection F between X and the replacement $A^\omega(I_u)$. This bijection is defined by the following rule: if $x \in I_u$, then $F(x) = (u, x)$.

It remains to show that there is a system of bijections $\{f_{u\to au} : I_u \to I_{au} : a \in A, u \in A^\omega\}$. There is a natural choice for $f_{u\to au}$, namely the map f_a, restricted to I_u. And this choice works, that is, $f_a[I_u] = I_{au}$. For we have

$$\begin{aligned} f_a[I_u] &= f_a[\textstyle\bigcap_n X_{u\restriction n}] \\ &= \textstyle\bigcap_n f_a[X_{u\restriction n}] \\ &= \textstyle\bigcap_n X_{a(u\restriction n)} \\ &= I_{au}. \end{aligned}$$

Hence $\{f_a \restriction I_u : a \in A, u \in A^\omega\}$ gives the desired system of bijections. It follows that the replacement $A^\omega(I_u)$ is of the form $A^\omega(I_{[u]})$. $\square$

2.1.3 Natural bijections $f : A^n \times X \to X$

If there exists a bijection $f : AX \to X$, then by iterating it one can obtain a bijection between $A^n X$ and X for any fixed $n > 1$. Similarly, if A and X are structures such that $AX \cong X$, then any given isomorphism can be iterated to witness $A^n X \cong X$. That is, for any fixed $n > 1$ we have $AX \cong X \implies A^n X \cong X$. The converse, however, is not necessarily true: if $A^n X \cong X$ for some $n > 1$, then it does not follow, formally at least, that $AX \cong X$. And indeed, there are many examples of structures A and X for which $A^n X \cong X$ for some $n > 1$ but $AX \ncong X$.

We will be interested in knowing under what conditions $A^n X \cong X$ *does* imply $AX \cong X$. In order to get at such conditions, we first seek to characterize those structures X such that $A^n X \cong X$. As in the case when $n = 1$, the first step toward such a characterization for a given class of structures is understanding the case when A and X are simply sets, and isomorphism means bijection.

We begin with some examples. Just as Proposition 1.1.2 yields examples of sets X for which there is a natural bijection $f : AX \to X$, Proposition 1.2.4 can be used to produce sets X naturally in bijection with $A^n X$. Suppose that for every $u \in A^\omega$ we fix a set I_u, with the restriction that if $u \sim_n v$ then $I_u = I_v$. Let $A^\omega(I_{[u]_n})$ denote the replacement $A^\omega(I_u)$. If we let $X = A^\omega(I_{[u]_n})$, then there is a natural bijection between $A^n X$ and X, namely the map defined by $(r, u, x) \mapsto (ru, x)$, where $r \in A^n$ denotes a sequence of length n. This is just the n-flattening map on the first $n+1$ coordinates, and the identity on the last. The reason this map is well-defined is because for any n-sequence r and any $u \in A^\omega$, we have $ru \sim_n u$, so that $I_u = I_{ru}$.

In analogy with our work in the previous section, let us generalize the above. Suppose that $I_u, u \in A^\omega$ is a collection of sets, and for every n-sequence $r \in A^n$ and sequence $u \in A^\omega$ we have a bijection $f_{u \to ru} : I_u \to I_{ru}$. Then if we let $X = A^\omega(I_u)$, there is a natural bijection between AX and X defined by $(r, u, x) \mapsto (ru, f_{u \to ru}(x))$. We use the notation $A^\omega(I_{[u]_n})$ to denote a replacement $A^\omega(I_u)$ for which we have such a system of bijections $\{f_{u \to ru}\}$. When all of these bijections are the identity, this reduces to the case in the previous paragraph.

As one might guess, this form is general for sets X for which there is a bijection $f : AX \to X$.

Theorem 2.1.3. Let A and X be nonempty sets, and n be a fixed integer greater than 1. Then there is a bijection $f : A^n X \to X$ if and only if there is a bijection F from X onto a replacement of the form $A^\omega(I_{[u]_n})$.

Proof. It remains to prove the forwards direction. Suppose that $f : A^n X \to X$ is a bijection. Then by Theorem 2.1.2, there is a replacement $(A^n)^\omega(I_u)$ of $(A^n)^\omega$ for which we have a bijection $F : X \to (A^n)^\omega(I_u)$, and along with this replacement, a system of bijections $f_{u \to ru} : I_u \to I_{ru}$ for every $r \in A^n$ and $u \in (A^n)^\omega$. If we identify every sequence $u = ((a_0, a_1, \ldots, a_{n-1}), (a_n, \ldots a_{2n-1}), \ldots)$ in $(A^n)^\omega$ with the corresponding $u = (a_0, a_1, \ldots)$ in A^ω, then this identifies $(A^n)^\omega$ with A^ω. Under this identification, the replacement $(A^n)^\omega(I_u)$ is identified with the replacement $A^\omega(I_u)$, and F becomes a bijection of X with $A^\omega(I_u)$. □

2.2 The isomorphism $A \times X \cong X$ over a class $(\mathfrak{C}, \times)$

In this section we will show how Theorem 2.1.2 can be used to characterize those structures X from a given class $\mathfrak{C}$ that are invariant (up to isomorphism) under left multiplication by a given $A \in \mathfrak{C}$. The idea is that if $AX \cong X$, as witnessed by an isomorphism $f : AX \to X$, then because f is in particular a bijection, we can use the proof of Theorem 2.1.2 to produce a bijection F from X onto a *set* of the form $A^\omega(I_{[u]})$. We will argue that in the cases we consider, this set can actually be viewed as a structure from $\mathfrak{C}$, and that this structure behaves "as expected" given that we denote it $A^\omega(I_{[u]})$. In particular, on its first ω coordinates, it behaves like the infinite product A^ω.

We will consider three cases: the first is when $(\mathfrak{C}, \times)$ is the class of linear orders under the lexicographical product, the second, when it is the class of topological spaces under the topological product, and the third, when it is the class of groups under the direct product. Analogous characterizations can be proved for many other classes, but these three are somewhat representative: linear orders stand in for relational structures, groups for algebraic structures, and topological spaces for topological structures (or more generally "second-order" structures).

2.2.1 $A \times X \cong X$ over the class of linear orders

If X and Y are linear orders, the *lexicographical product* $X \times Y$ is the order obtained by lexicographically ordering the cartesian product of X and Y. That is, $X \times Y = \{(x, y) : x \in X, y \in Y\}$ ordered by the rule $(x_0, y_0) < (x_1, y_1)$ if and only if $x_0 < x_1$ (in X), or $x_0 = x_1$ and $y_0 < y_1$ (in Y).

Visually, XY is the order obtained by replacing every point in X with a copy of Y. Every point $x \in X$ determines an interval of points in XY of order type Y, namely the set of pairs $(x, \cdot)$ with left entry x. One may also visualize XY as a tree with two ordered levels. The first level has X-many nodes, and each of these has Y-many descendants. The order type of the terminal nodes is XY.

The lexicographical product is associative, in the sense that $(X \times Y) \times Z$ is isomorphic to $X \times (Y \times Z)$ for all orders X, Y, Z. But it is not commutative. For example, let $\mathbb{Z}$ be the integers in their usual order, and let 2 be the unique linear order with two elements. Then $\mathbb{Z}2$ is isomorphic to $\mathbb{Z}$, but $2\mathbb{Z}$ is not, as the latter order contains a bounded infinite increasing sequence, whereas every infinite increasing sequence in $\mathbb{Z}$ is unbounded.

We identify the n-length product $X_0X_1 \ldots X_{n-1}$ with the set of n-tuples $\{(x_0, x_1, \ldots, x_{n-1}) : x_i \in X_i, i < n\}$ ordered lexicographically. If $X_i = X$ for all $i < n$, this order is abbreviated as X^n. We also define the ω-length product $X_0X_1 \ldots$ as the set of sequences $\{(x_0, x_1, \ldots) : x_i \in X_i, i \in \omega\}$ ordered lexicographically. If $X_i = X$ for all $i \in \omega$, this order is abbreviated as X^ω. One may think of X^n as a tree with n levels and X^ω as a tree with ω-many levels, but since in the latter case there are no terminal nodes, one must lexicographically order the branches of the tree to recover the order.

Given a linear order X, and for every $x \in X$ an order I_x, the *replacement* $X(I_x)$ is the linear order obtained by lexicographically ordering the replacement *set* $X(I_x)$. That is, $X(I_x) = \{(x, i) : x \in X, i \in I_x\}$ ordered by the rule $(x_0, i_0) < (x_1, i_1)$ if and only if $x_0 < x_1$ (in X), or $x_0 = x_1$ and $i_0 < i_1$ (in $I_{x_0} = I_{x_1}$). Visually, $X(I_x)$ is the order obtained by replacing every point $x \in X$ with the corresponding I_x. As in the case of sets, we allow that for a given $x_0 \in X$ we have $I_{x_0} = \emptyset$, and in forming $X(I_x)$ think of replacing x_0 with a gap.

Now, let A be a fixed, nonempty linear order. For which orders X do we have $AX \cong X$? We can adapt the results of the previous sections to completely characterize such orders, and the adaptation is straightforward. First, suppose that for every $u \in A^\omega$ we fix a linear order I_u, with the restriction that if $u \sim v$, then $I_u = I_v$. Let $X = A^\omega(I_u)$ be the replacement of A^ω by these orders. We also denote a replacement of this form by $A^\omega(I_{[u]})$.

The flattening map $(a, u, x) \mapsto (au, x)$ is order-preserving, and hence defines an isomorphism between AX and X. Thus for any order X of the form $A^\omega(I_{[u]})$, we have that $AX \cong X$.

More generally, even if we do not have identity, but only an isomorphism $f_{u \to v} : I_u \to I_v$ for every pair of tail-equivalent sequences u, v, we still have that $AX \cong X$, as witnessed by the map $(a, u, x) \mapsto (au, f_{u \to au}(x))$.

We will take the notation $A^\omega(I_{[u]})$ to mean a replacement $A^\omega(I_u)$ for which $u \sim v$ implies $I_u \cong I_v$. We have just observed that any order X of this form is isomorphic to AX. An easy adaptation of the proof of Theorem 2.1.2 shows that the converse is also true.

Theorem 2.2.1. Let A and X be nonempty linear orders. Then $AX \cong X$ if and only if X is isomorphic to an order of the form $A^\omega(I_{[u]})$.

Proof. It remains to prove the forward direction. Assume that $AX \cong X$, as witnessed by an isomorphism $f : AX \to X$. We will simply run the proof Theorem 2.1.2 using this f. The notation below is adopted from that proof.

Note that for any $a \in A$, the set aX is in fact an interval in AX. Moreover if $a < b$, then aX and bX are disjoint intervals with aX lying to the left of bX. For such intervals we write $aX < bX$. If $aX < bX$ in AX, then by taking their images under f we see that $X_a < X_b$ in X. For each $a \in A$, the map $f_a : X \to X_a$ is not only a bijection but an order isomorphism of X with the interval X_a.

Extending this, we have that for every finite sequence $r \in A^{<\omega}$, the set X_r is an interval in X, and $f_r : X \to X_r$ is an order isomorphism of X with this interval. If t extends r, then X_t is a subinterval of X_r. If neither one of r, t extends the other, and if $r_k < t_k$, where k is the leftmost entry at which the sequences differ, then $X_r < X_t$.

Since an intersection of intervals is an interval, each I_u is an interval in X, and these intervals partition X. It follows from the previous paragraph that if $u < v$ in A^ω then $I_u < I_v$ in X. That is, the intervals I_u partition X and respect the ordering of their indices. (It may be that for a given u, we have that I_u is in fact empty. Such an interval is not really an interval but a gap. However, if I_u is a gap, then it still is ordered correctly with respect to its index u: if v, w are sequences such that $v < u < w$, then I_v lies to the left of the gap I_u, and I_w lies to the right.)

Hence, if we consider $A^\omega(I_u)$ not only as a replacement set but as a replacement order, we have that the bijection F constructed in the proof of Theorem 2.1.2 defines an isomorphism of X with this order.

To complete the proof, suppose that u and v are tail-equivalent sequences in A^ω, as witnessed by a meeting representation $u = ru'$, $v = su'$. Then the map $f_s \circ f_r^{-1}$, which is an order isomorphism of X_r with X_s, sends the interval I_u onto I_v. Hence $I_u \cong I_v$ as linear orders, so that our replacement $A^\omega(I_u)$ is really of the form $A^\omega(I_{[u]})$. □

We conclude this section by stating, without proof, the corresponding characterization of those orders X satisfying the isomorphism $A^n X \cong X$ for a given $n > 1$. We single out the case when $n = 2$, since we will frequently be interested in this case specifically in Chapter 3.

Fix $n > 1$. Suppose that $A^\omega(I_u)$ is a replacement of A^ω with the restriction that if $u \sim_n v$, then the orders I_u and I_v are isomorphic. We use the notation $A^\omega(I_{[u]_n})$ to denote such an order. If X is an order of this form, then $A^n X \cong X$, as witnessed by the n-flattening map $(r, u, x) \mapsto (ru, x)$. Conversely, this form is general for orders satisfying the isomorphism $A^n X \cong X$.

Theorem 2.2.2. Let A and X be nonempty linear orders. Then $A^n X \cong X$ if and only if X is isomorphic to an order of the form $A^\omega(I_{[u]_n})$.

Theorem 2.2.3. Let A and X be nonempty linear orders. Then $A^2 X \cong X$ if and only if X is isomorphic to an order of the form $A^\omega(I_{[u]_2})$.

2.2.2 $A \times X \cong X$ over the class of topological spaces

Suppose now that $(\mathfrak{C}, \times)$ is the class of topological spaces under the topological product. If X and Y are topological spaces, then the space $X \times Y = XY$ is the cartesian product of X and Y, equipped with the topology generated by the collection of sets of the form $O \times P$, where O is an open subset of X and P is an open subset of Y. In this section, the notation $X \cong Y$ means "X is homeomorphic to Y."

Let A be a fixed topological space. Our aim is to characterize those spaces X such that AX is homeomorphic to X. We will show below that any such space is a replacement of A^ω up to tail-equivalence, equipped with a topology that is, roughly speaking, compatible with the topology of A.

Before considering replacements, let us note first that there are many topologies on A^ω itself that are compatible with the topology on A, that is, that make the flattening map $fl : A \times A^\omega \to A^\omega$ a homeomorphism. The two most common are the product topology and the box topology. The product topology is generated by the collection of open sets of the form $\Pi_{i \in \omega} O_i = O_0 \times O_1 \times \ldots$, where each O_i is open in A, and for all but finitely many i we have $O_i = A$. The box topology is also generated sets of the form $\Pi_{i \in \omega} O_i$, where now there is no restriction on the O_i except that they are all open in A. Clearly, the box topology refines the product topology.

We will verify later that under both of these topologies the flattening map is a homeomorphism, though this is not difficult to check. We will show that in fact these topologies are the minimal and maximal topologies on A^ω with this property, respectively.

If O is a subset of A and $\mathcal{O}$ is a subset of A^ω, we will take the notation $O \times \mathcal{O}$ to have two different meanings. The first is its usual meaning as a subset of $A \times A^\omega$. The second is as the subset of A^ω consisting of points $u = (u_0, u_1, \ldots)$ with first coordinate $u_0 \in O$ and tail-sequence $(u_1, u_2, \ldots) \in \mathcal{O}$. These meanings are the same, modulo the flattening map fl. When we do need to distinguish between these two senses of the notation $O \times \mathcal{O}$, we will do so verbally, referring to its usual meaning as the "first sense" and its flattened meaning as the "second sense."

Now suppose that $A^\omega(I_{[u]})$ is a replacement of A^ω up to tail-equivalence. Suppose that O is a subset of A and $\mathcal{O}$ is a subset of $A^\omega(I_{[u]})$. In analogy with the previous paragraph, we wish to define two meanings of the notation $O \times \mathcal{O}$. The purpose of doing so is to make the proof of Theorem 2.2.4 easy, but since in this case the confusion is less transparent, we will explain it carefully.

Recall that, as a set, $A^\omega(I_{[u]})$ is a replacement $A^\omega(I_u)$ equipped with a system of bijections $f_{u\to au} : I_u \to I_{au}$. By the "first sense" of $O \times \mathcal{O}$, we mean the subset of $A \times A^\omega(I_{[u]})$ consisting of points (a, u, x) with $a \in O$ and $(u, x) \in \mathcal{O}$. By the "second sense" of $O \times \mathcal{O}$, we mean the image of $O \times \mathcal{O}$ (in the first sense) under the flattening map $(a, u, x) \mapsto (au, f_{u\to au}(x))$.

Hence $(a, u, x) \in O \times \mathcal{O}$ in the first sense, if and only if $(au, f_{u\to au}(x)) \in O \times \mathcal{O}$ in the second sense. In the other direction, suppose $v = au$ and $y \in I_v$. Then $(v, y) \in O \times \mathcal{O}$ in the second sense, if and only if $(a, u, f^{-1}_{u\to au}(y)) \in O \times \mathcal{O}$ in the first sense, if and only if $a \in O$ and $(u, f^{-1}_{u\to au}(y)) \in \mathcal{O}$.

In the case when all the bijections $f_{u\to au}$ are the identity, the notation is significantly more transparent. In this case, we have that $I_u = I_v$ for all tail-equivalent pairs u, v. In particular, $I_u = I_{au}$ for all u, a. Then if O is a subset of A and $\mathcal{O}$ is a subset of $A^\omega(I_{[u]})$, we have $(a, u, x) \in O \times \mathcal{O}$ in the first sense, if and only if $(au, x) \in O \times \mathcal{O}$ in the second sense.

Now suppose that X, considered as a set, is a replacement of A^ω up to tail-equivalence, that is, $X = A^\omega(I_{[u]})$. Suppose further that X is equipped with a topology with the following two properties:

(1) if O is open in A, and $\mathcal{O}$ is open in X, then $O \times \mathcal{O}$ (in the second sense) is open in X.

(2) if $\mathcal{O}$ is open in X, then $\mathcal{O}$ is a union of open sets of the form $O_i \times \mathcal{O}_i$ (in the second sense). That is,

$$\mathcal{O} = \bigcup_{i \in I} O_i \times \mathcal{O}_i$$

where O_i is open in A and $\mathcal{O}_i$ is open in X for all i in some indexing set I.

We claim that if X carries such a topology, then the flattening map $(a, u, x) \mapsto (au, f_{u \to au}(x))$ defines a homeomorphism between AX and X. To see this, it is enough to check that this map is open and continuous, since we know already that it is a bijection. To check openness, suppose that $O \times \mathcal{O}$ (in the first sense) is a basic open subset of AX. Then the image of this map is just $O \times \mathcal{O}$ (in the second sense), which is open in X by (1). To check continuity, suppose that $\mathcal{O}$ is an open subset of X. Then by (2), $\mathcal{O}$ is a union of sets of the form $O_i \times \mathcal{O}_i$ (in the second sense), where each O_i is open in A and each $\mathcal{O}_i$ is open in X. The pre-image of each $O_i \times \mathcal{O}_i$ is simply $O_i \times \mathcal{O}_i$ (in the first sense), which is a basic open set in AX. Therefore the pre-image of $\mathcal{O}$ is a union of basic open sets, and hence is open.

We will now show that this form in general for spaces X such that $AX \cong X$.

Theorem 2.2.4. Let A and X be nonempty topological spaces. Then $AX \cong X$ if and only if X is homeomorphic to a space of the form $A^\omega(I_{[u]})$ that carries a topology satisfying conditions (1) and (2).

Proof. It remains to prove the forward direction. So suppose that A, X are topological spaces and $f : AX \to X$ is a homeomorphism. By the proof of Theorem 2.1.2, X can be relabeled as $A^\omega(I_{[u]})$ by way of the bijection F, and under this relabeling the map f becomes the flattening map $(a, u, x) \mapsto (au, f_{u \to au}(x))$. For simplicity, we forget F and simply identify X with the set $A^\omega(I_{[u]})$.

If O is open in A, and $\mathcal{O}$ is open in X, then the image of the basic open set $O \times \mathcal{O}$ (in the first sense) under the flattening map is just $O \times \mathcal{O}$ (in the second sense). Since f is open, this set is open in X. Thus the topology on X satisfies condition (1).

