UNIVERSITY OF CALIFORNIA, SAN DIEGO

Adaptive Methods in the Finite Element Exterior Calculus Framework

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics with a specialization in Computational Science

by

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2014

The dissertation of Adam Mihalik is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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University of California, San Diego

2014

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Michael Holst, for the time and energy he has dedicated in helping me grow as a mathematician. I owe him a great deal of gratitude for introducing me to such a captivating area of research and for his patience in guiding me through the challenges. I thank my committee members and appreciate the time and support they have given me, both in this process and as teachers. I also thank co-author Ryan Szypowski for his help, especially early on, and assistance on the computational aspects of the second paper.

I also thank the many friends I have made in this process. There was always someone around to discuss math, or take a break and explore the outdoors. Lastly, I thank my family for providing endless encouragement and support; most of all Norma for always finding a way to be here and make this period of life so enjoyable.

Chapter 2, in full, has been submitted for publication of the material as it may appear in Foundations of Computational Mathematics, M. Holst, A. Mihalik and R. Szypowski, 2014. The dissertation author was the primary investigator and author of this paper.

Chapter 3 is in the preprint stage and will, in full, be submitted for publication. The material may appear as M. Holst, A. Mihalik and R. Szypowski, 2014. The dissertation author was the primary investigator and author of this paper.

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M. Holst, A. Mihalik and R. Szypowski, "Convergence and Optimality of Adaptive Mixed Methods on Surfaces", *preprint stage,* 2014.
arXiv:1404.1956v1.

M. Holst, A. Mihalik and R. Szypowski, "Convergence and Optimality of Adaptive Methods in the Finite Element Exterior Calculus Framework", *submitted for publication* 2013.
arXiv:1306.1886v2.

ABSTRACT OF THE DISSERTATION

Adaptive Methods in the Finite Element Exterior Calculus Framework

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Doctor of Philosophy in Mathematics with a specialization in Computational Science

University of California, San Diego, 2014

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In this thesis we explore convergence theory for adaptive mixed finite element methods. In particular, we introduce an *a posteriori* error-indicator, and prove convergence and optimality results for the mixed formulation of the Hodge Laplacian posed on domains of arbitrary dimensionality and topology in $\mathbb{R}^n$. After developing this framework, we introduce a new algorithm and extend our theory and results to problems posed on Euclidean hypersurfaces.

We begin by introducing the finite element exterior calculus framework, which is the key tool allowing us to address the convergence proofs in such generality. This introduction focuses on the fundamentals of the well-developed *a priori* theory and the results needed to extend the core of this theory to problems posed on surfaces. A basic set of results needed to develop adaptivity in this framework is also established. We then introduce an adaptive algorithm, and show convergence using this infrastructure as a tool to generalize existing finite element theory. The algorithm is then shown to be computationally optimal through a series of complexity analysis arguments. Finally, a second algorithm is introduced for problems posed on surfaces, and our original convergence and optimality results are extended using properties of specific geometric maps between surfaces.

Chapter 1

Introduction

1.1 Overview

Ideas from three distinct areas of research are essential to the theory developed in this thesis. In this introduction we review the fundamentals of each of these fields and discuss the role each will play in developing our main results. The chapter then concludes with a discussion focusing on our research and the results proven in later chapters.

1.2 Finite Element Exterior Calculus

Mixed finite elements have become standard tools for solving a large class of partial differential equations (PDE), and practical applications require implementation of this method using computationally efficient algorithms. In Chapters 2 and 3 we develop such algorithms for a large class of PDE problems, and show they produce a sequence of approximating solutions that convergence in a computationally optimal manner to the correct solution. Such a framework was developed in [13] for problems posed on connected domains in $\mathbb{R}^2$, using properties of the standard vector calculus operators. For more generality we turn to the theory of *finite element exterior calculus* (FEEC), introduced by Arnold, Falk and Winther. Using established connections between mixed finite elements and the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [10, 35, 36, 22], they showed that Hilbert complexes were a natural setting for analysis and numerical approximation of mixed variational problems. One of the powerful aspects of this theory is that results are proven for abstract Hilbert complexes and then can be applied to a large class of differential equations. Recently, researchers have shown the power and elegance of this framework by using it to make further generalizations in mixed finite element theory [24, 25, 18]. This framework will be a key tool as we develop a convergence and optimality theory for domains of arbitrary dimension and topology, and for problems posed on Euclidean hyper-surfaces. In the first part of this section we introduce the heart of the abstract theory; Hilbert complexes and the abstract Hodge Laplacian. We then discuss the de Rham complex and apply the abstract results to a large class of applicable equations.

1.2.1 Hilbert Complexes and the abstract Hodge Laplacian

A *Hilbert complex* (W, d) is a sequence of Hilbert spaces W^k equipped with closed, densely defined linear operators, d^k, mapping $V^k \subset W^k$ to the kernel of d^{k+1} in W^{k+1}. A Hilbert complex is *bounded* if each d^k is a bounded linear map from W^k to W^{k+1}. A Hilbert complex is *closed* if the range of each d^k is closed in W^{k+1}. Given a Hilbert complex (W, d), the subspaces $V^k \subset W^k$ endowed with the graph inner product

$$\langle u, v\rangle_{V^k} = \langle u, v\rangle_{W^k} + \langle d^k u, d^k v\rangle_{W^{k+1}}, \tag{1.1}$$

form a Hilbert complex (V, d) known as the *domain complex.* By construction $d^{k+1} \circ d^k = 0$, thus (V, d) is a bounded Hilbert complex.

The range of d^{k-1} in V^k and the null space of d^k will be denoted $\mathfrak{B}^k$ and $\mathfrak{Z}^k$, respectively. By construction $\mathfrak{B}^k \subset \mathfrak{Z}^k$, and the elements of $\mathfrak{Z}^k \cap \mathfrak{B}^{k\perp}$ form the space of harmonic forms, denoted $\mathfrak{H}^k$. For a closed Hilbert complex we can write the *Hodge decomposition* of W^k and V^k,

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W}, \tag{1.2}$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}. \tag{1.3}$$

Another important Hilbert complex will be the *dual complex* (W, d^*), where $d_k^* : W^k \to W^{k-1}$ is the adjoint of d^{k-1}. The domain of d_k^* will be denoted by V_k^*. For closed Hilbert complexes, an important result will be the *Poincaré inequality*,

$$\|v\|_V \leq c_P \|d^k v\|_W, \;\; v \in \mathfrak{Z}^{k\perp}. \tag{1.4}$$

Given a Hilbert complex (W, d), the operator $L = dd^* + d^*d$, $W^k \to W^k$ will be referred to as the *abstract Hodge Laplacian.* For $f \in W^k$, the Hodge Laplacian problem can be formulated as the problem of finding $u \in W^k$ such that

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, \quad v \in V^k \cap V_k^*.$$

This formulation, however, has undesirable computational properties. The finite element spaces $V^k \cap V_k^*$ are often difficult to implement, and the problem is not well-posed in the presence of a non-trivial harmonic space. In order to circumvent these issues, a well posed (cf. [4, 5]) *mixed formulation of the abstract Hodge Laplacian* is introduced as finding $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, such that:

$$\begin{aligned} \langle \sigma, \tau\rangle - \langle d\tau, u\rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, && \forall v \in V^k, \\ \langle u, q\rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{1.5}$$

In [4, 5] a theory of approximate solutions to the Hodge-Laplace problem is developed by using finite dimensional approximating Hilbert complexes. For a Hilbert complex (W, d) with domain complex (V, d), an approximating subcomplex is a set of finite dimensional Hilbert spaces, $V_h^k \subset V^k$ with the property that $dV_h^k \subset V_h^{k+1}$. Since V_h is a Hilbert complex, V_h has a corresponding Hodge decomposition,

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp_V}.$$

By this construction, (V_h, d) is an abstract Hilbert complex with a well posed Hodge Laplace problem. Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, such that

$$\begin{aligned} \langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle &= \langle f, v\rangle, && \forall v \in V_h^k, \\ \langle u_h, q\rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{1.6}$$

Now, with this background in place, [4, 5] prove an *a priori* convergence result for the abstract mixed Hodge-Laplacian.

Theorem 1.2.1. *Let (V_h, d) be a family of subcomplexes of the domain complex (V, d) of a closed Hilbert complex, parameterized by h and admitting V-bounded* *cochain projection. Let* $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ *be the solution to the mixed variational problem and let* $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ *be the solution to discrete mixed variational problem. Then*

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \le C(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\| + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|)$$

where

$$\mu = \sup_{r \in \mathfrak{H}^h, \|r\|=1} \|(I - \pi_h^k) r\|.$$

This result shows that as long as the approximating sub-complexes approach the approximated complex then convergence will follow. As we will see in the next section, this assumption cannot be made for adaptive methods. In developing our generalized adaptive finite element method we build an analogous abstract infrastructure that does not rely on this assumption.

1.2.2 The de Rham Complex

Let d be the exterior derivative acting as an operator from $L^2\Lambda^k(\Omega)$ to $L^2\Lambda^{k+1}(\Omega)$. The L^2 inner-product will define the W-norm, and the V-norm will be defined as the graph inner-product

$$\langle u, \omega \rangle_{V^k} = \langle u, \omega \rangle_{L^2} + \langle du, d\omega \rangle_{L^2}.$$

This forms a Hilbert complex $(L^2\Lambda(\Omega), d)$, with domain complex $(H\Lambda(\Omega), d)$, where $H\Lambda^k(\Omega)$ is the set of elements in $L^2\Lambda^k(\Omega)$ with weak exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$. The domain complex can be described with the following diagram

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} \cdots \to H\Lambda^{n-1}(\Omega) \xrightarrow{d} L^2(\Omega) \to 0. \tag{1.7}$$

It can be shown that a specific compactness property is satisfied, and therefore the prior results shown on abstract Hilbert complexes can be applied.

The Hodge star operator, $\star : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$, is then defined using the wedge product. For $\omega \in \Lambda^k(\Omega)$,

$$\int_\Omega \omega \wedge \mu = \langle \star\omega, \mu \rangle_{L^2\Lambda^{n-k}}, \quad \forall \mu \in \Lambda^{n-k}(\Omega).$$

Next we introduce the coderivative operator, $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$,

$$\star\delta\omega = (-1)^k d \star \omega, . \tag{1.8}$$

which combined with Stokes theorem allow integration by parts to be written as

$$\langle d\omega, \mu \rangle = \langle \omega, \delta\mu \rangle + \int_{\partial\Omega} \mathrm{tr}\, \omega \wedge \mathrm{tr} \star \mu, \;\; \omega \in \Lambda^{k-1}, \;\; \mu \in \Lambda^k(\Omega). \tag{1.9}$$

Using this formulation and the following spaces,

$$\begin{aligned} \mathring{H}\Lambda^k(\Omega) &= \{\omega \in H\Lambda^k(\Omega) \big| \, \mathrm{tr}_{\partial\Omega}\omega = 0\}, \\ \mathring{H}^*\Lambda^k(\Omega) &:= \star \mathring{H}\Lambda^{n-k}(\Omega), \end{aligned}$$

the following theorem gives an important connection between the framework built for abstract Hilbert complexes and the de Rham complex.

Theorem 1.2.2. *(Theorem 4.1 from [5]) Let d be the exterior derivative viewed as an unbounded operator $L^2\Lambda^{k-1}(\Omega) \to L^2\Lambda^k(\Omega)$ with domain $H\Lambda^k(\Omega)$. The adjoint d^*, as an unbounded operator $L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega)$, has $\mathring{H}^*\Lambda^k(\Omega)$ as its domain and coincides with the operator δ defined in* (1.8).

Applying the results from the previous section and Theorem 1.2.2, we get the mixed Hodge Laplace problem on the de Rham complex: Find the unique $(\sigma, u, p) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega) \times \mathfrak{H}^k$ such that

$$\begin{aligned} \sigma = \delta u, \;\; d\sigma + \delta d u = f - p \quad &\text{in } \Omega, \\ \mathrm{tr} \star u = 0, \;\; \mathrm{tr} \star du = 0 \qquad &\text{on } \partial\Omega, \\ u \perp \mathfrak{H}^k. \qquad\qquad & \end{aligned} \tag{1.10}$$

The following sub-sections follow the outline of a similar discussion given in

[5]. The Hodge Laplacian in $\mathbb{R}^3$ is analyzed, and is related to classical differential equations using the terminology of vector calculus. In addition to the material in [5], this section aims to give a more detailed and applied discussion of the harmonics, while also expanding the discusion to the case of surfaces without boundary.

The Hodge Laplacian for case k = 0

The discussion of the harmonic 0-forms are the most straightforward. The space of harmonic forms, $\mathfrak{H}^0$, is simply the constant functions on each connected component of the domain. In this case $H\Lambda^{-1}$ is void and the Hodge Laplacian is the problem of finding $(u,p) \in H\Lambda^k \times \mathfrak{H}^k$, such that:

$$\begin{aligned} \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad \forall v \in H\Lambda^k, \\ \langle u, q \rangle &= 0, \qquad\quad \forall q \in \mathfrak{H}^k. \end{aligned} \tag{1.11}$$

This equation implies that

$$\langle du, dv \rangle = \langle f - p, v \rangle, \quad \forall v \in H\Lambda^k, \tag{1.12}$$

thus

$$-\operatorname{div}\operatorname{grad} u = f - p \quad \text{in } \Omega. \tag{1.13}$$

Since f is the adjoint of du, Theorem 1.2.2 can be applied to yield the boundary condition,

$$\operatorname{grad} u \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1.14}$$

This is simply the scalar Laplacian with Neumann boundary conditions. The role the harmonics play is simple. In order to guarantee existence of a solution, the harmonic component of f is calculated as p, and then subtracted from f when solving the equivalent of Poisson's equation. The second equation of this mixed formulation guarantees uniqueness. Otherwise one could add a constant function to any solution u to derive another solution. In classical differential equations these restrictions are taken care of with boundary conditions and restrictions to the function spaces.

A more intuitive discussion of the harmonics is also simple in this case. The space V^{k-1} equals 0, and thus $\mathfrak{B}^k$ equals zero. This implies $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^\perp = \mathfrak{Z}^k$. Since $H\Lambda^0$ is the set of H^1-functions, it is clear that the harmonics are the set functions which are constant on each connected component.

We can also expand this discussion to surfaces without boundary. Equation (1.17) is no longer constraint when the surface has no boundary, but now Stokes theorem implies $\int_\Omega du = 0$.

Hilbert complex isomorphisms do not generally map harmonic spaces of one complex to harmonic spaces of another complex. For a given manifold, M, the de Rahm complex on the manifold will be denoted by $(H\Omega(M), d)$. Any isomorphic map between the Hilbert complexes $H\Omega(M)$ and $H\Omega(M_A)$ must map $\mathfrak{B}^k_M \to \mathfrak{B}^k_{M_A}$, $\mathfrak{Z}^k_M \to \mathfrak{Z}^k_{M_A}$ and the reverse must also hold. But, for instance, $\mathfrak{B}^k_M \subset \mathfrak{Z}^k_M$, and therefore there is no guarantee that $\mathfrak{H}^k_M = \mathfrak{B}^{k\perp}_M \cap \mathfrak{Z}^k_M$ is mapped to $\mathfrak{H}^k_{M_A}$. In the case $k = 0$, the maps i_A and π_A introduced in Chapter 3 will provide such a map of harmonic spaces. These maps are constructed using the pull-backs and push-forwards, which in the case $k = 0$ simply maps function values point-wise back and forth between the surfaces M and M_A. Therefore, a constant function on one surface will remain constant when mapped to the other surface using these maps, and thus the harmonic spaces will map to each other. Another way to think about this case is that $\mathfrak{B}^0$ is void, and thus $\mathfrak{Z}^k_M \to \mathfrak{Z}^k_{M_A}$ implies $\mathfrak{H}^k_M \to \mathfrak{H}^k_{M_A}$.

The Hodge Laplacian for case k = 1

Now we have moved off the edge of the Hilbert complex and have the familiar Hodge Laplacian problem: Find $(\sigma, u, p) \in H\Omega^{k-1} \times H\Omega^k \times \mathfrak{H}$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, && \forall \tau \in H\Omega^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in H\Omega^k, \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{1.15}$$

This equation implies that

$$\langle u, d\tau \rangle = \langle \sigma, \tau \rangle, \quad \forall \tau \in H\Lambda^{k-1},$$

thus σ is the adjoint of u, and an application of Theorem 1.2.2 yields the following boundary condition,

$$u \cdot n = 0 \quad \text{on } \partial\Omega. \tag{1.16}$$

This equation also implies that

$$\langle du, dv \rangle = \langle f - d\sigma - p, v \rangle, \quad \forall \tau \in H\Lambda^{k-1},$$

thus $f - \sigma - p$ is the adjoint of du, and an application of Theorem 1.2.2 yields the following boundary condition,

$$\text{curl } u \times n = 0 \quad \text{on } \partial\Omega. \tag{1.17}$$

Since u is orthogonal to the harmonics, we have $du = 0$ and $\delta u = 0$, which are in vector calculus terminology adds the constraints,

$$\text{curl } u = 0, \quad \text{div } u = 0 \quad \text{in } \Omega. \tag{1.18}$$

Additionally, the first two equations of (1.15) imply

$$\sigma = -\text{div } u, \quad \text{grad } \sigma + \text{ curl curl } u = f - p \quad \text{in } \Omega. \tag{1.19}$$

In terms of vector calculus, the above equation is a boundary value problem for a formulation of the the vector Laplacian.

As seen above, the elements of $\mathfrak{H}^1$ must satisfy (1.17) and (1.18). The dimension of $\mathfrak{H}^1$ is the first Betti number, which is the number of handles [5]. To give an example, take the case where Ω is a solid cylinder. There are no handles to this domain, and thus the dimension of the harmonic space is zero. Now take the case where we remove a cylinder of equal height and a smaller radius from the domain. Now the dimension of the harmonic space equals 1 as a vector field can essentially wrap around the handle.

The Hodge Laplacian for case k = 2

As was the case for $k = 1$, we are dealing with the middle of the Hilbert complex, and (1.15) is our reference equation. Using the same logic as the $k = 1$ case, we get the boundary conditions,

$$u \times n = 0, \text{ div } u = 0 \quad \text{on } \partial\Omega. \tag{1.20}$$

In this case the first two equations of (1.15) imply

$$\sigma = \text{curl } u, \quad \text{curl } \sigma - \text{ grad div } u = f - p \quad \text{in } \Omega. \tag{1.21}$$

And again, since u is orthogonal to $\mathfrak{H}^2$, we have the constraints

$$\text{curl } u = 0, \quad \text{div } u = 0 \quad \text{in } \Omega \tag{1.22}$$

In terms of vector calculus, the above is then seen to be a slightly different boundary value problem for a formulation of the vector Laplacian.

The dimension of the harmonic space is the second Betti number, and can be seen to be the number of voids in the domain. Further discussion of this harmonic space is not of interest here, as the properties of interest were already discussed for the case $k = 1$.

The Hodge Laplacian for case k = 3

The harmonic space is void when dealing with polygonal domains in $\mathbb{R}^3$, and we have the Hodge Laplacian problem: Find $(\sigma, u) \in H\Lambda^{k-1} \times H\Lambda^k$ satisfying

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, & \forall \tau \in H\Lambda^{k-1}, \\ \langle d\sigma, v \rangle &= \langle f, v \rangle, & \forall v \in H\Lambda^k. \end{aligned} \tag{1.23}$$

In terms of vector calculus we have,

$$\sigma = -\text{grad } u, \ \text{div } \sigma = f \quad \text{in } \Omega. \tag{1.24}$$

Since σ is the adjoint of u, an application of Theorem 1.2.2 yields the boundary condition,

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.25}$$

This problem now can be seen to be Poisson equation with Dirichlet boundary conditions.

The situation is more complicated when the domain is a surface without boundary. Since $d\omega = 0 \ \forall \omega \in H\Lambda^n$, $\mathfrak{H}^3$ can be described as the set $\omega \in H\Lambda^n$ such that $\delta\omega = 0$. When $\delta\omega = 0$ it implies $d \star \omega = 0$, where d is the gradient operating on $H^1(\Omega)$. This implies the harmonics are constant forms and, if the domain has a boundary, (1.25) implies these constants must be zero. Thus the harmonic space is void in the case where there is a boundary. In the absence a boundary, (1.25) does not apply and thus constant volume forms on each connected component form the harmonics. We give an analogous lower-dimensional example of what is happening. Consider a one dimensional surface in $\mathbb{R}^2$ with boundary. Any 1-form in L^2 is the derivative of some 0-form in H^1, thus there are no harmonics. Now connect the two ends of the line. Now only functions which integrate to zero can be derived as derivatives of H^1 functions. This idea can be extended to the language of differential forms, and then dealt with in the cases of higher dimensions, and it can be seen that the harmonics are the constant volume forms on the surface.

The above fact brings the problem on surfaces to the more familiar Hodge Laplacian: Find $(\sigma, u, p) \in H\Omega^{k-1}(M) \times H\Omega^k(M) \times \mathfrak{H}$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, & \forall \tau \in H\Omega^{k-1}(M), \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, & \forall v \in H\Omega^k(M), \\ \langle u, q \rangle &= 0, & \forall q \in \mathfrak{H}^k. \end{aligned} \tag{1.26}$$

The boundary condition clearly can no longer exist and the emergence of the harmonic constraint is essentially replacing this condition.

Numerically the algorithms we have developed work specifically for the case where f is formulated as a volume form, and take advantage of the lack of harmonics. Thus our algorithm will not attack this problem directly. Since constant volume forms are easy to deal with, we can simply factor out the average of any volume form f, and essentially create a mean zero set of basis functions for calculating u. This will allow us to solve 1.26 using the same tools we built to solve 1.23.

Unlike the case $k = 0$, Hilbert complex isomorphisms do not generally map the harmonic forms on M to harmonic forms on M_A. Harmonic forms are the constant volume forms, and thus maps such as the pull-back scale by volume ratios point-wise between the surfaces. However, as mentioned above, we can deal with the harmonics in a preprocessing step thus this is more a point of interest rather than a computational concern.

1.3 Adaptive Finite Element Methods

Adaptive finite elements methods (AFEM) are a class of finite element methods (FEM) which aim to distribute computational power efficiently by refining the mesh of the domain non-uniformly. For a large class of important problems in science and engineering it is not practical to refine the mesh uniformly and AFEM based on *a posteriori* error estimators have become standard tools (cf. [1, 46, 38]). A fundamental difficulty with these adaptive methods is guaranteeing convergence of the solution sequence. The first convergence result was obtained by Babuska and Vogelius [7] for linear elliptic problems in one space dimension, and many improvements and generalizations to the theory have followed [19, 31, 34, 33, 41].

Given an initial triangulation of the domain, $\mathcal{T}_0$, the adaptive procedure will generate a nested sequence of triangulations $\mathcal{T}_k$ and discrete solutions (σ_k, u_k, p_k), by looping through the following steps:

$$\textsf{Solve} \longrightarrow \textsf{Estimate} \longrightarrow \textsf{Mark} \longrightarrow \textsf{Refine} \tag{1.27}$$

The following subsection will describe details of these steps.

Approximation Procedure

We assume access to a routine SOLVE, which can produce solution to (1.6) given a triangulation, problem data, and a desired level of accuracy. For the $\mathsf{ESTIMATE}$ step we will introduce *a posteriri* error indicators η_T on each element $T \in \mathcal{T}_k$. Two important properties of such indicators are *reliability* and *efficiency*. We say that an indicator is reliable if it bounds the error of interest up to a constant. We say that an indicator is efficient if, up to some positive constant and higher order terms, it provides a lower bound of the error. In the MARK step we will use Dörfler Marking strategy [20]. An essential feature of the marking process is that the summation of the error indicators on the marked elements exceeds a user defined marking parameter θ.

