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Duality, Entropy and ADM Mass in Supergravity

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ABSTRACT

We consider the Bekenstein-Hawking entropy-area formula in four dimensional extended ungauged supergravity and its electric-magnetic duality property.

Symmetries of both “large” and “small” extremal black holes are considered, as well as the ADM mass formula for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity, preserving different fraction of supersymmetry.

The interplay between BPS conditions and duality properties is an important aspect of this investigation.

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1 Introduction

In $d = 4$ extended *ungauged* supergravity theories based on scalar manifolds which are (*at least locally*) *symmetric* spaces

$$M = \frac{G}{H}, \tag{1.1}$$

it is known that the classification of static, spherically symmetric and asymptotically flat extremal black hole (BH) solutions is made in terms of *charge orbits* of the corresponding *classical* electric-magnetic duality group G [1, 2, 3, 4, 5, 6] (later called *U-duality*¹ in string theory) .

These orbits correspond to certain values taken by a *duality invariant*² combination of the “*dressed*” central charges and matter charges. Denoting such an invariant by \mathcal{I} , the set of scalars parametrizing the symmetric manifold M by ϕ , and the set of “*bare*” magnetic and electric charges of the (*dyonic*) BH configuration by the $2n \times 1$ symplectic vector

$$\mathcal{P} \equiv \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}, \quad \Lambda = 1, \dots, n, \tag{1.2}$$

then it holds that

$$\partial_\phi \mathcal{I}(\phi, \mathcal{P}) = 0 \Leftrightarrow \mathcal{I} = \mathcal{I}(\mathcal{P}). \tag{1.3}$$

¹Here *U-duality* is referred to as the “*continuous*” version, valid for large values of the charges, of the *U-duality* groups introduced by Hull and Townsend [7].

²By *duality invariant*, throughout our treatment we mean that such a combination is G -invariant. Thus, it is actually *independent* on the scalar fields, and it depends only on “*bare*” electric and magnetic (asymptotical) charges (defined in Eq. (1.2)).

In some cases, the relevant invariant \mathcal{I} is not enough to characterize the orbit, and additional constraints are needed. This is especially the case for the so-called³ “small” BHs, in which case $\mathcal{I} = 0$ on the corresponding orbit [3, 4, 8].

An explicit expression for the $E_{7(7)}$ -invariant [9] was firstly introduced in supergravity in [10], and then adopted in the study of BH entropy in [11]. The additional U -invariant constraints which specify charge orbits with higher supersymmetry were given in [3]. The corresponding (“large” and “small”) charge orbits for $\mathcal{N} = 8$ and exceptional $\mathcal{N} = 2$ supergravity were determined in [4], whereas the “large” orbits for all other *symmetric* $\mathcal{N} = 2$ supergravities were obtained in [6], and then in [12] for all $\mathcal{N} > 2$ -extended theories. Furthermore, the invariant for $\mathcal{N} = 4$ supergravity was earlier discussed in [13, 14].

The invariants play an important role in the *attractor mechanism* [15, 16, 17, 18, 19], because the Bekenstein-Hawking BH entropy [20], determined by evaluating the *effective black hole potential* ([17, 18, 19])

$$V_{BH}(\phi, \mathcal{P}) \equiv -\frac{1}{2}\mathcal{P}^T \mathcal{M}(\phi) \mathcal{P} \quad (1.4)$$

at its critical points, actually coincides with the relevant invariant:

$$\frac{S_{BH}}{\pi} = V_{BH}|_{\partial_\phi V_{BH}=0} = V_{BH}(\phi_H(\mathcal{P}), \mathcal{P}) = |\mathcal{I}(\mathcal{P})|^{1/2} \text{ (or } |\mathcal{I}(\mathcal{P})|). \quad (1.5)$$

In Eq. (1.4) \mathcal{M} stands for the $2n \times 2n$ real (negative definite) symmetric scalar-dependent symplectic matrix

$$\mathcal{M}(\phi) \equiv \begin{pmatrix} \text{Im}\mathcal{N}_{\Lambda\Sigma} + \text{Re}\mathcal{N}_{\Lambda\Xi} (\text{Im}\mathcal{N})^{-1|\Xi\Delta} \text{Re}\mathcal{N}_{\Delta\Sigma} & -\text{Re}\mathcal{N}_{\Lambda\Xi} (\text{Im}\mathcal{N})^{-1|\Xi\Sigma} \\ -(\text{Im}\mathcal{N})^{-1|\Lambda\Delta} \text{Re}\mathcal{N}_{\Xi\Sigma} & (\text{Im}\mathcal{N})^{-1|\Lambda\Sigma} \end{pmatrix}, \quad (1.6)$$

defined in terms of the normalization of the Maxwell and topological terms⁴

$$\text{Im}\mathcal{N}_{\Lambda\Sigma}(\phi) F^\Lambda F^\Sigma, \quad \text{Re}\mathcal{N}_{\Lambda\Sigma}(\phi) F^\Lambda \tilde{F}^\Sigma \quad (1.7)$$

of the corresponding supergravity theory (see *e.g.* [21, 22] and Refs. therein). Furthermore, in Eq. (1.5) $\phi_H(\mathcal{P})$ denotes the set of charge-dependent, stabilized horizon values of the scalars, solutions of the criticality conditions for V_{BH} :

$$\left. \frac{\partial V_{BH}(\phi, \mathcal{P})}{\partial \phi} \right|_{\phi=\phi_H(\mathcal{P})} \equiv 0. \quad (1.8)$$

For the case of charge orbits corresponding to “small” BHs, in the case of a *single-center* solution $\mathcal{I}(\mathcal{P}) = 0$, and thus the event horizon area vanishes, and the solution is

³Throughout the present treatment, we will respectively call “small” or “large” (extremal) BHs those BHs with vanishing or non-vanishing area of the event horizon (and therefore with vanishing or non-vanishing *Bekenstein-Hawking entropy* [20]). For symmetric geometries, they can be G -invariantly characterized respectively by $\mathcal{I} = 0$ or by $\mathcal{I} \neq 0$.

⁴Attention should be paid in order to distinguish between the notations of the number \mathcal{N} of *supercharges* of a supergravity theory and the *kinetic vector matrix* $\mathcal{N}_{\Lambda\Sigma}$ introduced in Eqs. (1.6) and (1.7).

singular (*i.e.* with vanishing Bekenstein-Hawking entropy). However, the charge orbits with vanishing duality invariant play a role for *multi-center* solutions as well as for elementary BH constituents through which “*large*” (*i.e.* with non-vanishing Bekenstein-Hawking entropy) BHs are made [23, 24, 25].

In the present investigation, we re-examine the duality invariant and the U -invariant classification of charge orbits of $\mathcal{N} = 8$, $d = 4$ supergravity, we give a complete analysis of the $\mathcal{N} = 4$ “*large*” and “*small*” charge orbits, and we also derive a diffeomorphism-invariant expression of the $\mathcal{N} = 2$ duality invariant, which is common to all symmetric spaces and which is completely independent on the choice of a symplectic basis.

The paper is organized as follows.

In Sect. 2 we recall some basic facts about electric-magnetic duality in \mathcal{N} -extended supergravity theories, firstly treated in [2]. The treatment follows from the general analysis of [1], and the dictionary between that paper and the present work is given.

In Sect. 3 we re-examine $\mathcal{N} = 8$, $d = 4$ supergravity and the $E_{7(7)}$ -invariant characterization of its charge orbits. This refines, re-organizes and extends the various results of [3, 4, 5, 8].

In Sect. 4 we reconsider *matter coupled* $\mathcal{N} = 4$, $d = 4$ supergravity. The $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant characterization of all its BPS and non-BPS charge orbits, firstly obtained in [3, 8], is the starting point of the novel results presented in this Section.

Sect. 5 is devoted to the analysis of the $\mathcal{N} = 2$, $d = 4$ case [3]. Beside the generalities on the special Kähler geometry of Abelian vector multiplets’ scalar manifold, the results of this Section are novel. In particular, a formula for the duality invariant is determined, which is *diffeomorphism-invariant* and holds true for all *symmetric* special Kähler manifolds (see *e.g.* [26] and Refs. therein), regardless of the considered symplectic basis.

Sect. 6, starting from the analysis of [3, 8], deals with the issue of the *ADM mass* [27] in $\mathcal{N} = 8$ (Subsect. 6.1) and $\mathcal{N} = 4$ (Subsect. 6.2), ungauged $d = 4$ supergravities. In general, for all supersymmetric orbits the *ADM mass* has a known explicit expression, depending on the number of supersymmetries preserved by the state which is supported by the considered orbit (saturating the *BPS* [28] bound).

2 Electric-Magnetic Duality in Supergravity : Basic Facts

The basic requirement for consistent coupling of a non-linear sigma model based on a *symmetric* manifold (1.1) to \mathcal{N} -extended, $d = 4$ supergravity (see *e.g.* [21] and Refs. therein) is that the vector field strengths and their duals (through *Legendre transform* with respect the Lagrangian density \mathcal{L})

$$F^\Lambda, \quad G_\Lambda \equiv \frac{\delta \mathcal{L}}{\delta F^\Lambda}, \quad (2.1)$$

belong to a *symplectic* representation \mathbf{R}_s of the global (*classical*, see Footnote 1) U -duality group G , given by $2n \times 2n$ matrices with block structure

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}), \quad (2.2)$$

where A , B , C and D are $n \times n$ real matrices. By defining the $2n \times 2n$ *symplectic* metric (each block being $n \times n$)

$$\Omega \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (2.3)$$

the *finite symplecticity condition* for a $2n \times 2n$ real matrix P

$$P^T \Omega P = \Omega \quad (2.4)$$

yields the following relations to hold for the block components of the matrix defined in Eq. (2.2):

$$A^T C - C^T A = 0; \quad (2.5)$$

$$B^T D - D^T B = 0; \quad (2.6)$$

$$A^T D - C^T B = 1. \quad (2.7)$$

An analogous, equivalent definition of the representation \mathbf{R}_s is the following one: \mathbf{R}_s is real and it contains the singlet in its 2-fold antisymmetric tensor product

$$(\mathbf{R}_s \times \mathbf{R}_s)_a \ni \mathbf{1}. \quad (2.8)$$

If the basic requirements (2.5)-(2.7) or (2.8) are met, the *coset representative* of M in the symplectic representation \mathbf{R}_s is given by the (*scalar-dependent*) $2n \times 2n$ matrix

$$S(\phi) \equiv \begin{pmatrix} A(\phi) & B(\phi) \\ C(\phi) & D(\phi) \end{pmatrix} \in Sp(2n, \mathbb{R}). \quad (2.9)$$

A particular role is played by the two (*scalar-dependent*) complex $n \times n$ matrices f and h , which do satisfy the properties

$$-f^\dagger h + h^\dagger f = i\mathbf{1}, \quad (2.10)$$

$$-f^T h + h^T f = 0. \quad (2.11)$$

The constraining relations (2.10) and (2.11) are equivalent to require that

$$S(\phi) = \sqrt{2} \begin{pmatrix} \text{Re}f & -\text{Im}f \\ \text{Re}h & -\text{Im}h \end{pmatrix}, \quad (2.12)$$

or equivalently:

$$f = \frac{1}{\sqrt{2}}(A - iB); \quad (2.13)$$

$$h = \frac{1}{\sqrt{2}}(C - iD). \quad (2.14)$$

In order to make contact with the formalism introduced by Gaillard and Zumino in [1], it is convenient to use another (complex) basis, namely the one which maps an element $S \in Sp(2n, \mathbb{R})$ into an element $U \in U(n, n) \cap Sp(2n, \mathbb{C})$. The change of basis is exploited through the matrix

$$\mathcal{A} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i1 & i1 \end{pmatrix}, \quad \mathcal{A}^{-1} = \mathcal{A}^\dagger. \quad (2.15)$$

The (*scalar-dependent*) matrix U is thus defined as follows:

$$U(\phi) \equiv \mathcal{A}^{-1} S \mathcal{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} f + ih & \bar{f} + i\bar{h} \\ f - ih & \bar{f} - i\bar{h} \end{pmatrix} \in U(n, n) \cap Sp(2n, \mathbb{C}). \quad (2.16)$$

This is the matrix named S in Eq. (5.1) of [1]. Correspondingly, the $Sp(2n, \mathbb{R})$ -covariant vector $(F^\Lambda, G_\Lambda)^T$ is mapped into the vector

$$\mathcal{A}^{-1} \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i1 \\ 1 & -i1 \end{pmatrix} \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} F^\Lambda + iG_\Lambda \\ F^\Lambda - iG_\Lambda \end{pmatrix}. \quad (2.17)$$

The kinetic vector matrix $\mathcal{N}_{\Lambda\Sigma}$ appearing in Eqs. (1.6) and (1.7) is given by (in matrix notation)

$$\mathcal{N}(\phi) = hf^{-1} = (f^{-1})^T h^T, \quad (2.18)$$

and it is named $-i\bar{K}$ in [1].

Thus, by introducing the $2n \times 1$ ($n \times n$ *matrix-valued*) complex vector

$$\Xi \equiv \begin{pmatrix} f \\ h \end{pmatrix} \quad (2.19)$$

and recalling the definition (1.6), the matrix \mathcal{M} can be written as

$$\begin{aligned} \mathcal{M}(\phi) &= -i\Omega + 2\Omega\Xi(\Omega\Xi)^\dagger = -i\Omega - 2\Omega\Xi\Xi^\dagger\Omega = \\ &= -i\Omega - 2 \begin{pmatrix} -h \\ f \end{pmatrix} (h^\dagger, -f^\dagger) = \\ &= -i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} hh^\dagger & -hf^\dagger \\ -fh^\dagger & ff^\dagger \end{pmatrix}. \end{aligned} \quad (2.20)$$

Eqs. (1.4), (1.6) and (2.20) imply that

$$\begin{aligned}
V_{BH}(\phi, \mathcal{P}) &\equiv -\frac{1}{2}\mathcal{P}^T \mathcal{M}(\phi) \mathcal{P} = \text{Tr}(\mathcal{Z}\mathcal{Z}^\dagger) = \text{Tr}(\mathcal{Z}^\dagger\mathcal{Z}) = \\
&= \sum_{A>B=1}^{\mathcal{N}} Z_{AB}\bar{Z}^{AB} + Z_I\bar{Z}^I = \frac{1}{2}Z_{AB}\bar{Z}^{AB} + Z_I\bar{Z}^I = \\
&= \frac{1}{2}\text{Tr}(ZZ^\dagger) + Z_I\bar{Z}^I = \frac{1}{2}\text{Tr}(Z^\dagger Z) + Z_I\bar{Z}^I,
\end{aligned} \tag{2.21}$$

where $(A, B = 1, \dots, \mathcal{N}$ and $I = 1, \dots, m$ throughout; recall $\Lambda = 1, \dots, n$)

$$\mathcal{Z} \equiv \mathcal{P}^T \Omega \Xi = qf - ph = (Z_{AB}(\phi, \mathcal{P}), Z_I(\phi, \mathcal{P})); \tag{2.22}$$

$$\begin{aligned}
&\Downarrow \\
\mathcal{Z}^\dagger &\equiv -\Xi^\dagger \Omega \mathcal{P} = f^\dagger q - h^\dagger p = \begin{pmatrix} \bar{Z}^{AB}(\phi, \mathcal{P}) \\ \bar{Z}^I(\phi, \mathcal{P}) \end{pmatrix};
\end{aligned} \tag{2.23}$$

$$Z_{AB}(\phi, \mathcal{P}) \equiv f_{AB}^\Lambda q_\Lambda - h_{AB|\Lambda} p^\Lambda; \tag{2.24}$$

$$Z_I(\phi, \mathcal{P}) \equiv \bar{f}_I^\Lambda q_\Lambda - \bar{h}_{I|\Lambda} p^\Lambda. \tag{2.25}$$

Thus, Eq. (2.21) yields the “*BH potential*” $V_{BH}(\phi, \mathcal{P})$ to be nothing but the sum of the squares of the “*dressed*” charges. It is here worth noticing that $(f_{AB}^\Lambda, \bar{f}_I^\Lambda)$ and $(h_{AB|\Lambda}, \bar{h}_{I|\Lambda})$ are $n \times n$ complex matrices, because it holds that⁵ $f_{AB}^\Lambda = f_{[AB]}^\Lambda$, $h_{AB|\Lambda} = h_{[AB]|\Lambda}$ (thus implying $Z_{AB} = Z_{[AB]}$), and

$$n = \frac{\mathcal{N}(\mathcal{N} - 1)}{2} + m, \tag{2.26}$$

where \mathcal{N} stands for the number of *spinorial supercharges* (see Footnote 4), and m denotes the number of *matter multiplets* coupled to the supergravity multiplet, except for $\mathcal{N} = 6$, $d = 4$ *pure* supergravity, for which $m = 1$.

Eqs. (2.24) and (2.25) are the basic relation between the (scalar-dependent) “*dressed*” charges Z_{AB} and Z_I and the (scalar-independent) “*bare*” charges \mathcal{P} . It is worth remarking that Z_{AB} is the “*central charge matrix function*”, whose asymptotical value appears in the right-hand side of the \mathcal{N} -extended ($d = 4$) supersymmetry algebra, pertaining to the asymptotical Minkowski space-time background:

$$\{Q_\alpha^A, Q_\beta^B\} = \epsilon_{\alpha\beta} Z^{AB}(\phi_\infty, \mathcal{P}), \tag{2.27}$$

where ϕ_∞ denotes the set of values taken by the scalar fields at *radial infinity* ($r \rightarrow \infty$) within the considered static, spherically symmetric and asymptotically flat *dyonic extremal* BH background. Notice that the indices A, B of the *central charge matrix* are raised and lowered

⁵Unless otherwise noted, square brackets denote antisymmetrization with respect to the enclosed indices.

with the metric of the relevant \mathcal{R} -*symmetry* group of the corresponding supersymmetry algebra.