Conversely, if $\mathcal{O}$ is open in X, then since f is continuous, the pre-image under f of $\mathcal{O}$ is open in AX. Hence this preimage is a union of basic open sets $O_i \times \mathcal{O}_i$ (in the first sense), so that $\mathcal{O}$ itself is a union of such sets (in the second sense). Thus the topology on X satisfies condition (2). $\square$

We conclude this section with some examples. Let A be a fixed nonempty topological space. For simplicity, instead of considering spaces of the form $A^\omega(I_{[u]})$, we simply consider the space $X = A^\omega$, and ask which topologies on this space satisfy (1) and (2) (this is equivalent to considering spaces $X = A^\omega(I_{[u]})$ for which each $I_{[u]}$ is a singleton). These are exactly the topologies for which the flattening map defines a homeomorphism between AX and X.

The product topology is easily seen to satisfy (1) and (2). In fact, any topology that satisfies (1) refines the product topology. This is because A^ω itself is open in any topology on A^ω. Therefore, if it carries a topology that satisfies (1), then any "shifted" set $O_0 \times O_1 \times \ldots \times O_n \times A^\omega$ is must also be open in this topology. But sets of this form are exactly the sets that generate the product topology.

Similarly, the box topology is easily seen to satisfy (1) and (2). In fact, any topology that satisfies (2) must be a coarsening of the box topology. For, if $\mathcal{O}$ is open in A^ω in some topology T satisfying (2), then $\mathcal{O}$ is a union of sets of the form $O_i \times \mathcal{O}_i$. Each of the "tail-sets" $\mathcal{O}_i$ can in turn be written as a union of sets of the form $O_{i,j} \times \mathcal{O}_{i,j}$. Hence $\mathcal{O}$ is actually a union of sets of the form $O_i \times O_{i,j} \times \mathcal{O}_{i,j}$. If we repeat this factoring process infinitely many times, then we see that $\mathcal{O}$ is actually a union of sets of the form $O_{i_0} \times O_{i_0,i_1} \times \ldots$, where each $O_{i_0,\ldots,i_n}$ is open in A. These sets are of course open in the box topology, and so it follows that every open set in T is open in the box topology. We cannot conclude that these sets themselves are open in T (though each such set is of course G_δ in T). Hence the box topology refines T.

There are topologies on A^ω that fall strictly between the product and box topologies and that satisfy (1) and (2). For example, consider the collection of sets of the form $\Pi_{i\in\omega} O_i$, where each O_i is open in A, and for all but finitely many pairs i, j, we have $O_i = O_j$. These "eventually constant" open sets form a basis for a topology on A^ω. It can be easily verified that this topology satisfies (1) and (2). However, in most any case when the topology on A is nontrivial, this topology will be strictly finer than the product topology on A^ω and strictly coarser than the box topology.

2.2.3 $A \times X \cong X$ over the class of groups: a partial characterization

Now suppose that our class $(\mathfrak{C}, \times)$ is the class of groups under the direct product. If X and Y are groups, then the group $X \times Y = XY$ is the cartesian product of X and Y whose group operation is coordinate-wise multiplication. If A is a group, then the infinite direct product A^ω also forms a group under coordinate-wise multiplication.

If a, b are elements in a group A, we write $a \cdot b$ to denote their product in A, whereas ab still denotes the 2-sequence of a followed by b. If u, v are elements in A^ω, we write $u \cdot v$ to denote their product in A^ω. Hence, if (a, u) and (b, v) are elements in $A \times A^\omega$, their product $(a, u) \cdot (b, v)$ is $(a \cdot b, u \cdot v)$. The image of this product under the flattening map is the sequence with first entry is $a \cdot b$ followed by the tail-sequence $u \cdot v$. For clarity, we use parentheses and write this sequences as $(a \cdot b)(u \cdot v)$. That is, $(a \cdot b)(u \cdot v) = (a \cdot b, u_0 \cdot v_0, u_1 \cdot v_1, \ldots)$. We write 1 for the identity of A. In A^ω, the identity is the sequence $(1, 1, \ldots)$. We denote this sequence by $\overline{1}$.

If $\varphi : X \to \mathrm{Aut}(N)$ is a homomorphism from X into the automorphism group of a group N, then we can define the semi-direct product $X \ltimes N$ with respect to the homomorphism φ. This is the group with underlying universe $X \times N$, whose group operation is defined by the rule $(x_0, n_0) \cdot (x_1, n_1) = (x_0 \cdot x_1, \varphi_{x_1}(n_0) \cdot n_1)$, where φ_{x_1} denotes the automorphism $\varphi(x_1)$.

While the operation of coordinate-wise multiplication in A^ω is well-defined and makes A^ω a group, we do not know a general way to put a sensible group structure on a set of the form $A^\omega(I_u)$ that extends the coordinate-wise multiplication on A^ω. It will follow from our work below that if there is a general way of defining such a group, there are at least significant restrictions on what the I_u can be. In particular, in any such "replacement group" it must be that $I_{\bar{1}}$ is a normal subgroup, and every nonempty I_u is a coset of this subgroup. In particular, all nonempty I_u must be of the same cardinality.

Let A be a fixed, non-empty group. Can we say for which groups X we have $AX \cong X$? Unlike in the case of linear orders and topological spaces, we are not able to give a complete characterization of such groups in terms of A. However, we can characterize certain quotients of such groups. This is Theorem 2.2.5 below.

Before proving this partial characterization, we give some examples. First, note that the flattening map $(a, u) \mapsto au$ is a group isomorphism of $A \times A^\omega$ with A^ω. We will call a subgroup $H \leq A^\omega$ a *good subgroup* of A^ω if it closed under tail-equivalence. If H is any good subgroup, then the flattening map witnesses $AH \cong H$. Similarly, given a good subgroup H and an arbitrary group N, if we let $X = HN$, then we have $AX \cong X$. The isomorphism is, as always, just the flattening map $(a, u, x) \mapsto (au, x)$, restricted here to the first two coordinates.

Hence any direct product of a good subgroup of A^ω with an arbitrary group is invariant under multiplication by A. The situation for semi-direct products, however, is less clear.

Suppose we have a homomorphism $\varphi : H \to \mathrm{Aut}(N)$ from a good subgroup H into the automorphism group of a group N. If we let $X = H \ltimes N$, do we have $AX \cong X$?

We do not know the answer to this question in general. However, if the homomorphism φ is of a particular kind, then we can give a positive answer. We say that φ is *shift-invariant* if for every $u \in H$ and $a \in A$ we have $\varphi_u = \varphi_{au}$. It follows that if $u \sim v$, then $\varphi_u = \varphi_v$. That is, a shift-invariant homomorphism is constant on tail-equivalence classes.

If φ is shift-invariant, and we form the semi-direct product $X = H \ltimes N$ with respect to φ, then we claim that $AX \cong X$, and that the isomorphism is witnessed by the flattening map $(a, u, x) \mapsto (au, x)$. We know that this map defines a bijection, so we need only check that it respects multiplication and inverses. If (a, u, x) and (b, v, y) are arbitrary elements of AX, then their product is $(a \cdot b, u \cdot v, \varphi_v(x) \cdot y)$. The image of this product under the flattening map is $((a \cdot b)(u \cdot v), \varphi_v(x) \cdot y)$. On the other hand, if we flatten first, our arbitrary elements become (au, x) and (bv, y). The product of these elements in X is $((a \cdot b)(u \cdot v), \varphi_{bv}(x) \cdot y)$. By shift-invariance, this is equal to $((a \cdot b)(u \cdot v), \varphi_v(x) \cdot y)$. Hence the flattening map respects multiplication. To see that it respects inverses, suppose (a, u, x) is an arbitrary element in AX. Then its inverse is $(a^{-1}, u^{-1}, \varphi_{u^{-1}}(x^{-1}))$. If we flatten this image we get $(a^{-1}u^{-1}, \varphi_{u^{-1}}(x^{-1}))$. On the other hand, if we flatten first to form (au, x), and then take the inverse we get $(a^{-1}u^{-1}, \varphi_{a^{-1}u^{-1}}(x^{-1}))$. By shift-invariance of φ this is equal to $(a^{-1}u^{-1}, \varphi_{u^{-1}}(x^{-1}))$. Hence the flattening map also respects inverses, and we have the claim.

Let us give an example of such a group $H \ltimes N$. Let $\mathbb{Z}_2 = \{-1, 1\}$ be the unique group on two elements, written multiplicatively. Let H be the subgroup of $\mathbb{Z}_2^\omega$ consisting of all sequences whose entries are either eventually 1, or eventually -1. Then $H = \lceil \overline{1} \rceil \cup \lceil \overline{-1} \rceil$. Thus H is closed under tail-equivalence and so forms a good subgroup of $\mathbb{Z}_2^\omega$. Let $N = \mathbb{Z}_3$. Then $\mathrm{Aut}(N)$ is isomorphic to $\mathbb{Z}_2$. We identify $\mathrm{Aut}(N)$ with $\mathbb{Z}_2$. We can define a homomorphism $\varphi : H \to \mathrm{Aut}(N)$ by mapping every element $u \in \lceil \overline{1} \rceil$ to 1, and every element $u \in \lceil \overline{-1} \rceil$ to -1. It is easily checked that this is a homomorphism, and it is also clearly shift-invariant. Then by above, $AX \cong X$. It can also be checked that X is a nonabelian group, and hence distinct from the corresponding direct product HN.

If we assume that X is a group such that $AX \cong X$, we do not know in general if X must be of the form $H \ltimes N$ for some good subgroup H, even if we do not assume that the semi-direct product is defined with respect to a shift-invariant homomorphism. What we can show is that any such X contains a normal subgroup N such that the quotient X/N is isomorphic to a good subgroup $H \leq A^\omega$.

Theorem 2.2.5. Suppose that A and X are nonempty groups. If $AX \cong X$, then there is a good subgroup $H \leq A^\omega$ and a normal subgroup $N \trianglelefteq X$ such that $X/N \cong H$.

Proof. Suppose that $f : AX \to X$ is an isomorphism. Then if we run the proof of Theorem 2.1.2, we obtain a bijection F of X onto a set of the form $A^\omega(I_{[u]})$. For simplicity, we identify the underlying universe of the group X with this set.

We claim:

(1) for $u, v \in A^\omega$, if (u, x) is an arbitrary element of I_u and (v, y) is an arbitrary element of I_v, then $(u, x) \cdot (v, y) \in I_{u \cdot v}$. That is, multiplication in $X = A^\omega(I_{[u]})$ respects multiplication in A^ω.

(2) for $u \in A^\omega$, if (u, x) is an arbitrary element of I_u, then $(u, x)^{-1} \in I_{u^{-1}}$. That is, the operation of taking inverses in X respects the operation of taking inverses in A^ω.

It follows from these claims that the map $(u, x) \mapsto u$ defines a homomorphism from X onto a subgroup H of A^ω. From Theorem 2.1.2 we know that H is a good subgroup: we have $u \in H$ if and only if $I_u \neq \emptyset$, if and only if $I_v \neq \emptyset$ for all $v \sim u$. The kernel of this homomorphism is $N = I_{[\bar{1}]}$. By the first isomorphism theorem for groups, $X/N \cong H$.

Hence it remains to prove the claims. If A, B are subsets of a group, let $A \cdot B$ denote the set $\{a \cdot b : a \in A, b \in B\}$. We also adopt the notation of the proof of Theorem 2.1.2. Since A, X are groups we have, for any $a, b \in A$, that $aX \cdot bX = (a \cdot b)X$. Hence $X_a \cdot X_b = X_{a \cdot b}$. Extending this, if r, s are finite sequences of the same length n, we have that $X_r \cdot X_s = X_{r \cdot s}$, where $r \cdot s$ denotes the n-sequence $(r_0 \cdot s_0, \ldots, r_{n-1} \cdot s_{n-1})$.

Fix elements $(u, x) \in I_u$ and $(v, y) \in I_v$. By definition, (u, x) is contained in $X_{u \restriction n}$ for every n, and (v, y) is contained in $X_{v \restriction n}$ for every n. Since $X_{u \restriction n} \cdot X_{v \restriction n} = X_{(u \cdot v) \restriction n}$, we have that $(u, x) \cdot (v, y) \in X_{(u \cdot v) \restriction n}$ for every n. Hence $(u, x) \cdot (v, y) \in I_{u \cdot v}$. This proves the first claim. A similar calculation proves the second. $\square$

Chapter 3

The Cube Problem for Linear Orders

3.1 Introduction

In this chapter we will prove that the *cube property* holds for the class $(LO, \times_{lex})$ of linear orders under the lexicographical product: if X is a linear order that is isomorphic to its lexicographically ordered cube X^3, then X is isomorphic to X^2. More generally we have the following.

Main Theorem. Let X be a linear order, and fix an integer $n > 1$. If $X^n \cong X$, then $X^2 \cong X$.

This is Theorem 3.3.15. Thus the answer to Sierpiński's question from *Cardinal and Ordinal Numbers* of whether there exists a linear order that is isomorphic to its cube but not its square is negative. See the introduction of this thesis for a detailed discussion of this problem and its history, as well as an overview of the proof.

3.1.1 Terminology

A *linear order* is a pair $(X, <_X)$ where X is a set and $<_X$ is a binary relation that totally orders X. We will always refer to linear orders by their underlying sets, and write $<$ without any subscript. Throughout the chapter, "order" always means linear order, and "isomorphism" means order isomorphism.

Given a linear order X, a subset $I \subseteq X$ is called an *interval* if for all points $x, y, z \in X$, if $x < y < z$ and $x, z \in I$, then $y \in I$. Every singleton is an interval, as is X itself. Given points $x, y \in X$ with $x < y$, the interval notation (x, y), $[x, y)$, $(x, y]$, and $[x, y]$ has its usual meaning. If I and J are intervals in X, and for all $x \in I$ and $y \in J$ we have $x < y$, then I lies to the left of J and we write $I < J$. An interval I is called an *initial segment* of X if whenever $x \in I$ and $y < x$ then $y \in I$. An interval J is called a *final segment* of X if J is the complement of an initial segment of X.

An order (or interval) may have endpoints. The terms minimal element, left endpoint, and bottom point will be used interchangeably, as will maximal element, right endpoint, and top point.

An order X is *dense* if between any two distinct points in X one may find a third that lies strictly between them. A subset $D \subseteq X$ is *dense in* X if for any two points in X, either one of them lies in D or there exists a point between them that lies in D. An order X is *complete* if every bounded monotonic sequence (of any ordinal length) in X converges to a point in X.

For X a linear order, X^* denotes the reverse order. That is, X and X^* share the same underlying set of points, but $x < y$ in X if and only if $y < x$ in X^*. To every function $f : X \to X$, there is a corresponding function on X^*, denoted f^*, which acts identically to

f on the underlying set of points shared by X and X^*. If f is an order automorphism of X, then f^* is an order automorphism of X^*.

The following definitions are copied from Section 2.2.1. If X and Y are linear orders, then their *lexicographical product* $X \times Y = XY$ is the order obtained by replacing every point in X with a copy of Y. Formally, XY is the cartesian product $\{(x, y) : x \in X, y \in Y\}$ ordered by the rule $(x_0, y_0) < (x_1, y_1)$ if and only if $x_0 < x_1$ (in X), or $x_0 = x_1$ and $y_0 < y_1$ (in Y). Longer finite products $X_0 X_1 \ldots X_{n-1}$, as well as infinite products $X_0 X_1 \ldots$, are defined similarly, as the lexicographically ordered set of n-sequences $\{(x_0, \ldots x_{n-1}) : x_i \in X_i\}$, and the lexicographically ordered set of ω-sequences $\{(x_0, x_1, \ldots) : x_i \in X_i\}$, respectively. In the case when all of the X_i are equal to a single order X, these orders are abbreviated X^n and X^ω, respectively.

If for every $x \in X$ we fix a linear order I_x, then the *replacement* $X(I_x)$ is the order obtained by replacing every point $x \in X$ with the corresponding I_x. Formally, $X(I_x) = \{(x, i) : x \in X, i \in I_x\}$ ordered by the rule $(x_0, i_0) < (x_1, i_1)$ if and only if $x_0 < x_1$ (in X), or $x_0 = x_1$ and $i_0 < i_1$ (in $I_{x_0} = I_{x_1}$). We allow that for a given $x_0 \in X$ we have $I_{x_0} = \emptyset$. Both of the notions of lexicographical product and replacement are central to the rest of this chapter. See Section 2.2.1 for a more detailed discussion of these definitions.

3.1.2 Examples of countable orders isomorphic to their squares

To begin, we give some examples of countable orders that are isomorphic to their own squares. In Section 3.4 we will construct uncountable examples.

Our examples here rely on Cantor's theorem characterizing the order types of countable dense linear orders, as well as a generalization due to Skolem.

Theorem. (Cantor) If X and Y are countable dense linear orders without endpoints, then X is isomorphic to Y.

Thus every countable dense linear order without endpoints is isomorphic to the set of rationals $\mathbb{Q}$ in their usual order.

Theorem. (Skolem) Fix some k, $1 \leq k \leq \omega$. Let X, Y be countable dense linear orders without endpoints. Fix a partition $X = \bigcup_{i<k} X_i$ such that each X_i is dense in X, and similarly $Y = \bigcup_{i<k} Y_i$. There is an isomorphism $f : X \to Y$ such that $f[X_i] = Y_i$ for every $i < k$.

This says that if we have two copies of the rationals, and color each of them with the same k colors, using every color densely often, then there is an isomorphism between the two orders that respects the colorings. The proof is an easy generalization of the usual back-and-forth proof of Cantor's theorem.

Now, if L is any countable order, then $L\mathbb{Q}$ is countable, dense, and without endpoints. Hence, $L\mathbb{Q} \cong \mathbb{Q}$. In particular, $\mathbb{Q}^2 \cong \mathbb{Q}$, yielding our first example of an order isomorphic to its square. To get another, let 2 denote the unique order (up to isomorphism) with two points, and let $X = \mathbb{Q}2$. This order is not isomorphic to $\mathbb{Q}$, since it contains many intervals isomorphic to 2, and thus is not dense. Yet if L is any countable order, then $LX = L(\mathbb{Q}2) \cong (L\mathbb{Q})2 \cong \mathbb{Q}2 = X$. In particular $X^2 \cong X$.

Using Skolem's theorem, we generalize these examples. Fix k, $1 \leq k \leq \omega$, and a partition $\mathbb{Q} = \bigcup_{i<k} \mathbb{Q}_i$ such that each $\mathbb{Q}_i$ is dense in $\mathbb{Q}$. For each i, fix some countable order I_i, and form the replacement $X = \mathbb{Q}(I_q)$, where if $q \in \mathbb{Q}_i$ then $I_q = I_i$. This is the order obtained by replacing each point in the rationals with one of the k countable orders I_i, such that each order appears densely often.

Let L be any countable order. Then $L\mathbb{Q}(I_q) \cong \mathbb{Q}(I_q)$, i.e. $LX \cong X$. The isomorphism follows from Skolem's theorem: $L\mathbb{Q}(I_q)$ is also a countable dense shuffling of the I_i, which is the same form as $\mathbb{Q}(I_q)$.

To see this explicitly, let us write $L\mathbb{Q}(I_q)$ as $L\mathbb{Q}(J_{(l,q)})$, where $J_{(l,q)} = I_q$ for all $l \in L$. Thus $J_{(l,q)} = I_i$ if $q \in \mathbb{Q}_i$. We partition $L\mathbb{Q}$ according to the partition of $\mathbb{Q}$: let $Q_i = \{(l,q) \in L\mathbb{Q} : q \in \mathbb{Q}_i\}$. Then the Q_i partition $L\mathbb{Q}$ and we have $J_{(l,q)} = I_i$ if $(l,q) \in Q_i$. Since each Q_i is dense in $L\mathbb{Q}$, there is an isomorphism $f : L\mathbb{Q} \to \mathbb{Q}$ such that $f[Q_i] = \mathbb{Q}_i$ for every i. But then $J_{(l,q)} = I_{f((l,q))}$ for every $(l,q) \in L\mathbb{Q}$. Thus the isomorphism lifts to give an isomorphism of $L\mathbb{Q}(J_{(l,q)})$ with $\mathbb{Q}(I_q)$, i.e. of LX with X. Since L was arbitrary, we have in particular that $X^2 \cong X$.

