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Stability of Current Density Impedance Imaging and Uniqueness for the Inverse Sturm-Liouville Problem

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Stability of Current Density Impedance Imaging and Uniqueness for the Inverse Sturm-Liouville Problem

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Robert Julius Lopez

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The text of this dissertation in part, is a reprint of the material as it appears in [19] and [20]. The co-author, Dr. Amir Moradifam, listed in that publication directed and supervised the research which forms the basis for this dissertation.

For my parents, Melodee and Jim, who have supported me morally, emotionally, and financially throughout my life.

ABSTRACT OF THE DISSERTATION

Stability of Current Density Impedance Imaging and Uniqueness for the Inverse Sturm-Liouville Problem

by

Robert Julius Lopez

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2021
Dr. Amir Moradifam, Chairperson

In a joint effort with my advisor, we study stability of reconstruction in current density impedance imaging (CDII), that is, the inverse problem of recovering the conductivity of a body from the measurement of the magnitude of the current density vector field in the interior of the object. Our results show that CDII is stable with respect to errors in interior measurements of the current density vector field, and confirm the stability of reconstruction which was previously observed in numerical simulations, and was long believed to be the case. Next, we show that CDII is stable with respect to errors in both measurement of the magnitude of the current density vector field in the interior and the measurement of the voltage potential on the boundary. This completes the authors study of the global stability of Current Density Independence Imaging. These results are accomplished through analysis on a related functional from the so-called least gradient problem as well as geometric arguments on the level sets of the induced voltage potential function. These geometric arguments are dependent upon some ad hoc conditions which are shown to be guaranteed by reasonable sufficient conditions.

Additionally, we study the Inverse Sturm-Liouville problem which is the problem of reconstructing the coefficient function q from the second order elliptic differential operator $-\nabla + q$ using the boundary spectral data. While there are several results in one dimension

and higher dimensions using complete spectral data and even finitely many terms omitted, none have explored results for a subsequence of spectral data. We aim to establish such results in one dimension and higher dimensions by using the asymptotic behavior of eigenfunctions on the boundary.

Contents

Chapter 1

Introduction

The aim of this thesis is two study two distinct problems in the field of Inverse Problems. This field is broadly concerned with determining the causes or conditions that led to certain effects or results. In partial differential equations this often occurs in the form of determining the coefficients of a differential operator from knowledge of the solution to a given equation (likely in tandem with additional information). There are myriad applications using the principles of inverse problems, many of which have important physical implications. Problems that arise in medical imaging often take the form of an inverse problem. The process involves some sort of external probing which results in a quantitative reading on the tissue in question. The goal is to then interpret this reading to the point where the conditions that caused it can be determined.

Another point of interest relating to inverse problems is stability. In other words, can we guarantee the parameters we wish to solve for will be continuous with respect to the observed quantities? Again, in terms of medical imaging this equates to determining the effects of "noise" or some sort of disturbance in the data on the resulting image.

1.1 Electrical Impedance Tomography

The classical Electrical Impedance Tomography (EIT) aims to obtain quantitative information on the electrical conductivity, σ, of a conductive body from measurements of voltages and corresponding currents at its boundary. Mathematics of EIT has been extensively studied, and many interesting results have been obtained about uniqueness, stability and reconstruction algorithms for this problem. See [4, 5, 6, 9] for excellent reviews of the results. It is well known that that EIT is severely ill-posed (with respect to initial conditions), and provides images with very low resolution away from the boundary [13, 21].

The method of EIT is based on the Calderón problem which can be stated in the following way: Suppose $\sigma(x)$ is the electrical conductivity for each $x \in \Omega$ where $\Omega \subset \mathbb{R}^n$ is a bounded, open set with C^∞ boundary. Imposing a voltage $f \in L^1(\partial\Omega)$ on $\partial\Omega$ then induces a potential u on the interior of Ω which solves the conductivity equation with Dirichlet boundary condition

$$\begin{gathered} \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega \\ u = f \quad \text{on} \quad \partial\Omega. \end{gathered} \tag{1.1}$$

Solving this problem employs the so called Dirichlet to Neumann map defined in the following way

$$\Lambda_\sigma(f) = \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

where ν denotes the outer normal vector to $\partial\Omega$. This is also sometimes also referred to as the voltage to current map (due to the scenario outlined previously).

In this problem, one assumes knowledge of the Dirichlet to Neumann map and, as a result, is able to recover the conductivity, σ. However, knowledge of the Dirichlet to Neumann map would require knowledge of the induced current for every possible voltage f imposed on $\partial\Omega$. The following method offers an alternative approach to obtaining σ.

1.2 Current Density Impedance Imaging

A more recent class of Inverse Problems seeks to provide images with high accuracy and by using data obtained from the interior of the region. Such methods are referred to as Hybrid Inverse Problems or Coupled-physics methods, as they usually involve the interaction of two kinds of physical fields. An example of such a problem is the method of Current Density Impedance Imaging (CDII). This is the inverse problem of recovering the conductivity of a body from the measurement of the magnitude of the current density vector field in the interior of the object. Interior measurements of current density is possible by Magnetic Resonance Imaging (MRI) due to the work of M. Joy and his collaborators [16, 17]. This problem has been studied in [26, 28, 30, 31, 32]. See also [33] for a comprehensive review.

Much like EIT, CDII involves the equation (1.1) and the recovery of σ. In contrast with EIT, the method of CDII requires a single voltage f to be imposed on $\partial\Omega$ and the magnitude of the induced current (denoted by J) to be known in order to recover σ. Thus, this method simply requires the knowledge of the pair of measurements $(f, |J|)$ as opposed to knowledge of the Dirichlet to Neumann map.

While the uniqueness of the reconstruction in CDII is established and a robust reconstruction algorithm is developed in [27], the global stability of CDII was an open problem until [19]. In this paper, my advisor and I were able to show a detailed analysis on the stability of this problem. The stability hinged on few assumptions which fit quite logically into a physical setting (which will be outlined in chapter 2).

1.3 The Inverse Sturm-Liouville problem

The inverse Sturm-Liouville problem is concerned with recovering the potential function for a second order equation based on knowledge of the eigenvalues and eigenfunctions of the second order elliptic operator. The author in [18] outlines that this is indeed possible in the one dimensional problem beginning with the following equation subject to Robin boundary conditions:

$$y'' + (\lambda - P)y = 0$$

for $x \in [0, \pi]$. Subject to

$$\begin{aligned} y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad & y(\pi)\cos\beta + y'(\pi)\sin\beta = 0 \\ y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad & y(\pi)\cos\gamma + y'(\pi)\sin\gamma = 0 \end{aligned}$$

In the multi-dimensional case the authors in [29] extend this result to a smooth bounded domain $\Omega \subset \mathbb{R}^n$ for the problem

$$\begin{aligned} -\Delta u + qu = \mu u \quad & \text{in} \quad \Omega \\ u = 0 \quad & \text{on} \quad \partial\Omega. \end{aligned}$$

The arguments in each of these papers do suggest that it may be possible to only consider partial spectral data. In [14], the author shows that the multi-dimensional results still hold when lacking finitely many terms from the spectrum of this operator. However, it is also suggested that one could take this idea further to only consider a subsequence of the spectral data. This will be further discussed in chapters 4 and 5.

Chapter 2

Stability of Current Density Impedance Imaging (CDII)

2.1 Introduction

Let σ be the isotropic conductivity of an object $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where Ω is a bounded open region in with connected boundary. Suppose J is the current density vector field generated by imposing a given boundary voltage f on $\partial\Omega$. Then the corresponding voltage potential u satisfies the second order elliptic equation

$$\nabla \cdot (\sigma \nabla u) = 0, \quad u|_{\partial\Omega} = f. \tag{2.1}$$

By Ohm's law $J = -\sigma \nabla u$, and u is the unique minimizer of the weighted least gradient problem

$$I(w) = \min_{w \in BV_f(\Omega)} \int_\Omega a|\nabla w| dx, \tag{2.2}$$

where $a = |J|$, and $BV_f(\Omega) = \{w \in BV(\Omega), w|_{\partial\Omega} = f\}$, see [26, 28, 30, 31, 32].

Remark 2.1.1. In general, the least gradient problem (2.2) may not have a minimizer [7, 35]. Throughout the paper we shall assume that (2.2) has a solution. For sufficient conditions for the existence of minimizers of weighted least gradient problems we refer to [8, 15, 25]. Note also that any voltage potential u solving the equation (2.1) is also a minimizer of (2.2). In particular, if $0 < a(x) \in C(\overline{\Omega})$ and $\partial\Omega$ satisfies a Barrier condition (see Definition 3.1 in [15]), then for every $f \in C(\partial\Omega)$ the least gradient problem (2.2) has a minimizer in $BV_f(\Omega)$. In other words, the set of weights for which the least gradient problem (2.2) has a solution is open in $C(\overline{\Omega})$ if $\partial\Omega$ satisfies a barrier condition.

Since any stability result trivially implies uniqueness, it is evident that one needs additional assumptions to prove any stability result. Indeed stability analysis of CDII is a challenging problem. The first stability result on CDII was proved by Montalto and Stefanov in [23].

Theorem 2.1.2 ([23]). *Let u solve equation (1) and let $\tilde{u}$ solve equation (1) for $\tilde{\sigma}$ with $|\nabla \tilde{u}| > 0$ in $\overline{\Omega}$. For any $0 < \alpha < 1$, there exists $s > 0$ such that if $\|\sigma\|_{H^s(\Omega)} < L$ for some $L > 0$ then there is an $\epsilon > 0$ such that if*

$$\|\sigma - \tilde{\sigma}\|_{C^2(\Omega)} < \epsilon, \tag{2.3}$$

then

$$\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} < C\||J| - |\tilde{J}|\|^{\alpha}_{L^2(\Omega)}$$

Later in [22], Montalto and Tamasan proved the following stability result.

Theorem 2.1.3 ([22]). *Let $\sigma \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, be positive in $\overline{\Omega}$. Let u solve equation (1) with $|\nabla u| > 0$ in $\overline{\Omega}$. There exists $\epsilon > 0$ depending on Ω and some $C > 0$ depending on ϵ such that if $\tilde{\sigma} \in C^{1,\alpha}(\overline{\Omega})$ with $\tilde{u}$ solving (1) for $\tilde{\sigma}$, $u = \tilde{u} = f$ on $\partial\Omega$, $\sigma = \tilde{\sigma}$ on $\partial\Omega$, and*

$$\|\sigma - \tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})} < \epsilon,$$

then

$$\|\sigma - \tilde{\sigma}\|_{L^2(\Omega)} \leq C\|\nabla \cdot (\Pi_{\nabla u}(J - \tilde{J}))\|^{\frac{\alpha}{2+\alpha}}_{L^2(\Omega)}$$

where $\Pi_{\nabla u}(J - \tilde{J})$ *is the projection of* $J - \tilde{J}$ *onto* ∇u.

Note that both of the above results assume a priori that σ and $\tilde{\sigma}$ are close, and a natural question which remains open is that whether there exists two distant conductivities σ and $\tilde{\sigma}$ which could induce the corresponding currents J and $\tilde{J}$ with $||J| - |\tilde{J}||$ arbitrarily small. In this paper we address the this question and show that the answer is negative, and hence show that CDII is actually stable. Under some natural assumption, we shall prove that in dimensions $n = 2, 3$ the following stability result holds

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \leq C\||J| - |\tilde{J}|\|^{\frac{1}{4}}_{L^\infty(\Omega)}, \tag{2.4}$$

for some constant C independent of $\tilde{\sigma}$ (see Theorems 2.4.6 and 2.4.7 for precise statements of the results).

This chapter is organized as follows. In Section 2, under very weak assumptions, we will prove that the structure of level sets of the least gradient problem (2.2) is stable. In Section 3, we will provide stability results for minimizers of (2.2) in L^1. In Section 4, we will prove stability of minimizers of (2.2) in $W^{1,1}$, and shall use them to prove Theorems 2.4.6 and 2.4.7 which are the main results of this paper.

2.2 Stability of level sets

In this section, we show that the structure of the level sets of minimizers of the least gradient problem (2.2) is stable. Throughout the paper, we will assume that $a, \tilde{a} \in C(\Omega)$ with

$$0 < m \leq a(x), \tilde{a}(x) \leq M, \quad \forall x \in \Omega, \tag{2.5}$$

for some positive constants m, M. The following theorem which was proved in [25] by the second author, shall play a crucial role in the proof of the results in this section.

Theorem 2.2.1 ([25]). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary and assume that* $a \in C(\overline{\Omega})$ *is a non-negative function, and* $f \in L^1(\partial\Omega)$. *Then there exists a* *divergence free vector field $J \in (L^\infty(\Omega))^n$ with $|J| \le a$ a.e. in Ω such that every minimizer w of (2.2) satisfies*

$$J \cdot \frac{Dw}{|Dw|} = |J| = a, \quad |Dw| - a.e. \quad in \quad \Omega, \tag{2.6}$$

where $\frac{Dw}{|Dw|}$ is the Radon-Nikodym derivative of Dw with respect to $|Dw|$.

Remark 2.2.2. Throughout chapters 2 and 3 we will assume that $\partial\Omega$ is Lipschitz at the very least (that is to say, the boundary is sufficiently regular).

Lemma 2.2.3. *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (2.2) with the weights a and $\tilde{a}$, respectively. Then*

$$\left| \int_\Omega a|Du|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx \right| \le C\|a - \tilde{a}\|_{L^\infty(\Omega)}, \tag{2.7}$$

for some constant $C = C(m, M, \Omega, f)$ independent of u and $\tilde{u}$.

Proof. First note that in view of (2.5) we have

$$m \int_\Omega |D\tilde{u}|dx \le \int_\Omega \tilde{a}|D\tilde{u}|dx \le \int_\Omega \tilde{a}|Dw|dx \le M \int_\Omega |Dw|$$

for any $w \in BV_f(\Omega)$. Thus $\int_\Omega |D\tilde{u}| \le C$, and similarly $\int_\Omega |Du| \le C$ for some constant C which depends only on m, M, and Ω. Hence

$$\max\left\{ \int_\Omega |D\tilde{u}|, \int_\Omega |Du| \right\} \le C, \tag{2.8}$$

for some $C(m, M)$ independent of $\tilde{u}$ and u. Since u, $\tilde{u}$ are the minimizers of (2.2) with the weights a and $\tilde{a}$,

$$\begin{aligned} \int_\Omega a|Du|dx - \int_\Omega \tilde{a}|Du|dx &\le \int_\Omega a|Du|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx \\ &\le \int_\Omega a|D\tilde{u}|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx. \end{aligned}$$

Thus

$$\int_\Omega (a-\tilde{a})|Du|dx \le \int_\Omega a|Du|dx - \int_\Omega \tilde{a}|Du|dx \le \int_\Omega (a-\tilde{a})|D\tilde{u}|dx,$$

and we get

$$\begin{aligned} -\|a-\tilde{a}\|_{L^\infty(\Omega)}\|Du\|_{L^1(\Omega)} &\le \int_\Omega a|Du|dx - \int_\Omega \tilde{a}|Du|dx \\ &\le \|a-\tilde{a}\|_{L^\infty(\Omega)}\|D\tilde{u}\|_{L^1(\Omega)}. \end{aligned}$$

Hence (2.7) follows from (5.4). □

Let ν_Ω denote the outer unit normal vector to $\partial\Omega$. Then for every $T \in (L^\infty(\Omega))^n$ with $\nabla \cdot T \in L^n(\Omega)$ there exists a unique function $[T, \nu_\Omega] \in L^\infty(\partial\Omega)$ such that

$$\int_{\partial\Omega} [T,\nu_\Omega]u\, d\mathcal{H}^{n-1} = \int_\Omega u\nabla\cdot T dx + \int_\Omega T\cdot Du dx, \quad u \in C^1(\bar{\Omega}). \tag{2.9}$$

Moreover, for $u \in BV(\Omega)$ and $T \in (L^\infty(\Omega))^n$ with $\nabla\cdot T \in L^n(\Omega)$, the linear functional $u \mapsto (T\cdot Du)$ gives rise to a Radon measure on Ω, and (2.9) holds for every $u \in BV(\Omega)$ (see [1, 3] for a proof). We shall need the weak integration by parts formula (2.9).

