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Authors
Concus, P.
Finn, R.

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Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

Physics Division

Mathematics Department

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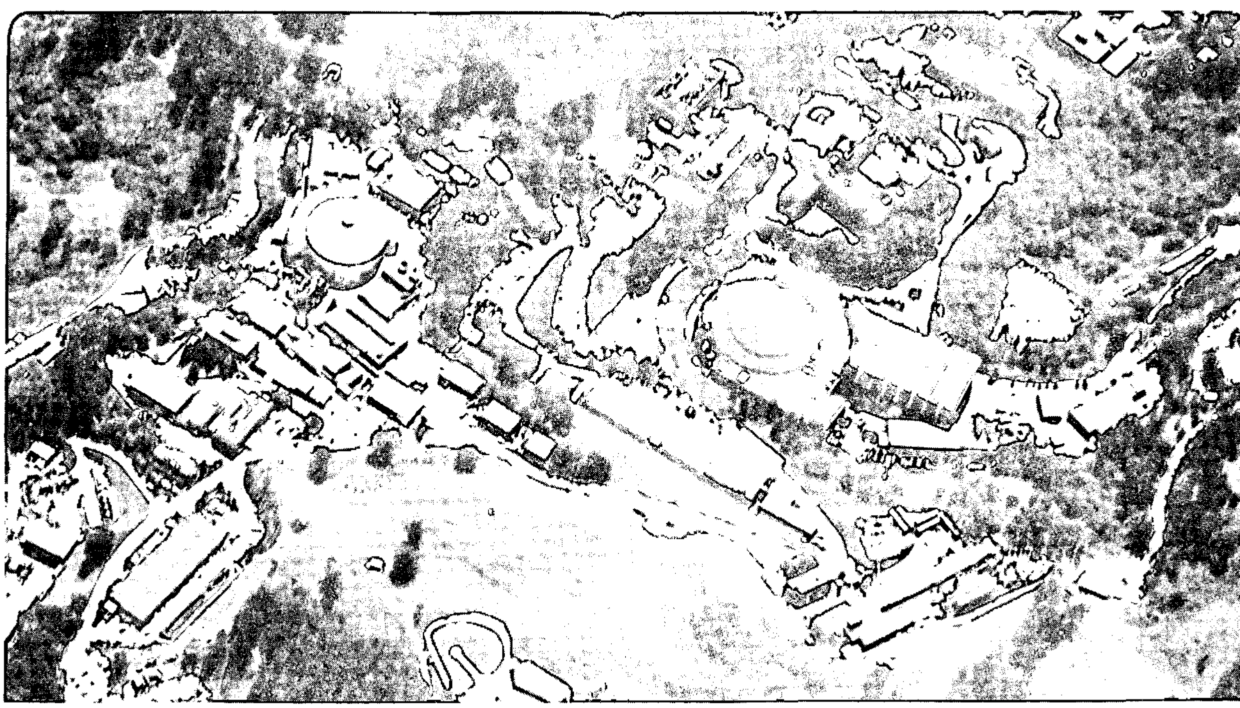

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COMMENTS ON A PAPER OF KELLER, KING, AND MERCHANT*

Paul Concus
Lawrence Berkeley Laboratory
and
Department of Mathematics
University of California
Berkeley, CA 94720

Robert Finn
Department of Mathematics
Stanford University
Stanford, California 94305

April 1995

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COMMENTS ON A PAPER OF KELLER, KING, AND MERCHANT

Paul Concus and Robert Finn

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Abstract. Some of the statements and procedures appearing in the indicated reference are discussed and interpreted, and alternatives to them are proposed.

1. In Sections 1 and 2 of the reference [1], the authors consider the behavior of a capillary surface in a "wedge shaped region" in the absence of gravity. Not all hypotheses and conclusions are clearly stated, so it is difficult to determine what are the precise contributions of the paper, and to some extent we have had to surmise the intent of the authors. Nevertheless, the underlying nature of the problem being attacked and the general form of the methods that were employed can be discerned with some certainty. In the Abstract of [1] appears the statement

"It is shown that the height of the free surface at the corner tends to infinity as the wedge angle $[2\alpha]$ *decreases to a critical value* $[2\alpha_{\mathrm{cr}}]$ *dependent upon the contact angle* $[\gamma]$*."*

