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Classical and Quantum Effective Theories for Gravitating Spinning Bodies

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Classical and Quantum Effective Field Theories for Gravitating Spinning Bodies

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by

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ABSTRACT OF THE DISSERTATION

Classical and Quantum Effective Field Theories for Gravitating Spinning Bodies

by

Trevor Scheopner
Doctor of Philosophy in Physics
University of California, Los Angeles, 2024
Professor Zvi Bern, Chair

Amplitudes methods have found increasingly varied applications across physics, in particular in the field of gravitational waves. In this manuscript we will present how amplitudes techniques apply to interactions of gravitating spinning bodies. We will analyze the amplitudes of generic spinning bodies and pay special attention to how the structure of these amplitudes simplifies when specializing to spinning black holes. In Chapter 1, we develop how several amplitudes tools such as the Kosower-Maybee-O'Connell formalism, eikonal methods, and effective field theory in the classical limit apply to the dynamics of classically colored particles. Classically colored particles share many of the theoretical features of classical spinning bodies, with the color directly analogous to the spin in many ways, and so serve as an effective proof of concept of how these same tools may be applied to spinning bodies. In Chapter 2, we directly apply these techniques to classical spinning electromagnetically interacting bodies, which share all of the spin related complications of the gravitational problem while being simpler due to the relative simplicity of electromagnetism compared to gravity. In doing so, we find that the effective field theory is capable of carrying an extra vector degree of freedom compared with the previously established worldline formalism, and that that vector degree of freedom allows for spin magnitude change in the theory. We also present a modification of the traditional worldline formalism which perfectly matches the effective field theory. In Chapter 3, we use the worldline formalism to compute generic spinning body Compton amplitudes through the fifth order in spin, at which several interesting complications occur. This Compton amplitude is essential for computing one-loop observables for the spinning binary system. We then use Dixon's multipole moment formalism to identify an effective source energy-momentum tensor for a spinning black hole, which if treated as appropriate in the effective theory determines several previously unconstrained Wilson coefficients which affect black hole observables. In Chapter 4, we extend the electromagnetic methods of Chapter 2 to gravity. We find that the nonminimal degrees of freedom persist in having matching physical effects between the worldline and field theory approaches. As well, we find that those degrees of freedom have observable effects on the gravitational waveform.

The dissertation of Trevor Scheopner is approved.

Eric D'Hoker

Michael Gutperle

Mikhail Pil Solon

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2024

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Contributions of Authors

Chapter 1 is based on work done with Leonardo de la Cruz and Andres Luna in Ref. [1]. Chapter 2 is based on work done with Zvi Bern, Dimitrios Kosmopoulos, Andres Luna, Radu Roiban, Fei Teng, and Justin Vines in Ref. [2]. Chapter 3 is based on work done with Justin Vines in Ref. [3]. Chapter 4 is based on work done with Mark Alaverdian, Zvi Bern, Dimitrios Kosmopoulos, Andres Luna, Radu Roiban, Fei Teng in Ref. [4].

Publications

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Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, **T. Scheopner**, F. Teng, and J. Vines, "Quantum Field Theory, Worldline Theory, and Spin Magnitude Change in Orbital Evolution," `arXiv:2308.14176 [hep-th]`

T. Scheopner and J. Vines, "Dynamical Implications of the Kerr Multipole Moments for Spinning Black Holes," `arXiv:2311.18421 [gr-qc]`

M. Alaverdian, Z. Bern, D. Kosmopoulos, A. Luna, R. Roiban, **T. Scheopner**, F. Teng, "Conservative Spin Magnitude Change in Orbital Evolution in General Relativity," `arXiv:2407.10928 [hep-th]`

Chapter 1

Yang-Mills observables: from KMOC to eikonal through EFT

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We obtain a conservative Hamiltonian describing the interactions of two charged bodies in Yang-Mills through $\mathcal{O}(\alpha^2)$ and to all orders in velocity. Our calculation extends a recently-introduced framework based on scattering amplitudes and effective field theory (EFT) to consider color-charged objects. These results are checked against the direct integration of the observables in the Kosower-Maybee-O'Connell (KMOC) formalism. At the order we consider we find that the linear and color impulses in a scattering event can be concisely described in terms of the eikonal phase, thus extending the domain of applicability of a formula originally proposed in the context of spinning particles.

1.1 Introduction

The Kosower-Maybee-O'Connell (KMOC) formalism [5, 6, 7] is a first principle approach to extract the classical limit, understood as the limit $\hbar \to 0$, from on-shell scattering amplitudes. It is based on the construction of certain observables which are well-defined at the quantum and classical levels. They can be defined by considering the expectation value of certain operators $\mathbb{O}$ evaluated at the beginning and at the end of the scattering event. Considering the two-to-two classical scattering the observable associated with the operator $\mathbb{O}$ is given by

$$\langle \Delta O \rangle = \langle \Psi | S^\dagger \mathbb{O} S | \Psi \rangle - \langle \Psi | \mathbb{O} | \Psi \rangle, \tag{1.1.1}$$

where $S = 1 + iT$. The "in" states $|\Psi\rangle$ are two-particle coherent states for momenta and color, whose function is to give the notion of point particles with a sharply-defined position, momenta, and color. To make this notion precise, the restoration of $\hbar$'s on couplings and color factors as well as the distinction between momenta p and wavenumber $\bar{p}$ for certain particles play an important role.

Employing unitarity the observables can be written as

$$\langle \Delta O \rangle = i\langle \Psi|[\mathbb{O}, T]|\Psi\rangle + \langle \Psi|T^\dagger[\mathbb{O}, T]|\Psi\rangle, \tag{1.1.2}$$

which can be used to derive general expressions for these observables in terms of amplitudes. In this chapter we will consider the color charge operator $\mathbb{C}_1^a$ and the momentum operator $\mathbb{P}_1^\mu$ of one of the particles, but of course the other particle can be chosen as well. The observables associated to these operators are called the color impulse Δc_1^a and the momentum impulse Δp_1^μ. The KMOC formalism has been applied to the study of waveforms [8], soft theorems [9], radiative gravitational observables at two-loops [10, 11] and adapted to study the classical limit of thermal currents [12].

On the other hand, the classical limit can also be described in the language of effective field theory (EFT). This idea was pioneered in Ref. [13], which proposed the application of the well-established scattering-amplitudes toolkit to the derivation of gravitational potentials. Later, an EFT of non-relativistic scalar fields was developed [14], and used to translate a one-loop scattering amplitude into the $\mathcal{O}(G^2)$ canonical Hamiltonian, which is equivalent to the results of Westpfahl [15]. This approach was later implemented to obtain novel results at $\mathcal{O}(G^3)$ order [16, 17, 18].

Besides making use of the KMOC formalism or non-relativistic EFTs, various approaches have been developed to extract the dynamics of compact objects from scattering data. These include making use of the Lippman-Schwinger equation [19, 20], a heavy black hole effective theory and its generalizations [21, 22, 23], developing a boundary-to-bound (B2B) dictionary [24, 25], implementing a post-Minkowskian EFT [26, 27, 28] and a worldline QFT [29]. More recently the conservative binary potential at $\mathcal{O}(G^4)$ was obtained by means of an amplitude-action relation that allows the calculation of physical observables directly from the scattering amplitude [30].

The techniques mentioned above have been extended in multiple directions in recent years, including the computation of observables in supergravity [31, 32, 33] and other generalizations of GR [34, 35], the study of three-body dynamics [36], incorporating the radiation emitted by the binary into their analysis [37, 38, 39, 40, 41, 42], and considering tidal deformations [43, 44, 45, 46, 47, 48, 49] and spin effects [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66] of the astrophysical objects.

A further relation between amplitudes and classical observables is given through the eikonal phase, which is obtained as the Fourier transform to impact parameter space of the scattering amplitude [67]. In turn, one can derive the scattering angle through differentiation of the eikonal phase. This subject has seen renewed interest [68, 69, 70, 71, 72, 73, 74, 32, 33, 75, 76, 77] and a recent calculation in Ref. [59] showed a surprising structure for the expression of the observables in terms of the eikonal phase. This formula was the first example of such a relation for arbitrary orientations of the spins[1]. This striking observation potentially implies that all physical observables are obtainable via simple manipulations of the scattering amplitude.

While most of the attention has been given to gravitational theories, Yang-Mills theory shares many important physical features with gravity, like non-linearity and a gauge structure. Furthermore, the double copy relates scattering amplitudes in both theories[2]. The connection has showed to be deeper than this, holding in a classical worldline setting [81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91], and extending to exact maps [92, 93][3]. Then, since perturbation theory in Yang-Mills is far simpler than in standard approaches of gravity, one may study Yang-Mills as a toy model for gravitational dynamics or as a building block that could be double copied to gravity. One may also note that, as already pointed out in Ref. [7], the dynamics of the color degrees of freedom in Yang-Mills, is in many respects analogous to spin (though actually simpler). This analogy with spin will be evidenced in a generalization of the formula of Ref. [59], now describing the dynamics of color charges.

The proliferation of approaches to extract classical information from quantum scattering amplitudes motivates us to strive for an understanding of the relations between them. The goal of this chapter is to use Yang-Mills theory as a toy model to study the connection between three such approaches. Namely, the KMOC formalism, the Hamiltonian approach to classical dynamics, and a formula directly relating the eikonal phase with classical observables.

The remainder of this chapter is structured as follows: In Section 1.2 we compute color and momentum impulse at NLO using the integrands obtained in Ref. [7]. Then, In Section 1.3, we develop the Hamiltonian approach to classical dynamics. First, we show the necessary full-theory amplitudes and use a matching procedure to an EFT to obtain the desired two-body Hamiltonian. Then we use the derived Hamiltonian to compute scattering observables, and check their match both to the KMOC approach of Section 1.2, as well as to the conjecture of Ref. [59], which directly relates these observables to the eikonal phase, and holds (almost unalteredly) when we include color effects. We present our concluding remarks in Section 1.4.

1.2 KMOC approach to color observables

In this Section we introduce the KMOC approach for color and introduce our notation and conventions. The classical scattering of two color-charged scalar particles of masses m_1 and m_2 can be modeled by the

[1] Before this, there was evidence for such a relation in the special kinematic configuration where the spins of the particles are parallel to the angular momentum of the system [53, 78, 79].

[2] The double copy has been reviewed thoroughly in Ref. [80].

[3] The classical double copy has also made contact with fluid dynamics, as shown in Refs. [94, 95].

action

$$S = \int \hat{\mathrm{d}}^4 x \Big[\sum_{i=1,2} ((D_\mu \varphi_i)^\dagger (D^\mu \varphi_i) - \frac{m_i^2}{\hbar^2} \varphi_i^\dagger \varphi_i) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \Big], \tag{1.2.1}$$

where $D_\mu = \partial_\mu + igA^a_\mu T^a_R$ and $\hat{\mathrm{d}}^n x = (\mathrm{d})^n x)/(2\pi)^n$. The generators T^R_a of the Lie algebra of $SU(N)$ are in some representation R. The color charge operators, obtained from the Noether procedure, satisfy the usual Lie algebra modified by a factor of $\hbar$

$$[\mathbb{C}^a, \mathbb{C}^b] = i\hbar f^{abc} \mathbb{C}^c, \tag{1.2.2}$$

emphasizing that $\mathbb{C}^a$ corresponds to an operator and

$$\langle p_i | \mathbb{C}^a | p^j \rangle \equiv (C^a)_i^{\ j} = \hbar (T^a_R)_i^{\ j}. \tag{1.2.3}$$

So the color factors $(C^a)_i^{\ j}$ are simply rescalings of the usual generators $(T^a_R)_i^{\ j}$. The classical color charges are then defined by

$$c^a \equiv \langle \psi | \mathbb{C}^a | \psi \rangle, \tag{1.2.4}$$

where the states $|\psi\rangle$ are coherent states for $SU(N)$, whose explicit form will not be relevant for our purposes[4]. These states ensure the correct behavior of color charges in the classical limit, namely

$$\langle \psi | \mathbb{C}^a | \psi \rangle = \text{finite}, \tag{1.2.5}$$

$$\langle \psi | \mathbb{C}^a \mathbb{C}^b | \psi \rangle = c^a c^b + \text{negligible}, \tag{1.2.6}$$

which is guaranteed by choosing the dimension of the representation R to be large. The factors of $\hbar$ in Eq.(1.2.2) produce a nontrivial interplay between color factors and kinematics in the classical limit. However ultimately classical quantities do not have any factors of $\hbar$ as it should be. Thus, for the purposes of this chapter we will quote the integrands derived in Ref. [7] dropping the bar notation for wavenumbers. We will also employ the notation $\Delta O^{(L)}$ to indicate the L-loop contribution to the observable such that the full result is given by

$$\Delta O = \Delta O^{(0)} + \Delta O^{(1)} + \dots. \tag{1.2.7}$$

We also introduce the following notation for the Dirac-delta

$$\hat{\delta}(x) = 2\pi \delta(x), \qquad \hat{\delta}'(x) = \frac{i}{(x - i\epsilon)^2} - \frac{i}{(x + i\epsilon)^2}. \tag{1.2.8}$$

[4]When considering the classical limit of multi-particle states, the full state is a tensor product of coherent states for the kinematics and coherent states for color. $SU(N)$ coherent states can be constructed using Schwinger bosons [7].

1.2.1 Leading order

Let us briefly review the LO calculation of Ref. [7] in order to introduce some notation. We define the integral

$$\mathcal{I}_{\succ\!\prec} \equiv \int \hat{\mathrm{d}}^4 q \frac{\hat{\delta}(q \cdot u_1)\hat{\delta}(q \cdot u_2)}{q^2} e^{-iq\cdot b}, \tag{1.2.9}$$

where $p_i^\mu = m_i u_i^\mu$ and b^μ is the impact parameter. Recalling that b^μ is spacelike we also define $|b| \equiv \sqrt{-b^2}$. The classical four velocities u_i are normalized to $u_i^2 = 1$. The divergent integral $I_{\succ\!\prec}$ can be regulated using a cut-off regulator L

$$\mathcal{I}_{\succ\!\prec} = \frac{1}{4\pi\sqrt{\sigma^2 - 1}} \log\left(\frac{|b|^2}{L^2}\right), \tag{1.2.10}$$

where σ is the standard Lorentz factor $\sigma = u_1 \cdot u_2$. The LO momentum impulse can then be written as

$$\Delta p_1^{(0),\mu} = -g^2 \sigma c_1 \cdot c_2 \frac{\partial \mathcal{I}_{\succ\!\prec}}{\partial b_\mu}, \tag{1.2.11}$$

where $c_1 \cdot c_2 \equiv c_1^a c_2^a$. So the momentum impulse is given by

$$\Delta p_1^{(0),\mu} = -2\alpha\, c_1 \cdot c_2 \frac{\sigma}{\sqrt{\sigma^2-1}} \frac{b^\mu}{b^2}, \tag{1.2.12}$$

where $\alpha \equiv g^2/(4\pi)$. Similarly the color impulse at leading order reads

$$\Delta c_1^{(0),a} = g^2 \sigma f^{abc} c_1^b c_2^c \mathcal{I}_{\succ\!\prec} = \alpha\, f^{abc} c_1^b c_2^c \frac{\sigma}{\sqrt{\sigma^2-1}} \log\left(\frac{|b|^2}{L^2}\right). \tag{1.2.13}$$

The divergence of the color impulse is the familiar divergence due to the long-range nature of $1/r^2$ forces in four-dimensions.