By denoting the *ADM mass* [27] of the considered BH background by $M_{ADM}(\phi_\infty, \mathcal{P})$, the *BPS bound* [28] implies that

$$M_{ADM}(\phi_\infty, \mathcal{P}) \geq |\mathbf{Z}_1(\phi_\infty, \mathcal{P})| \geq \dots \geq |\mathbf{Z}_{[\mathcal{N}/2]}(\phi_\infty, \mathcal{P})|, \quad (2.28)$$

where $\mathbf{Z}_1(\phi, \mathcal{P}), \dots, \mathbf{Z}_{[\mathcal{N}/2]}(\phi, \mathcal{P})$ denote the set of *skew-eigenvalues* of $Z_{AB}(\phi, \mathcal{P})$, and here square brackets denote the integer part of the enclosed number. If $1 \leq \mathbf{k} \leq [\mathcal{N}/2]$ of the bounds expressed by Eq. (2.28) are saturated, the corresponding extremal BH state is named to be $\frac{\mathbf{k}}{\mathcal{N}}$ -BPS. Thus, the minimal fraction of total supersymmetries (pertaining to the asymptotically flat space-time metric) preserved by the extremal BH background within the considered assumptions is $\frac{1}{\mathcal{N}}$ (for $\mathbf{k} = 1$), while the maximal one is $\frac{1}{2}$ (for $\mathbf{k} = \frac{\mathcal{N}}{2}$). See Sect. 6 for further details.

We end the present Section with some considerations on the issue of duality invariants.

A *duality* invariant \mathcal{I} is a suitable linear combination (in general with complex coefficients) of (ϕ -dependent) H -invariant combinations of $Z_{AB}(\phi, \mathcal{P})$ and $Z_I(\phi, \mathcal{P})$ such that Eq. (1.3) holds, *i.e.* such that \mathcal{I} is invariant under G , and thus ϕ -independent:

$$\mathcal{I} = \mathcal{I}(Z_{AB}(\phi, \mathcal{P}), Z_I(\phi, \mathcal{P})) = \mathcal{I}(\mathcal{P}). \quad (2.29)$$

In presence of *matter coupling*, a charge configuration \mathcal{P} (and thus a certain orbit of the symplectic representation of the U -duality group G , to which \mathcal{P} belongs) is called *supersymmetric iff*, by suitably specifying $\phi = \phi(\mathcal{P})$, it holds that

$$Z_I(\phi(\mathcal{P}), \mathcal{P}) = 0, \quad \forall I = 1, \dots, m. \quad (2.30)$$

Notice that the conditions (2.30) cannot hold *identically* in ϕ , otherwise such conditions would be G -invariant, which generally are *not*. Indeed, in order for the supersymmetry constraints (2.30) to be invariant (or covariant) under G , the following conditions must hold *identically* in ϕ :

$$\partial_\phi Z_I(\phi, \mathcal{P}) = 0, \quad \forall \phi \in M. \quad (2.31)$$

Therefore, supersymmetry conditions are *not* generally G -invariant (*i.e.* U -invariant), otherwise extremal BH attractors (which are “*large*”) supported by supersymmetric charge configurations would not exist.

Nevertheless, in some supergravities it is possible to give U -invariant supersymmetry conditions. In light of previous reasoning, such U -invariant supersymmetric conditions cannot stabilize the scalar fields in terms of charges (by implementing the *attractor mechanism* in the considered framework), because such U -invariant conditions are actually *identities*, and *not equations*, for the set of scalar fields ϕ . Actually, U -invariant supersymmetry conditions can be given for all supersymmetric charge orbits supporting “*small*” BHs (for which the classical *attractor mechanism* does not hold). This can be seen *e.g.* in $\mathcal{N} = 8$ (*pure*) and $\mathcal{N} = 4$ (*matter coupled*) $d = 4$ supergravities, respectively treated in Sects. 3 and 4.

3 $\mathcal{N} = 8$

The scalar manifold of the maximal, namely $\mathcal{N} = 8$, supergravity in $d = 4$ is the *symmetric* real coset

$$\left(\frac{G}{H}\right)_{\mathcal{N}=8,d=4} = \frac{E_{7(7)}}{SU(8)}, \quad \dim_{\mathbb{R}} = 70, \quad (3.1)$$

where the usual notation for non-compact forms of exceptional Lie groups is used, with subscripts denoting the difference “# non-compact generators – # compact generators”. This theory is *pure*, *i.e.* matter coupling is *not* allowed. The *classical* (see Footnote 1) *U*-duality group is $E_{7(7)}$. Moreover, the \mathcal{R} -symmetry group is $SU(8)$ and, due to the absence of *matter multiplets*, it is nothing but the stabilizer of the scalar manifold (3.1) itself.

The Abelian vector field strengths and their *duals*, as well the corresponding *fluxes* (charges), sit in the *fundamental* representation **56** of the global, *classical U*-duality group $E_{7(7)}$. Such a representation determines the embedding of $E_{7(7)}$ into the symplectic group $Sp(56, \mathbb{R})$, which is the largest symmetry acting linearly on charges. The **56** of $E_{7(7)}$ admits an *unique* invariant, which will be denoted by $\mathcal{I}_{4,\mathcal{N}=8}$ throughout. $\mathcal{I}_{4,\mathcal{N}=8}$ is *quartic* in charges, and it was firstly determined in [10].

More precisely, $\mathcal{I}_{4,\mathcal{N}=8}$ is the *unique* combination of $Z_{AB}(\phi, \mathcal{P})$ satisfying

$$\partial_{\phi} \mathcal{I}_{4,\mathcal{N}=8}(Z_{AB}(\phi, \mathcal{P})) = 0, \quad \forall \phi \in \frac{E_{7(7)}}{SU(8)}. \quad (3.2)$$

Eq. (3.2) can be computed by using the *Maurer-Cartan Eqs.* of the coset $\frac{E_{7(7)}}{SU(8)}$ (see *e.g.* [29] and Refs. therein):

$$\nabla Z_{AB} = \frac{1}{2} P_{ABCD} \bar{Z}^{CD}, \quad (3.3)$$

or equivalently by performing an infinitesimal $\frac{E_{7(7)}}{SU(8)}$ -transformation of the central charge matrix (see *e.g.* [29] and Refs. therein):

$$\delta_{\xi_{ABCD}} Z_{AB} = \frac{1}{2} \xi_{ABCD} \bar{Z}^{CD}, \quad (3.4)$$

where ∇ and P_{ABCD} respectively denote the covariant differential operator and the *Vielbein* 1-form in $\frac{E_{7(7)}}{SU(8)}$, and the infinitesimal $\frac{E_{7(7)}}{SU(8)}$ -parameters ξ_{ABCD} satisfy the reality constraint

$$\xi_{ABCD} = \frac{1}{4!} \epsilon_{ABCDEFGH} \bar{\xi}^{EFGH}. \quad (3.5)$$

As firstly found in [10] and rigorously re-obtained in [29], the unique solution of Eq. (3.2) reads:

$$\mathcal{I}_{4,\mathcal{N}=8} = \frac{1}{2^2} \left[2^2 \text{Tr} \left(\left(Z_{AC} \bar{Z}^{BC} \right)^2 \right) - \left(\text{Tr} \left(Z_{AC} \bar{Z}^{BC} \right) \right)^2 + 2^5 \text{Re} (Pf(Z_{AB})) \right], \quad (3.6)$$

where the *Pfaffian* of Z_{AB} is defined as [10]

$$Pf(Z_{AB}) \equiv \frac{1}{2^4 4!} \epsilon^{ABCDEFGH} Z_{AB} Z_{CD} Z_{EF} Z_{GH}, \quad (3.7)$$

and it holds that (see *e.g.* [29])

$$|Pf(Z_{AB})| = |\det(Z_{AB})|^{1/2}. \quad (3.8)$$

In [29] it was indeed shown that, although each of the three terms of the expression (3.6) is $SU(8)$ -invariant but *scalar-dependent*, only the combination given by the expression (3.6) is actually $E_{7(7)}$ -independent and thus *scalar-independent*, satisfying

$$\delta_{\xi_{ABCD}} \mathcal{I}_{4,\mathcal{N}=8} = 0, \quad (3.9)$$

with Eqs. (3.4) and (3.5) holding true.

It is here worth commenting a bit further about formula (3.6). The first two terms in its right-hand side are actually $U(8)$ -invariant, while the third one, namely $2^5 \text{Re}(Pf(Z_{AB}))$, is *only* $SU(8)$ -invariant. Such a third term introduces an $SU(8)$ -invariant phase φ_Z , defined as (one fourth of) the overall phase of the central charge matrix, when this latter is reduced to a skew-diagonal form in the so-called *normal frame* through an $SU(8)$ -transformation:

$$Z_{AB} \xrightarrow{SU(8)} Z_{AB,skew-diag.} \equiv e^{i\varphi_Z/4} \begin{pmatrix} e_1 & & & \\ & e_2 & & \\ & & e_3 & \\ & & & e_4 \end{pmatrix} \otimes \epsilon, \quad e_i \in \mathbb{R}^+, \quad \forall i = 1, \dots, 4, \quad (3.10)$$

where the ordering $e_1 \geq e_2 \geq e_3 \geq e_4$ can be performed without any loss of generality, and the 2×2 symplectic metric

$$\epsilon \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (3.11)$$

has been introduced (notice $\epsilon = \Omega$ for $n = 1$, as defined in Eq. (2.3)). For non-vanishing (in general all different) *skew-eigenvalues* e_i , the symmetry group of $Z_{AB,skew-diag.}$ is $(USp(2))^4 \sim (SU(2))^4$. Thus, beside the 4 *skew-eigenvalues* e_i and the phase φ_Z , the generic Z_{AB} is described by $51 = \dim_{\mathbb{R}} \left(\frac{SU(8)}{(SU(2))^4} \right)$ “*generalized angles*”. Consistently, the total number of parameters is $4 + 1 + 51 = 56$, which is the real dimension of the *fundamental* representation **56**, defining the embedding of $E_{7(7)}$ into $Sp(56, \mathbb{R})$.

Equivalently, φ_Z can be defined through the *Pfaffian* of Z_{AB} as follows:

$$e^{2i\varphi_Z} \equiv \frac{Pf(Z_{AB})}{Pf(\overline{Z}_{AB})}, \quad (3.12)$$

where clearly $Pf(\overline{Z}_{AB}) = \overline{Pf(Z_{AB})}$, as yielded by the definition (3.7). It is then immediate to compute φ_Z from Eq. (3.6):

$$\cos\varphi_Z(\phi, \mathcal{P}) = \frac{\left[2^2 \mathcal{I}_{4,\mathcal{N}=8}(\mathcal{P}) - 2^2 \text{Tr} \left(\left(Z_{AC} \overline{Z}^{BC} \right)^2 \right) + \left(\text{Tr} \left(Z_{AB} \overline{Z}^{AC} \right) \right)^2 \right]}{2^5 \left(\det \left(Z_{AC} \overline{Z}^{BC} \right) \right)^{1/4}}. \quad (3.13)$$

Notice that through Eq. (3.13) $(\cos)\varphi_Z$ is determined in terms of the scalar fields ϕ and of the BH charges \mathcal{P} , also along the “*small*” orbits where $\mathcal{I}_{4,\mathcal{N}=8} = 0$. However, Eq. (3.13) is not defined in the cases in which $\det\left(Z_{AC}\overline{Z}^{BC}\right) = 0$, *i.e.* when *at least* one of the eigenvalues of the matrix $Z_{AC}\overline{Z}^{BC}$ vanishes. In such cases, φ_Z is actually undetermined.

In $\mathcal{N} = 8$, $d = 4$ supergravity five distinct orbits of the **56** of $E_{7(7)}$ exist, as resulting from the analyses performed in [4] and [5]. They can be classified in “*large*” and “*small*” charge orbits, depending whether they correspond to $\mathcal{I}_{4,\mathcal{N}=8} \neq 0$ or $\mathcal{I}_{4,\mathcal{N}=8} = 0$, respectively.

Only two “*large*” charge orbits (for which $\mathcal{I}_{4,\mathcal{N}=8} \neq 0$, and the *attractor mechanism* holds) exist in $\mathcal{N} = 8$, $d = 4$ supergravity:

1. The “*large*” $\frac{1}{8}$ -BPS orbit [4, 5]

$$\mathcal{O}_{\frac{1}{8}\text{-BPS},\text{large}} = \frac{E_{7(7)}}{E_{6(2)}}, \quad \dim_{\mathbb{R}} = 55, \quad (3.14)$$

is defined by the $E_{7(7)}$ -invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=8} > 0. \quad (3.15)$$

At the event horizon of the extremal BH, the solution of the $\mathcal{N} = 8$, $d = 4$ *Attractor Eqs.* yields [3, 8, 30]

$$e_1 \in \mathbb{R}_0^+, \quad e_2 = e_3 = e_4 = 0, \quad (3.16)$$

implying $\det(Z_{AB}) = 0 \Leftrightarrow Pf(Z_{AB}) = 0$, and thus φ_Z to be *undetermined*. Thus, at the event horizon, the symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,\text{skew-diag}}$ defined in Eq. (3.10) gets *enhanced* as follows, revealing the maximal compact symmetry of $\mathcal{O}_{\frac{1}{8}\text{-BPS},\text{large}}$:

$$(USp(2))^4 \xrightarrow{r \rightarrow r_H^+} USp(2) \times SU(6) \sim SU(2) \times SU(6). \quad (3.17)$$

Indeed, $SU(2) \times SU(6)$ is the *maximal compact subgroup* (*mcs*, with symmetric embedding [31]) of $E_{6(2)}$ (stabilizer of $\mathcal{O}_{\frac{1}{8}\text{-BPS},\text{large}}$) itself.

2. The “*large*” non-BPS ($Z_{AB} \neq 0$) orbit [4, 5]

$$\mathcal{O}_{\text{non-BPS},Z_{AB} \neq 0} = \frac{E_{7(7)}}{E_{6(6)}}, \quad \dim_{\mathbb{R}} = 55, \quad (3.18)$$

is defined by the $E_{7(7)}$ -invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=8} < 0. \quad (3.19)$$

At the event horizon of the extremal BH, the solution of the $\mathcal{N} = 8$, $d = 4$ *Attractor Eqs.* yields [3, 8, 30]

$$e_1 = e_2 = e_3 = e_4 \in \mathbb{R}_0^+, \quad \varphi_Z = \pi + 2k\pi, \quad k \in \mathbb{Z}, \quad (3.20)$$

so the *skew-eigenvalues* of Z_{AB} at the horizon (see Eq. (3.10)) are complex. Thus, at the event horizon, the symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew-diag.}$ defined in Eq. (3.10) gets *enhanced* as follows, revealing the maximal compact symmetry of $\mathcal{O}_{non-BPS,Z_{AB}\neq 0}$:

$$(USp(2))^4 \xrightarrow{r \rightarrow r_H^+} USp(8). \quad (3.21)$$

Indeed, $USp(8)$ is the *mcs* (with symmetric embedding [31]) of $E_{6(6)}$ (stabilizer of $\mathcal{O}_{non-BPS,Z_{AB}\neq 0}$) itself.

As mentioned above, for such “*large*” charge orbits, corresponding to a non-vanishing quartic $E_{7(7)}$ -invariant $\mathcal{I}_{4,\mathcal{N}=8}$ and thus supporting “*large*” BHs, the *attractor mechanism* holds. Consequently, the computations of the Bekenstein-Hawking BH entropy can be performed by solving the criticality conditions for the “*BH potential*”

$$V_{BH,\mathcal{N}=8} = \frac{1}{2} Z_{AB} \bar{Z}^{AB}, \quad (3.22)$$

the result being

$$\frac{S_{BH}}{\pi} = V_{BH,\mathcal{N}=8}|_{\partial V_{BH,\mathcal{N}=8}=0} = V_{BH,\mathcal{N}=8}(\phi_H(\mathcal{P}), \mathcal{P}) = |\mathcal{I}_{4,\mathcal{N}=8}|^{1/2}, \quad (3.23)$$

where $\phi_H(\mathcal{P})$ denotes the set of solutions to the *criticality conditions* of $V_{BH,\mathcal{N}=8}$, namely the *Attractor Eqs.* of $\mathcal{N} = 8$, $d = 4$ supergravity:

$$\partial_\phi V_{BH,\mathcal{N}=8} = 0, \quad \forall \phi \in \frac{E_{7(7)}}{SU(8)}, \quad (3.24)$$

expressing the stabilization of the scalar fields purely in terms of supporting charges \mathcal{P} at the event horizon of the extremal BH. Through Eqs. (3.3) and (3.22), Eqs. (3.24) can be rewritten as follows (notice the strict similarity to Eq. (3.40) further below) [30]:

$$Z_{[AB} Z_{CD]} + \frac{1}{4!} \epsilon_{ABCDEFGH} \bar{Z}^{EF} \bar{Z}^{GH} = 0. \quad (3.25)$$

Actually, the critical potential $V_{BH,\mathcal{N}=8}|_{\partial V_{BH,\mathcal{N}=8}=0}$ exhibits some “*flat*” directions, so not all scalars are stabilized in terms of charges at the event horizon [32, 33]. Thus, Eq. (3.23) yields that the *unstabilized* scalars, spanning a related *moduli space* of the considered class of attractor solutions, do not enter in the expression of the BH entropy at all. The *moduli spaces*⁶ exhibited by the *Attractor Eqs.* (3.24)-(3.25) are [33]

$$\mathcal{M}_{\frac{1}{8}-BPS,large} = \frac{E_{6(2)}}{SU(2) \times SU(6)}, \quad \dim_{\mathbb{R}} = 40; \quad (3.26)$$

$$\mathcal{M}_{non-BPS,Z_{AB}\neq 0} = \frac{E_{6(6)}}{USp(8)}, \quad \dim_{\mathbb{R}} = 42. \quad (3.27)$$

⁶Results obtained by explicit computations within the $\mathcal{N} = 2$, $d = 4$ *symmetric* so-called *stu* model in [23] and [34] seem to point out that the *moduli spaces* should be present not only at the event horizon of the considered extremal BH (*i.e.* for $r \rightarrow r_H^+$), but also all along the scalar attractor *flow* (*i.e.* $\forall r \geq r_H$).