From this statement we may infer that the authors are studying a fluid surface interface described by a function $u(x,y)$, and which meets the bounding walls of the wedge in a prescribed angle γ; on the following page of [1] the further condition is introduced that the surface is to have a constant mean curvature κ. This latter condition can be expressed explicitly by the equation

$$\operatorname{div} Tu = 2\kappa, \qquad Tu = \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \tag{1}$$

over the wedge domain $\mathcal{W}$ of opening 2α, while the former one takes the form

$$\nu \cdot Tu = \cos\gamma \tag{2}$$

at boundary points distinct from the vertex P; here ν is the unit exterior normal vector to $\partial\mathcal{W}$.

We observe that the height $u(x,y)$ of the surface appears in the equation and boundary condition only in differentiated form; thus any solution is determined at most up to an additive constant. The italicized statement cannot have a meaning until this constant is determined. In physical situations, it is customary to fix the constant via a normalization; that can be done, for example, in a capillary tube closed at one end by normalizing the volume. However, the authors assume at a crucial point in their discussion that the region of definition is an entire infinite wedge, thus excluding the possibility of volume normalization. A reasonable alternative might be to normalize the height to vanish asymptotically at infinity in directions interior to the wedge and distinct from those of the bounding edges. This possibility is however excluded by the conclusion of the authors that the solution must be an inclined plane whenever $\gamma \neq \pi/2$. Another alternative is to normalize the height at one point, and one way to do that is to normalize it at the vertex P. It is not immediately clear that such a choice is feasible, as P is a singular boundary point and the solution height is not initially known to be defined there. In view additionally of the italicized statement, that particular normalization would not at first seem to have been what the authors had in mind; nevertheless, it is precisely what they do. Indeed, on the following page, they determine by an incomplete reasoning the possibly correct result that the solution must be a plane and write: "Let us write the equation of this plane as $z = -Ax$ where $x = r\cos\theta$ is the distance from the edge along the bisector of the wedge angle." Thus the height at the vertex is set to be zero, and so we are hard-pressed to understand how it could go to infinity as $\alpha \to \alpha_{\text{cr}}$, as they state.

One could attempt to justify the reasoning by taking the view that what the authors really had in mind is the local behavior at a boundary corner point P in a capillary tube with bounded section. One is interested only in behavior very close to P, and relative to such points the remainder of the boundary could be viewed as being effectively an infinite wedge. In such a configuration a volume normalization would be feasible, and experience with other physical problems might lead one to expect the local height to become large with decreasing wedge angle. In anticipation of the final conclusion that there can be no solution when $\alpha < \alpha_{\text{cr}} = \left|\frac{\pi}{2} - \gamma\right|$, one might guess that the local height should become unbounded as α decreases to α_{cr}. We may surmise that such thoughts were the motivation for the statement italicized above. But a brief reflection yields the following counterexample:

We observe that in the absence of gravity a lower spherical cap yields a formal solution to the capillarity equation (1). In the configuration of Figure 1, which indicates the section Ω of a cylindrical capillary tube, we choose for $u(x,y)$ the portion lying over Ω, of a lower hemisphere $\mathcal{S}_R$ for which Σ_R is the equatorial circle; Σ_R is positioned at a height above the base plane to be determined. Then $\mathcal{S}_R$ meets the boundary walls over $\Sigma = \partial\Omega$ in the constant angle $\gamma = \frac{\pi}{2} - \sin^{-1}(\rho/R)$, while $\alpha = \sin^{-1}(\rho/R_0)$.

Keeping ρ and R fixed, we let R_0 increase to R. Then γ remains constant while α decreases to $\alpha_{\text{cr}} = \frac{\pi}{2} - \gamma$, and it is clear that the height of Σ_R can be adjusted within fixed upper and lower bounds so that the volume remains constant throughout the deformation. It follows that $u(x,y)$