1.2.2 Next-to-Leading-Order

The NLO momentum impulse can be obtained from the QED one computed in Ref. [5] using the charge to color replacements $Q_1 Q_2 \to c_1 \cdot c_2$ and $e \to g$. That this replacement works follows from the color-decomposition of the QCD amplitude and $\hbar$-counting as detailed in [7]. The result reads

$$\begin{aligned} \Delta p_1^{\mu,(1)} = i\frac{g^4 (c_1\cdot c_2)^2}{2} \int \hat{\mathrm{d}}^4\ell\, \hat{\mathrm{d}}^4 q\, \frac{\hat{\delta}(u_1\cdot q)\hat{\delta}(u_2 \cdot q)}{\ell^2(\ell-q)^2} e^{-iq\cdot b} \Bigg[q^\mu \Bigg\{ \frac{\hat{\delta}(u_2\cdot\ell)}{m_1} + \frac{\hat{\delta}(u_1\cdot \ell)}{m_2} \\ + (u_1\cdot u_2)^2 \ell\cdot(\ell-q) \left(\frac{\hat{\delta}(u_1\cdot\ell)}{m_2(u_2\cdot\ell - i\epsilon)^2} + \frac{\hat{\delta}(u_2\cdot \ell)}{m_1(u_1\cdot\ell+i\epsilon)^2}\right)\Bigg\} \\ - i(u_1\cdot u_2)^2 \ell^\mu \ell\cdot(\ell-q)\left(\frac{\hat{\delta}'(u_1\cdot\ell)\hat{\delta}(u_2\cdot\ell)}{m_1} - \frac{\hat{\delta}(u_1\cdot\ell)\hat{\delta}'(u_2\cdot\ell)}{m_2}\right)\Bigg]. \end{aligned} \tag{1.2.14}$$

On the other hand the NLO color impulse is given by

$$
\begin{aligned}
\Delta c_1^{a,(1)} = g^4 \int \hat{\mathrm{d}}^4 q\, \hat{\mathrm{d}}^4 \ell\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q) e^{-iq\cdot b}\frac{1}{\ell^2(\ell-q)^2} \\
\times \Bigg\{ \hat{\delta}(u_1\cdot\ell)\Bigg[\frac{f^{acd}c_1^c c_2^d (c_1\cdot c_2)}{m_2}\Bigg[1+(u_1\cdot u_2)^2 \ell\cdot(\ell-q)\Bigg(\frac{1}{(u_2\cdot\ell - i\epsilon)^2} \\
+ i\hat{\delta}'(u_2\cdot\ell)\Bigg)\Bigg] - f^{acd}f^{dbe}c_1^b c_1^c c_2^e \frac{(u_1\cdot u_2)^2}{2}\hat{\delta}(u_2\cdot\ell)\Bigg] \\
+ \hat{\delta}(u_2\cdot\ell)\Bigg[\frac{f^{acd}c_1^c c_2^d (c_1\cdot c_2)}{m_1}\Bigg[1+(u_1\cdot u_2)^2 \ell\cdot(\ell-q)\Bigg(\frac{1}{(u_1\cdot\ell + i\epsilon)^2} \\
- i\hat{\delta}'(u_1\cdot\ell)\Bigg)\Bigg] + f^{acd}f^{dbe}c_1^e c_2^b c_2^c \frac{(u_1\cdot u_2)^2}{2}\hat{\delta}(u_1\cdot\ell)\Bigg]\Bigg\}.
\end{aligned}
\tag{1.2.15}
$$

Inspecting Eqs.(1.2.14) and (1.2.15) it is easy to see that the color and momentum impulses can be expressed in terms of the following "master integrals"

$$
\mathcal{I}^i_{\triangle}[\alpha,\beta,\gamma] = \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q) e^{-iq\cdot b}\int \hat{\mathrm{d}}^4\ell\, \frac{\hat{\delta}(u_i\cdot\ell)}{[\ell^2]^\alpha[(\ell-q)^2]^\beta[(\ell\cdot u_j + (-1)^i i\epsilon)]^\gamma}, j\neq i \tag{1.2.16}
$$

$$
\mathcal{I}_{\not\square}[\alpha,\beta] = \int \hat{\mathrm{d}}^4 q\, \hat{\delta}(u_1\cdot q)\hat{\delta}(u_2\cdot q) e^{-iq\cdot b}\int \hat{\mathrm{d}}^4\ell\, \frac{\hat{\delta}(u_1\cdot\ell)\hat{\delta}(u_2\cdot\ell)}{[\ell^2]^\alpha[(\ell-q)^2]^\beta}, \tag{1.2.17}
$$

where the vector dependence on the momentum transfer q^μ can be recovered by taking derivatives w.r.t. the impact parameter b^μ. Notice that we have excluded from the master integrals those involving $\hat{\delta}'(x)$ since they can be reduced to the above cases using the identity

$$
\int \hat{\mathrm{d}}x\; x\; \hat{\delta}'(x) f(x^2) = -\int \hat{\mathrm{d}}x\; \hat{\delta}(x) f(x^2). \tag{1.2.18}
$$

Following arguments by Kälin-Porto [26], the integrals below vanish due to the presence of a double pole on a convergent integral[5]

$$
\mathcal{I}^i_{\triangle}[1,1,2] = \mathcal{I}^i_{\triangle}[0,1,2] = \mathcal{I}^i_{\triangle}[1,0,2] = 0, \qquad i=1,2, \tag{1.2.19}
$$

and therefore only $I^i_{\triangle}[1,1,0]$ contributes to the observables. In the following we then simply write $\mathcal{I}^i_{\triangle}[1,1,0] \equiv \mathcal{I}^i_{\triangle}$ and for later purpose we write $\mathcal{I}_{\not\square}[1,1] \equiv \mathcal{I}_{\not\square}$. We also have that

$$
\mathcal{I}_{\not\square}[1,0] = \mathcal{I}_{\not\square}[0,1] = 0 \tag{1.2.20}
$$

since their loop integrals reduce to massless tadpole integrals. Now let us move on with the reductions of integrals of the form

$$
I^\mu = \int \hat{\mathrm{d}}^4\ell\, \ell^\mu \ell\cdot(\ell-q)\frac{\hat{\delta}'(u_1\cdot\ell)\hat{\delta}(u_2\cdot\ell)}{\ell^2(\ell-q)^2}, \tag{1.2.21}
$$

which appear in Eq. (1.2.14) and its mirror obtained by $1\leftrightarrow 2$. In contrast to the above vanishing integrals,

[5] This result can also be shown by first using the Dirac-delta constraint and then IBP identities. As emphasized by Kälin-Porto these integrals do contribute in $d>4$ [72].

the presence of the numerator makes this integral nonzero. Let us also recall that they are still integrated over the momentum transfer q and therefore in the integral reduction we can set to zero any term proportional to Eq.(1.2.19) or (1.2.20). Performing a simple Passarino-Veltman reduction we can write

$$I^\mu = K_1 u_1^\mu + K_2 u_2^\mu + K_3 q^\mu, \tag{1.2.22}$$

where setting up a system of equations the resulting coefficients are

$$K_1 = \frac{1}{1-\sigma^2} u_1 \cdot I, \qquad K_2 = -\frac{\sigma}{1-\sigma^2} u_1 \cdot I, \qquad K_3 = \frac{1}{q^2} q \cdot I, \tag{1.2.23}$$

where we have used the delta constraints $\hat{\delta}(q \cdot u_1)$ and $\hat{\delta}(q \cdot u_2)$ on which the integral is supported. The result thus depends only on two integrals, namely $u_1 \cdot I$ and $q \cdot I$. After cancellations, the product $q \cdot I$ leads to

$$q \cdot I = \frac{1}{4} \int \hat{\mathrm{d}}^4 \ell \left[\frac{2q^2}{\ell^2} - \frac{(q^2)^2}{\ell^2 (\ell - q)^2} \right] \hat{\delta}'(u_1 \cdot \ell) \hat{\delta}(u_2 \cdot \ell), \tag{1.2.24}$$

which can be set to zero after integration over q using Eq.(1.2.19). Therefore we can express Eq.(1.2.21) only in terms of the integral

$$\begin{aligned} u_1 \cdot I &= \int \hat{\mathrm{d}}^4 \ell \, u_1 \cdot \ell \;\; \ell \cdot (\ell - q) \frac{\hat{\delta}'(u_1 \cdot \ell) \hat{\delta}(u_2 \cdot \ell)}{\ell^2 (\ell - q)^2} \\ &= - \int \hat{\mathrm{d}}^4 \ell \, \ell \cdot (\ell - q) \frac{\hat{\delta}(u_1 \cdot \ell) \hat{\delta}(u_2 \cdot \ell)}{\ell^2 (\ell - q)^2}. \end{aligned} \tag{1.2.25}$$

Without loss of generality, the second equality can be checked by choosing a frame where $u_1 = (1, 0, 0, 0)$ and $u_2 = (\sigma, 0, 0, \sigma\beta)$ and β is defined from the condition $\sigma^2 - \sigma^2 \beta^2 = 1$. We can further reduce this integral ignoring vanishing terms (i.e., terms which have the form (1.2.20)) thus obtaining

$$u_1 \cdot I = \frac{1}{2} q^2 \int \hat{\mathrm{d}}^4 \ell \, \frac{\hat{\delta}(u_1 \cdot \ell) \hat{\delta}(u_2 \cdot \ell)}{\ell^2 (\ell - q)^2}. \tag{1.2.26}$$

The result for I^μ then reads

$$I^\mu = \frac{1}{2} q^2 \left(\frac{1}{1-\sigma^2} u_1^\mu - \frac{\sigma}{1-\sigma^2} u_2^\mu \right) \int \hat{\mathrm{d}}^4 \ell \, \frac{\hat{\delta}(u_1 \cdot \ell) \hat{\delta}(u_2 \cdot \ell)}{\ell^2 (\ell - q)^2}, \tag{1.2.27}$$

which implies that we can express our results only in terms of the integrals (1.2.16)-(1.2.17) as claimed. Therefore, excluding all vanishing contributions, the impulses in terms of the master integrals can be written as

$$\Delta p_1^{\mu,(1)} = \frac{g^4 (c_1 \cdot c_2)^2}{2} \left\{ -\frac{\partial}{\partial b_\mu} \left[\frac{\mathcal{I}_\triangle^1}{m_2} + \frac{\mathcal{I}_\triangle^2}{m_1} \right] - \left[\frac{\sigma^2}{2(1-\sigma^2)} \left(\frac{u_1^\mu}{m_1} - \frac{\sigma u_2^\mu}{m_1} \right) - (1 \leftrightarrow 2) \right] \frac{\partial}{\partial b_\nu} \frac{\partial}{\partial b^\nu} \mathcal{I}_{\not\square} \right\}$$

and

$$\Delta c_1^{a,(1)} = g^4 \left\{ f^{acd} c_1^c c_2^d (c_1 \cdot c_2) \left(\frac{\mathcal{I}_\triangle^1}{m_2} + \frac{\mathcal{I}_\triangle^2}{m_1} \right) + \frac{\sigma^2}{2} \left(f^{acd} f^{dbe} c_1^e c_2^b c_2^c - f^{acd} f^{dbe} c_1^b c_1^c c_2^e \right) \mathcal{I}_{\not\square} \right\}. \tag{1.2.28}$$

Let us now consider the integration of the master integrals. The triangle one is well-known (see e.g., Ref.

[96]) and we simply quote the result

$$\mathcal{I}_{\triangle}^{1} = \int \hat{\mathrm{d}}^4 q \int \hat{\mathrm{d}}^4 \ell \, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot q) e^{-iq\cdot b} \frac{\hat{\delta}(u_1 \cdot \ell)}{\ell^2(\ell - q)^2} = \frac{1}{16\pi} \frac{1}{\sigma\beta|b|} \,. \tag{1.2.29}$$

The loop integral inside $\mathcal{I}_{\boxslash}[1,1]$ can be computed using dimensional regularization [97], leading to

$$\int \hat{\mathrm{d}}^D \ell \, \frac{\hat{\delta}(u_1 \cdot \ell)\hat{\delta}(u_2 \cdot \ell)}{\ell^2(\ell - q)^2} = \frac{1}{2\pi\sigma\beta q^2} \left[\frac{1}{\varepsilon} - \log(-q^2) \right], \tag{1.2.30}$$

where $D = 4 - 2\varepsilon$ and the usual factors $\mu^{2\varepsilon} e^{\varepsilon\gamma_E}$ have been used to avoid the proliferation of the Euler-Mascheroni constant γ_E and factors of π. The divergent term leads to a contact term that can be discarded in the classical limit[6]. Therefore, keeping only the finite part we have

$$\frac{\partial}{\partial b_\nu} \frac{\partial}{\partial b^\nu} \mathcal{I}_{\boxslash} = \frac{1}{2\pi\sigma\beta} \int \hat{\mathrm{d}}^4 q \, \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot q) e^{-iq\cdot b} \log(-q^2) = \frac{1}{2\pi^2\sigma^2\beta^2} \frac{1}{b^2}. \tag{1.2.31}$$

It will also be convenient to use a cut-off regularization to evaluate the divergent integral $\mathcal{I}_{\boxslash}$. Exchanging the integration orders and introducing the change of variables $Q = -\ell + q$ we have

$$\mathcal{I}_{\boxslash} = \int \hat{\mathrm{d}}^4 \ell \, \frac{\hat{\delta}(u_1 \cdot \ell)\hat{\delta}(u_2 \cdot \ell)}{\ell^2} e^{-i\ell\cdot b} \int \hat{\mathrm{d}}^4 Q \, \hat{\delta}(u_1 \cdot Q)\hat{\delta}(u_2 \cdot Q) e^{-iQ\cdot b} \frac{1}{Q^2}, \tag{1.2.32}$$

which leads to the product of two integrals of the form (1.2.9). Hence the result is simply

$$\mathcal{I}_{\boxslash} = \mathcal{I}_{\succ\!\prec}^{2} = \frac{1}{16\pi^2\sigma^2\beta^2} \log^2\left(\frac{|b|^2}{L^2} \right). \tag{1.2.33}$$