We assume access to an algorithm REFINE in which marked elements are subdivided into two elements of the same size, resulting in a conforming, shape-regular mesh. Triangles outside of the original marked set may be refined in order to maintain conformity. Bounding the number of such refinements is important in showing optimality of the method. Along these lines, Stevenson [44] showed certain bisection algorithms developed in two-dimensions can be extended to n-simplices of arbitrary dimension satisfying

(1)$\{\mathcal{T}_k\}$ is shape regular and the shape regularity depends only on $\mathcal{T}_0$,

(2)$\#\mathcal{T}_k \leq \#\mathcal{T}_0 + C\#\mathcal{M}$,

where $\mathcal{M}$ is the collection of all marked triangles going from $\mathcal{T}_0$ to $\mathcal{T}_k$.

Convergence and Optimality

The idea of convergence in this context is straightforward; the approximations must approach the actual solution. We will show that our adaptive algorithm convergences linearly to any given error tolerance in a finite number of steps. Convergence, however, does not necessarily imply optimality of a method. This idea has led to the development of a theory related to the optimal computational complexity of AFEMs, and within this framework certain classes of adaptive methods have been shown to be optimal [8, 43, 12].

In order to make this discussion of optimality more concrete, we introduce the approximation classes as defined in [8]. Let σ be the solution to the to the Hodge Laplacian, and the $\sigma_{\mathcal{T}}$ be the solution to the discrete version of the problem on a given triangulation of the domain, $\mathcal{T}$. For a given refinement of $\mathcal{T}$, let N represent the number of elements in the triangulation added from an original conforming triangulation, $\mathcal{T}_0$. Also let $\hat{\mathcal{T}}_N$ represent the entire set of triangulations that can be formed for a given value of N. Another tool in developing the idea of the approximation class is the following norm,

$$\|\sigma\|_{\mathcal{A}^s} = \sup_{N \geq \#\mathcal{T}_0} \left(N^s \inf_{\mathcal{T} \in \hat{\mathcal{T}}_N} \|\sigma - \sigma_{\mathcal{T}}\| \right). \tag{1.28}$$

With these tools we now define the approximation class

$$\mathcal{A}^s = \{\sigma \in V^{k-1} : \|\sigma\|_{\mathcal{A}^s} < \infty\}. \tag{1.29}$$

A method is optimal if for any iterated triangulation there is a constant C, such that$\|\sigma - \sigma_N\| \leq CN^{-s}$. In proving the optimality of our method we follow [43, 13], which, after proving convergence, only requires orthogonality, a discrete upper bound and an efficient error indicator,

$$\|\sigma - \sigma_{k+1}\|^2 = \|\sigma - \sigma_k\|^2 - \|\sigma_{k+1} - \sigma_k\|^2, \tag{1.30}$$

$$\|\sigma_{k+1} - \sigma_k\|^2_{\hat{\mathcal{T}}_k} \leq C\eta^2(\sigma_k, \hat{\mathcal{T}}_k \subseteq \mathcal{T}_k), \tag{1.31}$$

$$C_2\eta^2(\sigma_k, \mathcal{T}_k) \leq \|\sigma - \sigma_k\|^2. \tag{1.32}$$

In [13], ideas of [8] related to optimal function approximation are used to show certain quasi-orthogonality are enough for optimality. The remainder of proving the optimality of our method follows [43]. The high level intuition behind the proof is as follows. Take any single refinement step of our algorithm. The collections of elements refined contribute to a minimal amount of error reduction, λ.

Then, combining the above equations with cardinality properties of the refinement strategy, the amount of elements added is bounded by $\lambda^{-1/s}$ multiplied with a multiple of the current error. Then this relationship can be combined over each iteration of the algorithm, and then used to get the desired optimality result.

In [13], Chen, Holst, and Xu used error indicators developed in [3] and establish convergence and optimality of an adaptive mixed finite element method for the Poisson equation on simply connected polygonal domains in two dimensions. Their argument used a type of quasi-orthogonality result, exploiting the fact that the error was orthogonal to the divergence free subspace, while the part of the error not divergence free was bounded by data oscillation through a discrete stability result. A generalization of this quasi-orthogonailty will be fundamental in proving convergence and optimality of our algorithm introduced in Chapter 2.

1.4 Surface Finite Element Methods

Many applicable problems are posed on embedded surfaces, and thus an extension of finite element theory developed in Chapter 2 is desired. Typically, surfaces of interest are not polygonal, and it is necessary to approximate the surfaces or introduce a map to the surface from an approximating polygonal manifold. This thesis investigates these methods, and develops convergence and optimality results for a class of surface PDE.

In a 1988 article, Dziuk [21] introduced a nodal finite element method for the Laplace-Beltrami equation on 2-surfaces approximated by a piecewise-linear triangulation, pioneering a line of research into surface finite element (SFEM) methods. Demlow and Dziuk [17] built on the original results, introducing an adaptive method for problems on 2-surfaces, and Demlow later extended the a priori theory to 3-surfaces and higher order elements [16]. While a posteriori error indicators are introduced and shown to have desirable properties in [17], a convergence and optimality theory related to problems on surfaces is a relatively undeveloped area, and developing such a theory is the main topic of Chapter 3. Key tools to our development will be ideas from [24], where Holst and Stern extend the FEEC theory to include problems in which the discrete complex is not a subcomplex of the approximated complex. This made it possible in [24] to reproduce the existing a priori theory for SFEM as a particular application, as well as to generalize SFEM theory in several directions.

Surfaces finite element methods, by their nature, have additional complications which make developing and all-encompassing algorithm difficult. Surfaces, for instance, can be described in different manners, and depending on the access to surface quantities, algorithms that are ideal in one situation may be infeasible in others. Also, when refining a mesh, element nodes are not necessarily required to lie on the approximated surface, or even alter the approximating surface. Continually improving the surface approximation has desirable features, but it also complicates the analysis of convergence and optimality. With these ideas in mind, we have explored three separate categories of surface finite element methods, and developed results with desirable features depending on the specifics of the problem of interest.

- Parametric Finite Element Approximations, where the basis elements are defined on an approximating surface and mapped back to the original domain.
- Algorithms that reduce the geometric error to specified levels prior to working on the error related to the PDE on the approximating surface.
- Standard surface finite element methods constrained to have the element nodes lying on the actual surface.

Ideas similar to the first method were discussed in the a priori setting in Dziuk [21] and Demlow [16]. Initially an approximate surface lying an a tubular neighborhood of the domain must be created. A mesh on the approximating surface is continually refined and the basis elements defined on this mesh are mapped back to the approximated surface. This creates a desirable situation where nested refinements produce function subspaces on the elements. In Chapter 3 we use this property, and further develop differential geometry tools introduced in [24], to prove convergence and optimality of an adaptive method on surfaces.

In the second type of method above, the user must define desired error tolerance prior to running the computation. The first step of the method is to create a surface with a geometric portion of the approximation error bounded below a specified tolerance. The second step of the algorithm applies standard adaptive methods to reduce the residual portion of the approximation error below the desired tolerance, and hence the overall error is reduced as desired. The advantage of this method is that calculation of the geometrical terms related to iterated values of the PDE are avoided and this is computationally desirable. This also yields an optimal algorithm in cases where error tolerance can be fixed a priori.

Once the fixed approximating surface is created, however, the additive geometric error does not decrease. If the desired error tolerance changes, then the algorithm must be started from scratch. This can be an undesirable feature which we have addressed by adding geometrical terms to the error indicator, and forcing the nodes of the mesh to lie on the approximated surface. The error indicators used in this method can be shown to be reliable and efficient, providing promise for a good adaptive method. Convergence and complexity results, however, are harder to obtain, as the evolving surface removes the desirable subspace properties present on non-evolving domains.

The third method above is in the realm of [21, 16, 17] and has many similarities to the first method. The biggest disadvantage of this method is that the refined spaces do not have the nested subspace property for the function spaces. This property is at the core of many traditional convergence arguments, and thus this problem requires a new technique. One advantage of this method is the surface estimates improve at each step, and therefor the bounding constant due to surface approximation decrease. We discuss topic this more detail and compare with our method in Chapter 3.

1.5 Main Results

In Chapter 2 we use the FEEC framework to extend the convergence and complexity arguments of [13] for simply connected domains in two dimensions to more general domains. Specifically we introduce an adaptive algorithm and show convergence and optimality of the method for the Hodge-Laplace problem ($k = n$) on domains of arbitrary topology and spatial dimension. While our results are shown for the specific case ($k = n$), many of the auxiliary results hold for arbitrary k, and we also discuss possible extensions to the general $\mathfrak{B}$-Hodge-Laplace problem. In Chapter 4 we apply this adaptive method and step through a detailed computational example.

In Chapter 3 we introduce an adaptive method for problems posed on smooth Euclidean hypersurfaces in which finite element spaces are mapped from a fixed approximating polygonal manifold. The mesh on the fixed approximating surface is refined using error indicators related to the original problem. Using tools developed in [24], the auxiliary results of Chapter 2 are modified to account for the surface mapping. In doing this we establish the optimality of a convergent algorithm for the Hodge Laplacian (case $k = m$) on hypersurfaces of arbitrary dimension.

Chapter 2

Convergence and Optimality of Adaptive Methods in the Finite Element Exterior Calculus Framework

2.1 Abstract

Finite Element Exterior Calculus (FEEC) was developed by Arnold, Falk, Winther and others over the last decade to exploit the observation that mixed variational problems can be posed on a Hilbert Complex, and Galerkin-type mixed methods can then be obtained by solving finite-dimensional subcomplex problems. Stability and consistency of the resulting methods then follow directly from the framework by establishing the existence of operators connecting the Hilbert complex with its subcomplex, essentially giving a "recipe" for well-behaved methods. In 2012, Demlow and Hirani developed a posteriori error indicators for driving adaptive methods in the FEEC framework. While adaptive techniques have been used successfully with mixed methods for years, convergence theory for such techniques has not been fully developed. The main difficulty is lack of error orthogonality. In 2009, Chen, Holst, and Xu established convergence and optimality of an adaptive mixed finite element method for the Poisson equation on simply connected polygonal domains in two dimensions. Their argument used a type of quasi-orthogonality result, exploiting the fact that the error was orthogonal to the divergence free subspace, while the part of the error not divergence free was bounded by data oscillation through a discrete stability result. In this paper, we use the FEEC framework to extend these convergence and complexity results for mixed methods on simply connected domains in two dimensions to more general domains. While our main results are for the Hodge-Laplace problem ($k = n$) on domains of arbitrarily topology and spatial dimension, a number of our supporting results also hold for the more general $\mathfrak{B}$-Hodge-Laplace problem ($k \neq n$).

2.2 Introduction

An idea that has had a major influence on the development of numerical methods for PDE applications is that of *mixed finite elements*, whose early success in areas such as computational electromagnetics was later found to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [10, 35, 36, 22]. A core idea underlying these developments is the *Helmholtz-Hodge* orthogonal decomposition of an arbitrary vector field $f \in L^2(\Omega)$ into curl-free, divergence-free, and harmonic functions:

$$f = \nabla p + \nabla \times q + h,$$

where h is harmonic (divergence- and curl-free). The mixed formulation is explicitly computing the decomposition for $h = 0$, and finite element methods based on mixed formulations exploit this. There is a connection between this decomposition and *de Rham cohomology*; the space of harmonic forms is isomorphic to the first *de Rham cohomology* of the domain Ω, with the number of holes in Ω giving the first Betti number, and creating obstacles to well-posed formulations of elliptic problems. A natural question is then: What is an appropriate mathematical framework for understanding this abstractly, that will allow for a methodical construction of "good" finite element methods for these types of problems? The answer turns out to be theory of *Hilbert Complexes*. Hilbert complexes were originally studied in [11] as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and other aspects of Hodge theory. The *Finite Element Exterior Calculus* (or *FEEC*) [4, 5] was developed to exploit this abstraction. A key insight was that from a functional-analytic point of view, a mixed variational problem can be posed on a Hilbert complex: a differential complex of Hilbert spaces, in the sense of [11]. Galerkin-type mixed methods are then obtained by solving the variational problem on a finite-dimensional subcomplex. Stability and consistency of the resulting method, often shown using complex and case-specific arguments, are reduced by the framework to simply establishing existence of operators with certain properties that connect the Hilbert complex with its subcomplex, essentially giving a "recipe" for the development of provably well-behaved methods.

Due to the pioneering work of Babuska and Rheinboldt [6], adaptive finite element methods (AFEM) based on *a posteriori* error estimators have become standard tools in solving PDE problems arising in science and engineering (cf. [1, 46, 38]). A standard adaptive algorithm has the general iterative structure:

$$\mathsf{Solve} \longrightarrow \mathsf{Estimate} \longrightarrow \mathsf{Mark} \longrightarrow \mathsf{Refine} \tag{2.1}$$

where Solve computes the discrete solution u_k in a subspace $X_k \subset X$; Estimate computes certain error estimators based on u_k, which are reliable and efficient in the sense that they are good approximation of the true error $u - u_k$ in the energy norm; Mark applies certain marking strategies based on the estimators; and finally, Refine divides each marked element and completes the mesh to to obtain a new partition, and subsequently an enriched subspace X_{k+1}. The fundamental problem with the adaptive procedure (2.1) is guaranteeing convergence of the solution sequence. The first convergence result for (2.1) was obtained by Babuska and Vogelius [7] for linear elliptic problems in one space dimension. The multi-dimensional case was open until Dörfler [19] proved convergence of (2.1) for Poisson equation, under the assumption that the initial mesh was fine enough to resolve the influence of data oscillation. This result was improved by Morin, Nochetto, and Siebert [31], in which the convergence was proved without conditions on the initial mesh, but requiring the so-called *interior node property*, together with an additional marking step driven by data oscillation. These results were then improved and generalized in several respects [34, 33, 41]. In another direction, it was shown by Binev, Dahmen and DeVore [8] for the first time that AFEM for Poisson equation in the plane has optimal computational complexity by using a special coarsening step. This result was improved by Stevenson [43] by showing the optimal complexity in general spatial dimension without a coarsening step. These error reduction and optimal complexity results were improved recently in several aspects in [12]. In their analysis, the artificial assumptions of interior node and extra marking due to data oscillation were removed, and the convergence result is applicable to general linear elliptic equations. The main ingredients of this new convergence analysis are the global upper bound on the error given by the *a posteriori* estimator, orthogonality (or possibly only quasi-orthogonality) of the underlying bilinear form arising from the linear problem, and a type of error indicator reduction produced by each step of AFEM. We refer to [37] for a recent survey of convergence analysis of AFEM for linear elliptic PDE problems which gives an overview of all of these results through late 2009. See also [26] or an overview of various extensions to nonlinear problems.

Of particular relevance here is the 2009 article of Chen, Holst, and Xu [13], where convergence and optimality of an adaptive mixed finite element method for the Poisson equation on simply connected polygonal domains in two dimensions was established. The main difficulty for mixed finite element methods is the lack of minimization principle, and thus the failure of orthogonality. A quasi-orthogonality property is proved on the $\|\sigma - \sigma_h\|_{L^2}$ error in [13] using the fact that the error is orthogonal to the divergence free subspace, while the part of the error that is not divergence free was bounded by the data oscillation using a discrete stability result. This discrete stability result was then also used to get a localized discrete upper bound, which was the key to giving a proof of optimality of the resulting adaptive method. A key technical tool was the use of the error indicator developed by Alonso in [3]. In this paper, we will generalize the approach taken in [13] by analyzing the $\|\sigma - \sigma_h\|_{L^2\Lambda^{k-1}(\Omega)}$ error in the FEEC framework, which will allow us extend the convergence and complexity results for simply connected domains in two dimensions in [13] to domains of arbitrary topology and spatial dimension. In FEEC terminology, the methods considered in [13] are equivalent to those for solving the Hodge-Laplace problem when $k = n = 2$. As described in more detail in Section 2.3 below, Hodge-Laplace problems on the complex $H\Lambda^k(\Omega)$, $k = n$ are a subset of the more general $\mathfrak{B}$-Hodge-Laplace problem. Our main result will apply to the $k = n$ case for arbitrary n and domains which are not necessarily simply connected. However, a number of our supporting results also hold for the more general $\mathfrak{B}$-Hodge-Laplace problem ($k \neq n$). For each result, we will indicate whether it holds for all $\mathfrak{B}$-Hodge-Laplace problems, or just the case $k = n$.

In mixed finite element methods σ is often the variable of interest, and the error measured in the natural norm can be broken into two components,

$$\|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} = \|\sigma - \sigma_h\|_{L^2\Lambda^{k-1}(\Omega)} + \|d(\sigma - \sigma_h)\|_{L^2\Lambda^{k-1}(\Omega)}.$$

In the general $\mathfrak{B}$ problems we have $d(\sigma - \sigma_h) = f - f_h$, and standard interpolation techniques can be used to efficiently reduce this error. Our results will focus on the first term involving $\sigma - \sigma_h$, the quantity that is often of interest, yet typically cannot be calculated explicitly.

The remainder of the paper is organized as follows. In Section 2.3 we introduce the notational and technical tools needed for the paper. The first part of Section 2.3 follows the ideas of [5] in introducing general Hilbert complexes, the de Rham complex, and properties of specific mappings between the complexes. We then give a brief overview of a standard adaptive finite element algorithm. In Section 2.4 we follow the ideas in [13] and develop a quasi-orthogonality result. In Section 2.5, we prove discrete stability (which was needed for proving quasi-orthogonality in Section 2.4), and also establish a continuous stability result, which will be needed for deriving an upper bound on the error. In Section 2.6 we begin by introducing an error indicator and then derive bounds and a type of continuity result for this indicator. An adaptive algorithm is then presented in Section 2.7, and we then combine the results from the previous sections to prove both convergence and optimality. Finally, we draw some conclusions in 2.8.

2.3 Preliminaries

In this section we first review abstract Hilbert complexes. We then examine the particular case of the de Rham complex. We follow closely the notation and the general development of Arnold, Faulk and Winther in [4, 5]. We also discuss results from Demlow and Hirani in [18]. (See also [24, 25] for a concise summary of Hilbert Complexes in a yet more general setting.) We then give an overview of the basics of Adaptive Finite Element Methods (AFEM), and the ingredients we will need to prove convergence and optimality within the FEEC framework.

2.3.1 Hilbert complexes

We begin with a quick summary of some basic concepts and definitions. A *Hilbert complex* (W, d) is a sequence of Hilbert spaces W^k equipped with closed, densely defined linear operators, d^k, which map their domain, $V^k \subset W^k$ to the kernel of d^{k+1} in W^{k+1}. A Hilbert complex is *bounded* if each d^k is a bounded linear map from W^k to W^{k+1} A Hilbert complex is *closed* if the range of each d^k is closed in W^{k+1}. Given a Hilbert complex (W, d), the subspaces $V^k \subset W^k$ endowed with the graph inner product

$$\langle u, v\rangle_{V^k} = \langle u, v\rangle_{W^k} + \langle d^k u, d^k v\rangle_{W^{k+1}},$$

form a Hilbert complex (V, d) known as the *domain complex*. By definition $d^{k+1} \circ d^k = 0$, thus (V, d) is a bounded Hilbert complex. Additionally, (V, d) is closed if (W, d) is closed.

The range of d^{k-1} in V^k will be represented by $\mathfrak{B}^k$, and the null space of d^k will be represented by $\mathfrak{Z}^k$. Clearly, $\mathfrak{B}^k \subset \mathfrak{Z}^k$. The elements of $\mathfrak{Z}^k$ orthogonal to $\mathfrak{B}^k$ are the space of harmonic forms, represented by $\mathfrak{H}^k$. For a closed Hilbert complex we can write the *Hodge decomposition* of W^k and V^k,

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W}, \tag{2.2}$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}. \tag{2.3}$$

Following notation common in the literature, we will write $\mathfrak{Z}^{k\perp}$ for $\mathfrak{Z}^{k\perp_W}$ or $\mathfrak{Z}^{k\perp_V}$, when clear from the context. Another important Hilbert complex will be the *dual complex* (W, d^*), where d_k^*, which is an operator from W^k to W^{k-1}, is the adjoint of d^{k-1}. The domain of d_k^* will be denoted by V_k^*. For closed Hilbert complexes, an important result will be the *Poincaré inequality*,

$$\|v\|_V \leq c_P \|d^k v\|_W, \;\; v \in \mathfrak{Z}^{k\perp}. \tag{2.4}$$

The de Rham complex is the practical complex where general results we show on an abstract Hilbert complex will be applied. The de Rham complex satisfies an important compactness property discussed in [5], and therefore this compactness property is assumed in the abstract analysis.

The abstract Hodge Laplacian

Given a Hilbert complex (W, d), the operator $L = dd^* + d^*d$, $W^k \to W^k$ will be referred to as the *abstract Hodge Laplacian*. For $f \in W^k$, the Hodge Laplacian problem can be formulated as the problem of finding $u \in W^k$ such that

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, v \in V^k \cap V_k^*.$$

The above formulation has undesirable properties from a computation perspective. The finite element spaces $V^k \cap V_k^*$ can be difficult to implement, and the problem will not be well-posed in the presence of a non-trivial harmonic space, $\mathfrak{H}^k$. In order to circumvent these issues, a well posed (cf. [4, 5]) *mixed formulation of the abstract Hodge Laplacian* is introduced as the problem of finding $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, such that:

$$\begin{aligned} \langle \sigma, \tau\rangle - \langle d\tau, u\rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle, && \forall v \in V^k, \\ \langle u, q\rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{2.5}$$

Sub-complexes and approximate solutions to the Hodge Laplacian

In [4, 5] a theory of approximate solutions to the Hodge-Laplace problem is developed by using finite dimensional approximating Hilbert complexes. Let (W, d) be a Hilbert complex with domain complex (V, d). An approximating subcomplex is a set of finite dimensional Hilbert spaces, $V_h^k \subset V^k$ with the property that $dV_h^k \subset V_h^{k+1}$. Since V_h is a Hilbert complex, V_h has a corresponding Hodge decomposition,

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp_V}.$$

By this construction, (V_h, d) is an abstract Hilbert complex with a well posed Hodge Laplace problem. Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, such that

$$\begin{aligned} \langle \sigma_h, \tau\rangle - \langle d\tau, u_h\rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle &= \langle f, v\rangle, && \forall v \in V_h^k, \\ \langle u_h, q\rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{2.6}$$

An assumption made in [5] in developing this theory is the existence of a bounded cochain projection, $\pi_h : V \to V_h$, which commutes with the differential operator.

In [5], an a priori convergence result is developed for the solutions on the approximating complexes. The result relies on the approximating complex getting sufficiently close to the original complex in the sense that $\inf_{v \in V_h^k} \|u - v\|_V$ can be assumed sufficiently small for relevant $u \in V^k$. Adaptive methods, on the other hand, gain computational efficiency by limiting the degrees of freedom used in areas of the domain where it does not significantly impact the quality of the solution.

2.3.2 The de Rham complex and approximation properties

The de Rham complex is a cochain complex where the abstract results from the previous section can be applied in developing practical computational methods. This section reviews concepts and definitions related to the de Rham complex that will be needed in our development of an adaptive finite element method. This introduction will be brief and and mostly follows the notation from the more in-depth discussion in [5].

For the remainder of the paper we assume a bounded Lipschitz polyhedral domain, $\Omega \in \mathbb{R}^n, n \geq 2$. Let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω, and let $L^2\Lambda^k(\Omega)$ be the completion of $\Lambda^k(\Omega)$ with respect to the L^2 inner-product. There are no non-zero harmonic forms in $L^2\Lambda^n(\Omega)$ (see [4], Theorem 2.4) which will often simplify the analysis in our primary case of interest, $k = n$. For general k such a property cannot be assumed, and therefore, since the $\mathfrak{B}$ problem deals with the spaces of k and $(k-1)$-forms, analysis of the harmonic spaces is still necessary. Note that the results in [13] hold only for polygonal and simply connected domains, therefore $\mathfrak{H}^{k-1}$ is also void in the case $k = n = 2$.