Lemma 2.2.4. *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of* (2.2) *with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2.1. Suppose $0 \le \sigma(x) \le \sigma_1$ in Ω for some constant $\sigma_1 > 0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du|$. Then*

$$\int_\Omega |J||\tilde{J}| - J\cdot\tilde{J}dx \le C\|a-\tilde{a}\|_{L^\infty(\Omega)}, \tag{2.10}$$

where $C = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$.

Proof. We have

$$\begin{aligned}
\int_\Omega |J||\tilde{J}| - J\cdot\tilde{J}dx &= \int_\Omega \sigma|\tilde{J}||Du| - \sigma\tilde{J}\cdot Dudx \\
&\leq \sigma_1\int_\Omega |\tilde{J}||Du| - \tilde{J}\cdot Dudx \\
&= \sigma_1\int_\Omega |\tilde{J}||Du|dx - \int_{\partial\Omega} f[\tilde{J},\nu_\Omega]dx \\
&= \sigma_1\int_\Omega |\tilde{J}||Du| - \tilde{J}\cdot D\tilde{u}dx \\
&= \sigma_1\int_\Omega |\tilde{J}||Du| - |\tilde{J}||D\tilde{u}|dx,
\end{aligned}$$

where we have used (2.6) and the integration by parts formula (2.9). On the other hand it follows from lemma 2.2.3 that

$$\begin{aligned}
\sigma_1\int_\Omega |\tilde{J}||Du| - |\tilde{J}||D\tilde{u}|dx &= \sigma_1\int_\Omega |\tilde{J}||Du| - |J||Du| + |J||Du| - |\tilde{J}||D\tilde{u}|dx \\
&= \sigma_1\left(\int_\Omega (a-\tilde{a})|Du|dx + \int_\Omega a|Du| - \tilde{a}|D\tilde{u}|dx\right) \\
&\leq \sigma_1(\|Du\|_{L^1(\Omega)}\|a-\tilde{a}\|_{L^\infty(\Omega)} + C\|a-\tilde{a}\|_{L^\infty(\Omega)}),
\end{aligned}$$

which yields the desired result. $\square$

Roughly speaking, Lemma 2.2.4 implies that as $a \to \tilde{a}$, $\frac{Du}{|Du|}(x)$ becomes parallel to $\frac{D\tilde{u}}{|D\tilde{u}|}(x)$ at points where the two gradients do not vanish. We are now ready to prove the main result of this section.

Theorem 2.2.5. *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (2.2) with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2.1. Suppose $0 \leq \sigma(x) \leq \sigma_1$ in Ω for some constant $\sigma_1 > 0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du|$. Then*

$$\|J-\tilde{J}\|_{L^1(\Omega)} \leq C\|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)}, \tag{2.11}$$

where $C = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$.

Proof. We have

$$
\begin{aligned}
\left(|J-\tilde{J}|^2\right)^{\frac{1}{2}} &= \left(|J|^2+|\tilde{J}|^2-2J\cdot\tilde{J}\right)^{\frac{1}{2}} \\
&= \left(\Big||J|-|\tilde{J}|\Big|^2+2(|J||\tilde{J}|-J\cdot\tilde{J})\right)^{\frac{1}{2}} \\
&\le \Big||J|-|\tilde{J}|\Big| + \left(2(|J||\tilde{J}|-J\cdot\tilde{J})\right)^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|J-\tilde{J}\|_{L^1(\Omega)} &= \int_\Omega \left(\Big||J|-|\tilde{J}|\Big|^2\right)^{\frac{1}{2}} dx \\
&\le \int_\Omega \Big||J|-|\tilde{J}|\Big| dx + \int_\Omega \left(2(|J||\tilde{J}|-J\cdot\tilde{J})\right)^{\frac{1}{2}} dx \\
&= \int_\Omega |a-\tilde{a}| dx + \int_\Omega \left(2(|J||\tilde{J}|-J\cdot\tilde{J})\right)^{\frac{1}{2}} dx \\
&\le |\Omega|\|a-\tilde{a}\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{2}} \left(\int_\Omega 2(|J||\tilde{J}|-J\cdot\tilde{J}) dx\right)^{\frac{1}{2}} \\
&\le |\Omega|\|a-\tilde{a}\|_{L^\infty(\Omega)} + (2|\Omega|)^{\frac{1}{2}}(C\|a-\tilde{a}\|_{L^\infty(\Omega)})^{\frac{1}{2}} \\
&= \left(|\Omega|\|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)} + (2C|\Omega|)^{\frac{1}{2}}\right)\|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)},
\end{aligned}
$$

where we have used the Holder's inequality and Lemma 2.2.4. □

Remark 2.2.6. In view of Theorem 2.2.1, $\frac{Du}{|Du|}$ and $\frac{D\tilde{u}}{|D\tilde{u}|}$ are parallel to J and $\tilde{J}$, respectively. So Theorem 5.1.3 implies that if $\tilde{a}$ is close to a, then the structure of level sets of $\tilde{u}$ is close to that of u.

2.3 L^1 stability of the minimizers

In this section, we establish stability of minimizers of the least gradient problem (2.2) in L^1. In general, (2.2) does not even have unique minimizers, so in order to prove any stability results further assumptions on the weights $a, \tilde{a}$, and on the corresponding minimizers are expected.

Definition 2.3.1. *Fix the positive constants $\sigma_0, \sigma_1 \in \mathbb{R}$. We say that $u \in C^1(\bar{\Omega})$ is admissible if it solves the conductivity equation* (2.1) *for some $\sigma \in C(\Omega)$ with*

$$0 < \sigma_0 < \sigma \leq \sigma_1,$$

and $m \leq |J| = |\sigma \nabla u| \leq M$, where m and M are positive constants as in (2.5)*. We shall denote the corresponding induced current by $J = -\sigma \nabla u$.*

Remark 2.3.2. Let $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ be a bounded Lipschitz domain and suppose $\partial\Omega$ satisfies the barrier condition defined in Definition 3.1 in [15]). A. Zuniga proved in [36] that if $0 < a \in C^2(\overline{\Omega})$, then for any boundary data $f \in C(\partial\Omega)$ the least gradient problem (2.2) has a minimizer $u \in C(\overline{\Omega})$. If $|\nabla u| > 0$ in $\overline{\Omega}$, then

$$\sigma = \frac{a}{|\nabla u|} \in C(\overline{\Omega}),$$

and by elliptic regularity $u \in C^1(\overline{\Omega})$, and therefore (2.2) has an admissible minimizer. To guarantee the condition $|\nabla u| > 0$ on $\overline{\Omega}$, in dimension $n = 2$ it suffices to assume that the boundary data $f \in \partial\Omega$ is two-to-one, i.e. f has only two critical points on $\partial\Omega$ (see Theorem 1.1 in [2]). In higher dimensions, it is still an open problem to find sufficient conditions under which $|\nabla u| > 0$ on $\overline{\Omega}$.

We will first prove our results in dimension $n = 2$ and then extend them to dimensions $n = 3$.

Let $u \in C^1(\Omega)$ with $|\nabla u| > 0$ in Ω. Then it follows from the regularity result of De Giorgi (see, e.g, Theorem 4.11 in [8]) that all level sets of u are C^1 curves. We will assume that the length of level sets of u in Ω is uniformly bounded, i.e.

$$\sup_{t \in \mathbb{R}} \int_{\{u=t\} \cap \Omega} 1 dl = L_M < \infty. \tag{2.12}$$

Theorem 2.3.3. *Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$, and corresponding current density vector fields J and $\tilde{J}$, respectively. If u satisfies* (2.12)*, then*

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C \parallel |J| - |\tilde{J}| \parallel^{\frac{1}{2}}_{L^\infty(\Omega)}, \tag{2.13}$$

for some constant $C(m, M, \sigma_0, \sigma_1, f, u, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. Since u is admissible,

$$|\nabla u(x)| = \frac{|J(x)|}{\sigma(x)} \geq \frac{m}{\sigma_1} > 0, \quad \forall x \in \Omega.$$

Using the coarea formula we get

$$\frac{m}{\sigma_1} \int_\Omega |u - \tilde{u}| dx \leq \int_\Omega |\nabla u||u - \tilde{u}| dx = \int_{\mathbb{R}} \int_{\{u=t\}\cap\Omega} |u - \tilde{u}| dl dt. \tag{2.14}$$

Since $|\nabla u| > 0$ in Ω, it follows from the regularity result of De Giorgi (Theorem 4.11 in [8]) that all level sets of u are C^1 curves. Now let Γ_t be a connected component of $\{x \in \Omega \colon u(x) = t\} \subset \Omega$, and $\gamma \colon [0, L] \to \Gamma_t$ to be a path parameterized by the arc length with $\gamma(0) \in \partial\Omega$. Define

$$h(s) := u(\gamma(s)) - \tilde{u}(\gamma(s)).$$

Then $h(0) = 0$. Moreover since $\nabla u(\gamma(s)) \cdot \gamma'(s) = 0$ on Γ_t,

$$\begin{aligned} h'(s) &= \nabla u(\gamma(s)) \cdot \gamma'(s) - \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) \\ &= \Big(\frac{\sigma}{\tilde{\sigma}}(\gamma(s)) \nabla u(\gamma(s) - \nabla \tilde{u}(\gamma(s)\Big) \cdot \gamma'(s). \end{aligned}$$

We can rewrite the above equality as

$$h'(s) = \frac{J(\gamma(s)) - \tilde{J}(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s).$$

Now let x_t^* be a point on Γ_t where the maximum distance between u and $\tilde{u}$ along the path γ occurs, i.e.

$$|u(x_t^*) - \tilde{u}(x_t^*)| = \max_{x \in \Gamma_t} |u(x) - \tilde{u}(x)|.$$

Then $x_t^* = \gamma(s_0)$ for some $s_0 \in [0, L]$, and

$$\begin{aligned} |u(x_t^*) - \tilde{u}(x_t^*)| = |h(s_0)| &= \left| \int_0^{s_0} \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau \right| \\ &\leq \int_0^{s_0} \frac{1}{\tilde{\sigma}(\gamma(\tau))} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau \\ &\leq \frac{1}{\sigma_0} \int_0^{s_0} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau. \end{aligned}$$

In particular for every $x \in \Gamma_t$

$$|u(x) - \tilde{u}(x)| \leq |u(x_t^*) - \tilde{u}(x_t^*)| \leq \frac{1}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau,$$

where L denotes the entire length of Γ_t. Hence

$$\begin{aligned} \int_{\Gamma_t} |u(x) - \tilde{u}(x)| dl &\leq |u(x_t^*) - \tilde{u}(x_t^*)| \int_{\Gamma_t} 1 dl \\ &\leq L_M |u(x_t^*) - \tilde{u}(x_t^*)| \\ &\leq \frac{L_M}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau \\ &= \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl, \end{aligned}$$

and therefore

$$\int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dl \leq \frac{L_M}{\sigma_0} \int_{\{u=t\} \cap \Omega} |J - \tilde{J}| dl. \tag{2.15}$$

Thus we have

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega}|u-\tilde{u}|dldt &\leq \frac{L_M}{\sigma_0}\int_{\mathbb{R}}\int_{\{u=t\}}|J-\tilde{J}|dldt \\
&= \frac{L_M}{\sigma_0}\int_{\Omega}|\nabla u||J-\tilde{J}|dx \\
&\leq \frac{L_M}{\sigma_0}\|\nabla u\|_{L^\infty(\Omega)}\int_{\Omega}|J-\tilde{J}|dx \\
&\leq C\|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)}
\end{aligned}
$$

$C(m, M, \sigma_0, \sigma_1, f, u, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$, where we have used (2.15) and Theorem 5.1.3. □

Next we generalize Theorem 2.3.3 to dimension $n = 3$. In order to do this, we need the following additional assumption on level sets of u.

Definition 2.3.4. *Let $u \in C^1(\bar{\Omega})$ be admissible. We say that level sets of u can be foliated to one-dimensional curves if for almost every $t \in range(u)$, every conected component Γ_t of $\{u = t\}$ (equipped with the metric induced from the Euclidean metric in $\mathbb{R}^3$) there exists a function $g_t(x) \in C^1(\Gamma_t)$ such that $0 < c_g \leq |\nabla g_t| \leq C_g$, for some constants c_g and C_g independent of t. Moreover, every connected component of $\{u = t\} \cap \{g_t = r\} \cap \Omega$ is a C^1 curve reaching the boundary $\partial\Omega$ for almost every $t \in range(u)$ and all $r \in \mathbb{R}$. Similar to the case $n = 2$, we assume that the length of connected components of $\{u = t\} \cap \{g_t = r\} \cap \Omega$ are uniformly bounded by some constant L_M.*

Remark 2.3.5. It follows from the regularity result of De Giorgi (see, e.g. Theorem 4.11 in [8]) that for a function $u \in BV(\Omega)$, level sets $\{u = t\}$ is a C^1-hypersurface for almost all $t \in range(u)$. Note also that every connected component of $\{u = t\}$ reaches the boundary $\partial\Omega$ (see [26, 28, 30, 31]), for almost every t. Now let Γ_t be a C^1 connected component of $\{u = t\}$. If f has only two critical points (one minimum and one maximum points) on $\partial\Omega$, then Γ_t is a simply-connected C^1 surface reaching the boundary $\partial\Omega$, and hence there exists a C^1 homeomorphism $\mathcal{F}_t$ from $\overline{B(0,1)} \subset \mathbb{R}^2$ to the closure of Γ_t in $\overline{\Omega}$ (see Theorem

3.7 and Theorem 2.9 in [11]). It is easy to see that the unit ball $B(0, 1)$ can be foliated to one dimensional curves by level sets of $g : B(0, 1) \to \mathbb{R}$ defined by $g(x, y) = y$. Consequently Γ_t can be foliated into one dimensional curves reaching the boundary of $\partial\Omega$ by level sets of $g_t(X) = g(\mathcal{F}_t^{-1}(X))$, $X \in \Gamma_t$. Note also that since g and $\mathcal{F}_t^{-1}$ are both C^1, and since $\overline{\Gamma_t}$ is compact, there exists constant $c(t), C(t) > 0$ such that

$$0 < c(t) < |\nabla g_t| < C(t) \quad \text{on } \Gamma_t. \tag{2.16}$$

Indeed the above argument shows that (2.16) holds for every connected components of almost every level sets of a function $u \in BV(\Omega)$, for some constant $c(t), C(t)$ depending on t. So in Definition 2.3.4 the only significant assumption is that the constants $c(t)$ and $C(t)$ are uniformly bounded from below and above by two positive constant c_g and C_g. In particular, if u is a C^1 function with $|\nabla u| > 0$ in Ω and $\{x \in \partial\Omega : f(x) = t\}$ has finitely many connected components for all t, then it follows from the implicit function theorem that every level set of u is a C^1 surface, and hence existence of c_g and C_g follows immediately from compactness of $range(u)$, and hence level sets of u can be foliated to one-dimensional curves in the sense of Definition 2.3.4.

Definition 2.3.6. *Let $t \in range(u)$ and suppose Γ_t^i, $i \in I$, are C^1 connected components of $\{u = t\}$, where I is countable. In view of Remark 2.3.5, there exists functions $g_t^i : \Gamma_t^i \to R$ whose level sets foliate Γ_t^i into one dimensional curves in the sense of Definition 2.3.4. We define $g_t : \{u = t\} \to R$ be the function with*

$$g_t|_{\Gamma_t^i} = g_t^i, \qquad i \in I. \tag{2.17}$$

We shall use this notation throughout the paper.