For later purposes we will express the color impulse in terms of the cut-off regulated integral. Our full integrated result for the NLO momentum impulse then reads

$$\begin{aligned} \Delta p_1^{\mu,(1)} = (c_1 \cdot c_2)^2 \frac{2\pi\alpha^2}{m_1 m_2} \Bigg\{ &- \frac{1}{4\sqrt{\sigma^2 - 1}} (m_1 + m_2) \frac{b^\mu}{|b|^3} \\ &- \frac{1}{\pi} \frac{1}{b^2} \frac{\sigma^2}{(\sigma^2 - 1)^2} \left[(m_2 + \sigma m_1) u_1^\mu - (m_1 + \sigma m_2) u_2^\mu \right] \Bigg\}, \end{aligned} \tag{1.2.34}$$

and for the NLO color impulse

$$\Delta c_1^{a,(1)} = \alpha^2 \left\{ \pi \frac{f^{acd} c_1^c c_2^d (c_1 \cdot c_2)}{\sqrt{\sigma^2 - 1}|b|} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) + \frac{1}{2} \frac{\sigma^2}{(\sigma^2 - 1)} \log^2\left(\frac{|b|^2}{L^2} \right) \left[f^{acd} f^{dbe} c_1^e c_2^b c_2^c - f^{acd} f^{dbe} c_1^b c_1^c c_2^e \right] \right\}. \tag{1.2.35}$$

1.3 Hamiltonian approach to color dynamics

In this Section we will compute the position-space Hamiltonian H that describes the classical dynamics of the two-to-two scattering of $SU(N)$ colored objects with masses m_1 and m_2 and color charges c_1 and c_2. The classical dynamics described by such a Hamiltonian must be consistent with Wong's equations [98] and its perturbative solutions and by extension to observables in the KMOC formalism. Let $\boldsymbol{r}$ and $\boldsymbol{p}$ be the

[6]Notice that the factor of q^2 in the denominator cancels after taking derivatives with respect to the impact parameter, so the singular term leads to $\delta^2(\mathbf{b})$ which we can set to zero because we assume $\mathbf{b} \neq 0$.

relative distance between the particles and the momentum vector in the center of mass frame, respectively. We are interested in a perturbative expansion of the Hamiltonian

$$H \equiv H(\boldsymbol{r}, \boldsymbol{p}, \mathcal{C}_i) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{r}^2, \boldsymbol{p}^2, \mathcal{C}_i) + \dots\,, \tag{1.3.1}$$

where the potential is an expansion up to the second power in the coupling constant α and the color structures $\mathcal{C}_i$ are all possible functions of the color charges that can appear in the amplitude. These charges are understood in the sense of Wong, i.e., as the classical limit of a quantum operator in a large representation of the gauge group so they can be treated as c-numbers.

1.3.1 Classical perturbation theory

Consider the general problem of an arbitrary Hamiltonian H describing the interaction of two particles with color charges c_1 and c_2 in their center of mass frame. While, as usual, $\boldsymbol{r}$ and $\boldsymbol{p}$ are canonically-conjugate to each other, color charges do not have a natural canonical conjugate. To derive the equations of motion we use the fact that they satisfy the relation [99, 100]

$$\{c_i^a, c_j^b\} = \delta_{ij} f^{abc} c_i^c\,, \qquad\qquad i, j = 1, 2\,, \tag{1.3.2}$$

where $\{A, B\}$ is the Poisson bracket of A and B. The equations of motion are then

$$\dot{\boldsymbol{r}} = \frac{\partial H}{\partial \boldsymbol{p}}\,, \qquad\qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{r}}\,, \qquad\qquad \dot{c}_i^a = f^{abc} c_i^b \frac{\partial H}{\partial c_i^c}\,, \quad i = 1, 2\,. \tag{1.3.3}$$

In the color equation of motion, no summation over i is implied on the right-hand side. For the purpose of finding the impulse $\Delta\boldsymbol{p}$ we find it convenient to use Cartesian coordinates. One can solve the equations of motion for coordinates, momenta, and colors as a function of time.

There are conservation laws that aid the construction of classical solutions. These fix the energy and the total angular momentum in terms of their asymptotic values. For example for the energy we have

$$E \equiv H(\boldsymbol{r}_\infty, \boldsymbol{p}_\infty, c_1, c_2) = \sqrt{\boldsymbol{p}_\infty^2 + m_1^2} + \sqrt{\boldsymbol{p}_\infty^2 + m_2^2}\,, \tag{1.3.4}$$

where $\boldsymbol{p}_\infty = p_\infty \boldsymbol{e}_z$ is the incoming momentum at infinity. We take the orbital angular momentum at infinity to be

$$\boldsymbol{L} \equiv \boldsymbol{b} \times \boldsymbol{p}_\infty = b \cdot p_\infty \boldsymbol{e}_y\,, \tag{1.3.5}$$

where $\boldsymbol{b} = -b\boldsymbol{e}_x$ and b is the impact parameter. We solve the equations of motion perturbatively in the coupling constant, i.e. we search for a solution for coordinates, momenta, and colors of the form

$$\begin{aligned}
\boldsymbol{r}(t) &= \boldsymbol{r}_0(t) + \alpha \boldsymbol{r}_1(t) + \alpha^2 \boldsymbol{r}_2(t) + \dots\,, \\
\boldsymbol{p}(t) &= \boldsymbol{p}_0(t) + \alpha \boldsymbol{p}_1(t) + \alpha^2 \boldsymbol{p}_2(t) + \dots\,, \\
c_i^a(t) &= c_{i,0}^a(t) + \alpha c_{i,1}^a(t) + \alpha^2 c_{i,2}^a(t) + \dots\,.
\end{aligned} \tag{1.3.6}$$

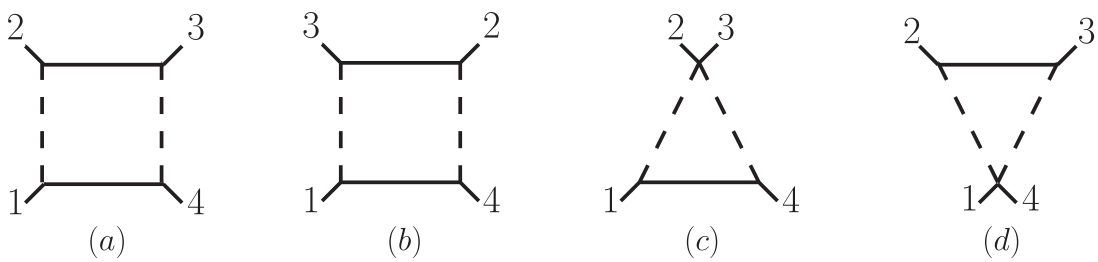


Figure 1.1: The one-loop scalar box integrals $I_{\square}$ (a) and $I_{\bowtie}$ (b) and the corresponding triangle integrals $I_{\triangle}$ (c) and $I_{\bigtriangledown}$ (d). The bottom (top) solid line corresponds to a massive propagator of mass m_1 (m_2). The dashed lines denote massless propagators.

Replacing them in the equations of motion (1.3.3) leads to iterative relations between the time derivative of the n-th term in the expansions above and all the lower-order terms. The $\mathcal{O}(\alpha^0)$ terms describe the motion of a free color-charged particle in flat space, i.e. a straight line fixed by the initial momentum, the impact parameter, and initial color charge. The first-order differential equations for the higher-order terms can be integrated; the relevant boundary conditions are that $\boldsymbol{r}_{n\geq 1}$, $\boldsymbol{p}_{n\geq 1}$ and $c^a_{i,n\geq 1}$ vanish at $t = -T$, where T is a cutoff time. It is necessary to introduce such a cutoff due to the same divergence identified in Eqs.(1.2.13) and (1.2.35); the cutoff T is proportional to the cutoff L in those equations. The contribution of each order in α to an observable O, such as the linear or color impulse, is then

$$\Delta O^{(n)} = \int_{-T}^{T} \mathrm{d}t \, \frac{\mathrm{d}O^{(n)}}{\mathrm{d}t} = O^{(n)}(t = T) - O^{(n)}(t = -T) \, , \tag{1.3.7}$$

with the complete result being their sum weighted with the appropriate powers of α.

1.3.2 Hamiltonian from effective field theory

The perturbative classical problem can be solved straightforwardly once the Hamiltonian is obtained. We then proceed to compute it following the EFT approach adapted to this case. In order to apply this approach we will decompose the amplitudes in some color basis and neglect contributions of higher orders in $\hbar$ using Eq. (1.2.2). Our amplitude expressions will be directly written in terms of classical color factors, i.e., we consider that the expectation value with respect to coherent states has already been taken[7].

Full theory amplitudes from unitarity

Let us first show the two-to-two scattering amplitudes between color-charged particles needed to construct the Hamiltonian. The information to determine the $\mathcal{O}(\alpha)$ Hamiltonian is contained in the tree-level amplitude. We take the incoming momenta of the color-charged particles to be $-p_1$ and $-p_2$ and their outgoing momenta

[7] This essentially amounts to the replacement $C_i \to c_i$ which is implemented in Ref.[7] by the double bracket notation.

to be p_3 and p_4. The amplitude is given by

$$\mathcal{A}^{\text{tree}} = -\frac{4\pi\alpha}{q^2}\,\lambda_1 \mathcal{C}\,(\text{tree}) + \ldots\,, \tag{1.3.8}$$

where we omit terms that do not contribute to the classical limit in the ellipsis, along with pieces proportional to q^2, since they cancel the propagator and do not yield long-range contributions. The color structure is given by

$$\mathcal{C}\,(\text{tree}) = c_1 \cdot c_2\,, \tag{1.3.9}$$

and the coefficient λ_1 takes the explicit form

$$\lambda_1 = -4m_1 m_2 \sigma\,, \tag{1.3.10}$$

where we use the kinematic variable

$$\sigma = \frac{p_1 \cdot p_2}{m_1 m_2}\,. \tag{1.3.11}$$

In order to construct the $\mathcal{O}(\alpha^2)$ Hamiltonian we further need the corresponding one-loop amplitude. It was shown in Ref. [7] that classically, the 1-loop scalar YM amplitude has a basis of only one color factor, and moreover depends on the same topologies as in electrodynamics, so it's given by

$$\mathcal{A}^{\text{1-loop}} = \mathcal{C}\left(\;\right) \mathcal{A}^{\text{1-loop, QED}} + \ldots\,, \tag{1.3.12}$$

in terms of the one-loop QED amplitude. The color structure is given by

$$\mathcal{C}\left(\;\right) = (c_1 \cdot c_2)^2\,. \tag{1.3.13}$$

We could express the latter one-loop amplitude as a linear combination of scalar box, triangle, bubble and tadpole integrals, but Refs. [14, 17] showed that the bubble and tadpole integrals do not contribute to the classical limit. Dropping these pieces we write

$$i\mathcal{A}^{\text{1-loop, QED}} = d_{\Box}\, I_{\Box} + d_{\bowtie}\, I_{\bowtie} + c_{\triangle}\, I_{\triangle} + c_{\bigtriangledown}\, I_{\bigtriangledown}\,, \tag{1.3.14}$$

where the coefficients $d_{\Box}$, $d_{\bowtie}$, $c_{\triangle}$ and $c_{\bigtriangledown}$ are rational functions of external momenta. The integrals $I_{\Box}$, $I_{\bowtie}$, $I_{\triangle}$ and $I_{\bigtriangledown}$ are shown in Fig. 1.1. The triangle integrals take the form [14]

$$I_{\triangle,\bigtriangledown} = -\frac{i}{32m_{1,2}}\frac{1}{\sqrt{-q^2}} + \cdots\,. \tag{1.3.15}$$

The box contributions do not contain any novel $\mathcal{O}(\alpha^2)$ information. They correspond to infrared-divergent pieces that cancel out when we equate the full-theory and EFT amplitudes [14, 17]. In this sense, the explicit values for the box coefficients serve only as a consistency check of our calculation and we do not show them. Instead, we give the result for

$$i\mathcal{A}^{\triangle+\bigtriangledown} \equiv (c_{\triangle}\, I_{\triangle} + c_{\bigtriangledown}\, I_{\bigtriangledown})\,\mathcal{C}\left(\;\right)\,. \tag{1.3.16}$$

As detailed in Ref.[59], we use the generalized-unitarity method to obtain the integral coefficients of

Figure 1.2: The Compton-amplitude Feynman diagrams. The straight line corresponds to the massive color-charged particle. The wiggly lines correspond to gluons.

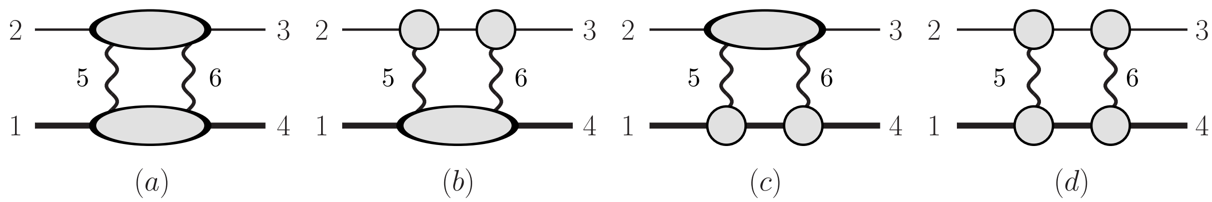


Figure 1.3: Appropriate residues of the two-particle cut (a) give the triple cuts (b) and (c), and the quadruple cut (d). The straight lines corresponds to the color-charged particles and the wiggly lines to the exchanged gluons. All exposed lines are taken on-shell.