In Section 3.2 it will be shown that if X is a countable order without endpoints and $X^n \cong X$ for some $n > 1$, then X has the same form as the order above: $X \cong \mathbb{Q}(I_q)$, where for each q there is a dense set of $p \in \mathbb{Q}$ such that $I_q = I_p$. It follows, as above, that $X^2 \cong X$ and hence $X^m \cong X$ for any $m > 1$. Of course, our main theorem is that $X^n \cong X \implies X^2 \cong X$ holds in general, but in the countable, no-endpoints case we have a complete classification: $X^n \cong X$ for some $n > 1$ if and only if $X \cong \mathbb{Q}(I_q)$. As we observed, such an order is not only invariant under left multiplication by itself, but by any countable order L.

3.1.3 A solution to the cube problem when X has both endpoints

It turns out that it is easy to prove $X^n \cong X \implies X^2 \cong X$ for linear orders X with both a left and right endpoint. The proof uses the following theorem of Lindenbaum, that could be called the Schroeder-Bernstein theorem for linear orders. Although the proof for the general case is substantially harder, a generalized version of Lindenbaum's theorem does play an important role.

Theorem. (Lindenbaum) Suppose X, Y are linear orders. If X is isomorphic to an initial segment of Y, and Y is isomorphic to a final segment of X, then $X \cong Y$.

Proof. Suppose $f : X \to Y$ is an isomorphism of X onto an initial segment of Y and $g : Y \to X$ is an isomorphism of Y onto a final segment of X. Then f, g are in particular injections. Let $h : X \to Y$ be the bijection constructed out of f, g as in the classical proof of the Schroeder-Bernstein theorem. Then the hypotheses guarantee that this bijection is order-preserving. □

Corollary 3.1.1. If X is a linear order with both a left endpoint and right endpoint, and $X^n \cong X$ for some $n > 1$, then $X^2 \cong X$.

Proof. Denote the minimal element of X by 0 and the maximal element by 1. Then X^2 contains an initial segment isomorphic to X, namely the interval of points of the form $(0, \cdot)$. Since $X^n \cong X$, it follows that X^2 contains an initial segment isomorphic to X^n. But the interval consisting of points $(1, 1, \ldots, 1, \cdot, \cdot)$ with $n - 2$ leading 1s is a final segment of X^n isomorphic to X^2. By Lindenbaum's theorem, $X^2 \cong X^n$. Hence $X^2 \cong X$. □

Thus any linear order with both endpoints that is isomorphic to its cube is isomorphic to its square, solving the cube problem in this case. The simplicity of the proof suggests that there may be a simple proof when X has either no endpoints or one endpoint. But the proof above uses crucially that X has both endpoints, and there does not seem to be an adaptation to the other cases. The isomorphisms constructed in Section 3.3 to take care of the other cases do make use of Schroeder-Bernstein style maps, but using them requires an understanding of the structure of orders X that satisfy isomorphisms of the form $A^n X \cong X$.

3.2 A sufficient condition for $A^n \times X \cong X \implies A \times X \cong X$

3.2.1 Review

Throughout the rest of the chapter, when it is not otherwise specified, A simply refers to some fixed order. We recall several of the relevant definitions and results from Chapters 1 and 2 concerning orders that are invariant under left multiplication by A or a finite power of A.

Suppose that $A^\omega(I_u)$ is a replacement of A^ω such that whenever $u \sim v$, we have $I_u \cong I_v$. We refer to such an order as a replacement of A^ω *up to tail-equivalence*, and denote it by $A^\omega(I_{[u]})$. If X is a linear order, then by Theorem 2.2.1, we have $AX \cong X$ if and only if X is isomorphic to an order of the form $A^\omega(I_{[u]})$.

We will adopt the following convention, since it simplifies notation: if $X = A^\omega(I_{[u]})$ is a replacement of A^ω up to tail-equivalence, and u and v are tail-equivalent sequences, then we will assume that in fact I_u *equals* I_v. This is safe to do, since any replacement in which we only have isomorphism between I_u and I_v is isomorphic to a replacement in which we have equality. Hence our convention does not change the statement of Theorem 2.2.1. For such an order X, the isomorphism between AX and X is witnessed by the flattening map $(a, u, x) \mapsto (au, x)$. We can disregard the isomorphisms $f_{u \to au}$ that we considered in Chapter 2, since they are irrelevant to our proof of the cube property.

Here is a concrete example of a replacement up to tail-equivalence. Let $\mathbb{Z}$ denote the integers in their usual order, and form the product $\mathbb{Z}^\omega$ (this order is isomorphic to the irrationals). Each tail-equivalence class $[u] \subseteq \mathbb{Z}^\omega$ is a countable dense subset of $\mathbb{Z}^\omega$, hence the number of classes is 2^ω. Enumerate them as $\{C_\alpha : \alpha < 2^\omega\}$. Let X be the order obtained by replacing every point in the αth class with the ordinal α, i.e. if $[u] = C_\alpha$ let $I_{[u]} = \alpha$, and let $X = \mathbb{Z}^\omega(I_{[u]})$. This order may be visualized as densely many copies of each ordinal less than 2^ω interspersed with one another. By our observations above, $\mathbb{Z}X \cong X$.

Similarly, if $A^\omega(I_u)$ is a replacement of A^ω such that whenever $u \sim_2 v$ we have $I_u = I_v$, then we refer to such an order as a replacement of A^ω *up to 2-tail-equivalence*, and denote it by $A^\omega(I_{[u]_2})$. (Here, we are again assuming equality of I_u and I_v instead of isomorphism.) For an order X, by Theorem 2.2.3, we have $A^2X \cong X$ if and only if X is isomorphic to an order of the form $A^\omega(I_{[u]_2})$.

Also recall that for a sequence $u \in A^\omega$, we have $[u] = [u]_2$ if and only if u is eventually periodic and the period of u is odd. This is Corollary 1.2.3. In all other cases, $[u]$ is the disjoint union of $[u]_2$ and $[au]_2$, where a is any fixed element of A.

3.2.2 When $A^2 \times X \cong X$ implies $A \times X \cong X$

We now turn to the question of when the isomorphism $A^2X \cong X$ implies $AX \cong X$. Notice that if $A^2X \cong X$ and we use Theorem 2.2.3 to decompose X as $A^\omega(I_{[u]_2})$, then there is no "obvious" isomorphism between AX and X. The natural guess for such an isomorphism would be the flattening map $(a, u, x) \mapsto (au, x)$. But the definition of this map may be meaningless: the interval consisting of tuples $(a, u, \cdot)$ in AX is of type $I_u = I_{[u]_2}$ whereas the interval $(au, \cdot)$ in X is of type $I_{[au]_2}$. If $u \not\sim_2 au$, these orders may be distinct.

If it so happens that $I_{[u]_2} = I_{[au]_2}$ for every u and a, then the flattening map makes sense and witnesses $AX \cong X$. Indeed if this is so we may denote the common order type of $I_{[u]_2}$ and $I_{[au]_2}$ by $I_{[u]}$, so that $X \cong A^\omega(I_{[u]})$.

But this need not be the case. For example, consider $\mathbb{Z}^\omega$ and let v denote the sequence $(1, 2, 3, \ldots)$. Then v is not eventually periodic and hence not 2-tail-equivalent to $0v = (0, 1, 2, \ldots)$. Let $I_{[v]_2} = 1$ and $I_{[0v]_2} = 2$. For all other $\sim_2$-classes $[u]_2$, let $I_{[u]_2} = \emptyset$. Let

$X = \mathbb{Z}^\omega(I_{[u]_2})$. Then the map $((a,b),u,x) \mapsto (abu,x)$ witnesses $\mathbb{Z}^2 X \cong X$, while the "map" $(a,u,x) \mapsto (au,x)$, which purports to send each copy of 1 in AX isomorphically onto a copy of 2 in X, and vice versa, is meaningless. In this sense it is not immediate that $\mathbb{Z}X \cong X$ (though, in this case, this turns out to be true).

The following proposition describes the precise relationship between AX and X when $A^2 X \cong X$.

Proposition 3.2.1. Suppose X is an order such that $A^2 X \cong X$, so that X is of the form $A^\omega(I_{[u]_2})$, and let $Y = AX$. Then $Y \cong A^\omega(J_{[u]_2})$, where for all $u \in A^\omega$ and $a \in A$, we have $J_{[u]_2} = I_{[au]_2}$.

Proof. Define $J_{[u]_2} = I_{[au]_2}$ as in the statement of the theorem, and assume for simplicity that X is not only isomorphic to but in fact equals $A^\omega(I_{[u]_2})$. Then the isomorphism between AX and $A^\omega(J_{[u]_2})$ is given by the flattening map $(a,u,x) \mapsto (au,x)$. This is an isomorphism since each interval in AX consisting of points $(a,u,\cdot)$ is of type $I_{[u]_2}$ and each interval $(au,\cdot)$ in $A^\omega(J_{[u]_2})$ is of type $J_{[au]_2} = I_{[aau]_2} = I_{[u]_2}$ as well. $\square$

Thus if $A^2 X \cong X$, we obtain AX by interchanging the role of each $I_{[u]_2}$ with $I_{[au]_2}$ in the decomposition $X \cong A^\omega(I_{[u]_2})$. It is worth noting that in the case when $[u]_2 = [au]_2 = [u]$, no interchange is needed: in this case the orders $I_{[u]_2}$, $I_{[au]_2}$, and $J_{[u]_2}$, $J_{[au]_2}$ are all identical, whereas in general only the equalities $J_{[u]_2} = I_{[au]_2}$ and $I_{[u]_2} = J_{[au]_2}$ hold.

The upshot of Proposition 3.2.1 is that, in the case where $A^2 X \cong X$, if we can find an order automorphism of A^ω that maps each $\sim_2$-class $[u]_2$ onto $[au]_2$, we can lift it to obtain an isomorphism of AX with X.

Definition 3.2.2. An order automorphism $f : A^\omega \to A^\omega$ is called a *parity-reversing automorphism* (abbreviated p.r.a.) if $f(u) \in [au]_2$ for every $u \in A^\omega$ and $a \in A$. Equivalently, an order automorphism f of A^ω is a p.r.a. if for every $u \in A^\omega$ and $a \in A$ the image of $[u]_2$ under f is $[au]_2$.

It follows that the image of $[au]_2$ under a parity-reversing automorphism f is $[u]_2$. Parity-reversing automorphisms are "idempotent on $\sim_2$-classes" in this sense.

Proposition 3.2.3. Suppose that A^ω admits a parity-reversing automorphism. Then for every order X, if $A^2X \cong X$ then $AX \cong X$.

Proof. Let $f : A^\omega \to A^\omega$ be a parity-reversing automorphism. Fix X and assume $A^2X \cong X$. Writing X as $A^\omega(I_{[u]_2})$ and AX as $A^\omega(J_{[u]_2})$, we may define an isomorphism from X to AX by the map $(u, x) \mapsto (f(u), x)$. This map is well-defined: the interval $(u, \cdot)$ in X is of type $I_{[u]_2}$, and the interval $(f(u), \cdot)$ in AX is of type $J_{[au]_2} = I_{[u]_2}$ since $f(u) \in [au]_2$. Hence $X \cong AX$. □

In the context of the cube problem, the case of interest is when $A = X$. If X is an order such that $X^3 \cong X$, then we may rewrite this as $X^2X \cong X$. By 3.2.3, if X^ω has a p.r.a., then $XX \cong X$ as well, i.e. $X^2 \cong X$. In Section 3.3, parity-reversing isomorphisms for A^ω are constructed for many different kinds of orders A, culminating in the proof that if $X^3 \cong X$ then indeed X^ω has a p.r.a., no matter the cardinality of X.

3.2.3 When $A^n \times X \cong X$ implies $A \times X \cong X$

We provide the analogous definitions and results (without proof) for analyzing orders that satisfy the isomorphism $A^nX \cong X$, for some fixed $n > 1$.

Recall from Chapter 2 that $A^\omega(I_{[u]_n})$ denotes a replacement $A^\omega(I_u)$ for which $u \sim_n v$ implies $I_u = I_v$. For an order X, by Theorem 2.2.2, we have $A^nX \cong X$ if and only if X is isomorphic to an order of the form $A^\omega(I_{[u]_n})$.

The next theorem describes the relationship between the various orders $A^k X$, $1 \leq k \leq n-1$, in the case where $A^n X \cong X$. The notation a^k denotes the sequence $aa \dots a \in A^{<\omega}$ in which a is repeated k times.

Proposition 3.2.4. Suppose X is an order such that $A^n X \cong X$, so that X is of the form $A^\omega(I_{[u]_n})$. For a fixed k, $1 \leq k \leq n-1$, let $Y = A^k X$. Then $Y \cong A^\omega(J_{[u]_n})$, where for all $u \in A^\omega$ and $a \in A$, we have $J_{[u]_n} = I_{[a^k u]_n}$.

In the above, the sequence a^k may be replaced with any k-sequence $a_0 a_1 \dots a_{k-1}$ of elements of A.

Definition 3.2.5. An order automorphism $f : A^\omega \to A^\omega$ is called an *n-revolving automorphism* (abbreviated n-r.a.) if for every $u \in A^\omega$ and $a \in A$ the image of $[u]_n$ under f is $[au]_n$.

Proposition 3.2.6. If $A^n X \cong X$ and there exists an n-revolving automorphism on A^ω, then $AX \cong X$ as well (and hence $A^k X \cong X$ for all k).

Hence if $X^{n+1} \cong X$ for some $n \geq 1$ and there is an n-revolving automorphism on X^ω, we have $X^2 \cong X$ as well.

3.2.4 A solution to the cube problem when X is countable, without endpoints

As a corollary to Theorem 2.2.1, we prove the characterization that was claimed in Section 3.1 for countable orders X without endpoints such that $X^n \cong X$ for some $n > 1$. A consequence is that the cube property holds for such orders. The argument differs substantially from the argument for the general case, and if desired this section can be skipped.

Corollary 3.2.7. Let X be a countable linear order without endpoints. If $X^n \cong X$ for some $n > 1$, then X is isomorphic to an order of the form $\mathbb{Q}(I_q)$, where for each q, there are densely many $p \in \mathbb{Q}$ with $I_q = I_p$. Hence for any countable order L we have $LX \cong X$, and in particular $X^2 \cong X$.

Proof. Let $A = X^{n-1}$. The hypothesis is that $AX \cong X$. By Theorem 2.2.1, X is isomorphic to an order of the form $A^\omega(I_{[u]})$. Note that since A is countable, each equivalence class $[u]$ is countable: each $v \in [u]$ is of the form ru' for some tail-sequence u' of u, and there are only countably many $r \in A^{<\omega}$, and countably many tail-sequences u'.

Since X is countable, it must be that for all but countably many of the equivalence classes $[u]$ we have $I_{[u]} = \emptyset$. Let k be the number of classes for which $I_{[u]}$ is nonempty, so that $1 \leq k \leq \omega$. Enumerate these classes as $C_i, i < k$, and let I_i denote $I_{[u]}$ if $[u] = C_i$. Let $C = \bigcup_{i<k} C_i$. Since C is the union of those classes $[u]$ for which $I_{[u]}$ is nonempty, we have that $X = C(I_u)$ where $I_u = I_i$ if $[u] = C_i$. We use the notation $X = C(I_i)$.

Split $\mathbb{Q}$ into k-many dense subsets $\mathbb{Q} = \bigcup_{i<k} \mathbb{Q}_i$ and form the order $\mathbb{Q}(I_q)$, where if $q \in \mathbb{Q}_i$, then $I_q = I_i$. Denote this order by $\mathbb{Q}(I_i)$. If we can show that $X = C(I_i)$ is isomorphic to $\mathbb{Q}(I_i)$, the proof will be complete. To do this, it is sufficient to show that there is an isomorphism of C with $\mathbb{Q}$ that sends each C_i onto $\mathbb{Q}_i$. By Skolem's theorem, it is enough to show that C is countable, dense and without endpoints, and each C_i is dense in C.

Certainly C is countable, since it is a countable union of countable classes. For the rest, note that since X is without endpoints, $A = X^{n-1}$ is also without endpoints. Fix $v < w$ in C and fix one of the C_i. Pick a representative $u \in C_i$, so that $C_i = [u]$. Since $v < w$ we may find an n so that $v \upharpoonright n < w \upharpoonright n$ lexicographically. Since A is without endpoints, we may find $a \in A$ such that $a > v_n$, and $b, c \in A$ such that $b < v_0$ and $c > w_0$. Let $x = (v \upharpoonright n)au$, $y = bu$, and $z = cu$. Then $y < v < x < w < z$ in A^ω, and clearly $x, y, z \in [u] = C_i$. But this proves the claims: C_i is dense in C, but further C is dense and without a top or bottom point. The proof is complete. □

Although we are going to show that the implication $X^n \cong X \implies X^2 \cong X$ holds for any linear order X, the advantage of the corollary is that it yields a complete classification in the countable, no endpoints case. Such an X is of the form $\mathbb{Q}(I_i)$. Hence to specify X it is enough to specify the number of parts k in the partition $\mathbb{Q} = \bigcup_{i<k} \mathbb{Q}_i$ and the countable orders I_i.

Under what other hypotheses can we carry out a similar proof to classify the orders satisfying the isomorphism $X^n \cong X$? We can always use Theorem 2.2.1 to decompose such an X into an order of the form $A^\omega(I_{[u]})$, where $A = X^{n-1}$. But the argument from there can break down in a number of ways.

When X is countable and has a single endpoint, the same proof goes through and yields an analogous classification. In the countable, two endpoints case, the proof does not carry over. The reason is that, while there is a unique (up to isomorphism) countable dense order with two endpoints, namely $Q = \mathbb{Q} \cap [0, 1]$, this order does not enjoy the same invariance under multiplication that $\mathbb{Q}$ enjoys. There exist countable orders L with both endpoints such that $LQ \not\cong Q$. In fact it is easy to see we have non-isomorphism whenever L is not dense. Moreover, this difference has consequences. Unlike for $\mathbb{Q}$, there are right products of Q that are not isomorphic to their squares. For example, $X = Q2$ is not isomorphic to its square. It follows that the analogous classification in this case fails, though it may be that some more detailed classification is possible.

In the uncountable case, the proof breaks down completely, and the results of Section 3.4 show that no similar classification is possible. In many models of set theory, there do not even exist analogues to the order $\mathbb{Q}$ on higher cardinals. Even when such orders do exist, nothing like "$X^n \cong X$ if and only if $X \cong \mathbb{Q}(I_i)$" is true for uncountable X.

For example, under the continuum hypothesis there exists the saturated order $\mathbb{Q}_1$ of size $\aleph_1$. This order has a characterization a la Cantor's characterization of $\mathbb{Q}$, and enjoys many analogous properties to those of $\mathbb{Q}$. For one, if L is any order of size $\aleph_1$ with uncountable cofinality and coinitiality, then $L\mathbb{Q}_1 \cong \mathbb{Q}_1$, and in particular $\mathbb{Q}_1^2 \cong \mathbb{Q}_1$. Furthermore, we have the result "any order X of the form $\mathbb{Q}_1(I_i)$ is invariant under left multiplication by any L of size $\aleph_1$ with uncountable cofinality and coinitiality. In particular, if the orders I_i are of size at most $\aleph_1$, then for any n we have $X^n \cong X$." (Here, each I_i replaces densely many points in $\mathbb{Q}_1$.) But the converse is false outright: if X is of size $\aleph_1$ and $X^n \cong X$ it need not be true that $X \cong \mathbb{Q}_1(I_i)$. For example, it is possible to use the methods of Section 3.4 to produce, without extra set theoretic hypotheses, an X of size $\aleph_1$ such that $X^2 \cong X$ but $(\omega_1^* + \omega_1)X \ncong X$.

3.3 Parity-reversing automorphisms on A^ω

In this section we show how to construct parity-reversing automorphisms of A^ω for orders A satisfying certain structural requirements. A consequence of our work is that if $X^3 \cong X$, then X^ω has a parity-reversing automorphism. By Proposition 3.2.3 it follows that $X^2 \cong X$ as well. At the end of the section, we sketch how to generalize the argument to get the implication $X^n \cong X \implies X^2 \cong X$ for any order X and $n \geq 2$, finishing the proof of the main theorem.