The de Rham complex

Let d be the exterior derivative acting as an operator from $L^2\Lambda^k(\Omega)$ to $L^2\Lambda^{k+1}(\Omega)$. The L^2 inner-product will define the W-norm, and the V-norm will be defined as the graph inner-product

$$\langle u, \omega \rangle_{V^k} = \langle u, \omega \rangle_{L^2} + \langle du, d\omega \rangle_{L^2}.$$

This forms a Hilbert complex $(L^2\Lambda(\Omega), d)$, with domain complex $(H\Lambda(\Omega), d)$, where $H\Lambda^k(\Omega)$ is the set of elements in $L^2\Lambda^k(\Omega)$ with exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$. The domain complex can be described with the following diagram

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} \cdots \to H\Lambda^{n-1}(\Omega) \xrightarrow{d} L^2(\Omega) \to 0. \tag{2.7}$$

It can be shown that the compactness property is satisfied, and therefore the prior results shown on abstract Hilbert complexes can be applied.

The importance of the adjoint operator is clear by the first equation of the mixed Hodge Laplace problem. Defining the coderivative operator, $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$, and two particular spaces, will be helpful in understanding the adjoint operator on the de Rham complex.

$$\star\delta\omega = (-1)^k d \star \omega, \tag{2.8}$$

$$\mathring{H}\Lambda^k(\Omega) = \{\omega \in H\Lambda^k(\Omega) | \operatorname{tr}_{\partial\Omega}\omega = 0\}, \tag{2.9}$$

$$\mathring{H}^*\Lambda^k(\Omega) := \star\mathring{H}\Lambda^{n-k}. \tag{2.10}$$

Combining δ with Stoke's theorem gives a useful version of integration by parts

$$\langle d\omega, \mu\rangle = \langle \omega, \delta\mu\rangle + \int_{\partial\Omega} \operatorname{tr} \omega \wedge \operatorname{tr} \star \mu, \;\; \omega \in \Lambda^{k-1}, \;\; \mu \in \Lambda^k. \tag{2.11}$$

The following result uses the above concepts and is helpful in understanding the mixed Hodge Laplace problem on the de Rham complex.

Theorem 2.3.1. *(Theorem 4.1 from [5]) Let d be the exterior derivative viewed as an unbounded operator $L^2\Lambda^{k-1}(\Omega) \to L^2\Lambda^k(\Omega)$ with domain $H\Lambda^k(\Omega)$. The adjoint d^*, as an unbounded operator $L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega)$, has $\mathring{H}^*\Lambda^k(\Omega)$ as its domain and coincides with the operator δ defined in* (2.8)*.*

Applying the results from the previous section and Theorem 2.3.1, we get the mixed Hodge Laplace problem on the de Rham complex: find the unique

$(\sigma, u, p) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega) \times \mathfrak{H}^k$ such that

$$\begin{aligned} \sigma = \delta u,\ d\sigma + \delta du = f - p \quad & \text{in } \Omega, \\ \operatorname{tr} \star u = 0,\ \operatorname{tr} \star du = 0 \quad & \text{on } \partial\Omega, \\ u \perp \mathfrak{H}^k. & \end{aligned} \tag{2.12}$$

Using proxy fields and symmetric properties of the problem, a generic method for solving (3.9) in the case $k = n$ equivalently solves the Poisson equation with natural Dirichlet boundary conditions. In this case $du = 0$ and $p = 0$, thus the mixed Hodge Laplace problem on the de Rham complex simplifies to: find the unique $(\sigma, u, p) \in H\Lambda^{n-1}(\Omega) \times H\Lambda^n(\Omega) \times \mathfrak{H}^n$ such that

$$\begin{aligned} \sigma = \delta u,\ d\sigma = f \quad & \text{in } \Omega, \\ \operatorname{tr} \star u = 0, \quad & \text{on } \partial\Omega. \end{aligned} \tag{2.13}$$

Let (Λ_n, d) be a finite dimensional subcomplex of the de Rham complex, then a discrete version of (2.13) can be written: find the unique $(\sigma_h, u_h, p_h) \in \Lambda_h^{n-1}(\Omega) \times \Lambda_h^n(\Omega) \times \mathfrak{H}_h^n$ such that

$$\begin{aligned} \sigma_h = \delta_h u_h,\ d\sigma_h = P_h f \quad & \text{in } \Omega, \\ \operatorname{tr} \star u_h = 0, \quad & \text{on } \partial\Omega, \end{aligned} \tag{2.14}$$

where $P_h f$ is the L^2 projection of f on to the discrete space parameterized by h. Note that δ_h is distinct from δ, and follows from the definition of the abstract discrete problem.

Finite element differential forms

For the remainder of the paper it is assumed that all approximating subcomplexes of the de Rham complex are constructed as combinations of the polynomial spaces of k-forms, $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$. For a detailed discussion on these spaces and construction of Hilbert complexes using these spaces, see [5]. We also have useful properties in the case $k = n$,

$$\mathcal{P}_r^- \Lambda^n = \mathcal{P}_{r-1}\Lambda^n,$$
$$\mathcal{P}_r^- \Lambda^0 = \mathcal{P}_r\Lambda^0.$$

For a shape-regular, conforming triangulation $\mathcal{T}_h$ of $\Omega, \Lambda_h^k(\Omega) \subset L^2\Lambda^k(\Omega)$ will denote a space of k-forms constructed using specific combinations of the these spaces on $\mathcal{T}_h$. For an element $T \in \mathcal{T}_h$, we set $h_T := \text{diam}(T)$. We do not discuss the details of these spaces further, but specific properties will be explained when necessary.

Bounded Cochain Projections

Bounded cochain projections and their approximation properties are necessary in the analysis of both uniform and adaptive FEMs in the FEEC framework. Properties of three different interpolation operators will be important in our analysis. The three operators and respective notation that we will use are as follows: the canonical projections I_h defined in [4, 5], the smoothed projection operator π_h from [5], and the commuting quasi-interpolant Π_h, as defined in [18] with ideas similar to [39, 40, 14]. Some cases will require a simple projection, and $P_h f$ also written f_h, will denote the L^2-projection of f on to the discrete space parameterized by h. $f_{\mathfrak{B}_h}$ will denote the L^2 projection of f onto the $\mathfrak{B}$ component of the discrete space parameterized by h. Note $f_{\mathfrak{B}_h} = f_h$ when $k = n$.

For the remainder of the paper, $\|\cdot\|$ will denote the $L^2\Lambda^k(\Omega)$ norm, and when taken on specific elements of the domain, T, we write $\|\cdot\|_T$. For all other norms, such as $H\Lambda^k(\Omega)$ and $H^1\Lambda^k(\Omega)$, we write $\|\cdot\|_{H\Lambda^k(\Omega)}$ and $\|\cdot\|_{H^1\Lambda^k(\Omega)}$ respectively.

Lemma 2.3.2. *Suppose $\tau \in H^1\Lambda^{n-1}(\Omega)$, and I_h is the canonical projection operator defined in [4, 5]. Let $\Lambda_h^{n-1}(\Omega)$ and $\Lambda_h^n(\Omega)$ be taken as above. Then I_h is a projection onto $\Lambda_h^n(\Omega), \Lambda_h^{n-1}(\Omega)$ and satisfies*

$$\|\tau - I_h\tau\|_T \leq Ch_T\|\tau\|_{H^1\Lambda^{n-1}(T)}, \quad \forall T \in \mathcal{T}_h, \tag{2.15}$$

$$I_h d = dI_h \tag{2.16}$$

Proof. The first part is comes from Equation (5.4) in [4]. The second part follows the construction of I_h. □

Lemmas 2.3.3 and 2.3.4 deal with important properties of the canonical projections. In each case we assume $f_h, u_h \in \Lambda_h^n(\Omega)$, and let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$.

Lemma 2.3.3. *Let $T \in \mathcal{T}_H$, then*

$$\int_T (f_h - I_H f_h) = 0. \tag{2.17}$$

Proof. In the case $k = n$, the canonical projections are L^2 bounded. Let $\omega \in P_r\Lambda^n(T)$. This is sufficient since $P_r^-\Lambda^n(T) = P_{r-1}\Lambda^n(T)$. By definition [4]:

$$\int_T (\omega - I_H\omega) \wedge \eta = 0, \;\; \eta \in P_r^-\Lambda^0(T) = P_r\Lambda^0(T).$$

Set $\eta = 1$ and this completes proof. □

Lemma 2.3.4. *Let $T \in \mathcal{T}_H$, then*

$$\langle (I_h - I_H)u_h, f_h\rangle_T = \langle u_h, (I_h - I_H)f_h\rangle_T. \tag{2.18}$$

Proof.

$$\int_T (u_h - I_H u_h) \wedge \eta = 0, \quad \eta \in P_r\Lambda^0(T) = \star P_r\Lambda^n(T).$$

Thus,

$$\langle (I_h - I_H)u_h, f_h\rangle_T = \langle (I_h - I_H)u_h, (I_h - I_H)f_h\rangle_T.$$

The proof is completed using the same logic on the $I_H u_h$ term. □

The next lemma is taken directly from [18], and will be a key tool in developing an upper bound for the error.

Lemma 2.3.5. *Assume $1 \leq k \leq n$, and $\phi \in H\Lambda^{k-1}(\Omega)$ with $\|\phi\| \leq 1$. Then there exists $\varphi \in H^1\Lambda^{k-1}(\Omega)$ such that $d\varphi = d\phi, \Pi_H d\phi = d\Pi_H\phi = d\Pi_H\varphi$, and*

$$\sum_{T\in T_h} h_T^{-2}\|\varphi - \Pi_H\varphi\|_T^2 + h_T^{-1}\|\mathrm{tr}(\varphi - \Pi_H\varphi)\|_{\partial T}^2 \le C.$$

Proof. See Lemma 6 in [18]. □

The following theorem is a special case of Theorem 3.5 from [5]. Rather than showing the result on an abstract Hilbert Complex with a general cochain projection, we use the de Rham complex and the smoothed projection operator π_h in order to use uniform boundedness of the cochain projection.

Theorem 2.3.6. *Assume $\Lambda_h^k(\Omega)$ is a subcomplex of $H\Lambda^k(\Omega)$ as described above, and let π_h be the smoothed projection operator. Then*

$$\|(I - P_{\mathfrak{H}^k})q\|_V \le \|(I - \pi_h^k)P_{\mathfrak{H}^k}q\|_V, \quad q \in \mathfrak{H}_h^k, \tag{2.19}$$

then combining the above with the triangle inequality,

$$\|q\|_V \le c\|P_{\mathfrak{H}^k}q\|_V, \quad q \in \mathfrak{H}_h^k. \tag{2.20}$$

Proof. Since the de Rham complex is a bounded closed Hilbert complex, (2.19) is directly from [5]. (2.19) with the triangle inequality implies

$$\|q\|_V - \|P_{\mathfrak{H}^k}q\|_V \le \|(I - \pi_h^k)\|\|P_{\mathfrak{H}^k}q\|_V, \quad q \in \mathfrak{H}_h^k,$$

thus,

$$\|q\|_V \le (\|(I - \pi_h^k)\| + 1)\|P_{\mathfrak{H}^k}q\|_V, \quad q \in \mathfrak{H}_h^k.$$

By construction π_h is uniformly bounded with respect to h and therefore $(\|(I - \pi_h^k)\| + 1)$ can be replaced with a generic constant not dependent on the triangulation. In Corollary 2.3.8 a similar result is used to relate the harmonics on two discrete complexes. In this case the canonical projection I_h can be used as a map between the two complexes, and $(I - I_h)$ is clearly uniformly bounded with respect to h. □

Theorem 2.3.7 will be essential in dealing with the harmonic forms in the proof of a continuous upper-bound. The corollary will be used identically when proving a discrete upper-bound. For use in our next two results we introduce an operator δ and one of its important properties. Let A, B be $n < \infty$ dimensional, closed subspaces of a Hilbert space W, and let

$$\delta(A, B) = \sup_{x \in A, \|x\|=1} \|x - P_B x\|,$$

then [18], Lemma 2 which takes the original ideas from [27], shows

$$\delta(A, B) = \delta(B, A). \tag{2.21}$$

Theorem 2.3.7. *Assume $\mathfrak{H}_H^k$ and $\mathfrak{H}^k$ have the same finite dimensionality. The there exist a constant $C_{\mathfrak{H}^k}$ dependent only on $\mathcal{T}_0$, such that*

$$\delta(\mathfrak{H}^k, \mathfrak{H}_H^k) = \delta(\mathfrak{H}_H^k, \mathfrak{H}^k) \leq C_{\mathfrak{H}^k} < 1. \tag{2.22}$$

Proof. Given that $\mathfrak{H}_H^k$ and $\mathfrak{H}^k$ have the same finite dimensionality we can apply (2.21) to prove the first equality.

For the second part,

$$\delta(\mathfrak{H}_H^k, \mathfrak{H}^k) = \sup_{x \in \mathfrak{H}_H^k, \|x\|=1} \|x - P_{\mathfrak{H}^k} x\|,$$

for any $x \in \mathfrak{H}_H^k$ with $\|x\| = 1$, (2.20) implies,

$$C \leq \|P_{\mathfrak{H}^k} x\|, \quad 0 < C < 1.$$

Now, by orthogonality of the projection, we have

$$\delta(\mathfrak{H}_H^k, \mathfrak{H}^k) \leq \sqrt{1 - C^2} = C_{\mathfrak{H}^k} < 1.$$

□

Corollary 2.3.8.

$$\delta(\mathfrak{H}_h^k, \mathfrak{H}_H^k) = \delta(\mathfrak{H}_H^k, \mathfrak{H}_h^k) \leq \tilde{C}_{\mathfrak{H}^k} < 1. \tag{2.23}$$

Proof. The proof follows the same logic as Theorem 2.3.7. The only difference is that the harmonics are compared on two discrete complexes $\mathfrak{H}_h^k$ and $\mathfrak{H}_H^k$, and therefore I_h is used rather than π_h. $\square$

2.3.3 Adaptive Finite Elements Methods

Given an initial triangulation, $\mathcal{T}_0$, the adaptive procedure will generate a nested sequence of triangulations $\mathcal{T}_k$ and discrete solutions σ_k, by looping through the following steps:

$$\mathsf{Solve} \longrightarrow \mathsf{Estimate} \longrightarrow \mathsf{Mark} \longrightarrow \mathsf{Refine} \tag{2.24}$$

The following subsection will describe details of these steps.

Approximation Procedure

We assume access to a routine SOLVE, which can produce solution to (2.6) given a triangulation, problem data, and a desired level of accuracy. For the ESTIMATE step we will introduce error indicators η_T on each element $T \in \mathcal{T}_k$. In the MARK step we will use Dörfler Marking strategy [20]. An essential feature of the marking process is that the summation of the error indicators on the marked elements exceeds a user defined marking parameter θ.

We assume access to an algorithm REFINE in which marked elements are subdivided into two elements of the same size, resulting in a conforming, shape-regular mesh. Triangles outside of the original marked set may be refined in order to maintain conformity. Bounding the number of such refinements is important in showing optimality of the method. Along these lines, Stevenson [44] showed certain bisection algorithms developed in two-dimensions can be extended to n-simplices of arbitrary dimension satisfying

(1)$\{\mathcal{T}_k\}$ is shape regular and the shape regularity depends only on $\mathcal{T}_0$,

(2)$\#\mathcal{T}_k \leq \#\mathcal{T}_0 + C\#\mathcal{M}$, where $\mathcal{M}$ is the collection of all marked triangles going from $\mathcal{T}_0$ to $\mathcal{T}_k$.

Approximation of the Data

A measure of data approximation will be necessary in establishing a quasi-orthogonality result. Following ideas of [31], data oscillation will be defined as follows,

Definition 2.3.9. (*Data oscillation*) Let f $\in L^2\Lambda^k(\Omega)$, and $\mathcal{T}_h$ be a conforming triangulation of Ω. Let h_T be the mesh-size for a given $T \in \mathcal{T}_h$. We define

$$osc(f, \mathcal{T}_h) := \Big(\sum_{T \in \mathcal{T}_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2 \Big)^{1/2}.$$

Stevenson [44] generalized the ideas of [8] to show that approximation of data can be done in an optimal way regardless of dimension. Using the approximation spaces $(\mathcal{A}^s, \|\cdot\|_{\mathcal{A}^s})$ and $(\mathcal{A}^s_o, \|\cdot\|_{\mathcal{A}^s_o})$ as in [8] we recall the result.

Theorem 2.3.10. (Generalized Binev, Dahmen and DeVore) *Given a tolerance ϵ, $f \in L^2\Lambda^n(\Omega)$ and a shape regular triangulation $\mathcal{T}_0$, there exists an algorithm*

$$\mathcal{T}_H = \mathsf{APPROX}(f, \mathcal{T}_0, \epsilon),$$

such that

$$\operatorname{osc}(f, \mathcal{T}_H) \leq \epsilon, \quad \textit{and} \quad \#\mathcal{T}_H - \#\mathcal{T}_0 \leq C\|f\|^{1/s}_{\mathcal{A}^{1/s}_o} \epsilon^{-1/s}.$$

As in the case of [13], the analysis of convergence and procedure will follow [12], and the optimality will follow [43].

2.4 Quasi-Orthogonality

The main difficulty for mixed finite element methods is the lack of minimization principle, and thus the failure of orthogonality. In [13], a quasi-orthogonality property is proven using the fact that the error is orthogonal to the divergence free subspace. In this section we follow a similar line of reasoning to prove a quasi-orthogonality result between the solutions to (2.13) and (2.14). Analogous to [13], our result uses the fact that $\sigma - \sigma_h$ is orthogonal to the subspace $\mathfrak{Z}^{n-1}_h \subset H\Lambda^{n-1}_h(\Omega)$.

Solutions of Hodge Laplace problems on nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ will frequently be compared. Nested in the sense that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. For a given $f \in H\Lambda^k(\Omega)$, let $\mathcal{L}^{-1}f$ denote the solutions of (2.12). Let $\mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$ and $\mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$ denote the solutions to the discrete problems on $\mathcal{T}_h$ and $\mathcal{T}_H$ respectively. Set the following triples, $(u, \sigma, p) = \mathcal{L}^{-1}f$, $(u_h, \sigma_h, p_h) = \mathcal{L}_h^{-1}f_{\mathfrak{B}_h}$, $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) = \mathcal{L}_h^{-1}f_{\mathfrak{B}_H}$ and $(u_H, \sigma_H, p_H) = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$. In the case $k = n$, as with general $\mathfrak{B}$ problems, the harmonic component will be zero in each of these solutions. When we are only interested in σ we will abuse this notation by witting $\sigma = \mathcal{L}^{-1}f$.

Lemma 2.4.1. *Given $f \in L^2\Lambda^k(\Omega)$, such that $f \in \mathfrak{B}^k$, and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then*

$$\langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle = 0. \tag{2.25}$$

Proof. Since $\tilde{\sigma}_h - \sigma_H \in V_h^{k-1} \subset V^{k-1}$, (2.5) implies

$$\langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle = \langle u - u_h, d(\tilde{\sigma}_h - \sigma_H) \rangle,$$

and the harmonic terms are zero since these are $\mathfrak{B}$ problems,

$$\begin{aligned} &= \langle u - u_h, f_{\mathfrak{B}_H} - f_{\mathfrak{B}_H} \rangle \\ &= 0. \end{aligned}$$

□

Theorem 2.4.2. *Given $f \in L^2\Lambda^n(\Omega)$ and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then*

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle \leq \sqrt{C_0} \|\sigma - \sigma_h\| \operatorname{osc}(f_{\mathfrak{B}_h}, \mathcal{T}_H), \tag{2.26}$$

and for any $\delta > 0$,

$$(1 - \delta)\|\sigma - \sigma_h\|^2 \leq \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C_0}{\delta} \operatorname{osc}^2(f_{\mathfrak{B}_h}, \mathcal{T}_H). \tag{2.27}$$

Proof. By (2.25) we have

$$\begin{aligned}\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle &= \langle \sigma - \sigma_h, \sigma_h - \tilde{\sigma}_h \rangle + \langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle \\ &= \langle \sigma - \sigma_h, \sigma_h - \tilde{\sigma}_h \rangle \\ &\leq \|\sigma - \sigma_h\| \|\sigma_h - \tilde{\sigma}_h\|.\end{aligned}$$

And then by the discrete stability result, Theorem 2.5.4, we have

$$\leq \sqrt{C_0} \|\sigma - \sigma_h\| \text{osc}(f_{\mathfrak{B}_h}, \mathcal{T}_H).$$

(2.27) follows standard arguments and is identical to [13] (3.4) □

2.5 Continuous/Discrete Stability

In this section we will prove stability results for approximate solutions to the σ portion of the Hodge Laplace problem. Theorem 2.5.1 gives a stability result for particular solutions of the Hodge de Rham problem that will be useful in bounding the approximation error in Section 2.6. Theorem 2.5.4 will prove the discrete stability result used in Theorem 2.4.2. The basic structure of these arguments will follow [13], but key modifications are introduced in order to generalize the dimensionality and topology of the main results.

Theorem 2.5.1. (Continuous Stability Result) *Given $f \in L^2\Lambda^n(\Omega)$, let $\mathcal{T}_h$ be a triangulation of Ω. Set $(\sigma, u, p) = \mathcal{L}^{-1} f$ and $(\tilde{\sigma}, \tilde{u}, \tilde{p}) = \mathcal{L}^{-1} f_{\mathfrak{B}_h}$, then*

$$\|\sigma - \tilde{\sigma}\| \leq C \text{osc}(f, \mathcal{T}_h). \tag{2.28}$$

Proof. The harmonic terms are zero since $f, f_{\mathfrak{B}_h} \in \mathfrak{B}^k$, thus

$$\|\sigma - \tilde{\sigma}\|^2 = \langle d(\sigma - \tilde{\sigma}), u - \tilde{u} \rangle = \langle f - f_{\mathfrak{B}_h}, u - \tilde{u} \rangle.$$

Let $v = u - \tilde{u}$. Since $v \in \mathfrak{B}^k$ and $\|\delta v\| = \|\text{grad } v\| = \|\sigma - \tilde{\sigma}\|$, we have $v \in H^1\Lambda^n(\Omega)$.

Restricting v to an element $T \in \mathcal{T}_h$, we have $v \in H^1\Lambda^n(T)$, thus

$$\|\sigma - \tilde{\sigma}\|^2 = \langle f - f_{\mathfrak{B}_h}, v\rangle = \sum_{T\in\mathcal{T}_h} \langle f - f_{\mathfrak{B}_h}, v - I_h v\rangle_T.$$

Applying (2.15),

$$\begin{aligned}
&\leq C \sum_{T\in\mathcal{T}_h} h_T \|f - f_{\mathfrak{B}_h}\|_T \|v\|_{H^1\Lambda^n(T)} \\
&= C \sum_{T\in\mathcal{T}_h} h_T \|f - f_{\mathfrak{B}_h}\|_T (\|u - \tilde{u}\|_T + \|\delta(u - \tilde{u})\|_T) \\
&\leq C (\sum_{T\in\mathcal{T}_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2} (\sum_{T\in\mathcal{T}_h} (\|u - \tilde{u}\|_T + \|\delta(u - \tilde{u})\|_T)^2)^{1/2},
\end{aligned}$$

and $v \in H^1\Lambda^n(\Omega)$ allows us to to combine terms of the summation,

$$\leq C (\sum_{T\in\mathcal{T}_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2} (\|u - \tilde{u}\| + \|\delta(u - \tilde{u})\|).$$

Since $u - \tilde{u} \in \mathfrak{B}^k$, $\|u - \tilde{u}\| = \langle u - \tilde{u}, d\tau\rangle$ for some $\tau \in \mathfrak{Z}^\perp$ with $\|d\tau\| = 1$, thus

$$= C (\sum_{T\in\mathcal{T}_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2} \langle(\sigma - \tilde{\sigma}), \tau\rangle_\Omega + \|\sigma - \tilde{\sigma}\|).$$

Then applying Poincaré on τ:

$$= C\|\sigma - \tilde{\sigma}\| (\sum_{T\in\mathcal{T}_h} \|h_T(f - f_{\mathfrak{B}_h})\|_T^2)^{1/2}.$$

Divide through by $\|\sigma - \tilde{\sigma}\|$ to complete proof. □

The following is Lemma 4 in [18], and is a special case of Theorem 1.5 of [30]. It is related to the bounded invertibility of d, and will be an important tool in proving discrete stability.