Theorem 2.3.7. *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$ and corresponding current density vector fields J and $\tilde{J}$, respectively. Suppose the level sets of u*

can be foliated to one-dimensional curves in the sense of Definition 2.3.4. Then

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C\||J| - |\tilde{J}|\|^{\frac{1}{2}}_{L^\infty(\Omega)}, \tag{2.18}$$

where $C(m, M, \sigma_0, \sigma_1, f, u, L_M, c_g, C_g)$ is independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. The proof is similar to the proof of Theorem 2.3.3, and we provide the details for the sake of the reader. Since u is admissible,

$$\frac{m}{\sigma_1}\int_\Omega |u - \tilde{u}|dx \leq \int_\Omega |\nabla u||u - \tilde{u}|dx = \int_\mathbb{R}\int_{\{u=t\}\cap\Omega} |u - \tilde{u}|dSdt. \tag{2.19}$$

The level sets of u can be foliated into one-dimensional curves by level sets of some function g_t in the sense of Definition 2.3.4. Thus

$$\begin{aligned}
\int_\mathbb{R}\int_{\{u=t\}\cap\Omega} |u - \tilde{u}|dSdt &= \int_\mathbb{R}\int_{\{u=t\}\cap\Omega} \frac{|\nabla g_t|}{|\nabla g_t|}|u - \tilde{u}|dSdt \\
&= \int_\mathbb{R}\int_\mathbb{R}\int_{\{u=t\}\cap\{g_t=r\}\cap\Omega} \frac{1}{|\nabla g_t|}|u - \tilde{u}|dldrdt \\
&\leq \frac{1}{c_g}\int_\mathbb{R}\int_\mathbb{R}\int_{\{u=t\}\cap\{g_t=r\}\cap\Omega} |u - \tilde{u}|dldrdt.
\end{aligned}$$

Similar to the two dimensional case, we parameterize every connected component Γ_t of $\{u = t\} \cap \{g_t = r\} \cap \Omega$ by arc length, $\gamma\colon [0, L] \to \Gamma_t$ with $\gamma(0) \in \partial\Omega$, and let $h(s) = u(\gamma(s)) - \tilde{u}(\gamma(s))$. Let x_t^* be the point that maximizes $|u - \tilde{u}|$ on Γ_t and suppose $\gamma(s_0) = x_t^*$ for some $s_0 \in (0, L)$, where L is the length of Γ_t. Then by an argument similar to the one in the proof of Theorem 2.3.3 we get

$$|u(x_t^*) - \tilde{u}(x_t^*)| \leq \frac{1}{\sigma_0}\int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))|d\tau,$$

and consequently

$$\int_{\Gamma_t} |u(x) - \tilde{u}(x)| dl \quad \le \quad \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl.$$

Hence,

$$\int_{\{u=t\}\cap\{g_t=r\}\cap\Omega} |u - \tilde{u}| dl \le \frac{L_M}{\sigma_0} \int_{\{u=t\}\cap\{g_t=r\}\cap\Omega} |J - \tilde{J}| dl. \tag{2.20}$$

Using this estimate and the coarea formula we have

$$\begin{aligned}
\frac{m}{\sigma_1} \int_{\Omega} |u - \tilde{u}| dx &\le \int_{\mathbb{R}} \int_{\{u=t\}\cap\Omega} |u - \tilde{u}| dS dt \\
&\le \frac{1}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\}\cap\{g_t=r\}\cap\Omega} |u - \tilde{u}| dl dr dt \\
&\le \frac{L_M}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\}\cap\{g_t=r\}\cap\Omega} |J - \tilde{J}| dl dr dt \\
&= \frac{L_M}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\{u=t\}} |\nabla g_t| |J - \tilde{J}| dS dt \\
&\le \frac{L_M C_g}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\{u=t\}} |J - \tilde{J}| dS dt \\
&= \frac{L_M C_g}{c_g \sigma_0} \int_{\Omega} |\nabla u| |J - \tilde{J}| dx \\
&\le \frac{L_M C_g}{c_g \sigma_0} \|\nabla u\|_{L^\infty(\Omega)} \left(C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)}^{\frac{1}{2}} \right) \\
&\le \frac{L_M C_g M}{c_g \sigma_0^2} \left(C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)}^{\frac{1}{2}} \right),
\end{aligned}$$

where we have applied Theorem 5.1.3. □

2.4 $W^{1,1}$ stability of the minimizers

In this section, we prove stability of minimizers of (2.2) in $W^{1,1}$. As mentioned in previously, in general (2.2) does not even have unique minimizers, so in order to prove stability results in $W^{1,1}$, it is natural to expect stronger assumptions on on the minimizers.

Lemma 2.4.1. *Let $n = 2, 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f \in L^\infty(\partial\Omega)$ and corrsponding conductivities σ and $\tilde{\sigma}$, and current density vector fields J and $\tilde{J}$,* *respectively. Suppose* $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ *with*

$$\| \sigma \|_{C^2(\Omega)}, \| \tilde{\sigma} \|_{C^2(\Omega)} \leq \sigma_2 \tag{2.21}$$

for some $\sigma_2 \in \mathbb{R}$*. Let*

$$G(x) := \frac{\tilde{J}(x) - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega, \tag{2.22}$$

with $G = (G_1, G_2)$ *for* $n = 2$ *and* $G = (G_1, G_2, G_3)$ *for* $n = 3$*. Then*

$$\|\nabla G_i\|_{L^1(\Omega)} \leq C_1 \|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}}, \tag{2.23}$$

for some constant C_1 *which depends only on* Ω*,* σ_0*,* σ_2 *and* $\| f \|_{L^\infty(\Omega)}$*.*

Proof. Since u and $\tilde{u}$ satisfy (2.1), it follows from elliptic regularity that

$$\| u \|_{H^3(\Omega)}, \| \tilde{u} \|_{H^3(\Omega)} \leq C_1 \| f \|_{L^2(\Omega)} \leq C_1 |\Omega|^{\frac{1}{2}} \| f \|_{L^\infty(\Omega)}, \tag{2.24}$$

for some constant C_1 depending only on σ_0, σ_2, and Ω. Now note that

$$G(x) = \nabla \tilde{u} - \frac{\sigma}{\tilde{\sigma}} \nabla u.$$

Thus it follows from (2.21) and (2.24) that

$$\| D^2 G_i \|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}} \| D^2 G_i \|_{L^2(\Omega)} \leq C, \quad 1 \leq i \leq n, \tag{2.25}$$

for some constant C which only depends on σ_0, σ_2, Ω and $\| f \|_{L^\infty(\Omega)}$. On the other hand it follows from Gagliardo-Nirenberg interpolation inequality that

$$\|\nabla G_i\|_{L^1(\Omega)} \leq C_2 \|D^2 G_i\|_{L^1(\Omega)}^{\frac{1}{2}} \|G_i\|_{L^1(\Omega)}^{\frac{1}{2}}, \tag{2.26}$$

for some C_2 which only depends on Ω. Combining (2.25), (2.26), and

$$\| G_i \|_{L^1(\Omega)} \leq \frac{\| J - \tilde{J} \|_{L^1(\Omega)}}{\sigma_0}, 1 \leq i \leq n,$$

we arrive at the inequality (2.23). □

Next we prove that u and $\tilde{u}$ are close in $W^{1,1}(\Omega)$. In order to do so, we need additional assumptions on the structure of level sets of u.

Definition 2.4.2. *Suppose u is admissible, $n = 2$, and $x \in \Omega$. Pick $h \in \mathbb{R}^2$ with $|h| = 1$, and $t \in \mathbb{R}$ small enough such that $x + th \in \Omega$. Let Γ and Γ_t be the level sets of u passing through x and $x + th$, respectively. Parametrize Γ and Γ_t by the arc length such that $\gamma(0), \gamma_t(0) \in \partial\Omega$, and denote these parametrizations by γ and γ_t, respectively.*

Similarly in dimension $n = 3$, let u be admissible and suppose level sets of u can be foliated to one-dimensional curves in the sense of Definition 2.3.4. Pick $x \in \Omega$ and $h \in \mathbb{R}^3$ with $|h| = 1$, and choose t small enough such that $x + th \in \Omega$. Let Γ and Γ_t be the unique curves in

$$\{\{u = \tau\} \cap \{g_\tau = r\} \quad \tau, r \in \mathbb{R}\}$$

which pass through x and $x + th$, respectively, and let γ and γ_t be the parametrization of these curves with respect to arc length with $\gamma(0), \gamma_t(0) \in \partial\Omega$.

We say that level sets of u are well structured *if the following conditions are satisfied*

(a) There exists $K \geq 0$ such that

$$\left| \frac{\gamma_t'(s) - \gamma'(s)}{t} \right| \leq K \tag{2.27}$$

for every $s \in [0, L]$, $t \in \mathbb{R}$, $x \in \Omega$ and $h \in S^{n-1}$. In particular,

$$\gamma_t'(s) \to \gamma'(s) \quad as \quad t \to 0, \tag{2.28}$$

where $\gamma'(s) = \frac{d\gamma(s)}{ds}$ and $\gamma_t'(s) = \frac{d\gamma_t(s)}{ds}$.

(b) There exists a bounded function $F_{x,h}(s) = F(x, h; s) \in L^\infty(\Omega \times S^{n-1} \times [0, L_M])$ such that

$$\lim_{t\to 0} \frac{\gamma_t(s) - \gamma(s)}{t} = F_{x,h}(s) \tag{2.29}$$

for every $s \in [0, L]$, $x \in \Omega$ and $h \in S^{n-1}$.

Remark 2.4.3. Let $x \in \Omega$, $h \in \mathbb{R}^2$ with $|h| = 1$, and $t \in \mathbb{R}$ be small enough such that $x + th \in \Omega$. Also, as in Definition 2.4.2, let γ, and γ_t be the parametrization of the curves passing through x and $x + th$. In view of Remark 2.3.5 we have

$$\gamma(s) = \mathcal{F}_{u(x)}(\bar{\gamma}(s)) \quad \text{and} \quad \gamma_t(s) = \mathcal{F}_{u(x+th)}(\bar{\gamma}_t(s)), \tag{2.30}$$

where $\bar{\gamma}(s)$ and $\bar{\gamma}_t(s)$ are parametrization of two level sets of the function $g(x, y) = y = \Pi_y(\mathcal{F}^{-1}(x))$ and $g(x, y) = y = \Pi_y(\mathcal{F}^{-1}(x + th))$, respectively. Here Π_y is the projection operator on y-axis, and $\mathcal{F}_{u(x)}$ and $\mathcal{F}_{u(x+th)}$ are C^1 diffeomorphisms from $B(0, 1)$ to the connected components of the level sets of u passing through x and $x + th$, respectively. It is easy to see that $\bar{\gamma}_t(s)$ is continuously differentiable with respect to t, for each fixed s.

Now let Γ_{x_0} be the connected component of the level set of u that passes through x_0, and assume that $|\nabla u| > 0$ on Ω. Then in a neighborhood of $r_0 = u(x_0)$ we can find C^1 diffeomorphisms F_r so that $F_r(y)$ is continuously differentiable with respect to r, for each fixed y. Indeed let $y \in B(0, 1)$ and consider the gradient flow

$$\dot{z}_y(q) = \nabla u(z_y(q)), \quad z_y(0) = F_0(y), \tag{2.31}$$

which has a unique solution as long as $z_y(q) \in \Omega$. Let $r \in range(u)$ be and Γ_r be a connected component of $\{u = r\}$. Define $F_r : B(0, 1) \to \Gamma_r$ by

$$F_r(y) = F_{r_0}(z_y(q_r)),$$

where $q_r \in \mathbb{R}$ is the unique point where $z_y(q_r) \in \Gamma_r$. Also observe that the set

$$\mathcal{R} = \{\ r \in range(u) : \quad \mathcal{F} \ \text{ is well defined on } \ \{u = r\}\},$$

is both open and closed in $range(u)$, and hence $\mathcal{R} = range(u)$ and therefore F_r could be defined globally as above for all $r \in range(u)$.

Since u, F_{r_0}, and z_y are all C^1, it is easy to see that $F_r(y)$ is continuously differentiable with respect to r, for each fixed $y \in B(0,1)$. Now notice that the level sets of the function $g(x,y) : B(0,1) \to \mathbb{R}$ defined by $g(x,y) = y$ are well structured in the sense of Definition 2.4.2. In view of the above arguments, it follows from the chain rule that $\gamma_t(s) = \mathcal{F}_t(\bar{\gamma}_t(s))$, where $\bar{\gamma}_t(s)$ is a parametrization of the level set $g(x,y) = y$ passing through $\mathcal{F}^{-1}(x+th)$, and $\mathcal{F}_t$ and $\bar{\gamma}_t$ are both continuously differentiable with respect to t. Therefore, since (2.27), (2.28), (2.29) hold for any parametrization of level sets of $g(x,y) = y$, an application of the chain rule implies that (2.27), (2.28), (2.29) also hold under the assumptions of Definition (2.4.2). In particular, if u is a C^1 function with $|\nabla u| > 0$ in Ω and $\{x \in \partial\Omega : f(x) = t\}$ has finitely many connected components for all t, then level sets of u are well structured in the sense of Definition 2.4.2.

Theorem 2.4.4. *Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy* (2.21)*. If u satisfies* (2.12)*, and the level sets of u are well-structured in the sense of Definition 2.4.2, then*

$$\|\nabla\tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C \parallel |J| - |\tilde{J}| \parallel_{L^\infty(\Omega)}^{\frac{1}{4}}, \tag{2.32}$$

for some constant $C(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. Fix $x \in \Omega$ and $h \in \mathbb{R}^2$ with $|h| = 1$. Then

$$\mathcal{L}(x,h) := (\nabla \tilde{u}(x) - \nabla u(x)) \cdot h = \lim_{t \to 0} \frac{[\tilde{u}(x+th) - u(x+th)] - [\tilde{u}(x) - u(x)]}{t}.$$

First we estimate the above limit. Since all level sets of u reach the boundary $\partial\Omega$, there exist $z, z_t \in \partial\Omega$ such that

$$u(x) = u(z) = \tilde{u}(z),$$

$$u(x+th) = u(z_t) = \tilde{u}(z_t).$$

Thus,

$$[\tilde{u}(x+th) - u(x+th)] - [\tilde{u}(x) - u(x)] = [\tilde{u}(x+th) - \tilde{u}(z_t)] - [\tilde{u}(x) - \tilde{u}(z)].$$

Let γ and γ_t be the curves passing through x and $x + th$, described in Definition 2.4.2 with $\gamma(0) = z$ and $\gamma_t(0) = z_t$. Suppose $\gamma(s_0) = x$ and reparamterize γ_t so that $\gamma_t(s_0) = x+th$. Then we have

$$[\tilde{u}(x+th) - \tilde{u}(z)] - [\tilde{u}(x) - \tilde{u}(z)] = [\tilde{u}(\gamma_t(s_0)) - \tilde{u}(\gamma_t(0))] - [\tilde{u}(\gamma(s_0)) - \tilde{u}(\gamma(0))]$$

$$= \int_0^{s_0} \nabla\tilde{u}(\gamma_t(s)) \cdot \gamma_t'(s) ds - \int_0^{s_0} \nabla\tilde{u}(\gamma(s)) \cdot \gamma'(s) ds.$$

Hence

$$\mathcal{L}(x,h) = \lim_{t \to 0} \frac{1}{t} \left(\int_0^{s_0} \nabla\tilde{u}(\gamma_t(s)) \cdot \gamma_t'(s) ds - \int_0^{s_0} \nabla\tilde{u}(\gamma(s)) \cdot \gamma'(s) ds \right).$$

Substituting $\nabla\tilde{u}$ by $\frac{\tilde{J}}{\tilde{\sigma}}$ and using the fact that J is perpendicular to γ' and γ_t' we get

$$\mathcal{L}(x,h) = \lim_{t \to 0} \frac{1}{t} \left(\int_0^{s_0} \frac{\tilde{J}(\gamma_t(s)) - J(\gamma_t(s))}{\tilde{\sigma}(\gamma_t(s))} \cdot \gamma_t'(s) ds - \int_0^{s_0} \frac{\tilde{J}(\gamma(s)) - J(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s) ds \right).$$

Now define

$$G(x) := \frac{\tilde{J}(x) - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega.$$

Hence

$$\mathcal{L}(x,h) = \lim_{t\to 0} \frac{1}{t}\left(\int_0^{s_0} G(\gamma_t(s))\cdot \gamma_t'(s)ds - \int_0^{s_0} G(\gamma(s))\cdot\gamma'(s)ds\right).$$