For a sequence $u = (u_0, u_1, u_2, \ldots) \in A^\omega$, let σu denote the once shifted tail-sequence $(u_1, u_2, \ldots)$ and $\sigma^n u$ the n-times shifted tail-sequence $(u_n, u_{n+1}, \ldots)$.

As a first step toward constructing a p.r.a. for A^ω, we show that any closed interval in A^ω with periodic endpoints of odd period has a parity-reversing automorphism. It is then possible to write down conditions under which a collection of these partial maps can be stitched together to get a full p.r.a. on A^ω.

A technical note: if $r, s \in A^{<\omega}$ are finite sequences, and if $\overline{r} < \overline{s}$ in A^ω, then it always holds that $\overline{r} < \overline{rs} < \overline{sr} < \overline{s}$, even when one of the sequences is an initial sequence of the other. This is Proposition 3.3.1 below. This fact will be used repeatedly throughout the rest of the chapter without explicit mention, in particular in Lemma 3.3.2. However, in every case in the subsequent work where we actually consider specific sequences $\overline{r} < \overline{s}$, it will be apparent that we actually have $\overline{r} < \overline{rs} < \overline{sr} < \overline{s}$. As such the following proof can be skipped, if desired.

Proposition 3.3.1. Suppose $r, s \in A^{<\omega}$ are finite sequences such that $\overline{r} < \overline{s}$ in A^ω. Then $\overline{r} < \overline{rs} < \overline{sr} < \overline{s}$.

Proof. The proof is essentially an instance of the Euclidean algorithm. For any pair of finite sequences $q, p \in A^{<\omega}$, either one sequence is an initial sequence of the other, or for some $n < \min(|p|, |q|)$, we have $q_n \neq p_n$. In the latter case, if n is the minimal place where we have inequality and $q_n < p_n$ we will write, for the purpose of this argument, $q < p$. If one sequence is an initial sequence of the other, we do not define an order between them. For k a non-negative integer, for the purposes of this argument only, we let kq denote the sequence $qq \dots q$ consisting of k-many copies of q. (In the rest of the thesis, we denote this sequence q^k.) We will sometimes employ parentheses for clarity: $(kq)p$ means the sequence beginning with k-copies of q followed by a copy of p.

Now, suppose $r, s \in A^{<\omega}$ are finite sequences such that $\overline{r} < \overline{s}$. We are trying to establish the three inequalities $\overline{r} < \overline{rs}, \overline{rs} < \overline{sr}$, and $\overline{sr} < \overline{s}$. Note that by cancelling the leading r in both sequences, the first inequality is equivalent to $\overline{r} < \overline{sr}$. Similarly, the third inequality is equivalent to $\overline{rs} < \overline{s}$. We will show the first and third inequality hold by establishing these rewritten forms.

We assume that $|r| \leq |s|$. The case when $|r| > |s|$ is similar. The proof proceeds algorithmically.

Stage 0: if r can be compared lexicographically to s, then it must be that $r < s$. It follows immediately that the three inequalities hold, and our algorithm terminates. If r cannot be compared lexicographically to s, we go to Stage 1.

Stage 1: in this case, it must be that one of r, s is an initial sequence of the other. Since we are assuming $|r| \leq |s|$, we have that r is an initial sequence of s. But it may be that s begins with more than one copy of r. Suppose that s begins with exactly k_1 copies of r, that is, $s = (k_1 r)q_1$, where $k_1 \geq 1$, and q_1 is the "remainder sequence" that does not begin with r. Note that q_1 is not the empty sequence, since otherwise s would be multiple of r and we would have $\overline{r} = \overline{s}$.

Then we have

$$\begin{aligned} \overline{s} &= s\overline{s} \\ &= (k_1 r)q_1\overline{s}. \end{aligned}$$

We also have

$$\overline{r} = (k_1 r)r\overline{r}.$$

If r can be compared lexicographically to q_1, it must be that $r < q_1$, since $\overline{r} < \overline{s}$. Assume this is the case. We have that

$$\begin{aligned} \overline{rs} &= rs\overline{rs} \\ &= r(k_1 r)q_1\overline{rs} \\ &= (k_1 r)rq_1\overline{rs}, \end{aligned}$$

and also that

$$\begin{aligned}\overline{sr} &= sr\overline{sr}\\ &= (k_1r)q_1r\overline{sr}.\end{aligned}$$

Comparing these expansions and using the fact that $r < q_1$, we see easily that the inequalities $\overline{r} < \overline{sr}, \overline{rs} < \overline{sr}$, and $\overline{rs} < \overline{s}$ all hold, and our algorithm terminates.

In the case where r and q_1 cannot be compared lexicographically, we proceed to Stage 2.

Stage 2: in this case we must have that $|q_1| < |r|$, since otherwise r would be comparable to q_1. Hence q_1 is an initial sequence of r, that is $r = (k_2q_1)q_2$ for some strictly positive integer k_2 and a remainder sequence q_2 that does not begin with q_1. Again, the remainder cannot be the empty sequence, since if it were r and s would both be multiples of q_1 and we would have $\overline{r} = \overline{s}$.

Continuing our expansion of $\overline{s}$ we have

$$\begin{aligned}\overline{s} &= (k_1r)q_1\overline{s}\\ &= (k_1r)q_1r\ldots\end{aligned}$$

where we have split off the r in the last line from the initial s in the tail sequence $\overline{s}$ from the previous line (that is, the ellipsis denotes the sequence $((k_1-1)r)q_1\overline{s})$,

$$\begin{aligned}&= (k_1r)q_1(k_2q_1)q_2\ldots\\ &= (k_1r)(k_2q_1)q_1q_2\ldots\end{aligned}$$

Further expanding $\overline{r}$, we have

$$\begin{aligned}\overline{r} &= (k_1r)(k_2q_1)q_2\overline{r}\\ &= (k_1r)(k_2q_1)q_2q_1\ldots\end{aligned}$$

where, in the last line, we have split off the initial copy of q_1 from the first r in the tail $\overline{r}$ from the previous line.

If q_2 is comparable to q_1, then from the above expansions it follows that $q_2 < q_1$, since $\overline{r} < \overline{s}$. Assume this is the case.

We also have

$$\begin{aligned}
\overline{rs} &= (k_1 r)(k_2 q_1) q_2 q_1 \overline{rs}, \\
\overline{sr} &= (k_1 r) q_1 (k_2 q_1) q_2 \overline{sr} \\
&= (k_1 r)(k_2 q_1) q_1 q_2 \overline{sr}.
\end{aligned}$$

From these expansions and the fact that $q_2 < q_1$ we see that $\overline{r} < \overline{sr}, \overline{rs} < \overline{sr}$, and $\overline{rs} < \overline{s}$ all hold, and our algorithm terminates.

If on the other hand q_2 is not comparable to q_1, then reasoning as before it must be that q_2 is a proper initial sequence of q_1. In this case $q_1 = (k_3 q_2) q_3$ for some $k_3 > 0$ and non-empty remainder q_3 not beginning with q_2, and the algorithm proceeds to Stage 3. And so on.

This algorithm must eventually terminate: if at Stage n we cannot lexicographically compare q_n and q_{n-1}, then q_n is strictly shorter in length than q_{n-1}. So at worst we continue until $|q_n| = 1$, in which case it must be comparable to q_{n+1}, since it can never be an initial sequence of q_{n+1}. At whatever stage q_n is first comparable to q_{n+1}, we are able to expand the expressions above for $\overline{r}, \overline{rs}, \overline{sr}$, and $\overline{s}$ and conclude the desired inequalities. □

Now, suppose that we have finite sequences $r, s \in A^{<\omega}$ of lengths m, n respectively, such that $\overline{r} < \overline{s}$ in A^ω. The shift map $u \mapsto \sigma^m u$ restricted to the interval $[\overline{r}, \overline{rs}]$ is an isomorphism of this interval with the interval $[\overline{r}, \overline{sr}]$. The inverse map is given by $u \mapsto ru$, restricted to $[\overline{r}, \overline{sr}]$. Similarly the map $u \mapsto su$, when restricted to $[\overline{rs}, \overline{s}]$, is an isomorphism of this interval with $[\overline{sr}, \overline{s}]$ whose inverse is given by $u \mapsto \sigma^n u$. Thus we have the following proposition.

Lemma 3.3.2. Suppose $r, s \in A^{<\omega}$ are finite sequences, respectively of lengths m, n, such that $\overline{r} < \overline{s}$ in A^ω. Then the map $f : [\overline{r}, \overline{s}] \to [\overline{r}, \overline{s}]$ defined by

$$f(u) = \begin{cases} \sigma^m u & u \in [\overline{r}, \overline{rs}] \\ su & u \in [\overline{rs}, \overline{s}] \end{cases}$$

is an order automorphism of the interval $[\overline{r}, \overline{s}]$. Its inverse is given by

$$f^{-1}(u) = \begin{cases} ru & u \in [\overline{r}, \overline{sr}] \\ \sigma^n u & u \in [\overline{sr}, \overline{s}]. \end{cases}$$

Note that there is no ambiguity in the definition of $f(\overline{rs})$ in the statement of the lemma, since $\sigma^m \overline{rs} = s\overline{rs} = \overline{sr}$.

Given an interval of the form $[\overline{r}, \overline{s}]$ in A^ω, we call the automorphism f defined in Lemma 3.3.2 a *standard map* on $[\overline{r}, \overline{s}]$. Standard maps are the essential tool in our proof of the cube property for $(LO, \times)$.

Standard maps are defined with respect to given sequences $r, s \in A^{<\omega}$, and an interval can carry more than one standard map. For example, if $r_0 = a$, $r_1 = aa$, and $s = b$ for some points $a < b$ in A, then we have $[\overline{r_0}, \overline{s}] = [\overline{r_1}, \overline{s}] = [\overline{a}, \overline{b}]$. However, the standard map on this interval defined with respect to the sequences r_0, s is different than the standard map defined with respect to the sequences r_1, s. We adopt the convention that the phrase "the standard map on $[\overline{r}, \overline{s}]$" means the one defined with respect to the sequences r, s.

Lemma 3.3.3. Fix $r, s \in A^{<\omega}$ of lengths m, n respectively. If m, n are both odd, then the standard map $f : [\overline{r}, \overline{s}] \to [\overline{r}, \overline{s}]$ is parity-reversing, in the sense that for all $u \in [\overline{r}, \overline{s}]$ and $a \in A$, we have $f(u) \in [au]_2$.

Proof. Either $f(u) = \sigma^m u$ or $f(u) = su$. In either case, the obvious meeting representation between au and $f(u)$ witnesses $f(u) \sim_2 au$. □

Theorem 3.3.4. Suppose A has both a left and right endpoint. Then A^ω has a parity-reversing automorphism.

Proof. Let 0 denote the left endpoint of A, and 1 denote the right endpoint. Then A^ω also has left and right endpoints, namely $\overline{0}$ and $\overline{1}$. That is, $A^\omega = [\overline{0}, \overline{1}]$. Thus the standard map on this interval is actually an automorphism of A^ω. It is parity-reversing by Lemma 3.3.3, since $\overline{0}$ and $\overline{1}$ are both of period 1. □

Thus if A has both a left and right endpoint, and X is an order such that $A^2X \cong X$, then $AX \cong X$ as well. Decomposing X as $A^\omega(I_{[u]_2})$ and AX as $A^\omega(J_{[u]_2})$, the isomorphism is given by $(u, x) \mapsto (f(u), x)$, where f is the standard map on $A^\omega = [\overline{0}, \overline{1}]$.

In this case, we may alternatively apply Lindenbaum's version of the Schroeder-Bernstein theorem to get an isomorphism between X and AX: X, which is isomorphic to A^2X, contains an initial copy of AX by virtue of the fact that A has a left endpoint, and AX contains a final copy of X since A has a right endpoint. Thus $AX \cong X$ by Lindenbaum's theorem. If $A = X$, we recover Corollary 3.1.1.

These approaches are actually the same. The isomorphism one gets from the proof of Lindenbaum's theorem (which is really just the classical proof of the Schroeder-Bernstein theorem), when viewed as an isomorphism of $A^\omega(I_{[u]_2})$ with $A^\omega(J_{[u]_2})$, turns out to be exactly the isomorphism $(u, x) \to (f(u), x)$, where f is the standard map on $[\overline{0}, \overline{1}] = A^\omega$. In this sense, standard maps are generalized instances of the classical Schroeder-Bernstein bijection in the context of orders of the form A^ω.

The standard map f fixes the endpoints $\overline{0}$ and $\overline{1}$ of A^ω—necessarily so, since f is an order automorphism. It is easily checked that these are the only fixed points of f. This does not change the fact that f is parity-reversing: both $\overline{0}$ and $\overline{1}$ are periodic sequences of period 1, so that $[a\overline{0}]_2 = [\overline{0}]_2 = [\overline{0}]$ and $[a\overline{1}]_2 = [\overline{1}]_2 = [\overline{1}]$ for any $a \in A$. Thus the parity-reversing requirement on f at these points is simply that $f(\overline{0}) \in [\overline{0}]$ and $f(\overline{1}) \in [\overline{1}]$, which certainly holds. Viewing f as defining an isomorphism between some $X \cong A^2X \cong A^\omega(I_{[u]_2})$ and $AX \cong A^\omega(J_{[u]_2})$, we have that this isomorphism maps $I_{\overline{0}}$ onto $J_{\overline{0}}$, which is legitimate as these intervals are identical, as noted in the discussion following 3.2.1. Similarly for $I_{\overline{1}}$ and $J_{\overline{1}}$.

If A does not have any endpoints, or only a single endpoint, then A^ω is not of the form $[\overline{r}, \overline{s}]$. Thus no standard map defines a p.r.a. on all of A^ω. However, because they fix the endpoints of the intervals on which they are defined, standard maps can be stitched together to obtain automorphisms of longer intervals. For example, suppose $r, s, t \in A^{<\omega}$ are sequences such that $\overline{r} < \overline{s} < \overline{t}$. Let f denote the standard map on $[\overline{r}, \overline{s}]$ and g the standard map on $[\overline{s}, \overline{t}]$. Then $f \cup g$ is an automorphism of $[\overline{r}, \overline{t}]$. Its fixed points are exactly $\overline{r}, \overline{s}$, and $\overline{t}$. If $|r|, |s|$, and $|t|$ are all odd, then f and g, and thus $f \cup g$, are parity-reversing.

The standard map h on $[\overline{r}, \overline{t}]$ also serves as an automorphism of this interval. This map is different than the two-piece map $f \cup g$ (e.g. h does not fix $\overline{s}$). The advantage of the piecewise construction is that it can be extended to get parity-reversing maps on intervals without endpoints, as well as orders without endpoints. To deal with such intervals, we introduce some terminology.

The *cofinality* of an interval I is the minimum length λ of an increasing sequence of points $\{x_i : i < \lambda\} \subseteq I$ such that for every $y \in I$ there is $i < \lambda$ with $y \leq x_i$. Similarly, the *coinitiality* of I is the minimum length κ of a decreasing sequence in I that eventually goes below every element of I. The cofinality is 1 if the interval has a maximal element, and if it has a minimal element the coinitiality is 1. When these cardinals are not 1, they are infinite and regular. An interval with coinitiality κ and cofinality λ is called a (κ, λ)-interval. We usually write these as ordinals, e.g. ω, ω_1, etc., to emphasize that they refer to sequences.

We refer to $(1,1)$-intervals as 2-intervals, $(1,\omega)$-intervals as ω-intervals, $(\omega,1)$-intervals as ω^*-intervals, and (ω,ω)-intervals as $\mathbb{Z}$-intervals, since these intervals are, respectively, spanned by sequences of order type 2, ω, ω^*, and $\mathbb{Z}$. Viewing the order A as itself an interval, we may speak of (κ,λ)-orders, or 2, ω, ω^*, and $\mathbb{Z}$-orders.

A *cover* for an order A is a collection of disjoint intervals $\mathcal{C} = \{C_a : a \in A\}$ so that $a \in C_a$ for all a. Indexing by the elements of A is for convenience: it is not assumed that $a \neq b$ implies $C_a \neq C_b$, since this would give only trivial covers, but only that either $C_a = C_b$ or $C_a \cap C_b = \emptyset$.

A cover $\mathcal{C}$ is called a $\mathbb{Z}$-cover if every $C \in \mathcal{C}$ is a $\mathbb{Z}$-interval. Not every order A admits a $\mathbb{Z}$-cover: if, for example, A has a left endpoint 0, then in any cover $\mathcal{C}$ for A, the interval C_0 cannot be a $\mathbb{Z}$-interval. Less trivially, suppose A is a complete (ω_1, ω_1)-order, and $\mathcal{C}$ is a cover for A. Suppose $\mathcal{C}$ contains a $\mathbb{Z}$-interval C. This interval is open. Since A is "long" to both the left and right, C is bounded. Since A is complete, C has a greatest lower bound x and least upper bound y. The intervals C_x and C_y must be disjoint from C, hence x must be the maximal element in C_x and y the minimal element in C_y. Thus neither is a $\mathbb{Z}$-interval, and it follows that A admits no $\mathbb{Z}$-cover.

A cover $\mathcal{C}$ is called a $\{\mathbb{Z},\omega\}$-cover if every $C \in \mathcal{C}$ is either a $\mathbb{Z}$-interval or an ω-interval. Similarly, $\mathcal{C}$ is called a $\{\mathbb{Z},\omega^*\}$-cover if every $C \in \mathcal{C}$ is either a $\mathbb{Z}$-interval or ω^*-interval. Given a cover $\mathcal{C}$, let $\mathcal{C}_X \subseteq \mathcal{C}$ denote the collection of intervals in $\mathcal{C}$ of type X. For example, if $\mathcal{C}$ is a $\{\mathbb{Z},\omega\}$-cover, then $\mathcal{C} = \mathcal{C}_{\mathbb{Z}} \cup \mathcal{C}_\omega$.

Theorem 3.3.5.

1. If A admits a $\mathbb{Z}$-cover, then A^ω has a parity-reversing automorphism.

2. If A has a left endpoint and admits a $\{\mathbb{Z},\omega\}$-cover, then A^ω has a parity-reversing automorphism.

3. If A has a right endpoint and admits a $\{\mathbb{Z}, \omega^*\}$-cover, then A^ω has a parity-reversing automorphism.

Thus in any of these three cases, if X is an order such that $A^2X \cong X$, then $AX \cong X$ as well.

Proof. (1.) Assume first that A has a $\mathbb{Z}$-cover $\mathcal{C}$. It follows that A has no endpoints. For every $C \in \mathcal{C}$, fix a $\mathbb{Z}$-sequence $\ldots < x_{-1}^C < x_0^C < x_1^C < x_2^C < \ldots$ spanning C. To each of the points x_k^C in A there is the corresponding periodic sequence $\overline{x}_k^C$ in A^ω. For $C, D \in \mathcal{C}$ and $k, l \in \mathbb{Z}$, the intervals $[x_k^C, x_{k+1}^C)$ and $[x_l^D, x_{l+1}^D)$ intersect if and only if they are identical. The same is true of the corresponding intervals $[\overline{x}_k^C, \overline{x}_{k+1}^C)$ and $[\overline{x}_l^D, \overline{x}_{l+1}^D)$ in A^ω.

These intervals actually cover A^ω, that is, for every $u \in A^\omega$ there is a unique $C \in \mathcal{C}$ and $k \in \mathbb{Z}$ such that $u \in [\overline{x}_k^C, \overline{x}_{k+1}^C)$. To see this, note that since the intervals C cover A, u's first entry u_0 (viewed as a point in A) falls in one of the C. There are two possibilities: either $x_k^C < u_0 < x_{k+1}^C$ for some $k \in \mathbb{Z}$, or $u_0 = x_k^C$ for some $k \in \mathbb{Z}$. In the first case, we have that $\overline{x}_k^C < u < \overline{x}_{k+1}^C$ so that $u \in [\overline{x}_k^C, \overline{x}_{k+1}^C)$. In the second, either $\overline{x}_{k-1}^C < u < \overline{x}_k^C$ or $\overline{x}_k^C \leq u < \overline{x}_{k+1}^C$ depending on whether the first entry of u differing from x_k^C (if it exists) is greater or less than x_k^C. Thus either $u \in [\overline{x}_{k-1}^C, \overline{x}_k^C)$ or $u \in [\overline{x}_k^C, \overline{x}_{k+1}^C)$.