Eq. (1.3.14). We start by calculating the Compton amplitude for the color-charged particle, using Feynman rules. Subsequently, we construct the two-particle cut. The residues of the two-particle cut on the matter poles give the triple cuts, and localizing both matter poles gives the quadruple cut. We obtain the triangle and box coefficients from the triple and quadruple cuts respectively. Our result reads

$$\mathcal{A}^{\triangle+\bigtriangledown} = \frac{2\pi^2\alpha^2}{\sqrt{-q^2}}\lambda_2\mathcal{C}\left(\square\right) + \dots\,, \tag{1.3.17}$$

where the coefficient is given by

$$\lambda_2 = 2m\,, \tag{1.3.18}$$

and $m = m_1 + m_2$. In preparation for the matching procedure in the following Section, we specialize our expressions to the center-of-mass frame. In this frame, the independent four-momenta read

$$p_1 = -(E_1, \boldsymbol{p})\,, \qquad p_2 = -(E_2, -\boldsymbol{p})\,, \qquad q = (0, \boldsymbol{q})\,, \qquad \boldsymbol{p}\cdot\boldsymbol{q} = \boldsymbol{q}^2/2\,. \tag{1.3.19}$$

Using the above expressions, our amplitudes take the form

$$\frac{\mathcal{A}^{\text{tree}}}{4E_1E_2} = \frac{4\pi\alpha}{\boldsymbol{q}^2}\Lambda_1\mathcal{C}\,(\text{tree})\,, \qquad \frac{\mathcal{A}^{\triangle+\bigtriangledown}}{4E_1E_2} = \frac{2\pi^2\alpha^2}{|\boldsymbol{q}|}\Lambda_2\mathcal{C}\left(\square\right)\,. \tag{1.3.20}$$

The coefficients Λ_i are given in terms of the λ_i of Eqs. (1.3.10) and (1.3.18) by

$$\Lambda_1 = -\frac{\nu\sigma}{\gamma^2\xi}, \qquad \Lambda_2 = \frac{1}{2m\gamma^2\xi}\,, \tag{1.3.21}$$

where in addition to the definition in Eq. (1.3.11) we use

$$\nu = \frac{m_1 m_2}{m^2} \qquad \gamma = \frac{E}{m}\,, \qquad E = E_1 + E_2\,, \qquad \xi = \frac{E_1 E_2}{E^2}\,. \tag{1.3.22}$$

Construction of the EFT amplitudes

With the full theory amplitudes in hand, we now turn our attention to the task of translating the scattering amplitudes of color-charged fields to a two-body conservative Hamiltonian. We do this by matching the scattering amplitude computed above to the two-to-two amplitude of an EFT of the positive-energy modes of fields. Ref. [14] developed this matching procedure for higher orders in the coupling constants and all orders in velocity, and we adapt it here to describe the color-charged fields ξ_1 and ξ_2. We follow closely the construction for classical spin in Ref. [59]. The action of the effective field theory (supressing representation indices) for ξ_1 and ξ_2 is given by

$$\begin{aligned} S = &\int \hat{\mathrm{d}}^{D-1}\boldsymbol{k} \sum_{a=1,2} \xi_a^\dagger(-\boldsymbol{k}) \left(i\partial_t - \sqrt{\boldsymbol{k}^2 + m_a^2}\right) \xi_a(\boldsymbol{k}) \\ &- \int \hat{\mathrm{d}}^{D-1}\boldsymbol{k} \int \hat{\mathrm{d}}^{D-1}\boldsymbol{k}'\, \xi_1^\dagger(\boldsymbol{k}')\xi_2^\dagger(-\boldsymbol{k}')\, V(\boldsymbol{k}', \boldsymbol{k}, \hat{\mathcal{C}}_i)\, \xi_1(\boldsymbol{k})\xi_2(-\boldsymbol{k})\,, \end{aligned} \tag{1.3.23}$$

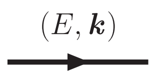
where the interaction potential $V(\boldsymbol{k}', \boldsymbol{k}, \hat{\mathcal{C}}_i)$ is a function of the incoming and outgoing momenta $\boldsymbol{k}$ and $\boldsymbol{k}'$ and the color-structure operators $\hat{\mathcal{C}}_i$. We consider kinematics in the center-of-mass frame. As on the full theory side, one could construct the color asymptotic states of ξ_i using $SU(N)$ coherent states (analogous to the spin coherents states of [59]) so color operators satisy the defining properties eqs. (1.2.5)-(1.2.6). We obtain the classical color charge vector as the expectation value of the color operator with respect to these on-shell states.

We build the most general potential containing only long-range classical contributions. This will be in terms of color operators, whose expectation values with respect to $SU(N)$ coherent states are in correspondence with the classical color structures in the full theory amplitude, Eq. (1.3.20). We use the following ansatz for the potential operator

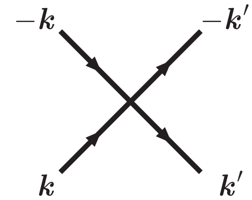
$$\hat{V}(\boldsymbol{k}', \boldsymbol{k}, \hat{\mathcal{C}}_i) = \frac{4\pi\alpha}{\hat{\boldsymbol{q}}^2} d_1\left(\hat{\boldsymbol{p}}^2\right) \hat{\mathcal{C}}\left(\text{tree}\right) + \frac{2\pi^2\alpha^2}{|\hat{\boldsymbol{q}}|} d_2\left(\hat{\boldsymbol{p}}^2\right) \hat{\mathcal{C}}\left(\square\right) + \mathcal{O}(\alpha^3)\,, \tag{1.3.24}$$

where $\hat{\boldsymbol{p}}^2 \equiv (\boldsymbol{k}^2 + \boldsymbol{k}'^2)/2$.

We now evaluate the EFT two-to-two scattering amplitude. To this end we use the Feynman rules derived from the EFT action (Eq. (1.3.23)),



$$= \frac{i\,\mathbb{I}}{E - \sqrt{\boldsymbol{k}^2 + m^2} + i\epsilon}\,,$$



$$= -iV(\boldsymbol{k}', \boldsymbol{k}, \hat{\mathcal{C}}_i)\,. \tag{1.3.25}$$

Using these rules we compute the amplitude up to $\mathcal{O}(\alpha^2)$ directly evaluating the relevant Feynman diagrams, omitting terms that do not contribute to long range interactions. The color factors must be treated as operators, and thus their ordering is important. After carrying out the energy integration, we obtain an expression for the amplitude

$$\hat{\mathcal{A}}^{\mathrm{EFT}} = -\,\hat{V}(\boldsymbol{p}', \boldsymbol{p}, \hat{\mathcal{C}}_i) - \int \hat{\mathrm{d}}^{D-1}\boldsymbol{k}\,\frac{\hat{V}(\boldsymbol{p}', \boldsymbol{k}, \hat{\mathcal{C}}_i)\,\hat{V}(\boldsymbol{k}, \boldsymbol{p}, \hat{\mathcal{C}}_i)}{E_1 + E_2 - \sqrt{\boldsymbol{k}^2 + m_1^2} - \sqrt{\boldsymbol{k}^2 + m_2^2}}\,. \tag{1.3.26}$$

We can now take the expectation value with respect to coherent states. At $\mathcal{O}(\alpha)$ the EFT amplitude receives a contribution only from the first term of Eq. (1.3.26), and after taking the expectation value with respect to coherent states the result is

$$\mathcal{A}^{\mathrm{EFT}}_{\mathcal{O}(\alpha)} = -\frac{4\pi\alpha}{\boldsymbol{q}^2} d_1 \mathcal{C}\,(\mathrm{tree})\,, \tag{1.3.27}$$

which is a c-number. On the other hand, the EFT amplitude at $\mathcal{O}(\alpha^2)$ receives contributions from both terms in Eq. (1.3.26) and can be written as

$$\mathcal{A}^{\mathrm{EFT}}_{\mathcal{O}(\alpha^2)} = \frac{2\pi^2\alpha^2}{|\boldsymbol{q}|}\Lambda_2\,\mathcal{C}\left(\square\right) + (4\pi\alpha)^2\,\Lambda_{\mathrm{iter}}\,\mathcal{C}\,(\mathrm{tree})^2 \int \hat{\mathrm{d}}^{D-1}\boldsymbol{\ell}\,\frac{2\xi E}{\boldsymbol{\ell}^2(\boldsymbol{\ell} + \boldsymbol{q})^2(\boldsymbol{\ell}^2 + 2\boldsymbol{p}\cdot\boldsymbol{\ell})}\,, \tag{1.3.28}$$

where $\boldsymbol{\ell} = \boldsymbol{k} - \boldsymbol{p}$ and we only keep terms that are relevant in the classical limit. Anticipating the matching, we write the amplitude in terms of Λ_2, which is given directly in terms of the momentum-space potential coefficient by

$$\Lambda_2 = -d_2 + \frac{1 - 3\xi}{2\xi E}\, d_1^2 + \xi E \partial_{\boldsymbol{p}^2} d_1^2\,, \tag{1.3.29}$$

The second term in Eq. (1.3.28) is infrared divergent and we have explicitly verified that it cancels out when we equate the full-theory and EFT amplitudes. The potential takes the form

$$V(\boldsymbol{r}^2, \boldsymbol{p}^2, \mathcal{C}_i) = \frac{\alpha}{|\boldsymbol{r}|} d_1(\boldsymbol{p}^2)\mathcal{C}\,(\mathrm{tree}) + \left(\frac{\alpha}{|\boldsymbol{r}|}\right)^2 d_2(\boldsymbol{p}^2)\mathcal{C}\left(\square\right) + \mathcal{O}(\alpha^3)\,. \tag{1.3.30}$$

We obtain the position-space Hamiltonian by taking the Fourier transform of the momentum-space[8] Hamiltonian with respect to the momentum transfer $\boldsymbol{q}$, which is the conjugate of the separation between the particles $\boldsymbol{r}$. We determine the momentum-space coefficient d_i in terms of the amplitudes coefficients Λ_i by a matching

[8]The position-space coefficients are trivially related to the momentum-space coefficients. This is unlike the case for spinning particles, where a set of linear relations was established between them.

procedure, i.e. by demanding that the EFT amplitude matches the full-theory one,

$$\mathcal{A}^{\text{EFT}}_{\mathcal{O}(\alpha)} = \frac{\mathcal{A}^{\text{tree}}}{4E_1E_2}\ , \qquad \mathcal{A}^{\text{EFT}}_{\mathcal{O}(\alpha^2)} = \frac{\mathcal{A}^{\text{1-loop}}}{4E_1E_2}\ , \tag{1.3.31}$$

where the factors of the energy account for the non-relativistic normalization of the EFT amplitude. Using Eq. (1.3.21) we relate Λ_i to λ_i, which are explicitly shown in Eqs. (1.3.10) and (1.3.18). Putting everything together, we obtain expressions for the position-space coefficients

$$d_1 = -\frac{\nu\sigma}{\gamma^2\xi}, \tag{1.3.32}$$

$$d_2 = \frac{1}{m\xi}\left(\frac{1}{2\gamma^2} - \frac{\nu\sigma}{\xi\gamma^3} + \frac{(1-\xi)\nu^2\sigma^2}{2\xi^2\gamma^5}\right). \tag{1.3.33}$$

This finishes the computation of the effective Hamiltonian. The classical equations of motion can now be solved iteratively using the Eqs.(1.3.3), (1.3.6) and the defintion of the observables (1.3.7). Following this procedure we have found agreement with the results of Section 1.2.

1.3.3 Observables from the eikonal phase

The conservative Hamiltonian we obtained in previous Sections enables the calculation of physical observables for a scattering of compact objects interacting through gluon exchange. Ref. [59] conjectured a formula that expresses physical observables in terms of derivatives of the eikonal phase for the spinning case. In this Section we extend that analysis.

Let us start by obtaining the eikonal phase via a Fourier transform of our amplitudes. Then, following Ref. [59] we can solve Hamilton's equations for the impulse and color impulse and relate them to derivatives of the eikonal phase. The eikonal phase $\chi = \chi_1 + \chi_2 + \mathcal{O}(\alpha^3)$ is given by

$$\begin{aligned}\chi_1 &= \frac{1}{4m_1m_2\sqrt{\sigma^2-1}}\int \hat{\mathrm{d}}^2\boldsymbol{q}\; e^{-i\boldsymbol{q}\cdot\boldsymbol{b}}\mathcal{A}^{\text{tree}}(\boldsymbol{q})\,,\\ \chi_2 &= \frac{1}{4m_1m_2\sqrt{\sigma^2-1}}\int \hat{\mathrm{d}}^2\boldsymbol{q}\; e^{-i\boldsymbol{q}\cdot\boldsymbol{b}}\mathcal{A}^{\triangle+\triangledown}(\boldsymbol{q})\,.\end{aligned} \tag{1.3.34}$$

Using our amplitudes expressed in the center-of-mass frame (see Eq. (1.3.20)) we find

$$\chi_1 = -\frac{\xi E\alpha}{|\boldsymbol{p}|}\Lambda_1\left(\ln\frac{\boldsymbol{b}^2}{L^2}\right)\mathcal{C}\left(\text{tree}\right)\ , \tag{1.3.35}$$

$$\chi_2 = \frac{\pi\xi E\alpha^2}{|\boldsymbol{p}|}\,\frac{\Lambda_2}{|\boldsymbol{b}|}\mathcal{C}\left(\quad\right)\ , \tag{1.3.36}$$

where in the first order eikonal phase we include a cutoff regulator L as we did in Section 1.2. In the case without color, the integration is regulated via dimensional regularization, and the divergence is ignored, because the derivative of the eikonal phase is always taken and they don't contribute. This is no longer the case here.

We may now use the eikonal phase to obtain classical observables. Generalizing the conjecture of Ref. [59]

to the color-charged case, the observables in question are the impulse $\Delta \boldsymbol{p}$ and color impulse Δc_i^a, where

$$\begin{aligned} \boldsymbol{p}(t=\infty) &= \boldsymbol{p} + \Delta \boldsymbol{p}\,, \qquad \boldsymbol{p}(t=-\infty) = \boldsymbol{p}\,, \\ c_i^a(t=\infty) &= c_i^a + \Delta c_i^a\,, \qquad c_i^a(t=-\infty) = c_i^a\,. \end{aligned} \tag{1.3.37}$$

Inspired by the gravitational spinning case let us decompose the impulse as

$$\Delta \boldsymbol{p} = \Delta p_{\parallel} \frac{\boldsymbol{p}}{|\boldsymbol{p}|} + \Delta \boldsymbol{p}_{\perp}\,, \tag{1.3.38}$$

where $\Delta p_{\parallel}$ can be obtained from the on-shell condition $(\boldsymbol{p} + \Delta \boldsymbol{p})^2 = \boldsymbol{p}^2$. Therefore, ignoring the mixing of spin and orbital angular momentum—which is absent in our case since the particle is spinless—the impulse and color impulse through $\mathcal{O}(\alpha^2)$ satisfy

$$\begin{aligned} \Delta \boldsymbol{p}_{\perp} &= -\{\boldsymbol{p}_{\perp}, \chi\} - \frac{1}{2}\{\chi, \{\boldsymbol{p}_{\perp}, \chi\}\}\,, \\ \Delta c_1^a &= -\{c_1^a, \chi\} - \frac{1}{2}\{\chi, \{c_1^a, \chi\}\}\,, \end{aligned} \tag{1.3.39}$$

where in Eq. (1.3.39) we use the definitions

$$\{\boldsymbol{p}_{\perp}, g\} \equiv -\frac{\partial g}{\partial \boldsymbol{b}}\,, \qquad \{c_1^a, g\} \equiv f^{abc} \frac{\partial g}{\partial c_1^b} c_1^c\,. \tag{1.3.40}$$

The second term in the linear impulse doesn't contribute because the tree color structure commutes with itself but we leave it there to keep the suggestive structure. It is then straightforward to show that the linear impulse will be reproduced here, the same way it was for the spinless QED case, simply by taking a replacement of electric for color charges. We have compared both the impulse and the color impulse, to the solution of the EOM, and the integrated result of the NLO color impulse finding full agreement.

Our calculation extends the conjecture of Ref. [59] to the domain of color. We may note that in this setting the momentum and the color are separately conserved. This is unlike the case for spinning particles, where only the sum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ is conserved. Due to the mixing of spin and orbital angular momentum, it was possible to define the object $\mathcal{D}_{SL}(f,g) \equiv -\boldsymbol{S}_1 \cdot \left(\frac{\partial f}{\partial \boldsymbol{S}_1} \times \frac{\partial g}{\partial \boldsymbol{L}_{\boldsymbol{b}}}\right)$ (where $\boldsymbol{S}_1$ is the spin vector and $\boldsymbol{L}_{\boldsymbol{b}} \equiv \boldsymbol{b} \times \boldsymbol{p}$). Such an object was necessary to add terms of the form $\mathcal{D}_{SL}(\chi, \{\boldsymbol{o}, \chi\})$ and $\{\boldsymbol{o}, \mathcal{D}_{SL}(\chi, \chi)\}$. In consequence, the form of Eq. (1.3.39) is indeed simpler than its spin counterpart.