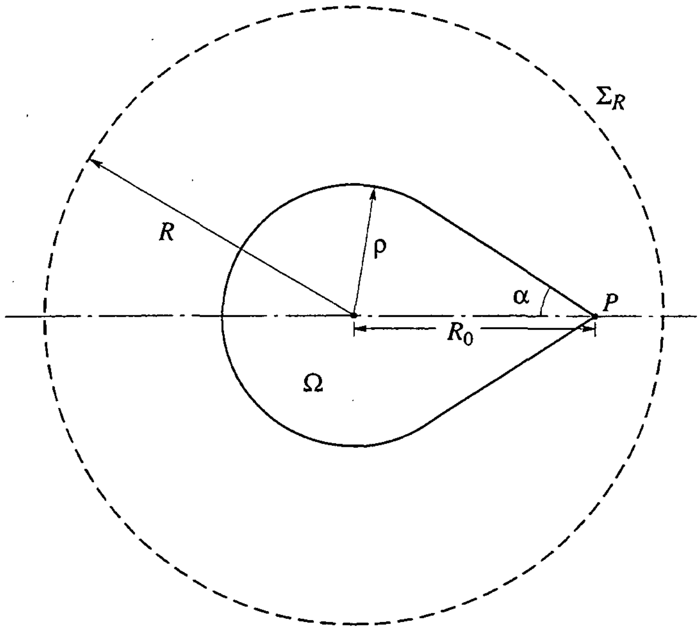

Figure 1. Construction of example.

remains bounded above and below, and thus the surface cannot tend to infinity in the corner.

Note that in this example the surface continues to exist as a formal solution of the capillarity equation, up to and including the limiting configuration $\alpha = \alpha_{\mathrm{cr}}$, with the boundary condition satisfied at all smooth points of Σ (it is not possible to specify a boundary angle at P).

Thus, even for a bounded container and fixed fluid volume, the statement italicized above cannot be justified.

It will be shown in Sec. 3 below that the solution just determined disappears *discontinuously* if α is decreased beyond α_{cr}. Physically, the fluid presumably flows out along the edge over P until it either reaches the top of the container or uncovers a region on the base. But the fluid remains at a height over the base uniformly bounded above and below for all $\alpha \geq \alpha_{\mathrm{cr}}$.

2. The infinite wedge considered in [1] has an independent mathematical interest, and it seems worthwhile to examine the line of reasoning offered by the authors. The reasoning given on p. 162 that $\kappa = 0$ is suggestive but not rigorous. A rigorous justification follows from a theorem of S. Bernstein [2], that a surface whose height $u(x, y)$ satisfies (1) with $\kappa \neq 0$ cannot be defined in a disk of radius exceeding $1/|\kappa|$. (Sharper forms of this result, not needed in the present context, are given in [3] and in [4].) Thus, the only possibility for a solution of (1) in an infinite wedge is that $u(x, y)$ represent a minimal surface, for which $\kappa = 0$. The authors then attempt to use the invariance of the domain and of the boundary condition under dilation to show that the surface is ruled. Unfortunately, this reasoning depends essentially on the uniqueness of the solution, which for the indicated configuration is currently an open problem under active study (see, for example, [5] and [6]). But in the expectation of an ultimate positive outcome for this step, let us go on to the next one, which is that because the surface is minimal and ruled it must be a plane. For this last statement, the helicoid $u(x,y) = \tan^{-1}(y/x)$ provides a counterexample.

Nevertheless, the situation is not wholly lost. It was shown by E. Catalan [7] in 1842 that the helicoids are the *only* ruled minimal surfaces other than the planes. But a helicoid cannot satisfy the boundary condition (2), and we arrive finally—subject to the uncertainty about uniqueness—at the stated conclusion that $u(x,y)$ represents a plane. From this point, we may safely follow the authors' reasoning to obtain their conclusion that $\alpha > \left|\frac{\pi}{2} - \gamma\right|$.

3. The authors state on p. 161 that their procedure provides a simple derivation of our earlier result. But beyond the incompleteness of their derivation, they assume a solution defined in an entire infinite rectilinear wedge, whereas we require only that the solution be defined in a neighborhood of a protruding corner in a general domain. There is a further important distinction, in that our original conclusion was that $\alpha \geq \left|\frac{\pi}{2} - \gamma\right|$. The possibility of achieving the equality sign is illustrated in the example given above; it is an important case, as it demonstrates concretely the discontinuous dependence on the boundary data.