We have shown that

$$A^\omega = \bigcup_{\substack{k \in \mathbb{Z} \\ C \in \mathcal{C}}} [\overline{x}_k^C, \overline{x}_{k+1}^C].$$

The intervals in this union are pairwise disjoint unless they are consecutive, in which case they share a single endpoint.

For every $k \in \mathbb{Z}$ and $C \in \mathcal{C}$, let f_k^C denote the standard map on $[\overline{x}_k^C, \overline{x}_{k+1}^C]$. Since the endpoints of this interval are of period 1, we have that f_k^C is parity-reversing.

The function

$$f = \bigcup_{\substack{k\in\mathbb{Z}\\ C\in\mathcal{C}}} f_k^C$$

is well-defined, since the domains of two different f_k^C share at most one point and at that point the functions agree. Since each f_k^C is a parity-reversing automorphism of $[\overline{x}_k^C, \overline{x}_{k+1}^C]$ and A^ω is the union of these intervals, f is a p.r.a. for A^ω.

(2.) Now assume that A has a left endpoint 0 and a $\{\mathbb{Z}, \omega\}$-cover $\mathcal{C}$. Then A has no right endpoint. The argument is similar to the previous one, but with a catch. For every $C \in \mathcal{C}_\mathbb{Z}$, fix a sequence $\ldots < x_{-1}^C < x_0^C < x_1^C < x_2^C < \ldots$ spanning C. Similarly, for every $C \in \mathcal{C}_\omega$ fix a sequence $x_0^C < x_1^C < x_2^C < \ldots$ spanning C, where x_0^C is the left endpoint of C.

As before, the intervals $[\overline{x}_k^C, \overline{x}_{k+1}^C)$ are pairwise disjoint. The difference is that now there may be points in A^ω that do not fall in any of these intervals.

To see this, suppose $u \in A^\omega$. Then there is a unique $C \in \mathcal{C}$ and k such that $x_k^C \leq u_0 < x_{k+1}^C$. If in fact $x_k^C < u_0 < x_{k+1}^C$, then certainly $u \in [\overline{x}_k^C, \overline{x}_{k+1}^C)$. So suppose $u_0 = x_k^C$. If either $C \in \mathcal{C}_\mathbb{Z}$, or $C \in \mathcal{C}_\omega$ and $k > 0$, then u is contained in either $[\overline{x}_{k-1}^C, \overline{x}_k^C)$ or $[\overline{x}_k^C, \overline{x}_{k+1}^C)$, depending on whether the first entry of u differing from x_k^C (if it exists) is greater than or less than x_k^C.

So suppose $u_0 = x_0^C$ for some $C \in \mathcal{C}_\omega$. Then either $u \geq \overline{x}_0^C$ or $u < \overline{x}_0^C$. In the first case we have $u \in [\overline{x}_0^C, \overline{x}_1^C)$.

The issue occurs in the second case when $u < \overline{x}_0^C$. Assume we are in this case, and for notational simplicity denote x_0^C by x. It must be then, that x (viewed as a point in A) is greater than the left endpoint 0, and further that $u_n < x$, where u_n is the leftmost entry of u differing from x. However, since $u_0 = x$ it must be that $u \geq x000\ldots = x\overline{0}$. Thus $u \in [x\overline{0}, \overline{x})$.

This interval is disjoint from all of the $[\overline{x}_k^C, \overline{x}_{k+1}^C)$, and by the same argument, any $v \in A^\omega$ that is not contained in one of the $[\overline{x}_k^C, \overline{x}_{k+1}^C)$ must be contained in an interval of this form.

Thus

$$A^\omega = \bigcup_{\substack{k\in\mathbb{Z}\\ C\in\mathcal{C}_\mathbb{Z}}} [\overline{x}_k^C, \overline{x}_{k+1}^C] \;\cup \bigcup_{\substack{k\in\omega\\ C\in\mathcal{C}_\omega}} [\overline{x}_k^C, \overline{x}_{k+1}^C] \;\cup \bigcup_{\substack{x=x_0^C\\ C\in\mathcal{C}_\omega}} [x\overline{0}, \overline{x}].$$

These intervals are pairwise disjoint up to endpoints. The intervals $[\overline{x}_k^C, \overline{x}_{k+1}^C]$ have parity-reversing standard maps f_k^C. If we can show that for every interval $[x\overline{0}, \overline{x}]$, there is a parity-reversing automorphism $f_x : [x\overline{0}, \overline{x}] \to [x\overline{0}, \overline{x}]$, then the map

$$f = \bigcup_{\substack{k\in\mathbb{Z}\\ C\in\mathcal{C}_\mathbb{Z}}} f_k^C \;\cup \bigcup_{\substack{k\in\omega\\ C\in\mathcal{C}_\omega}} f_k^C \;\cup \bigcup_{\substack{x=x_0^C\\ C\in\mathcal{C}_\omega}} f_x$$

is a parity-reversing automorphism of A^ω.

So fix $C \in \mathcal{C}_\omega$ and let $x = x_0^C$. If C is the leftmost interval in $\mathcal{C}$, then $x = 0$ and $[x\overline{0}, \overline{x}] = \{\overline{0}\}$ is just the left endpoint of A^ω. In this case let f_x be the map that fixes $\overline{0}$ and is undefined elsewhere.

Otherwise $x > 0$. The interval $[x\overline{0}, \overline{x}]$ has a periodic right endpoint, but only an eventually periodic left endpoint. We have not defined a standard map for such an interval. Note however that the shift map $u \mapsto \sigma u$ restricted to $[x\overline{0}, \overline{x}]$ is an isomorphism of this interval with $[\overline{0}, \overline{x}]$. Let g denote this shift map, and let f denote the standard map on $[\overline{0}, \overline{x}]$. Then $f_x = g^{-1} \circ f \circ g$ is a parity-reversing automorphism of $[x\overline{0}, \overline{x}]$, as desired.

Now f, as defined above, is a p.r.a. for A^ω.

(3.) Finally, suppose A has a right endpoint and admits a $\{\mathbb{Z}, \omega^*\}$-cover. Then A^* has a left endpoint and admits a $\{\mathbb{Z}, \omega\}$-cover. Thus $(A^*)^\omega$, which is isomorphic to $(A^\omega)^*$, has a p.r.a. f^*. The corresponding automorphism f on A^ω is still parity-reversing since the requirement $f(u) \in [au]_2$ does not depend on the ordering of A^ω, but only on the underlying set of points. □

Suppose for the moment that A is an order without endpoints. The content of the theorem is that if A has a $\mathbb{Z}$-cover, then A^ω can be covered disjointly (up to endpoints) by intervals of the form $[\overline{x}, \overline{y}]$, with $x, y \in A$. Since we have parity-reversing standard maps on these intervals, we can use this decomposition to build a p.r.a. on all of A^ω.

The essence of what can go wrong when A does not admit a $\mathbb{Z}$-cover can be illustrated by an example. Suppose that A is a complete (ω_1, ω_1)-order. We have seen that A has no $\mathbb{Z}$-cover. We might still attempt to build a p.r.a. for A^ω piecewise: begin with some $x_0 < x_1$ in A and consider the corresponding periodic points $\overline{x}_0 < \overline{x}_1$ in A^ω. Put the standard map on $[\overline{x}_0, \overline{x}_1]$. Then pick some $\overline{x}_2 > \overline{x}_1$, and so on. After ω-many steps we will have defined a p.r.a. on the interval spanned by $\overline{x}_0 < \overline{x}_1 < \overline{x}_2 < \dots$.

Now, since A is complete and of cofinality ω_1, the sequence $x_0 < x_1 < x_2 < \dots$ is bounded in A and converges to some point x. However, the sequence of $\overline{x}_n$ does *not* converge to $\overline{x}$ in A^ω. Rather, there is a nonempty open interval I sitting above all the $\overline{x}_n$ and below $\overline{x}$. This interval I consists of points u with first coordinate $u_0 = x$ but with some later coordinate $u_i < x$. Notice that there are no periodic points of period 1 in this interval. We might hope to bridge this gap (perhaps up to some collection of legally fixable points) with intervals whose endpoints are only eventually periodic, as we did in the proof of (2.) of Theorem 3.3.5. There, however, it was crucial for the argument that A had a left endpoint 0.

It turns out it is impossible to bridge this gap, in the sense that any automorphism on A^ω extending the partial automorphism described above cannot be parity-reversing on I.

Moreover, no alternative construction works. We will show in Section 3.5 that any complete (ω_1, ω_1)-order does not admit a parity-reversing automorphism.

Here is an immediate corollary of Theorem 3.3.5.

Corollary 3.3.6. If A is a 2-order, ω-order, ω^*-order, or $\mathbb{Z}$-order, then A^ω has a parity-reversing automorphism. Thus for any order X, if $A^2X \cong X$ then $AX \cong X$.

Proof. When A is a 2-order, this is simply a restatement of Theorem 3.3.4. In the other three cases A has, respectively, a $\{\mathbb{Z}, \omega\}$-cover, $\{\mathbb{Z}, \omega^*\}$-cover, or $\mathbb{Z}$-cover given by $\mathcal{C} = \{A\}$. □

Among the orders satisfying the hypotheses of the corollary are all countable orders, since any countable order has a cofinality and coinitiality at most ω. Thus for any countable A and any X we have $A^2X \cong X \implies AX \cong X$. This gives in particular that any countable X isomorphic to its cube is isomorphic to its square, recovering Corollaries 3.1.1 and 3.2.7 and also giving the result for countable orders with a single endpoint.

The order $\mathbb{R}$ of the real numbers gives an example of an uncountable $\mathbb{Z}$-order. Hence $\mathbb{R}^2X \cong X \implies \mathbb{R}X \cong X$ for any X. One of the simplest examples of an order that is not a $\mathbb{Z}$-order but admits a $\mathbb{Z}$-cover is $\omega_1\mathbb{Z}$. Here, the cover consists of the intervals $C_\alpha = \{(\alpha, z) : z \in \mathbb{Z}\}$. Similarly, $\omega_1\mathbb{R}$ admits a $\mathbb{Z}$-cover, since each copy of $\mathbb{R}$ is spanned by a $\mathbb{Z}$-sequence.

The order ω_1 has a left endpoint, but admits no $\{\mathbb{Z}, \omega\}$-cover, as can be shown by a similar argument to the one showing any complete (ω_1, ω_1)-order has no $\mathbb{Z}$-cover. Thus it does not follow from 3.3.5 that there is a p.r.a. on ω_1^ω. It turns out that ω_1^ω does have a p.r.a., though we will not prove this. However, if $A = \omega_1^* + \omega_1$ is the order obtained by putting a copy of ω_1 to the right of a copy of ω_1^*, then A is a complete (ω_1, ω_1)-order. It follows from the results of Section 3.5 that there is no p.r.a. on A^ω. We will show in fact that there exists an X such that $A^2X \cong X$ but $AX \ncong X$.

It remains to show that any order X that is isomorphic to its cube admits the right kind of cover to get a p.r.a. on X^ω. This is Theorem 3.3.7 below. When combined with Theorem 3.3.5, it shows that for any X without endpoints or with a single endpoint, if $X^3 \cong X$ then $X^2 \cong X$. Since we have already dealt with the two endpoint case, we have $X^3 \cong X \implies X^2 \cong X$ for any X, finishing the proof of the cube property.

Theorem 3.3.7. Suppose X is an order such that $X^3 \cong X$.

1. If X has no endpoints, then X admits a $\mathbb{Z}$-cover.
2. If X has a left endpoint but no right endpoint, then X admits a $\{\mathbb{Z}, \omega\}$-cover.
3. If X has a right endpoint but no left endpoint, then X admits a $\{\mathbb{Z}, \omega^*\}$-cover.

The proof of the theorem is in two lemmas. Before we can state these lemmas, we need some more terminology.

Up to this point we have only studied invariance of orders under *left* multiplication (by a fixed A or power of A). Orders X such that $X^n \cong X$ fit into this context since the isomorphism $X^n \cong X$ can be rewritten as $A^{n-1}X \cong X$, where $A = X$. However, such orders also display an invariance under right multiplication, since $X^n \cong X$ can just as well be rewritten $XA^{n-1} \cong X$, where again $A = X$. It is this right-sided invariance under multiplication that is needed to get the covers of X in Theorem 3.3.7.

Forgetting the isomorphism $X^3 \cong X$ for now, we consider the single power version of this right-sided invariance. That is, for an order A, we analyze the structure of orders X such that $XA \cong X$.

In the case of invariance under left multiplication by A, the space A^ω of right-infinite sequences plays a crucial role. In the case of right-sided invariance, it is the space of left-infinite sequences that is relevant.

Let $A^{\omega^*} = \{(\ldots, a_2, a_1, a_0) : a_i \in A, i \in \omega\}$ denote the set of left-infinite sequences of elements of A. Notice that we still index the entries of such sequences by elements in ω. We copy over the notation from the right-sided case, reversing it when necessary. The letters $u, v, \ldots$ will now be used to denote elements of A^{ω^*}. The nth entry of u is still denoted u_n.

Let $A^{<\omega^*} = \{(a_{n-1}, \ldots, a_1, a_0) : a_i \in A, n \in \omega\}$ denote the set of left-growing finite sequences. The letters $r, s, \ldots$ will for now denote elements of $A^{<\omega^*}$. For $r = (a_{n-1}, \ldots, a_1, a_0) \in A^{<\omega^*}$ and $u = (\ldots, u_1, u_0) \in A^{\omega^*}$, we use ur to denote the sequence $(\ldots, u_1, u_0, a_{n-1}, \ldots, a_1, a_0)$.

It is impossible to order A^{ω^*} lexicographically, in the sense that two left-infinite sequences may not have a leftmost place in which they differ. If, however, two such sequences eventually agree, it is possible to compare them lexicographically.

Definition 3.3.8. For $u, v \in A^{\omega^*}$, we say u is *eventually equal* to v, and write $u \sim_\infty v$, if there exists $N \in \omega$ such that for all $n > N$ we have $u_n = v_n$.

Equivalently, $u \sim_\infty v$ if there exist finite sequences r, s with $|r| = |s|$ and a sequence $u' \in A^{\omega^*}$ such that $u = u'r$ and $v = u's$. This is just the usual eventual equality relation, considered here for left-infinite sequences. The $\sim_\infty$-class of u is denoted $[u]_\infty$.

Observe that for a given $u \in A^{\omega^*}$, the class $[u]_\infty$ can be ordered lexicographically. If $u \sim_\infty v$, define $u < v$ if and only if $u_N < v_N$, where N is the leftmost place such that $u_N \neq v_N$. It is immediate that $u \not< u$, and one of $u < v$, $u = v$, $u > v$ always holds. Transitivity can be checked as well.

The classes $[u]_\infty$ are the largest subsets of A^{ω^*} that can possibly be ordered lexicographically, in the sense that if $u \not\sim_\infty v$, then u and v have no leftmost place of difference. In what follows, whenever we refer to an order on $[u]_\infty$ we mean the lexicographical order. If we write $u < v$ it is assumed that $u \sim_\infty v$.

Here are our lemmas.

Lemma 3.3.9. Fix $u \in A^{\omega^*}$.

1. If A has no endpoints, the class $[u]_\infty$ is a $\mathbb{Z}$-order.

2. If A has a left endpoint but no right endpoint, the class $[u]_\infty$ is either an ω-order or $\mathbb{Z}$-order.

3. If A has a right endpoint but no left endpoint, the class $[u]_\infty$ is either an ω^*-order or $\mathbb{Z}$-order.

Proof. (1.) Assume first A has no endpoints. We define a sequence $\ldots < v^{-1} < v^0 < v^1 < \ldots$ spanning $[u]_\infty$. Let $v^0 = u$. For n a fixed positive integer, define $v^n = (\ldots, v_1^n, v_0^n)$ to be any sequence such that $v_m^n = u_m$ for all $m > n$ but $v_n^n > u_n$. It is always possible to find such a v_n^n, since A does not have a top point. On the other side, again for n a fixed positive integer, let v^{-n} be a sequence such that $v_m^{-n} = u_m$ for all $m > n$, but now $v_n^{-n} < u_n$. This is possible since A does not have a bottom point.

It is clear that $\ldots < v^{-1} < v^0 < v^1 < \ldots$ since the ordering is lexicographical. Further if $v \in [u]_\infty$, then there is some N such that for all $m \geq N$ we have $v_m = u_m$. Thus $v^{-N} < v < v^N$, and so the sequence v^n spans $[u]_\infty$, as desired.

(2.) Now assume A has a left endpoint 0 but no right endpoint. The class $[\overline{0}]_\infty$ has a left endpoint, namely $\overline{0}$ itself. Let $v^0 = \overline{0}$ and v^n be any sequence that is 0 beyond the nth place, but is greater than 0 in the nth place. Then we have $v^0 < v^1 < \ldots$ as before, and in fact this sequence spans $[\overline{0}]_\infty$. Thus $[\overline{0}]_\infty$ is an ω-order.

So assume $u \not\sim_\infty \overline{0}$. We show $[u]_\infty$ is a $\mathbb{Z}$-order. As before, since A has no top point, we may find $v^1 < v^2 < \ldots$ cofinal in $[u]_\infty$.

On the other side, observe that since $u \not\sim_\infty \overline{0}$, there are infinitely many places n such that $u_n > 0$. Enumerate these places as n_k, $k \in \omega$. Let v^{n_k} be any sequence that agrees with u beyond the n_kth place, but such that $v^{n_k}_{n_k} < u_{n_k}$. This is possible since $u_{n_k} > 0$. Then $\ldots < v^{n_1} < v^{n_0}$, and further this sequence is coinitial in $[u]_\infty$. Hence $[u]_\infty$ is a $\mathbb{Z}$-order, as claimed.

The case (3.) is symmetric to (2.). □

Lemma 3.3.10. Suppose that X is an order such that $XA \cong X$.

1. If A has neither a left nor right endpoint, then X admits a $\mathbb{Z}$-cover.
2. If A has a left endpoint but no right endpoint, then X admits a $\{\mathbb{Z}, \omega\}$-cover.
3. If A has a right endpoint but no left endpoint, then X admits a $\{\mathbb{Z}, \omega^*\}$-cover.

Proof. The intuition of the proof is simple. If $XA \cong X$, then X can be organized into X-many intervals of type A. Since there are X-many of them, these intervals in turn may be organized into XA-many copies of A, that is, X-many copies of A^2. And so on. Now consider the situation from the point of view of some fixed $x \in X$. At the first stage x is included in a copy of A, and in the second, in some copy of A^2 containing the initial copy of A, etc. These larger and larger intervals surrounding x, consecutively isomorphic to A, A^2, A^3, ..., have a limit, which is isomorphic to $[u]_\infty$ for some $u \in A^{\omega^*}$. The conclusion follows.

To make this explicit, let $f : X \to XA$ be an isomorphism. Let f_l and f_r denote the left and right components of f, that is, the unique functions such that $f(x) = (f_l(x), f_r(x))$ for all $x \in X$. For every $x \in X$, we define a sequence of points $x_0, x_1, \ldots$ in X and a sequence $a^x_0, a^x_1, \ldots$ in A. Let $x_0 = x$ and recursively define $x_{n+1} = f_l(x_n)$. Let $a^x_n = f_r(x_n)$.

With this notation, we have $f(x) = (x_1, a_0^x)$. By repeatedly factoring the x_i, we get an isomorphism from X onto XA^n defined by $x \mapsto (x_n, a_{n-1}^x, \dots, a_1^x, a_0^x)$. Although it is not literally an n-fold composition of f, we denote this isomorphism by f^n.

For $n \in \omega$, define I_n^x to be the set of $y \in X$ such that $y_n = x_n$. Then the image of I_n^x under f^n is the set of points in XA^n of the form $(x_n, b_{n-1}, \dots, b_0)$, $b_i \in A$. Hence I_n^x is an interval in X, and it is isomorphic to A^n. Furthermore, we have the containments $\{x\} = I_0^x \subseteq I_1^x \subseteq I_2^x \subseteq \dots$ since if $y_N = x_N$ then for every $n \geq N$ we have $y_n = x_n$ as well.