Lemma 2.5.2. *Assume that B is a bounded Lipschitz domain in $\mathbb{R}^n$ that is homeomorphic to a ball. Then the boundary value problem $d\varphi = g \in L_2\Lambda^k(B)$ in*

$B, \mathrm{tr}\,\varphi = 0$ *on* ∂B *has a solution* $\varphi \in H_0^1\Lambda^{k-1}(B)$ *with* $\|\varphi\|_{H^1\Lambda^{k-1}(B)} \leq C\|g\|_B$ *if and only if* $dg = 0$ *in* B*, and in addition,* $\mathrm{tr}\; g = 0$ *on* ∂B *if* $0 \leq k \leq n-1$ *and* $\int_B g = 0$ *if* $k = n$ *.*

The next lemma is an intermediate step in proving the discrete stability result. The general structure follows [13] and applies Lemma 2.5.2 in order to find a sufficiently smooth function that is essentially a bounded inverse of d for the approximation error of u_h on $\mathcal{T}_H$.

Lemma 2.5.3. *Let* $\mathcal{T}_h$*,* $\mathcal{T}_H$ *be nested conforming triangulations and let* σ_h, σ_H *be the respective solutions to* (2.6) *with data* $f \in L^2\Lambda^n(\Omega)$*. Then for any* $T \in \mathcal{T}_H$

$$\|u_h - I_H u_h\|_T \leq \sqrt{C_0} h_T \|\sigma_h\|_T. \tag{2.29}$$

Proof. Let $g_\Omega = u_h - I_H u_h = (I_h - I_H)u_h \in L^2\Lambda^n(\Omega)$. Then, for any $T \in \mathcal{T}_H$ let $g = \mathrm{tr}_T g_\Omega \in L^2\Lambda^n(T)$, and by Lemma 2.3.3, $\int_T g = 0$. Thus Lemma 2.5.2 can be applied to find $\tau \in H_0^1\Lambda^{n-1}(T)$, such that:

$$d\tau = (I_h - I_H)u_h, \text{ on } T$$
$$\|\tau\|_{H^1\Lambda^{n-1}(T)} \leq C\|(I_h - I_H)u_h\|_T.$$

Extend τ to $H^1\Lambda^{n-1}(\Omega)$ by zero and then, by Lemma 2.3.4,

$$\|(I_h - I_H)u_h\|_T^2 = \langle (I_h - I_H)u_h, d\tau\rangle_T = \langle u_h, (I_h - I_H)d\tau\rangle_T.$$

Then by Lemma 2.3.2, and locality of τ,

$$= \langle u_h, d(I_h - I_H)\tau\rangle_\Omega = \langle \sigma_h, (I_h - I_H)\tau\rangle_\Omega.$$

Then again by locality of τ,

$$= \langle \sigma_h, (I_h - I_H)\tau\rangle_T \leq \|\sigma_h\|_T(\|\tau - I_h\tau\|_T + \|\tau - I_H\tau\|_T).$$

Since I_h is uniformly bounded with respect to h on $\Lambda_h^n(\Omega)$,

$$\le Ch_T\|\sigma_h\|_{0,T}\|\tau\|_{H^1\Lambda^{n-1}(T)} \le Ch_T\|\sigma_h\|_T\|(I_h - I_H)u_h\|_T.$$

Cancel one power of $\|(I_h - I_H)u_h\|_T$ to complete the proof.

□

Theorem 2.5.4. (Discrete Stability Result) *Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be nested conforming triangulations. Let $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) = \mathcal{L}_h^{-1} f_{\mathfrak{B}_H}$ and $(u_h, \sigma_h, p_h) = \mathcal{L}_h^{-1} f_{\mathfrak{B}_h}$, with $f \in L^2\Lambda^n(\Omega)$. Then there exists a constant such that*

$$\|\sigma_h - \tilde{\sigma}_h\| \le C\mathrm{osc}(f_{\mathfrak{B}_h}, \mathcal{T}_H) \tag{2.30}$$

Proof. From 2.6, and since $p_h, \tilde{p}_h = 0$, we have

$$\langle \sigma_h - \tilde{\sigma_h}, \tau_h\rangle = \langle u_h - \tilde{u}_h, d\tau_h\rangle, \quad \forall \tau_h \in \Lambda_h^{k-1}, \tag{2.31}$$

$$\langle d(\sigma_h - \tilde{\sigma_h}), v_h\rangle = \langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h\rangle, \quad \forall v_h \in \Lambda_h^k. \tag{2.32}$$

Next set $\tau_h = \sigma_h - \tilde{\sigma}_h$ in (2.31), and $v_h = u_h - \tilde{u}_h$ in (2.32) to obtain:

$$\|\sigma_h - \tilde{\sigma}_h\|^2 = \langle u_h - \tilde{u}_h, d(\sigma_h - \tilde{\sigma}_h)\rangle = \langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h\rangle,$$

and since $(f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}) \perp L^2\Lambda_H^k(\Omega)$ when $k = n$, we have

$$\langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h\rangle = \langle f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}, v_h - I_H v_h\rangle.$$

Then by Lemma 2.5.3, we have:

$$\begin{aligned}
\|\sigma_h - \tilde{\sigma}_h\|^2 &= \sum_{T\in\mathcal{T}_H} \langle v_h - I_H v_h, f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\rangle_T \\
&\le \sum_{T\in\mathcal{T}_H} \|f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\|_T \|v_h - I_H v_h\|_T \\
&\le C \sum_{T\in\mathcal{T}_H} h_T \|f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\|_T \|\sigma_h - \tilde{\sigma}_h\|_T \\
&\le C (\sum_{T\in\mathcal{T}_H} h_T^2 \|f_{\mathfrak{B}_h} - f_{\mathfrak{B}_H}\|_T^2)^{1/2} \|\sigma_h - \tilde{\sigma}_h\|
\end{aligned}$$

Then cancel one $\|\sigma_h - \tilde{\sigma}_h\|$ to complete the proof.

□

2.6 Error Estimator, Upper and Lower bounds

In this section we introduce the a posteriori error estimators used in our adaptive algorithm. The first two terms of the estimator follow [3, 13], and a third term is introduced in order to construct a more practical and efficient algorithm. Next, we prove bounds on these estimators and a continuity result, both of which are key ingredients in showing the convergence and optimality of our adaptive method.

2.6.1 Error Estimator: Definition, Lower bound and Continuity

Definition 2.6.1. *(Element Error Estimator)* Let $T \in \mathcal{T}_H, f \in L^2\Lambda^k(\Omega)$, and $\sigma_H = \mathcal{L}^{-1} f_{\mathfrak{B}_H}$. Let the jump in τ over an element face be denoted by $[[\tau]]$. For element faces on $\partial\Omega$ we set $[[\tau]] = \tau$. The element error indicator is defined as

$$\eta_T^2(\sigma_H) = h_T \|[[tr \ \star \sigma_H]]\|_{\partial T}^2 + h_T^2 \|\delta\sigma_H\|_T^2 + h_T^2 \|f - d\sigma_H\|_T^2$$

For a subset $\tilde{\mathcal{T}}_H \subset \mathcal{T}_H$, define

$$\eta^2(\sigma_H, \tilde{\mathcal{T}}_H) := \sum_{T \in \tilde{\mathcal{T}}_H} \eta_T^2(\sigma_H)$$

Theorem 2.6.2. (Lower Bound) *Given $f \in L^2\Lambda^k(\Omega)$ and a shape regular triangulation $\mathcal{T}_H$, let $\sigma = \mathcal{L}^{-1} f$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then there exists a constant dependent only on the shape regularity of $\mathcal{T}_H$ such that*

$$C_2 \eta^2(\sigma_H, \mathcal{T}_H) \leq \|\sigma - \sigma_H\|^2 + C_2 \text{osc}^2(f, \mathcal{T}_H). \tag{2.33}$$

Proof. In proving a lower bound, in [18] it is shown that

$$h_T \|\delta \sigma_H\|_T \leq C \|\sigma - \sigma_H\|_T,$$
$$h_T^{1/2} \|[[\text{tr} \star \sigma_H]]\|_{\partial T} \leq C \|\sigma - \sigma_H\|_{\mathcal{T}_t},$$

where $\mathcal{T}_t$ is the set of all triangles sharing a boundary with T. The first is equation (5.7) and the second is a result of equation (5.12) in [18].

Sum terms and add oscillation to both sides to complete the error-indicator term. Notice, by conformity of the triangulation, the summation of the $\|\sigma - \sigma_H\|_{\mathcal{T}_t}$ terms can at most be some fixed multiple of $\|\sigma - \sigma_H\|$ depending on the dimensionality of the problem.

□

The following lemma will be important in proving a continuity result used in showing convergence of our adaptive algorithm. It is nearly identical to an estimator efficiency proof in [18], but the subtle difference is that we make use of σ_H, the solution on the less refined mesh, and σ is not used in our arguments.

Lemma 2.6.3. *Given $f \in L^2\Lambda^k(\Omega)$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $\sigma_h = \mathcal{L}_h^{-1} f_{\mathfrak{B}_h}$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then for $T \in \mathcal{T}_h$*

$$C_2 \sum_{T \in \mathcal{T}_h} (h_T \|[[\text{tr} \ \star (\sigma_h - \sigma_H)]]\|_{\partial T}^2 + h_T^2 \|\delta(\sigma_h - \sigma_H)\|_T^2) \leq \|\sigma_h - \sigma_H\|^2. \tag{2.34}$$

Proof. Here we will closely follow [18] in applying the “bubble function” technique of Verfürth[45] in order to bound residual terms in the FEEC framework. For $T \in \mathcal{T}_h$ one can construct a corresponding bubble function $b_T \in W^1_\infty(\Omega)$ with $\operatorname{supp}(b_T) = T$, and the property that for any polynomial form v of arbitrary but uniformly bounded degree defined on T, we have

$$\|v\|_T \simeq \|\sqrt{b_T} v\|_T. \tag{2.35}$$

For $n-1$ dimensional faces $e = T_1 \cap T_2$, with $T_1, T_2 \in \mathcal{T}_h$, and T_2 void (see [18]) on $\partial\Omega$, one can construct a corresponding edge bubble function $b_e \in W^1_\infty(\Omega)$ with $\operatorname{supp}(b_e) = T_1 \cup T_2$ and the property that for any polynomial form v of arbitrary but uniformly bounded degree defined on e, we have

$$\|v\|_e \simeq \|\sqrt{b_e} v\|_e. \tag{2.36}$$

Given a k-form v defined on an $n-1$ dimensional face $e = T_1 \cap T_2$, one can construct χ_v to be a polynomial extension of v to $T_1 \cup T_2$ such that

$$\|\chi_v\|_{(T_1 \cup T_2)} \le C h_T^{1/2} \|v\|_e, \tag{2.37}$$

where h_T can be either h_{T_1} or h_{T_2} since they are neighbors which have sizes related by a shape regularity constant

Let $\psi = b_T(\delta(\sigma_h - \sigma_H))$, which by construction of b_T will be zero on ∂T. Applying integration by parts, we have

$$\|\delta(\sigma_h - \sigma_H)\|_T^2 \simeq \langle \delta(\sigma_h - \sigma_H), \psi \rangle = \langle \sigma_h - \sigma_H, d\psi \rangle, \tag{2.38}$$

and then applying an inverse inequality $\|d\psi\|_T \le C h_T^{-1} \|\psi\|_T$,

$$h_T \|\delta(\sigma_h - \sigma_H)\| \le C \|\sigma_h - \sigma_H\|_T. \tag{2.39}$$

For an element face e, shared by elements T_1, T_2, we have

$$\begin{aligned}\|[[\text{tr } \star(\sigma_h - \sigma_H)]]\|_e^2 &\simeq \langle b_e \star \psi, [[\text{tr } \star(\sigma_h - \sigma_H)]]\rangle \\ &= \int_e \text{tr } (b_e \chi_\psi) \wedge [[\text{tr } \star(\sigma_h - \sigma_H)]] \\ &= \langle d(b_e\chi_\psi), \sigma_h - \sigma_H\rangle_{T_1\cup T_2} - \langle b_e\chi_\psi, \delta_h(\sigma_h - \sigma_H)\rangle_{T_1\cup T_2},\end{aligned} \tag{2.40}$$

where δ_h is defined to be δ evaluated elementwise on elements of $\mathcal{T}_h$. The necessity of this additional definition is that neither σ_h or σ_H are in $H^*\Lambda^{k-1}(\Omega)$ globally, but σ_h and σ_H are in $H^*\Lambda^{k-1}$ when restricted to individual elements of $\mathcal{T}_h$. Next, using the inverse inequality $\|d(b_e\chi_\psi)\|_T \le Ch_T^{-1}\|b_e\chi_\psi\|_T$, we have

$$\le C\|[[\text{tr} \star(\sigma_h - \sigma_H)]]\|_e(h_T^{-1/2}\|\sigma_h - \sigma_H\|_{T_1\cup T_2} + h_T^{1/2}\|\delta_h(\sigma_h - \sigma_H)\|_{T_1\cup T_2}), \tag{2.41}$$

where h_T can be either h_{T_1} or h_{T_2} for the same reasons as mentioned above. Applying (2.39) we have

$$h_T^{1/2}\|[[\text{tr} \star(\sigma_h - \sigma_H)]]\|_e \le C\|\sigma_h - \sigma_H\|_{T_1\cup T_2}. \tag{2.42}$$

Squaring and summing (2.39) and (2.42) for every element will complete the proof. The edges not on the boundary of Ω will be included twice in the summation, and the overlap of the $C\|\sigma_h - \sigma_H\|_{T_1\cup T_2}$ terms can be bounded by a multiple depending on n. □

Theorem 2.6.4. (Continuity of the Error Estimator) *Given $f \in L^2\Lambda^k(\Omega)$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $\sigma_h = \mathcal{L}_h^{-1} f_{\mathfrak{B}_h}$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then we have:*

$$\beta(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h)) \le \|\sigma_h - \sigma_H\|^2 + \text{osc}^2(f_{\mathfrak{B}_h}, \mathcal{T}_H) \tag{2.43}$$

Proof. Applying the triangle inequality to (2.34) gives

$$\|\sigma_h - \sigma_H\|^2 \geq C\big(\sum_{T\in\mathcal{T}_h}(h_T\|[[\mathrm{tr}\ \star(\sigma_h)]]\|^2_{\partial T} + h_T^2\|\delta(\sigma_h)\|^2_T) \\ - \sum_{T\in\mathcal{T}_h}(h_T\|[[\mathrm{tr}\ \star(\sigma_H)]]\|^2_{\partial T} + h_T^2\|\delta(\sigma_H)\|^2_T)\big).$$

In terms of the error indicator this can be written

$$\|\sigma_h - \sigma_H\|^2 \geq C\big(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h) \\ - \sum_{T\in\mathcal{T}_h} h_T^2\|f - d\sigma_h\|^2_T + \sum_{T\in\mathcal{T}_h} h_T^2\|f - d\sigma_H\|^2_T\big).$$

An additional application of the triangle inequality yields

$$\|\sigma_h - \sigma_H\|^2 \geq C(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h) - \sum_{T\in\mathcal{T}_h} h_T^2\|d(\sigma_h - \sigma_H)\|^2_T)).$$

Since $f_{\mathfrak{B}_h}, f_{\mathfrak{B}_H} \in L^2\Lambda^k(\Omega)$ globally, using the summation on the coarser mesh completes the proof

$$\|\sigma_h - \sigma_H\|^2 + \sum_{T\in\mathcal{T}_H} h_T^2\|d(\sigma_h - \sigma_H)\|^2_T \geq C(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h)).$$

□

2.6.2 Continuous and Discrete Upper Bounds

The following proofs have a similar structure to the continuous and discrete upper bounds proved in [3, 13]. A key element of the proof will be comparisons between the discrete solution $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$ and the solution to the intermediate problem, $\tilde{\sigma} = \mathcal{L}^{-1} f_{\mathfrak{B}_H}$. We begin by looking the orthogonal decomposition of $\tilde{\sigma} - \sigma_H$,

$$\tilde{\sigma} - \sigma_H = (\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H) - P_{\mathfrak{B}^{k-1}}\sigma_H - P_{\mathfrak{H}^{k-1}}\sigma_H$$

which allows the norm to be rewritten

$$\|\tilde{\sigma} - \sigma_H\|^2 = \|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 + \|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2 + \|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2.$$

Lemmas 2.6.5, 2.6.6 and 2.6.7 will each bound a portion of this orthogonal decomposition. Then Theorem 2.6.8 will combine these results in proving the desired error bound.

Lemma 2.6.5. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\tilde{\sigma} = \mathcal{L}^{-1} f_{\mathfrak{B}_H}$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then*

$$\|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 = 0. \tag{2.44}$$

Proof. Since we are only dealing with $\mathfrak{Z}^\perp$, we have

$$\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H = \delta v,\ v \in H\Lambda^k(\Omega).$$

Thus,

$$\|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 = \langle \tilde{\sigma} - \sigma_H, \delta v\rangle = \langle d(\tilde{\sigma} - \sigma_H), v\rangle.$$

In the case of $\mathfrak{B}$ problems the harmonics are void and

$$\langle d(\tilde{\sigma} - \sigma_H), v\rangle = \langle f_{\mathfrak{B}_H} - f_{\mathfrak{B}_H}, v\rangle = 0.$$

□

The next lemma uses the quasi-interpolant Π_H described in [18], and also applies integration by parts in the same standard fashion that [18] use when bounding error measured in the natural norm, $\|u - u_h\|_{H\Lambda^k(\Omega)} + \|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|p - p_h\|$. In [18], coercivity of the bilinear-form is used to separate components of the error, whereas here we simply analyze the orthogonal decomposition of $\sigma - \sigma_H$. In [18], the Galerkin orthogonality implied by taking the difference between the continuous and discrete problems is employed in order to make use of Π_h. Here we are able to introduce the quasi-interpolant by simply using the fact that $\sigma_H \perp \mathfrak{B}_H^{k-1}$.

Lemma 2.6.6. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then*

$$\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2 \leq C\eta^2(\sigma_H, \mathcal{T}_H). \tag{2.45}$$

Proof.

$$\|P_{\mathfrak{B}^{k-1}}\sigma_H\| = \langle \sigma_H, \frac{P_{\mathfrak{B}^{k-1}}\sigma_H}{\|P_{\mathfrak{B}^{k-1}}\sigma_H\|}\rangle = \langle -\sigma_H, d\phi\rangle, \ \ \phi \in \mathfrak{Z}^{\perp k-2}$$

By the the Poincaré inequality $\|\phi\|$ can be bounded from above by a constant. ϕ can then be replaced with φ satisfying the properties of Lemma 2.3.5, and noting $\sigma_H \perp \mathfrak{B}_H^{k-1}$,

$$= \langle -\sigma_H, d(\varphi - \Pi_H\varphi)\rangle.$$

The problem is now reduced to a case handled in [18], when they bound a portion of their η_{-1} estimator. We follow their ideas to complete to proof. Applying the integration by parts formula we have

$$= \sum_{T\in\mathcal{T}_H} [\int_{\partial T} (\text{tr} \star \sigma_H \wedge \text{tr } (\varphi - \Pi_H\varphi)) + \langle \delta\sigma_H, \varphi - \Pi_H\varphi\rangle_T]$$

Noting $\text{tr}(\varphi-\Pi_H\varphi)$ is single-valued on the element boundaries, this can be reduced to

$$\leq C \sum_{T\in\mathcal{T}_H} \|\text{tr}(\varphi - \Pi_H\varphi)\|_{\partial T}\|[[\text{tr} \star \sigma_H]]\|_{\partial T} + \|\varphi - \Pi_H\varphi\|_T\|\delta\sigma_H\|_T$$
$$\leq C \sum_{T\in\mathcal{T}_H} (h_T^{\frac{1}{2}}\|[[\text{tr} \star \sigma_H]]\|_{\partial T} + h_T\|\delta\sigma_H\|_T)(h_T^{-\frac{1}{2}}\|\text{tr}(\varphi - \Pi_H\varphi)\|_{\partial T} + h_T^{-1}\|\varphi - \Pi_H\varphi\|_T)$$

Which using the definition of the error indicator simplifies to

$$\leq C\eta(\sigma_H, \mathcal{T}_H) \sum_{T\in\mathcal{T}_H} (h_T^{-1}\|\text{tr}(\varphi - \Pi_H\varphi)\|_{\partial T}^2 + h_T^{-2}\|\varphi - \Pi_H\varphi\|_T^2)^{1/2}.$$

The proof is then complete by applying the bounds from Lemma 2.3.5, and squaring both sides.

$\square$

Lemma 2.6.7. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\tilde{\sigma} = \mathcal{L}^{-1} f_{\mathfrak{B}_H}$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$.*

Then

$$\|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2 \le C\|\tilde{\sigma} - \sigma_H\|^2, \qquad C < 1. \tag{2.46}$$

Proof. Since $\tilde{\sigma} \perp \mathfrak{Z}^{k-1}$ and $\sigma_H \perp \mathfrak{Z}_H^{k-1}$, we follow [18] and write

$$\begin{aligned}\|P_{\mathfrak{H}^{k-1}}\sigma_H\| &= \sup\nolimits_{v\in\mathfrak{H}^{k-1},\|v\|=1}(\sigma_H - \tilde{\sigma}, v - P_{\mathfrak{H}_H^{k-1}}v)\\ &\le \sup\nolimits_{v\in\mathfrak{H}^{k-1},\|v\|=1}(\|v - P_{\mathfrak{H}_H^{k-1}}v\|)\|\sigma_H - \tilde{\sigma}\|\\ &= \delta(\mathfrak{H}^{k-1}, \mathfrak{H}_H^{k-1})\|\sigma_H - \tilde{\sigma}\|.\end{aligned}$$

Applying Theorem 2.3.7, and then squaring both sides we get

$$\|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2 \le C\|\tilde{\sigma} - \sigma_H\|^2, \quad C < 1.$$

□

Now we have the tools to prove the continuous upper bound for the $\mathfrak{B}$ problems.