The expression in the right hand side can be rewritten as

$$\frac{1}{t}\int_0^{s_0}[G(\gamma_t(s)) - G(\gamma(s))]\cdot\gamma_t'(s)ds + \frac{1}{t}\int_0^{s_0} G(\gamma(s))\cdot[\gamma_t'(s)-\gamma'(s)]ds. \tag{2.33}$$

It follows from the assumption (a) in Definition 2.4.2 that

$$\left|\frac{\gamma_t'(s)-\gamma'(s)}{t}\right| \le K,$$

and hence

$$\left|\frac{1}{t}\int_0^{s_0} G(\gamma(s))\cdot[\gamma_t'(s)-\gamma'(s)]ds\right| \le \frac{K}{\sigma_0}\int_0^L |\tilde{J}(\gamma(s)) - J(\gamma(s))|ds. \tag{2.34}$$

Now we turn our attention to the first term in (2.33). Let $G = (G_1, G_2)$. Since

$$\lim_{t\to 0}\frac{\gamma_t(s)-\gamma(s)}{t} = F_{x,h}(s)$$

for $i = 1, 2$ we have

$$\begin{aligned} \lim_{t\to 0}\frac{G_i(\gamma_t(s)) - G_i(\gamma(s))}{t} &= \lim_{t\to 0}\frac{G_i(\gamma(s)+tF(s)) - G_i(\gamma(s))}{t} \\ &= \nabla G_i(\gamma(s))\cdot F(s). \end{aligned}$$

Thus the first term of (2.33) can be rewritten as

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\int_0^{s_0}[G(\gamma_t(s))-G(\gamma(s))]\cdot\gamma_t'(s)dl\\
&=\int_0^{s_0}(\nabla G_1(\gamma(s))\cdot F(s),\nabla G_2(\gamma(s))\cdot F(s))\cdot\gamma'(s)dl\\
&\leq \| F \|_{L^\infty}\int_0^{s_0}|\nabla G_1(\gamma(s))|+|\nabla G_2(\gamma(s))|dl\\
&\leq \| F \|_{L^\infty}\int_0^{L}|\nabla G_1(\gamma(s))|+|\nabla G_2(\gamma(s))|dl, \qquad (2.35)
\end{aligned}$$

where we have used the assumption (b) in Definition 2.4.2. Combining (2.34) and (2.35) we obtain

$$\begin{aligned}
|\nabla\tilde{u}(x)-\nabla u(x)| &\leq \sup_{h\in\mathbb{R}^2,|h|=1}\mathcal{L}(x,h)\\
&\leq \frac{K}{\sigma_0}\int_0^L|\tilde{J}(\gamma(s))-J(\gamma(s))|dl\\
&\quad+\| F \|_{L^\infty}\int_0^L|\nabla G_1(\gamma(s))|+|\nabla G_2(\gamma(s))|dl.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_\Gamma|\nabla\tilde{u}(x)-\nabla u(x)|dl &\leq \frac{KL_M}{\sigma_0}\int_\Gamma|\tilde{J}(x)-J(x)|dl\\
&\quad+L_M\| F \|_{L^\infty}\int_\Gamma|\nabla G_1(x)|+|\nabla G_2(x)|dl,
\end{aligned}$$

and consequently

$$\begin{aligned}
\int_{\{u=\tau\}\cap\Omega}|\nabla\tilde{u}(x)-\nabla u(x)|dl &\leq \frac{KL_M}{\sigma_0}\int_{\{u=\tau\}\cap\Omega}|\tilde{J}(x)-J(x)|dl \qquad (2.36)\\
&\quad+L_M\| F \|_{L^\infty}\int_{\{u=\tau\}\cap\Omega}|\nabla G_1(x)|+|\nabla G_2(x)|dl.
\end{aligned}$$

Using (2.36) and the coarea formula we have

$$\begin{aligned}
\frac{m}{\sigma_1}\|\nabla\tilde{u}-\nabla u\|_{L^1(\Omega)} &\leq \int_\Omega |\nabla u||\nabla\tilde{u}-\nabla u|dx \\
&= \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\nabla\tilde{u}-\nabla u|dld\tau \\
&\leq \frac{KL_M}{\sigma_0}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\tilde{J}-J|dld\tau \\
&+ L_M \parallel F \parallel_{L^\infty} \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\nabla G_1|+|\nabla G_2|dld\tau \\
&\leq \frac{KL_M M}{(\sigma_0)^2}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} \frac{|\tilde{J}-J|}{|\nabla u|}dld\tau \\
&+ \frac{L_M \parallel F \parallel_{L^\infty} M}{\sigma_0}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} \frac{|\nabla G_1|+|\nabla G_2|}{|\nabla u|}dld\tau \\
&= \frac{KL_M M}{(\sigma_0)^2}\int_\Omega |\tilde{J}-J|dx \\
&+ \frac{L_M \parallel F \parallel_{L^\infty} M}{\sigma_0}\int_\Omega |\nabla G_1|+|\nabla G_2|dx \\
&\leq \frac{KL_M M}{(\sigma_0)^2} \parallel J-\tilde{J} \parallel_{L^1(\Omega)} \\
&+ \frac{2L_M C_1 \parallel F \parallel_{L^\infty} M}{\sigma_0} \parallel J-\tilde{J} \parallel_{L^1(\Omega)}^{\frac{1}{2}}
\end{aligned}$$

where we have used (2.4.1) to obtain the last inequality. Applying Theorem 5.1.3, and noting that

$$\parallel J-\tilde{J} \parallel_{L^1(\Omega)}^{\frac{1}{2}} \leq 2M,$$

where M is defined in (2.5), we arrive at (2.32). □

Now we prove three dimensional version of this theorem.

Theorem 2.4.5. *Let $n=3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega}=\tilde{u}|_{\partial\Omega}=f$, corresponding conductivities $\sigma,\tilde{\sigma}\in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma,\tilde{\sigma}\in C^2(\bar{\Omega})$ and satisfy (2.21). In addition suppose u satisfies (2.12), the level sets of u can be foliated to one-dimensional curves in the sense of Definition 3.4, and the level sets of u are well-structured in the sense of Definition 4.2. Then*

$$\|\nabla\tilde{u}-\nabla u\|_{L^1(\Omega)} \le C\|a-\tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}}, \tag{2.37}$$

for some constant $C(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M, c_g, C_g)$ *is independent of* $\tilde{u}$ *and* $\tilde{\sigma}$*.*

Proof. With an argument similar to the one used in the proof of Theorem 2.4.4 we get

$$\begin{aligned} \int_{U_{\tau,r}} |\nabla\tilde{u}(x)-\nabla u(x)|dl \;&\le\; \frac{KL_M}{\sigma_0}\int_{U_{\tau,r}}|\tilde{J}(x)-J(x)|dl \\ &\quad +L_M \parallel F \parallel_{L^\infty}\int_{U_{\tau,r}}|\nabla G_1(x)|+|\nabla G_1(x)|+|\nabla G_3(x)|dl, \end{aligned} \tag{2.38}$$

where $U_{\tau,r} := \{u=\tau\}\cap\{g_\tau = r\}\cap\Omega$ and $G=(G_1,G_2,G_3)$ is defined in (3.16).

It follows follows from (2.38) and the coarea formula that

$$\begin{aligned} \frac{m}{\sigma_1}\|\nabla\tilde{u}-\nabla u\|_{L^1(\Omega)} &\le \int_\Omega |\nabla u||\nabla\tilde{u}-\nabla u|dx \\ &= \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega}|\nabla\tilde{u}-\nabla u|dSd\tau \\ &= \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega}\frac{|\nabla g_\tau|}{|\nabla g_\tau|}|\nabla\tilde{u}-\nabla u|dSd\tau \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}}\frac{1}{|\nabla g_\tau|}|\nabla\tilde{u}-\nabla u|dldrd\tau \\ &\le \frac{KL_M}{\sigma_0 c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}}|\tilde{J}-J|dldrdt \\ &+ \frac{L_M \parallel F \parallel_{L^\infty}}{c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}}|\nabla G_1|+|\nabla G_2|+|\nabla G_3|dldrdt \\ &\le \frac{KL_M M C_g}{(\sigma_0)^2 c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}}\frac{|\tilde{J}-J|}{|\nabla u||\nabla g_\tau|}dldrdt \\ &+ \frac{L_M M \parallel F \parallel_{L^\infty} C_g}{\sigma_0 c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}}\frac{|\nabla G_1|+|\nabla G_2|+|\nabla G_3|}{|\nabla u||\nabla g_t|}dldrdt \end{aligned}$$

$$
\begin{aligned}
&= \frac{KL_M M C_g}{(\sigma_0)^2 c_g}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega}\frac{|\tilde{J}-J|}{|\nabla u|}dSdt \\
&+ \frac{L_M M \parallel F \parallel_{L^\infty} C_g}{\sigma_0 c_g}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega}\frac{|\nabla G_1|+|\nabla G_2|+|\nabla G_3|}{|\nabla u|}dSdt \\
&= \frac{KL_M M C_g}{(\sigma_0)^2 c_g}\int_{\Omega}|\tilde{J}-J|dx \\
&+ \frac{L_M M \parallel F \parallel_{L^\infty} C_g}{\sigma_0 c_g}\int_{\Omega}|\nabla G_1|+|\nabla G_2|+|\nabla G_3|dx \\
&\leq \frac{KL_M M C_g}{(\sigma_0)^2 c_g}\parallel J-\tilde{J}\parallel_{L^1(\Omega)} \\
&+ \frac{3L_M C_1 M \parallel F \parallel_{L^\infty(\Omega)} C_g}{\sigma_0 c_g}\parallel J-\tilde{J}\parallel^{\frac{1}{2}}_{L^1(\Omega)},
\end{aligned}
$$

where we have used (2.4.1) to obtain the last inequality. Applying Theorem 5.1.3, and noting that

$$
\parallel J-\tilde{J}\parallel^{\frac{1}{2}}_{L^1(\Omega)}\leq 2M,
$$

we obtain the inequality (2.32). □

Now, we are ready to prove our main stability results.

Theorem 2.4.6. *Let $n=2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega}=\tilde{u}|_{\partial\Omega}=f$, corresponding conductivities $\sigma,\tilde{\sigma}\in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma,\tilde{\sigma}\in C^2(\bar{\Omega})$ and satisfy* (2.21)*. If u satisfies* (2.12) *and level sets of u are well-structured in the sense of Definition 2.4.2, then*

$$
\|\sigma-\tilde{\sigma}\|_{L^1(\Omega)}\leq C\parallel |J|-|\tilde{J}|\parallel^{\frac{1}{4}}_{L^\infty(\Omega)},
$$

for some constant $C(m,M,\sigma_0,\sigma_1,\sigma_2,\sigma,f,L_M)$ independent of $\tilde{\sigma}$.

Proof. Using Theorem 2.4.4 we have

$$
\begin{aligned}
\int_\Omega |\sigma - \tilde{\sigma}| dx &= \int_\Omega \left| \frac{|J|(|\nabla \tilde{u}| - |\nabla u|)}{|\nabla u||\nabla \tilde{u}|} + \frac{|J| - |\tilde{J}|}{|\nabla \tilde{u}|} \right| dx \\
&\le \int_\Omega \frac{|J|}{|\nabla u||\nabla \tilde{u}|} \left||\nabla u| - |\nabla \tilde{u}|\right| dx + \int_\Omega \frac{1}{|\nabla \tilde{u}|} \left||J| - |\tilde{J}|\right| dx \\
&\le \int_\Omega \frac{|J|}{|\nabla u||\nabla \tilde{u}|} |\nabla u - \nabla \tilde{u}| dx + \int_\Omega \frac{1}{|\nabla \tilde{u}|} \left||J| - |\tilde{J}|\right| dx \\
&\le \frac{M\sigma_1^2 C}{m^2} \| \, |J| - |\tilde{J}| \, \|_{L^\infty(\Omega)}^{\frac{1}{4}} + \frac{\sigma_1 |\Omega|}{m} \| \, |J| - |\tilde{J}| \, \|_{L^\infty(\Omega)} \\
&\le \left[\frac{M\sigma_1^2 C}{m^2} + \frac{\sigma_1 |\Omega| (2M)^{\frac{3}{4}}}{m} \right] \| \, |J| - |\tilde{J}| \, \|_{L^\infty(\Omega)}^{\frac{1}{4}} .
\end{aligned}
$$

□

Theorem 2.4.7. *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = f$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (2.21). If u satisfies (2.12), the level sets of u can be foliated to one-dimensional curves in the sense of Definition 2.3.4, and the level sets of u are well-structured in the sense of Definition 2.4.2, then*

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \le C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)}^{\frac{1}{4}}, \tag{2.39}$$

for some constant $C(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, f, L_M, c_g, C_g)$ independent of $\tilde{\sigma}$.

Proof. The proof follows from Theorem 2.4.5 and a calculation similar to that of the proof of Theorem 2.4.6. □

Chapter 3

Stability of CDII with Boundary Errors

3.1 Introduction

A natural question which remains open is how the presence of errors in measurements of the boundary voltage f together with errors in measurements of $|J|$ affect reconstruction of the conductivity σ in the interior? That is to say, if we consider the pair of interior and boundary measurements $(f, |J|)$ and $(\tilde{f}, |\tilde{J}|)$ will the resulting σ and $\tilde{\sigma}$ be close in some sense? In this Chapter, I will discuss the extension of the CDII problem outlined in the previous chapter to the inclusion of boundary errors, which is covered in the work done in [20]. The original setup will largely be the same, however we generalize our approach in [19] to prove that in dimensions $n = 2, 3$ the following stability result holds

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \leq C_1 \||J| - |\tilde{J}|\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\Omega)}^{\frac{1}{4}},$$

for some constants C_1, C_2 independent of $\tilde{\sigma}$ (see Theorems 3.4.4 and 3.4.5 for precise statements of the results). The proofs are generalizations of the arguments developed in [19].

This chapter is organized similarly to the previous. First, under very weak assumptions, we will prove that the structure of level sets of the least gradient problem (2.2) is stable under these new initial conditions. Next, we will provide stability results for minimizers of (2.2) in L^1. Finally, we will prove stability of minimizers of (2.2) in $W^{1,1}$, and shall use them to prove Theorems 3.4.4 and 3.4.5 which are the main results of this chapter.

3.2 Stability of level sets

In this section, we show that the structure of the level sets of minimizers of the least gradient problem (2.2) is stable. Throughout the chapter, we will assume that $a, \tilde{a} \in C(\Omega)$ and $f, \tilde{f} \in L^\infty(\partial\Omega)$ with

$$0 < m \leq a(x), \tilde{a}(x) \leq M \quad \forall x \in \Omega \qquad \text{and} \qquad |f(y)|, |\tilde{f}(y)| \leq M \quad \forall y \in \partial\Omega \tag{3.1}$$

for some positive constants m, M.

Lemma 3.2.1. *Let f, $\tilde{f} \in L^1(\partial\Omega)$. Suppose u solves* (2.1) *for $u|_{\partial\Omega} = f$, and $\tilde{u}$ solves* (2.1) *for $\tilde{u}|_{\partial\Omega} = \tilde{f}$. Then there exists $C(m, M, \Omega, f) > 0$ such that*

$$\max\left\{\int_\Omega |D\tilde{u}|, \int_\Omega |Du|\right\} \leq C. \tag{3.2}$$

Proof. Fix $w \in BV_f(\Omega)$ and let $\tilde{w} \in BV_{\tilde{f}}(\Omega)$. Then in view of (3.1) we have

$$\begin{aligned}
m\int_\Omega |D\tilde{u}|dx &\leq \int_\Omega \tilde{a}|D\tilde{u}|dx \leq \int_\Omega \tilde{a}|D\tilde{w}|dx \leq M\int_\Omega |D\tilde{w}| \\
&\leq M\int_\Omega |Dw| + M\int_\Omega |D(w-\tilde{w})| \\
&\leq M\int_\Omega |Dw| + MC_1||f-\tilde{f}||_{L^1(\partial\Omega)} \\
&\leq M\int_\Omega |Dw| + M^2C_1|\Omega| =: C(m.M, \Omega, f),
\end{aligned}$$

where we have used Theorem 2.16 in [8] to get the fifth inequality above. Similarly we can establish an analogous estimate for u and show that $\int_\Omega |Du| \le C$, where C is the constant appearing in the above estimates. Hence

$$\max\left\{\int_\Omega |D\tilde{u}|, \int_\Omega |Du|\right\} \le C,$$

for some $C(m, M, \Omega, f)$ independent of $\tilde{u}$, u, and $\tilde{f}$. $\square$

Lemma 3.2.2. *Let $f, \tilde{f} \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are the corresponding minimizers of* (2.2) *with the weights a and $\tilde{a}$, respectively. Then*

$$\left|\int_\Omega a|Du|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx\right| \le C_1\|a - \tilde{a}\|_{L^\infty(\Omega)} + C_2\|f - \tilde{f}\|_{L^1(\partial\Omega)}, \tag{3.3}$$

for some constants $C_i = C(m, M, \Omega, f)$ independent of u, $\tilde{u}$, and $\tilde{f}$.