Let $I_\infty^x = \bigcup_n I_n^x$. Then since the I_n^x form a chain of intervals in X, I_∞^x is also an interval in X. Notice that by definition, for every $N \in \omega$ and $x, y \in X$, either $I_N^x \cap I_N^y = \emptyset$ or $I_N^x = I_N^y$, and in this latter case $I_n^x = I_n^y$ for all $n \geq N$. Hence I_∞^x and I_∞^y are either equal or disjoint as well.

For $x \in X$, define u_x to be the sequence $(\dots, a_1^x, a_0^x)$. Then if $y \in I_\infty^x$, it must be that for all sufficiently large n we have $y_n = x_n$. Thus for all sufficiently large n we have $a_n^x = a_n^y$, which gives $u_x \sim_\infty u_y$. On the other hand, suppose $v \in [u_x]_\infty$, say $v = (\dots, a_{n+1}^x, a_n^x, b_{n-1}, \dots, b_1, b_0)$ for some $b_i \in A$, $i < n$. There is a unique $y \in X$ such that $f^n(y) = (y_n, a_{n-1}^y, \dots, a_0^y) = (x_n, b_{n-1}, \dots, b_0)$. But then $u_y = v$.

This shows that the map $F : I_\infty^x \to [u_x]_\infty$ defined by $F(y) = u_y$ is a bijection. This map is clearly order-preserving as well, and hence an isomorphism of I_∞^x with $[u_x]_\infty$. Thus $\{I_\infty^x : x \in X\}$ is a cover of X by intervals of the form $[u]_\infty$. The conclusion of the lemma now follows from Lemma 3.3.9. □

Proof of Theorem 3.3.7. Suppose X is an order such that $X^3 \cong X$. Then $XA \cong X$, where $A = X^2$. If X has no endpoints, then A has no endpoints. By Lemma 3.3.10, X admits a $\mathbb{Z}$-cover, and therefore, by Lemma 3.3.9, X^ω admits a parity-reversing automorphism. If X has a left endpoint, but not a right one, then similarly A has a left endpoint, but no right one. Hence X admits a $\{\mathbb{Z}, \omega\}$-cover, and again X^ω admits a parity-reversing automorphism. The right endpoint case is symmetric. □

We conclude this section with a sketch of the proof that $X^n \cong X$ implies $X^2 \cong X$ for all orders X and all $n \geq 2$. When $n = 2$ the statement is trivial, and we have just finished the proof for $n = 3$. The argument for larger n is a straightforward adaptation of the case when $n = 3$.

For convenience, we consider orders satisfying the (only notationally distinct) isomorphism $X^{n+1} \cong X$, and we assume $n > 2$. This isomorphism can be rewritten as $A^n X \cong X$, where $A = X$. We know by the results at the end of Section 3.2 that if A^ω has an n-revolving automorphism (n-r.a.), then $AX \cong X$. So we turn to the question of when A^ω admits such an automorphism.

We build n-r.a.'s as we built p.r.a.'s, that is, as unions of standard maps. In our previous argument, to ensure that a map f was parity-reversing, it was enough to have that for every u either $f(u) = \sigma^n u$ or $f(u) = ru$, with n and $|r|$ both odd. This is because deleting or adding an initial sequence of odd length always sends u into $[au]_2$. For $n > 2$, the situation is not symmetric.

Lemma 3.3.11. Suppose $f : A^\omega \to A^\omega$ is an order automorphism, and for every $u \in A^\omega$, either $f(u) = \sigma^k u$ for some $k \equiv n-1 \pmod{n}$ or there exists a finite sequence r with $|r| \equiv 1 \pmod{n}$ and $f(u) = ru$. Then f is an n-r.a.

Proof. In either case, the obvious meeting representation witnesses $f(u) \sim_n au$. □

Thus if we have two sequences $r, s \in A^{<\omega}$ such that $|r| \equiv n-1 \pmod{n}$ and $|s| \equiv 1 \pmod{n}$, the standard map on $[\overline{r}, \overline{s}]$ is n-revolving. As before, when A has both a left and right endpoint, a single standard map serves to get an n-r.a. of A^ω.

Theorem 3.3.12. Suppose A has both a left and right endpoint. Then A^ω has an n-revolving automorphism.

Proof. Let 0 and 1 denote the left and right endpoints of A. Then $A^\omega = [\overline{0^{n-1}}, \overline{1}]$. By Lemma 3.3.11, the standard map on this interval is n-revolving. □

In particular, if X has both endpoints and $X^{n+1} \cong X$, then $X^2 \cong X$. For the cases with one or neither endpoint, we need the generalizations of Theorems 3.3.5 and 3.3.7.

Theorem 3.3.13.

1. If A admits a $\mathbb{Z}$-cover, then A^ω has an n-r.a.
2. If A has a left endpoint and admits a $\{\mathbb{Z}, \omega\}$-cover, then A^ω has an n-r.a.
3. If A has a right endpoint and admits a $\{\mathbb{Z}, \omega^*\}$-cover, then A^ω has an n-r.a.

Thus in any of these three cases, if X is an order such that $A^n X \cong X$, then $AX \cong X$ as well.

Theorem 3.3.14. Suppose X is an order such that $X^{n+1} \cong X$.

1. If X has neither a left nor right endpoint, then X admits a $\mathbb{Z}$-cover.
2. If X has a left endpoint but no right endpoint, then X admits a $\{\mathbb{Z}, \omega\}$-cover.
3. If X has a right endpoint but no left endpoint, then X admits a $\{\mathbb{Z}, \omega^*\}$-cover.

The conjunction of these theorems along with 3.3.12 gives that $X^{n+1} \cong X \implies X^2 \cong X$ for all X. Theorem 3.3.14 follows immediately from Lemmas 3.3.9 and 3.3.10. The proof of 3.3.13 (1.) is essentially the same as 3.3.5 (1.): if A has a $\mathbb{Z}$-cover, then A^ω can be covered (disjointly up to endpoints) by intervals of the form $[\overline{a}, \overline{b}]$ for $a, b \in A$. Writing such intervals as $[\overline{a^{n-1}}, \overline{b}]$, we have that the associated standard maps are n-revolving. The union of these maps then gives an n-r.a. on A^ω.

The proof of 3.3.13 (2.) is also very similar to that of 3.3.5 (2.). Given a $\{\mathbb{Z}, \omega\}$-cover of A we obtain a cover of A^ω by intervals of the form $[\overline{a}, \overline{b}]$ and $[x\overline{0}, \overline{x}]$, where 0 denotes the left endpoint of A. Intervals of the first type have n-revolving standard maps, as above. Intervals of the second type shift onto intervals of the first type, and therefore also have n-revolving automorphisms. The union of these maps yields an n-r.a. for A^ω.

The proof of (3.) is symmetric. The main theorem follows:

Theorem 3.3.15. Suppose X is an order such that $X^n \cong X$ for some $n > 1$. Then $X^2 \cong X$. In particular, the cube property holds for $(LO, \times)$.

3.4 Constructing orders X such that $X^n \cong X$

The purpose of this section is to justify the previous work, in two ways. First, we will show that the cube problem for linear orders, and more generally the problem of showing $X^n \cong X \implies X^2 \cong X$, is not vacuous, in the sense that for every n there exist many orders X such that $X^n \cong X$. We construct such orders below as direct limits, and show in particular that they can be of any infinite cardinality.

Secondly, we wish to justify the need for the machinery developed in Section 3.3 for solving the cube problem. When we say "$X^3 \cong X$," what is meant implicitly is that there exists an isomorphism $f : X^3 \to X$. All of our analysis of the relation $X^3 \cong X$ has really been with respect to a fixed isomorphism f. Associated to such an f is an order of the form $X^\omega(I_{[u]_2})$ and an isomorphism $F : X \to X^\omega(I_{[u]_2})$ built using f. If we view f, via the relabeling F, as an isomorphism of $X^2 \times X^\omega(I_{[u]_2})$ with $X^\omega(I_{[u]_2})$, then f is just the flattening isomorphism fl_2 on the first three coordinates. This follows from Theorem 2.2.1 and its corollary 2.2.3.

Conversely, if we can ever construct an isomorphism F from X onto an order of the form $X^\omega(I_{[u]_2})$, we immediately have that $X^3 \cong X$ as witnessed by the flattening isomorphism.

Suppose it were the case that for every X for which there exists an isomorphism F of X onto an order of the form $X^\omega(I_{[u]_2})$, we had that $I_{[u]_2} \cong I_{[au]_2}$ for every $a \in X$ and $u \in X^\omega$. Let us call such a decomposition *trivial*. Then letting $I_{[u]}$ denote the common order type of $I_{[u]_2}$ and $I_{[au]_2}$, we would have that $X \cong X^\omega(I_{[u]})$ and hence $X^2 \cong X$. We would have this isomorphism of X with X^2 without any need for a parity-reversing automorphism of X^ω, and the work in Section 3.3 showing that X^ω has such an automorphism would be unnecessary to solve the cube problem. However, we shall show that this is not the case. There are pairs (X, F) where X is a linear order and F is an isomorphism of X onto an order of the form $X^\omega(I_{[u]_2})$ such that for many a and u we have $I_{[au]_2} \ncong I_{[u]_2}$. This follows from Theorem 3.4.2 below. For such an X we have $X^3 \cong X$ naturally, but in order to show $X^2 \cong X$ we need the device of a p.r.a. for X^ω.

Although Theorem 3.4.1 can be derived from 3.4.2, it has a proof which is easier to understand and thereby serves as a warmup for the proof of 3.4.2.

Given orders X and Y, an *embedding* of X into Y is an injective order-preserving map $f : X \to Y$. Given embeddings $f : X_0 \to Y_0$ and $g : X_1 \to Y_1$ the map defined by $(x, y) \mapsto (f(x), g(y))$ is an embedding of $X_0 \times X_1$ into $Y_0 \times Y_1$. We denote this map by (f, g).

Given a sequence of orders $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ the order $X = \bigcup_{i\in\omega} X_i$ is well-defined. Similarly, given a directed system

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots$$

where each f_i is an embedding of X_i into X_{i+1}, we may form the direct limit

$$X = \varinjlim_{i} X_i.$$

If the f_i are inclusion maps, the direct limit of the X_i is isomorphic to their union. We will sometimes confuse the direct limit construction with the union construction, and speak of each X_i as a suborder of X_j for $j \geq i$, and as a suborder of $X = \varinjlim X_i$.

Given two systems

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots$$

$$Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \dots$$

we obtain a system

$$X_0 \times Y_0 \xrightarrow{(f_0,g_0)} X_1 \times Y_1 \xrightarrow{(f_1,g_1)} \dots$$

It is a standard fact that if $Z = \varinjlim(X_i \times Y_i)$, then $Z \cong X \times Y$, where $X = \varinjlim X_i$ and $\varinjlim Y_i$. Written shortly, we have

$$\varinjlim(X_i \times Y_i) \cong (\varinjlim X_i) \times (\varinjlim Y_i).$$

Given orders X and Y, we say that X *spans* Y if there is an embedding f of X into Y such that for every $y \in Y$, there exist y_0, y_1 in the image $f[X]$ such that $y_0 \leq y \leq y_1$. The following theorem says that any order can be expanded to an order of the same cardinality that is isomorphic to its square.

Theorem 3.4.1. Let X_0 be any order. Then there exists an order X such that

1. $X^2 \cong X$,

2. X_0 spans X,

3. $|X| = |X_0| + \aleph_0$.

Proof. For every $i \in \omega$, let $X_{i+1} = X_i \times X_i$. Then $X_n = X_0^{2^n}$. Let $f_0 : X_0 \to X_1$ be the embedding defined by $f_0(x) = (x, x)$. For every $i > 0$, let $f_i = (f_{i-1}, f_{i-1})$. Then f_i is an embedding of X_i into X_{i+1}.

Let $X = \varinjlim X_i$. Then we have

$$\begin{aligned}
X &= \varinjlim X_i && (i \geq 0) \\
&\cong \varinjlim X_i && (i > 0) \\
&= \varinjlim (X_{i-1} \times X_{i-1}) && (i > 0) \\
&= \varinjlim (X_i \times X_i) && (i \geq 0) \\
&\cong \varinjlim (X_i) \times \varinjlim (X_i) && (i \geq 0) \\
&= X \times X.
\end{aligned}$$

This proves (1.). The cardinality claim (3.) is clear. To verify (2.), let us view the f_i as inclusions, so that each X_i is included in X_j for $j \geq i$, and in X. Then X_0 is included in $X_1 = X_0 \times X_0$ as the set of points of the form (a, a), and more generally in $X_n = X_0^{2^n}$ as the set of points of the form $(a, a, \ldots, a)$.

Fix $x \in X$. Then $x \in X_n$ for some n. Clearly, for some a, b we have $(a, a, \ldots, a) \leq x \leq (b, b, \ldots, b)$. That is, x lies between two points of X_0, provided we view X_0 as a subset of X_n. Hence we also have that x lies between two points of X_0, now viewing X_0 as a subset of X. □

Note in particular that X has the same endpoint configuration as X_0: if X_0 has both endpoints, then so does X, and likewise for the other cases. Hence there are orders of any cardinality and any endpoint configuration isomorphic to their squares.

We can get "$X^3 \cong X$" instead of "$X^2 \cong X$" in the conclusion of the theorem by letting $X_{i+1} = X_i^3$ and letting f_0 be the embedding $x \mapsto (x, x, x)$. Similarly, we can get orders X such that $X^n \cong X$.

However, with this particular construction the resulting isomorphisms turn out to be trivial. That is, if one uses the proof of 3.4.1 to produce an order X isomorphic to X^3, and then analyzes the isomorphism $f : X^3 \to X$ yielded by the proof and the associated decomposition $X^\omega(I_{[u]_2})$, one finds that $I_{[u]_2} = 1$ if and only if u is eventually constant. Otherwise $I_{[u]_2} = \emptyset$. Since for eventually constant u we have $[u]_2 = [au]_2 = [u]$, such a decomposition already witnesses $X^2 \cong X$ without the need for a p.r.a. of X^ω.

We shall now show how to directly build an order X along with an isomorphism F of X onto an order of the form $X^\omega(I_{[u]})$. More generally we can construct X and F with $F : X \to X^\omega(I_{[u]_n})$. It will follow from the construction that these decompositions can be arranged to be non-trivial.

We have observed that if the embedding from $X_i \times Y_i$ into $X_{i+1} \times Y_{i+1}$ is of the form (f_i, g_i), we have that the limit of the products $X_i \times Y_i$ is isomorphic to the product of the limits. An analogous fact holds for limits of replacements, as well as for limits of infinite products. Let us spell these out.

Suppose we are given a system

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots$$

Let $X = \varinjlim X_i$. We again view each X_i as included in all subsequent X_j as well as in X. Suppose further that for every $x \in X$ we are given a system

$$I_{0,x} \xrightarrow{g_{0,x}} I_{1,x} \xrightarrow{g_{1,x}} \ldots$$

For each $x \in X$, let $I_x = \varinjlim I_{i,x}$.

Naturally we obtain a system

$$X_0(I_{0,x}) \xrightarrow{F_0} X_1(I_{1,x}) \xrightarrow{F_1} \ldots$$

The embedding F_i is defined by $F_i(x, y) = (f_i(x), g_{i,x}(y))$, and in the replacement $X_i(I_{i,x})$, it is understood that the index x of each $I_{i,x}$ only ranges over X_i. Then we have that $\varinjlim X_i(I_{i,x}) = \varinjlim(X_i)(\varinjlim(I_{i,x})) = X(I_x)$.

Similarly, if we are given

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \ldots$$

then we obtain a system

$$X_0^\omega \xrightarrow{F_0} X_1^\omega \xrightarrow{F_1} \ldots$$

where the embedding F_i is defined by $F_i((x_0, x_1, \ldots)) = (f_i(x_0), f_i(x_1), \ldots)$. We use the notation $F_i = (f_i, f_i, \ldots)$. Viewing each X_i as included in X_{i+1} by way of f_i, we may view X_i^ω as included in X_{i+1}^ω by way of F_i. Letting $Y = \varinjlim X_i^\omega$, observe that it is not the case (naturally, at least) that Y is isomorphic to X^ω, where $X = \varinjlim X_i$. Rather, Y is isomorphic to the subset of X^ω consisting of sequences of "bounded rank," that is, sequences $(x_0, x_1, \ldots)$

such that for some i, for all n we have $x_n \in X_i$. If the F_i are true inclusions then $Y = \bigcup_i X_i^\omega$. Note that Y is a union of tail-equivalence classes.

Theorem 3.4.2. Let $\{L_j : j \in J\}$ by any collection of nonempty, pairwise non-isomorphic orders, indexed by some indexing set J. Then there exists an X such that $X \cong X^\omega(I_{[u]})$ for some collection of orders $I_{[u]}$ (and hence $X^2 \cong X$), and further such that

1. For $u, v \in X^\omega$, if $[u] \neq [v]$ then either $I_{[u]} = I_{[v]} = \emptyset$ or $I_{[u]}$ and $I_{[v]}$ are non-isomorphic.
2. For every $j \in J$, there exists a unique tail-equivalence class $[u]$ such that $I_{[u]} = L_j$.

Proof. Let X_0 be any order such that the number of tail-equivalence classes in X_0^ω is at least $|J|$.

Fix an injection $\iota : J \to \{[u] : u \in X_0^\omega\}$. If $\iota(j) = [u]$, define $I_{0,[u]} = L_j$. For those $[u]$ not assigned an index $j \in J$, pick orders $I_{0,[u]}$ so that the final collection $\{I_{0,[u]} : u \in X_0^\omega\}$ consists of pairwise non-isomorphic orders.

Denote the order $X_0^\omega(I_{0,[u]})$ by X_1. Fix an embedding $f_0 : X_0 \to X_1$. For example, we may define f_0 by $f_0(x) = (x, x, x, \ldots, a_x)$, where a_x is any element in $I_{0,[\overline{x}]}$.

By way of this f_0, view X_0 as included in X_1. Then as in the discussion preceding the theorem, we may view X_0^ω as included in X_1^ω by way of $F_0 = (f_0, f_0, \ldots)$. For each $u \in X_1^\omega$, if $[u] \cap X_0^\omega \neq \emptyset$, define $I_{1,[u]} = I_{0,[u]}$. The remaining u are exactly those sequences with infinitely many terms from $X_1 \setminus X_0$. For these $[u]$, iteratively choose orders $I_{1,[u]}$ so that the final collection $\{I_{1,[u]} : u \in X_1^\omega\}$ consists of pairwise non-isomorphic orders.

Let $X_2 = X_1^\omega(I_{1,[u]})$. The "inclusion" $F_0 : X_0^\omega \to X_1^\omega$ naturally determines an "inclusion" $f_1 : X_1 \to X_2$. Namely, if $(u, a) \in X_1 = X_0^\omega(I_{0,[u]})$, with say $u = (u_0, u_1, \ldots)$, we define $f_1((u, a)) = (F_0(u), a) = (f_0(u_0), f_0(u_1), \ldots, a)$. It is legal to let f_1 be the identity on the last coordinate, since the interval $(u, \cdot)$ in X_1 is of type $I_{0,[u]}$, and the interval $(F_0(u), \cdot)$ in X_2 is of type $I_{1,[u]} = I_{0,[u]}$.

Now repeat this process. View X_1 as included (by way of f_1) in X_2, and X_1^ω as included in X_2^ω. For each $u \in X_2^\omega$, if $u \in X_1^\omega$, let $I_{2,[u]} = I_{1,[u]}$. For the remaining u, fix orders $I_{2,[u]}$ so that the collection $\{I_{2,[u]} : u \in X_2^\omega\}$ consists of pairwise non-isomorphic orders.

Continuing in this way, we get a system

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

Let X be the limit. Then we have

$$\begin{aligned} X &= \varinjlim X_i \\ &\cong \varinjlim X_{i+1} \\ &= \varinjlim X_i^\omega(I_{i,[u]}) \\ &= X^\omega(I_{[u]}) \end{aligned}$$

where if $u \in X_i^\omega$ for some i, then $I_{[u]} = I_{i,[u]}$, and if u is in none of the X_i^ω (i.e. if u has terms of unboundedly high rank), then $I_{[u]} = \emptyset$.