As found in [33], the general structure of the *moduli spaces* of attractor solutions in supergravities based on *symmetric* scalar manifolds $\frac{G}{H}$ is

$$\frac{\mathcal{H}_{nc}}{\mathbf{h}}, \quad (3.28)$$

where \mathcal{H}_{nc} is the *non-compact* stabilizer of the charge orbit $\frac{G}{\mathcal{H}_{nc}}$ (apart from eventual $U(1)$ factors, \mathcal{H}_{nc} is a non-compact, real form of H), and $\mathbf{h} = mcs(\mathcal{H}_{nc})$. As justified in [29] and then in [32], $\mathcal{M}_{\frac{1}{8}\text{-BPS},large}$ is a *quaternionic* symmetric manifold. Furthermore, $\mathcal{M}_{non\text{-BPS},Z_{AB}\neq 0}$ given by Eq. (3.27) is nothing but the scalar manifold of $\mathcal{N} = 8$, $d = 5$ supergravity. The stabilizers of $\mathcal{M}_{\frac{1}{8}\text{-BPS},large}$ and $\mathcal{M}_{non\text{-BPS},Z_{AB}\neq 0}$ exploit the maximal compact symmetry of the corresponding charge orbits; this symmetry becomes fully manifest through the enhancement of the compact symmetry group of $Z_{AB,skew\text{-diag}}$ at the event horizon of the extremal BH, respectively given by Eqs. (3.17) and (3.21).

It is now convenient to denote with λ_i ($i = 1, \dots, 4$) the four real non-negative eigenvalues of the matrix $Z_{AB}\bar{Z}^{CB} = (ZZ^\dagger)_A^C$. By recalling Eq. (3.10), one can notice that

$$\lambda_i = e_i^2, \quad (3.29)$$

and one can order them as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$, without any loss of generality. The explicit expression of λ_i in terms of $U(8)$ -invariants (namely of $Tr(ZZ^\dagger)$, $Tr((ZZ^\dagger)^2)$, $Tr((ZZ^\dagger)^3)$ and $Tr((ZZ^\dagger)^4)$, and suitable powers) is given by Eqs. (4.74), (4.75), (4.86) and (4.87) of [8], and it will be used in Sect. 6 to determine the *ADM mass* for $\frac{k}{8}$ -BPS ($k = 1, 2, 4$) extremal BH states.

Three distinct “*small*” charge orbits (all with $\mathcal{I}_{4,\mathcal{N}=8} = 0$) exist, and they all are supersymmetric :

1. The generic “*small*” *lightlike* orbit is $\frac{1}{8}$ -BPS, it is defined by the $E_{7(7)}$ -invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=8} = 0, \quad (3.30)$$

and it reads [4, 5]

$$\mathcal{O}_{\frac{1}{8}\text{-BPS},small} = \frac{E_{7(7)}}{F_{4(4)} \times_s T_{26}}, \quad dim_{\mathbb{R}} = 55. \quad (3.31)$$

Generally, it yields four different λ_i 's, and in this case Eq. (3.13) reduces to

$$\cos\varphi_Z(\phi, \mathcal{P})|_{\mathcal{I}_{4,\mathcal{N}=8}=0} = - \left. \frac{\left[2^2 Tr \left(\left(Z_{AC} \bar{Z}^{BC} \right)^2 \right) - \left(Tr \left(Z_{AB} \bar{Z}^{AC} \right) \right)^2 \right]}{2^5 \left(\det \left(Z_{AC} \bar{Z}^{BC} \right) \right)^{1/4}} \right|_{\mathcal{I}_{4,\mathcal{N}=8}=0}. \quad (3.32)$$

In agreement with the results of [4] and [5], the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew\text{-diag}}$ all along the $\frac{1}{8}$ -BPS “*small*”

flow is the generic one: $(SU(2))^4$. The counting of the parameters of $\mathcal{O}_{\frac{1}{8}\text{-BPS},small}$ consistently reads: $55 = 4$ skew-eigenvalues $\lambda_i + 1$ phase $\varphi_Z + 51$ $\left(= \dim_{\mathbb{R}} \left(\frac{SU(8)}{(SU(2))^4} \right) \right)$ “generalized angles” -1 defining constraint (3.30).

2. The “small” critical orbit is $\frac{1}{4}$ -BPS. It reads [4, 5]

$$\mathcal{O}_{\frac{1}{4}\text{-BPS}} = \frac{E_{7(7)}}{(SO(6,5) \times_s T_{32}) \times T_1}, \quad \dim_{\mathbb{R}} = 45, \quad (3.33)$$

and it is defined by the following differential constraint on $\mathcal{I}_{4,\mathcal{N}=8}$ [3, 8]:

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=8}}{\partial Z_{AB}} = 0, \quad (3.34)$$

which, due to the reality of $\mathcal{I}_{4,\mathcal{N}=8}$, is actually $E_{7(7)}$ -invariant. Let us also notice that, due to the homogeneity of $\mathcal{I}_{4,\mathcal{N}=8}$ of degree four in \mathcal{P} , Eq. (3.34) implies the constraint (3.30). In particular, along the $\frac{1}{4}$ -BPS orbit it holds that (the labelling does not yield any loss of generality)

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 \geq 0. \quad (3.35)$$

If $Pf(Z_{AB}) \neq 0$ then

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > 0, \quad (3.36)$$

and Eq. (3.13) yields $\varphi_Z = k\pi$, $k \in \mathbb{Z}$, so the skew-eigenvalues of Z_{AB} (see Eq. (3.10)) are real and the (maximal) compact symmetry of $Z_{AB,skew-diag.}$ is $(USp(4))^2$. On the other hand, if $Pf(Z_{AB}) = 0$ then

$$\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 = 0, \quad (3.37)$$

and φ_Z is undetermined. In this case, the (maximal compact) symmetry of the skew-diagonalized central charge matrix $Z_{AB,skew-diag.}$ is $USp(4) \times SU(4) \sim SO(5) \times SO(6)$, which is the *mcs* of the non-translational part of the stabilizer of $\mathcal{O}_{\frac{1}{4}\text{-BPS}}$, expressing the maximal compact symmetry of $\mathcal{O}_{\frac{1}{4}\text{-BPS}}$ itself. In agreement with the results of [4] and [5], the maximal (compact) symmetry of the skew-diagonalized central charge matrix $Z_{AB,skew-diag.}$ along the $\frac{1}{4}$ -BPS “small” flow (fully manifest in the particular solution (3.37)) is $USp(4) \times SU(4)$. The counting of the parameters of $\mathcal{O}_{\frac{1}{4}\text{-BPS}}$ consistently reads: $45 = 2$ skew-eigenvalues λ_1 and $\lambda_2 + 43$ $\left(= \dim_{\mathbb{R}} \left(\frac{SU(8)}{(USp(4))^2} \right) \right)$ “generalized angles”.

3. The “small” doubly-critical orbit is $\frac{1}{2}$ -BPS, and it reads [4, 5]

$$\mathcal{O}_{\frac{1}{2}\text{-BPS}} = \frac{E_{7(7)}}{E_{6(6)} \times_s T_{27}}, \quad \dim_{\mathbb{R}} = 28. \quad (3.38)$$

It can be defined in an $E_{7(7)}$ -invariant way by performing the following two-step procedure [8]. One starts by considering the requirement that the second derivative of $\mathcal{I}_{4,\mathcal{N}=8}$

(with respect to Z_{AB}) projected along the adjoint representation $\mathbf{Adj}(SU(8)) = \mathbf{63}$ of $SU(8)$ vanishes, yielding [8]

$$\left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=8}}{\partial Z_{AB} \partial \bar{Z}^{BC}} \right|_{\mathbf{Adj}(SU(8))} = 0 \iff Z_{AC} \bar{Z}^{BC} = \frac{1}{2^3} \delta_A^B Z_{DE} Z^{DE}. \quad (3.39)$$

This is a mixed rank-2 $SU(8)$ -covariant condition. By further differentiating with respect to the scalars ϕ parametrizing $\frac{E_{7(7)}}{SU(8)}$ and using the Maurer-Cartan Eqs. (3.3), one obtains another $SU(8)$ -covariant relation (notice the strict similarity to the $\mathcal{N} = 8$, $d = 4$ *Attractor Eqs.* (3.25)) [8]:

$$Z_{[AB} Z_{CD]} - \frac{1}{4!} \epsilon_{ABCDEFGH} \bar{Z}^{EF} \bar{Z}^{GH} = 0. \quad (3.40)$$

Actually, Eq. (3.40) form with Eq. (3.39) an $E_{7(7)}$ -invariant set of differential conditions defining $\mathcal{O}_{\frac{1}{2}\text{-BPS}}$. Indeed, as noticed in [8], Eq. (3.40) can be rewritten as

$$\frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=8}}{\partial Z_{[AB} \partial Z_{CD]}} - \frac{1}{4!} \epsilon_{ABCDEFGH} \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=8}}{\partial \bar{Z}^{[EF} \partial \bar{Z}^{GH]}} = 0. \quad (3.41)$$

Thus, by using the notation $Z_{\mathbf{56}} \equiv (\mathcal{Z}, \mathcal{Z}^T) = (Z_{AB}, \bar{Z}^{AB})$ (recall Eqs. (2.22) and (2.23)), Eqs. (3.39) and (3.40)-(3.41) can be rewritten in the manifestly $E_{7(7)}$ -invariant fashion

$$\left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=8}}{\partial Z_{\mathbf{56}} \partial Z_{\mathbf{56}}} \right|_{\mathbf{Adj}(E_{7(7)})} = 0, \quad (3.42)$$

where $\mathbf{Adj}(E_{7(7)}) = \mathbf{133}$ is the adjoint representation of $E_{7(7)}$. Notice that $\frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=8}}{\partial Z_{\mathbf{56}} \partial Z_{\mathbf{56}}}$ is a rank-2 symmetric true-tensor $E_{7(7)}$ -tensor, thus sitting in the symmetric product representation $(\mathbf{56} \times \mathbf{56})_s = \mathbf{1596}$ of $E_{7(7)}$, which in turns enjoys the following branching with respect to $E_{7(7)}$ [31, 8]:

$$(\mathbf{56} \times \mathbf{56})_s = \mathbf{1596} \longrightarrow \mathbf{1463} + \frac{\mathbf{133}}{\mathbf{Adj}(E_{7(7)})}. \quad (3.43)$$

It is here worth remarking that the constraints (3.39) and (3.40)-(3.41) (or equivalently ((3.42))) imply the constraint (3.34), because in fact they are stronger constraints.

Along the $\frac{1}{2}$ -BPS orbit it holds that

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4. \quad (3.44)$$

Furthermore, it can be shown that $\varphi_Z = 2k\pi$, $k \in \mathbb{Z}$, so the *skew-eigenvalues* of Z_{AB} (see Eq. (3.10)) are real. In agreement with the results of [4] and [5], the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB, skew-diag.}$ all along the $\frac{1}{2}$ -BPS “small” flow is $USp(8)$, which is the *mcs* of the non-translational part of the stabilizer of $\mathcal{O}_{\frac{1}{2}\text{-BPS}}$, expressing the maximal compact symmetry of $\mathcal{O}_{\frac{1}{2}\text{-BPS}}$ itself. The counting of the

parameters of $\mathcal{O}_{\frac{1}{2}-BPS}$ consistently reads: $28 = 1$ skew-eigenvalue $\lambda_1 + 27$ ($= \dim_{\mathbb{R}} \left(\frac{SU(8)}{USp(8)} \right)$) “generalized angles”.

Interestingly, $USp(8)$ also is the *enhanced* compact symmetry of $Z_{AB,skew-diag.}$ at the event horizon of the “large” non-BPS $Z_{AB} \neq 0$ attractor scalar flow (see Eq. (3.21) above). Indeed, the charge orbits $\mathcal{O}_{non-BPS, Z_{AB} \neq 0}$ and $\mathcal{O}_{\frac{1}{2}-BPS}$ (respectively given by Eqs. (3.18) and (3.38)) coincide, up to the translational factor T_{27} in the stabilizer, and thus they have the same maximal compact symmetry.

As given by the analysis of [3], the classification of “large” and “small” orbits of the **56** of $E_{7(7)}$ can be performed also considering the symplectic basis composed by the fluxes q_{Λ} ($\Lambda = 1, \dots, 56$). In general, the symplectic basis of charges is useful in order to determine, through constraints imposed on the relevant U -invariant, the number and typology of orbits of the relevant representation of the U -duality group. On the other hand, using the manifestly H -covariant basis of central charges and matter charges one can achieve a symplectic-invariant characterization of charge orbits, and also study the related supersymmetry-preserving features.

Finally, it is worth pointing out once again that there is a crucial difference among the various constraints defining the two “large” and the three “small” charge orbits of $\mathcal{N} = 8$, $d = 4$ supergravity listed above:

- The “large” charge orbits $\mathcal{O}_{\frac{1}{8}-BPS, large}$ and $\mathcal{O}_{non-BPS, Z_{AB} \neq 0}$, respectively given by Eqs. (3.14) and (3.18), are in order defined by the $E_{7(7)}$ -invariant conditions $\mathcal{I}_{4, \mathcal{N}=8} > 0$ and $\mathcal{I}_{4, \mathcal{N}=8} < 0$. Due to their $E_{7(7)}$ -invariance, these conditions are *identities* for the scalar fields ϕ spanning $\frac{E_{7(7)}}{SU(8)}$. However, the *classical attractor mechanism* does hold for “large” extremal BHs, and the scalars ϕ are stabilized purely in terms of charges \mathcal{P} at the event horizon ($r \rightarrow r_H^+$) through the only two independent solutions (3.16) and (3.20) to the $\mathcal{N} = 8$, $d = 4$ Attractor Eqs. (3.24)-(3.25).
- The “small” charge orbits $\mathcal{O}_{\frac{1}{8}-BPS, small}$, $\mathcal{O}_{\frac{1}{4}-BPS}$ and $\mathcal{O}_{\frac{1}{2}-BPS}$, respectively given by Eqs. (3.31), (3.33) and (3.38), are in order defined by the $E_{7(7)}$ -invariant conditions (3.30), (3.34) and (3.42). Due to their $E_{7(7)}$ -invariance, these conditions are *identities* for the scalars ϕ , which thus are *not* stabilized along such orbits. Indeed, the *classical attractor mechanism* does *not* hold for “small” BHs.

4 $\mathcal{N} = 4$

In $\mathcal{N} = 4$, $d = 4$ supergravity, unlike the $\mathcal{N} = 8$ case, matter (*vector*) multiplets appear (see *e.g.* [35, 36]). By denoting their number with M , the related scalar manifold is the *symmetric* coset

$$\left(\frac{G}{H} \right)_{\mathcal{N}=4, d=4} = \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)}, \quad \dim_{\mathbb{R}} = 6M + 2. \quad (4.1)$$

The Abelian vector field strengths and their *duals*, as well the corresponding *fluxes* (charges), sit in the bi-fundamental $(\mathbf{2}, \mathbf{6} + \mathbf{M})$ representation of the global, *classical* (see Footnote 1) U -duality group $SL(2, \mathbb{R}) \times SO(6, M)$ [37]. Such a representation determines the embedding of $SL(2, \mathbb{R}) \times SO(6, M)$ into the symplectic group $Sp(12 + 2M, \mathbb{R})$. The representation $(\mathbf{2}, \mathbf{6} + \mathbf{M})$ is endowed with a natural symplectic metric

$$\Omega \equiv \epsilon_{\alpha\beta}\eta_{\Lambda\Sigma}, \quad (4.2)$$

where $\epsilon_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) is the (inverse of the) $SL(2, \mathbb{R})$ skew-symmetric metric defined in Eq. (3.11), and $\eta_{\Lambda\Sigma}$ ($\Lambda, \Sigma = 1, \dots, 6 + M = n$; recall Eq. (2.26)) is the Lorentzian metric of $SO(6, M)$. Moreover, the \mathcal{R} -symmetry group is $U(4)$.