Theorem 2.6.8. (Continuous Upper-Bound) *Given an* $f \in L^2\Lambda^k(\Omega)$ *in* $\mathfrak{B}^k$*. Let* $\tilde{\sigma} = \mathcal{L}^{-1}f_{\mathfrak{B}_H}$ *and* $\sigma_H = \mathcal{L}_H^{-1}f_{\mathfrak{B}_H}$*. Then*

$$\|\sigma - \sigma_H\|^2 \le C_1\eta^2(\sigma_H, \mathcal{T}_H). \tag{2.47}$$

Proof. Since these are $\mathfrak{B}$ problems, $p, \tilde{p} = 0$, and

$$\begin{aligned}\langle\sigma - \tilde{\sigma}, \tilde{\sigma} - \sigma_H\rangle &= \langle u - \tilde{u}, d(\tilde{\sigma} - \sigma_H)\rangle\\ &= \langle u - \tilde{u}, f_{\mathfrak{B}_H} - f_{\mathfrak{B}_H}\rangle\\ &= 0.\end{aligned}$$

Thus, by applying (2.46), (2.44), (2.45) and Theorem 2.5.1,

$$\begin{aligned}
\|\sigma - \sigma_H\|^2 &= \|\tilde{\sigma} - \sigma_H\|^2 + \|\sigma - \tilde{\sigma}\|^2 \\
&\leq C(\|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 + \|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2) + \|\sigma - \tilde{\sigma}\|^2 \\
&\leq C(\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2) + \|\sigma - \tilde{\sigma}\|^2 \\
&\leq C_1(\eta^2(\sigma_H, \mathcal{T}_H)) + C_0 \mathrm{osc}^2(f, \mathcal{T}_H) \\
&\leq C\eta^2(\sigma_H, \mathcal{T}_H).
\end{aligned}$$

□

Theorem 2.6.9. (Discrete Upper-Bound) *Given $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, let $\sigma_h = \mathcal{L}_h^{-1} f_{\mathfrak{B}_h}$ and $\sigma_H = \mathcal{L}_H^{-1} f_{\mathfrak{B}_H}$. Then*

$$\|\sigma_h - \sigma_H\|^2 \leq C_1\eta^2(\sigma_H, T_H). \tag{2.48}$$

Proof. The proof requires the same ingredients needed to prove the continuous upper bound. The same intermediate steps are taken by performing analysis on the W_h^{k-1} orthogonal decomposition of $\tilde{\sigma}_h - \sigma_H$.

$$\tilde{\sigma}_h - \sigma_H = (\tilde{\sigma}_h - P_{\mathfrak{Z}_h^\perp}\sigma_H) - P_{\mathfrak{B}_h^{k-1}}\sigma_H - P_{\mathfrak{H}_h^{k-1}}\sigma_H.$$

The discrete version of Lemma 2.6.5 uses δ_h rather than δ, but is otherwise identical. The discrete version of Lemma 2.6.6 is identical. The discrete version of Lemma 2.6.7 follows the same structure but makes use of Corollary 2.3.8. The final step in the proof uses the discrete stability result, Theorem 2.5.4.

□

2.7 Convergence of AMFEM

After presenting the adaptive algorithm, the remainder of this section proves convergence and then optimality. The results in this section follow ideas already in the literature [43, 31, 32, 20, 13], with Theorem 2.7.3 building on these ideas by proving reduction in a quasi-error using relationships between data oscillation and reduction of a second type of quasi-error. The following algorithm and analysis of convergence deal specifically with the case $k = n$. In presenting our algorithm we replace h with an iteration counter k.

Algorithm:$[\mathcal{T}_N, \sigma_N\ ] = $ AMFEM$(\mathcal{T}_0, f, \epsilon, \theta)$: Given a initial shape-regular triangulation $\mathcal{T}_0$ and marking parameter θ, set $k = 0$ and iterate the following steps until a desired decrease in the error-estimator is achieved:

$$(1)(u_k, \sigma_k, p_k) = SOLVE(f, \mathcal{T}_k)$$
$$(2)\{\eta_T\} = ESTIMATE(f, \sigma_k, \mathcal{T}_k)$$
$$(3)\mathcal{M}_k = MARK(\{\eta_T\}, \mathcal{T}_k, \theta)$$
$$(4)\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$$

2.7.1 Convergence of AMFEM

The following notation will be used in the proofs and discussion of this section:

$$e_k = \|\sigma - \sigma_k\|^2, \qquad E_k = \|\sigma_{k+1} - \sigma_k\|^2, \qquad \eta_k = \eta^2(\sigma_k, \mathcal{T}_k),$$

$$o_k = \mathrm{osc}^2(f, \mathcal{T}_k), \qquad \hat{o}_k = \mathrm{osc}^2(f_{k+1}, \mathcal{T}_k),$$

where $f_k = P_k f = P_{\mathfrak{B}_k} f$ since $k = n$.

Lemma 2.7.1.

$$\beta\eta_{k+1} \leq \beta(1 - \lambda\theta)\eta_k + E_k + \hat{o}_k. \tag{2.49}$$

Proof. This follows from continuity of the error estimator (2.43), and properties of the marking strategy, i.e. reduction of the summation on a finer mesh due to smaller element sizes on refined elements. The proof can be found in [13]. $\lambda < 1$ is a constant dependent on the dimensionality of the problem. □

For convenience, we recall the quasi-orthogonality (2.27) the continuous upper-bound (2.47) equations,

$$(1-\delta)e_{k+1} \le e_k - E_k + C_0\hat{o}_k, \text{ for any } \delta > 0,$$
$$e_k \le C_1\eta_k.$$

With these three ingredients, basic algebra leads to the following result,

Theorem 2.7.2. *When*

$$0 < \delta < \min\{\frac{\beta}{2C_1}\theta, 1\}, \tag{2.50}$$

there exists $\alpha \in$ (0,1) *and* C_δ *such that*

$$(1-\delta)e_{k+1} + \beta\eta_{k+1} \le \alpha[(1-\delta)e_k + \beta\eta_k] + C_\delta\hat{o}_k. \tag{2.51}$$

Proof. Follows the same steps as [13]. □

With the above result we next prove convergence.

Theorem 2.7.3. (Termination in Finite Steps) *Let* σ_k *be the solution obtained in the kth loop in the algorithm AMFEM, then for any* $0 < \delta < \min\{\frac{\beta}{2C_1}\theta, 1\}$*, there exists positive constants* C_δ *and* 0< $\gamma_\delta < 1$ *depending only on given data and the initial grid such that,*

$$(1-\delta)\|\sigma - \sigma_k\|^2 + \beta\eta^2(\sigma_k, \mathcal{T}_k) + \zeta\text{osc}^2(f, \mathcal{T}_k) \le C_q\gamma_\delta^k,$$

and the algorithm will terminate in finite steps.

Proof. The following proof will be broken into two cases, depending on the relative size of $\hat{o}_k$. For ease of reading, let $q_k = (1-\delta)\|\sigma - \sigma_k\|^2 + \beta\eta^2(\sigma_k, \mathcal{T}_k)$.

Case 1. Suppose the case $C_\delta\hat{o}_k \le (\frac{1-\alpha}{2})q_k$. Thus for an arbitrary positive constant C, (2.51) yields

$$q_{k+1} + Co_{k+1} \le (\alpha + \frac{1-\alpha}{2})q_k + Co_k.$$

Since $\beta o_k \le q_k$,

$$q_{k+1} + Co_{k+1} \le (\hat{\alpha} + \frac{1-\hat{\alpha}}{2})q_k + \frac{C - \frac{\beta(1-\hat{\alpha})}{2}}{C} Co_k, \tag{2.52}$$

where

$$\hat{\alpha} = (\alpha + \frac{1-\alpha}{2}) < 1.$$

Case 2. Suppose the case $C_\delta \hat{o}_k \ge (\frac{1-\alpha}{2})q_k$. We then have,

$$o_{k+1} \le \kappa o_k, \kappa < 1,$$
$$\hat{o}_k \le o_k.$$

This implies

$$o_{k+1} \le (\kappa + \frac{1-\kappa}{2})o_k - \frac{1-\kappa}{2}\hat{o}_k.$$

Combined with (2.51) we have

$$q_{k+1} + \frac{2C_\delta}{1-\kappa} o_{k+1} \le \alpha q_k + \hat{\kappa}\frac{2C_\delta}{1-\kappa} o_k, \tag{2.53}$$

where $\hat{\kappa} = (\kappa + \frac{1-\kappa}{2}) < 1$. The proof is completed by taking $\frac{2C_\delta}{1-\kappa}$ for the constant in (2.52), and then combining with (2.53). The rate of decay will be determined by

$$\gamma_\delta = \max\left\{\hat{\kappa}, \frac{C - \frac{\beta(1-\hat{\alpha})}{2}}{C}, \hat{\alpha} + \frac{1-\hat{\alpha}}{2}\right\} < 1. \tag{2.54}$$

□

The methods used above to prove convergence have many similarities to prior work. Our treatment of oscillation, however, uses properties of $\hat{o}_k$ that create distinct implementation and efficiency improvements. To clarify this point, next we compare our convergence proof with two from the literature, [12, 13]. In order to make the differences clear, we focus on the basic properties of the three equations that are at the core of the convergence analysis.

Convergence is essentially proved by manipulating the equations for quasi-orthogonality, continuity of the error estimator and the upper-bound. For this reason they will be the focus of our discussion, and for ease of comparison, we present our results together in a simplified form.

$$\begin{aligned} (1-\delta)e_{k+1} &\leq e_k - E_k + \tfrac{C_0}{\delta}\hat{o}_k, \quad \text{for any } \delta > 0, \\ \beta\eta_{k+1} &\leq \beta\lambda\eta_k + E_k + \hat{o}_k, \quad \lambda < 1, \\ e_k &\leq C_1\eta_k. \end{aligned} \tag{2.55}$$

In [12], an orthogonality result, $e_{k+1} = e_k - E_k$, is possible since they are not working with a mixed merthod. In addition, a similar estimator continuity result is proved without the need for the $\hat{o}_k$ term. Since oscillation is not present, convergence is proved without the additional analysis used in the proof of Theorem 2.7.3.

For the purpose of comparison, we now present a simplified version of the equivalent equations from [13],

$$\begin{aligned} (1-\delta)e_{k+1} &\leq e_k - E_k + \tfrac{C_0}{\delta}o_k, \quad \text{for any } \delta > 0, \\ \beta\eta_{k+1} &\leq \beta\lambda\eta_k + E_k, \qquad \lambda < 1, \\ e_k &\leq C_1\eta_k + C_0 o_k. \end{aligned} \tag{2.56}$$

In [13], oscillation is not included in the error indicator and therefore it is needed in the upper bound. Once o_k is used for the upper-bound, it is used out of simplicity in the quasi-orthogonality result as $o_k \geq \hat{o}_k$. The issue with including o_k versus $\hat{o}_k$ is that o_k can be significant in steps where oscillation is not reduced. Whereas the value of $\hat{o}_k$ indicates oscillation reduction and thus reduces the impact of data oscillation on remaining iterations. In order to manage this situation, the algorithm in [13] marks separately for η and oscillation. This is a disadvantage from an implementation point of view, and is also inefficient in cases when η and oscillation are different orders of magnitude.

2.7.2 Optimality of AMFEM

Once Theorem 2.5.1, Theorem 2.3.10, and the Lemma 2.7.1 are established, optimality can be proved independent of dimension following the proof of Theorem

5.3 in [43].

Theorem 2.7.4. (Optimality) *For any* $f \in L^2\Lambda^n(\Omega)$*, shape regular* $\mathcal{T}_0$ *and* $\epsilon > 0$*, let* $\sigma = \mathcal{L}^{-1}f$ *and* $[\ \sigma_N, \mathcal{T}_N\ ] = \mathsf{AMFEM}(\ \mathcal{T}_H, f_H, \epsilon/2, \theta)$*. Where* $[\mathcal{T}_H, f_H] = \mathsf{APPROX}(\ f, \mathcal{T}_0, \epsilon/2\ )$*. If* $\sigma \in \mathcal{A}^s$ *and* $f \in \mathcal{A}^s_o$*, then*

$$\|\sigma - \sigma_N\| \leq C(\|\sigma\|_{\mathcal{A}^s} + \|f\|_{\mathcal{A}^s_o})(\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}. \tag{2.57}$$

Proof. Follows directly from [13]. □

2.8 Conclusion and Future Work

In this paper, we have focused on the error $\|\sigma - \sigma_h\|$ for the Hodge Laplacian in the specific case $k = n$. Next, we outline how this work relates to further generalizations.

Extending the convergence results to general $\mathfrak{B}$ problems is of particular interest. With the exception of the stability results, the methods used to prove convergence relied only on properties inherent to all $\mathfrak{B}$ problems. The issue with stability is that we cannot assume $H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \subset H^1\Lambda^k(\Omega)$ [5]. However, $\mathring{H}^*\Lambda^n(\Omega) \subset H^1\Lambda^n(\Omega)$, thus we have the desired interpolation properties of $u - \tilde{u}$ in the case $k = n$. Since σ and σ_h depend only on the $\mathfrak{B}$ component of f, a convergent method for $\mathfrak{B}$ problems might be useful in extending results to general Hodge Laplace problems. The issue with this extension, however, is that only the $\mathfrak{B}$ component of f should be considered when resolving data oscillation, and access to this quantity is not presumed. Thus extending results from $\mathfrak{B}$ problems to general Hodge Laplace problems would require analysis of the error caused by using an approximation of $f_{\mathfrak{B}}$.

Adaptivity focusing on the natural norm, $\|u - u_h\|_{H\Lambda^k(\Omega)} + \|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|p - p_h\|$, is another direction of interest. Error indicators related to this norm are analyzed in [18], yet difficulties still arise in an attempt to gain full generality (see [18] for a detailed discussion). Additionally, a quasi-orthogonality result would be useful. The quasi-orthogonality proved here used analysis tailored specifically to the norm of interest. A generalized quasi-orthogonality and convergence result would likely require a different line of of reasoning and a specific analysis regarding a refinement strategy that takes into account the approximation of the harmonic forms.

2.9 Acknowledgments

MH was supported in part by NSF Awards 1217175 and 1065972. AM and RS was supported in part by NSF Award 1217175.

Chapter 2, in full, has been submitted for publication of the material as it may appear in Foundations of Computational Mathematics, M. Holst, A. Mihalik and R. Szypowski, 2014. The dissertation author was the primary investigator and author of this paper.

Chapter 3

Convergence and Optimality of Adaptive Mixed Methods on Surfaces

3.1 Abstract

In a 1988 article, Dziuk introduced a nodal finite element method for the Laplace-Beltrami equation on 2-surfaces approximated by a piecewise-linear triangulation, initiating a line of research into surface finite element methods (SFEM). Demlow and Dziuk built on the original results, introducing an adaptive method for problems on 2-surfaces, and Demlow later extended the *a priori* theory to 3-surfaces and higher order elements. In a separate line of research, the Finite Element Exterior Calculus (FEEC) framework has been developed over the last decade by Arnold, Falk and Winther and others as a way to exploit the observation that mixed variational problems can be posed on a Hilbert complex, and Galerkin-type mixed methods can be obtained by solving finite dimensional subproblems. In 2011, Holst and Stern merged these two lines of research by developing a framework for variational crimes in abstract Hilbert complexes, allowing for application of the FEEC framework to problems that violate the subcomplex assumption of Arnold, Falk and Winther. When applied to Euclidean hypersurfaces, this new framework recovers the original *a priori* results and extends the theory to problems posed on surfaces of arbitrary dimensions. In yet another seemingly distinct line of research, Holst, Mihalik and Szypowski developed a convergence theory for a specific class of adaptive problems in the FEEC framework. Here, we bring these ideas together, showing convergence and optimality of an adaptive finite element method for the mixed formulation of the Hodge Laplacian on hypersurfaces.

3.2 Introduction

Adaptive finite element methods (AFEM) based on *a posteriori* error estimators have become standard tools in solving PDE problems arising in science and engineering (cf. [1, 46, 38]). A fundamental difficulty with these adaptive methods is guaranteeing convergence of the solution sequence. The first convergence result was obtained by Babuska and Vogelius [7] for linear elliptic problems in one space dimension, and many improvements and generalizations to the theory have followed [19, 31, 34, 33, 41]. Convergence, however, does not necessarily imply optimality of a method. This idea has led to the development of a theory related to the optimal computational complexity of AFEM, and within this framework certain classes of adaptive methods have been shown to be optimal [8, 43, 12].

In a 1988 article, Dziuk [21] introduced a nodal finite element method for the Laplace-Beltrami equation on 2-surfaces approximated by a piecewise-linear triangulation, pioneering a line of research into *surface finite element (SFEM)* methods. Demlow and Dziuk [17] built on the original results, introducing an adaptive method for problems on 2-surfaces, and Demlow later extended the *a priori* theory to 3-surfaces and higher order elements [16]. While *a posteriori* error indicators are introduced and shown to have desirable properties in [17], a convergence and optimality theory related to problems on surfaces is a relatively undeveloped area, and developing such a theory is the main topic of this article.

A separate idea that has had a major influence on the development of numerical methods for PDE applications is that of *mixed finite elements*, whose early success in areas such as computational electromagnetics was later found to have surprising connections with the calculus of exterior differential forms, including de Rham cohomology and Hodge theory [10, 35, 36, 22]. Around the same time period, Hilbert complexes were studied as a way to generalize certain properties of elliptic complexes, particularly the Hodge decomposition and Hodge theory [11]. These ideas came together with the introduction of the theory of *finite element exterior calculus* (FEEC), where Arnold, Falk and Winther showed that Hilbert complexes were a natural setting for analysis and numerical approximation of mixed variational problems by mixed finite elements. This theory has proved a powerful tool in developing general results related to mixed finite elements. In [24, 25], Holst and Stern extend the theory to include problems in which the discrete complex is not a subcomplex of the approximated complex, and applying these results they develop an *a priori* theory for the Hodge Laplacian on hypersurfaces, and to nonlinear problems. This made it possible in [24] to reproduce the existing *a priori* theory for SFEM as a particular application, as well as to generalize SFEM theory in several directions. In [23], we used the FEEC framework as a critical tool for developing an AFEM convergence theory for a class of adaptive methods for linear problems posed on domains in $\mathbb{R}^n$. The aim of this paper is to build upon these results and develop a convergence theory for a class of problems that violate the subcomplex assumption of Arnold, Falk and Winther, allowing for the treatment of problems on surfaces.

More specifically, we introduce an adaptive method for problems posed on smooth Euclidean hypersurfaces in which finite element spaces are mapped from a fixed approximating polygonal manifold. The mesh on the fixed approximating surface will be refined using error indicators related to the original problem. Using tools developed in [24, 25], the auxiliary results of [23] are modified to account for the surface mapping, yielding an adaptive method whose main results mirror those of [23]. In doing this we establish the optimality of a convergent algorithm for the Hodge Laplacian (case $k = m$) on hypersurfaces of arbitrary dimension.

The remainder of the paper is organized as follows. In Section 3.3 we introduce the notational and technical tools essential for the paper. We begin by discussing the fundamental framework of abstract Hilbert complexes and in particular the de Rham complex [5], ideas which are critical in the development of the theory of finite element exterior calculus. We then finish the section with a brief overview of a standard adaptive finite element algorithm. Next, Section 3.4 follows [24, 25] by introducing geometric tools and ideas that tie the general theory developed in [4, 5] to problems on Euclidean hypersurfaces. Additionally we prove some basic results for an interpolant built on the approximating surface. In Section 3.5.2 we closely follow the ideas in [23] and develop a similar quasi-orthogonality result, specifically tailoring our results for application on surfaces. Section 3.5.3 again closely follows [23], and we prove a discrete stability result applicable to problems on surfaces (which is needed for proving quasi-orthogonality in Section 3.5.2), and also establish a continuous stability result, which will be needed for deriving an upper bound on the error. In Section 3.6 we begin by introducing an error indicator and then derive bounds and a type of continuity result for this indicator. An adaptive algorithm is then presented in Section 3.7, for which convergence and optimality are proved using the auxiliary results from the previous sections. Finally, we close in Section 3.8 with a discussion on related future directions and alternative methods for solving numerical PDE on surfaces. The results in this paper follow [23] in a natural manner. It is the same convergence idea, but the results are adapted to account for the geometry of the surface and the mapping between the surfaces.

3.3 Notation and Framework

The algorithm developed in this article will rely heavily on the methods introduced on polygonal domains in [23]. In order to keep this work self contained, this section will provide a similar introduction to that of [23], from which we quote freely. We begin with an introduction of some basic concepts of abstract Hilbert complexes. Next, we examine the particular case of the de Rham complex, closely following the notation and general development of Arnold, Falk and Winther in [4, 5]. We also discuss results from Demlow and Hirani in [18]. (See also [24, 25] for a concise summary of Hilbert complexes in a yet more general setting.) We then give an overview of the basics of adaptive finite element methods (AFEM), and the ingredients we will need to prove convergence and optimality within the FEEC framework.

3.3.1 Hilbert Complexes

A *Hilbert complex* (W, d) is a sequence of Hilbert spaces W^k equipped with closed, densely defined linear operators, d^k, which map their domain, $V^k \subset W^k$ to the kernel of d^{k+1} in W^{k+1}. A Hilbert complex is *bounded* if each d^k is a bounded linear map from W^k to W^{k+1} A Hilbert complex is *closed* if the range of each d^k is closed in W^{k+1}. Given a Hilbert complex (W, d), the subspaces $V^k \subset W^k$ endowed with the graph inner product

$$\langle u, v\rangle_{V^k} = \langle u, v\rangle_{W^k} + \langle d^k u, d^k v\rangle_{W^{k+1}},$$

form a Hilbert complex (V, d) known as the *domain complex*. By definition $d^{k+1} \circ d^k = 0$, thus (V, d) is a bounded Hilbert complex. Additionally, (V, d) is closed if

(W, d) is closed.

The range of d^{k-1} in V^k will be represented by $\mathfrak{B}^k$, and the null space of d^k will be represented by $\mathfrak{Z}^k$. Clearly, $\mathfrak{B}^k \subset \mathfrak{Z}^k$. The elements of $\mathfrak{Z}^k$ orthogonal to $\mathfrak{B}^k$ are the space of harmonic forms, represented by $\mathfrak{H}^k$. For a closed Hilbert complex we can write the *Hodge decomposition* of W^k and V^k,

$$W^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_W}, \tag{3.1}$$

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp_V}. \tag{3.2}$$

Following notation common in the literature, we will write $\mathfrak{Z}^{k\perp}$ for $\mathfrak{Z}^{k\perp_W}$ or $\mathfrak{Z}^{k\perp_V}$, when clear from the context. Another important Hilbert complex will be the *dual complex* (W, d^*), where d_k^*, which is an operator from W^k to W^{k-1}, is the adjoint of d^{k-1}. The domain of d_k^* will be denoted by V_k^*. For closed Hilbert complexes, an important result will be the *Poincaré inequality*,

$$\|v\|_V \le c_P \|d^k v\|_W, \;\; v \in \mathfrak{Z}^{k\perp}. \tag{3.3}$$

The de Rham complex is the practical complex where general results we show on an abstract Hilbert complex will be applied. The de Rham complex satisfies an important compactness property discussed in [5], and therefore this compactness property is assumed in the abstract analysis.

The Abstract Hodge Laplacian

Given a Hilbert complex (W, d), the operator $L = dd^* + d^*d$, $W^k \to W^k$ will be referred to as the *abstract Hodge Laplacian*. For $f \in W^k$, the Hodge Laplacian problem can be formulated weakly as the problem of finding $u \in W^k$ such that

$$\langle du, dv\rangle + \langle d^*u, d^*v\rangle = \langle f, v\rangle, v \in V^k \cap V_k^*.$$

The above formulation has undesirable properties from a computation perspective. The finite element spaces $V^k \cap V_k^*$ can be difficult to implement, and the problem will not be well-posed in the presence of a non-trivial harmonic space,

$\mathfrak{H}^k$. In order to circumvent these issues, a well posed (cf. [4, 5]) *mixed formulation of the abstract Hodge Laplacian* is introduced as the problem of finding $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$, such that:

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, && \forall \tau \in V^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in V^k, \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}^k. \end{aligned} \tag{3.4}$$

Sub-Complexes and Approximate Solutions to the Hodge Laplacian

In [4, 5] a theory of approximate solutions to the Hodge-Laplace problem is developed by using finite dimensional approximating Hilbert complexes. Let (W, d) be a Hilbert complex with domain complex (V, d). An approximating subcomplex is a set of finite dimensional Hilbert spaces, $V_h^k \subset V^k$ with the property that $dV_h^k \subset V_h^{k+1}$. Since V_h is a Hilbert complex, V_h has a corresponding Hodge decomposition,

$$V_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp_V}.$$

By this construction, (V_h, d) is an abstract Hilbert complex with a well posed Hodge Laplace problem. Find $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$, such that

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle d\tau, u_h \rangle &= 0, && \forall \tau \in V_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, && \forall v \in V_h^k, \\ \langle u_h, q \rangle &= 0, && \forall q \in \mathfrak{H}_h^k. \end{aligned} \tag{3.5}$$

An assumption made in [5] in developing this theory is the existence of a bounded cochain projection, $\pi_h : V \to V_h$, which commutes with the differential operator.