Proof. Let $w \in BV(\Omega)$ such that $w|_{\partial\Omega} = f - \tilde{f}$. Suppose u, $\tilde{u}$ are the minimizers of (2.2) with the weights a and $\tilde{a}$ and boundary data f and $\tilde{f}$, respectively. Note:

$$u - w \in BV_{\tilde{f}}(\Omega),\ \tilde{u} + w \in BV_f(\Omega)$$

We have

$$\int_\Omega a|Du|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx = \int_\Omega (a - \tilde{a})(|Du| + |D\tilde{u}|)dx + \int_\Omega \tilde{a}|Du| - a|D\tilde{u}|dx \tag{3.4}$$

Hence,

$$\begin{aligned}\int_\Omega a|Du|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx &\le \left(\|Du\|_{L^1(\Omega)} + \|D\tilde{u}\|_{L^1(\Omega)}\right)\|a - \tilde{a}\|_{L^\infty(\Omega)} + \int_\Omega \tilde{a}|Du| - a|D\tilde{u}|dx \\ &\le 2C\|a - \tilde{a}\|_{L^\infty(\Omega)} + \int_\Omega \tilde{a}|Du| - a|D\tilde{u}|dx\end{aligned}$$

Where we have applied Lemma 3.2.2 to the first term. Focusing on the second term, we have

$$\begin{aligned}\int_\Omega \tilde{a}|Du| - a|D\tilde{u}|dx &= \int_\Omega \tilde{a}|Du| - a|D(\tilde{u}+w) - Dw|dx \\ &\le \int_\Omega \tilde{a}|Du| - a|D(\tilde{u}+w)| + a|Dw|dx \\ &\le \int_\Omega \tilde{a}|Du| - a|Du| + a|Dw|dx \\ &\le \|Du\|_{L^1(\Omega)}\|a-\tilde{a}\|_{L^\infty(\Omega)} + M\int_\Omega |Dw|dx\end{aligned}$$

This comes from the triangle inequality and the fact that u is a minimizer for (2.2) on $BV_f(\Omega)$. Now, by invoking the extension Theorem 2.16 in [8] we get:

$$\|Dw\|_{L^1(\Omega)} \le C'\|f-\tilde{f}\|_{L^1(\partial\Omega)} \tag{3.5}$$

and subsequently

$$\int_\Omega a|Du|dx - \int_\Omega \tilde{a}|D\tilde{u}|dx \le 2C\|a-\tilde{a}\|_{L^\infty(\Omega)} + MC'\|f-\tilde{f}\|_{L^1(\partial\Omega)} \tag{3.6}$$

Similarly, we can prove

$$\int_\Omega \tilde{a}|D\tilde{u}|dx - \int_\Omega a|Du|dx \le 2C\|a-\tilde{a}\|_{L^\infty(\Omega)} + MC'\|f-\tilde{f}\|_{L^1(\partial\Omega)},$$

and hence (3.3) follows. $\square$

For the following, we will again need the weak integration by parts formula (2.9) (as outlined in Chapter 2).

Lemma 3.2.3. *Let $f, \tilde{f} \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (2.2) with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2.1. Suppose $0 \le \sigma(x) \le \sigma_1$ in Ω for some constant $\sigma_1 > 0$, where σ is the*

Radon-Nikodym derivative of $|J|dx$ with respect to $|Du|$. Then

$$\int_\Omega |J||\tilde{J}| - J \cdot \tilde{J} dx \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)} + C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}, \tag{3.7}$$

where $C_i = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$ and $\tilde{f}$.

Proof. We have

$$\begin{aligned}
\int_\Omega |J||\tilde{J}| - J \cdot \tilde{J} dx &= \int_\Omega \sigma |\tilde{J}||Du| - \sigma \tilde{J} \cdot Du dx \\
&\leq \sigma_1 \int_\Omega |\tilde{J}||Du| - \tilde{J} \cdot Du dx \\
&= \sigma_1 \left(\int_\Omega |\tilde{J}||Du| dx - \int_{\partial\Omega} f [\tilde{J}, \nu_\Omega] dx \right) \\
&= \sigma_1 \left(\int_\Omega |\tilde{J}||Du| dx + \int_{\partial\Omega} (\tilde{f} - f)[\tilde{J}, \nu_\Omega] dx - \int_{\partial\Omega} \tilde{f}[\tilde{J}, \nu_\Omega] dx \right) \\
&\leq \sigma_1 \left(\int_\Omega |\tilde{J}||Du| - \tilde{J} \cdot D\tilde{u} dx + \|[\tilde{J}, \nu_\Omega]\|_{L^\infty(\partial\Omega)} \|f - \tilde{f}\|_{L^1(\partial\Omega)} \right) \\
&\leq \sigma_1 \left(\int_\Omega |\tilde{J}||Du| - |\tilde{J}||D\tilde{u}| dx + \|\tilde{a}\|_{L^\infty(\Omega)} \|f - \tilde{f}\|_{L^1(\partial\Omega)} \right) \\
&\leq \sigma_1 \left(\int_\Omega |\tilde{J}||Du| - |\tilde{J}||D\tilde{u}| dx + M \|f - \tilde{f}\|_{L^1(\partial\Omega)} \right)
\end{aligned}$$

where we have used the integration by parts formula (2.9) to get the second inequality above. On the other hand, it follows from Lemma 3.2.2 that

$$\begin{aligned}
\sigma_1 \int_\Omega |\tilde{J}||Du| - |\tilde{J}||D\tilde{u}| dx &= \sigma_1 \int_\Omega |\tilde{J}||Du| - |J||Du| + |J||Du| - |\tilde{J}||D\tilde{u}| dx \\
&= \sigma_1 \left(\int_\Omega (a - \tilde{a})|Du| dx + \int_\Omega a|Du| - \tilde{a}|D\tilde{u}| dx \right) \\
&\leq \sigma_1 \|Du\|_{L^1(\Omega)} \|a - \tilde{a}\|_{L^\infty(\Omega)} \\
&+ \sigma_1 C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)} + \sigma_1 C_2 \|f - \tilde{f}\|_{L^1(\partial\Omega)}.
\end{aligned}$$

Hence,

$$\int_\Omega |J||\tilde{J}| - J\cdot\tilde{J}dx \le \sigma_1\left(\|Du\|_{L^1(\Omega)} + C_1\right)\|a-\tilde{a}\|_{L^\infty(\Omega)}$$
$$+\sigma_1\left(M + C_2\right)\|f-\tilde{f}\|_{L^1(\partial\Omega)},$$

which yields the desired result. $\square$

Roughly speaking, Lemma 3.2.3 implies that as $a \to \tilde{a}$ and $f \to \tilde{f}$, $\frac{Du}{|Du|}(x)$ becomes parallel to $\frac{D\tilde{u}}{|D\tilde{u}|}(x)$ at points where the two gradients do not vanish. We are now ready to prove the main result of this section.

Theorem 3.2.4. *Let $f, \tilde{f} \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (2.2) with the weights a and $\tilde{a}$ and boundary data f and $\tilde{f}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2.1. Suppose $0 \le \sigma(x) \le \sigma_1$ in Ω for some constant $\sigma_1 > 0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du|$. Then*

$$\|J-\tilde{J}\|_{L^1(\Omega)} \le C_1\|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)} + C_2\|f-\tilde{f}\|^{\frac{1}{2}}_{L^1(\partial\Omega)}, \tag{3.8}$$

where $C_i = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$ and $\tilde{f}$.

Proof. The second line following from the argument outlined in the beginning of Theorem 2.5 in [19] we have:

$$
\begin{aligned}
\|J-\tilde{J}\|_{L^1(\Omega)} &= \int_\Omega \left(|J-\tilde{J}|^2\right)^{\frac{1}{2}} dx \\
&\leq \int_\Omega \Big||J|-|\tilde{J}|\Big| \, dx + \int_\Omega \left(2(|J||\tilde{J}| - J\cdot\tilde{J})\right)^{\frac{1}{2}} dx \\
&= \int_\Omega |a-\tilde{a}| dx + \int_\Omega \left(2(|J||\tilde{J}| - J\cdot\tilde{J})\right)^{\frac{1}{2}} dx \\
&\leq |\Omega| \|a-\tilde{a}\|_{L^\infty(\Omega)} + |\Omega|^{\frac{1}{2}} \left(\int_\Omega 2(|J||\tilde{J}| - J\cdot\tilde{J}) dx\right)^{\frac{1}{2}} \\
&\leq |\Omega| \|a-\tilde{a}\|_{L^\infty(\Omega)} + (2|\Omega|)^{\frac{1}{2}} \left[C_1\|a-\tilde{a}\|_{L^\infty(\Omega)} + C_2\|f-\tilde{f}\|_{L^\infty(\partial\Omega)}\right]^{\frac{1}{2}} \\
&\leq |\Omega| \|a-\tilde{a}\|_{L^\infty(\Omega)} \\
&\quad + (2|\Omega|)^{\frac{1}{2}} \left[(C_1\|a-\tilde{a}\|_{L^\infty(\Omega)})^{\frac{1}{2}} + (C_2\|f-\tilde{f}\|_{L^1(\partial\Omega)})^{\frac{1}{2}}\right] \\
&\leq \left[|\Omega|(2M)^{\frac{1}{2}} + (2C_1|\Omega|)^{\frac{1}{2}}\right] \|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)} \\
&\quad + [2C_2|\Omega|]^{\frac{1}{2}} \|f-\tilde{f}\|^{\frac{1}{2}}_{L^1(\partial\Omega)},
\end{aligned}
$$

where we have used the Holder's inequality and Lemma 3.2.3. □

Remark 3.2.5. In view of Theorem 2.2.1, $\frac{Du}{|Du|}$ and $\frac{D\tilde{u}}{|D\tilde{u}|}$ are parallel to J and $\tilde{J}$, respectively. So Theorem 3.2.4 implies that if $\tilde{a}$ is close to a and $\tilde{f}$ is close to f, then the structure of level sets of $\tilde{u}$ is close to that of u.

3.3 L^1 stability of the minimizers

In this section, we establish stability of minimizers of the least gradient problem (2.2) in L^1 with respect to our new initial conditions. Yet again, in order to prove any stability results further assumptions on the weights $a, \tilde{a}$ as well as the corresponding minimizers are expected. Similarly to the previous chapter, we will need to make use of the admissibility condition (Definition 2.3.1) as well as the foliation condition outlined in (2.12) for dimension $n = 2$.

Theorem 3.3.1. *Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$, and corresponding current density vector fields J and $\tilde{J}$, respectively. If u satisfies* (2.12), *then*

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C_1 \||J| - |\tilde{J}|\|^{\frac{1}{2}}_{L^\infty(\Omega)} + C_2 \|f - \tilde{f}\|^{\frac{1}{2}}_{L^\infty(\partial\Omega)}, \tag{3.9}$$

for some constants $C_i(m, M, \sigma_0, \sigma_1, f, u, L_M)$ independent of $\tilde{u}$, $\tilde{\sigma}$, and $\tilde{f}$.

Proof. Since u is admissible,

$$|\nabla u(x)| = \frac{|J(x)|}{\sigma(x)} \geq \frac{m}{\sigma_1} > 0, \quad \forall x \in \Omega.$$

Using the coarea formula we get

$$\frac{m}{\sigma_1} \int_\Omega |u - \tilde{u}| dx \leq \int_\Omega |\nabla u||u - \tilde{u}| dx = \int_{\mathbb{R}} \int_{\{u=t\}\cap\Omega} |u - \tilde{u}| dS dt. \tag{3.10}$$

Since $|\nabla u| > 0$ in Ω, it follows from the regularity result of De Giorgi (Theorem 4.11 in [8]) that all level sets of u are C^1 curves. Now let Γ_t be a connected component of $\{x \in \Omega \colon u(x) = t\} \subset \Omega$, and $\gamma \colon [0, L] \to \Gamma_t$ to be a path parametrized by the arc length with $\gamma(0) \in \partial\Omega$. We will henceforth denote $\gamma(0)$ by x_t^0. Define

$$h(s) := u(\gamma(s)) - \tilde{u}(\gamma(s)).$$

Since $\nabla u(\gamma(s)) \cdot \gamma'(s) = 0$ on Γ_t, we have

$$\begin{aligned} h'(s) &= \nabla u(\gamma(s)) \cdot \gamma'(s) - \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) \\ &= \Big(\frac{\sigma}{\tilde{\sigma}}(\gamma(s)) \nabla u(\gamma(s)) - \nabla \tilde{u}(\gamma(s))\Big) \cdot \gamma'(s). \end{aligned}$$

We can rewrite the above equality as

$$h'(s) = \frac{J(\gamma(s)) - \tilde{J}(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s).$$

Note that

$$h(0) = u(\gamma(0)) - \tilde{u}(\gamma(0)) = f(x_t^0) - \tilde{f}(x_t^0).$$

Consequently, we have that

$$h(s) - h(0) = \int_0^s \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau$$

and, moreover,

$$h(s) = \int_0^s \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau + f(x_t^0) - \tilde{f}(x_t^0).$$

Now let x_t^* be a point on Γ_t where the maximum distance between u and $\tilde{u}$ along the path γ occurs, i.e.

$$|u(x_t^*) - \tilde{u}(x_t^*)| = \max_{x \in \Gamma_t} |u(x) - \tilde{u}(x)|.$$

Then $x_t^* = \gamma(s_0)$ for some $s_0 \in [0, L]$, and

$$\begin{aligned} |u(x_t^*) - \tilde{u}(x_t^*)| = |h(s_0)| &= \left| \int_0^{s_0} \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau + f(x_t^0) - \tilde{f}(x_t^0) \right| \\ &\leq \int_0^{s_0} \frac{1}{\tilde{\sigma}(\gamma(\tau))} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)| \\ &\leq \frac{1}{\sigma_0} \int_0^{s_0} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)|. \end{aligned}$$

In particular, for every $x \in \Gamma_t$

$$|u(x) - \tilde{u}(x)| \leq |u(x_t^*) - \tilde{u}(x_t^*)| \leq \frac{1}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + |f(x_t^0) - \tilde{f}(x_t^0)|,$$

where L denotes the entire length of Γ_t. Hence

$$\begin{aligned}
\int_{\Gamma_t} |u(x) - \tilde{u}(x)| dl &\leq |u(x_t^*) - \tilde{u}(x_t^*)| \int_{\Gamma_t} 1 dl \\
&\leq L_M |u(x_t^*) - \tilde{u}(x_t^*)| \\
&\leq \frac{L_M}{\sigma_0} \int_0^L |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau + L_M |f(x_t^0) - \tilde{f}(x_t^0)| \\
&= \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)|,
\end{aligned}$$

and therefore

$$\int_{\{u=t\}\cap\Omega} |u - \tilde{u}| dl \leq \frac{L_M}{\sigma_0} \int_{\{u=t\}\cap\Omega} |J - \tilde{J}| dl + L_M |f(x_t^0) - \tilde{f}(x_t^0)|. \tag{3.11}$$

Since $u \in C(\overline{\Omega})$ solves (2.1), by maximum and minimum principles for solutions to elliptic equations,

$$\max_{\Omega} u = \max_{\partial\Omega} f := C_f$$

$$\min_{\Omega} u = \min_{\partial\Omega} f := c_f$$

and hence $c_f \leq u \leq C_f$, with $-M \leq c_f, C_f \leq M$. Thus we have

$$
\begin{aligned}
\int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega}|u-\tilde{u}|dldt &= \int_{c_f}^{C_f}\int_{\{u=t\}\cap\Omega}|u-\tilde{u}|dldt \\
&\le L_M\int_{c_f}^{C_f}\left(\int_{\{u=t\}\cap\Omega}\frac{1}{\sigma_0}|J-\tilde{J}|dl + L_M|f(x_t^0)-\tilde{f}(x_t^0)|\right)dt \\
&\le \frac{L_M}{\sigma_0}\int_{c_f}^{C_f}\int_{\{u=t\}\cap\Omega}|J-\tilde{J}|dldt + L_M\|f-\tilde{f}\|_{L^\infty(\partial\Omega)}\int_{c_f}^{C_f}dt \\
&= \frac{L_M}{\sigma_0}\int_{\Omega}|\nabla u||J-\tilde{J}|dx + L_M(C_f-c_f)\|f-\tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\le \frac{L_M}{\sigma_0}\|\nabla u\|_{L^\infty(\Omega)}\int_{\Omega}|J-\tilde{J}|dx + L_M(C_f-c_f)\|f-\tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\le \frac{L_M}{\sigma_0}\|\nabla u\|_{L^\infty(\Omega)}\left[C_1\|a-\tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2\|f-\tilde{f}\|_{L^1(\partial\Omega)}^{\frac{1}{2}}\right] \\
&+ L_M(C_f-c_f)\|f-\tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\le \frac{L_M M C_1}{\sigma_0^2}\|a-\tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} \\
&+ \left[\frac{L_M M C_2}{\sigma_0^2}|\partial\Omega|^{\frac{1}{2}} + L_M(C_f-c_f)(2M)^{\frac{1}{2}}\right]\|f-\tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}},
\end{aligned}
$$

where we have used (3.11) and Theorem 3.2.4. Hence, (3.9) follows.