Furthermore, $(\mathbf{2}, \mathbf{6} + \mathbf{M})$ admits an *unique* invariant, which will be denoted by $\mathcal{I}_{4, \mathcal{N}=4}$ throughout. $\mathcal{I}_{4, \mathcal{N}=4}$ is *quartic* in charges, and it was firstly determined in [13, 18, 38].

More precisely, $\mathcal{I}_{4, \mathcal{N}=4}$ is the *unique* combination of “*dressed*” charges $Z_{AB} = Z_{[AB]}(\phi, \mathcal{P})$ (*central charge matrix*, $A, B = 1, \dots, 4$) and $Z_I(\phi, \mathcal{P})$ (*matter charges*, $I = 1, \dots, M$) satisfying

$$\partial_\phi \mathcal{I}_{4, \mathcal{N}=4}(Z_{AB}(\phi, \mathcal{P}), Z_I(\phi, \mathcal{P})) = 0, \quad \forall \phi \in \left(\frac{G}{H} \right)_{\mathcal{N}=4, d=4}. \quad (4.3)$$

Eq. (4.3) can be computed by using the *Maurer-Cartan Eqs.* of the coset $\frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)}$ (see *e.g.* [29], and Refs. therein):

$$\nabla Z_{AB} = \frac{1}{2} P \epsilon_{ABCD} \bar{Z}^{CD} + P_{ABI} \bar{Z}^I; \quad (4.4)$$

$$\nabla Z_I = \frac{1}{2} P_{ABI} \bar{Z}^{AB} + P \eta_{IJ} \bar{Z}^J, \quad (4.5)$$

or equivalently by performing an infinitesimal $\frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)}$ -transformation of the central charge matrix and of matter charges (see *e.g.* [29], and Refs. therein):

$$\delta_{(\xi, \xi_{AB|I})} Z_{AB} = \frac{1}{2} \xi \epsilon_{ABCD} \bar{Z}^{CD} + \xi_{AB|I} Z^I; \quad (4.6)$$

$$\delta_{(\xi, \xi_{AB|I})} Z_I = \bar{\xi} \eta_{IJ} \bar{Z}^J + \frac{1}{2} \xi_{AB|I} \bar{Z}^{AB}, \quad (4.7)$$

where ∇ stands for the covariant differential operator in $\frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)}$. P and P_{ABI} respectively are the *Vielbein* 1-forms of $\frac{SL(2, \mathbb{R})}{U(1)}$ and $\frac{SO(6, M)}{SO(6) \times SO(M)}$, with P_{ABI} satisfying the reality condition:

$$P_{ABI} = \frac{1}{2} \eta_{IJ} \epsilon_{ABCD} \bar{P}^{CDJ}. \quad (4.8)$$

Moreover, ξ is the infinitesimal $\frac{SL(2, \mathbb{R})}{U(1)}$ -parameter and $\xi_{AB|I}$ are the infinitesimal $\frac{SO(6, M)}{SO(6) \times SO(M)}$ -parameters, satisfying the reality condition

$$\bar{\xi}^{AB|I} = \frac{1}{2} \eta^{IJ} \epsilon^{ABCD} \xi_{CD|J}. \quad (4.9)$$

As found in [13, 18, 38] and rigorously re-obtained in [29], in terms of Z_{AB} and Z_I the unique solution of Eq. (4.3) reads:

$$\mathcal{I}_{4,\mathcal{N}=4} = \mathcal{S}_1^2 - |\mathcal{S}_2|^2, \quad (4.10)$$

where one can identify $\mathcal{S}_1 \equiv L_0$, $\mathcal{S}_2 = L_1 + iL_2$, with $L \equiv (L_0, L_1, L_2)$ being an $SL(2, \mathbb{R}) \sim SO(1, 2)$ -vector with square norm

$$L^2 = L_0^2 - L_1^2 - L_2^2 = \mathcal{S}_1^2 - |\mathcal{S}_2|^2. \quad (4.11)$$

\mathcal{S}_1 and \mathcal{S}_2 are defined as [29]

$$\mathcal{S}_1 \equiv \frac{1}{2} Z_{AB} \bar{Z}^{AB} - Z_I \bar{Z}^I \in \mathbb{R}; \quad (4.12)$$

$$\mathcal{S}_2 \equiv \frac{1}{4} \epsilon^{ABCD} Z_{AB} Z_{CD} - \bar{Z}_I \bar{Z}^I \in \mathbb{C}. \quad (4.13)$$

In [29] it was indeed shown that $\mathcal{I}_{4,\mathcal{N}=4}$ given by Eq. (4.10) is the unique combination of $SO(6, M)$ -invariant and scalar-dependent quantities, which is actually *also* $SL(2, \mathbb{R})$ -independent and thus *scalar-independent*, satisfying

$$\delta_\xi \mathcal{I}_{4,\mathcal{N}=4} = 0; \quad (4.14)$$

$$\delta_{\xi_{AB|I}} \mathcal{I}_{4,\mathcal{N}=4} = 0, \quad (4.15)$$

with Eqs. (4.6), (4.7) and (4.9) holding true.

On the other hand, the expression of $\mathcal{I}_{4,\mathcal{N}=4}$ in terms of the “bare” charges \mathcal{P} reads [13, 14, 17, 18]

$$\mathcal{I}_{4,\mathcal{N}=4} = p^2 q^2 - (p \cdot q)^2 = \frac{1}{2} (p_\Lambda q_\Sigma - p_\Sigma q_\Lambda) (p_\Xi q_\Omega - p_\Omega q_\Xi) \eta^{\Lambda\Xi} \eta^{\Sigma\Omega} = \frac{1}{2} T_{\Lambda\Sigma}^{(a)} T^{(a)|\Lambda\Sigma}, \quad (4.16)$$

where

$$p^2 \equiv p \cdot p \equiv p_\Lambda p_\Sigma \eta^{\Lambda\Sigma}, \quad q^2 \equiv q \cdot q \equiv q_\Lambda q_\Sigma \eta^{\Lambda\Sigma}, \quad p \cdot q \equiv p_\Lambda q_\Sigma \eta^{\Lambda\Sigma}, \quad (4.17)$$

and the tensor

$$T_{\Lambda\Sigma}^{(a)} \equiv p_\Lambda q_\Sigma - p_\Sigma q_\Lambda = T_{[\Lambda\Sigma]}^{(a)} \quad (4.18)$$

has been introduced (the upperscript “(a)” stands for “*anti-symmetric*”).

The classification of charge orbits, in particular the BPS ones, was performed in [3] and [8]. By performing a suitable $U(1) \times SO(6) (\sim U(4))$ -transformation, the *central charge matrix* Z_{AB} can be *skew-diagonalized* in the *normal frame* (recall definition (3.11)):

$$Z_{AB} \xrightarrow{U(4)} Z_{AB,skew-diag.} \equiv \begin{pmatrix} z_1 & \\ & z_2 \end{pmatrix} \otimes \epsilon, \quad z_1, z_2 \in \mathbb{R}^+, \quad (4.19)$$

where the ordering $z_1 \geq z_2$ does not imply any loss of generality. Furthermore, by performing a suitable $SO(M)$ -transformation, the vector Z_I of *matter charges* can be reduced to have only two non-vanishing entries, one real positive and the other one complex, say (without loss of generality, with the subscript “*red.*” standing for “*reduced*”)

$$Z_I \xrightarrow{SO(M)} Z_{I,red.} \equiv (\rho_1 e^{i\theta}, \rho_2, 0, \dots, 0), \quad \rho_1, \rho_2 \in \mathbb{R}^+, \quad \theta \in \mathbb{R}. \quad (4.20)$$

For non-vanishing (in general different) *skew-eigenvalues* z_1 and z_2 , the symmetry group of $Z_{AB,skew-diag.}$ is $(USp(2))^2 \sim (SU(2))^2$. Analogously, for non-vanishing (in general different) ρ_1 and ρ_2 (and non-vanishing phase θ) the symmetry group of $Z_{I,red.}$ is $SO(M-2)$. Thus, beside z_1, z_2, ρ_1, ρ_2 and θ the generic Z_{AB} and Z_I are described by $7 + 2M = \dim_{\mathbb{R}} \left(\frac{U(4) \times SO(M)}{(SU(2))^2 \times SO(M-2)} \right)$ “*generalized angles*”. Consistently, the total number of parameters is $2 + 2 + 1 + 7 + 2M = 12 + 2M$, which is the real dimension of the bi-fundamental representation $(\mathbf{2}, \mathbf{6} + \mathbf{M})$, defining the embedding of $SL(2, \mathbb{R}) \times SO(6, M)$ into $Sp(12 + 2M, \mathbb{R})$.

In $\mathcal{N} = 4, d = 4$ *matter coupled* supergravity three distinct “*large*” charge orbits of the $(\mathbf{2}, \mathbf{6} + \mathbf{M})$ of $SL(2, \mathbb{R}) \times SO(6, M)$ (for which $\mathcal{I}_{4, \mathcal{N}=4} \neq 0$, and the *attractor mechanism* holds) exist, as resulting from the analysis performed in⁷ [12]:

1. The “*large*” $\frac{1}{4}$ -BPS orbit

$$\mathcal{O}_{\frac{1}{4}\text{-BPS,large}} = SL(2, \mathbb{R}) \times \frac{SO(6, M)}{SO(4, M) \times SO(2)}, \quad \dim_{\mathbb{R}} = 11 + 2M, \quad (4.21)$$

is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraint

$$\mathcal{I}_{4, \mathcal{N}=4} > 0. \quad (4.22)$$

Thus, the corresponding horizon solution of the $\mathcal{N} = 4, d = 4$ *Attractor Eqs.* yields [3, 8, 12]

$$z_1 \in \mathbb{R}_0^+, \quad z_2 = 0, \quad \rho_1 = \rho_2 = 0, \quad \theta \text{ undetermined}; \quad (4.23)$$

$$\mathcal{S}_1 = z_1^2 > 0, \quad \mathcal{S}_2 = 0. \quad (4.24)$$

Therefore, at the event horizon, the symmetry group of $Z_{AB,skew-diag.}$ defined in Eq. (4.19) does not get enhanced, while the symmetry group of $Z_{i,red.}$ defined in Eq. (4.20) gets *enhanced* as follows:

$$SO(M-2) \xrightarrow{r \rightarrow r_H^+} SO(M). \quad (4.25)$$

As a consequence, the horizon attractor solution exploits the maximal compact symmetry $SU(2) \times SU(2) \times SO(M) \times SO(2)$, which is the *mcs* [31] of the stabilizer of $\mathcal{O}_{\frac{1}{4}\text{-BPS,large}}$ itself.

2. The “*large*” non-BPS $Z_{AB} = 0$ orbit (existing for $M \geq 2$) [12]

$$\mathcal{O}_{non\text{-BPS}, Z_{AB}=0, large} = SL(2, \mathbb{R}) \times \frac{SO(6, M)}{SO(6, M-2) \times SO(2)}, \quad \dim_{\mathbb{R}} = 11 + 2M, \quad (4.26)$$

is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraint

$$\mathcal{I}_{4, \mathcal{N}=4} > 0. \quad (4.27)$$

⁷Consistent with the analysis of [12], Eqs. (4.21), (4.26) and (4.31) fix a slightly misleading notation for the “*large*” charge orbits of $\mathcal{N} = 4, d = 4$ *matter coupled* supergravity, as given by Table 1 of [39].

Thus, the corresponding attractor solution of the $\mathcal{N} = 4$, $d = 4$ *Attractor Eqs.* yields (for $M \geq 2$) [3, 8, 12]

$$z_1 = z_2 = 0, \quad \rho_1^2 e^{2i\theta} + \rho_2^2 = 0 \Leftrightarrow \rho_1 = \rho_2 \in \mathbb{R}_0^+, \quad \theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}; \quad (4.28)$$

$$\mathcal{S}_1 = -2\rho_1^2 < 0, \quad \mathcal{S}_2 = 0. \quad (4.29)$$

Therefore, at the event horizon, the symmetry group of $Z_{AB,skew-diag.}$ defined in Eq. (4.19) gets enhanced as follows:

$$(SU(2))^2 \xrightarrow{r \rightarrow r_H^+} SU(4), \quad (4.30)$$

and the symmetry group of $Z_{i,red.}$ defined in Eq. (4.20) does not get *enhanced*. Consequently, the horizon attractor solution exploits the maximal compact symmetry $SU(4) \times SO(M-2) \times SO(2)$, which is the *mcs* [31] of the stabilizer of $\mathcal{O}_{non-BPS, Z_{AB}=0, large}$ itself.

3. The “*large*” non-BPS $Z_{AB} \neq 0$ orbit (existing for $M \geq 1$) [12]

$$\mathcal{O}_{non-BPS, Z_{AB} \neq 0, large} = SL(2, \mathbb{R}) \times \frac{SO(6, M)}{SO(5, M-1) \times SO(1, 1)}, \quad dim_{\mathbb{R}} = 11 + 2M, \quad (4.31)$$

is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraint

$$\mathcal{I}_{4, \mathcal{N}=4} < 0. \quad (4.32)$$

At the event horizon of the extremal BH, the solution of the $\mathcal{N} = 4$, $d = 4$ *Attractor Eqs.* yields (for $M \geq 1$) [3, 8, 12]

$$z_1 = z_2 = \frac{\rho_1}{\sqrt{2}} \in \mathbb{R}_0^+, \quad \rho_2 = 0, \quad \theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}; \quad (4.33)$$

$$\mathcal{S}_1 = 0, \quad \mathcal{S}_2 = 3z_1^2 > 0. \quad (4.34)$$

Thus, at the event horizon, the symmetry group of $Z_{AB,skew-diag.}$ defined in Eq. (4.19) gets enhanced as follows:

$$(SU(2))^2 \xrightarrow{r \rightarrow r_H^+} USp(4), \quad (4.35)$$

and the symmetry group of $Z_{i,red.}$ defined in Eq. (4.20) gets also *enhanced* as

$$SO(M-2) \xrightarrow{r \rightarrow r_H^+} SO(M-1). \quad (4.36)$$

As a consequence, the horizon attractor solution exploits the maximal compact symmetry $USp(4) \times SO(M-1)$ which, due to the isomorphism $USp(4) \sim SO(5)$, is the *mcs* [31] of the stabilizer of $\mathcal{O}_{non-BPS, Z_{AB} \neq 0, large}$ itself.

As mentioned above, for such “large” charge orbits, corresponding to a non-vanishing quartic $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant $\mathcal{I}_{4, \mathcal{N}=4}$ and thus supporting “large” BHs, the *attractor mechanism* holds. Consequently, the computations of the Bekenstein-Hawking BH entropy can be performed by solving the criticality conditions for the “BH potential”

$$V_{BH, \mathcal{N}=4} = \frac{1}{2} Z_{AB} \bar{Z}^{AB} + Z_I \bar{Z}^I, \quad (4.37)$$

the result being

$$\frac{S_{BH}}{\pi} = V_{BH, \mathcal{N}=4} |_{\partial V_{BH, \mathcal{N}=4}=0} = V_{BH, \mathcal{N}=4}(\phi_H(\mathcal{P}), \mathcal{P}) = |\mathcal{I}_{4, \mathcal{N}=4}|^{1/2}, \quad (4.38)$$

where $\phi_H(\mathcal{P})$ denotes the set of solutions to the *criticality conditions* of $V_{BH, \mathcal{N}=4}$, namely the *Attractor Eqs.* of $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity:

$$\partial_\phi V_{BH, \mathcal{N}=4} = 0, \quad \forall \phi \in \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SO(6, M)}{SO(6) \times SO(M)}, \quad (4.39)$$

expressing the stabilization of the scalar fields purely in terms of supporting charges \mathcal{P} at the event horizon of the extremal BH. Through Eqs. (4.4)-(4.5) and (4.37), Eqs. (4.39) can be rewritten as follows [12]:

$$\begin{cases} \left(\bar{Z}^{AB} + \frac{1}{2} \epsilon^{ABCD} Z_{CD} \right) Z^I = 0; \\ Z^I Z^J \delta_{IJ} + \frac{1}{4} \epsilon_{ABCD} \bar{Z}^{AB} \bar{Z}^{CD} = 0. \end{cases} \quad (4.40)$$

Actually, the critical potential $V_{BH, \mathcal{N}=4} |_{\partial V_{BH, \mathcal{N}=4}=0}$ exhibits some “flat” directions, so not all scalars are stabilized in terms of charges at the event horizon [39]. Thus, Eq. (4.38) yields that the *unstabilized* scalars, spanning a related *moduli space* of the considered class of attractor solutions, do not enter in the expression of the BH entropy at all. The *moduli spaces* exhibited by the *Attractor Eqs.* (4.39)-(4.40) are [39]

$$\mathcal{M}_{\frac{1}{4}\text{-BPS, large}} = \frac{SO(4, M)}{SU(2) \times SU(2) \times SO(M)}, \quad \dim_{\mathbb{R}} = 4M; \quad (4.41)$$

$$\mathcal{M}_{\text{non-BPS, } Z_{AB}=0, \text{ large}} = \frac{SO(6, M-2)}{SU(4) \times SO(M-2)}, \quad \dim_{\mathbb{R}} = 6(M-2); \quad (4.42)$$

$$\mathcal{M}_{\text{non-BPS, } Z_{AB} \neq 0, \text{ large}} = SO(1, 1) \times \frac{SO(5, M-1)}{USp(4) \times SO(M-1)}, \quad \dim_{\mathbb{R}} = 5(M-1) + 1. \quad (4.43)$$

As justified in [29] and then in [39], $\mathcal{M}_{\frac{1}{4}\text{-BPS, large}}$ is a *quaternionic* symmetric manifold. Furthermore, $\mathcal{M}_{\text{non-BPS, } Z_{AB} \neq 0, \text{ large}}$ given by Eq. (4.43) is nothing but the scalar manifold of $\mathcal{N} = 4$, $d = 5$ *matter coupled* supergravity. The stabilizers of $\mathcal{M}_{\frac{1}{4}\text{-BPS, large}}$, $\mathcal{M}_{\text{non-BPS, } Z_{AB}=0, \text{ large}}$ and $\mathcal{M}_{\text{non-BPS, } Z_{AB} \neq 0, \text{ large}}$ exploit the maximal compact symmetry of the corresponding charge orbits; this symmetry becomes fully manifest through the enhancement of the compact symmetry group of $Z_{AB, \text{skew-diag.}}$ and $Z_{I, \text{red.}}$ at the event horizon of the extremal BH, respectively given by Eqs. (4.25), (4.30) and (4.35)-(4.36).