In [5], an *a priori* convergence result is developed for the solutions on the approximating complexes. The result relies on the approximating complex getting sufficiently close to the original complex in the sense that $\inf_{v \in V_h^k} \|u - v\|_V$ can be assumed sufficiently small for relevant $u \in V^k$. Adaptive methods, on the other hand, gain computational efficiency by limiting the degrees of freedom used in areas of the domain where it does not significantly impact the quality of the solution.

3.3.2 The de Rham Complex and its Approximation Properties

The de Rham complex is a cochain complex where the abstract results from the previous section can be applied in developing practical computational methods. This section reviews concepts and definitions related to the de Rham complex necessary in our development of an adaptive finite element method. This introduction will be brief and and mostly follows the notation from the more in-depth discussion in [5].

In order to introduce the ideas of [23], we first assume a bounded Lipschitz polyhedral domain, $\Omega \in \mathbb{R}^n, n \geq 2$. Let $\Lambda^k(\Omega)$ be the space of smooth k-forms on Ω, and let $L^2\Lambda^k(\Omega)$ be the completion of $\Lambda^k(\Omega)$ with respect to the L^2 inner-product. There are no non-zero harmonic forms in $L^2\Lambda^n(\Omega)$ (see [4], Theorem 2.4) which will often simplify the analysis in our primary case of interest, $k = n$. For general k such a property cannot be assumed, and therefore, since the $\mathfrak{B}$ problem deals with the spaces of k and $(k-1)$-forms, analysis of the harmonic spaces is still necessary. Note that the results in [13] hold only for polygonal and simply connected domains, therefore $\mathfrak{H}^{k-1}$ is also void in the case $k = n = 2$.

The de Rham Complex

Let d be the exterior derivative acting as an operator from $L^2\Lambda^k(\Omega)$ to $L^2\Lambda^{k+1}(\Omega)$. The L^2 inner-product will define the W-norm, and the V-norm will be defined as the graph inner-product

$$\langle u, \omega \rangle_{V^k} = \langle u, \omega \rangle_{L^2} + \langle du, d\omega \rangle_{L^2}.$$

This forms a Hilbert complex $(L^2\Lambda(\Omega), d)$, with domain complex $(H\Lambda(\Omega), d)$, where $H\Lambda^k(\Omega)$ is the set of elements in $L^2\Lambda^k(\Omega)$ with exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$. The domain complex can be described with the following diagram

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} \cdots \to H\Lambda^{n-1}(\Omega) \xrightarrow{d} L^2(\Omega) \to 0. \tag{3.6}$$

It can be shown that the compactness property is satisfied, and therefore the prior results shown on abstract Hilbert complexes can be applied.

The Hodge star operator, $\star : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$, is then defined using the wedge product. For $\omega \in \Lambda^k(\Omega)$,

$$\int_\Omega \omega \wedge \mu = \langle \star\omega, \mu \rangle_{L^2\Lambda^{n-k}}, \;\; \forall \mu \in \Lambda^{n-k}(\Omega).$$

Next we introduce the coderivative operator, $\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$,

$$\star\delta\omega = (-1)^k d \star \omega, . \tag{3.7}$$

which combined with Stokes theorem allow integration by parts to be written as

$$\langle d\omega, \mu \rangle = \langle \omega, \delta\mu \rangle + \int_{\partial\Omega} \mathrm{tr}\; \omega \wedge \mathrm{tr} \star \mu, \;\; \omega \in \Lambda^{k-1}, \;\; \mu \in \Lambda^k(\Omega). \tag{3.8}$$

Using this formulation and the following spaces,

$$\begin{aligned} \mathring{H}\Lambda^k(\Omega) &= \{\omega \in H\Lambda^k(\Omega) \big| \; \mathrm{tr}_{\partial\Omega}\omega = 0\}, \\ \mathring{H}^*\Lambda^k(\Omega) &:= \star\mathring{H}\Lambda^{n-k}(\Omega), \end{aligned}$$

the following theorem connects the framework built for abstract Hilbert complexes to the de Rham complex.

Theorem 3.3.1. *(Theorem 4.1 from [5]) Let d be the exterior derivative viewed as an unbounded operator $L^2\Lambda^{k-1}(\Omega) \to L^2\Lambda^k(\Omega)$ with domain $H\Lambda^k(\Omega)$. The adjoint d^*, as an unbounded operator $L^2\Lambda^k(\Omega) \to L^2\Lambda^{k-1}(\Omega)$, has $\mathring{H}^*\Lambda^k(\Omega)$ as its domain and coincides with the operator δ defined in* (3.7).

Applying the results from the previous section and Theorem 3.3.1, we get the mixed Hodge Laplace problem on the de Rham complex: find the unique

$(\sigma, u, p) \in H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega) \times \mathfrak{H}^k$ such that

$$\begin{aligned} \sigma = \delta u,\ d\sigma + \delta du = f - p \quad & \text{in } \Omega, \\ \operatorname{tr} \star u = 0,\ \operatorname{tr} \star du = 0 \quad & \text{on } \partial\Omega, \\ u \perp \mathfrak{H}^k. & \end{aligned} \tag{3.9}$$

Finite Element Differential Forms

For the remainder of the paper it is assumed that all approximating subcomplexes of the de Rham complex are constructed as combinations of the polynomial spaces of k-forms, $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$. For a detailed discussion on these spaces and construction of Hilbert complexes using these spaces, see [5]. We also have useful properties in the case $k = n$,

$$\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n, \tag{3.10}$$

$$\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0. \tag{3.11}$$

For a shape-regular, conforming triangulation $\mathcal{T}_h$ of $\Omega, \Lambda_h^k(\Omega) \subset L^2\Lambda^k(\Omega)$ will denote a space of k-forms constructed using specific combinations of the these spaces on $\mathcal{T}_h$. For an element $T \in \mathcal{T}_h$, we set $h_T := \operatorname{diam}(T)$. We do not discuss the details of these spaces further, but specific properties will be explained when necessary.

Bounded Cochain Projections

Bounded cochain projections and their approximation properties are necessary in the analysis of both uniform and adaptive FEMs in the FEEC framework. Properties of three different interpolation operators will be important in our analysis. The three operators and respective notation that we will use are as follows: the canonical projections I_h defined in [4, 5], the smoothed projection operator π_h from [5], and the commuting quasi-interpolant Π_h, as defined in [18] with ideas similar to [39, 40, 14]. Some cases will require a simple projection, and $P_h f$ also written f_h, will denote the L^2-projection of f on to the discrete space parameterized by h.

For the remainder of the paper, $\|\cdot\|$ will denote the $L^2\Lambda^k(\Omega)$ norm, and when taken on specific elements of the domain, T, we write $\|\cdot\|_T$. For all other norms, such as $H\Lambda^k(\Omega)$ and $H^1\Lambda^k(\Omega)$, we write $\|\cdot\|_{H\Lambda^k(\Omega)}$ and $\|\cdot\|_{H^1\Lambda^k(\Omega)}$ respectively.

Lemma 3.3.2. *Suppose $\tau \in H^1\Lambda^k(\Omega)$, where $k = n-1$ or $k = n$. Let I_h be the canonical projection operator defined in [4, 5] and let $\Lambda_h^{n-1}(\Omega)$ and $\Lambda_h^n(\Omega)$ be defined as above. Then I_h is a projection onto $\Lambda_h^n(\Omega), \Lambda_h^{n-1}(\Omega)$ and satisfies*

$$\|\tau - I_h\tau\|_T \leq Ch_T\|\tau\|_{H^1\Lambda^k(T)}, \quad \forall T \in \mathcal{T}_h, \tag{3.12}$$

$$I_h d = dI_h \tag{3.13}$$

Proof. The first part is comes from Equation (5.4) in [4]. The second part follows the construction of I_h. □

Lemmas 3.3.3 and 3.3.4 deal with important properties of the canonical projections. In each case we assume $f_h, u_h \in \Lambda_h^n(\Omega)$, and let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$.

Lemma 3.3.3. *Let $T \in \mathcal{T}_H$, then*

$$\int_T (f_h - I_H f_h) = 0. \tag{3.14}$$

Proof. See [23] □

Lemma 3.3.4. *Let $T \in \mathcal{T}_H$, then*

$$\langle (I_h - I_H)u_h, f_h\rangle_T = \langle u_h, (I_h - I_H)f_h\rangle_T. \tag{3.15}$$

Proof. See [23]. □

The next lemma is taken directly from [18], and will be a key tool in developing an upper bound for the error.

Lemma 3.3.5. *Assume $1 \leq k \leq n$, and $\phi \in H\Lambda^{k-1}(\Omega)$ with $\|\phi\| \leq 1$. Then there exists $\varphi \in H^1\Lambda^{k-1}(\Omega)$ such that $d\varphi = d\phi, \Pi_H d\phi = d\Pi_H\phi = d\Pi_H\varphi$, and*

$$\sum_{T \in T_h} h_T^{-2} \|\varphi - \Pi_H \varphi\|_T^2 + h_T^{-1} \|tr(\varphi - \Pi_H \varphi)\|_{\partial T}^2 \leq C.$$

Proof. See Lemma 6 in [18]. □

The following theorem is a special case of Theorem 3.5 from [5]. Rather than showing the result on an abstract Hilbert Complex with a general cochain projection, we use the de Rham complex and the smoothed projection operator π_h in order to use uniform boundedness of the cochain projection.

Theorem 3.3.6. *Assume $\Lambda_h^k(\Omega)$ is a subcomplex of $H\Lambda^k(\Omega)$ as described above, and let π_h be the smoothed projection operator. Then*

$$\|(I - P_{\mathfrak{H}^k})q\|_V \leq \|(I - \pi_h^k)P_{\mathfrak{H}^k} q\|_V, \quad q \in \mathfrak{H}_h^k, \tag{3.16}$$

then combining the above with the triangle inequality,

$$\|q\|_V \leq c\|P_{\mathfrak{H}^k} q\|_V, \quad q \in \mathfrak{H}_h^k. \tag{3.17}$$

Proof. See [23]. □

Theorem 3.3.7 will be essential in dealing with the harmonic forms in the proof of a continuous upper-bound. The corollary will be used identically when proving a discrete upper-bound. For use in our next two results we introduce an operator δ and one of its important properties. Let A, B be $n < \infty$ dimensional, closed subspaces of a Hilbert space W, and let

$$\delta(A, B) = \sup_{x \in A, \|x\|=1} \|x - P_B x\|,$$

then [18], Lemma 2 which takes the original ideas from [27], shows

$$\delta(A, B) = \delta(B, A). \tag{3.18}$$

Theorem 3.3.7. *Assume $\mathfrak{H}_H^k$ and $\mathfrak{H}^k$ have the same finite dimensionality. The there exist a constant $C_{\mathfrak{H}^k}$ dependent only on $\mathcal{T}_0$, such that*

$$\delta(\mathfrak{H}^k, \mathfrak{H}_H^k) = \delta(\mathfrak{H}_H^k, \mathfrak{H}^k) \leq C_{\mathfrak{H}^k} < 1. \tag{3.19}$$

Proof. See [23] □

Corollary 3.3.8.

$$\delta(\mathfrak{H}_h^k, \mathfrak{H}_H^k) = \delta(\mathfrak{H}_H^k, \mathfrak{H}_h^k) \leq \tilde{C}_{\mathfrak{H}^k} < 1. \tag{3.20}$$

Proof. The proof follows the same logic as Theorem 3.3.7. The only difference is that the harmonics are compared on two discrete complexes $\mathfrak{H}_h^k$ and $\mathfrak{H}_H^k$, and therefore I_h is used rather than π_h. □

3.3.3 Adaptive Finite Elements Methods

This section gives a concise introduction to key concepts and notation used in developing our AFEM. Our methods will follow [43, 31, 32, 20, 13], which give more a more complete discussion on AFEM.

Given an initial triangulation, $\mathcal{T}_0$, the adaptive procedure will generate a nested sequence of triangulations $\mathcal{T}_k$ and discrete solutions σ_k, by looping through the following steps:

$$\textsf{Solve} \longrightarrow \textsf{Estimate} \longrightarrow \textsf{Mark} \longrightarrow \textsf{Refine} \tag{3.21}$$

The following subsection will describe details of these steps.

Approximation Procedure

We assume access to a routine SOLVE, which can produce solution to (3.5) given a triangulation, problem data, and a desired level of accuracy. For the ESTIMATE step we will introduce error indicators η_T on each element $T \in \mathcal{T}_k$. In the MARK step we will use Dörfler Marking strategy [20]. An essential feature of the marking process is that the summation of the error indicators on the marked elements exceeds a user defined marking parameter θ.

We assume access to an algorithm REFINE in which marked elements are subdivided into two elements of the same size, resulting in a conforming, shape-regular mesh. Triangles outside of the original marked set may be refined in order to maintain conformity. Bounding the number of such refinements is important in showing optimality of the method. Along these lines, Stevenson [44] showed certain bisection algorithms developed in two-dimensions can be extended to n-simplices of arbitrary dimension satisfying

$$(1)\{\mathcal{T}_k\} \text{ is shape regular and the shape regularity depends only on } \mathcal{T}_0,$$
$$(2)\#\mathcal{T}_k \leq \#\mathcal{T}_0 + C\#\mathcal{M},$$

where $\mathcal{M}$ is the collection of all marked triangles going from $\mathcal{T}_0$ to $\mathcal{T}_k$.

Approximation of the Data

A measure of data approximation will be necessary in establishing a quasi-orthogonality result. Following ideas of [31], data oscillation will be defined as follows,

Definition 3.3.9. (*Data oscillation*) Let f $\in L^2\Lambda^k(\Omega)$, and $\mathcal{T}_h$ be a conforming triangulation of Ω. Let h_T be the diameter for a given $T \in \mathcal{T}_h$. We define

$$osc(f, \mathcal{T}_h) := \big(\sum_{T\in\mathcal{T}_h} \|h_T(f - f_h)\|_T^2\big)^{1/2}.$$

Stevenson [44] generalized the ideas of [8] to show that approximation of data can be done in an optimal way regardless of dimension. Using the approximation spaces $(\mathcal{A}^s, \|\cdot\|_{\mathcal{A}^s})$ and $(\mathcal{A}^s_o, \|\cdot\|_{\mathcal{A}^s_o})$ as in [8] we recall the result.

Theorem 3.3.10. *(Generalized Binev, Dahmen and DeVore) Given a tolerance* ϵ*,* $f \in L^2\Lambda^n(\Omega)$ *and a shape regular triangulation* $\mathcal{T}_0$*, there exists an algorithm*

$$\mathcal{T}_H = \mathsf{APPROX}(f, \mathcal{T}_0, \epsilon),$$

such that

$$osc(f, \mathcal{T}_H) \leq \epsilon, \quad and \quad \#\mathcal{T}_H - \#\mathcal{T}_0 \leq C\|f\|^{1/s}_{\mathcal{A}^{1/s}_o}\epsilon^{-1/s}.$$

As in the case of [13], the analysis of convergence and procedure will follow [12], and the optimality will follow [43].

3.4 The de Rham Complex on Approximating Manifold

3.4.1 Hodge-de Rham Theory and Diffeomorphic Riemannian Manifolds

We next introduce the Hodge-de Rham complex of differential forms on a compact oriented Riemannian manifold. This discussion will be minimal and closely follows [24, 25], where a more complete development can be found.

We assume M is a smooth, oriented, compact m-dimensional manifold equipped with a Riemannian metric, g. Let $\Omega^k(M)$ be the space of smooth k-forms on M, and define the L^2 *inner product* for any $u, v \in \Omega^k(M)$ as

$$\langle u, v \rangle_{L^2\Omega(M)} = \int_M u \wedge \star_g v = \int_M \langle\!\langle u, v \rangle\!\rangle_g \mu_g,$$

where $\star_g : \Omega^k(M) \to \Omega^{m-k}(M)$ is the Hodge star operator associated to the metric, $\langle\!\langle \cdot, \cdot \rangle\!\rangle_g$ is the pointwise inner product induced by g, and μ_g is the Riemannian volume form. For each k, define $L^2\Omega^k(M)$ as the Hilbert space formed by the completion of $\Omega^k(M)$ with respect to the L^2-inner product.

Combined with the exterior derivative, $d^k : \Omega^k(M) \to \Omega^{k+1}(M)$, these spaces form a Hilbert complex, $(L^2\Omega^k(M), d)$, with domain complex $(H\Omega^k(M), d)$. Here $H\Omega^k(M) \subset L^2\Omega^k(M)$ is the set of elements in $L^2\Omega^k(M)$ with a weak exterior derivative in $L^2\Omega^{k+1}(M)$. Each space $H\Omega^k(M)$ is endowed with a graph inner-product,

$$\langle u, v \rangle_{H\Omega(M)} = \langle u, v \rangle_{L^2\Omega(M)} + \langle du, dv \rangle_{L^2\Omega(M)},$$

and the complex can be described with the following diagram,

$$0 \to H\Omega^0(M) \xrightarrow{d} H\Omega^1(M) \xrightarrow{d} \cdots \to H\Omega^m(M) \xrightarrow{d} 0. \tag{3.22}$$

Next, assume M_A is a polygonal, oriented, compact Riemannian manifold equipped with a metric g_A and an orientation preserving differmorphism $\varphi_A : M_A \to M$. For any point $x \in M_A$, let $\{e_1, ..., e_m\}$ and $\{f_1, ..., f_m\}$ be positively-oriented orthonormal (with respect to the given metric) bases for the tangent spaces $T_x M_A$ and $T_{\varphi(x)}M$. The tangent map $T_x\varphi_A : T_x M_A \to T_{\varphi(x)}M$ can be represented by an $m \times m$ matrix with m strictly positive singular values independent of the choice of basis,

$$\alpha_1(x) \geq \cdots \geq \alpha_m > 0.$$

The next theorem, from [24, 42], describes a useful property of these singular values; see also [15] for the classical version of the result in the case of domains in $\mathbb{R}^n$.

Theorem 3.4.1. *Let (M_A, g_h) and (M, g) be oriented m-dimensional Riemannian manifolds, and let $\varphi_h : M_A \to M$ be an orientation-preserving diffeomorphism with singular values $\alpha_1(x) \geq \cdots \geq \alpha_n(x) > 0$ at each $x \in M_A$. Given $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, and some $k = 0, \ldots, m$, suppose that the product $(\alpha_1 \ldots \alpha_k)^{1/p}(\alpha_{m-k+1} \ldots \alpha_m)^{-1/q}$ is bounded uniformly on M_A. Then, for any $\omega \in L^p\Omega^k(M_A)$,*

$$\begin{aligned} &\|(\alpha_1 \cdots \alpha_k)^{1/q}(\alpha_{k+1} \cdots \alpha_m)^{-1/p}\|_\infty^{-1}\|\omega\|_p \\ &\qquad \leq \|\varphi_{h*}\omega\|_p \leq \|(\alpha_1 \cdots \alpha_{m-k})^{1/p}(\alpha_{m-k+1} \cdots \alpha_m)^{-1/q}\|_\infty \|\omega\|_p. \end{aligned}$$

Holst and Stern then use the above theorem with $q = p = 2$, noting the compactness of the manifolds yields the uniform boundedness condition, to show that φ_A induces Hilbert complex isomorphisms $\varphi_{A*} : L^2\Omega(M_A) \to L^2\Omega(M)$ and $\varphi_A^* : L^2\Omega(M) \to L^2\Omega(M_A)$.

3.4.2 Signed Distance Functions and Euclidean Hypersurfaces

Let $M \subset \mathbb{R}^{m+1}$ be a compact, oriented, m dimensional Euclidean hypersurface. It is then possible to construct an open neighborhood, U, encompassing the surface with a well-defined mapping along normals to the surface, $a : U \to M$. Furthermore, associated to any such surface is a value $\delta_0 > 0$ such that the set of points whose Riemanian distance from M is less than δ_0 forms such a neighborhood. Given an adequate U, let $\delta : U \to \mathbb{R}$ be the standard signed distance function. Then for every $x \in U, \nabla\delta(x) = \nu(x)$ is the outward facing unit normal vector to the surface at $a(x)$, and

$$x = a(x) + \delta(x)\nu(x),$$

and the normal projection $a : U \to M$ can be expressed

$$a(x) = x - \delta(x)\nu(x).$$

Thus for any approximating surface, $M_h \subset U$, $a(x)$ restricted to M_h gives an orientation-preserving diffeomorphism, $\varphi_h(x) = a|_{M_h} : M_h \to M$, with well defined singular values. Therefore Theorem 3.4.1 can be applied.

The approximating surface is introduced as a computational tool used to approximate solutions to the Hodge Laplacian which are then mapped to the smooth approximated surface. In order to do this it is necessary to develop a map of k-forms between the two surfaces. In doing this we follow a subset of the ideas of [24, 25]. Letting $P = I - v \otimes v$ and $S = -\nabla v$, we have

$$\nabla a = I - \nabla\delta \otimes v - \delta\nabla v = I - v \otimes v - \delta\nabla v = P + \delta S.$$

This leads to the following theorem from [24] allowing for the computation of the pullback map, $a^* : \Omega^1(M) \to \Omega^1(U)$.

Theorem 3.4.2. *(Holst and Stern [24] Theorem 4.3) Let M be an oriented, compact, m-dimensional hyper surface of $\mathbb{R}^{m+1}$ with a tubular neighborhood U.*

If $Y \in T_y M$ and $x \in a^{-1}(y) \subset U$. them the lifted vector $a^*Y \in T_x U$ satisfies

$$a^*Y = (I + \delta S)Y$$

Proof. See [24]. □

Let $j : M \hookrightarrow \mathbb{R}^{m+1}$ and $j_h : M_h \hookrightarrow \mathbb{R}^{m+1}$ be inclusions of the submanifolds endowed with metrics $g = j^*\gamma$ and $g_h = j_h^*\gamma$, where γ is the standard Euclidean metric. For a point $x \in M_h$, the mapping can be restricted to $T_x M_h$ by composing a^*Y with the adjoint of j_h, yielding the adjoint of the restricted tangent map $T\varphi_h = j_h^* a^*$, satisfying

$$Y_h = j_h^* a^* Y = P_h(I + \delta S)Y.$$

3.4.3 Discrete Problem on a Euclidean Surface

The Hodge Laplacian defined on a Euclidean hypersurface is our main problem of interest: Find $(\sigma, u, p) \in H\Omega^{k-1}(M) \times H\Omega^k(M) \times \mathfrak{H}^k$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, && \forall \tau \in H\Omega^{k-1}(M), \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, && \forall v \in H\Omega^k(M), \\ \langle u, q \rangle &= 0, && \forall q \in \mathfrak{H}'^k. \end{aligned} \tag{3.23}$$

For the remainder of the paper, let M_A be an approximating surface satisfying assumptions of the previous section. Then φ_{A*} and φ_A^* act as the isomorphisms between $H\Omega(M)$ and $H\Omega(M_A)$. For ease of discussion, and similarity to the general maps in the FEEC frameworks, we use the notation i_A and π_A respectively for φ_{A*} and φ_A^*. This notation is also consistent with the current literature in the sense that φ_{A*} and φ_A^* are an injective and projective Hilbert complex morphisms. Since we have an isomorphism of Hilbert complexes we can define an equivalent problem on M_A using the map i_A. Find $(\sigma', u', p') \in H\Omega^{k-1}(M_A) \times H\Omega^k(M_A) \times \mathfrak{H}'^k$

such that

$$\begin{aligned}
\langle i_A\sigma', i_A\tau\rangle - \langle i_A d\tau, i_A u'\rangle &= 0, && \forall \tau \in H\Omega^{k-1}(M_A),\\
\langle i_A d\sigma', i_A v\rangle + \langle i_A du', i_A dv\rangle + \langle i_A p', i_A v\rangle &= \langle f, i_A v\rangle, && \forall v \in H\Omega^{k}(M_A),\\
\langle i_A u', i_A q\rangle &= 0, && \forall q \in \mathfrak{H}'^k.
\end{aligned}$$

This equivalent reformulation is helpful in defining a practical discrete problem: find $(\sigma'_h, u'_h, p'_h) \in V_h^{k-1}(M_A) \times V_h^k(M_A) \times \mathfrak{H}'^k_h$ such that

$$\begin{aligned}
\langle i_A\sigma'_h, i_A\tau\rangle - \langle d(i_A\tau), i_A u'_h\rangle &= 0, && \forall \tau \in V_h^{k-1}(M_A),\\
\langle d(i_A\sigma'_h), i_A v\rangle + \langle d(i_A u'_h), d(i_A v)\rangle + \langle i_A p'_h, i_A v\rangle &= \langle f, i_A v\rangle, && \forall v \in V_h^{k}(M_A),\\
\langle i_A u'_h, i_A q\rangle &= 0, && \forall q \in \mathfrak{H}'^k_h.
\end{aligned}$$

Here $\mathfrak{H}'^k$ and $\mathfrak{H}'^k_h$ are the spaces which i_A maps to harmonic forms in $H\Omega^k(M)$. The properties of these spaces will not affect our analysis, but it is worth noting that these spaces are distinct from $\mathfrak{H}^k$ and $\mathfrak{H}^k_h$ (see [23] for a detailed discussion).