1.4 Conclusions

In this chapter we have used the KMOC formalism and a matching procedure with a non-relativistic EFT to evaluate classical Yang-Mills observables. Using these approaches we have found that the eikonal phase conjecture of Ref. [59] to the case of color is realized at NLO. On the KMOC side we have used the integrands already computed in Ref. [7] and performed a direct integration, while on the EFT side we have used unitarity adapting the formalism by Cheung-Rothstein-Solon [14] to the case of color charges.

The integration of the color and momentum impulses follows from a simple integral reduction and techniques successfully applied in gravity, e.g., in Ref. [26]. We have found that, as in the case of gravity, the integrals related only with the box and crossed box vanish. However those related with the cut box contribute as expected. In order to expose the exponentiation of the NLO color impulse we have used a cut-off regulator as in Ref. [7] to evaluate cut-box integrals.

Once the color decomposition has been performed and the classical relevant parts identified the matching procedure follows essentially the QED case. The Hamiltonian thus constructed was used to solve the equations of motion and obtain the classical linear impulse and color impulse by direct integration. The results were in complete agreement to the evaluation using KMOC integrands. Finally, the eikonal phase construction matches the result of the KMOC and of EOM in a rather elegant way giving more evidence of the observation Ref. [59] that all physical observables are obtainable via simple manipulations of the scattering amplitude.

For the case of impulses it is also worth mentioning that the intricacies due to the mixing of color and kinematics in the KMOC calculation are absent in the rather straightforward construction based on unitarity and EFT. However, for the construction of the EFT it was crucial to employ coherent states to obtain the classical limit, so this aspect is common to both approaches as is the use of the Lie algebra of the rescaled color factors. Obtaining higher order corrections in the KMOC formalism for Yang-Mills observables would be perhaps more efficient using unitarity from the beginning as done in Refs. [10, 11] (for the gravitational case), benefiting from advances in relativistic integration.

Our results provide evidence in favor of the eikonal phase conjecture of Ref. [59], and so they call for the calculation of the 2-loop color impulse as a toy example towards the gravitational spin. Besides being a toy model for gravitational dynamics, the classical limit of Yang-Mills theory is useful to describe non-equilibrium plasma through kinetic theory, where color is treated as a continuous classical variable. In Ref. [12] solutions of kinetic equations were interpreted as classical limits of certain off-shell currents so it would be interesting to explore a Hamiltonian perspective to this problem.

Chapter 2

Quantum Field Theory, Worldline Theory, and Spin Magnitude Change in Orbital Evolution

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A previous paper [101] identified a puzzle stemming from the amplitudes-based approach to spinning bodies in general relativity: additional Wilson coefficients appear compared to current worldline approaches to conservative dynamics of generic astrophysical objects, including neutron stars. In this chapter we clarify the nature of analogous Wilson coefficients in the simpler theory of electrodynamics. We analyze the original field-theory construction, identifying definite-spin states some of which have negative norms, and relating the additional Wilson coefficients in the classical theory to transitions between different quantum spin states. We produce a new version of the theory which also has additional Wilson coefficients, but no negative-norm states. We match, through $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(S^2)$, the Compton amplitudes of these field theories with those of a modified worldline theory with extra degrees of freedom introduced by releasing the spin supplementary condition. We build an effective two-body Hamiltonian that matches the impulse and spin kick of the modified field theory and of the worldline theory, displaying additional Wilson coefficients compared to standard worldline approaches. The results are then compactly expressed in terms of an eikonal formula. Our key conclusion is that, contrary to standard approaches, while the magnitude of the spin tensor is still conserved, the magnitude

of the spin vector can change under conserved Hamiltonian dynamics and this change is governed by the additional Wilson coefficients. For specific values of Wilson coefficients the results are equivalent to those from a definite spin obeying the spin supplementary condition, but for generic values they are physically inequivalent. These results warrant detailed studies of the corresponding issues in general relativity.

2.1 Introduction

2.1.1 General Overview

The landmark detection of gravitational waves by the LIGO/Virgo collaboration [102, 103] opened a new era in astronomy, cosmology and perhaps even particle physics. As gravitational-wave detectors become more sensitive [104, 105, 106], the spin of objects such as black holes and neutron stars will play an increasingly important role in identifying and interpreting signals. Spin also leads to much richer three-dimensional dynamics because of the exchange of angular momentum between bodies and their orbital motion. Its precise definition leads to interesting and subtle theoretical questions, some of which we address here.

The study of the dynamics of spinning objects in general relativity [107, 108, 109, 110] has a long history, in both the post-Newtonian (PN) framework [111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 51, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163], where observables are simultaneously expanded in Newton's constant G and in the velocity v, and the post-Minkowskian (PM) framework [164, 165, 6, 57, 56, 52, 78, 21, 22, 58, 53, 54, 55, 59, 61, 60, 65, 62, 166, 167, 168, 169, 66, 170, 171, 172, 173, 174, 101, 168, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184], where observables are expanded only in Newton's constant with exact velocity dependence. In both approaches the interaction of spinning objects with the gravitational field is described in terms of a set of higher-dimension operators whose Wilson coefficients encode the detailed properties of the objects. For the interesting case of black holes, the values of these coefficients at $\mathcal{O}(G)$ are known [58], with proposals for the additional coefficients at $\mathcal{O}(G^2)$ recently given based on a shift symmetry [173, 101, 179, 180] already present at $\mathcal{O}(G)$.[1] The electromagnetic case is similar in structure [187, 188, 189, 5, 190, 191, 192] (see also Refs. [7, 1] for non-abelian generalisations), with the post-Coulombian (PC) and post-Lorentzian (PL) expansions being the respective analogs of the gravitational PN and PM expansions.

A primary purpose of this chapter is to explore puzzles identified in Ref. [101] regarding the description of spinning bodies in general relativity. In that paper, results for the conservative two-body scattering angle

[1] Through $O(S^4)$ Refs. [185, 186] find that the Compton amplitude derived by solving the Teukolsky equation agrees with with these previous results. However, the predictions based on shift symmetry at $\mathcal{O}(S^5)$ are in tension with results from the Teukolsky equation, though the latter involve a subtle analytic continuation between the black-hole and naked-singularity regimes.

were obtained through fifth power in the spin using a scattering-amplitudes-based method. A rather striking outcome, which follows from the fact that the field-theory Lagrangian is not directly expressed in terms of particles' spin tensor, is that the field-theory approach of Ref. [59] has a larger number of independent Wilson coefficients for a given power of spin than standard (worldline) methods. While at 1PM (tree) level the number of independent Wilson coefficients is identical in the two approaches, matching of physical observables starting at 2PM (one loop) and third power of the spin can only be attained by setting some of the field-theory Wilson coefficients to definite numerical values, so that they are no longer independent. This implies that the field theory contains a larger number of physically-relevant independent Wilson coefficients. For the special case of Kerr black holes it appears that the additional Wilson coefficients present in the field theory are not needed [58, 167, 168, 173, 101]. In electrodynamics we find a similar situation for the root-Kerr solution [193], related to the Kerr solution via the double copy [194, 195, 196, 92, 87, 80].

The connection between scattering amplitudes and effective two-body interactions has been known for some time [197, 198, 199, 200, 201, 50, 13, 202, 203, 204, 96]. Recent years have seen the construction of new systematic methods for extracting potentials and physical observables at high orders from scattering amplitudes [14, 5, 16, 17, 19, 20, 205], which leverage modern methods for calculating scattering amplitudes, including generalized unitarity [206, 207, 208, 209, 210, 211], the double copy [194, 195, 196, 80] and advanced integration techniques [212, 213, 214, 215, 215, 216, 33]. The extraction of classical physics from quantum scattering is greatly simplified by concepts from effective field theories (EFTs), systematized for the gravitational-wave problem in Ref. [217] and applied to the PM framework in Ref. [14]. By manifestly maintaining Lorentz invariance, the amplitudes approach fits naturally in the PM or PL frameworks, and produced the first conservative spinless two-body Hamiltonian at $\mathcal{O}(G^3)$ and $\mathcal{O}(G^4)$ [16, 17, 30, 218] (see also Refs. [18, 27, 28, 219, 220, 221, 222, 181]). Such methods also led to new perspectives on the gravitational interactions of spinning particles [223, 54, 53, 57, 55, 78, 21, 59, 22, 167, 101] and on tidal effects [224, 26, 47, 43, 46, 49, 48, 44].

Here we use both the amplitudes-based method and the more standard worldline approach [217, 140, 225, 145, 226, 227] to study the interactions of spinning particles. Since they describe the same physics, one may expect that there is a (usually nontrivial) correspondence between the operators (as well as between their Wilson coefficients) describing these interactions in the two approaches. Each type of object, whether a Kerr black hole or neutron star, is described by particular values for the Wilson coefficients, which are determined by an appropriate matching calculation. In the worldline approach one imposes a spin supplementary condition (SSC) [228] that identifies the three physical spin degrees of freedom. This condition has been interpreted in terms of a spin-gauge symmetry which formalizes the freedom to shift the worldline in the ambient space [229, 145, 230] without changing the physics. An important aspect of an SSC is that it reduces the number of possible independent operators—and consequently the number of Wilson coefficients—by equating operators whose difference is proportional to the SSC. Here we use the dynamical mass function formalism of Ref. [226] to explore the consequences of relaxing the SSC and to help interpret the additional degrees of freedom.

An interesting subtlety in the amplitudes approach is whether the complete description of a spinning compact body is provided by a single quantum spin $s \gg 1$ or by a suitable combination of multiple quantum spins, with possible transitions between them. For the sake of simplicity, the field theory of Ref. [59]—meant to be valid only in the classical limit—is based on the matter states forming an irreducible representation of the Lorentz group but a reducible representation of the rotation group; some of its components have negative norm. One might worry these negative-norm states might lead to some difficulties in the classical limit [231]. In addition, projecting onto the physical states of a quantum spin s [231, 179] appears to effectively remove the additional Wilson coefficients, leaving only those included in the worldline framework, which we affirm here. Field-theory approaches [167, 168, 173, 180] based on the massive-spinor-helicity amplitudes [193] are a convenient means for restricting the propagation to a single irreducible quantum spin. Here we use physical state projectors [232, 233] for the same purpose.

The results of Ref. [101] raises several questions:

1. What is a complete description of a spinning body in general relativity?
2. Can one construct a worldline theory that matches field-theory descriptions containing extra independent Wilson coefficients? If so, what extra degrees of freedom are needed?
3. The field-theory construction of Ref. [101] uses propagating reducible representations of the rotation group (spin representations), some with negative norm. In the context of this construction, what happens if only a single quantum spin propagates?
4. Can one build a field theory based on positive-norm irreducible representations of the rotation group that also contain extra independent Wilson coefficients?
5. Should a classical spin be modeled as a definite-spin field or as a superposition of fields with different spins? A related question on the latter case is whether transitions between different spins are allowed that change the magnitude of the spin vector even in the conservative sector.[2]
6. Can one build an effective two-body Hamiltonian with extra degrees of freedom whose physical observables match field-theory results containing extra Wilson coefficients?

[2]With dissipation and absorption included the spin magnitude is, of course, not preserved (see e.g. Refs. [234, 235, 236] for recent discussions).

7. What is the physical interpretation of the operators associated with additional Wilson coefficients?

To address these questions we turn to electrodynamics, which has been useful as a toy model for gravity [187, 188, 189, 5, 190, 191, 192]. While electrodynamics cannot answer all questions about gravity, the overlap is more than sufficient to make this a useful test case. In addition to the absence of photon self-interactions, electrodynamics is particularly helpful for our questions because the additional independent operators and their Wilson coefficients affect observables already at the first order spin, rather than at third order as for gravity, greatly simplifying the analysis.

We use various field theories, worldline theories and effective two-body Hamiltonians, comparing and contrasting the results from each. In particular, to help identify the origin of the extra Wilson coefficients we evaluate Compton amplitudes and scattering angles for three related but distinct field theories of electrodynamics coupled with higher-spin fields:

FT1: The setup from Refs. [59, 101], except for electrodynamics instead of general relativity. The matter states of this theory form an irreducible representation of the Lorentz group and a reducible representation of the rotation group, thereby as a quantum theory it carries more degrees of freedom than those of a fixed-spin particle, including negative-norm states. In this theory we consider FT1s with classical asymptotic states having spin tensors obeying the covariant spin supplementary condition (SSC), $S_{\mu\nu}p^{\nu} = 0$, and FT1g with classical asymptotic states having unconstrained spin tensors. This is equivalent to relaxing the covariant SSC, so that the resulting amplitudes explicitly contain factors of $S_{\mu\nu}p^{\nu}$. When we do not need to distinguish between FT1s and FT1g, we collectively refer to them as FT1. The results of FT1s are obtained from those of FT1g simply by imposing the covariant SSC on the initial and final spin tensors.

FT2: The higher-spin field is constrained to contain a single irreducible spin-s representation of the rotation group [232]. The external massive states are traceless and transverse due to the equation of motion. In contrast to FT1s and FT1g, only positive-norm states propagate, and as we shall see, the covariant SSC is automatically imposed on the spin tensors.

FT3: The same construction as for FT2 except that two positive-norm irreducible representations of the rotation group, one with spin-s and the other with spin-$(s-1)$, are considered. While this field content allows us to reliably capture effects linear in spin, it is sufficient to demonstrate that such field theories support more Wilson coefficients than FT2. We include suitable couplings between matter fields of different spin. Similarly to FT1, we consider FT3s with asymptotic states having spin tensors obeying the covariant SSC and FT3g with asymptotic states being a particular combination of the asymptotic states of the two fields. When we do not need to distinguish between FT3s and FT3g, we collectively refer to them as FT3.

The above field-theory constructions do not exhaust the ways to adjust the spectrum of propagating states. For example, one can use the chiral construction of Ref. [237], based on the representation $(2s, 0)$ of the Lorentz group leading to the same number $(2s+1)$ of propagating degrees as a quantum spin-s particle. We note that FT1 are not fully consistent as quantum theories because of the appearance of propagating negative-norm states. Because of this we use them only in the classical limit, as envisioned in Ref. [59]. We moreover see that there is a close relation between them and FT3, which is constructed using only positive-norm states.

To address the question of what kind of worldline theory has the same observables as field theories with extra Wilson coefficients we consider two worldline theories:

WL1: The standard worldline construction with the covariant SSC imposed. We use the formalism of Ref. [226].

WL2: A modified worldline construction with no SSC imposed and consequently with extra degrees of freedom. In the absence of an SSC we can include additional operators and Wilson coefficients equivalent to the additional ones that can be included in FT1g through the constructed orders.