Let us now analyze the “*small*” charge orbits of the $(\mathbf{2}, \mathbf{6} + \mathbf{M})$ of $SL(2, \mathbb{R}) \times SO(6, M)$, associated to $\mathcal{I}_{4, \mathcal{N}=4} = 0$, for which the *attractor mechanism* does not hold. The analysis performed below completes the one given in [3] and [8].

While in $\mathcal{N} = 8, d = 4$ supergravity all three “*small*” charge orbits are BPS (with various degrees of supersymmetry-preservation), in the considered $\mathcal{N} = 4, d = 4$ theory there are six “*small*” charge orbits, one of them being $\frac{1}{2}$ -BPS and the other five being non-BPS. Such an abundance of different charge orbits can be traced back to the *factorized* nature of the U -duality group $SL(2, \mathbb{R}) \times SO(6, M)$. Furthermore, it should be remarked that in $\mathcal{N} = 4, d = 4$ supergravity the $\frac{1}{(\mathcal{N}=4)}$ -BPS charge orbit exists only in its “*large*” version, differently from the $d = 4$ maximal theory, in which both “*large*” and “*small*” $\frac{1}{(\mathcal{N}=8)}$ -BPS charge orbits exist.

It is now convenient to denote with α_1 and α_2 the two real non-negative eigenvalues of the matrix $Z_{AB} \bar{Z}^{CB} = (ZZ^\dagger)_A^C$. By recalling Eq. (4.19), one can notice that ($i = 1, 2$)

$$\alpha_i = z_i^2. \quad (4.44)$$

and one can order them as $\alpha_1 \geq \alpha_2$, without any loss of generality. The explicit expression of α_i in terms of $U(4) \times SO(M)$ -invariants (namely of $Tr(ZZ^\dagger)$, $Tr((ZZ^\dagger)^2)$, and suitable powers) is given by Eqs. (5.108) and (5.109) of [8].

Firstly, let us observe that from Eqs. (4.16) and (4.11) the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant “*degeneracy*” condition can be written in the “*dressed*” (\mathcal{R} -symmetry- and $SO(M)$ -covariant) and “*bare*” (symplectic-, *i.e.* $Sp(12 + 2M, \mathbb{R})$ -covariant) charges’ bases respectively as follows:

$$\mathcal{I}_{4, \mathcal{N}=4} = 0 \Leftrightarrow \mathcal{S}_1^2 = |\mathcal{S}_2|^2 \Leftrightarrow p^2 q^2 = (p \cdot q)^2 \geq 0. \quad (4.45)$$

Then, in order to determine the number and typology of “*small*” orbits, it is convenient to start differentiating $\mathcal{I}_{4, \mathcal{N}=4}$ in the symplectic “*bare*” charges’ basis $\mathcal{P} \equiv (p^\Lambda, q_\Lambda)^T$ (recall definition (1.2)). Eqs. (4.16) and (4.18) yield the constraints defining the “*small*” *critical* orbits to read

$$\frac{\partial \mathcal{I}_{4, \mathcal{N}=4}}{\partial p_\Lambda} = 2 [q^2 p^\Lambda - (q \cdot p) q^\Lambda] = 2T^{(a)|\Lambda\Sigma} q_\Sigma = 0; \quad (4.46)$$

$$\frac{\partial \mathcal{I}_{4, \mathcal{N}=4}}{\partial q_\Lambda} = 2 [p^2 q^\Lambda - (q \cdot p) p^\Lambda] = -2T^{(a)|\Lambda\Sigma} p_\Sigma = 0. \quad (4.47)$$

Due to the definition (4.18), or equivalently to the homogeneity (of degree four) in charges of $\mathcal{I}_{4, \mathcal{N}=4}$, it is worth noticing that the “*criticality*” constraints (4.46) and (4.47) imply the “*degeneracy*” condition (4.45).

Beside the trivial one ($p_\Lambda = 0 = q_\Lambda \forall \Lambda$), all the solutions to the “*criticality*” constraints

(4.46) and (4.47) list as follows:

$$A] \begin{cases} T_{\Lambda\Sigma}^{(a)} = 0; \\ \begin{cases} p^2 q^2 = (p \cdot q)^2 > 0 : \begin{cases} \text{A.1]} & p^2 > 0, q^2 > 0; \\ \text{aut} & \\ \text{A.2]} & p^2 < 0, q^2 < 0; \end{cases} \\ \text{A.3]} & p^2 q^2 = (p \cdot q)^2 = 0 : p^2 = 0, q^2 = 0; \end{cases} \end{cases} \quad (4.48)$$

$$B] \begin{cases} T_{\Lambda\Sigma}^{(a)} \neq 0; \\ p^2 = q^2 = p \cdot q = 0. \end{cases} \quad (4.49)$$

Notice that each set (**A.1**, **A.2**, **A.3** and **B**) of constraints is $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant, but formulated in terms of the symplectic charge basis \mathcal{P} .

The solutions (4.48)-(4.49) can be rewritten by noticing that $\frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}}$, *i.e.* the tensor of second derivatives of $\mathcal{I}_{4, \mathcal{N}=4}$ with respect to \mathcal{P} , sits in the symmetric product representation $((\mathbf{2}, \mathbf{6} + \mathbf{M}) \times (\mathbf{2}, \mathbf{6} + \mathbf{M}))_s$ of the U -duality group $SL(2, \mathbb{R}) \times SO(6, M)$, which decomposes as follows [8]:

$$((\mathbf{2}, \mathbf{6} + \mathbf{M}) \times (\mathbf{2}, \mathbf{6} + \mathbf{M}))_s \xrightarrow{SL(2, \mathbb{R}) \times SO(6, M)} \begin{matrix} (\mathbf{3}, \mathbf{1}) & + & (\mathbf{3}, \mathbf{TrSym}(SO(6, M))) & + & (\mathbf{1}, \mathbf{Adj}(SO(6, M))) \\ T^{(0)} & & T_{\Lambda\Sigma}^{(tr-s)} & & T_{\Lambda\Sigma}^{(a)} \end{matrix} \quad (4.50)$$

The antisymmetric tensor

$$T_{\Lambda\Sigma}^{(a)} \equiv \left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \right|_{(\mathbf{1}, \mathbf{Adj}(SO(6, M)))} \quad (4.51)$$

was already introduced in Eq. (4.18). **TrSym** and **Adj** respectively denote the *traceless symmetric* and *adjoint* representations, and [8]

$$\begin{aligned} T_{\Lambda\Sigma}^{(tr-s)} &\equiv \left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \right|_{(\mathbf{3}, \mathbf{TrSym}(SO(6, M)))} \equiv \\ &\equiv \left(q_{\Lambda} q_{\Sigma} - \frac{q^2}{6+M} \eta_{\Lambda\Sigma}, p_{\Lambda} p_{\Sigma} - \frac{p^2}{6+M} \eta_{\Lambda\Sigma}, \frac{1}{2} (q_{\Lambda} p_{\Sigma} + q_{\Sigma} p_{\Lambda}) - \frac{q \cdot p}{6+M} \eta_{\Lambda\Sigma} \right); \end{aligned} \quad (4.52)$$

$$\begin{aligned} T^{(0)} &\equiv \left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \right|_{(\mathbf{3}, \mathbf{1})} \equiv Tr_{SO(6, M)} \left(T_{\Lambda\Sigma}^{(s)} \right) \equiv Tr_{SO(6, M)} \left(\left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \right|_{(\mathbf{3}, \mathbf{Sym}(SO(6, M)))} \right) = \\ &= (q^2, p^2, q \cdot p) = \begin{pmatrix} q^2 & q \cdot p \\ q \cdot p & p^2 \end{pmatrix}. \end{aligned} \quad (4.53)$$

The definition (4.53) of $T^{(0)}$ implies that (recall Eq. (4.16))

$$\mathcal{I}_{4,\mathcal{N}=4} = \det(T^{(0)}) = \det\left(\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \Big|_{(\mathbf{3},\mathbf{1})}\right), \quad (4.54)$$

in turn yielding another, equivalent $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant characterization of the “degeneracy” condition (4.45):

$$\det(T^{(0)}) = \det\left(\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \Big|_{(\mathbf{3},\mathbf{1})}\right) = 0. \quad (4.55)$$

Thus, Eqs. (4.48)-(4.49) can be recast as follows:

$$A] \left\{ \begin{array}{l} T_{\Lambda\Sigma}^{(a)} = 0; \\ \det(T^{(0)}) = 0, \left\{ \begin{array}{l} A.1] \text{ } Tr(T^{(0)}) > 0; \\ \textit{aut} \\ A.2] \text{ } Tr(T^{(0)}) < 0; \\ \textit{aut} \\ A.3] \text{ } Tr(T^{(0)}) = 0; \end{array} \right. \end{array} \right. \quad (4.56)$$

$$B] \left\{ \begin{array}{l} T_{\Lambda\Sigma}^{(a)} \neq 0; \\ \det(T^{(0)}) = 0, \text{ } Tr(T^{(0)}) = 0. \end{array} \right. \quad (4.57)$$

As mentioned above, each set (**A.1**, **A.2**, **A.3** and **B**) of constraints is $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant, but formulated in terms of the symplectic charge basis \mathcal{P} .

It is interesting to point out that, differently from $\mathcal{N} = 8$, $d = 4$ supergravity treated in Sect. 3, in $\mathcal{N} = 4$, $d = 4$ supergravity there are no “small” doubly-critical (or with higher degree of criticality) charge orbits *independent* from the “small” critical ones. This can be easily seen by noticing that the solutions (4.56)-(4.57) to the “criticality” constraints (4.46) and (4.47) can actually be rewritten in a doubly-critical fashion, *i.e.* through $\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}}$ and related projections (according to decomposition (4.50)). For completeness’ sake, we report here the second order derivatives of $\mathcal{I}_{4,\mathcal{N}=4}$ with respect to the “bare” symplectic charges:

$$\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial p_\Sigma \partial p_\Lambda} = 2(q^2 \eta^{\Lambda\Sigma} - q^\Lambda q^\Sigma); \quad (4.58)$$

$$\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial q_\Sigma \partial q_\Lambda} = 2(p^2 \eta^{\Lambda\Sigma} - p^\Lambda p^\Sigma); \quad (4.59)$$

$$\frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial q_\Sigma \partial p_\Lambda} = 4T^{(a)|\Lambda\Sigma}. \quad (4.60)$$

In order to determine the “small” orbits of the bi-fundamental representation (**2**, **6** + **M**) of the U -duality group $SL(2, \mathbb{R}) \times SO(6, M)$ and to study their supersymmetry-preserving

properties, it is now convenient to switch to the basis of “*dressed*” charges (recall Eqs. (2.22) and (2.23))

$$\mathcal{U} \equiv (\mathcal{Z}, \bar{\mathcal{Z}})^T = \left(Z_{AB}, Z^I, \bar{Z}_{AB}, \bar{Z}^I \right)^T. \quad (4.61)$$

From the analysis of [8], one obtains the following equivalence:

$$T_{\Lambda\Sigma}^{(a)} \equiv \left. \frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{P} \partial \mathcal{P}} \right|_{\mathbf{Adj}(SO(6,M))} = 0 \Leftrightarrow \left. \frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{U} \partial \mathcal{U}} \right|_{\mathbf{Adj}(SO(6,M))} = 0. \quad (4.62)$$

The $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraint (4.62) is common to the “*small*” *critical* charge orbits determined by the solutions **A.1**, **A.2** and **A.3** of Eqs. (4.56). It also implies that $\alpha_1 = \alpha_2$ [8]. Then, the further $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints $Tr(T^{(0)}) \gtrless 0$ can equivalently be rewritten as (recall definition (4.12))

$$Tr(T^{(0)}) \gtrless 0 \Leftrightarrow \mathcal{S}_1 \gtrless 0. \quad (4.63)$$

Therefore, one can characterize the “*small*” *critical* orbits **A.1**, **A.2** and **A.3** of Eqs. (4.48) and (4.56) as follows:

$$A] \left\{ \begin{array}{l} \left. \frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{U} \partial \mathcal{U}} \right|_{\mathbf{Adj}(SO(6,M))} = 0; \\ \mathcal{S}_1^2 = |\mathcal{S}_2|^2, \left\{ \begin{array}{l} A.1] \mathcal{S}_1 > 0; \\ \text{aut} \\ A.2] \mathcal{S}_1 < 0; \\ \text{aut} \\ A.3] \mathcal{S}_1 = 0. \end{array} \right. \end{array} \right. \quad (4.64)$$

Notice that each set (**A.1**, **A.2**, **A.3** and **B**) of constraints is $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant but, differently from Eqs. (4.48) and (4.56), it is also independent from the symplectic basis eventually considered.

On the other hand, the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints (4.49) and (4.57) defining the “*small*” *critical* orbit **B** can be recast in a form which (differently from Eqs. (4.49) and (4.57)) is independent from the symplectic basis eventually considered, as follows:

$$B] \left\{ \begin{array}{l} \left. \frac{\partial^2 \mathcal{I}_{4,\mathcal{N}=4}}{\partial \mathcal{U} \partial \mathcal{U}} \right|_{\mathbf{Adj}(SO(6,M))} \neq 0; \\ \mathcal{S}_1^2 = |\mathcal{S}_2|^2 = 0. \end{array} \right. \quad (4.65)$$

Thus, five distinct “*small*” charge orbits (all with $\mathcal{I}_{4,\mathcal{N}=4} = 0$) exist:

1. The *critical* orbit **A.1** is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution (exhibiting maximal symmetry):

$$z_1 = z_2 \in \mathbb{R}_0^+, \quad \rho_1 = \rho_2 = 0, \quad \theta \text{ undetermined}. \quad (4.66)$$

Thus, from the reasoning performed at the end of Sect. 2 and the analysis of [8], the considered “*small*” *critical* orbit is $\frac{1}{2}$ -BPS. Along the corresponding “*small*” *critical* $\frac{1}{2}$ -BPS flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew-diag.}$ defined in Eq. (4.19) is $USp(4)$, whereas the one of $Z_{I,red.}$ defined in Eq. (4.20) is $SO(M)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **A.1** is $USp(4) \times SO(M)$.

2. The *critical* orbit **A.2** is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution, existing for $M \geq 1$ (and exhibiting maximal symmetry)

$$z_1 = z_2 = 0, \quad \rho_1 \in \mathbb{R}_0^+, \quad \rho_2 = 0. \quad (4.67)$$

Thus, the considered “*small*” *critical* orbit is non-BPS $Z_{AB} = 0$. Along the corresponding “*small*” *critical* non-BPS $Z_{AB} = 0$ flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew-diag.}$ defined in Eq. (4.19) is $SU(4)$, whereas the one of $Z_{I,red.}$ defined in Eq. (4.20) is $SO(M-1)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **A.2** is $SU(4) \times SO(M-1)$.

3. The *critical* orbit **A.3** is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints (4.48) (or (4.56), or (4.64)). Such constraints are solved by the following flow solution, existing for $M \geq 1$ (and exhibiting maximal symmetry)

$$z_1 = z_2 = \frac{1}{\sqrt{2}}\rho_1 \in \mathbb{R}_0^+, \quad \rho_2 = 0. \quad (4.68)$$

Thus, the considered “*small*” *critical* orbit is non-BPS $Z_{AB} \neq 0$. Along the corresponding “*small*” *critical* non-BPS $Z_{AB} \neq 0$ flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew-diag.}$ defined in Eq. (4.19) is $USp(4)$, whereas the one of $Z_{I,red.}$ defined in Eq. (4.20) is $SO(M-1)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **A.3** is $USp(4) \times SO(M-1)$.

4. The *critical* orbit **B** is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints (4.49) (or (4.57), or (4.65)). Such constraints are solved by the following flow solution, existing for $M \geq 2$ (and exhibiting maximal symmetry)

$$z_1 \in \mathbb{R}_0^+, \quad z_2 = 0, \quad \rho_1 = \rho_2 = \frac{z_1}{\sqrt{2}}; \quad (4.69)$$

$$\theta = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}. \quad (4.70)$$

Thus, the considered “*small*” *critical* orbit is non-BPS $Z_{AB} \neq 0$. Along the corresponding “*small*” *critical* non-BPS $Z_{AB} \neq 0$ flow, the (maximal compact) symmetry of the *skew-diagonalized* central charge matrix $Z_{AB,skew-diag.}$ defined in Eq. (4.19) is $(SU(2))^2$, whereas the one of $Z_{I,red.}$ defined in Eq. (4.20) is $SO(M-2)$. Therefore, the resulting maximal compact symmetry of the *critical* orbit **B** is $(SU(2))^2 \times SO(M-2)$.