Using this discrete formulation is equivalent to defining finite element spaces on the polygonal approximating surface and mapping them to M. The spaces on M_A can be refined with standard techniques yielding a refined mapped space. Using this discrete formulation, we will prove a convergent and optimal algorithm for solving (3.23). Notationally we will use $\mathcal{T}$ to represent a triangulation of the linear approximating surface, and $a(\mathcal{T})$ to represent the triangulation mapped to the approximated surface.

Unlike [23] we are dealing with manifolds which may not have a boundary, and thus harmonics may be present in the case $k = m$. However, in the case $k = m$, the harmonic component of f on a surface without boundary is simply the constant volume form. In this situation the harmonic component of f can be calculated efficiently, essentially reducing the problem to a $\mathfrak{B}$ problem. Therefore we focus on $\mathfrak{B}$ problems in the case $k = m$ for the remainder of the paper.

3.5 Approximation, Orthogonality, and Stability Properties

3.5.1 Approximation Properties

Before proceeding, we prove similar results to those of Section 2 for cases in which the finite element space is no longer constructed on triangulations of polygonal domains in $\mathbb{R}^n$. Here, and for the remainder of our analysis, we will use a triangulation of a polynomial approximating surface, M_A, and pull the spaces $\mathcal{P}_r\Omega^k(M_A)$ or $\mathcal{P}_r^-\Omega^k(M_A)$ to the surface M.

Next, we define $i_A^* : L^2\Omega(M) \to L^2\Omega(M_A)$ as the the adjoint of i_A, such that

$$\langle i_A u, i_A v\rangle_M = \langle i_A^* i_A u, v\rangle_{M_A}, \quad \forall u, v \in L^2\Omega(M_A).$$

The H^1 boundedness of i_A^* will be important in our convergence analysis. We prove this boundedness in Lemma 3.5.2, but first introduce an intermediary lemma.

Lemma 3.5.1. *Given $\tau \in L^2\Omega^k(M)$ we have*

$$i_A^* \tau = (-1)^{k(m-k)} \star_{M_A} \pi_A \star_M \tau \tag{3.24}$$

where $\star_M$ and $\star_{M_A}$ are the Hodge star operators related to the surfaces M and M_A.

Proof.

$$\begin{aligned}
\langle i_A^* \tau, \sigma\rangle_{M_A} &= \langle \tau, i_A \sigma\rangle_M, \\
&= \int_M \langle\langle \tau, i_A\sigma\rangle\rangle \mu_M, \\
&= \int_M i_A\sigma \wedge \star_M \tau, \\
&= \int_M i_A\sigma \wedge (i_A \pi_A) \star_M \tau
\end{aligned}$$

Next, since the pullback commutes with the wedge product and $\star_{M_A}\star_{M_A} = (-1)^{k(m-k)}$, we have

$$
\begin{aligned}
&= \int_{M_A} \sigma \wedge (\pi_A) \star_M \tau, \\
&= (-1)^{k(m-k)} \int_{M_A} \sigma \wedge (\star_{M_A} \star_{M_A}) \pi_A \star_M \tau, \\
&= (-1)^{k(m-k)} \int_{M_A} \langle\!\langle \sigma, \star_{M_A} \pi_A \star_M \tau \rangle\!\rangle \mu_{M_A}, \\
&= \langle \sigma, (-1)^{k(m-k)} \star_{M_A} \pi_A \star_M \tau \rangle_{M_A}.
\end{aligned}
$$

Given the construction of the Hilbert spaces this is sufficient to complete the proof. □

Lemma 3.5.2. *Let $\tau \in H^1\Omega^m(M)$, and let i_A be defined as above. Then $i_A^*\tau \in H^1\Omega^m(M_A)$, and*

$$
C_1 \| i_A^* \tau \|_{H^1\Omega^m(M_A)} \le \| \tau \|_{H^1\Omega^m(M)} \le C_2 \| i_A^* \tau \|_{H^1\Omega^m(M_A)}. \tag{3.25}
$$

Proof. Lemma 3.5.1 shows, in the case $k = m$, that the bounds on i_A^* are the same as those used for π_A in the case $k = 0$. In the case $k = 0$, π_A maps $H^1\Omega(M) \to H^1\Omega(M_A)$, with bounds introduced earlier. □

Next we introduce a new interpolant, I_{M_h} which is related to the canonical interpolant introduced earlier.

Definition 3.5.3. Let I_h be the canonical projection operator on $\mathcal{T}_h$, a triangulation of M_A. Then for $\tau \in H\Omega^k(M)$, we define I_{M_h} as

$$
I_{M_h} \tau = i_A I_h (\pi_A \tau)
$$

Lemma 3.5.4. *Suppose $\tau \in H^1\Omega^k(M)$, where $k = m - 1$ or $k = m$. Let I_{M_h} be the altered canonical projection operator introduced above. Then*

$$
I_{M_h} d = d I_{M_h} \tag{3.26}
$$

Proof. Using Lemma 3.3.2 and the fact that the pull-back and push-forward commute with d, we have

$$\begin{aligned} I_{M_h} d &= i_A I_h \pi_A d \\ &= i_A I_h d \pi_A \\ &= d(i_A I_h \pi_A). \end{aligned}$$

□

Lemma 3.5.5. *Let $f \in H\Omega^m(M)$, and let $T \in \mathcal{T}_h$. Then*

$$\int_{a(T)} (f_h - I_{M_H} f_h) = 0 \tag{3.27}$$

Proof. From (3.3.3) we know

$$\int_T (I_h(\pi_A f) - I_H(I_h \pi_A f)) = 0$$

and since $k = m$, we have

$$\int_{a(T)} (i_A I_h(\pi_A f) - i_A I_H(I_h \pi_A f)) = 0$$

yielding

$$\int_{a(T)} I_{M_h} f - (i_A I_H \pi_A) I_{M_h} f = 0$$

□

3.5.2 Quasi-Orthogonality

The main difficulty for mixed finite element methods is the lack of minimization principle, and thus the failure of orthogonality. In [23] results from [13] are generalized, and a quasi-orthogonality property is proven using the fact that $\sigma - \sigma_h$ is orthogonal to the subspace $\mathfrak{Z}_h^{n-1} \subset H\Lambda_h^{n-1}(\Omega)$. In this section we show that this same orthogonality result holds for finite elements spaces mapped to the smooth surface from the approximating surface.

Solutions of Hodge Laplace problems on nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$ will frequently be compared. Nested in the sense that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. For a given $f \in L^2\Omega^m(M)$, let $\mathcal{L}^{-1}f$ denote the solutions of (3.23). Let $\mathcal{L}_h^{-1}f_h$ and $\mathcal{L}_H^{-1}f_H$ denote the solutions to the discrete problems on $a(\mathcal{T}_h)$ and $a(\mathcal{T}_H)$ respectively. Set the following triples, $(u,\sigma,p) = \mathcal{L}^{-1}f$, $(u_h,\sigma_h,p_h) = \mathcal{L}_h^{-1}f_h$, $(\tilde{u}_h,\tilde{\sigma}_h,\tilde{p}_h) = \mathcal{L}_h^{-1}f_H$ and $(u_H,\sigma_H,p_H) = \mathcal{L}_H^{-1}f_H$. The following analysis deals with the $\mathfrak{B}$ problem and thus the harmonic component will be zero in each of these solutions. When we are only interested in σ we will abuse this notation by writing $\sigma = \mathcal{L}^{-1}f$.

Lemma 3.5.6. *Given $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$, and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then*

$$\langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle_M = 0. \tag{3.28}$$

Proof. See [23]. □

The next result is similar to Theorem 4.2 in [23]. We present the proof in order to clarify the impact of the surface mapping.

Theorem 3.5.7. *Given $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$, and two nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, then*

$$\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle \le \sqrt{C_0}\|\sigma - \sigma_h\| osc(\pi_A f_h, \mathcal{T}_H), \tag{3.29}$$

and for any $\delta > 0$,

$$(1-\delta)\|\sigma - \sigma_h\|^2 \le \|\sigma - \sigma_H\|^2 - \|\sigma_h - \sigma_H\|^2 + \frac{C_0}{\delta} osc^2(\pi_A f_h, \mathcal{T}_H). \tag{3.30}$$

Proof. By (3.28) we have

$$\begin{aligned}
\langle \sigma - \sigma_h, \sigma_h - \sigma_H \rangle &= \langle \sigma - \sigma_h, \sigma_h - \tilde{\sigma}_h \rangle + \langle \sigma - \sigma_h, \tilde{\sigma}_h - \sigma_H \rangle \\
&= \langle \sigma - \sigma_h, \sigma_h - \tilde{\sigma}_h \rangle \\
&\le \|\sigma - \sigma_h\| \|\sigma_h - \tilde{\sigma}_h\|.
\end{aligned}$$

And then by the discrete stability result, Theorem 3.5.11, we have

$$\leq \sqrt{C_0}\|\sigma - \sigma_h\|\text{osc}(\pi_A f_h, \mathcal{T}_H).$$

(3.30) follows standard arguments and is identical to [13] (3.4) □

3.5.3 Continuous and Discrete Stability

In this section we will prove stability results for approximate solutions to the σ portion of the Hodge Laplace problem. Theorem 3.5.8 gives a stability result for particular solutions of the Hodge de Rham problem that will be useful in bounding the approximation error in Section 3.6. Theorem 3.5.11 will prove the discrete stability result used in Theorem 3.5.7. These proofs follow the same structure as [23], with additional steps that take care of the mapping between the surfaces.

Theorem 3.5.8. *(Continuous Stability Result) Given $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$, let $\mathcal{T}_h$ be a triangulation of M_A. Set $(\sigma, u, p) = \mathcal{L}^{-1} f$ and $(\tilde{\sigma}, \tilde{u}, \tilde{p}) = \mathcal{L}^{-1} f_h$, then*

$$\|\sigma - \tilde{\sigma}\| \leq C osc(\pi_A f, \mathcal{T}_h). \tag{3.31}$$

Proof. The harmonic terms are vacuous, thus

$$\|\sigma - \tilde{\sigma}\|_M^2 = \langle d(\sigma - \tilde{\sigma}), u - \tilde{u}\rangle_M = \langle f - f_h, u - \tilde{u}\rangle_M = \langle \pi_A(f - f_h), i_A^*(u - \tilde{u})\rangle_{M_A}.$$

Let $v = u - \tilde{u}$. Since $v \in \mathfrak{B}^k$ and $\|\delta v\| = \|\text{grad } v\| = \|\sigma - \tilde{\sigma}\|$, we have $v \in H^1\Omega^m(M)$. Restricting v to an element $a(T) \in a(\mathcal{T}_h)$, we have $v \in H^1\Omega^m(a(T))$, thus

$$\begin{aligned}\|\sigma - \tilde{\sigma}\|^2 = \langle f - f_h, v\rangle &= \sum_{T\in\mathcal{T}_h} \langle \pi_A(f - f_h), i_A^* v\rangle_T. \\ &= \sum_{T\in\mathcal{T}_h} \langle \pi_A f - \pi_A f_h, i_A^* v - I_h(i_A^* v)\rangle_T.\end{aligned}$$

Applying (3.12) and then (3.5.2),

$$\begin{aligned}
&\leq C \sum_{T\in\mathcal{T}_h} h_T \|\pi_A f - \pi_A f_h\|_T \|i_A^* v\|_{H^1\Lambda^n(T)} \\
&\leq C \sum_{T\in\mathcal{T}_h} h_T \|\pi_A f - \pi_A f_h\|_T \|v\|_{H^1\Lambda^n(a(T))} \\
&= C \sum_{T\in\mathcal{T}_h} h_T \|\pi_A f - \pi_A f_h\|_T (\|u-\tilde{u}\|_{a(T)} + \|\delta(u-\tilde{u})\|_{a(T)}) \\
&\leq C (\sum_{T\in\mathcal{T}_h} \|h_T(\pi_A f - \pi_A f_h)\|_T^2)^{1/2} (\sum_{T\in T_h} (\|u-\tilde{u}\|_{a(T)} + \|\delta(u-\tilde{u})\|_{a(T)})^2)^{1/2},
\end{aligned}$$

and $v \in H^1\Omega^m(M)$ allows us to to combine terms of the summation,

$$\leq C(\sum_{T\in\mathcal{T}_h} \|h_T(\pi_A f - \pi_A f_h)\|_T^2)^{1/2}(\|u-\tilde{u}\|_M + \|\delta(u-\tilde{u})\|_M).$$

Since $u - \tilde{u} \in \mathfrak{B}^k$, $\|u-\tilde{u}\| = \langle u-\tilde{u}, d\tau\rangle$ for some $\tau \in \mathfrak{Z}^\perp$ with $\|d\tau\| = 1$, thus

$$= C(\sum_{T\in\mathcal{T}_h} \|h_T(\pi_A f - \pi_A f_h)\|_T^2)^{1/2}\langle(\sigma-\tilde{\sigma}), \tau\rangle_M + \|\sigma-\tilde{\sigma}\|_M).$$

Then applying Poincaré on τ:

$$= C\|\sigma-\tilde{\sigma}\|(\sum_{T\in\mathcal{T}_h} \|h_T(\pi_A f - \pi_A f_h)\|_T^2)^{1/2}.$$

Divide through by $\|\sigma - \tilde{\sigma}\|$ to complete proof. □

The following is Lemma 4 in [18], and is a special case of Theorem 1.5 of [30]. It is related to the bounded invertibility of d, and will be an important tool in proving discrete stability.

Lemma 3.5.9. *Assume that B is a bounded Lipschitz domain in $\mathbb{R}^n$ that is homeomorphic to a ball. Then the boundary value problem $d\varphi = g \in L_2\Lambda^k(B)$ in $B, tr\ \varphi = 0$ on ∂B has a solution $\varphi \in H_0^1\Lambda^{k-1}(B)$ with $\|\varphi\|_{H^1\Lambda^{k-1}(B)} \leq C\|g\|_B$ if and only if $dg = 0$ in B, and in addition, $tr\ g = 0$ on ∂B if $0 \leq k \leq n-1$ and $\int_B g = 0$ if $k = n$.*

The next lemma is an intermediate step in proving the discrete stability result. The general structure follows [13] and applies Lemma 3.5.9 in order to find a sufficiently smooth function that is essentially a bounded inverse of d for the approximation error of u_h on $\mathcal{T}_H$.

In [23], Lemma 3.3.3 is applied to shape regular polygonal elements, and thus the multiplicative constant can be bounded. In this case, however, the elements are not necessarily polygonal, and thus we are forced to map the proof to M_A, where the regularity is clear, and then map back to M.

Lemma 3.5.10. *Let $\mathcal{T}_h$, $\mathcal{T}_H$ be nested conforming triangulations and let σ_h, σ_H be the respective solutions to* (3.5) *with data $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$. Then for any $T \in \mathcal{T}_H$*

$$\|I_h i_A^* u_h - I_H i_A^* u_h\|_T \leq \sqrt{C_0} h_T \|\sigma_h\|_{a(T)}. \tag{3.32}$$

Proof. Let $g_\Omega = I_h i_A^* u_h - I_H i_A^* u_h = (I_h - I_H) i_A^* u_h \in L^2\Omega^m(M_A)$. Then, for any $T \in \mathcal{T}_H$ let $g = \mathrm{tr}_T g_\Omega \in L^2\Omega^m(T)$, and by Lemma 3.3.3, $\int_T g = 0$. Thus Lemma 3.5.9 can be applied to find $\tau \in H_0^1\Lambda^{n-1}(T)$, such that:

$$d\tau = (I_h - I_H) i_A^* u_h, \text{ on } T$$
$$\|\tau\|_{H^1\Lambda^{n-1}(T)} \leq C \|(I_h - I_H) i_A^* u_h\|_T.$$

Extend τ to $H^1\Lambda^{n-1}(\Omega)$ by zero and then, by Lemma 3.3.4,

$$\|(I_h - I_H) i_A^* u_h\|_T^2 = \langle (I_h - I_H) i_A^* u_h, d\tau \rangle_T = \langle i_A^* u_h, d(I_h - I_H)\tau \rangle_T$$

Then by Lemma 3.3.2, and locality of τ,

$$= \langle i_A^* u_h, d(I_h - I_H)\tau \rangle_{M_A} = \langle u_h, d(i_A(I_h - I_H)\tau) \rangle_M = \langle \sigma_h, i_A(I_h - I_H)\tau \rangle_M.$$

Then again by locality of τ and Theorem 3.4.1,

$$\begin{aligned} = \langle \sigma_h, i_A(I_h - I_H)\tau \rangle_{a(T)} &\leq \|\sigma_h\|_{a(T)} (\|i_A(\tau - I_h\tau)\|_{a(T)} + \|i_A(\tau - I_H\tau)\|_{a(T)}), \\ &\leq \|\sigma_h\|_{a(T)} (\|\tau - I_h\tau\|_T + \|\tau - I_H\tau\|_T). \end{aligned}$$

And by (3.12),

$$\leq Ch_T\|\sigma_h\|_{a(T)}\|\tau\|_{H^1(T)} \leq Ch_T\|\sigma_h\|_{a(T)}\|(I_h - I_H)i_A^* u_h\|_T.$$

Cancel one power of $\|(I_{M_h} - I_{M_H})u_h\|_T$ to complete the proof.

□

Theorem 3.5.11. *(Discrete Stability Result) Let $\mathcal{T}_h$ and $\mathcal{T}_H$ be nested conforming triangulations. Let $(\tilde{u}_h, \tilde{\sigma}_h, \tilde{p}_h) = \mathcal{L}_h^{-1} f_H$ and $(u_h, \sigma_h, p_h) = \mathcal{L}_h^{-1} f_h$, with $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$. Then there exists a constant such that*

$$\|\sigma_h - \tilde{\sigma}_h\| \leq C osc(\pi_A f_h, \mathcal{T}_H) \tag{3.33}$$

Proof. From 3.5, and since $p_h, \tilde{p}_h = 0$, we have

$$\langle \sigma_h - \tilde{\sigma_h}, \tau_h \rangle = \langle u_h - \tilde{u}_h, d\tau_h \rangle, \quad \forall \tau_h \in \Lambda_h^{k-1}, \tag{3.34}$$

$$\langle d(\sigma_h - \tilde{\sigma_h}), v_h \rangle = \langle f_h - f_H, v_h \rangle, \quad \forall v_h \in \Lambda_h^k. \tag{3.35}$$

Next set $\tau_h = \sigma_h - \tilde{\sigma}_h$ in (3.34), and $v_h = u_h - \tilde{u}_h$ in (3.35) to obtain:

$$\|\sigma_h - \tilde{\sigma}_h\|^2 = \langle u_h - \tilde{u}_h, d(\sigma_h - \tilde{\sigma}_h) \rangle = \langle f_h - f_H, v_h \rangle,$$

Then by Lemma 3.5.10, we have:

$$\begin{aligned}
\|\sigma_h - \tilde{\sigma}_h\|^2 &= \sum_{T\in\mathcal{T}_H} \langle v_h, f_h - f_H\rangle_{a(T)} \\
&= \sum_{T\in\mathcal{T}_H} \langle i_A^* v_h, I_h\pi_A f - I_H\pi_A f\rangle_T \\
&= \sum_{T\in\mathcal{T}_H} \langle I_h(i_A^* v_h), I_h\pi_A f - I_H\pi_A f\rangle_T \\
&= \sum_{T\in\mathcal{T}_H} \langle I_h(i_A^* v_h) - I_H(i_A^* v_h), I_h\pi_A f - I_H\pi_A f\rangle_T \\
&\leq C \sum_{T\in\mathcal{T}_H} \|I_h\pi_A f - I_H\pi_A f\|_T \|I_h i_A^* v_h - I_H i_A^* v_h\|_T \\
&\leq C \sum_{T\in\mathcal{T}_H} h_T \|I_h\pi_A f - I_H\pi_A f\|_T \|(\sigma_h - \tilde{\sigma}_h)\|_{a(T)} \\
&\leq C (\sum_{T\in\mathcal{T}_H} h_T^2 \|I_h\pi_A f - I_H\pi_A f\|_T^2)^{1/2} \|\sigma_h - \tilde{\sigma}_h\|_M
\end{aligned}$$

Then cancel one $\|\sigma_h - \tilde{\sigma}_h\|$ to complete the proof.

□

3.6 *A Posteriori* Error Indicator and Bounds

In this section we introduce the *a posteriori* error estimators used in our adaptive algorithm. The estimator follows from [23] which follows [3, 13]. The difference here is that we estimate the error on the fixed approximating surface. Next, applying ideas [23] to the surface estimator, we prove bounds on these estimators and a continuity result, both of which are key ingredients in showing the convergence and optimality of our adaptive method.

3.6.1 Error Indicator: Definition, Lower bound and Continuity

Definition 3.6.1. *(Element Error Indicator)* Let $T \in \mathcal{T}_H, f \in L^2\Omega^m(M)$ in $\mathfrak{B}$, and $\sigma_H = \mathcal{L}^{-1} f_H$. Let the jump in τ over an element face be denoted by $[[\tau]]$. For element faces on $\partial\Omega$ we set $[[\tau]] = \tau$. The element error indicator is defined as

$$\eta_T^2(\sigma_H) = h_T \|[[tr\ \star(\pi_A\sigma_H)]]\|_{\partial T}^2 + h_T^2\|\delta(\pi_A\sigma_H)\|_T^2 + h_T^2\|\pi_A f - d(\pi_A\sigma_H)\|_T^2$$

For a subset $\tilde{\mathcal{T}}_H \subset \mathcal{T}_H$, define

$$\eta^2(\sigma_H, \tilde{\mathcal{T}}_H) := \sum_{T\in\tilde{\mathcal{T}}_H} \eta_T^2(\sigma_H)$$

Theorem 3.6.2. *(Lower Bound) Given $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$ and a shape regular triangulation $\mathcal{T}_H$, let $\sigma = \mathcal{L}^{-1}f$ and $\sigma_H = \mathcal{L}_H^{-1}f_H$. Then there exists a constant dependent only on the shape regularity of $\mathcal{T}_H$ and the surface mapping, such that*

$$C_2\eta^2(\sigma_H, \mathcal{T}_H) \le \|\sigma - \sigma_H\|^2 + C_2\, osc^2(\pi_A f, \mathcal{T}_H). \tag{3.36}$$

Proof. In proving a lower bound, in [18] it is shown that

$$h_T\|\delta\sigma_H\|_T \le C\|\sigma - \sigma_H\|_T,$$
$$h_T^{1/2}\|[[\mathrm{tr}\star\sigma_H]]\|_{\partial T} \le C\|\sigma - \sigma_H\|_{\mathcal{T}_t},$$

where $\mathcal{T}_t$ is the set of all triangles sharing a boundary with T. The first is equation (5.7) and the second is a result of equation (5.12) in [18]. Substituting $\pi_A\sigma_H$ for σ, noting that $\sigma \in \mathfrak{Z}^\perp(M)$ implies $\pi_A\sigma \in \mathfrak{Z}^\perp(M_A)$, similar results for $\pi_A\sigma_H$ follow [18]. Then, using the boundedness of i_A, the remainder of the proof is identical to [23]. □

The following lemma will be important in proving a continuity result used in showing convergence of our adaptive algorithm. It is nearly identical to an estimator efficiency proof in [18], but the subtle difference is that we make use of σ_H, the solution on the less refined mesh, and σ is not used in our arguments.