Finally, we construct two two-body effective field-theory Hamiltonians by matching the amplitudes of field theories with different number of internal and asymptotic degrees of freedom. This allows us to directly construct observables for these field theories and compare them with worldline theories:

EFT1: The two-body Hamiltonian of the type in Ref. [59] containing only the spin vector $\mathbf{S}$ for each body. The parameters of this Hamiltonian can be adjusted to match either FT2 or WL1. We may also match this Hamiltonian to FT1, FT3, and WL2 when the additional Wilson coefficients are set to specific values.

EFT2: A two-body Hamiltonian containing both a spin vector $\mathbf{S}$ and a Lorentz boost vector $\mathbf{K}$, interpreted as a mass dipole and inducing an electric dipole. With suitable parameters this Hamiltonian matches FT1g, FT3g and WL2.

2.1.2 Summary of Results

We compute and compare electrodynamics Compton amplitudes, impulses, spin kicks and scattering angles in the theories outlined above. With α denoting the fine structure constant, the results of these computations through $\mathcal{O}(\alpha^2 S)$ for two-body observables and through $\mathcal{O}(\alpha S^2)$ for Compton amplitudes yield the following findings:

1. In electrodynamics with the massive propagating degrees of freedom of a single spin-s particle realized as a symmetric traceless transverse s-index tensor, as in FT2 and following Ref. [232], the number of Wilson coefficients agrees with the standard worldline construction [145], in accord with Refs. [231, 179].

2. By including additional degrees of freedom either by relaxing the transversality constraint of fields or/and by replacing the s-index symmetric tensor by a more general (l, r) representation of the Lorentz group, as in FT1, additional Wilson coefficients can appear in the classical limit. Thus, the additional Wilson coefficients reflect the additional degrees of freedom present in nontransverse fields.

3. We demonstrate that additional propagating positive-norm degrees of freedom in the form of symmetric traceless transverse lower-rank tensor, as in FT3, also lead to additional Wilson coefficients in the classical limit. Thus, the additional Wilson coefficients are not tied specifically to nontransverse fields, but are a manifestation of additional propagating degrees of freedom.

4. By relaxing the SSC constraint on the worldline, the Compton amplitudes as well as two-body physical observables such as the impulse and spin kick, match the corresponding results of field theories FT1 and FT3.

5. To match the worldline and field-theory amplitudes with additional asymptotic degrees of freedom and Wilson coefficients, a two-body EFT Hamiltonian with both spin and boost degrees of freedom are required.

6. In the systems with additional degrees of freedom and additional Wilson coefficients, the magnitudes of spin vectors are not preserved[3] in the scattering process while the magnitudes of spin tensors are preserved.

7. For specific choices of Wilson coefficients, such as the root-Kerr solution [193], the extra degrees of freedom decouple and the system can be described by removing the boost degrees of freedom.

These results are rather striking. Dropping the SSC would seem to contradict the standard interpretation of the worldline spin gauge symmetry, where local shifts in the worldline are interpreted as a symmetry [229, 145, 230]. Here we are reinterpreting this in terms of certain degrees of freedom of extended nonrigid objects, in much the same way as the spin is interpreted as an internal degree of freedom. As we discuss in section 2.4, in the electromagnetic case there is a natural explanation in terms of an induced electric dipole moment correlated to the mass dipole.

[3] We note that the non-conservation of the magnitude of the intrinsic angular momentum of subsystems of gravitationally-interacting conservative many-body systems has been known for some time, see e.g. [238, 239, 240].

This chapter is organized as follows: In section 2.2 we present the field-theory constructions FT1. FT2 and FT3 for electrodynamics, giving a nonminimal Lagrangian that contains additional Wilson coefficients compared to the standard worldline approaches. We also describe the classical asymptotic states in terms of coherent states and discuss the effect of using different Lorentz representations. The purpose of the various field theories is to identify the source of the extra Wilson coefficients. section 2.3 then gives the field-theory amplitudes associated with these theories, including the Compton tree amplitudes needed to build the one-loop two-body amplitudes, which are also presented. To interpret these results in the context of the more standard worldline framework, in section 2.4 we construct the two worldline theories WL1 and WL2 and compare their Compton amplitudes with the field-theory ones. In section 2.5 we construct two-body EFT Hamiltonians so that the scattering amplitudes of the corresponding EFTs match those of the various field theories. One Hamiltonian contains only the usual spin operator and the other also contains a boost operator. The impulse and spin kick derived from the latter are the same as those following from the SSC-less worldline theory. A remarkably compact form of physical observables is given in terms of an eikonal formula. Section 2.6 describes the link between extra Wilson coefficients and the degrees of freedom that propagate in the field theory. In section 2.7 we summarize our conclusions.

2.2 Field Theory

In this section we construct the field theories FT1, FT2 and FT3 listed in section 2.1 that we use to track the source of additional degrees of freedom and Wilson coefficients. We begin by discussing the covariantization of the free matter Lagrangians, which we refer to as the "minimal" Lagrangians, first in the framework of Refs. [59, 101] where the propagating states form a reducible representation of the rotation group, and then in the framework of Ref. [232], in which the only propagating states are only the $2s+1$ physical states of a spin-s field. After summarizing the coherent-state description of the classical asymptotic states and the propagators, we then discuss nonminimal interactions which are linear in the photon field strength and the corresponding three-point amplitudes. The scaling of massive momenta p, massless transferred momentum q, impact parameter b and spins S for obtaining the classical limit are [59]

$$p \to p, \qquad q \to \lambda q, \qquad b \to \lambda^{-1} b, \qquad S \to \lambda^{-1} S \,, \tag{2.2.1}$$

and the classical part of the L-loop two-body amplitude scales as λ^{-2+L} while Compton amplitudes scale as λ^0.[4] The connection of field theories FT1, FT2 and FT3 to worldline theories will be discussed in section 2.4.

[4]This scaling enforces the correspondence principle and the scaling parameter λ can be related to $\hbar$, see e.g. Ref. [241].

2.2.1 Minimal Lagrangian in Electrodynamics

The extension of the construction of Refs. [59] to QED and thus the definition of the covariantization of the free Lagrangians for FT1 is straightforward, with the main difference from the gravitational case being that the fields must be complex. The minimal coupling involves only the standard two-derivative kinetic term[5]

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \qquad \mathcal{L}_{\text{min}} = -(-1)^s\phi_s(D^2+m^2)\bar{\phi}_s\,, \tag{2.2.2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the covariant derivative is defined as

$$D_\mu\phi_s = \partial_\mu\phi_s - iQA_\mu\phi_s\,, \qquad D_\mu\bar{\phi}_s = \partial_\mu\bar{\phi}_s + iQA_\mu\bar{\phi}_s\,. \tag{2.2.3}$$

Without the loss of generality, we take all the massive bodies as carrying the same charge Q, and define the effective "fine structure constant"[6] as $\alpha = Q^2/(4\pi)$. The PL framework expands observables in powers of α keeping the exact velocity dependence. In $\mathcal{L}_{\text{min}}$, the fields ϕ_s and $\bar{\phi}_s$ can be in generic representations of the Lorentz group as long as their product is a Lorentz-singlet. The most general choice is that both fields are in the (l, r) representation, i.e. they are represented as

$$\phi_s = \phi_{\alpha_1\alpha_2\ldots\alpha_l}^{\dot{\beta}_1\dot{\beta}_2\ldots\dot{\beta}_r}\,, \qquad \bar{\phi}_s = \bar{\phi}^{\alpha_1\alpha_2\ldots\alpha_l}_{\dot{\beta}_1\dot{\beta}_2\ldots\dot{\beta}_r}\,, \tag{2.2.4}$$

where $l+r = 2s$ and ϕ_s and $\bar{\phi}_s$ are symmetric in the α_i and $\dot{\beta}_i$ indices, which transform in the two-dimensional representation of $SU(2)_L$ and $SU(2)_R$, respectively. The covariantized free Lagrangian $\mathcal{L}_{\text{min}}$ in (2.2.2) treats uniformly all the representations of the rotation group that are part of ϕ_s. Thus, the propagator derived from $\mathcal{L}_{\text{min}}$ is proportional to the identity operator $\mathbb{1}_{(l,r)}$ in the (l, r) representation. For ϕ_s in the (s, s) representation, it is

$$\underset{\mu(s)\qquad\qquad\nu(s)}{\overline{\qquad\qquad}} = \frac{(-1)^s i\,\delta^{\nu(s)}_{\mu(s)}}{p^2-m^2}\,, \qquad \delta^{\nu(s)}_{\mu(s)} \equiv \delta^{(\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\ldots\delta^{\nu_s)}_{\mu_s} \equiv \mathbb{1}_{(s,s)}\,. \tag{2.2.5}$$

Consequently, there is no explicit dependence on the value of s in the amplitudes that follow from $\mathcal{L}_{\text{min}}$, making the large-spin limit appropriate for classical physics convenient in this construction.

When evaluated on an (s, s) representation, the Lagrangian (2.2.2) contains propagating degrees of freedom beyond the $2s + 1$ associated with a single massive spin-s particle and some of them have negative norm in Minkowskian signature. While such a theory is not consistent as a quantum theory because of difficulties with unitarity, we use this Langangian and its nonminimal extension described below to find only classical observables, so that the issue is not directly relevant. One may nevertheless worry that the negative norm states might cause some inconsistency even in the classical limit, and very likely they are the origin of

[5]We are using the mostly minus signature. The $(-1)^s$ factor makes the spin-s component physical.

[6]Note that this differs from the standard definition of the fine structure constant in terms of the electron charge. To simplify subsequent formulae, we absorb in α the charge of macroscopic bodies.

the additional Wilson coefficients [231]. As we will see in 2.6, the key to the additional Wilson coefficients is the presence of propagating degrees of freedom beyond those of a single quantum spin-s particle. This is independent of the sign of the norm of the extra states. Moreover, there is a direct simple map which connects amplitudes in this theory with amplitudes in a theory in which all states have positive norm.

FT2 is designed to probe the relation between the extra Wilson coefficients and the presence of states beyond those of a spin-s representation of the rotation group. To define it and to compare straightforwardly with the Singh-Hagen Lagrangian [232] for a single spin-s particle it is convenient to choose ϕ_s in the (s, s) representation, which is realized as a symmetric traceless rank-s tensor,[7]

$$\phi_s \equiv \phi^{\dot\beta_1\dot\beta_2\ldots\dot\beta_s}_{\alpha_1\alpha_2\ldots\alpha_s} \propto \phi^{(\mu_1\mu_2\ldots\mu_s)}(\sigma_{\mu_1})_{(\alpha_1}{}^{(\dot\beta_1}\ldots(\sigma_{\mu_s})_{\alpha_s)}{}^{\dot\beta_s)}\,, \tag{2.2.6}$$

where as usual the parenthesis on the indices signify that they are symmetrized. We primarily focus on this representation in subsequent sections, especially when carrying out calculations at fixed values of the spin.

We ensure that only the $2s+1$ states of a spin-s field are propagating by imposing the requisite constraints with auxiliary fields, following the strategy of Ref. [232]. The net effect of imposing transversality is that the minimal Lagrangian $\mathcal{L}_{\rm min}$ in Eq. (2.2.2) is modified to

$$\mathcal{L}^s_{\rm min} = -(-1)^s\Big[\phi_s(D^2+m^2)\bar\phi_s + s(D\phi_s)(D\bar\phi_s) + \ldots\Big]\,, \tag{2.2.7}$$

where $(D\phi_s) \equiv D_\mu\phi^{\mu\mu_2\cdots\mu_s}$ and the ellipsis stand for terms that remove unwanted states, as explained below.

The coupling $s(D\phi_s)(D\bar\phi_s)$ originates from integrating out an auxiliary ϕ_{s-1} field that impose transversality via the equation of motion. To see this, we add to free part of $\mathcal{L}_{\rm min}$ the term $a\phi_{s-1}\partial\phi_s$ as well as a standard quadratic term for ϕ_{s-1}, where a is a normalization. The equations of motion are

$$(\partial^2+m^2)\phi_s^{\mu_1\mu_2\cdots\mu_s} = a\,\partial^{(\mu_1}\phi_{s-1}^{\mu_2\cdots\mu_s)}\,, \qquad (b\,\partial^2 + c\,m^2)\phi_{s-1} = (\partial\phi_s)\,, \tag{2.2.8}$$

where we introduced two additional normalization constants b and c. A solution to the equation of motion is $\phi_{s-1} = \partial\phi_s = 0$. Requiring that this is the only solution gives $b=0$ and $a = scm^2$ such that

$$(\partial^2+m^2)\phi_s^{\mu_1\mu_2\cdots\mu_s} = s\,\partial^{(\mu_1}(\partial\phi_s)^{\mu_2\cdots\mu_s)}\,. \tag{2.2.9}$$

Covariantization with respect to the photon gauge symmetry follows as usual, by replacing the partial derivatives with the appropriate covariant derivatives, leading to the $s(D\phi_s)(D\bar\phi_s)$ term in (2.2.7).

The process continues, as transversality of ϕ_s implies $\partial\partial\phi_s = 0$, which must also be imposed through an equation of motion. More auxiliary fields are therefore needed, and this process can be carried out recursively [232]. The resulting couplings involving traces, multiple-divergences like $D_\mu D_\nu\phi^{\mu\nu\mu_3\cdots\mu_s}$, and auxiliary fields with lower spins are collected in the ellipsis in (2.2.7). Up to $s=3$, the Lagrangians generated

[7] Throughout the chapter, the symmetrization is defined as $f_{(\mu_1\mu_2\ldots\mu_s)} \equiv \frac{1}{s!}(f_{\mu_1\mu_2\ldots\mu_s} + \text{ permutations})$.

by this procedure are

$$\mathcal{L}_{s=1} = \phi^{\mu_1}(D^2+m^2)\bar{\phi}_{\mu_1} + (D_\mu\phi^\mu)(D^\nu\bar{\phi}_\nu)\,, \tag{2.2.10a}$$

$$\begin{aligned}\mathcal{L}_{s=2} = &-\phi^{\mu_1\mu_2}(D^2+m^2)\bar{\phi}_{\mu_1\mu_2} - 2(D_\mu\phi^{\mu\mu_2})(D^\nu\bar{\phi}_{\nu\mu_2}) + \phi_\mu{}^\mu(D^2+m^2)\bar{\phi}^\nu{}_\nu\\ &-\phi_\mu{}^\mu D^\rho D^\lambda\bar{\phi}_{\rho\lambda} - \bar{\phi}^\mu{}_\mu D_\rho D_\lambda\phi^{\rho\lambda}\,,\end{aligned} \tag{2.2.10b}$$

$$\begin{aligned}\mathcal{L}_{s=3} = &\,\phi^{\mu_1\mu_2\mu_3}(D^2+m^2)\bar{\phi}_{\mu_1\mu_2\mu_3} + 3(D_\mu\phi^{\mu\mu_2\mu_3})(D^\nu\bar{\phi}_{\nu\mu_2\mu_3})\\ &-3\phi_\mu{}^{\mu\mu_3}(D^2+m^2)\bar{\phi}^\nu{}_{\nu\mu_3} + 3\phi_\mu{}^{\mu\mu_3}D^\rho D^\lambda\bar{\phi}_{\rho\lambda\mu_3} + 3\bar{\phi}^\mu{}_{\mu\mu_3}D_\rho D_\lambda\phi^{\rho\lambda\mu_3}\\ &+\frac{3}{2}(D_\mu\phi^{\mu\rho}{}_\rho)(D_\nu\bar{\phi}^{\nu\lambda}{}_\lambda) + 2\varphi(D^2+4m^2)\bar{\varphi} + m(\varphi D_\mu\bar{\phi}^{\mu\lambda}{}_\lambda + \bar{\varphi}D_\mu\phi^{\mu\lambda}{}_\lambda)\,,\end{aligned} \tag{2.2.10c}$$

where φ and $\bar{\varphi}$ in $\mathcal{L}_{s=3}$ are ghost-like scalar auxiliary fields. The $\mathcal{L}_{s=1}$ and $\mathcal{L}_{s=2}$ here are the Proca [242] and Fierz-Pauli Lagrangian [243], respectively, and $\mathcal{L}_{s=3}$ was first obtained by Chang [244]. We note that the construction in Ref. [232] uses only symmetric and traceless fields, and we have absorbed certain auxiliary fields into the trace of ϕ_s. We use (2.2.7) — which is the arbitrary-spin generalization of Eq. (2.2.10) — as the covariantization of the free Lagrangian of FT2.