This X satisfies the conclusion of the theorem. □

In particular, any fixed order L can appear as an interval in the X constructed in the proof, by simply including L among the L_j.

By an analogous construction, for any $n \geq 1$ we can get an X such that $X \cong X^\omega(I_{[u]_n})$ (and hence $X^{n+1} \cong X$), where the nonempty $I_{[u]_n}$ are pairwise non-isomorphic and fill up the classes of every X_i^ω for some increasing sequence $X_0 \subseteq X_1 \subseteq \dots$ converging to X. Although not all of the n-tail-equivalence classes $[u]_n$ are filled in the final decomposition $X^\omega(I_{[u]_n})$, those that are filled are filled with pairwise non-isomorphic orders. In particular, there will be many examples of where $I_{[u]_n} \ncong I_{[au]_n}$, and therefore these decompositions will be nontrivial. By doing a longer induction, it is possible to get $X \cong X^\omega(I_{[u]_n})$ where every $I_{[u]_n}$ is nonempty. Of course, it is also possible to arrange during the construction that some (or all) of the $I_{[u]_n}$ are isomorphic.

3.5 Constructing A and X such that $X \cong A^2 \times X \ncong A \times X$

Our main theorem gives that the cube property holds for the class of linear orders under the lexicographical product. In view of the proof, it is natural to ask if we could have established the stronger result, that for all orders A and X, if $A^2X \cong X$ then $AX \cong X$. By 3.2.3, if it were possible to construct a p.r.a. for *any* order of the form A^ω then the answer would be yes. This raises the subquestion of whether constructing a p.r.a. for A^ω is always possible.

We show in this section that the answer to both questions is no. In fact, if A is a complete (κ, λ)-order for cardinals κ and λ of uncountable cofinality, then A^ω does not admit a p.r.a. Furthermore, the converse to Theorem 3.2.3 holds: if A^ω does not have a p.r.a., then there exists an X such that $A^2X \cong X$ but $AX \ncong X$. Thus in particular, there is such an X when $A = \omega_1^* + \omega_1$. In the language of the introduction to this thesis, this says that the (left-sided) weak Schroeder-Bernstein property fails for the class of linear orders. (The right-sided weak Schroeder-Bernstein property also fails, that is, there exist X, A with $XA^2 \cong X$ but $XA \ncong X$. This is easier to prove, but we will not do so here.)

Given a linear order X, a *cut* in X is a pair of intervals (I, J) such that $I \cup J = X$, $I \cap J = \emptyset$, and $I < J$. Thus I is an initial segment of X, and J is a final segment. A cut (I, J) is called a *gap* if I has no maximal element, and J has no minimal element.

The *Dedekind completion* of X, denoted $\overline{X}$, is the order obtained from X by filling every gap (I, J) with a single point $x_{(I,J)}$. Viewing X as a suborder of $\overline{X}$, we have that X is dense in $\overline{X}$, and in particular the cofinality (respectively, coinitiality) of $\overline{X}$ coincides with that of X. The Dedekind completion $\overline{X}$ is always a complete linear order, and if X is complete to begin with, then $X = \overline{X}$.

Any automorphism $f : X \to X$ can be extended uniquely to an automorphism $\overline{f} : \overline{X} \to \overline{X}$. For, if (I, J) is a gap in X, then $(f[I], f[J])$ must also be a gap. The automorphism $\overline{f}$ is defined by setting $\overline{f}(x_{(I,J)}) = x_{(f[I],f[J])}$ for every gap (I, J), and $\overline{f}(x) = f(x)$ for every $x \in X$.

A subset $C \subseteq X$ is called *closed* if every monotone sequence in C (of any ordinal length) is either unbounded in X or converges to some point in C. For C to be closed it is necessary that C is complete in the order inherited from X. However, completeness is not sufficient for closure. For example, $\{\frac{1}{n} : n \geq 1\} \cup \{0\}$ is closed as a subset of $\mathbb{R}$, whereas $\{\frac{1}{n} : n \geq 1\} \cup \{-1\}$ is not. Note that if X is not complete, then X is not closed as a subset of itself.

A subset $C \subseteq X$ is called *left-unbounded* if for every $x \in X$ there exists $c_0 \in C$ such that $c_0 \leq x$, and *right-unbounded* if for every x one can find $c_1 \in C$ with $c_1 \geq x$. If C is unbounded in both directions, we simply say C is *unbounded.* If C is closed and left-unbounded, then C is called a *left club*, and C is a *right club* if it is closed and right-unbounded. If C is both a left and right club, then we say simply that C is a *club.* Similarly, if $I \subseteq X$ is an interval, then viewing I as an order in itself we may speak of a right club in I, left club in I, and club in I.

It is straightforward to check that if X has uncountable cofinality, then the intersection of two right clubs is a right club, and if X has uncountable coinitiality, then the intersection of two left clubs is a left club. Hence if X has both uncountable cofinality and coinitiality, the intersection of two clubs is a club, the intersection of a club with a right club is a right club, and the intersection of a club with a left club is a left club.

An order X may not contain any club suborders, but its completion $\overline{X}$ always contains at least one club, namely $\overline{X}$ itself.

Suppose that X is a (κ, λ)-order, and both κ and λ have uncountable cofinality (so that X has uncountable cofinality and coinitiality). Let f be an automorphism of X, and $\overline{f}$ its extension to $\overline{X}$. Let us check that C, the set of fixed points of $\overline{f}$, is a club in $\overline{X}$. It is clear that C is closed. To see that it is unbounded, fix $x \in \overline{X}$. If $\overline{f}(x) = x$, then $x \in C$ and there is nothing to check. So suppose $x \notin C$. Then either $\overline{f}(x) > x$ or $\overline{f}(x) < x$. Assume without loss of generality that we are in the former case. Then the positive iterates of x under $\overline{f}$ form an increasing sequence, that is, we have

$$x < \overline{f}(x) < \overline{f}^2(x) < \ldots$$

Since $\overline{X}$ has uncountable cofinality, this sequence is bounded, and hence converges (by completeness) to some point b. It is easy to see that b must be a fixed point of $\overline{f}$. Symmetrically, since the coinitiality of $\overline{X}$ is also uncountable, the negative iterates of x converge to some a, and this a must be fixed by $\overline{f}$. We have found $a, b \in C$ with $a < x < b$, and so C is unbounded as claimed.

Until further notice, let A denote the order $\omega_1^* + \omega_1$. If $\alpha \in \omega_1$ is an ordinal, we denote the corresponding element in ω_1^* by $-\alpha$. We identify the 0 of ω_1 with the 0 of ω_1^*. Thus

$$A = \ldots < -\alpha < \ldots < -1 < 0 < 1 < \ldots < \alpha < \ldots$$

Theorem 3.5.1. Let $A = \omega_1^* + \omega_1$. Then A^ω does not admit a parity-reversing automorphism.

Proof. For every finite sequence $r \in A^{<\omega}$, let I_r denote the interval in A^ω consisting of sequences beginning with r. We confuse sequences of length 1 with elements of A, so that if

$a \in A$ then I_a means the interval of points with first entry a. Note that I_r is isomorphic to A^ω for every r, and in particular has neither a left nor right endpoint.

While A is complete, A^ω is not, since for every r the interval I_r has a gap to its immediate left and right. To see this, let J be the interval consisting of points in I_r and above, and let I be the complement of J. Then J has no minimum, since I_r does not. On the other side, I has no maximum, since any $u \in I$ must begin with a finite sequence s, of the same length as r, but with some entry $s_i < r_i$. Writing $u = su'$, pick a sequence $u'' > u'$, which is always possible since A^ω has no top point. Let $v = su''$. Then $v > u$ but v is still in I since it lies below I_r. Thus the cut (I, J) is in fact a gap, and it lies to the immediate left of I_r. Symmetrically (since A^ω has no bottom point), I_r has a gap to the right.

For every $r \in A^{<\omega}$, let r^- and r^+ denote the elements of $\overline{A^\omega}$ that fill the gaps to the left and right of I_r respectively. (These points are not pairwise distinct: if $a, b \in A$ and $b = a + 1$ then for any $r \in A^{<\omega}$ we have $ra^+ = rb^-$.)

Let $f : A^\omega \to A^\omega$ be an automorphism of A^ω, and $\overline{f}$ its extension to $\overline{A^\omega}$. Let $C \subseteq \overline{A^\omega}$ denote the club of fixed points of $\overline{f}$.

We shall show that f has a fixed point of the form $u = (\alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \ldots)$ for some collection of ordinals $\alpha_i \in \omega_1$ with $\alpha_i \neq 0$ for all i. Since such a u cannot be periodic of odd period, we have that $[u]_2 \neq [au]_2$, and thus f is not parity-reversing.

Consider the ω_1-length increasing sequence of intervals

$$I_0 < I_1 < \ldots < I_\alpha < \ldots$$

If α is a limit ordinal, then there are no points in A^ω that lie below I_α but above each I_β for $\beta < \alpha$. This means that the ω_1-sequence of left endpoints

$$0^- < 1^- < \ldots < \alpha^- < \ldots$$

is closed in $\overline{A^\omega}$. Since this sequence is right unbounded, it forms a right club in $\overline{A^\omega}$. Denote this right club by D_0.

Then $D_0 \cap C$ is a right club in $\overline{A^\omega}$. Fix $\alpha_0^- \in D_0 \cap C$ with $\alpha_0 \neq 0$. Let J_1 denote the interval of points in $\overline{A^\omega}$ lying strictly above α_0^-. Then since α_0^- is a fixed point of $\overline{f}$, we have that $\overline{f}$ restricted to J_1 is an automorphism of J_1. The coinitiality of J_1 is ω_1 since I_{α_0} is an initial segment of J_1, and its cofinality is also ω_1, since it is a final segment of A^ω. Hence J_1 is a complete (ω_1, ω_1)-order. It follows that the set of fixed points of $\overline{f}$ in this interval, $J_1 \cap C$, is a club in J_1.

Now consider the descending sequence of intervals

$$I_{(\alpha_0,0)} > I_{(\alpha_0,-1)} > \ldots > I_{(\alpha_0,-\alpha)} > \ldots$$

and the corresponding closed sequence of *right* endpoints

$$(\alpha_0, 0)^+ > (\alpha_0, -1)^+ > \ldots > (\alpha_0, -\alpha)^+ > \ldots$$

Denote this sequence by D_1. Then D_1 is left unbounded in J_1, and hence a left club in J_1. Thus $D_1 \cap C$ is a left club in J_1, and we may fix $(\alpha_0, -\alpha_1)^+ \in D_1 \cap C$. We may choose this point so that $\alpha_1 \neq 0$.

At the next stage we define J_2 to be the subinterval of J_1 consisting of points that lie strictly below $(\alpha_0, -\alpha_1)^+$. Then $\overline{f}$ restricted to J_2 is an automorphism of J_2. This interval ends with $I_{(\alpha_0,-\alpha_1)}$ and hence has cofinality ω_1. Since it also has coinitiality ω_1, we have again that the set of fixed points in this interval, $J_2 \cap C$, is a club in J_2. Hence there must be a fixed point $(\alpha_0, -\alpha_1, \alpha_2)^-$ (with $\alpha_2 \neq 0$) in the right club sequence

$$(\alpha_0, -\alpha_1, 0)^- < (\alpha_0, -\alpha_1, 1)^- < \ldots < (\alpha_0, -\alpha_1, \alpha)^- < \ldots$$

Continuing in this way, we obtain a sequence of fixed points $(\alpha_0)^-$, $(\alpha_0, -\alpha_1)^+$, $(\alpha_0, -\alpha_1, \alpha_2)^-$, $(\alpha_0, -\alpha_1, \alpha_2, -\alpha_3)^+, \ldots$, where at each stage we ensure $\alpha_i \neq 0$. While these points all lie outside of A^ω, they converge (in the obvious sense) to the point $u = (\alpha_0, -\alpha_1, \alpha_2, \ldots) \in A^\omega$. Since the set of fixed points of $\overline{f}$ is closed, it must then be that u is fixed by $\overline{f}$. Since $\overline{f}$ agrees with f on A^ω, we have in fact that u is a fixed point of f. As observed already, it follows that f is not parity-reversing. □

It is easy to generalize the proof to get that if A is any complete (κ, λ)-order, where κ and λ have uncountable cofinality, then A^ω does not have a parity-reversing automorphism.

We will now prove the converse to 3.2.3. Recall that a linear order is called *scattered* if it does not contain an infinite, dense suborder. In particular, every ordinal α, considered as a linear order, is scattered.

Theorem 3.5.2. Suppose that A^ω does not have a parity-reversing automorphism. Then there exists an order X such that $A^2X \cong X$ but $AX \not\cong X$.

Proof. Since A^ω does not have a p.r.a., by Theorem 3.3.4 it must be that A either has no endpoints, or only a single endpoint. In either case A^ω is dense. Hence any interval in A^ω, considered as a linear order itself, is dense. Suppose that $A^\omega(I_u)$ is any replacement of A^ω such that for densely many u we have $I_u \neq \emptyset$. Then since A^ω is dense, for any interval $I \subseteq A^\omega(I_u)$, we have that either $I \subseteq I_u$ for some u, or I contains a dense suborder. (The "or" here is non-exclusive: both conditions may hold, but not neither.) In the latter case, I is by definition non-scattered.

Let κ denote the number of $\sim_2$-equivalence classes in A^ω. Enumerate these classes as $\{C_\alpha : \alpha \in \kappa\}$. Let X be the replacement of A^ω obtained by replacing every point in the αth class with a copy of α, that is, $X = A^\omega(I_{[u]_2})$ where $I_{[u]_2} = \alpha$ if $[u]_2 = C_\alpha$. (It is inessential, though convenient, that we replace the points with ordinals. Any κ-sized collection of pairwise non-isomorphic scattered orders also works to define the $I_{[u]_2}$.)

By construction we have that $A^2X \cong X$. We know from 3.2.1 that $AX \cong A^\omega(J_{[u]_2})$ where $J_{[u]_2} = I_{[au]_2}$ for all u and a. Suppose toward a contradiction that there is an isomorphism f of X with AX, that is, of $A^\omega(I_{[u]_2})$ with $A^\omega(J_{[u]_2})$.

For a fixed u, the order type of the interval I_u is an ordinal α, and in particular is scattered. Its image $f[I_u]$ must be an interval of type α in $A^\omega(J_{[u]_2})$, and so from our observation above, it must be that $f[I_u] \subseteq J_v$ for some v. By the same argument, $f^{-1}[J_v]$ must be contained in I_w for some w. But then since $f^{-1}[J_v]$ contains I_u, it must be that in fact $v = w$, and $f[I_u] = J_v$. But then $J_v = \alpha$, and hence it must be that $v \in [au]_2$. Moreover, f must be the identity on I_u, since the identity is the only isomorphism of α with itself.

Thus the isomorphism f may be factored as (g, id), where $g : A^\omega \to A^\omega$ is a parity-reversing automorphism. But no such g exists, by hypothesis. Hence there is no isomorphism between X and AX. □

3.6 Related problems

In *Cardinal and Ordinal Numbers*, Sierpiński poses several other questions concerning the multiplication of linear orders aside from the cube problem. On page 232 he writes, "We do not know so far any example of two types φ and ψ, such that $\varphi^2 = \psi^2$ but $\varphi^3 \neq \psi^3$, or types γ and δ such that $\gamma^2 \neq \delta^2$ but $\gamma^3 = \delta^3$." Later, on page 251, "We do not know whether there exist two different denumerable order types which are left-hand divisors of each other.

Neither do we know whether there exist two different order types which are both left-hand and right-hand divisors of each other."

Since Sierpiński ordered products anti-lexicographically, "left-hand divisor" for him means "right-hand divisor" in the convention of this thesis. Writing them out using our convention, his questions are,

1. Do there exist orders X and Y such that $X^2 \cong Y^2$ but $X^3 \ncong Y^3$?
2. Do there exist orders X and Y such that $X^2 \ncong Y^2$, but $X^3 \cong Y^3$?
3. Do there exist *countable* orders X and Y such that $X \ncong Y$ but for some orders A, B we have $AY \cong X$ and $BX \cong Y$?
4. Do there exist orders X and Y such that $X \ncong Y$ but for some orders A_0, A_1, B_0, B_1 we have $A_0Y \cong YA_1 \cong X$ and $B_0X \cong XB_1 \cong Y$?

In comparison with the questions from the thesis's introduction, these questions are phrased negatively, asking for counterexamples to the corresponding properties. Questions 3 and 4 were mentioned already in the thesis's introduction.

Sierpiński was aware of counterexamples to the unique square root property for linear orders, that is, of non-isomorphic orders X and Y such that $X^2 \cong Y^2$. These examples are due to Morel; see [12]. Question 2 is the natural generalization of the unique square root problem, and could be called the unique cube root problem. More generally, one may ask if there exist orders X and Y such that $X^n \cong Y^n$ but $X^k \ncong Y^k$ for $k < n$.

Question 1 is motivated by the fact that Morel's examples X, Y of non-isomorphic orders with isomorphic squares have the property that $X^n \cong X$ for all $n \geq 1$ and $Y^n \cong X$ for all $n > 1$. In particular, not only is it the case that $X^2 \cong Y^2$ but actually that $X^n \cong Y^n$ for all

$n > 1$. Question 1 asks whether this kind of collapsing is necessary, or if it is possible that two orders have isomorphic squares but non-isomorphic cubes.

Both Questions 1 and 2 are related to a generalization of the cube problem. Suppose it were possible to find an order X such that $X^5 \cong X$ but the powers X, X^2, X^3, and X^4 were pairwise non-isomorphic. Then X and $Y = X^3$ would give a positive answer to Question 2. Similarly, if it were possible to find an X isomorphic to X^7 but whose intermediate powers were pairwise non-isomorphic, then X and $Y = X^3$ would give a positive answer to Question 1. By our main theorem, there are no such X, but it may still be that these two questions have positive answers. These questions are, to the author's knowledge, still open.

Question 4 might be called the *two-sided Schroeder-Bernstein problem* for the class $(LO, \times)$. It is a sensible problem to ask given that the lexicographical product is non-commutative and that there exist examples witnessing the failure of left-sided Schroeder-Bernstein property and right-sided Schroeder-Bernstein property for linear orders. It is closely related to the cube problem, in the sense that if there existed an X isomorphic to its cube but not its square, then X and $Y = X^2$ would give a positive answer. There is no such X, but it turns out that Question 4 still has a positive answer. The proof of this is given in the next chapter. This gives further evidence that the cube property for $(LO, \times)$ is "close" to being false.

Question 3 is the left-sided Schroeder-Bernstein problem for *countable* linear orders. Sierpiński was aware of *uncountable* orders X, Y, A, B satisfying the relations in the problem. In Section 3.5 we constructed such orders with $A = B$. From the discussion following 3.3.6, we know that in the case when $A = B$, any orders X and Y satisfying the relations of Question 3 *must* be uncountable. We will now strengthen this result, and show that Question 3 has a negative answer.

Suppose that X, Y, A, B are countable orders, and $AY \cong X$ and $BX \cong Y$. We prove $X \cong Y$. Notice first that the hypotheses give $ABX \cong X$. Let $C = AB$. There are three cases. If C

has both endpoints, then both A and B must also have both endpoints. But then X, which is isomorphic to AY, contains an initial copy of Y, and Y, which is isomorphic to BX, contains a final copy of X. Hence $X \cong Y$ by Lindenbaum's theorem. If C has neither endpoint, then by the proof of 3.2.7, since $CX \cong X$ we have that X must be of the form $\mathbb{Q}(I_i)$. Thus X is invariant under left multiplication by any countable order. In particular, $BX \cong X$, that is, $Y \cong X$. Finally, suppose C has a single endpoint. Without loss of generality, assume it is the left endpoint. Then both A and B have a left endpoint (and at least one of them is missing the right endpoint). By the adaption of the proof of 3.2.7 to the left endpoint case, X must be of the form $Q(I_i)$, where $Q = \mathbb{Q} \cap [0, 1)$, and where the order replacing the left endpoint 0 also has a left endpoint. It can be shown that such an order is invariant under left multiplication by any countable order with a left endpoint. In particular, $BX \cong X$, that is, $Y \cong X$.