5. The generic “*small*” *lightlike* case is defined by the $SL(2, \mathbb{R}) \times SO(6, M)$ -invariant constraints (4.45) (or (4.55)). In this case, it is more convenient to consider the symplectic

basis of “bare” charges \mathcal{P} and, in order to determine the maximal compact symmetry of the flow solution(s), one can consider the saturation of the bound (4.45), namely:

$$p^2 q^2 = (p \cdot q)^2 = 0. \quad (4.71)$$

This is in general solved by $p^2 = 0$, $p \cdot q = 0$ and $q^2 \neq 0$ (or equivalently by $q^2 = 0$, $p \cdot q = 0$ and $p^2 \neq 0$). It is easy to realize that the maximal compact symmetry of the flow solution is $SO(4) \times SO(M-1)$ in the case $q^2 > 0$, and $SO(5) \times SO(M-2)$ in the case $q^2 < 0$. In the first case the solution exists for $M \geq 1$, whereas in the second case the solution exists for $M \geq 2$. Thus, one actually gets two generic “small” *lightlike* orbits, both non-BPS $Z_{AB} \neq 0$, with maximal compact symmetry respectively given by $SO(4) \times SO(M-1)$ and $SO(5) \times SO(M-2)$.

Mutatis mutandis, the same considerations made at the end of Sect. 3 for $\mathcal{N} = 8$, $d = 4$ supergravity also hold for $\mathcal{N} = 4$, $d = 4$ *matter coupled* supergravity.

Finally, it is worth noticing that the $U(1)$ (stabilizer of the factor $\frac{SL(2, \mathbb{R})}{U(1)}$ of the scalar manifold (4.1)) is broken both in “large” and “small” charge orbits, because both the *central charge matrix* Z_{AB} and the *matter charges* Z_I are charged with respect to it.

5 $\mathcal{N} = 2$

In $\mathcal{N} = 2$, $d = 4$ supergravity one can repeat the analysis of [1, 40] (see also [41]), by using the properties of *special Kähler geometry* (SKG, see *e.g.* [22], and Refs. therein). Indeed, in SKG one can define an $Sp(2n, \mathbb{R})$ matrix over the scalar manifold (as in Eq. (2.9)), as well complex matrices f and h (as in Eqs. (2.10)-(2.14)), without the need for the manifold to be necessarily a(n *at least locally*) symmetric space (see *e.g.* [21, 12]).

The basic identities of SKG applied to the (covariantly holomorphic) $\mathcal{N} = 2$, $d = 4$ *central charge* section

$$Z \equiv e^{K/2} (X^\Lambda q_\Lambda - F_\Lambda p^\Lambda) \quad (5.1)$$

of the $U(1)$ Kähler-Hodge bundle (with Kähler weights $(1, -1)$) read as follows [19] ($i, \bar{j} = 1, \dots, n-1$, with $n-1$ denoting the number of Abelian vector multiplets coupled to the supergravity one)

$$\bar{D}_{\bar{i}} Z = 0; \quad (5.2)$$

$$D_i D_j Z = i C_{ijk} g^{k\bar{k}} \bar{D}_{\bar{k}} \bar{Z}; \quad (5.3)$$

$$\bar{D}_{\bar{j}} D_i Z = g_{i\bar{j}} Z, \quad (5.4)$$

where (X^Λ, F_Λ) are the holomorphic symplectic sections of the $U(1)$ Kähler-Hodge bundle (with Kähler weights $(2, 0)$), and K denotes the Kähler potential of the Abelian vector multiplets’ scalar manifold, with metric $g_{i\bar{j}} = \bar{\partial}_{\bar{j}} \partial_i K$. C_{ijk} is the rank-3 symmetric and covariantly holomorphic *C-tensor* of SKG (see *e.g.* [22], and Refs. therein):

$$\bar{D}_{\bar{l}} C_{ijk} = 0; \quad (5.5)$$

$$D_{[l} C_{i]jk} = 0. \quad (5.6)$$

Thus, in $\mathcal{N} = 2$, $d = 4$ supergravity coupled to $n - 1$ Abelian vector multiplets, the “*BH potential*” is given by [17, 18]

$$V_{BH}(\phi, \mathcal{P}) = Z\bar{Z} + g^{i\bar{j}}(D_i Z)\bar{D}_{\bar{j}}\bar{Z}, \quad (5.7)$$

and the *Attractor Eqs.* read [19]

$$\partial_i V_{BH} = 0 \Leftrightarrow 2\bar{Z}D_i Z + iC_{ijk}g^{j\bar{j}}g^{k\bar{k}}(\bar{D}_{\bar{j}}\bar{Z})\bar{D}_{\bar{k}}\bar{Z} = 0. \quad (5.8)$$

1. The ($\frac{1}{2}$ -BPS) supersymmetric solution to *Attractor Eqs.* (5.8) is determined by

$$(D_i Z)_{\frac{1}{2}\text{-BPS}} = 0, \quad \forall i, \quad (5.9)$$

and therefore Eq. (5.7) yields

$$V_{BH, \frac{1}{2}\text{-BPS}} = |Z|_{\frac{1}{2}\text{-BPS}}^2, \quad (5.10)$$

and the corresponding Hessian matrix of V_{BH} has block components given by [19]

$$(D_i \partial_j V_{BH})_{\frac{1}{2}\text{-BPS}} = (\partial_i \partial_j V_{\partial BH})_{\frac{1}{2}\text{-BPS}} = 0; \quad (5.11)$$

$$(\partial_i \bar{\partial}_{\bar{j}} V_{BH})_{\frac{1}{2}\text{-BPS}} = 2g_{ij, \frac{1}{2}\text{-BPS}} |Z|_{\frac{1}{2}\text{-BPS}}^2, \quad (5.12)$$

showing that there are no “*flat*” *directions* for such the ($\frac{1}{2}$ -)BPS class of solutions to *Attractor Eqs.* (5.8) [33].

2. Non-supersymmetric (non-BPS) solutions to *Attractor Eqs.* (5.8) have $D_i Z \neq 0$ (at least) for some $i \in \{1, \dots, n - 1\}$. Generally, such solutions fall into two class [6], and they exhibit “*flat*” *directions* of V_{BH} itself [33]. The non-BPS, $Z = 0$ class is defined by the following constraints:

$$D_i Z = \partial_i Z \neq 0, \quad \text{for some } i, \quad Z = 0, \quad (5.13)$$

thus yielding (from Eqs. (5.8))

$$\left[C_{ijk} g^{j\bar{j}} g^{k\bar{k}} (\bar{\partial}_{\bar{j}} \bar{Z}) \bar{\partial}_{\bar{k}} \bar{Z} \right]_{\text{non-BPS}, Z=0} = 0. \quad (5.14)$$

Thus, Eqs. (5.7) and (5.13) yield

$$V_{BH, \text{non-BPS}, Z=0} = \left[g^{i\bar{j}} (D_i Z) \bar{D}_{\bar{j}} \bar{Z} \right]_{\text{non-BPS}, Z=0} = \left[g^{i\bar{j}} (\partial_i Z) \bar{\partial}_{\bar{j}} \bar{Z} \right]_{\text{non-BPS}, Z=0}. \quad (5.15)$$

3. The non-BPS, $Z \neq 0$ class is defined by the following constraints:

$$D_i Z \neq 0, \quad \text{for some } i, \quad Z \neq 0. \quad (5.16)$$

It is worth remarking that Eqs. (5.8) and the non-BPS $Z \neq 0$ defining constraints (5.16) imply the following relations to hold at the non-BPS $Z \neq 0$ critical points of V_{BH} [12]:

$$\left[g^{i\bar{j}} (D_i Z) \bar{D}_{\bar{j}} \bar{Z} \right]_{non-BPS, Z \neq 0} = -\frac{i}{2} \left[\frac{N_3(\bar{Z})}{\bar{Z}} \right]_{non-BPS, Z \neq 0} = \frac{i}{2} \left[\frac{\bar{N}_3(Z)}{Z} \right]_{non-BPS, Z \neq 0}, \quad (5.17)$$

where the cubic form $N_3(\bar{Z})$ is defined as [12]

$$N_3(\bar{Z}) \equiv C_{ijk} \bar{Z}^i \bar{Z}^j \bar{Z}^k \Leftrightarrow \bar{N}_3(Z) \equiv \bar{C}_{\bar{i}\bar{j}\bar{k}} Z^{\bar{i}} Z^{\bar{j}} Z^{\bar{k}}. \quad (5.18)$$

For an arbitrary SKG, it is in general hard to compute

$$\frac{S_{BH}}{\pi} = V_{BH}|_{\partial_\phi V_{BH}=0} = V_{BH}(\phi_H(\mathcal{P}), \mathcal{P}), \quad (5.19)$$

where $\phi_H(\mathcal{P})$ are the horizon scalar configurations solving the *Attractor Eqs.* (5.8). However, the situation dramatically simplifies for *symmetric* SK manifolds

$$\frac{G_4}{H_4}, \quad (5.20)$$

in which case a classification, analogous to the one available for $\mathcal{N} > 2$ -extended, $d = 4$ supergravities (see *e.g.* [12] and Refs. therein; see also Sects. 3 and 4) can be performed [6].

In the treatment below, we are going to give a remarkable general *topological* formula for $V_{BH}(\phi_H(\mathcal{P}), \mathcal{P})$ for *symmetric* SKG, which is manifestly *invariant* under diffeomorphisms of the SK scalar manifold, and which holds for *any* choice of symplectic basis of “bare” charges \mathcal{P} and of *special coordinates* (see *e.g.* [22] and Refs. therein) of the SK manifold itself. Indeed, such a formula by no means does refer to *special coordinates*, which may not even exist for certain parametrizations of $\frac{G_4}{H_4}$ itself.

It should be pointed out that a general formula for the G_4 -invariant $\mathcal{I}_{4, \mathcal{N}=2}$ is known for the so-called d -SK homogeneous *symmetric* manifolds [26], and it reads ($a = 1, \dots, n-1$) [4]:

$$\mathcal{I}_{4, \mathcal{N}=2}(\mathcal{P}) = - (p^0 q_0 + p^a q_a)^2 + 4 [q_0 \mathcal{I}_{3, \mathcal{N}=2}(p) - p^0 \mathcal{I}_{3, \mathcal{N}=2}(q) + \{\mathcal{I}_{3, \mathcal{N}=2}(p), \mathcal{I}_{3, \mathcal{N}=2}(q)\}], \quad (5.21)$$

where

$$\mathcal{I}_{3, \mathcal{N}=2}(p) \equiv \frac{1}{3!} d_{abc} p^a p^b p^c; \quad (5.22)$$

$$\mathcal{I}_{3, \mathcal{N}=2}(q) \equiv \frac{1}{3!} d^{abc} q_a q_b q_c; \quad (5.23)$$

$$\{\mathcal{I}_{3, \mathcal{N}=2}(p), \mathcal{I}_{3, \mathcal{N}=2}(q)\} \equiv \frac{\partial \mathcal{I}_{3, \mathcal{N}=2}(p)}{\partial p^a} \frac{\partial \mathcal{I}_{3, \mathcal{N}=2}(q)}{\partial q_a}, \quad (5.24)$$

in which the constant (number) rank-3 symmetric tensor d_{abc} has been introduced (and d^{abc} is its suitably defined completely contravariant form). However, such a formula holds for a particular symplectic basis (namely the one inherited from the $\mathcal{N} = 2$, $d = 5$ theory, *i.e.* the one of *special coordinates*), in which the holomorphic prepotential $F(X)$ of SKG can be written as

$$F(X) \equiv \frac{1}{3!} d_{abc} \frac{X^a X^b X^c}{X^0}. \quad (5.25)$$

In such a symplectic basis, the manifest symmetry is the $d = 5$ U -duality G_5 , under which G_4 branches as $G_4 \rightarrow G_5 \times SO(1, 1)$. Indeed, $\mathcal{I}_{3, \mathcal{N}=2}(p)$ and $\mathcal{I}_{3, \mathcal{N}=2}(q)$ are nothing but respectively the *magnetic* and *electric* invariants (both *cubic* in \mathcal{P}) of the relevant symplectic representations of G_5 .

Eq. (5.21) excludes the so-called *quadratic* (or *minimally coupled* [42]) sequence of symmetric SK manifolds (particular *complex Grassmannians*)

$$\frac{SU(1, n-1)}{SU(n-1) \times U(1)}, \quad n \in \mathbb{N} \quad (5.26)$$

(not upliftable to $d = 5$), for which $F(X)$ is given by (in the symplectic basis exhibiting the maximal non-compact symmetry $SU(1, n-1)$)

$$F(X) = -\frac{i}{2} \left[(X^0)^2 - \sum_{i=1}^{n-1} (X^i)^2 \right], \quad (5.27)$$

and the invariant of the symplectic representation of $G_4 = SU(1, n-1)$ reads as follows (notice it is *quadratic* in \mathcal{P}) [29]:

$$\mathcal{I}_{2, \mathcal{N}=2}(\mathcal{P}) = (p^0)^2 + q_0^2 - \sum_{i=1}^{n-1} \left((p^i)^2 + q_i^2 \right) = |Z|^2 - g^{ij} (D_i Z) \overline{D_{\bar{j}} \overline{Z}}. \quad (5.28)$$

Due to the *quadratic* nature of the G_4 -invariant $\mathcal{I}_{2, \mathcal{N}=2}(\mathcal{P})$ given by Eq. (5.28), the *quadratic* sequence of symmetric SK manifolds (5.26) exhibits only one “*small*” charge orbit, namely the *lightlike* one, beside the two “*large*” charge orbits determined in [6].

The symmetric SK manifolds whose geometry is determined by the holomorphic prepotential function (5.25) and the *minimally coupled* ones determined by Eq. (5.27) are *all* the possible symmetric SK manifolds. After [43], from the geometric perspective of SKG, symmetric SK manifolds can be characterized in the following way.

In SKG the Riemann tensor obeys to the following constraint (see *e.g.* [22] and Refs. therein):

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j}} g_{k\bar{l}} - g_{i\bar{l}} g_{k\bar{j}} + C_{ikm} \overline{C_{\bar{l}\bar{j}\bar{n}}} g^{m\bar{n}}. \quad (5.29)$$

The requirement that the manifold to be symmetric demands the Riemann to be covariantly constant:

$$D_m R_{i\bar{j}k\bar{l}} = 0. \quad (5.30)$$

Due to the SKG constraint (5.29) and to covariant holomorphicity of the C -tensor (expressed by Eq. (5.5)), Eq. (5.30) generally implies (for non-vanishing C_{ijk})

$$D_l C_{ijk} = D_{(l} C_{i)jk} = 0, \quad (5.31)$$

where in the last step Eq. (5.6) was used. Thus, in a SK symmetric space both the Riemann tensor and the C -tensor are covariantly constant. Eq. (5.31) implies the following relation [6]

$$C_{j(lm} C_{pq)k} \bar{C}_{\bar{i}\bar{j}\bar{k}} g^{\bar{j}\bar{j}} g^{k\bar{k}} = \frac{4}{3} C_{(lmp} g_{q)\bar{i}}, \quad (5.32)$$

which is nothing but the “*dressed*” form of the analogous relation holding for the d -tensor itself [44, 43]

$$d_{j(lm} d_{pq)k} d^{ijk} = \frac{4}{3} d_{(lmp} \delta_{q)}^i. \quad (5.33)$$

The quadratic sequence of symmetric manifolds (5.26) whose SKG is determined by the prepotential (5.27) has

$$C_{ijk} = 0, \quad (5.34)$$

whereas the remaining symmetric SK manifolds, whose prepotential in the special coordinates is given by Eq. (5.25) (with d_{abc} constrained by Eq. (5.33)), correspond to

$$C_{abc} = e^K d_{abc}. \quad (5.35)$$

By using Eqs. (5.31) and (5.32), as well as the SKG identities (5.2)-(5.4) (which, for symmetric SKG, are equivalent to the *Maurer-Cartan Eqs.*, as Eqs. (3.3) and (4.4)-(4.5) for $\mathcal{N} = 8$ and $\mathcal{N} = 4$, $d = 4$ supergravities, respectively; see *e.g.* [29, 21]), one can prove that the following *quartic* expression is a *duality* invariant for all symmetric SK manifolds :

$$\begin{aligned} \mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{P}) &= \left(Z \bar{Z} - Z_i \bar{Z}^i \right)^2 + \\ &+ \frac{2}{3} i \left(Z N_3(\bar{Z}) - \bar{Z} \bar{N}_3(Z) \right) + \\ &- g^{\bar{i}\bar{i}} C_{ijk} \bar{C}_{\bar{i}\bar{j}\bar{k}} \bar{Z}^{\bar{j}} \bar{Z}^{\bar{k}} Z^{\bar{l}} Z^{\bar{m}}, \end{aligned} \quad (5.36)$$

where the *matter charges* have been re-noted as $Z_i \equiv D_i Z$, $Z^{\bar{i}} = g^{\bar{j}\bar{i}} Z_{\bar{j}}$, and definition (5.18) was recalled.