FT3 is constructed to probe whether the extra Wilson coefficients in FT1 are due to the unphysical nature of the extra states of this theory. Thus, we define the covariantization of the free part of FT3 as being given, up to nonminimal terms, by the sum of Lagrangians for physical transverse fields with spins s, $s-1$, $\dots, 0$. For simplicity, here we consider a Lagrangian that involves only spin s and $s-1$,

$$\begin{aligned}\mathcal{L}^{s,s-1}_{\text{min}} = \mathcal{L}^{s}_{\text{min}} + \mathcal{L}^{s-1}_{\text{min}} = -(-1)^s\Big[\phi_s(D^2+m^2)\bar{\phi}_s + s(D\phi_s)(D\bar{\phi}_s) + \dots\Big]\\ -(-1)^{s-1}\Big[\phi_{s-1}(D^2+m^2)\bar{\phi}_{s-1} + (s-1)(D\phi_{s-1})(D\bar{\phi}_{s-1}) + \dots\Big]\,.\end{aligned} \tag{2.2.11}$$

We show below that this Lagrangian is sufficient to describe classical physics at $\mathcal{O}(S^1)$ up to the one-loop order. We assume that ϕ_s and ϕ_{s-1} have the same minimal coupling to the photon. Somewhat loosely, one may interpret this Lagrangian as being obtained from Eq. (2.2.2) upon separating ϕ_s into fields obeying transversality constraints and dropping the derivative factors that are responsible for the negative norms of the $s-(2k+1)$ components.

The minimal Lagrangians of FT2 and FT3 make explicit reference to the value of s, as can be seen in the explicit expressions in (2.2.10), and consequently the propagators (and vertices) have the same property. The propagators for massive $s=1$ and $s=2$ fields can be easily derived from the quadratic part of $\mathcal{L}_{s=1}$ and $\mathcal{L}_{s=2}$. They are

$$\overset{\mu\qquad\quad\nu}{\overline{\qquad\quad}} = \frac{-i\mathcal{P}_{\mu,\nu}}{p^2-m^2} = \frac{-i\Theta_{\mu\nu}}{p^2-m^2} = \frac{-i}{p^2-m^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)\,, \tag{2.2.12}$$

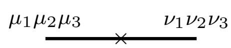
$$\overset{\mu_1\mu_2\qquad\nu_1\nu_2}{\overline{\qquad\quad}} = \frac{i\mathcal{P}_{\mu_1\mu_2,\nu_1\nu_2}}{p^2-m^2} = \frac{i}{p^2-m^2}\,\frac{1}{2}\left[\Theta_{\mu_1\nu_1}\Theta_{\mu_2\nu_2} + \Theta_{\mu_1\nu_2}\Theta_{\mu_2\nu_1} - \frac{2}{3}\Theta_{\mu_1\mu_2}\Theta_{\nu_1\nu_2}\right]\,, \tag{2.2.13}$$

where $\Theta_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}$. The numerators are instances of the spin-s state projector $\mathcal{P}$; its general closed-form expression [232],

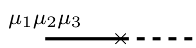
$$\mathcal{P}^{\nu(s)}_{\mu(s)} = \sum_{j=0}^{\lfloor s/2 \rfloor} \frac{(-1)^j s!(2s-2j-1)!!}{2^j j!(s-2j)!(2s-1)!!} \Theta_{(\mu_1\mu_2} \Theta^{(\nu_1\nu_2} \ldots \Theta_{\mu_{2j-1}\mu_{2j}} \Theta^{\nu_{2j-1}\nu_{2j}} \Theta^{\nu_{2j+1}}_{\mu_{2j+1}} \ldots \Theta^{\nu_s)}_{\mu_s)} \,, \tag{2.2.14}$$

is manifestly symmetric, transverse and traceless on-shell.

Beyond $s=2$ the off-diagonal nature of the quadratic terms in $\mathcal{L}^s_{\text{min}}$ makes the construction of propagators more involved. For example, the $\mathcal{L}^{s=3}_{\text{min}}$ Lagrangian contains quadratic mixing between $\phi_{\mu_1\mu_2\mu_3}$ and the auxiliary scalar φ; thus to derive the propagators it is necessary to diagonalize the quadratic terms, effectively summing over all possible insertions of such two-point vertices. We represent the resummed propagators by a cross in the middle,



$$= \frac{-i\,\mathcal{P}^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}}{p^2-m^2} + \frac{i\,\mathcal{Q}^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}}{40m^6} \,, \tag{2.2.15a}$$



$$= \frac{(p^2-m^2)\eta_{(\mu_1\mu_2}p_{\mu_3)} - 4p_{\mu_1}p_{\mu_2}p_{\mu_3}}{40m^5} \,, \tag{2.2.15b}$$

$$= \frac{i(p^2+5m^2)}{40m^4} \,. \tag{2.2.15c}$$

Apart from the non-local term encoding the energy-momentum relation, the propagator for the physical $s=3$ particle also has an additional local contribution, with tensor structure

$$\mathcal{Q}^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} = \eta_{(\mu_1\mu_2}p_{\mu_3)}\eta^{(\nu_1\nu_2}p^{\nu_3)}(p^2-7m^2) - 4p_{\mu_1}p_{\mu_2}p_{\mu_3}\eta^{(\nu_1\nu_2}p^{\nu_3)} - 4\eta_{(\mu_1\mu_2}p_{\mu_3)}p^{\nu_1}p^{\nu_2}p^{\nu_3} \,. \tag{2.2.16}$$

Meanwhile, the contribution from the auxiliary field is completely local, indicating that they carry no physical (asymptotic) degrees of freedom, as expected.

2.2.2 Classical Asymptotic States and Coherent States

The standard description of the asymptotic states of (massive) spinning fields is in terms of Lorentz tensors labeled by the (massive) little group. Extending Ref. [59], we first consider the asymptotic state $\mathcal{E}$ and its conjugate $\bar{\mathcal{E}}$ for general (l,r) representations of the Lorentz group with $l+r=2s$. Subsequently, we specialize to integer s and consider the (s,s) representation, in which we identify general consequences of transversality. In the classical limit, these states are chosen to minimize the dispersion of the Lorentz generators, $\mathcal{E}\cdot M^{\mu\nu}\cdot\bar{\mathcal{E}}$, where $M^{\mu\nu}$ satisfies the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\lambda}] = -i(\eta^{\mu\rho}M^{\lambda\nu} + \eta^{\nu\rho}M^{\mu\lambda} - \eta^{\mu\lambda}M^{\rho\nu} - \eta^{\nu\lambda}M^{\mu\rho}) \,. \tag{2.2.17}$$

In the rest frame, the state $\mathcal{E}$ generalizes the spin coherent states of $SU(2)$ implicit in the construction of Ref. [59] to those of $SU(2)_L \times SU(2)_R$. We start by representing the states in terms of spinors such that[8]

$$\begin{aligned}
\mathcal{E}(p)_{\alpha(l)\dot\beta(r)} &= \xi(p)_{\alpha_1}\ldots\xi(p)_{\alpha_l}\chi(p)_{\dot\beta_1}\ldots\chi(p)_{\dot\beta_r}\,,\\
\bar{\mathcal{E}}(p)^{\alpha(l)\dot\beta(r)} &= \tilde\xi(p)^{\alpha_1}\ldots\tilde\xi(p)^{\alpha_l}\tilde\chi(p)^{\dot\beta_1}\ldots\tilde\chi(p)^{\dot\beta_r}\,.
\end{aligned} \tag{2.2.18}$$

Here, we choose $\mathcal{E}$ to be null for convenience. We note that the final result does not rely on $\mathcal{E}$ being null. As with the spin coherent states, the spinors $\xi(p)$, $\chi(p)$, $\tilde\xi(p)$ and $\tilde\chi(p)$ are constructed by boosting their rest frame counterparts ξ_0, χ_0, $\tilde\xi_0$ and $\tilde\chi_0$,

$$\begin{aligned}
\xi(p)_\alpha &= \exp(i\eta\hat p^k\hat K^k_L)_\alpha{}^\beta\xi_{0\beta}\,, & \tilde\xi(p)_\alpha &= \exp(i\eta\hat p^k\hat K^k_L)_\alpha{}^\beta\tilde\xi_{0\beta}\,,\\
\chi(p)^{\dot\alpha} &= \exp(i\eta\hat p^k\hat K^k_R)^{\dot\alpha}{}_{\dot\beta}\chi_0^{\dot\beta}\,, & \tilde\chi(p)^{\dot\alpha} &= \exp(i\eta\hat p^k\hat K^k_R)^{\dot\alpha}{}_{\dot\beta}\tilde\chi_0^{\dot\beta}\,,
\end{aligned} \tag{2.2.19}$$

where $(\hat K^k_L)_\alpha{}^\beta = (i/2)(\sigma^k)_\alpha{}^\beta$ and $(\hat K^k_R)^{\dot\alpha}{}_{\dot\beta} = (-i/2)(\sigma^k)^{\dot\alpha}{}_{\dot\beta}$ are the left/right-handed boost operators, η is the rapidity and $\hat p^k$ are the components of the unit vector along the spatial part of the momentum.

The rest frame coherent-state spinors are [245]

$$\begin{aligned}
\xi_{0\alpha} &= \exp(z_L\hat N^L_+ - z_L^*\hat N^L_-)_\alpha{}^\beta\xi^+_{0\beta}\,, & \tilde\xi_{0\alpha} &= \exp(z_L\hat N^L_+ - z_L^*\hat N^L_-)_\alpha{}^\beta\xi^-_{0\beta}\,,\\
\chi_0^{\dot\alpha} &= \exp(z_R\hat N^R_+ - z_R^*\hat N^R_-)^{\dot\alpha}{}_{\dot\beta}\chi_0^{+,\dot\beta}\,, & \tilde\chi_0^{\dot\alpha} &= \exp(z_R\hat N^R_+ - z_R^*\hat N^R_-)^{\dot\alpha}{}_{\dot\beta}\chi_0^{-,\dot\beta}\,,
\end{aligned} \tag{2.2.20}$$

where $(\hat N^L_\pm)_\alpha{}^\beta = (1/2)(\sigma^1 \pm i\sigma^2)_\alpha{}^\beta$ and $(\hat N^R_\pm)^{\dot\alpha}{}_{\dot\beta} = (1/2)(\sigma^1 \pm i\sigma^2)^{\dot\alpha}{}_{\dot\beta}$ are the generators of $SU(2)_L$ and $SU(2)_R$, $\xi_0^\pm$ and $\chi_0^\pm$ are the eigenvectors of σ^3 with eigenvalues ±1, and

$$z_{L,R} \equiv -(\theta_{L,R}/2)e^{-i\phi_{L,R}}\,, \tag{2.2.21}$$

are coherent-state parameters. The rest frame spinors are normalized as $\xi_0^\alpha\tilde\xi_{0\alpha} = \chi_{\dot\alpha}\tilde\chi^{\dot\alpha} = -1$, such that $\mathcal{E}(p)\cdot\bar{\mathcal{E}}(p) = (-1)^r$. They are related to unit vectors via

$$\begin{aligned}
n^i_L &= \xi_0^\alpha(\sigma^i)_\alpha{}^\beta\tilde\xi_{0\beta} \equiv \xi_0\sigma^i\tilde\xi_0\,, & \boldsymbol{n}_L &= (\sin\theta_L\cos\phi_L, \sin\theta_L\sin\phi_L, \cos\theta_L)\,,\\
n^i_R &= \chi_{0\dot\alpha}(\sigma^i)^{\dot\alpha}{}_{\dot\beta}\tilde\chi_0^{\dot\beta} \equiv \chi_0\sigma^i\tilde\chi_0\,, & \boldsymbol{n}_R &= (\sin\theta_R\cos\phi_R, \sin\theta_R\sin\phi_R, \cos\theta_R)\,.
\end{aligned} \tag{2.2.22}$$

The rotation and boost generators in the (l,r) representation is given by

$$\begin{aligned}
&\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_L + \hat{\boldsymbol{S}}_R\,, \qquad \hat{\boldsymbol{K}} = \hat{\boldsymbol{K}}_L + \hat{\boldsymbol{K}}_R\,,\\
&\hat S^k_L = \frac{1}{2}\sum_{m=1}^{l}\underbrace{\mathbb{1}\otimes\ldots\mathbb{1}}_{m-1}\otimes\sigma^k\otimes\mathbb{1}\cdots\otimes\mathbb{1}\,, \qquad \hat S^k_R = \frac{1}{2}\sum_{m=1}^{r}\underbrace{\mathbb{1}\otimes\ldots\mathbb{1}}_{m-1}\otimes\sigma^k\otimes\mathbb{1}\cdots\otimes\mathbb{1}\,,
\end{aligned} \tag{2.2.23}$$