All of Sierpiński's questions are instances of a much more general question. If we distinguish the structures in a given class $\mathfrak{C}$ only up to isomorphism type, then $(\mathfrak{C}, \times)$ may be viewed as a (possibly very large) semigroup, where the semigroup operation is given by the product. For a given semigroup $(S, \cdot)$, we say that S can be *represented* in $\mathfrak{C}$ if there is a map $\iota : S \to \mathfrak{C}$ such that $\iota(a \cdot b) \cong \iota(a) \times \iota(b)$, and if $a \neq b$ then $\iota(a) \ncong \iota(b)$.

The failure of the cube property for a given class $\mathfrak{C}$ is equivalent to the statement that the group $\mathbb{Z}_2$ can be represented in $\mathfrak{C}$. Thus our main theorem gives that $\mathbb{Z}_2$, and more generally $\mathbb{Z}_n$, cannot be represented in $(LO, \times)$.

5. Which semigroups can be represented in $(LO, \times)$?

6. Can any nontrivial group be represented in $(LO, \times)$?

Question 6 is of course a subquestion of Question 5. By our results, if Question 6 has a positive answer, then any non-identity element in the witnessing group must be of infinite order.

Questions 1, 2, and 4 all concern relations that can be realized in certain semigroups. Thus a complete answer to Question 5 would yield answers to all of these questions.

Chapter 4

The Two-Sided Schroeder-Bernstein Problem for Linear Orders

4.1 Introduction

In this chapter we will solve the *two-sided Schroeder-Bernstein problem* for the class $(LO, \times)$ of linear orders under the lexicographical product. The problem asks whether there exist two non-isomorphic orders X and Y such that X is both a left-hand and right-hand divisor of Y, and Y is both a left-hand and right-hand divisor of X. We will show that the answer is positive by constructing such orders directly. See Section 3.6 of the previous chapter as well as the introduction to this thesis for a discussion of this problem and its history.

Specifically, we will construct non-isomorphic orders X and Y, such that (1) $X \cong AY$ and $X \cong Y\omega$, and (2) $Y \cong AX$ and $Y \cong X\omega$. Here A denotes the order $\omega_1^* + \omega_1$. It follows from these isomorphisms that $X \cong A^2X$. Hence from Theorem 2.2.3 we know that X is of the form $A^\omega(I_{[u]_2})$, and from Proposition 3.2.1 that Y is its shift $A^\omega(J_{[u]_2})$. Our construction will closely resemble our construction from Theorem 3.5.2 from Section 3.5. The difference will be in our choice of the $I_{[u]_2}$, which instead of being ordinals as in 3.5.2, will be what Jullien [8] called *surordinals*.

4.2 The construction

4.2.1 Review

Our terminology and notation is much the same as in the rest of the thesis. As always, $X \times Y = XY$ denotes the lexicographical product of X and Y, and $X(I_x)$ denotes the replacement of X by the orders I_x. If A is a linear order, then $A^\omega(I_{[u]})$ denotes a replacement $A^\omega(I_u)$ such that whenever $u \sim v$, we have $I_u = I_v$. We call such a replacement a replacement up to tail-equivalence. We similarly define replacements up to 2-tail-equivalence, denoted $A^\omega(I_{[u]_2})$. See Sections 2.2.1 and 3.1.1 for more on these notions.

Also as before, a cut in X is a pair (I, J) such that I is an initial segment of X and $J = X \setminus I$. If (I, J) is a cut in X, we will sometimes write $X = I + J$. Conversely, if X and Y are linear orders, we define the *sum* $X + Y$ to be the order obtained by placing a copy of X to the left of a copy of Y. Then (X, Y) is a cut in $X + Y$. We will informally refer to this cut as "the cut at the + sign."

In Section 3.3 we defined the cofinality and coinitiality of an interval. Cuts, like intervals, also have coinitialities and cofinalities. Suppose that (I, J) is a cut in X. A strictly increasing sequence of points $(x_i)_{i<\delta}$ in I is said to be *cofinal* in I if for every $y \in I$ there is an x_i such that $y \leq x_i$. If κ is the minimum length of a cofinal sequence in I we say that I has *cofinality* κ. Similarly, a strictly decreasing sequence of points in J is said to be *coinitial* in J if it eventually goes below every point in J, and J has *coinitiality* λ if λ is the minimum length of such a sequence. The cofinality of I is 1 if I has a top point, and the coinitiality of J

is 1 if J has a bottom point. When these cardinals are not 1 they are infinite and regular. If I has cofinality κ and J has coinitiality λ, we say that (I, J) is a (κ, λ)-*cut*. Although κ and λ are cardinals, just as in the case of intervals we will often write them as ordinals, to emphasize they refer to sequences. For example, saying (I, J) is an (ω, ω_1)-cut means that I has a countable cofinal sequence, whereas J has a coinitial sequence of length ω_1.

One may also define longer finite sums of orders, as well as infinite sums. Because the sum operation, like the product, is associative, there is no danger of confusion in ignoring parentheses. Given orders $X_i, i \in \mathbb{Z}$, we write $X_0 + X_1 + \dots$ to denote the order obtained by placing a copy of X_1 to the right of a copy of X_0, followed by a copy of X_2 to the right of a copy of X_1, etc. We similarly define $\dots + X_{-1} + X_0$ and $\dots + X_{-1} + X_0 + X_1 + \dots$ These orders are really just replacements of ω, ω^*, and $\mathbb{Z}$, respectively, but it will sometimes be convenient to write them out as infinite sums.

The lexicographical product is right-distributive over the sum, but it is not left-distributive. That is, for all orders X, Y, Z we have $(X + Y) \times Z \cong XZ + YZ$, but it is usually not the case that $Z \times (X + Y) \cong ZX + ZY$. In fact, the product distributes on the right over replacements. In particular, we have distributivity over sums of any finite length, as well as the sums of type ω, ω^*, and $\mathbb{Z}$ described above.

A word of warning: there will be one place in our construction where the ordinal ω^ω appears. Here, ω^ω has its traditional meaning as $\sup_{n<\omega} \omega^n$, and not as the set of infinite sequences with entries from ω as has been our convention throughout the thesis. We will point out when this happens so as to avoid any confusion that might arise from the ambiguity of notation. We also note that while traditional ordinal exponents of this kind behave as expected with respect to the *anti*-lexicographical product (the product usually used when studying ordinals), with regard to the lexicographical product there is some awkwardness. Namely, if α, γ, δ are ordinals, and if α^γ and α^δ have their traditional meanings and $\times$ is the lexicographical product, then $\alpha^\gamma \times \alpha^\delta = \alpha^{\delta+\gamma}$ (note the reversal in the exponent).

However, for spaces of *δ-sequences* X^δ, exponents behave as expected with respect to the lexicographical ordering and product. When there is no word to the contrary, X^δ is always assumed to mean the set of δ-length sequences on X ordered lexicographically, even when X is an ordinal.

4.2.2 The final lap

For the remainder of this chapter, let A denote the order $\omega_1^* + \omega_1$. We adopt the following conventions from Section 3.5. If α is a point in ω_1, we denote the corresponding element of ω_1^* by $-\alpha$. We also identify the 0 of ω_1^* and the 0 of ω_1, so that

$$A = \ldots < -\alpha < \ldots < -1 < 0 < 1 < \ldots < \alpha < \ldots$$

Though it does not follow directly from the statement of Theorem 3.5.1, the *proof* of 3.5.1 actually shows that we have the following theorem.

Theorem 4.2.1. Suppose that $f : A^\omega \to A^\omega$ is an order-automorphism. Then f has a fixed point of the form $(\alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \ldots)$, for some collection of ordinals $\alpha_i \in \omega_1$, where for all i we have $\alpha_i \neq 0$.

Using this, we can give our solution to the two-sided Schroeder-Bernstein problem for $(LO, \times)$.

Theorem 4.2.2. There exist non-isomorphic linear orders X and Y that are left-hand and right-hand divisors of one another. Specifically, X and Y satisfy the four isomorphisms $X \cong AY$, $X \cong Y\omega$, $Y \cong AX$, and $Y \cong X\omega$, where $A = \omega_1^* + \omega_1$.

Proof. It follows from the isomorphisms $X \cong AY$ and $Y \cong AX$ that $X \cong A^2X$. Hence, by Theorem 2.2.3 our X will be of the form $A^\omega(I_{[u]_2})$ for some collection of orders $I_{[u]_2}$. By

Proposition 3.2.1, it must be that $Y \cong AX$ will be of the form $A^\omega(J_{[u]_2})$, where for every $u \in A^\omega$ and $a \in A$ we have $J_{[u]_2} = I_{[au]_2}$. Since for such X, Y we automatically have that $Y \cong AX$ and $X \cong AY$, it remains only to specify the orders $I_{[u]_2}$, show that $X\omega \cong Y$ and $Y\omega \cong X$, and prove $X \ncong Y$.

In what follows, ω^ω has its traditional meaning as an ordinal, that is, as $\sup_{n<\omega} \omega^n$ and not as the collection of ω-length sequences on ω. The ordinals ω^n also appear, though in this case there is no ambiguity in the notation, since ω^n (as an ordinal) is isomorphic to ω^n (as the collection of n-sequences on ω, ordered lexicographically).

We first define a collection of orders $L_i, i \in \mathbb{Z}$. For $n \geq 0$, let

$$
\begin{aligned}
L_0 &= \ldots + 3\omega^3 + 2\omega^2 + \omega + \omega^\omega \\
L_1 &= \ldots + 3\omega^4 + 2\omega^3 + \omega^2 + \omega^\omega \\
&\vdots \\
L_n &= \ldots + 3\omega^{n+3} + 2\omega^{n+2} + \omega^{n+1} + \omega^\omega \\
&\vdots
\end{aligned}
$$

On the other side, let

$$
\begin{aligned}
L_{-1} &= \ldots + 4\omega^3 + 3\omega^2 + 2\omega + \omega^\omega \\
L_{-2} &= \ldots + 5\omega^3 + 4\omega^2 + 3\omega + \omega^\omega \\
&\vdots \\
L_{-n} &= \ldots (n+3)\omega^3 + (n+2)\omega^2 + (n+1)\omega + \omega^\omega \\
&\vdots
\end{aligned}
$$

We claim that for $i, j \in \mathbb{Z}$ with $i \neq j$ we have $L_i \ncong L_j$. This follows from a more general result due to Jullien [8] and independently Slater [16]. We argue from Slater's paper. Suppose that we have orders L and M such that

$$\begin{aligned} L &= \ldots + l_2\omega^{k_2} + l_1\omega^{k_1} + \omega^\omega \\ M &= \ldots + l'_2\omega^{k'_2} + l'_1\omega^{k'_1} + \omega^\omega, \end{aligned}$$

where the l_n, l'_n, k_n, k'_n are all positive integers, and furthermore $k_1 < k_2 < \ldots$ and $k'_1 < k'_2 < \ldots$ are strictly increasing sequences. In the terminology of Slater's paper, L and M are RJ types of type 4 (see Theorem 2 of [16]). By Theorem 4 of [16], if $L \cong M$, then there exists an $r \geq 0$ and N, such that for every $n \geq N$, we either have that $k'_n = k_{n+r}$ and $l'_n = l_{n+r}$, or we have that $k_n = k'_{n+r}$ and $l_n = l'_{n+r}$. That is, for L and M to be isomorphic, it is necessary that the coefficients l_n, l'_m and exponents k_n, k'_m eventually agree, up to some shift of index.

If we compare L_i and L_j for $i \neq j$, we see that while these orders are of the same form as L and M, they do not satisfy the condition given in Slater's paper necessary for their isomorphism. Hence $L_i \ncong L_j$, as claimed.

Though they are pairwise non-isomorphic, the L_i are all closely related. Namely, we claim $L_i\omega = L_{i+1}$ for all $i \in \mathbb{Z}$. There are three cases to verify. If $i \geq 0$, we have

$$\begin{aligned} L_i\omega &= (\ldots + 2\omega^{i+2} + \omega^{i+1} + \omega^\omega)\omega \\ &\cong \ldots + 2\omega^{i+2}\omega + \omega^{i+1}\omega + \omega^\omega\omega \\ &\cong \ldots + 2\omega^{i+3} + \omega^{i+2} + \omega^\omega \\ &\cong \ldots + 2\omega^{(i+1)+2} + \omega^{(i+1)+1} + \omega^\omega \\ &= L_{i+1}, \end{aligned}$$

where, in going from the second to third line, we have used the fact that $\omega^\omega\omega \cong \omega^{1+\omega}$ (reversing the exponent, as noted in the review) $\cong \omega^\omega$.

For $i = -1$ we have

$$\begin{aligned} L_{-1}\omega &= (\ldots + 3\omega^2 + 2\omega + \omega^\omega)\omega \\ &\cong \ldots + 3\omega^3 + 2\omega^2 + \omega^\omega \\ &\cong \ldots + 3\omega^3 + 2\omega^2 + \omega + \omega^\omega \\ &= L_0. \end{aligned}$$

where, in going from the second to third line, we have used the fact that $\alpha + \omega^\omega \cong \omega^\omega$ for any $\alpha < \omega^\omega$. Similarly, if $i < -1$, so that $i = -n$ for some $n > 1$ we have

$$\begin{aligned} L_i\omega &= L_{-n}\omega \\ &= (\ldots + (n+2)\omega^2 + (n+1)\omega + \omega^\omega)\omega \\ &\cong \ldots + (n+2)\omega^3 + (n+1)\omega^2 + \omega^\omega \\ &\cong \ldots + ((n-1)+3)\omega^3 + ((n-1)+2)\omega^2 + ((n-1)+1)\omega + \omega^\omega \\ &= L_{-(n-1)} \\ &= L_{i+1}, \end{aligned}$$

where, in going from the third to fourth line, we have split off the $((n-1)+1)\omega$ from the final segment ω^ω. Hence $L_i\omega = L_{i+1}$ in all cases, as claimed.

We are now almost ready to define the orders $I_{[u]_2}$ that will appear in the replacement $A^\omega(I_{[u]_2})$. These orders will each be one of the three orders I_{even}, I_{odd}, and I, defined as follows:

$$\begin{aligned} I_{even} &= \ldots + L_{-2} + L_0 + L_2 + \ldots \\ I_{odd} &= \ldots + L_{-1} + L_1 + L_3 + \ldots \\ I &= \ldots + L_{-1} + L_0 + L_1 + \ldots \end{aligned}$$

Before defining the $I_{[u]_2}$, we prove that these three orders are pairwise non-isomorphic. Note first that for a given i, every cut in L_i is either a $(1,1)$-cut or $(\omega, 1)$-cut. The only cuts in the orders I_{even}, I_{odd}, and I that do not fall in the midst of an L_i occur at the + signs, and these cuts are (ω, ω)-cuts. Hence these are the only (ω, ω)-cuts appearing in these orders.

Now suppose, for example, that there exists an isomorphism $f : I_{even} \to I_{odd}$. It must be, then, that $f[L_0] \subseteq L_k$ for some odd integer k. This is because $f[L_0]$ is an interval in I_{odd}, and every interval in I_{odd} is either a subinterval of some L_k or contains an (ω, ω)-gap. It cannot be that $f[L_0]$ contains an (ω, ω)-gap, since L_0 does not. But then we must actually have $f[L_0] = L_k$, since by a symmetric argument $f^{-1}[L_k]$ must be a subinterval of L_m for some even integer m, and the only possible m is $m = 0$.

This is a contradiction. It cannot be that $f[L_0] = L_k$ since this would mean that the orders L_0 and L_k are isomorphic. But L_0 is never isomorphic to L_k for k odd. Hence $I_{even} \ncong I_{odd}$. By similar arguments, $I_{even} \ncong I$ and $I_{odd} \ncong I$, as claimed.

However, it is easy to see that we have $I_{even}\omega \cong I_{odd}$, $I_{odd}\omega \cong I_{even}$, and $I\omega \cong I$. For example, to verify the first isomorphism, we check

$$\begin{aligned} I_{even}\omega &= (\ldots + L_{-2} + L_0 + L_2 + \ldots)\omega \\ &\cong +\ldots + L_{-2}\omega + L_0\omega + L_2\omega + \ldots \\ &\cong +\ldots + L_{-1} + L_1 + L_3 + \ldots \\ &= I_{odd}, \end{aligned}$$

and similarly for the other two isomorphisms. It follows that all three orders are invariant under right multiplication by ω^2, that is $I_{even}\omega^2 \cong I_{even}, I_{odd}\omega^2 \cong I_{odd}$, and $I\omega^2 \cong I$.

Now we can define the $I_{[u]_2}$. For every tail-equivalence class $C \subseteq A^\omega$, fix a representative u_C (so that $C = [u_C]$). There are two cases. If $u_C \not\sim_2 au_C$, so that $[u_C]_2 \cap [au_C]_2 = \emptyset$, we let $I_{[u_C]_2} = I_{even}$ and $I_{[au_C]_2} = I_{odd}$. If $u_C \sim_2 au_C$, so that $[u_C]_2 = [au_C]_2 = [u_C]$, we let $I_{[u_C]_2} = I$. Then by above, we have that $I_{[u]_2}\omega \cong J_{[u]_2}$ for all $u \in A^\omega$: depending on the u, this is just the isomorphism $I_{even}\omega \cong I_{odd}$, $I_{odd}\omega \cong I_{even}$, or $I\omega \cong I$.

Let $X = A^\omega(I_{[u]_2})$, and let $Y = AX$. Then $Y \cong A^\omega(J_{[u]_2})$, where for every $u \in A^\omega$ and $a \in A$ we have $J_{[u]_2} = I_{[au]_2}$. It is automatic from Theorem 2 that $X \cong AY$. We claim that these orders have the remaining desired properties, namely, that $X \cong Y\omega$, $Y \cong X\omega$, and $X \ncong Y$.

The first two properties are easy to verify. First, we have

$$\begin{aligned} X\omega &= A^\omega(I_{[u]_2})\omega \\ &\cong A^\omega(I_{[u]_2}\omega) \\ &\cong A^\omega(J_{[u]_2}) \\ &= Y, \end{aligned}$$

and similarly to show $Y\omega \cong X$.

So it remains to prove $X \ncong Y$. First note that since A has no endpoints, the order A^ω is dense. Thus every interval of A^ω is dense. It follows that, in general, if $A^\omega(I_u)$ and $A^\omega(J_u)$ are replacements of A^ω with none of the I_u, J_u empty, and $g : A^\omega(I_u) \to A^\omega(J_u)$ is an isomorphism, then for a given I_u we must have that either $f[I_u] \subseteq J_v$ for some v, or that $f[I_u]$ (and hence I_u) contains an infinite dense suborder. (The "or" here is non-exclusive.)

Now, suppose that $f : X \to Y$ is an isomorphism. We view f as an isomorphism of $A^\omega(I_{[u]_2})$ with $A^\omega(J_{[u]_2})$. None of the orders L_k contains an infinite dense suborder, and so neither do the $I_{[u]_2}$, $J_{[u]_2}$, as these are just $\mathbb{Z}$-sums of the L_k. By our observation above, it must be that for every $u \in A^\omega$ there is a v such that $f[I_u] \subseteq J_v$. Conversely, for every $v \in A^\omega$ there must be a u such that $f^{-1}[J_v] \subseteq I_u$. Combining these observations gives that in fact for every u there is a v such that $f[I_u] = J_v$. In particular, for such a pair u, v we have that $I_u \cong J_v$ as linear orders. We will assume for convenience that f is actually the identity on each I_u, that is, that $f((u, x)) = (v, x)$, since if f ever acts non-trivially on the right coordinates we can replace f with another isomorphism that does not, but still sends I_u onto J_v.

But then we have that f actually factors as (g, id), where $g : A^\omega \to A^\omega$ is an automorphism. By Theorem 4, the automorphism g has a fixed point $u = (\alpha_0, -\alpha_1, \alpha_2, -\alpha_3, \ldots)$, where the α_i are non-zero ordinals in ω_1. For such a u we have $u \not\sim_2 au$, and hence $I_{[u]_2} \ncong I_{[au]_2}$: one of these orders is I_{even}, and the other is I_{odd}. Hence one of I_u, J_u is I_{even} and the other is I_{odd}. But this is a contradiction. Since g fixes u it must be that $f[I_u] = J_u$, an impossibility, as these orders are non-isomorphic. Hence $X \ncong Y$, and the theorem follows. □

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