As claimed above, $\mathcal{I}_{4,\mathcal{N}=2,symm}$ given by Eq. (5.36) is ϕ -dependent *only apparently*, *i.e.* it is *topological*, merely charge-dependent:

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=2,symm}(\phi, \mathcal{P})}{\partial \phi} = 0 \Leftrightarrow \mathcal{I}_{4,\mathcal{N}=2,symm} = \mathcal{I}_{4,\mathcal{N}=2,symm}(\mathcal{P}). \quad (5.37)$$

Thus, by recalling Eq. (1.5), the general entropy-area formula [20] for extremal BHs in $\mathcal{N} = 2$, $d = 4$ supergravity coupled to Abelian vector multiplets whose scalar manifold is a symmetric (SK) space reads as follows:

$$\frac{S_{BH}}{\pi} = V_{BH}|_{\partial_\phi V_{BH}=0} = V_{BH}(\phi_H(\mathcal{P}), \mathcal{P}) = |\mathcal{I}_{4,\mathcal{N}=2,symm}(\mathcal{P})|^{1/2}. \quad (5.38)$$

Let us briefly analyze Eq. (5.36).

As for the case of $\mathcal{N} = 8$, $d = 4$ supergravity treated in Sect. 3, one can introduce a phase ϑ as follows (recall definitions (5.18)):

$$e^{2i\vartheta} \equiv -\frac{ZN_3(\bar{Z})}{\bar{Z}N_3(Z)} = \frac{iZC_{ijk}\bar{Z}^i\bar{Z}^j\bar{Z}^k}{-i\bar{C}_{\bar{l}\bar{m}\bar{n}}Z^{\bar{l}}Z^{\bar{m}}Z^{\bar{n}}}. \quad (5.39)$$

Thus, ϑ is the phase of the quantity $iZN_3(\bar{Z})$: $\vartheta \equiv \vartheta_{iZN_3(\bar{Z})}$. It is then immediate to compute ϑ from Eq. (5.36):

$$\cos\vartheta(\phi, \mathcal{P}) = \frac{3 \left[\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}(\mathcal{P}) - \left(Z\bar{Z} - Z_i\bar{Z}^i \right)^2 + g^{i\bar{i}}C_{ijk}\bar{C}_{\bar{i}\bar{l}\bar{m}}\bar{Z}^j\bar{Z}^k Z^{\bar{l}}Z^{\bar{m}} \right]}{2^2 |ZN_3(\bar{Z})|}. \quad (5.40)$$

Notice that through Eq. (5.40) $(\cos)\vartheta$ is determined in terms of the scalar fields ϕ and of the BH charges \mathcal{P} , also along the “small” orbits where $\mathcal{I}_{4,\mathcal{N}=2,\text{symm}} = 0$. However, Eq. (5.40) is not defined in the cases in which $ZN_3(\bar{Z}) = 0$. In such cases, ϑ is actually undetermined. It should be clearly pointed out that the phase ϑ has nothing to do with the phase of the $U(1)$ bundle over the SK-Hodge vector multiplets’ scalar manifold (see *e.g.* [22] and Refs. therein).

1. For $\frac{1}{2}$ -BPS attractors (defined by the constraints (5.9)), Eq. (5.36) yields

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\frac{1}{2}\text{-BPS}} = (Z\bar{Z})_{\frac{1}{2}\text{-BPS}}^2 = |Z|_{\frac{1}{2}\text{-BPS}}^4, \quad (5.41)$$

as in turn also implied by Eqs. (5.10) and (1.5) (or equivalently (5.38)). Notice that Eqs. (5.10) and (5.41) are *general, i.e.* they hold for any SKG, regardless the symmetric nature of the SK vector multiplets’ scalar manifold. Furthermore, the constraints (5.9) imply that at the event horizon of $\frac{1}{2}$ -BPS extremal BHs it holds

$$[N_3(\bar{Z})]_{\frac{1}{2}\text{-BPS}} = 0 \Rightarrow \vartheta_{\frac{1}{2}\text{-BPS}} \text{ undetermined}. \quad (5.42)$$

2. For non-BPS $Z = 0$ attractors (defined by the constraints (5.13) which, through Eqs. (5.8), imply Eq. (5.14)), Eq. (5.36) yields

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\text{non-BPS},Z=0} = \left(Z_i\bar{Z}^i \right)_{\text{non-BPS},Z=0}^2 = \left[g^{i\bar{j}}(\partial_i Z)\bar{\partial}_{\bar{j}}\bar{Z} \right]_{\text{non-BPS},Z=0}^2. \quad (5.43)$$

Notice that Eqs. (5.15) and (5.43) are *general, i.e.* they hold for any SKG, regardless the symmetric nature of the SK vector multiplets’ scalar manifold. Furthermore, the constraints (5.9) imply that at the event horizon of non-BPS $Z = 0$ extremal BHs it holds

$$Z_{\text{non-BPS},Z=0} = 0 \Rightarrow \vartheta_{\text{non-BPS},Z=0} \text{ undetermined}. \quad (5.44)$$

3. For non-BPS $Z \neq 0$ attractors (defined by the constraints (5.16) as well as by Eqs. (5.8)), Eqs. (5.36) and (5.17) yield

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\text{non-BPS},Z \neq 0} = -16 |Z|_{\text{non-BPS},Z \neq 0}^4, \quad (5.45)$$

thus implying, through Eq. (5.7) [46, 30, 6, 12]

$$Z_i \bar{Z}^i \Big|_{\text{non-BPS},Z \neq 0} = 3 |Z|_{\text{non-BPS},Z \neq 0}^2 \Leftrightarrow V_{BH,\text{non-BPS},Z \neq 0} = 4 |Z|_{\text{non-BPS},Z \neq 0}^2. \quad (5.46)$$

By plugging Eqs. (5.8), (5.16), (5.17) and (5.45) into Eq. (5.40), it follows that at the event horizon of non-BPS $Z \neq 0$ extremal BHs it holds that

$$\vartheta_{\text{non-BPS},Z \neq 0} = \pi + 2k\pi, \quad k \in \mathbb{Z}. \quad (5.47)$$

It should be remarked that, differently from the results (5.10)-(5.12), (5.41)-(5.42) (holding for $\frac{1}{2}$ -BPS attractors) and from the results (5.14)-(5.15), (5.43)-(5.44) (holding for non-BPS $Z = 0$ attractors), Eqs. (5.45)-(5.47) are not *general*: *i.e.* they hold at the event horizon of extremal non-BPS $Z \neq 0$ BHs for symmetric SK manifolds, but they do not hold true for generic SKG. However, when going *beyond* the symmetric SK case (and thus encompassing both homogeneous non-symmetric [26, 45] and non-homogeneous SK spaces), one can compute both $V_{BH,\text{non-BPS},Z \neq 0}$ and $\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\text{non-BPS},Z \neq 0}$, and express the deviation from the symmetric case considered above in terms of the complex quantity [12]

$$\Delta \equiv -\frac{3}{4} \frac{E_{i\bar{j}k\bar{l}m} \bar{Z}^i Z^{\bar{j}} Z^{\bar{k}} Z^{\bar{l}} Z^{\bar{m}}}{\bar{N}_3(Z)}, \quad (5.48)$$

where the tensor $E_{i\bar{j}k\bar{l}m}$ was firstly introduced in [26] (see also [12]). The results of straightforward computations read as follows:

$$V_{BH,\text{non-BPS},Z \neq 0} = 4 |Z|_{\text{non-BPS},Z \neq 0}^2 + \Delta_{\text{non-BPS},Z \neq 0}; \quad (5.49)$$

$$\mathcal{I}_{4,\mathcal{N}=2,\text{symm}}|_{\text{non-BPS},Z \neq 0} = \left[-16 |Z|^4 + \Delta^2 - \frac{8}{3} \Delta |Z|^2 \right]_{\text{non-BPS},Z \neq 0}. \quad (5.50)$$

Notice that, as yielded *e.g.* by Eq. (5.49), Δ is real at the non-BPS $Z \neq 0$ critical points of V_{BH} . For symmetric SK manifolds $E_{i\bar{j}k\bar{l}m} = 0$ globally, and thus Eqs.(5.49) and (5.50) respectively reduce to Eqs. (5.46) and (5.45). On the other hand, the results (5.45)-(5.46) hold also for those non-symmetry SK spaces ($E_{i\bar{j}k\bar{l}m} \neq 0$) such that

$$\Delta_{\text{non-BPS},Z \neq 0} = 0 \Leftrightarrow \left(E_{i\bar{j}k\bar{l}m} \bar{Z}^i Z^{\bar{j}} Z^{\bar{k}} Z^{\bar{l}} Z^{\bar{m}} \right)_{\text{non-BPS},Z \neq 0}, \quad (5.51)$$

where in the implication “ \Rightarrow ” the assumption $[\bar{N}_3(Z)]_{\text{non-BPS},Z \neq 0} \neq 0$ was made. The condition (5.51) might explain some results obtained for generic ($d-$)SKGs in some particular supporting BH charge configurations in [46] (see also the treatment in [12] and [39]).

Consistently, for the quadratic *minimally coupled* sequence (5.26), for which Eq. (5.34) holds, Eq. (5.36) formally reduces to

$$\begin{aligned} \mathcal{I}_{4,\mathcal{N}=2,symm}|_{C_{ijk}=0} &= \left(Z\bar{Z} - Z_i\bar{Z}^i \right)^2; \\ &\Downarrow \\ \left| \mathcal{I}_{4,\mathcal{N}=2,symm}|_{C_{ijk}=0} \right|^{1/2} &= |\mathcal{I}_{2,\mathcal{N}=2}|, \end{aligned} \quad (5.52)$$

where $\mathcal{I}_{2,\mathcal{N}=2}$ is given by Eq. (5.28).

Remarkably, Eq. (5.36) turns out to be directly related to the quantity $-h$ given by Eq. (2.31) of [26] (see also the treatment of [47]). This is seen by noticing that Eq. (4.42) of [26] coincides with Eq. (5.21) (along with definitions (5.22)-(5.24)). Note that the mapping of quaternionic coordinates $(A^\Lambda, B_\Lambda)^T$ into the charges $\mathcal{P}^T = (p^\Lambda, q_\Lambda)^T$ (in *special coordinates*) is related to the $d = 3$ attractor flows (see *e.g.* [48, 49, 50]).

For *symmetric* SK manifolds, “*small*” charge orbits of the symplectic representation of G_4 are known to exist since [4] and [5]. As evident from Eq. (5.36), “*small*” charge orbits in *symmetric* SK manifolds all are non-BPS. This is different from the cases of $\mathcal{N} = 8$ and $\mathcal{N} = 4, d = 4$ supergravities. As reviewed in Sect. 3, for $\mathcal{N} = 8$ all “*small*” orbits are BPS, whereas, as treated in Sect. 4, for $\mathcal{N} = 4$ “*small*” orbits can be either BPS or non-BPS.

- “*Small*” *lightlike* charge orbits are defined by the G_4 -invariant constraint

$$\mathcal{I}_{4,\mathcal{N}=2,symm} = 0; \quad (5.53)$$

\Downarrow

$$\left(Z\bar{Z} - Z_i\bar{Z}^i \right)^2 + \frac{2}{3}i \left(ZN_3(\bar{Z}) - \bar{Z}N_3(Z) \right) = g^{\bar{i}i} C_{ijk} \bar{C}_{\bar{i}\bar{m}} \bar{Z}^j \bar{Z}^k Z^{\bar{l}} Z^{\bar{m}}. \quad (5.54)$$

In this case, Eq. (5.40) reduces to

$$\cos\vartheta(\phi, \mathcal{P})|_{\mathcal{I}_{4,\mathcal{N}=2,symm}=0} = - \frac{3 \left[\left(Z\bar{Z} - Z_i\bar{Z}^i \right)^2 - g^{\bar{i}i} C_{ijk} \bar{C}_{\bar{i}\bar{m}} \bar{Z}^j \bar{Z}^k Z^{\bar{l}} Z^{\bar{m}} \right]}{2^2 |ZN_3(\bar{Z})|} \Bigg|_{\mathcal{I}_{4,\mathcal{N}=2,symm}=0}. \quad (5.55)$$

- Beside the constraint (5.53)-(5.54), “*small*” *critical* charge orbits are defined by the following G_4 -invariant set of first order differential constraints, as well:

$$\frac{\partial \mathcal{I}_{4,\mathcal{N}=2,symm}}{\partial Z} = 0 = \frac{\partial \mathcal{I}_{4,\mathcal{N}=2,symm}}{\partial Z_i}. \quad (5.56)$$

- Beside the constraints (5.53)-(5.54) and (5.56), “*small*” *doubly-critical* charge orbits are also defined by the following set of second order differential constraints, as well:

$$\mathcal{D}_{i\bar{j}} \mathcal{I}_{4,\mathcal{N}=2,symm} = 0 = \mathcal{D}_i \mathcal{I}_{4,\mathcal{N}=2,symm}, \quad (5.57)$$

where the second-order differential operators $\mathcal{D}_{i\bar{j}}$ and \mathcal{D}_i have been introduced:

$$\mathcal{D}_{i\bar{j}} \equiv R_{i\bar{j}k}{}^l \frac{\partial}{\partial Z_k} \frac{\partial}{\partial \bar{Z}^l}; \quad (5.58)$$

$$\mathcal{D}_i \equiv C_{ijk} \frac{\partial}{\partial Z_j} \frac{\partial}{\partial Z_k}. \quad (5.59)$$

Notice that, through the definitions (5.58) and (5.59), the constraints (5.57) are G_4 -invariant, because they are equivalent to the following constraint:

$$\left. \frac{\partial^2 \mathcal{I}_{4, \mathcal{N}=2, \text{symm}}}{\partial Z_{\text{symp}(G_4)} \partial \bar{Z}_{\text{symp}(G_4)}} \right|_{\text{Adj}(G_4)} = 0, \quad (5.60)$$

where

$$Z_{\text{symp}(G_4)} \equiv (Z, \bar{Z}_{i\bar{j}}, \bar{Z}, Z_i)^T, \quad (5.61)$$

and the change of charge basis between the manifestly H_4 -covariant (in “flat” local coordinates) basis $Z_{\text{symp}(G_4)}$ and the manifestly $Sp(2n, \mathbb{R})$ -covariant basis \mathcal{P} (defined by Eq. (1.2)) is expressed by the fundamental *identities* of the SKG (see *e.g.* [51, 22] and Refs. therein). Indeed, by considering the *Cartan decomposition* of the Lie algebra of G_4 :

$$\mathfrak{g}_4 = \mathfrak{h}_4 + \mathfrak{k}_4, \quad (5.62)$$

and switching to “flat” local coordinates in the scalar manifold (here denoted by capital Latin indices), it holds that \mathcal{D}_I (“flat” version of the operator defined in Eq. (5.59)) is \mathfrak{k}_4 -valued. Furthermore, in symmetric manifolds $R_{I\bar{J}K}{}^L$ is a two-form (in the first two “flat” local indices) which is Lie algebra-valued in \mathfrak{h}_4 , and thus $\mathcal{D}_{I\bar{J}}$ (“flat” version of the operator defined in Eq. (5.58)) turns out to be \mathfrak{h}_4 -valued. Notice that Eq. (5.60), G_4 -invariantly defining the “small” doubly-critical charge orbit(s) of the $\mathcal{N} = 2$, $d = 4$ vector multiplets’ symmetric SK scalar manifolds, is the analogue of Eq. (3.42), which defines in an $E_{7(7)}$ -invariant way the “small” doubly-critical charge orbit of $\mathcal{N} = 8$, $d = 4$ pure supergravity. It should be also recalled that in $\mathcal{N} = 4$, $d = 4$ matter coupled supergravity “small” doubly-critical (or higher-order-critical) charge orbits (independent from the “small” critical ones) are absent. As treated in Sect. 4, all “small” critical charge orbits of the $\mathcal{N} = 4$ theory actually are doubly-critical, and the analogues of Eqs. (3.42) and (5.60) are given, through Eq. (4.50) and definitions (4.51) and (4.53), by the rich case study exhibited by Eqs. ((4.48)-(4.49) and (4.56)-(4.57)).

The classification of “small” charge orbits of the relevant symplectic representation of G_4 for $\mathcal{N} = 2$, $d = 4$ supergravity coupled to Abelian vector multiplets whose scalar manifold $\frac{G_4}{H_4}$ is (SK) symmetric, performed in accordance to their “order of criticality” (*lightlike*, *critical*, *doubly-critical*), will be given elsewhere.

6 ADM Mass for BPS Extremal Black Hole States

For BPS BH states in $d = 4$ ungauged⁸ supergravity theories, the *ADM mass* [27] $M_{ADM}(\phi_\infty, \mathcal{P})$ is defined as the largest (of the absolute values) of the *skew-eigenvalues* of the (spatially asymptotically) *central charge matrix* $Z_{AB}(\phi_\infty, \mathcal{P})$ which saturate the *BPS bound* (2.28). The *skew-diagonalization* of Z_{AB} is made by performing a suitable transformation of the \mathcal{R} -*symmetry*, and thus by going to the so-called *normal frame*. In such a frame, the *skew-eigenvalues* of Z_{AB} can be taken to be real and positive (up to an eventual overall *phase*). By saturating the *BPS bound* (2.28), it therefore holds that

$$M_{ADM}(\phi_\infty, \mathcal{P}) = |\mathbf{Z}_1(\phi_\infty, \mathcal{P})| \geq \dots \geq |\mathbf{Z}_{[\mathcal{N}/2]}(\phi_\infty, \mathcal{P})|, \quad (6.1)$$

where $\mathbf{Z}_1(\phi, \mathcal{P}), \dots, \mathbf{Z}_{[\mathcal{N}/2]}(\phi, \mathcal{P})$ denote the set of *skew-eigenvalues* of $Z_{AB}(\phi, \mathcal{P})$, and square brackets denote the integer part of the enclosed number. As mentioned at the end of Sect. 2, if $1 \leq \mathbf{k} \leq [\mathcal{N}/2]$ of the bounds expressed by Eq. (2.28) are saturated, the corresponding extremal BH state is named to be $\frac{\mathbf{k}}{\mathcal{N}}$ -BPS. Thus, the minimal fraction of total supersymmetries (pertaining to the asymptotically flat space-time metric) preserved by the extremal BH background within the considered assumptions is $\frac{1}{\mathcal{N}}$ (for $\mathbf{k} = 1$), while the maximal one is $\frac{1}{2}$ (for $\mathbf{k} = \frac{\mathcal{N}}{2}$).