Note that $C_i(m, M, \sigma_0, \sigma_1, f, u, L_M)$ are independent of $\tilde{u}$, $\tilde{\sigma}$, and $\tilde{f}$. □

Next we generalize Theorem 3.3.1 to dimension $n = 3$. In order to do this, we need the additional assumptions on level sets of u that were outlined in the previous chapter (see Definition 2.3.4 as well as Remark 2.3.5).

Theorem 3.3.2. *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ and corresponding current density vector fields J and $\tilde{J}$, respectively. Suppose the level sets of u can be foliated to one-dimensional curves in the sense of Definition 2.3.4. Then*

$$
\|u-\tilde{u}\|_{L^1(\Omega)} \le C_1\|a-\tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2\|f-\tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}}, \tag{3.12}
$$

where $C(m, M, \sigma_0, \sigma_1, f, u, L_M, c_g, C_g, g)$ is independent of $\tilde{u}$, $\tilde{\sigma}$, and $\tilde{f}$.

Proof. The proof is similar to the proof of Theorem 3.3.1, and we provide the details for the sake of the reader. Since u is admissible,

$$\frac{m}{\sigma_1}\int_\Omega |u-\tilde{u}|dx \le \int_\Omega |\nabla u||u-\tilde{u}|dx = \int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega} |u-\tilde{u}|dSdt. \tag{3.13}$$

The level sets of u can be foliated into one-dimensional curves by level sets of some function g in the sense of Definition 2.3.4. Thus

$$\begin{aligned}\int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega} |u-\tilde{u}|dSdt &= \int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega} \frac{|\nabla g_t|}{|\nabla g_t|}|u-\tilde{u}|dSdt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{u=t\}\cap\{g=r\}\cap\Omega} \frac{1}{|\nabla g_t|}|u-\tilde{u}|dldrdt \\ &\le \frac{1}{c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{u=t\}\cap\{g=r\}\cap\Omega} |u-\tilde{u}|dldrdt.\end{aligned}$$

Similar to the two dimensional case, we parameterize every connected component Γ_t of $\{u=t\}\cap\{g=r\}\cap\Omega$ by arc length, $\gamma\colon [0,L]\to\Gamma_t$ with $\gamma(0)=x_t^0\in\partial\Omega$, and let $h(s)=u(\gamma(s))-\tilde{u}(\gamma(s))$. Let x_t^* be the point that maximizes $|u-\tilde{u}|$ on Γ_t and suppose $\gamma(s_0)=x_t^*$ for some $s_0\in(0,L)$, where L is the length of Γ_t. Then by an argument similar to the one in the proof of Theorem 3.3.1 we get

$$|u(x_t^*)-\tilde{u}(x_t^*)| \le \frac{1}{\sigma_0}\int_0^L |J(\gamma(\tau))-\tilde{J}(\gamma(\tau))|d\tau + |f(x_t^0)-\tilde{f}(x_t^0)|,$$

and consequently

$$\int_{\Gamma_t} |u(x)-\tilde{u}(x)|dl \quad\le\quad \frac{L_M}{\sigma_0}\int_{\Gamma_t} |J-\tilde{J}|dl + L_M|f(x_t^0)-\tilde{f}(x_t^0)|.$$

Hence,

$$\int_{\{u=t\}\cap\{g=r\}\cap\Omega} |u-\tilde{u}|dl \le \frac{L_M}{\sigma_0}\int_{\{u=t\}\cap\{g=r\}\cap\Omega} |J-\tilde{J}|dl + L_M|f(x_t^0)-\tilde{f}(x_t^0)|. \tag{3.14}$$

Using this estimate and the coarea formula we have

$$
\begin{aligned}
\frac{m}{\sigma_1}\int_\Omega |u-\tilde{u}|dx &\le \int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega}|u-\tilde{u}|dSdt \\
&\le \frac{1}{c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{u=t\}\cap\{g=r\}\cap\Omega}|u-\tilde{u}|dldrdt \\
&\le \frac{1}{c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{L_M}{\sigma_0}\int_{\{u=t\}\cap\{g=r\}\cap\Omega}|J-\tilde{J}|dl + L_M|f(x_t^0)-\tilde{f}(x_t^0)|\right)drdt \\
&= \frac{L_M}{c_g\sigma_0}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\{u=t\}\cap\{g=r\}\cap\Omega}|J-\tilde{J}|dldrdt \\
&\quad + \frac{L_M}{c_g}\int_{\min\limits_{\partial\Omega} f}^{\max\limits_{\partial\Omega} f}\int_{\min\limits_{\Omega} g}^{\max\limits_{\Omega} g}|f(x_t^0)-\tilde{f}(x_t^0)|drdt \\
&\le \frac{L_M C_g}{c_g\sigma_0}\int_{\mathbb{R}}\int_{\{u=t\}\cap\Omega}|\nabla g_t||J-\tilde{J}|dSdt \\
&\quad + \frac{2ML_M}{c_g}(2\|g\|_{L^\infty(\Omega)})\|f-\tilde{f}\|_{L^\infty(\partial\Omega)} \\
&= \frac{L_M C_g}{c_g\sigma_0}\int_\Omega|\nabla u||J-\tilde{J}|dx + \frac{4ML_M\|g\|_{L^\infty(\Omega)}}{c_g}\|f-\tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\le \frac{L_M C_g}{c_g\sigma_0}\|\nabla u\|_{L^\infty(\Omega)}\left(C_1\||J|-|\tilde{J}|\|_{L^\infty(\Omega)}^{\frac{1}{2}} + C_2\|f-\tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}}\right) \\
&\quad + \frac{4ML_M\|g\|_{L^\infty(\Omega)}}{c_g}\|f-\tilde{f}\|_{L^\infty(\partial\Omega)} \\
&\le \frac{L_M C_g M C_1}{c_g\sigma_0^2}\||J|-|\tilde{J}|\|_{L^\infty(\Omega)}^{\frac{1}{2}} \\
&+ \left[\frac{L_M C_g C_2}{c_g\sigma_0} + \frac{4ML_M\|g\|_{L^\infty(\Omega)}}{c_g}(2M|\Omega|)^{\frac{1}{2}}\right]\|f-\tilde{f}\|_{L^\infty(\partial\Omega)}^{\frac{1}{2}},
\end{aligned}
$$

where we have applied Theorem 3.2.4. □

3.4 $W^{1,1}$ stability of the minimizers

In this section, we prove stability of minimizers of (2.2) in $W^{1,1}$. As mentioned in Section previously, in general, (2.2) does not even have unique minimizers. Additionally, we have introduced a new source of error in this chapter. Consequently, in order to prove stability results in $W^{1,1}$, it is natural to expect stronger assumptions on the minimizers.

Lemma 3.4.1. *Let $n = 2, 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f, \tilde{u}|_{\partial\Omega} = \tilde{f}$ the respective traces of functions $f, \tilde{f} \in H^3(\Omega)$ and corresponding conductivities σ and $\tilde{\sigma}$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ with*

$$\| \sigma \|_{C^2(\Omega)}, \| \tilde{\sigma} \|_{C^2(\Omega)} \leq \sigma_2 \tag{3.15}$$

for some $\sigma_2 \in \mathbb{R}$. Let

$$G(x) := \frac{\tilde{J}(x) - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega, \tag{3.16}$$

with $G = (G_1, G_2)$ for $n = 2$ and $G = (G_1, G_2, G_3)$ for $n = 3$. Then

$$\|\nabla G_i\|_{L^1(\Omega)} \leq C_1 \|J - \tilde{J}\|_{L^1(\Omega)}^{\frac{1}{2}}, \tag{3.17}$$

for some constant C_1 which depends only on Ω, σ_0, σ_2 and $\| f \|_{L^\infty(\Omega)}$.

Proof. The proof is similar to that of Lemma 2.4.1 in Chapter 2 and we omit it. □

Next we prove that u and $\tilde{u}$ are close in $W^{1,1}(\Omega)$ under these initial conditions. In order to do so, we need additional assumptions on the structure of level sets of u. Namely, we will invoke Definition 2.4.2 for the remainder of this section.

Theorem 3.4.2. *Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy* (3.15). *If u satisfies* (2.12), *and the level sets of u are well-structured in the sense of Definition 2.4.2, then*

$$\|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C_1 \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2 \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}}, \tag{3.18}$$

for some constant $C(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. Fix $x \in \Omega$ and $h \in \mathbb{R}^2$ with $|h| = 1$. Then

$$\mathcal{L}(x,h) := (\nabla \tilde{u}(x) - \nabla u(x)) \cdot h = \lim_{t \to 0} \frac{[\tilde{u}(x+th) - u(x+th)] - [\tilde{u}(x) - u(x)]}{t}.$$

First we estimate the above limit. Since all level sets of u reach the boundary $\partial\Omega$, there exist $z, z_t \in \partial\Omega$ such that

$$u(x) = u(z),$$

$$u(x+th) = u(z_t).$$

Thus

$$[\tilde{u}(x+th) - u(x+th)] - [\tilde{u}(x) - u(x)] = [\tilde{u}(x+th) - u(z_t)] - [\tilde{u}(x) - u(z)]$$

$$= [\tilde{u}(x+th) - \tilde{u}(z_t)] - [\tilde{u}(x) - \tilde{u}(z)] + [\tilde{u}(z_t) - u(z_t)] - [\tilde{u}(z) - u(z)]$$

Let γ and γ_t be the curves passing through x and $x + th$, described in Definition 2.4.2 with $\gamma(0) = z$ and $\gamma_t(0) = z_t$. Suppose $\gamma(s_0) = x$ and reparamterize γ_t so that $\gamma_t(s_0) = x+th$. Then we have

$$[\tilde{u}(x+th) - \tilde{u}(z)] - [\tilde{u}(x) - \tilde{u}(z)] = [\tilde{u}(\gamma_t(s_0)) - \tilde{u}(\gamma_t(0))] - [\tilde{u}(\gamma(s_0)) - \tilde{u}(\gamma(0))]$$

$$= \int_0^{s_0} \nabla \tilde{u}(\gamma_t(s)) \cdot \gamma_t'(s) ds - \int_0^{s_0} \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) ds.$$

Hence

$$\mathcal{L}(x,h) = \lim_{t \to 0} \frac{1}{t} \left(\int_0^{s_0} \nabla \tilde{u}(\gamma_t(s)) \cdot \gamma_t'(s) ds - \int_0^{s_0} \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s) ds \right) \tag{3.19}$$

$$+ \lim_{t \to 0} \frac{1}{t} \left([\tilde{u}(z_t) - u(z_t)] - [\tilde{u}(z) - u(z)] \right) \tag{3.20}$$

Now, we can focus on the last term here by noticing

$$[\tilde{u}(z_t) - u(z_t)] - [\tilde{u}(z) - u(z)] = [\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)].$$

Also, we denote the tangential direction along $\partial\Omega$ at z by θ_z and we get,

$$\begin{aligned}
&\lim_{t\to 0} \frac{[\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)]}{t} \\
&\qquad = \lim_{t\to 0} \left(\frac{[\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)]}{|z_t - z|} \right) \lim_{t\to 0} \frac{|z_t - z|}{t} \\
&\qquad \leq |F_{x,h}(0)| \lim_{t\to 0} \left(\frac{[\tilde{f}(z_t) - f(z_t)] - [\tilde{f}(z) - f(z)]}{|z_t - z|} \right) \\
&\qquad = |F_{x,h}(0)| \frac{\partial}{\partial\theta_z} (\tilde{f} - f) \\
&\qquad \leq \|F\|_{L^\infty(\Omega\times S^{n-1}\times[0,L_M])} \|\nabla(f - \tilde{f})\|_{L^\infty(\partial\Omega)} \\
&\qquad \leq \|F\|_{L^\infty(\Omega\times S^{n-1}\times[0,L_M])} \|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}.
\end{aligned} \tag{3.21}$$

We can now shift our focus onto the first term (3.19). Substituting $\nabla\tilde{u}$ by $\frac{\tilde{J}}{\tilde{\sigma}}$ and using the fact that J is perpendicular to γ' and γ_t' we get

$$\lim_{t\to 0} \frac{1}{t} \left(\int_0^{s_0} \frac{\tilde{J}(\gamma_t(s)) - J(\gamma_t(s))}{\tilde{\sigma}(\gamma_t(s))} \cdot \gamma_t'(s) ds - \int_0^{s_0} \frac{\tilde{J}(\gamma(s)) - J(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s) ds \right).$$

Now define

$$G(x) := \frac{\tilde{J(x)} - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega.$$

Hence we get

$$\lim_{t\to 0} \frac{1}{t} \left(\int_0^{s_0} G(\gamma_t(s)) \cdot \gamma_t'(s) ds - \int_0^{s_0} G(\gamma(s)) \cdot \gamma'(s) ds \right).$$

This term can bounded in the same way as in the proof of Theorem 4.4 in [19], so we omit the calculation as it is identical. Hence we have

$$
\begin{aligned}
|\nabla\tilde{u}(x)-\nabla u(x)| &\leq \sup_{h\in\mathbb{R}^n, |h|=1} \mathcal{L}(x,h)\\
&\leq \frac{K}{\sigma_0}\int_0^L |\tilde{J}(\gamma(s))-J(\gamma(s))|dl\\
&\quad + \parallel F\parallel_{L^\infty} \int_0^L |\nabla G_1(\gamma(s))| + |\nabla G_2(\gamma(s))|dl\\
&\quad + \parallel F\parallel_{L^\infty} \|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_\Gamma |\nabla\tilde{u}(x)-\nabla u(x)|dl &\leq \frac{KL_M}{\sigma_0}\int_\Gamma |\tilde{J}(x)-J(x)|dl\\
&\quad + L_M \parallel F\parallel_{L^\infty} \int_\Gamma |\nabla G_1(x)| + |\nabla G_2(x)|dl\\
&\quad + L_M \parallel F\parallel_{L^\infty} \|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)},
\end{aligned}
$$

and consequently

$$
\begin{aligned}
\int_{\{u=\tau\}\cap\Omega} |\nabla\tilde{u}(x)-\nabla u(x)|dl &\leq \frac{KL_M}{\sigma_0}\int_{\{u=\tau\}\cap\Omega} |\tilde{J}(x)-J(x)|dl\\
&\quad + L_M \parallel F\parallel_{L^\infty} \int_{\{u=\tau\}\cap\Omega} |\nabla G_1(x)| + |\nabla G_2(x)|dl\\
&\quad + L_M \parallel F\parallel_{L^\infty} \|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)}.
\end{aligned}
\tag{3.22}
$$