Lemma 3.6.3. *Given $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$ and nested triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$,* *let* $\sigma_h = \mathcal{L}_h^{-1} f_h$ *and* $\sigma_H = \mathcal{L}_H^{-1} f_H$. *Then for* $T \in \mathcal{T}_h$

$$C_2 \sum_{T\in\mathcal{T}_h} (h_T \| [[tr \star (\pi_A \sigma_h - \pi_A \sigma_H)]] \|^2_{\partial T} + h_T^2 \| \delta(\pi_A \sigma_h - \pi_A \sigma_H) \|^2_T) \leq \| \pi_A \sigma_h - \pi_A \sigma_H \|^2. \tag{3.37}$$

Proof. Follows [23]. □

Theorem 3.6.4. *(Continuity of the Error Estimator) Given* $f \in L^2\Omega^m(M)$ *in* $\mathfrak{B}$ *and nested triangulations* $\mathcal{T}_h$ *and* $\mathcal{T}_H$, *let* $\sigma_h = \mathcal{L}_h^{-1} f_h$ *and* $\sigma_H = \mathcal{L}_H^{-1} f_H$. *Then we have:*

$$\beta(\eta^2(\sigma_h, \mathcal{T}_h) - \eta^2(\sigma_H, \mathcal{T}_h)) \leq \| \pi_A \sigma_h - \pi_A \sigma_H \|^2 + osc^2(\pi_A f_h, \mathcal{T}_H) \tag{3.38}$$

Proof. Follows [23]. □

3.6.2 Continuous and Discrete Upper Bounds

The following proofs have a similar structure to the continuous and discrete upper bounds proved in [3, 13]. A key element of the proof will be comparisons between the discrete solution $\sigma_H = \mathcal{L}_H^{-1} f_H$ and the solution to the intermediate problem, $\tilde{\sigma} = \mathcal{L}^{-1} f_H$. We begin by looking the orthogonal decomposition of $\tilde{\sigma} - \sigma_H$,

$$\tilde{\sigma} - \sigma_H = (\tilde{\sigma} - P_{\mathfrak{Z}^\perp} \sigma_H) - P_{\mathfrak{B}^{k-1}} \sigma_H - P_{\mathfrak{H}^{k-1}} \sigma_H$$

which allows the norm to be rewritten

$$\| \tilde{\sigma} - \sigma_H \|^2 = \| (\tilde{\sigma} - P_{\mathfrak{Z}^\perp} \sigma_H) \|^2 + \| P_{\mathfrak{B}^{k-1}} \sigma_H \|^2 + \| P_{\mathfrak{H}^{k-1}} \sigma_H \|^2.$$

Lemmas 3.6.5, 3.6.6 and 3.6.7 will each bound a portion of this orthogonal decomposition. Then Theorem 3.6.8 will combine these results in proving the desired error bound.

Lemma 3.6.5. *Given an* $f \in L^2\Omega^m(M)$ *in* $\mathfrak{B}$. *Let* $\tilde{\sigma} = \mathcal{L}^{-1} f_H$ *and* $\sigma_H = \mathcal{L}_H^{-1} f_H$.

Then

$$\|(\tilde{\sigma} - P_{\mathfrak{Z}^\perp}\sigma_H)\|^2 = 0. \tag{3.39}$$

Proof. See [23]. □

The next lemma uses the quasi-interpolant Π_H described in [18], and also applies integration by parts in the same standard fashion that [18] use when bounding error measured in the natural norm, $\|u - u_h\|_{H\Lambda^k(\Omega)} + \|\sigma - \sigma_h\|_{H\Lambda^{k-1}(\Omega)} + \|p - p_h\|$. In [18], coercivity of the bilinear-form is used to separate components of the error, whereas here we simply analyze the orthogonal decomposition of $\sigma - \sigma_H$. In [18], the Galerkin orthogonality implied by taking the difference between the continuous and discrete problems is employed in order to make use of Π_h. Here we are able to introduce the quasi-interpolant by simply using the fact that $\sigma_H \perp \mathfrak{B}_H^{k-1}$.

Lemma 3.6.6. *Given an $f \in L^2\Omega^m(M)$ in $\mathfrak{B}$. Let $\sigma_H = \mathcal{L}_H^{-1} f_H$. Then*

$$\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2 \leq C\eta^2(\sigma_H, \mathcal{T}_H). \tag{3.40}$$

Proof. Follow the same steps as [23] using $\|\pi_A(P_{\mathfrak{B}^{k-1}}\sigma_H)\|^2$. Next use the boundedness of π_A to relate to $\|P_{\mathfrak{B}^{k-1}}\sigma_H\|^2$. □

Lemma 3.6.7. *Given an $f \in L^2\Lambda^k(\Omega)$ in $\mathfrak{B}^k$. Let $\tilde{\sigma} = \mathcal{L}^{-1} f_H$ and $\sigma_H = \mathcal{L}_H^{-1} f_H$. Then*

$$\|P_{\mathfrak{H}^{k-1}}\sigma_H\|^2 \leq C\|\tilde{\sigma} - \sigma_H\|^2, \qquad C < 1. \tag{3.41}$$

Proof. See [23]. □

Now we have the tools to prove the continuous upper bound for the $\mathfrak{B}$ problems.

Theorem 3.6.8. *(Continuous Upper-Bound) Given an $f \in L^2\Omega^m(M)$ in $\mathfrak{B}^m$. Let $\tilde{\sigma} = \mathcal{L}^{-1} f_H$ and $\sigma_H = \mathcal{L}_H^{-1} f_H$. Then*

$$\|\sigma - \sigma_H\|^2 \leq C_1\eta^2(\sigma_H, \mathcal{T}_H). \tag{3.42}$$

Proof. See [23]. □

Theorem 3.6.9. *(Discrete Upper-Bound) Given* $f \in L^2\Omega^m(M)$ *in* $\mathfrak{B}$ *and nested triangulations* $\mathcal{T}_h$ *and* $\mathcal{T}_H$*, let* $\sigma_h = \mathcal{L}_h^{-1} f_h$ *and* $\sigma_H = \mathcal{L}_H^{-1} f_H$*. Then*

$$\|\sigma_h - \sigma_H\|^2 \leq C_1 \eta^2(\sigma_H, T_H). \tag{3.43}$$

Proof. The proof requires the same ingredients needed to prove the continuous upper bound. The same intermediate steps are taken by performing analysis on the W_h^{k-1} orthogonal decomposition of $\tilde{\sigma}_h - \sigma_H$.

$$\tilde{\sigma}_h - \sigma_H = (\tilde{\sigma}_h - P_{\mathfrak{Z}_h^\perp} \sigma_H) - P_{\mathfrak{B}_h^{k-1}} \sigma_H - P_{\mathfrak{H}_h^{k-1}} \sigma_H.$$

The discrete version of Lemma 3.6.5 uses δ_h rather than δ, but is otherwise identical. The discrete version of Lemma 3.6.6 is identical. The discrete version of Lemma 3.6.7 follows the same structure but makes use of Corollary 3.3.8. The final step in the proof uses the discrete stability result, Theorem 3.5.11.

□

3.7 Convergence of AMFEM

After presenting the adaptive algorithm, the remainder of this section proves convergence and then optimality. The results in this section follow ideas already in the literature [43, 31, 32, 20, 13], with Theorem 3.7.3 following [23] in proving reduction in a quasi-error using relationships between data oscillation and the decay of a second type of quasi-error. The following algorithm and analysis of convergence deal specifically with the case $k = m$. In presenting our algorithm we replace h with an iteration counter k.

Algorithm:$[\mathcal{T}_N, \sigma_N\ ] = \mathsf{AMFEM}(\mathcal{T}_0, f, \epsilon, \theta)$: Given a fixed approximating surface of M, an initial shape-regular triangulation $\mathcal{T}_0$, and a marking parameter θ, set $k = 0$ and iterate the following steps until a desired decrease in the error-estimator is achieved:

$$(1)(u_k, \sigma_k, p_k) = SOLVE(f, \mathcal{T}_k)$$
$$(2)\{\eta_T\} = ESTIMATE(f, \sigma_k, \mathcal{T}_k)$$
$$(3)\mathcal{M}_k = MARK(\{\eta_T\}, \mathcal{T}_k, \theta)$$
$$(4)\mathcal{T}_{k+1} = REFINE(\mathcal{T}_k, \mathcal{M}_k)$$

3.7.1 Convergence of AMFEM

The following notation will be used in the proofs and discussion of this section:

$$e_k = \|\sigma - \sigma_k\|^2, \qquad E_k = \|\sigma_{k+1} - \sigma_k\|^2, \qquad \eta_k = \eta^2(\sigma_k, \mathcal{T}_k),$$

$$o_k = \text{osc}^2(f, \mathcal{T}_k), \qquad \hat{o}_k = \text{osc}^2(f_{k+1}, \mathcal{T}_k),$$

where $f_k = P_k f = P_{\mathfrak{B}_k} f$ since $k = m$.

Lemma 3.7.1.

$$\beta\eta_{k+1} \leq \beta(1 - \lambda\theta)\eta_k + E_k + \hat{o}_k. \tag{3.44}$$

Proof. This follows from continuity of the error estimator (3.38), and properties of the marking strategy, i.e. reduction of the summation on a finer mesh due to smaller element sizes on refined elements. The proof can be found in [13]. $\lambda < 1$ is a constant dependent on the dimensionality of the problem. □

For convenience, we recall the quasi-orthogonality (3.30) the continuous upper-bound (3.42) equations,

$$(1 - \delta)e_{k+1} \leq e_k - E_k + C_0\hat{o}_k, \text{ for any } \delta > 0,$$
$$e_k \leq C_1\eta_k.$$

With these three ingredients, basic algebra leads to the following result,

Theorem 3.7.2. *When*

$$0 < \delta < min\{\frac{\beta}{2C_1}\theta, 1\}, \tag{3.45}$$

there exists $\alpha \in$ *(0,1) and* C_δ *such that*

$$(1-\delta)e_{k+1} + \beta\eta_{k+1} \leq \alpha[(1-\delta)e_k + \beta\eta_k] + C_\delta \hat{o}_k. \tag{3.46}$$

Proof. Follows the same steps as [13]. □

With the above result we next prove convergence.

Theorem 3.7.3. *(Termination in Finite Steps) Let* σ_k *be the solution obtained in the kth loop in the algorithm AMFEM, then for any* $0 < \delta < min\{\frac{\beta}{2C_1}\theta, 1\}$*, there exists positive constants* C_δ *and 0<* $\gamma_\delta < 1$ *depending only on given data and the initial grid such that,*

$$(1-\delta)\|\sigma - \sigma_k\|^2 + \beta\eta^2(\sigma_k, \mathcal{T}_k) + \zeta osc^2(f, \mathcal{T}_k) \leq C_q \gamma_\delta^k,$$

and the algorithm will terminate in finite steps.

Proof. See [23]. □

3.7.2 Optimality of AMFEM

Once Theorem 3.5.8, Theorem 3.3.10, (3.43), (3.30) and (3.36) are established, optimality can be proved independent of dimension following the proof of Theorem 5.3 in [43].

Theorem 3.7.4. *(Optimality) For any* $f \in L^2\Omega^m(M)$ *in* $\mathfrak{B}$*, shape regular* $\mathcal{T}_0$ *and* $\epsilon > 0$*, let* $\sigma = \mathcal{L}^{-1}f$ *and [* $\sigma_N, \mathcal{T}_N$ *] =* AMFEM*(* $\mathcal{T}_H, f_H, \epsilon/2, \theta$*). Where [*$\mathcal{T}_H, f_H$*] =* APPROX*(f,* $\mathcal{T}_0, \epsilon/2$ *). If* $\sigma \in \mathcal{A}^s$ *and* $f \in \mathcal{A}_o^s$*, then*

$$\|\sigma - \sigma_N\| \leq C(\|\sigma\|_{\mathcal{A}^s} + \|f\|_{\mathcal{A}_o^s})(\#\mathcal{T}_N - \#\mathcal{T}_0)^{-s}. \tag{3.47}$$

Proof. Follows directly from [13]. □

The key components in the optimality of a method are the rate of the convergence and the decaying constant, and (3.47) is a good model equation for analysis. Placing no restrictions on node placement substantially increases degrees of freedom and computational cost to the subspace approximation. We thus restrict our discussion of optimality to the two basic cases; evolving surfaces with element nodes lying on the approximated surface, and the scheme used above.

The rate rate of decay, s, is an intrinsic property related to a functions approximation class for a given refinement method. The map between surfaces is a Hilbert complex isomorphism, and thus $H\Omega^k(M)$ will be mapped to $H\Omega^k(M_A)$, analogous to mapping between similar Sobelev spaces. For example, when $k = m - 1$ elements in $H(div)$ on M will be mapped to $H(div)$ on M_A. Also, since the mapping is smooth between the two surfaces, preserving the differentiability properties of the forms. The relationship between the smoothness of the solution and data f to their approximation class is discussed in [8, 9].

Next we look at the multiplicative constant. One advantage of the evolving surface approximation is that the multiplicative bounds in Theorem 3.4.1 improve with better surface approximations. If the initial surface approximation is good, however, this constant is negligible in terms of computation cost. Other inefficiencies may arise by building the initial surface approximation without much analysis of the PDE. Interpolation of the surface can be done in a standard efficient manner, and as long as the initially surface isn't excessively precise, then the impact on C in (3.47) will not be significant. The other portion of the multiplicative constant is related to the norm of f and σ mapped to the approximating surface, and this value should be reasonable by the same arguments used for the rate of decay.

3.8 Conclusion and Future Work

Surface finite element methods, in their nature, have additional complexities which introduce difficulties developing a generic adaptive algorithm. Surfaces, for instance, can be described in different manners and, depending on the access to surface quantities, algorithms that are ideal in one case may be infeasible in others.

Also, when refining a mesh, element nodes are not necessarily required to lie on the approximated surface, or even alter the approximating surface between iterations. Continually improving the surface approximation has desirable features, but it also complicates the analysis of convergence and optimality. Along these lines, developing a method similar to [17] where the nodes of the mesh are require to lie on M, and thus the surface approximation continually improves, would be of interest.

As was the case in [23], in this paper we have focused on the error $\|\sigma - \sigma_h\|$ for the Hodge Laplacian in the specific case $k = m$. The results in [23], with the exception of the stability results, applied to general $\mathfrak{B}$ problems and such a generalizing the theory to this class of problems would be a desirable results. The proofs above introduce no additional complications in generalizing methods for $\mathfrak{B}$ problems to surfaces, and therefore an extension of the results in [23] would likely generalize to surfaces.

Analysis of adaptivity in the natural norm, $\|u-u_h\|_{H\Omega^k(M)}+\|\sigma-\sigma_h\|_{H\Omega^{k-1}(M)}+\|p-p_h\|$, is another direction of interest. Such indicators on polygonal domains are analyzed in [18], and using the results from [24, 25], these results can be extended to surfaces with additive geometrical terms. Analysis of algorithms using these indicators is another area of interest.

3.9 Acknowledgments

MH was supported in part by NSF Awards 1217175 and 1065972. AM and RS was supported in part by NSF Award 1217175.

Chapter 3 is in the preprint stage and will, in full, be submitted for publication. The material may appear as M. Holst, A. Mihalik and R. Szypowski, 2014. The dissertation author was the primary investigator and author of this paper.

Chapter 4

A Basic Computational Example

4.1 Introduction

In this chapter we give a basic computational example of the method introduced in Chapter 2. Implementation, in the cases $n = 2, 3$, is accomplished with the help of a collection of tools from the FEniCS Project[28, 29, 2]. The infrastructure is developed in a general fashion, and can be extended to higher dimensions in the presence of higher-dimensional elements. In the following section we will step through an example in the case $n = 2$, and show the algorithm achieves an optimal rate of convergence.

4.2 Computational Example

The following example uses data $f = -\Delta u$, in the case

$$u = -\sin(\pi x) \times \sin(\pi y) \times \frac{1}{(x-1)^2 + (y-1)^2 + .001}.$$

The domain will be the combination of unit squares meeting at $(1, 1)$. Figure 4.1 shows a plot of the data f on the given domain. It is clear that specific areas of the domain have much different features with respect to the differentiation of u. As we will show, this allows the adaptive method to gain efficiency when compared to refining the elements uniformly.

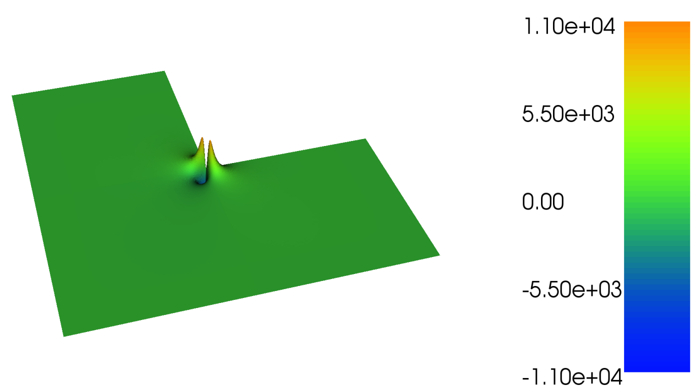

Figure 4.1: Example Problem: Data f on a domain with corners.

Figure 4.2 shows the initial mesh of the domain and the mesh generated adaptively with 12,254 elements. It is clear that the adaptive method is allocating more degrees of freedom near the point $(1, 1)$. This follows intuition, as the values of σ in the region are larger and change rapidly.

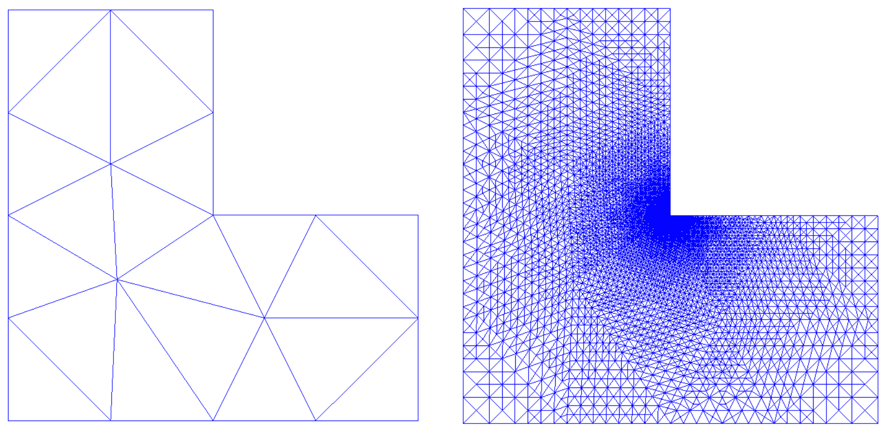

Figure 4.2: Initial mesh and adaptively refined (13,254 elements) mesh

Figure 4.3 and Figure 4.4 show plots of $\sigma = du$, in the cases using $3K$ and $161K$ elements. These plots help show why an accurate approximation of σ requires a finer mesh in certain sections of the domain.

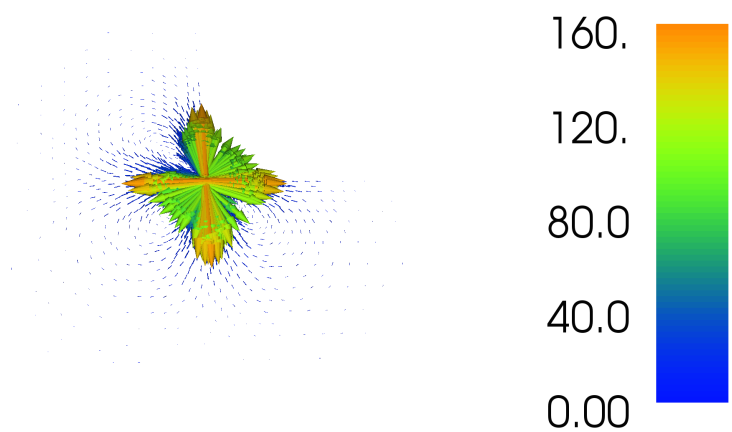

Figure 4.3: Approximation of σ with 3K Elements.

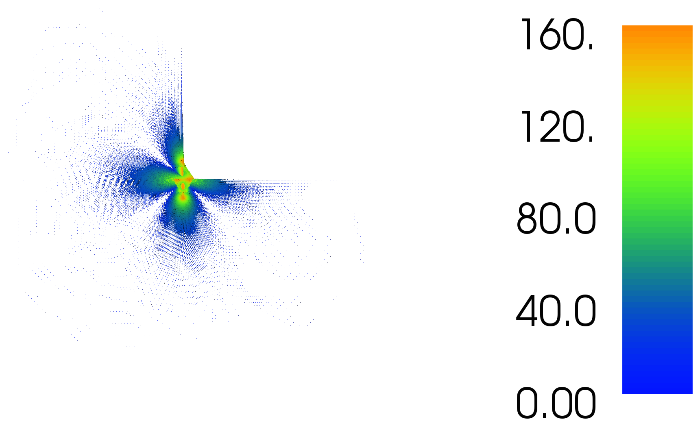

Figure 4.4: Approximation of σ with 161K Elements.

Figure 4.5 demonstrates the advantage of the adaptive algorithm over uniform refinement, and also shows that the adaptive method achieves optimal order of convergence. Initially, when using a small number of elements, the adaptive method and uniform refinement produce similar results. As the number of iterations increases, however, it becomes clear that the adaptive method can achieve equivalent error reduction with less than 5-percent of the elements needed using uniform refinement. The optimal order of convergence using the most basic linear elements is $N^{-\frac{1}{2}}$, and the red line shows that the adaptive method is achieving this rate of convergence.

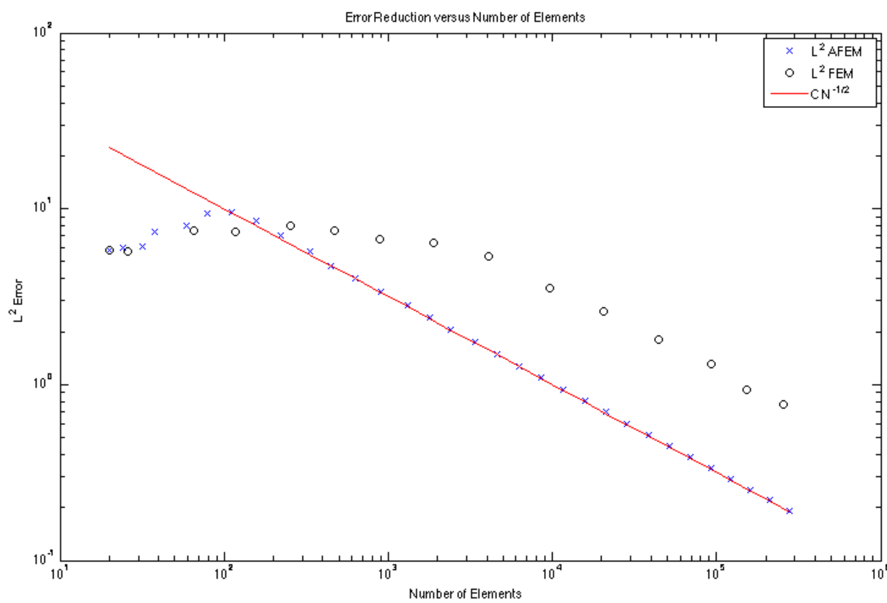

Figure 4.5: Error reduction: $\|\sigma - \sigma_h\|$, adaptive versus uniform refinement.

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