The ADM mass and its symmetries are different, depending on \mathbf{k} .

6.1 $\mathcal{N} = 8$

In $\mathcal{N} = 8$, $d = 4$ supergravity (treated in Sect. 3), the $E_{7(7)}$ U -duality symmetry only allows the cases [3] $\mathbf{k} = 1, 2, 4$. By recalling the review given in Sect. 3, the maximal compact symmetries of the supporting charge orbits respectively read [3, 4, 30, 12, 32, 33]

$$\mathbf{k} = 1 : SU(2) \times SU(6); \quad (6.2)$$

$$\mathbf{k} = 2 : USp(4) \times SU(4); \quad (6.3)$$

$$\mathbf{k} = 4 : USp(8), \quad (6.4)$$

and they hold all along the respective scalar flows. While cases $\mathbf{k} = 2$ and 4 are “*small*” (thus not enjoying the *attractor mechanism*), case $\mathbf{k} = 1$ can be either “*large*” or “*small*”.

In the “*large*” $\mathbf{k} = 1$ case, the *attractor mechanism* makes the maximal compact symmetry $SU(2) \times SU(6)$ of the supporting charge orbit $\mathcal{O}_{\frac{1}{8}\text{-BPS, large}}$ fully manifest as a symmetry of the central charge matrix Z_{AB} through the *symmetry enhancement* (3.17) at the event horizon of the considered extremal BH.

Furthermore, the $\frac{1}{4}$ -BPS saturation of the $\mathcal{N} = 8$ BPS bound (all along the $\frac{1}{4}$ -BPS scalar flow) has the following peculiar structure (recall Eq. (3.35)) [3]

$$|\mathbf{Z}_1(\phi, \mathcal{P})| = |\mathbf{Z}_2(\phi, \mathcal{P})| > |\mathbf{Z}_3(\phi, \mathcal{P})| = |\mathbf{Z}_4(\phi, \mathcal{P})|, \quad (6.5)$$

⁸In the present paper only ungauged supergravities are treated. It is here worth remarking that the definition of the *ADM mass* for (eventually rotating) asymptotically non-flat black holes in *gauged* supergravities is a fairly subtle issue, addressed by various studies in literature (see *e.g.* [53, 54], and Refs. therein).

where it should be recalled that in Sect. 3 the notation $e_i \equiv |\mathbf{Z}_i|$ ($i = 1, \dots, 4$) was used.

As done in Sect. 3, let us denote with λ_i ($i = 1, \dots, 4$) the four real non-negative eigenvalues of the 8×8 Hermitian matrix $Z_{AB}\bar{Z}^{CB} = (ZZ^\dagger)_A^C \equiv A_A^C$. Their relation with the absolute values of the complex *skew-eigenvalues* e_i of Z_{AB} is given by Eq. (3.29). As mentioned, the ordering $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ does not imply any loss of generality. After [8] (see in particular Eqs. (4.74), (4.75), (4.86) and (4.87) therein), the explicit expression of λ_i in terms of $U(8)$ -invariants (namely of TrA , $Tr(A^2)$, $Tr(A^3)$ and $Tr(A^4)$, and suitable powers) is known, and it can be thus be used in order to compute the ADM mass of $\frac{k}{8}$ -BPS extremal BH states of $\mathcal{N} = 8$, $d = 4$ supergravity.

The λ_i 's are solution of the (square root of) *characteristic equation* [8]

$$\sqrt{\det(A - \lambda\mathbb{I})} = \prod_{i=1}^4 (\lambda - \lambda_i) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \quad (6.6)$$

where [8]

$$a \equiv -\frac{1}{2}TrA = -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4); \quad (6.7)$$

$$\begin{aligned} b &\equiv \frac{1}{4} \left[\frac{1}{2} (TrA)^2 - Tr(A^2) \right] = \\ &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4; \end{aligned} \quad (6.8)$$

$$\begin{aligned} c &\equiv -\frac{1}{6} \left[\frac{1}{8} (TrA)^3 + Tr(A^3) - \frac{3}{4} Tr(A^2) TrA \right] = \\ &= -(\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4); \end{aligned} \quad (6.9)$$

$$\begin{aligned} d &\equiv \frac{1}{4} \left[\frac{1}{96} (TrA)^4 + \frac{1}{8} Tr^2(A^2) + \frac{1}{3} Tr(A^3) TrA + \right. \\ &\quad \left. - \frac{1}{2} Tr(A^4) - \frac{1}{8} Tr(A^2) Tr^2A \right] = \\ &= \sqrt{\det A} = \lambda_1\lambda_2\lambda_3\lambda_4. \end{aligned} \quad (6.10)$$

The system (6.7)-(6.10) can be inverted, yielding

$$\lambda_{1,2} = -\frac{a}{4} + \frac{s}{2} \pm \frac{1}{2} \sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{(a^3 - 4ab + 8c)}{4s} - \frac{u}{3w} - \frac{w}{3}}; \quad (6.11)$$

$$\lambda_{3,4} = -\frac{a}{4} - \frac{s}{2} \pm \frac{1}{2} \sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{(a^3 - 4ab + 8c)}{4s} - \frac{u}{3w} - \frac{w}{3}}; \quad (6.12)$$

where

$$u \equiv b^2 + 12d - 3ac; \quad (6.13)$$

$$v \equiv 2b^3 + 27c^2 - 72bd - 9abc + 27a^2d; \quad (6.14)$$

$$w \equiv \left(\frac{v + \sqrt{v^2 - 4u^3}}{2} \right)^{1/3}; \quad (6.15)$$

$$s \equiv \sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{u}{3w} + \frac{w}{3}}. \quad (6.16)$$

Notice that the positivity of quantities under square root in Eqs. (6.11), (6.12), (6.15) and (6.16) always holds. Furthermore, Eq. (6.6) is at most of fourth order (for $\mathbf{k} = 1$), of second order for $\mathbf{k} = 2$, and of first order for $\mathbf{k} = 1$.

1. $\mathbf{k} = 1$ ($\frac{1}{8}$ -BPS, either “*large*” or “*small*”). The $\frac{1}{8}$ -BPS extremal BH square ADM mass is

$$M_{ADM, \frac{1}{8}\text{-BPS}}^2(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}), \quad (6.17)$$

where $\lambda_1(> \lambda_2 > \lambda_3 > \lambda_4$, since $a < 0$ and $s > 0$) is given by Eq. (6.11). In the “*large*” $\mathbf{k} = 1$ case $\lambda_2 = \lambda_3 = \lambda_4 = 0$ at the event horizon of the extremal BH, as given by Eq. (3.16).

2. $\mathbf{k} = 2$ ($\frac{1}{4}$ -BPS, “*small*”). As given by Eq. (3.35), the eigenvalues are equal *in pairs*. By suitably renaming the two non-coinciding λ 's, one gets

$$\lambda_{1,2} = \frac{1}{8}TrA \pm \frac{1}{2}\sqrt{\frac{1}{2}Tr(A^2) - \frac{1}{16}(TrA)^2}. \quad (6.18)$$

As mentioned above, the maximal (compact) symmetry is manifest when λ_2 (in the renaming of Eq. (6.18)) vanishes (see treatment in Sect. 3). Eq. (3.35) implies [8]

$$c = \frac{1}{2}a \left(b - \frac{1}{4}a^2 \right); \quad (6.19)$$

$$d = \frac{1}{4} \left(b - \frac{1}{4}a^2 \right)^2. \quad (6.20)$$

In [8] Eqs. (6.19)-(6.20) were shown to be consequences of the criticality constraints (3.34). Thus, the $\frac{1}{4}$ -BPS extremal BH square ADM mass is

$$M_{ADM, \frac{1}{4}\text{-BPS}}^2(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}), \quad (6.21)$$

where $\lambda_1(> \lambda_2)$ is given by Eq. (6.18):

$$M_{ADM, \frac{1}{4}\text{-BPS}}^2(\phi_\infty, \mathcal{P}) = \frac{1}{8}TrA(\phi_\infty, \mathcal{P}) + \frac{1}{2}\sqrt{\frac{1}{2}Tr(A^2)(\phi_\infty, \mathcal{P}) - \frac{1}{16}(TrA(\phi_\infty, \mathcal{P}))^2}. \quad (6.22)$$

3. $\mathbf{k} = 4$ ($\frac{1}{2}$ -BPS, “small”). This case can be obtained from the $\frac{1}{4}$ -BPS considered at point 2 by further putting $\lambda_1 = \lambda_2$ in Eq. (6.18). Thus, *all* eigenvalues of the Hermitian 8×8 matrix A are equal:

$$A_A^C = \frac{1}{8} (Tr A) \delta_A^C, \quad (6.23)$$

which implies

$$Tr (A^2) = \frac{1}{8} (Tr A)^2. \quad (6.24)$$

Therefore, $\frac{1}{2}$ -BPS extremal BH square ADM mass is given by

$$M_{ADM, \frac{1}{2}\text{-BPS}}^2 (\phi_\infty, \mathcal{P}) = \frac{1}{8} Tr A (\phi_\infty, \mathcal{P}) = \frac{1}{16} Z_{AB} (\phi_\infty, \mathcal{P}) \bar{Z}^{AB} (\phi_\infty, \mathcal{P}). \quad (6.25)$$

6.2 $\mathcal{N} = 4$

In $\mathcal{N} = 4$, $d = 4$ supergravity (treated in Sect. 4), the $SL(2, \mathbb{R}) \times SO(6, M)$ U -duality symmetry only allows the cases [3] $\mathbf{k} = 1, 2$. By recalling the treatment of Sect. 4, the respective maximal compact symmetries read [3, 4, 12, 39]

$$\mathbf{k} = 1 : (SU(2))^2 \times SO(M) \times SO(2); \quad (6.26)$$

$$\mathbf{k} = 2 : USp(4) \times SO(M), \quad (6.27)$$

and they hold all along the respective scalar flows. While case $\mathbf{k} = 1$ is “large”, case $\mathbf{k} = 2$ is “small” (thus not enjoying the *attractor mechanism*).

In the “large” $\mathbf{k} = 1$ case, the *attractor mechanism* makes the maximal compact symmetry $(SU(2))^2 \times SO(M) \times SO(2)$ of the supporting charge orbit $\mathcal{O}_{\frac{1}{4}\text{-BPS, large}}$ fully manifest as a symmetry of the central charge matrix Z_{AB} through the *symmetry enhancement* (recall Eq. (4.25))

$$(SU(2))^2 \times SO(M-2) \times SO(2) \xrightarrow{r \rightarrow r_H^+} (SU(2))^2 \times SO(M) \times SO(2) \quad (6.28)$$

at the event horizon of the considered extremal BH.

As done in Sect. 4 and in the treatment of case $\mathcal{N} = 8$, $d = 4$ above, let us denote with λ_1 and λ_2 the two real non-negative eigenvalues of the 4×4 Hermitian matrix $Z_{AB} \bar{Z}^{CB} = (ZZ^\dagger)_A^C \equiv A_A^C$. Their relation with the absolute values of the complex *skew-eigenvalues* e_i of Z_{AB} is given by Eq. (3.29). As mentioned, the ordering $\lambda_1 \geq \lambda_2$ does not imply any loss of generality. After [8], the explicit expression of λ_1 and λ_2 in terms of $(U(4) \times SO(M))$ -invariants (namely of $Tr A$, $Tr (A^2)$ and $(Tr A)^2$) is known, and it can be thus be used in order to compute the ADM mass of $\frac{\mathbf{k}}{4}$ -BPS extremal BH states of $\mathcal{N} = 4$, $d = 4$ supergravity.

Indeed, λ_1 and λ_2 are solution of the (square root of) *characteristic equation* [8]

$$\sqrt{\det (A - \lambda \mathbb{I})} = \prod_{i=1}^2 (\lambda - \lambda_i) = \lambda^2 - \frac{1}{2} (Tr A) \lambda + (\det A)^{1/2} = 0, \quad (6.29)$$

whose solution reads

$$\lambda_{1,2} = \frac{1}{2} \left(\frac{1}{2} \text{Tr} A \pm \sqrt{\text{Tr} (A^2) - \frac{1}{4} (\text{Tr} A)^2} \right). \quad (6.30)$$

Notice that the positivity of quantities under square root in Eq. (6.30) always holds. Furthermore, Eq. (6.29) is at most of second order (for $\mathbf{k} = 1$) and of first order for $\mathbf{k} = 2$.

1. $\mathbf{k} = 1$ ($\frac{1}{4}$ -BPS “*large*”). The $\frac{1}{4}$ -BPS extremal BH square ADM mass is

$$\begin{aligned} M_{ADM, \frac{1}{4}\text{-BPS}}^2(\phi_\infty, \mathcal{P}) &= \lambda_1(\phi_\infty, \mathcal{P}) = \\ &= \frac{1}{2} \left(\frac{1}{2} \text{Tr} A(\phi_\infty, \mathcal{P}) + \sqrt{\text{Tr} (A^2)(\phi_\infty, \mathcal{P}) - \frac{1}{4} (\text{Tr} A(\phi_\infty, \mathcal{P}))^2} \right), \end{aligned} \quad (6.31)$$

where $\lambda_1 > \lambda_2$. Notice that $\lambda_2 = 0$ at the event horizon of the extremal BH, as given by Eq. (4.23).

2. $\mathbf{k} = 2$ ($\frac{1}{2}$ -BPS, “*small*”). This case can be obtained from the $\frac{1}{4}$ -BPS considered at point 1 by further putting $\lambda_1 = \lambda_2$ in Eq. (6.30). Thus, *all* eigenvalues of the Hermitian 4×4 matrix A are equal:

$$A_A^C = \frac{1}{4} (\text{Tr} A) \delta_A^C, \quad (6.32)$$

which implies

$$\text{Tr} (A^2) = \frac{1}{4} (\text{Tr} A)^2. \quad (6.33)$$

Thus, the $\frac{1}{2}$ -BPS extremal BH square ADM mass is

$$M_{ADM, \frac{1}{2}\text{-BPS}}^2(\phi_\infty, \mathcal{P}) = \lambda_1(\phi_\infty, \mathcal{P}) = \lambda_2(\phi_\infty, \mathcal{P}) = \frac{1}{4} \text{Tr} A(\phi_\infty, \mathcal{P}). \quad (6.34)$$

It should be here remarked that the \mathcal{R} -*symmetry* of the $\frac{\mathbf{k}}{\mathcal{N}}$ -BPS extremal BH states, *i.e.* the compact symmetry of the solution in the *normal frame* (determining the automorphism group of the supersymmetry algebra in the *rest frame*) gets broken as follows:

$$\mathcal{R} \longrightarrow USp(2\mathbf{k}) \times \dots. \quad (6.35)$$

This is precisely the symmetry of the $\frac{\mathbf{k}}{\mathcal{N}}$ -BPS saturated massive multiplets of the \mathcal{N} -extended, $d = 4$ Poincaré supersymmetry algebra [52].

We end this Section by finally commenting about the ADM mass for non-BPS extremal BH states.

In non-BPS cases, ADM mass of extremal BH states is not directly related to the *skew-eigenvalues* of the *central charge matrix* Z_{AB} . For some non-BPS extremal BHs a “*fake supergravity (first order) formalism*” [55] can be consistently formulated in terms of a “*fake superpotential*” $\mathcal{W}(\phi, \mathcal{P})$ [56, 57, 58, 59] such that (also recall Eq. (1.5))

$$\begin{aligned} \mathcal{W}_{non-BPS}^2(\phi, \mathcal{P}) \Big|_{\frac{\partial \mathcal{W}}{\partial \phi}=0} &\equiv \mathcal{W}_{non-BPS}^2(\phi_{H,non-BPS}(\mathcal{P}), \mathcal{P}) = \\ &= V_{BH}(\phi, \mathcal{P}) \Big|_{\frac{\partial V_{BH}}{\partial \phi}=0} \equiv V_{BH}(\phi_{H,non-BPS}(\mathcal{P}), \mathcal{P}) = \\ &= \frac{S_{BH,non-BPS}(\mathcal{P})}{\pi}, \end{aligned} \tag{6.36}$$

with $\mathcal{W}_{non-BPS}$ varying, dependently on whether $Z_{AB} = 0$ or not. In such frameworks, the general expression of the non-BPS ADM mass reads as follows [56, 57, 58]

$$M_{ADM,non-BPS}(\phi_\infty, \mathcal{P}) = \mathcal{W}_{non-BPS}(\phi_\infty, \mathcal{P}). \tag{6.37}$$

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