Using (3.22) and the coarea formula we have

$$\begin{aligned}
\frac{m}{\sigma_1}\|\nabla\tilde{u}-\nabla u\|_{L^1(\Omega)} &\leq \int_\Omega |\nabla u||\nabla\tilde{u}-\nabla u|dx \\
&= \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\nabla\tilde{u}-\nabla u|dld\tau \\
&\leq \frac{KL_M}{\sigma_0}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\tilde{J}-J|dld\tau \\
&+ L_M \parallel F \parallel_{L^\infty} \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\nabla G_1|+|\nabla G_2|dld\tau \\
&+ L_M \parallel F \parallel_{L^\infty} (2M)\|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \frac{KL_M M}{(\sigma_0)^2}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} \frac{|\tilde{J}-J|}{|\nabla u|}dld\tau \\
&+ \frac{L_M \parallel F \parallel_{L^\infty} M}{\sigma_0}\int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} \frac{|\nabla G_1|+|\nabla G_2|}{|\nabla u|}dld\tau \\
&+ 2ML_M \parallel F \parallel_{L^\infty} \|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)} \\
&= \frac{KL_M M}{(\sigma_0)^2}\int_\Omega |\tilde{J}-J|dx \\
&+ \frac{L_M \parallel F \parallel_{L^\infty} M}{\sigma_0}\int_\Omega |\nabla G_1|+|\nabla G_2|dx \\
&+ 2ML_M \parallel F \parallel_{L^\infty} \|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \frac{KL_M M}{(\sigma_0)^2} \parallel J-\tilde{J} \parallel_{L^1(\Omega)} \\
&+ \frac{2L_M C_1 \parallel F \parallel_{L^\infty} M}{\sigma_0} \parallel J-\tilde{J} \parallel_{L^1(\Omega)}^{\frac{1}{2}} \\
&+ 2ML_M \parallel F \parallel_{L^\infty} \|\tilde{f}-f\|_{W^{1,\infty}(\partial\Omega)}
\end{aligned}$$

where we have used Lemma 3.4.1 to obtain the last inequality. Applying Theorem 3.2.4, and noting that

$$\parallel J-\tilde{J} \parallel_{L^1(\Omega)}^{\frac{1}{2}} \leq (2M|\Omega|)^{\frac{1}{2}},$$

where M is defined in (3.1), we arrive at (3.18). □

Now we prove three dimensional version of this theorem.

Theorem 3.4.3. *Let $n=3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega}=f$, $\tilde{u}|_{\partial\Omega}=\tilde{f}$ corresponding conductivities $\sigma,\tilde{\sigma}\in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$,* *respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy* (3.15)*. In addition suppose u satisfies* (2.12)*, the level sets of u can be foliated to one-dimensional curves in the sense of Definition 2.3.4, and the level sets of u are well-structured in the sense of Definition 2.4.2. Then*

$$\|\nabla\tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C_1\|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{1}{4}} + C_2\|f - \tilde{f}\|_{W^{1,\infty}(\partial\Omega)}^{\frac{1}{4}}, \tag{3.23}$$

for some constant $C_i(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M, c_g, C_g)$ is independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. With an argument similar to the one used in the proof of Theorem 3.4.2 we get

$$\begin{aligned}
\int_{U_{\tau,r}} |\nabla\tilde{u}(x) - \nabla u(x)|dl \quad \leq \quad & \frac{KL_M}{\sigma_0}\int_{U_{\tau,r}} |\tilde{J}(x) - J(x)|dl \\
& + L_M \parallel F \parallel_{L^\infty} \int_{U_{\tau,r}} |\nabla G_1(x)| + |\nabla G_1(x)| + |\nabla G_3(x)|dl \\
& + L_M \parallel F \parallel_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)}
\end{aligned} \tag{3.24}$$

where $U_{\tau,r} := \{u = \tau\} \cap \{g_\tau = r\} \cap \Omega$ and $G = (G_1, G_2, G_3)$ is defined in (3.16).

It follows from (3.24) and the coarea formula that

$$\begin{aligned}
\frac{m}{\sigma_1}\|\nabla\tilde{u} - \nabla u\|_{L^1(\Omega)} \quad & \leq \quad \int_\Omega |\nabla u||\nabla\tilde{u} - \nabla u|dx \\
& = \quad \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} |\nabla\tilde{u} - \nabla u|dSd\tau \\
& = \quad \int_{\mathbb{R}}\int_{\{u=\tau\}\cap\Omega} \frac{|\nabla g_\tau|}{|\nabla g_\tau|}|\nabla\tilde{u} - \nabla u|dSd\tau \\
& = \quad \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}} \frac{1}{|\nabla g_\tau|}|\nabla\tilde{u} - \nabla u|dldrd\tau \\
& \leq \quad \frac{KL_M}{\sigma_0 c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}} |\tilde{J} - J|dldrdt \\
& + \quad \frac{L_M \parallel F \parallel_{L^\infty}}{c_g}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{U_{\tau,r}} |\nabla G_1| + |\nabla G_2| + |\nabla G_3|dldrdt
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\|g\|_{L^\infty(\Omega)} L_M \parallel F \parallel_{L^\infty} (2M)\|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \quad \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{|\tilde{J} - J|}{|\nabla u||\nabla g_\tau|} dldrdt \\
&+ \quad \frac{L_M M \parallel F \parallel_{L^\infty} C_g}{\sigma_0 c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u||\nabla g_t|} dldrdt \\
&+ \quad 4M\|g\|_{L^\infty(\Omega)} L_M \parallel F \parallel_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&= \quad \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \int_{\mathbb{R}} \int_{\{u=\tau\}\cap\Omega} \frac{|\tilde{J} - J|}{|\nabla u|} dSdt \\
&+ \quad \frac{L_M M \parallel F \parallel_{L^\infty} C_g}{\sigma_0 c_g} \int_{\mathbb{R}} \int_{\{u=\tau\}\cap\Omega} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u|} dSdt \\
&+ \quad 4M\|g\|_{L^\infty(\Omega)} L_M \parallel F \parallel_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&= \quad \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \int_{\Omega} |\tilde{J} - J| dx \\
&+ \quad \frac{L_M M \parallel F \parallel_{L^\infty} C_g}{\sigma_0 c_g} \int_{\Omega} |\nabla G_1| + |\nabla G_2| + |\nabla G_3| dx \\
&+ \quad 4M\|g\|_{L^\infty(\Omega)} L_M \parallel F \parallel_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)} \\
&\leq \quad \frac{KL_M MC_g}{(\sigma_0)^2 c_g} \parallel J - \tilde{J} \parallel_{L^1(\Omega)} \\
&+ \quad \frac{3L_M C_1 M \parallel F \parallel_{L^\infty(\Omega)} C_g}{\sigma_0 c_g} \parallel J - \tilde{J} \parallel_{L^1(\Omega)}^{\frac{1}{2}} \\
&+ \quad 4M\|g\|_{L^\infty(\Omega)} L_M \parallel F \parallel_{L^\infty} \|\tilde{f} - f\|_{W^{1,\infty}(\partial\Omega)},
\end{aligned}
$$

where we have used (3.17) to obtain the last inequality. Applying Theorem 3.2.4, and noting that

$$
\parallel J - \tilde{J} \parallel_{L^1(\Omega)}^{\frac{1}{2}} \leq (2M|\Omega|)^{\frac{1}{2}},
$$

we obtain the inequality (3.18). □

Now, we are ready to prove our main stability results.

Theorem 3.4.4. *Let $n = 2$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (2.21). If u satisfies (2.12) and level sets of*

u are well-structured in the sense of Definition 2.4.2, then

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \le C_1 \parallel a - \tilde{a} \parallel^{\frac{1}{4}}_{L^\infty(\Omega)} + C_2 \parallel f - \tilde{f} \parallel^{\frac{1}{4}}_{W^{1,\infty}(\partial\Omega)},$$

for some constants $C_i(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, f, L_M)$ independent of $\tilde{\sigma}$.

Proof. Using Theorem 3.4.2 we have

$$\begin{aligned}
\int_\Omega |\sigma - \tilde{\sigma}| dx &= \int_\Omega \left| \frac{|J|(|\nabla \tilde{u}| - |\nabla u|)}{|\nabla u||\nabla \tilde{u}|} + \frac{|J| - |\tilde{J}|}{|\nabla \tilde{u}|} \right| dx \\
&\le \int_\Omega \frac{|J|}{|\nabla u||\nabla \tilde{u}|} \, ||\nabla u| - |\nabla \tilde{u}|| \, dx + \int_\Omega \frac{1}{|\nabla \tilde{u}|} \left| |J| - |\tilde{J}| \right| dx \\
&\le \int_\Omega \frac{|J|}{|\nabla u||\nabla \tilde{u}|} |\nabla u - \nabla \tilde{u}| dx + \int_\Omega \frac{1}{|\nabla \tilde{u}|} \left| |J| - |\tilde{J}| \right| dx \\
&\le \frac{M\sigma_1^2}{m^2} \left(C_1 \parallel a - \tilde{a} \parallel^{\frac{1}{4}}_{L^\infty(\Omega)} + C_2 \parallel f - \tilde{f} \parallel^{\frac{1}{4}}_{W^{1,\infty}(\partial\Omega)} \right) \\
&+ \frac{\sigma_1 |\Omega|}{m} \parallel a - \tilde{a} \parallel_{L^\infty(\Omega)} \\
&\le \left[\frac{M\sigma_1^2 C_1}{m^2} + \frac{\sigma_1 |\Omega| (2M)^{\frac{3}{4}}}{m} \right] \parallel a - \tilde{a} \parallel^{\frac{1}{4}}_{L^\infty(\Omega)} \\
&+ \frac{M\sigma_1^2 C_2}{m^2} \parallel f - \tilde{f} \parallel^{\frac{1}{4}}_{W^{1,\infty}(\partial\Omega)}
\end{aligned}$$

□

Theorem 3.4.5. *Let $n = 3$, and suppose u and $\tilde{u}$ are admissible with $u|_{\partial\Omega} = f$, $\tilde{u}|_{\partial\Omega} = \tilde{f}$ corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields J and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (2.21). If u satisfies (2.12), the level sets of u can be foliated to one-dimensional curves in the sense of Definition 2.3.4, and the level sets of u are well-structured in the sense of Definition 2.4.2, then*

$$\|\sigma - \tilde{\sigma}\|_{L^1(\Omega)} \le C_1 \parallel |J| - |\tilde{J}| \parallel^{\frac{1}{4}}_{L^\infty(\Omega)} + C_2 \parallel f - \tilde{f} \parallel^{\frac{1}{4}}_{W^{1,\infty}(\partial\Omega)},$$

for some constants $C_i(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, f, L_M, g)$ independent of $\tilde{\sigma}$.

Proof. The proof follows from Theorem 3.4.3 and a calculation similar to that of the proof of Theorem 3.4.4. □

Chapter 4

The Inverse Sturm-Liouville Problem with Partial Spectral Data

4.1 One Dimensional Results

Consider the problem as outlined by the author in [18]:

$$y'' + (\lambda - P)y = 0 \tag{4.1}$$

for $x \in [0, \pi]$. Subject to the boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\beta + y'(\pi)\sin\beta = 0 \tag{4.2}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad y(\pi)\cos\gamma + y'(\pi)\sin\gamma = 0 \tag{4.3}$$

Now, let $y = u(x, \lambda)$ be the solution to (4.1) subject to (4.2) at $x = 0$, and $y = v(x, \lambda)$ be the solution to (4.1) subject to (4.2) at $x = \pi$. Hence, u and v satisfy

$$u(0, \lambda) = \sin \alpha, \quad u'(0, \lambda) = -\cos \alpha \tag{4.4}$$

$$v(\pi, \lambda) = \sin \beta, \quad v'(\pi, \lambda) = -\cos \beta \tag{4.5}$$

In order for our solution u to satisfy the condition at $x = \pi$ from (4.2), we must also require that

$$u(\pi, \lambda) \cos \beta + u'(\pi, \lambda) \sin \beta = 0.$$

This leads us to define the following function

$$w(\lambda) = -u(\pi, \lambda) \cos \beta - u'(\pi, \lambda) \sin \beta \tag{4.6}$$

whose zeros are the eigenvalues of (4.1) subject to (4.2). It then follows from Lemma 2.0 in [18] that w is an entire function of order $\frac{1}{2}$, and thus, by Hadamard's factorization theorem has the form

$$\prod_{m=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_m}\right). \tag{4.7}$$

This leads us to prove the following which is an analogous result to the main one in [18] with slightly weakened assumptions.

Theorem 4.1.1. *If a subsequence of the spectral data for* (4.1) *is given for the boundary conditions* (4.2)*, the entire spectrum of* (4.3) *is known and if* $\sin(\gamma - \beta) \neq 0$ *then* P *is uniquely determined.*

Proof. Suppose we have the problem (4.1) with coefficient functions P_1 and P_2 respectively. Thus, we have functions u_1, v_1 and u_2, v_2 corresponding to the condition (4.2) respectively. Let us denote the eigenvalues of the problem subject to (4.2) corresponding to P_1 by $\{\mu_{1,m}\}$ and those corresponding to P_2 by $\{\mu_{2,m}\}$ for $0 \leq m < \infty$. It is well outlined in [18] that we have the following:

$$u_1(x, \mu_{1,m}) = C_m v_1(x, \mu_{1,m})$$
$$u_2(x, \mu_{2,m}) = C_m v_2(x, \mu_{2,m})$$

where $C_m \neq 0$ is shown to be uniquely determined by the spectrum of (4.3). Referring back to the previous argument leading up to (4.7), we have two resultant functions w_1 and w_2 having the form for $j = 1, 2$

$$w_j(\lambda) = \prod_{m=1}^{\infty} \left(1 - \frac{\lambda}{\mu_{j,m}}\right).$$

Since we have assumed knowledge of only a subsequence of the spectral data (i.e. the spectra corresponding to P_1 and P_2 for (4.2) agree on a subsequence), let us consider

$$w_*(\lambda) = \prod_{m_k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{m_k}}\right) \tag{4.8}$$

where $\mu_{1,m_k} = \mu_{2,m_k} = \lambda_{m_k}$. Note w_* has zeros only on the given subsequence of eigenvalues for (4.2). Similarly to [18], we will define for any $f \in C_0^1([0, \pi])$ the function

$$H(x, \lambda) = \frac{1}{w_*(\lambda)} v_2(x, \lambda) \int_0^x u_1(x, \xi) f(\xi) d\xi$$

and we consider

$$\int_\Gamma H(x, \lambda) d\lambda$$

where Γ is a sufficiently large circle in the λ plane centered at the origin. Since the estimates proved in Lemma 2.0 of [18] all hold for any solution to the boundary value problem (4.1), this allows us to conclude yet again that

$$\int_\Gamma H(x, \lambda) d\lambda - \pi i f(x) \to 0 \tag{4.9}$$

as the radius of Γ becomes arbitrarily large. By residue theorem

$$\int_\Gamma H(x,\lambda)d\lambda = 2\pi i \sum_{m_k=1}^{N} Res(H;\lambda_{m_k})$$

where λ_{m_k} are the zeros of w_* lying inside of Γ. Combining this with (4.9) and using the fact that the entire argument is identical regardless of using any of u_1, u_2, v_1, v_2 we arrive at

$$f(x) = \sum_{m_k=1}^{\infty} \frac{1}{C_{m_k} w_*'(\lambda_{m_k})} u_2(x,\lambda_{m_k}) \int_0^\pi u_1(\xi,\lambda_{m_k}) f(\xi) d\xi \tag{4.10}$$

$$f(x) = \sum_{m_k=1}^{\infty} \frac{1}{C_{m_k} w_*'(\lambda_{m_k})} u_2(x,\lambda_{m_k}) \int_0^\pi u_2(\xi,\lambda_{m_k}) f(\xi) d\xi$$

and we have that

$$0 = \sum_{m_k=1}^{\infty} \frac{1}{C_{m_k} w_*'(\lambda_{m_k})} u_2(x,\lambda_{m_k}) \int_0^\pi [u_1(\xi,\lambda_{m_k}) - u_2(\xi,\lambda_{m_k})] f(\xi) d\xi$$

uniformly on closed intervals. Now, by the orthogonality of eigenfunctions, multiplying $u_2(x,\lambda_1)$ to the above equation and integrating gives

$$\int_0^\pi [u_1(\xi,\lambda_1) - u_2(\xi,\lambda_1)] f(\xi) d\xi = 0$$

for any $f \in C_0^1([0,\pi])$. Which gives $u_1(\xi,\lambda_1) = u_2(\xi,\lambda_1)$, and consequently $P_1 = P_2$ almost everywhere. $\square$

Remark 4.1.2. As we can see by this argument, the spectrum of (4.3) was only necessary in establishing that the normalizing constants C_{m_k} were the same for both P_1 and P_2. In [29] the authors assume that these normalizing constants coincide for each of P_1 and P_2 in the statements of both Theorem 1.1 and Corollary 1.2. This suggests they can be reproved for a subsequence of the spectrum with minimal adjustment to the methods they have used. The following are the results by the authors in [29] (setting $\alpha = \pi$, $\beta = 0$ in (4.2)):

Theorem 4.1.3. *Suppose $P_1, P_2 \in L^\infty(0,\pi)$, their spectra subject to* (4.2) *agree*

$$\mu_{1,m} = \mu_{2,m} = \lambda_m,$$

and that the normalizing constants agree, i.e. $c_{1,m} = c_{2,m}$, where

$$c_{j,m} = \int_0^\pi u_j^2(x,\lambda_m)dx = \|u_j(\cdot,\lambda_m)\|^2_{L^2(0,\pi)} \quad j = 1,2$$

for $u_j(\cdot,\lambda_m)$ the eigenfunctions corresponding to P_j, λ_m and $0 \le m < \infty$, $j = 1,2$. Then $P_1 = P_2$.