In order to compare the two reasonings from the points of view of completeness and of simplicity, we outline our proof. We consider a domain with a protruding corner P, and a solution of (1), (2) defined in a neighborhood $\mathcal{N}$ of P and smooth up to the boundary at points distinct from P. We choose a segment Γ within $\mathcal{N}$, and cut off the vertex with a parallel segment Λ, as indicated in Figure 2. In the resulting domain Ω^*_Λ, we integrate (1) and use (2) and the divergence theorem to obtain

$$2\kappa|\Omega^*_\Lambda| = \int_\Gamma \nu \cdot Tu \; ds + \int_\Lambda \nu \cdot Tu \; ds + |\Sigma^*_\Lambda| \cos\gamma.$$

Since $|\nu \cdot Tu| < 1$, we may let Λ move to P, and the integral over Λ vanishes in the limit. Writing $\ell = \lim_{\Lambda \to P} |\Sigma^*_\Lambda|$, we find

$$\ell \cos\gamma = -\int_\Gamma \nu \cdot Tu \; ds + O(\ell^2).$$

Using again that $|\nu \cdot Tu| < 1$ and letting $\ell \to 0$, we obtain $|\cos\gamma| \leq \sin\alpha$; that is, $\alpha \geq \left|\frac{\pi}{2} - \gamma\right|$. We are done. As one sees from the example above, the result is best possible.

4. It is not known whether every minimal surface $u(x,y)$ defined over an infinite rectilinear wedge and meeting the boundary walls in a constant angle $\gamma \neq \pi/2$ is necessarily a plane. (It has

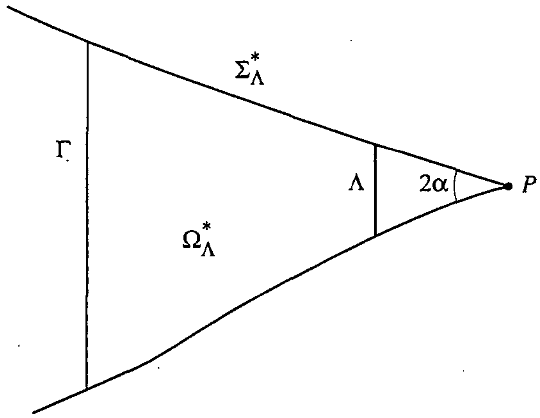

Figure 2. Portion of wedge.

only recently been proved that the special case $\gamma = \pi/2$ admits an affirmative answer, see [6].) We can show, however, that *there is no such minimal surface, even in a neighborhood of P, when the half-opening* $\alpha = \alpha_{\mathrm{cr}}$. In fact, if there were such a solution, we could introduce a segment Γ as above, and a formal integration would yield

$$\ell \cos\gamma = \int_\Gamma \nu \cdot Tu \, ds < \ell \sin\alpha$$

and also $\ell\cos\gamma > -\ell\sin\alpha$, which is not possible when $\alpha = \left|\frac{\pi}{2} - \gamma\right|$.

Thus, in the case of an infinite wedge, for which the surface is necessarily minimal, the extremal configuration cannot be realized by any solution; this differs from the behavior when only a portion of the wedge is occupied by the fluid, as follows from the above example.

5. The authors in [1] proceed on p. 163 to take up the case in which the contact angles on the two sides of the wedge may differ, concluding with the same reasoning as in the single angle case that the surface must be a plane and hence obtaining a condition on the range of possible opening angles. In this situation, however, they lose much more than particular solutions in a limiting configuration. There is an entire family of solutions, whose behavior near the vertex differs essentially from what can occur in the equal angle case. These surfaces appear when the two contact angles satisfy $|\gamma_1 - \gamma_2| > \pi - 2\alpha$, although the method of [1] predicts nonexistence for such configurations. In [8] and in [9] sectional geometries are characterized for which the surfaces exist globally for all such γ_1, γ_2 as formal solutions of the boundary value problem, and it must be expected that they will also be observed experimentally. Thus in our view—even apart from the uncertainties about the procedures—the condition (2.7) introduced in [1] is not a good indication of the behavior to be expected of solutions of the capillary equation in domains with protruding corners, when the contact angles differ on the two sides of the corner. Correspondingly, the experiment proposed in that reference, if interpreted by (2.7), would, we believe, lead to erroneous conclusions in many cases of physical interest.

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