Remark 4.1.4. Note that the normalizing constants satisy $c_{j,m} = \frac{C_m}{w'(\lambda_m)}$ where this relation comes from the Sturm-Liouville expansion seen in [18].

Corollary 4.1.5. *Suppose that $P_1, P_2 \in L^\infty(0,\pi)$, their spectra subject to* (4.2) *agree. That is to say:*

$$\mu_{1,m} = \mu_{2,m} = \lambda_m,$$
$$u_1'(\pi,\lambda_m) = u_2'(\pi,\lambda_m)$$

then $P_1 = P_2$.

4.2 n-Dimensional Results

We hope to continue working with this problem to establish similar results in n-dimensions. The authors in [29] proved several results using the entire spectral data. The author in [14] was able to extend the main multidimensional result to the case where finite boundary data is lacking. However, the author does remark that many of the arguments used are depending only on the asymptotic behavior of eigenfunctions. This leads us to believe similar results should hold in higher dimensions when only considering a subsequence of spectral data.

Chapter 5

Conclusion

5.1 Future Directions for CDII

After showing global stability for (CDII) with respect to both interior and boundary data, there are several directions in which the problem could be extended. Another important direction for this problem would be to develop a similar method to provide stability for anisotropic conductivities (such as that of the heart) from minimal interior measurements. In contrast with an isotropic conductivity which is represented as a scalar function on Ω, an anisotropic conductivity is directionally dependent and is represented as a Riemannian metric (or $n \times n$ matrix). It would certainly be a challenging problem worth investigation considering that the question of unique reconstruction is relatively unexplored. My advisor along with collaborators have taken the first step in [12]. Once unique reconstruction is more fully developed, one could then ask the question of stability which is likely to require stricter hypotheses.

Stability on a more complex Least Gradient Problem

Additionally, we may be able to prove similar stability results for solutions to the following equation related to mean curvature:

$$\nabla \cdot \left(a \frac{\nabla u + F}{|\nabla u + F|} \right) = H, \quad u|_{\partial\Omega} = f. \tag{5.1}$$

Where we can let $J = -a\frac{\nabla u+F}{|\nabla u+F|}$, and u is the unique minimizer of the weighted least gradient problem

$$I(w) = \min_{w \in BV_f(\Omega)} \int_\Omega a|\nabla w + F| + Hw\, dx, \tag{5.2}$$

where $a = |J|$, and $BV_f(\Omega) = \{w \in BV(\Omega), w|_{\partial\Omega} = f\}$.

One would could start by proving similar stability results for J and extending these results in a similar fashion to L^1 and $W^{1,1}$ stability for minimizers of $I(w)$.

Indeed we have taken some initial steps in this direction:

Lemma 5.1.1. *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of* (5.2) *with the weights a and $\tilde{a}$, respectively. Then*

$$\left| \int_\Omega a|Du + F| + Hu\, dx - \int_\Omega \tilde{a}|D\tilde{u} + F| + H\tilde{u}\, dx \right| \le C\|a - \tilde{a}\|_{L^\infty(\Omega)}, \tag{5.3}$$

for some constant $C = C(m, M, \Omega, f)$ independent of u and $\tilde{u}$.

Proof. First note that in view of (2.5) (bounds on $a, \tilde{a}$) we have

$$m\int_\Omega |D\tilde{u} + F|\, dx \le \int_\Omega \tilde{a}|D\tilde{u} + F|\, dx \le \int_\Omega \tilde{a}|Dw + F|\, dx \le M\int_\Omega |Dw + F|\, dx$$

for any $w \in BV_f(\Omega)$. Thus $\int_\Omega |D\tilde{u} + F| \le C$, and similarly $\int_\Omega |Du + F| \le C$ for some constant C which depends only on m, M, and Ω. Hence

$$\max\left\{ \int_\Omega |D\tilde{u} + F|, \int_\Omega |Du + F| \right\} \le C, \tag{5.4}$$

for some $C(m, M)$ independent of $\tilde{u}$ and u. Since u, $\tilde{u}$ are the minimizers of (5.2) with the weights a and $\tilde{a}$,

$$\begin{aligned}\int_\Omega a|Du+F| + Hu\,dx - \int_\Omega \tilde{a}|Du+F| + Hu\,dx \\ &\le \int_\Omega a|Du+F| + Hu\,dx - \int_\Omega \tilde{a}|D\tilde{u}+F| + H\tilde{u}\,dx \\ &\le \int_\Omega a|D\tilde{u}+F| + H\tilde{u}\,dx - \int_\Omega \tilde{a}|D\tilde{u}+F| + H\tilde{u}\,dx.\end{aligned}$$

Thus

$$\begin{aligned}\int_\Omega (a-\tilde{a})|Du+F|dx &\le \int_\Omega a|Du+F| + Hu\,dx \; - \int_\Omega \tilde{a}|D\tilde{u}+F| + H\tilde{u}\,dx \\ &\le \int_\Omega (a-\tilde{a})|D\tilde{u}+F|dx,\end{aligned}$$

and we get

$$\begin{aligned}-\|a-\tilde{a}\|_{L^\infty(\Omega)}\|Du+F\|_{L^1(\Omega)} &\le \int_\Omega a|Du+F| + Hu\,dx - \int_\Omega \tilde{a}|D\tilde{u}+F| + H\tilde{u}\,dx \\ &\le \|a-\tilde{a}\|_{L^\infty(\Omega)}\|D\tilde{u}+F\|_{L^1(\Omega)}.\end{aligned}$$

Hence (5.3) follows from (5.4). $\square$

Lemma 5.1.2. *Let $f \in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (5.2) with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2.1. Suppose $0 \le \sigma(x) = \frac{a(x)}{|Du+F|} \le \sigma_1 = \frac{\|a\|_{L^\infty(\Omega)}}{\delta}$ in Ω for some constant δ, such that $|Du+F| > \delta > 0$, where σ is the Radon-Nikodym derivative of $|J|\,dx$ with respect to $|Du+F|$. Then*

$$\int_\Omega |J||\tilde{J}| - J\cdot\tilde{J}\,dx \le C\|a-\tilde{a}\|_{L^\infty(\Omega)}, \tag{5.5}$$

where $C = C(m, M, \sigma_1, \Omega, f, u)$ is a constant independent of $\tilde{a}$.

Proof. We have

$$
\begin{aligned}
\int_\Omega |J||\tilde{J}| - J\cdot\tilde{J}dx &= \int_\Omega \frac{a}{|Du+F|}|Du+F||\tilde{J}| - \frac{a}{|Du+F|}(Du+F)\cdot\tilde{J}dx \\
&\le \frac{\|a\|_{L^\infty(\Omega)}}{\delta}\int_\Omega |Du+F||\tilde{J}| - (Du+F)\cdot\tilde{J}dx \\
&= \sigma_1\left(\int_\Omega |Du+F||\tilde{J}|dx - \int_\Omega F\cdot\tilde{J}dx - \int_{\partial\Omega} f[\tilde{J},\nu_\Omega]dx + \int_\Omega Hudx\right) \\
&= \sigma_1\left(\int_\Omega |Du+F||\tilde{J}|dx - \int_\Omega (D\tilde{u}+F)dx\cdot\tilde{J} + \int_\Omega H(u-\tilde{u})dx\right) \\
&= \sigma_1\left(\int_\Omega |Du+F||\tilde{J}| - |D\tilde{u}+F||\tilde{J}|dx + \int_\Omega H(u-\tilde{u})dx\right) \\
&= \sigma_1\left(\int_\Omega \tilde{a}|Du+F| - \tilde{a}|D\tilde{u}+F|dx + \int_\Omega H(u-\tilde{u})dx\right. \\
&\qquad\qquad \left. + \int_\Omega a|Du+F| - a|Du+F|dx\right) \\
&\le \sigma_1\left|\int_\Omega a|Du+F| + Hu\,dx - \int_\Omega \tilde{a}|D\tilde{u}+F| + H\tilde{u}\,dx\right| + \sigma_1\left|\int_\Omega (\tilde{a}-a)|Du+F|dx\right|,
\end{aligned}
$$

where we have used (2.6) and the integration by parts formula (2.9). Finally, the inequality below follows from Lemma 5.1.1,

$$
\le \sigma_1\left(C\|a-\tilde{a}\|_{L^\infty(\Omega)} + \|Du+F\|_{L^1(\Omega)}\|a-\tilde{a}\|_{L^\infty(\Omega)}\right),
$$

which yields the desired result. ◻

Theorem 5.1.3. *Let $f\in L^1(\partial\Omega)$, and assume u and $\tilde{u}$ are minimizers of (5.2) with the weights a and $\tilde{a}$, respectively. Let J and $\tilde{J}$ be the divergence free vector fields guaranteed by Theorem 2.2.1. Suppose $0\le\sigma(x)=\frac{a(x)}{|Du+F|}\le\sigma_1=\frac{\|a\|_{L^\infty(\Omega)}}{\delta}$ in Ω for some constant δ, such that $|Du+F|>\delta>0$, where σ is the Radon-Nikodym derivative of $|J|dx$ with respect to $|Du+F|$. Then*

$$
\|J-\tilde{J}\|_{L^1(\Omega)} \le C\|a-\tilde{a}\|^{\frac{1}{2}}_{L^\infty(\Omega)}, \tag{5.6}
$$

where $C=C(m,M,\sigma_1,\Omega,f,u)$ is a constant independent of $\tilde{a}$.

Note that the proof of this theorem is equivalent to that of Theorem 2.5 in [19]. The only difference being that we use our Lemma 5.1.2 to justify one of the inequalities in the proof.

One could now try to expand upon these results and use them in a similar way to chapters 2 and 3 in order to prove stability results on the minimizers themselves. However, this may be much more difficult in this setting.

5.2 Future directions for the Inverse Sturm-Liouville Problem

In view of Corollary 4.1.5, we will furthermore consider the n-dimensional analog of the one dimensional problem. Let $\Omega \in \mathbb{R}^n$ be bounded domain with smooth boundary and q be a real valued scalar function on Ω. Consider the problem

$$\begin{aligned} -\Delta u + qu &= \mu u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{5.7}$$

Denote the eigenvalues of (5.7) by $\{\mu_m\}_{m=1}^{\infty}$ with corresponding eigenfunctions $\{\varphi_m\}_{m=1}^{\infty}$. We then hope to extend the following results from [29]:

Theorem 5.2.1. [29] *Let $q_1, q_2 \in C^\infty(\overline{\Omega})$ be the coefficient functions in* (5.7)*, and suppose the respective spectra corresponding to each agree for all k*

$$\mu_{1,k} = \mu_{2,k},$$

and that the outward normal derivatives of eigenfunctions agree on the boundary

$$\frac{\partial \varphi_{1,k}}{\partial \nu} = \frac{\partial \varphi_{2,k}}{\partial \nu} \quad on \quad \partial\Omega$$

then $q_1 = q_2$ in Ω.

As detailed in [29] this result requires the use of the Dirichlet to Neumann map defined in the following way: let u be a solution to

$$-\Delta u + qu = 0 \quad \text{in} \quad \Omega \tag{5.8}$$
$$u = f \quad \text{on} \quad \partial\Omega.$$

We define the Dirichlet to Neumann map

$$\Lambda_q f = \frac{\partial u}{\partial \nu}. \tag{5.9}$$

The proof of 5.2.1 hinges upon the following

Theorem 5.2.2. *([29]) Let $q_1, q_2 \in L^\infty(\Omega)$ and suppose that, as meromorphic functions of $\lambda \in \mathbb{C}$*

$$\Lambda_{q_1-\lambda} = \Lambda_{q_2-\lambda}$$

then $q_1 = q_2$.

The proof of this theorem is unchanged by considering only partial spectral data. What remains to be shown is the connection between Theorem (5.2.2) and the hypotheses in Theorem (5.2.1). In other words, we must show that partial spectral data in the form of a subsequence can still give equality of the Dirichlet to Neumann maps. This would be achieved by establishing results similar to the following:

Lemma 5.2.3. *([29]) For m sufficiently large and $f \in C^\infty(\partial\Omega)$,*

$$\left(\frac{d}{d\lambda}\right)^m (\Lambda_{q-\lambda}(f)) = \int_{\partial\Omega} g(x,y) f(y) dS(y)$$

where g is continuous in $\overline{\Omega} \times \overline{\Omega}$ given by

$$g(x,y) = \sum_{i=1}^{\infty} \left[\frac{1}{(\mu_i - \lambda)^{m+1}}\right] \frac{\partial \varphi_i}{\partial \nu}(x) \frac{\partial \varphi_i}{\partial \nu}(y) m!$$

Lemma 5.2.4. *([29]) For $f \in C^\infty(\partial\Omega)$; $q_1, q_2 \in C^\infty(\Omega)$; and $0 \le t < \frac{1}{2}$ we have*

$$\lim_{\lambda \to -\infty} \|(\Lambda_{q_1-\lambda} - \Lambda_{q_2-\lambda})(f)\|_{H^t(\partial\Omega)} = 0$$

In [14], the author has this main result for partial spectral data:

Theorem 5.2.5. *([14]) Let $q_1, q_2 \in C^\infty(\overline{\Omega})$ be real valued. Suppose there exists an $N > 0$ such that*

$$\mu_{1,k} = \mu_{2,k}, \quad k \ge N$$
$$W_{1,k} = W_{2,k}, \quad k \ge N.$$

Then $q_1 = q_2$.

Where $W_{j,k}$ denotes the eigenspace corresponding to $\mu_{j,k}$. This result depends on proving the following:

Lemma 5.2.6. *([14]) Under the assumptions of the previous theorem, there exists a constant $C > 0$ such that*

$$\|\Lambda_{q_1-\lambda} - \Lambda_{q_2-\lambda}\|_{\boldsymbol{B}(L^2(\partial\Omega))} \le \frac{C}{|\lambda|}$$

for large $|\lambda|$ where $\|\cdot\|_{\boldsymbol{B}(L^2(\partial\Omega))}$ denotes the operator norm for an operator on $L^2(\partial\Omega)$.

If we can prove similar results to this for a subsequence of spectral data, then we would have the main result as a consequence.

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