[8]We note that the $SU(2)$ indices are raised and lowered by

$$\epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\hat{K}^k_L = \frac{i}{2}\sum_{m=1}^{l} \underbrace{\mathbb{1}\otimes\ldots\mathbb{1}}_{m-1}\otimes\,\sigma^k\otimes\mathbb{1}\cdots\otimes\mathbb{1}\,, \qquad \hat{K}^k_R = -\frac{i}{2}\sum_{m=1}^{r} \underbrace{\mathbb{1}\otimes\ldots\mathbb{1}}_{m-1}\otimes\,\sigma^k\otimes\mathbb{1}\cdots\otimes\mathbb{1}\,,$$

where the summation is over the position of σ^k. With these definitions, the expectation values of the rotation and boost generator under the rest frame spin coherent states are

$$\mathcal{E}_0\cdot\hat{\boldsymbol{S}}\cdot\bar{\mathcal{E}}_0 = \frac{1}{2}(l\,\boldsymbol{n}_L + r\,\boldsymbol{n}_R) \equiv \boldsymbol{S}\,, \qquad \mathcal{E}_0\cdot\hat{\boldsymbol{K}}\cdot\bar{\mathcal{E}}_0 = \frac{i}{2}(l\,\boldsymbol{n}_L - r\,\boldsymbol{n}_R) \equiv i\boldsymbol{K}\,. \tag{2.2.24}$$

We identify the former with the classical rest-frame spin vector $S^\mu_0 = (0,\boldsymbol{S})$ and the latter with the boost vector $K^\mu_0 = (0,\boldsymbol{K})$. If $\mathcal{E}$ is not null, we view (2.2.24) as the definition for the classical spin and boost vector. The classical rest-frame spin tensor given by

$$\mathsf{S}^{\mu\nu}_0 = \mathcal{E}_0\cdot M^{\mu\nu}\cdot\bar{\mathcal{E}}_0 = S^{\mu\nu}_0 + iK^{\mu\nu}_0\,, \tag{2.2.25}$$

does not obey the SSC, where

$$S^{\mu\nu}_0 = \frac{1}{m}\varepsilon^{\mu\nu\rho\lambda}p_{0\rho}S_\lambda\,, \qquad K^{\mu\nu}_0 = \frac{1}{m}(p^\mu_0 K^\nu_0 - p^\nu_0 K^\mu_0)\,. \tag{2.2.26}$$

It contains[9] an SSC-obeying component $S^{\mu\nu}_0$ and an SSC-violating one $K^{\mu\nu}_0$, where $p^\mu_0 = (m,0)$ is the rest-frame momentum. An important feature for generic (l,r) representations is that $\boldsymbol{K}$ no longer vanishes identically, so that $\mathsf{S}^{\mu\nu}_0$ no longer satisfies the convariant SSC condition. By suitably choosing l and r, the norm $|\boldsymbol{K}|$ can be subleading in the classical limit or commensurate with that of the spin vector. In this way, the appearance of $\boldsymbol{K}$ in the classical limit appears natural, simply by adjusting the Lorentz representation in the underlying quantum system. For generic values, the classical limit is independent of the details of the representation. However, for the special case of the irreducible transverse (s,s) representation then $\boldsymbol{K}$ vanishes, as noted in Appendix C of Ref. [55].

The next step is to restore the momentum dependence of various quantities by boosting the particle out of the rest frame. It is somewhat tedious but straightforward to use (2.2.19) and the properties of the Pauli matrices to boost products of polarization tensors and Lorentz generators for any (l,r) representation, as well as (2.2.25) and its two components to arbitrary frames. To leading order in the classical limit, we find

$$\mathcal{E}_1\cdot\{M^{\mu_1\nu_1},\ldots,M^{\mu_n\nu_n}\}\cdot\bar{\mathcal{E}}_2 = \mathsf{S}(p_1)^{\mu_1\nu_1}\ldots\mathsf{S}(p_n)^{\mu_n\nu_n}\mathcal{E}_1\cdot\bar{\mathcal{E}}_2 + \mathcal{O}(q^{1-n})\,, \tag{2.2.27}$$

where $\mathcal{E}_i \equiv \mathcal{E}(p_i)$, $q = p_2 - p_1$ is the momentum transfer and $\mathsf{S}(p_i)^{\mu\nu}$ is the boost of (2.2.25) to the frame moving with momentum p_i. The spin tensor scales as $\mathsf{S}\sim q^{-1}$, and we neglect all the subleading $\mathcal{O}(q^{1-n})$ terms. The symmetric product of Lorentz generators $\{M^{\mu_1\nu_1},\ldots,M^{\mu_n\nu_n}\}$ is defined as

$$\{M_{\mu_1\nu_1},M_{\mu_2\nu_2}\} = \frac{1}{2}(M_{\mu_1\nu_1}M_{\mu_2\nu_2} + M_{\mu_2\nu_2}M_{\mu_1\nu_1})\,,$$

[9] Our convention for the Levi-Civita tensor is $\epsilon_{0123} = +1$.

$$\{M_{\mu_1\nu_1}, M_{\mu_2\nu_2}, \ldots, M_{\mu_n\nu_n}\} = \frac{1}{n!}(M_{\mu_1\nu_1}M_{\mu_2\nu_2}\ldots M_{\mu_n\nu_n} + \text{permutations})\,. \tag{2.2.28}$$

They form a basis for arbitrary product of Lorentz generators under the Lorentz algebra. The factorization (2.2.27) of the expectation value of the product of Lorentz generators into the product of individual expectation values is a reflection of the classical nature of the asymptotic states $\mathcal{E}$ and $\bar{\mathcal{E}}$. In (2.2.27), the product of polarization tensors is given by

$$(-1)^r\mathcal{E}_1\cdot\bar{\mathcal{E}}_2 = \exp\left[-\frac{1}{m}\boldsymbol{q}\cdot\boldsymbol{K}\right]\exp\left[-i\frac{\epsilon_{rijk}u_1^i q^j S^k}{m(1+\sqrt{1+\boldsymbol{u}_1^2})} + \mathcal{O}(q^2)\right] + \mathcal{O}(q)\,, \tag{2.2.29}$$

where $u_i^k = p_i^k/m$, generalizing the corresponding expression in Ref. [59] to general $\boldsymbol{K}$. Eq. (2.2.29) captures the leading terms in the classical limit and, apart from the sign on the left-hand side, it is agnostic to the (l,r) representation chosen for the fields.

2.2.3 The Transverse (s,s) Representation

We now consider the special case of the (s,s) representation, which corresponds to symmetric-traceless fields. The coherent-state polarization tensors have an equal number of dotted and undotted indices; in the rest frame, they can be written as

$$(\mathcal{E}_0^{(s)})_{\alpha(s)\dot{\beta}(s)} = (\mathcal{E}_0^{(s)})^{\mu_1\mu_2\ldots\mu_s}(\sigma_{\mu_1})_{\alpha_1\dot{\beta}_1}\ldots(\sigma_{\mu_s})_{\alpha_s\dot{\beta}_s} = \xi_{0\alpha_1}\ldots\xi_{0\alpha_s}\chi_{0\dot{\beta}_1}\cdots\chi_{0\dot{\beta}_s}\,. \tag{2.2.30}$$

With this definition, we can explore the additional restrictions on the coherent states required by the transversality of $\mathcal{E}_0^{(s)}$. It suffices to analyze it in the rest frame, where it reads,[10]

$$p_{0\mu}\mathcal{E}_0^{\mu\mu_2\ldots\mu_s} = 0 \qquad \Longleftrightarrow \qquad (p_{0\mu}\sigma^\mu)^\alpha{}_{\dot{\beta}}(\mathcal{E}_0)^{\dot{\beta}\dot{\beta}_2\ldots\dot{\beta}_s}_{\alpha\alpha_2\ldots\alpha_s} = 0\,. \tag{2.2.31}$$

Using the explicit form of the rest-frame momentum, $p_0 = (m,0,0,0)$, and that $(\sigma^0)^\alpha{}_{\dot{\beta}}$ is numerically equal to the 2×2 Levi-Civita, it follows that

$$0 = (p_{0\mu}\sigma^\mu)^\alpha{}_{\dot{\beta}}(\mathcal{E}_0)^{\dot{\beta}\dot{\beta}_2\ldots\dot{\beta}_s}_{\alpha\alpha_2\ldots\alpha_s} \propto \xi_{0\alpha}\epsilon^\alpha{}_{\dot{\alpha}}\chi_0^{\dot{\alpha}}\,. \tag{2.2.32}$$

The solution, accounting for normalization, is

$$\xi_{0\alpha} = \chi_0^{\dot{\alpha}} \quad \text{as column vectors,} \tag{2.2.33}$$

which in turn implies $z_L = z_R$ and hence the equality of the left-handed and right-handed unit vectors $\boldsymbol{n}_L$ and $\boldsymbol{n}_R$ in (2.2.22). Together with (2.2.24) this implies that

$$\boldsymbol{K} = 0 \quad \Longleftrightarrow \quad \mathsf{S}_0^{\mu\nu} = S_0^{\mu\nu}\,, \tag{2.2.34}$$

[10]In this form transversality can be imposed on the polarization tensor of a general $(l\neq 0, r\neq 0)$ state.

for the transverse (s,s) representation, and therefore, cf. (2.2.25), $\mathcal{E}_0^{(s)} \cdot M^{\mu\nu} \cdot \bar{\mathcal{E}}_0^{(s)}$ becomes an SSC-satisfying spin tensor. On the other hand, if we do not impose transversality, then the discussion for a generic (l,r) representation also applies to (s,s), such that $\boldsymbol{K}$ does not vanish and hence that covariant SSC is not obeyed. We thus see that the (s,s) transverse asymptotic states chosen in Ref. [59] can be replaced with more general nontransverse ones. The polarization tensor for the transverse (s,s) representation can be written as a direct product of transverse $s=1$ coherent state vectors

$$\mathcal{E}^{(s)}(p)^{\mu_1\mu_2\ldots\mu_s} = \varepsilon(p)^{\mu_1}\varepsilon(p)^{\mu_2}\ldots\varepsilon(p)^{\mu_s}\,, \qquad \varepsilon(p)^{\mu}(\sigma_\mu)_{\alpha\dot{\beta}} = \xi_\alpha(p)\chi_{\dot{\beta}}(p)\,, \tag{2.2.35}$$

where the spinors are boosted from the rest frame ones that satisfy the condition (2.2.33), and we normalize the polarization vectors as $\varepsilon \cdot \bar{\varepsilon} = -1$. For such external states, the expectation value of (2.2.27) becomes

$$K^{\mu\nu}(p) = 0 \quad \Longleftrightarrow \quad \mathsf{S}^{\mu\nu}(p) = S^{\mu\nu}(p)\,, \tag{2.2.36}$$

which is simply the counterpart of the rest frame relation (2.2.34). The product (2.2.29) simplifies to

$$(-1)^s \mathcal{E}_1^{(s)} \cdot \bar{\mathcal{E}}_2^{(s)} = (-\varepsilon_1 \cdot \bar{\varepsilon}_2)^s = \exp\left[-i\frac{\epsilon_{rsk}u_1^r q^s S^k}{m(1+\sqrt{1+\boldsymbol{u}_1^2})} + \mathcal{O}(q^2)\right] + \mathcal{O}(q)\,. \tag{2.2.37}$$

The transverse (s,s) representation is used in FT2 and FT3. Because the Lagrangian depends explicitly on s, we need to use the explicit form of the Lorentz generators,

$$(M^{\mu\nu})_{\alpha(s)}{}^{\beta(s)} = -2i\delta^{[\mu}_{(\alpha_1}\eta^{\nu](\beta_1}\delta^{\beta_2}_{\alpha_2}\ldots\delta^{\beta_s)}_{\alpha_s)}\,. \tag{2.2.38}$$

Consequently, the results are given in terms of various symmetric and antisymmetric combinations of the polarization vector ε and momenta. To convert them into spin tensors, we need to compute the left-hand side of (2.2.27) and identify the resulting structures with spin tensors.

We first consider the transverse $\mathcal{E}^{(s)}$. Starting with $\mathcal{O}(S^1)$, we have

$$\mathcal{E}_1^{(s)} \cdot M^{\mu\nu} \cdot \bar{\mathcal{E}}_2^{(s)} = -2is(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1}\varepsilon_1^{[\mu}\bar{\varepsilon}_2^{\nu]}\,. \tag{2.2.39}$$

According to (2.2.27), this combination should be identified with $S(p_1)^{\mu\nu}$, such that

$$(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1}\varepsilon_1^{[\mu}\bar{\varepsilon}_2^{\nu]} = \frac{iS(p_1)^{\mu\nu}}{2s}(\varepsilon_1 \cdot \bar{\varepsilon}_2)^s + \mathcal{O}(q^0)\,. \tag{2.2.40}$$

In amplitudes, we can use this relation to turn antisymmetric combination of polarization vectors into spin tensors. The classical amplitude is obtained by further taking the $s \to \infty$ limit. Similarly, at $\mathcal{O}(S^2)$, we can use the following identity,

$$\begin{aligned}\mathcal{E}_1^{(s)} \cdot \{M^{\mu\nu}, M^{\rho\lambda}\} \cdot \bar{\mathcal{E}}_4^{(s)} &= -4s(s-1)(\varepsilon_1 \cdot \bar{\varepsilon}_4)^{s-2}\varepsilon_1^{[\mu}\bar{\varepsilon}_4^{\nu]}\varepsilon_1^{[\rho}\bar{\varepsilon}_4^{\lambda]} \\ &\quad - s(\varepsilon_1 \cdot \bar{\varepsilon}_4)^{s-1}\left(\eta^{\mu\lambda}\varepsilon_1^{(\nu}\bar{\varepsilon}_4^{\rho)} + \eta^{\nu\rho}\varepsilon_1^{(\mu}\bar{\varepsilon}_4^{\lambda)} - \eta^{\mu\rho}\varepsilon_1^{(\nu}\bar{\varepsilon}_4^{\lambda)} - \eta^{\nu\lambda}\varepsilon_1^{(\mu}\bar{\varepsilon}_4^{\rho)}\right).\end{aligned} \tag{2.2.41}$$

In the large s limit, the second term is subleading, such that we have

$$(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-2} \varepsilon_1^{[\mu} \bar{\varepsilon}_2^{\nu]} \varepsilon_1^{[\rho} \bar{\varepsilon}_2^{\lambda]} \xrightarrow{\text{large } s} -\frac{(\varepsilon_1 \cdot \bar{\varepsilon}_2)^s}{4s^2} S(p_1)^{\mu\nu} S(p_1)^{\rho\lambda} + \mathcal{O}(q^{-1}) \,, \tag{2.2.42}$$

which is equivalent to applying (2.2.40) twice. Contracting the Lorentz indices on the spin tensors once leads to

$$(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1} \varepsilon_1^{(\mu} \bar{\varepsilon}_2^{\nu)} \xrightarrow{\text{large } s} -\frac{(\varepsilon_1 \cdot \bar{\varepsilon}_2)^s}{2s^2} S(p_1)^{\mu\rho} S(p_1)_\rho{}^\nu + \mathcal{O}(q^{-1}) \,. \tag{2.2.43}$$

Similar identities have been previously used in e.g. Refs. [246, 51]. They are sufficient for amplitudes up to $\mathcal{O}(S^2)$.

2.2.4 The Nontransverse (s,s) Representation

For a nontransverse field in the (s,s) representation we can use the general results obtained for an (l,r) representation. In particular, it has $\boldsymbol{K} \neq 0$. It is instructive to identify the origin of $\boldsymbol{K}$, and thus the structures governing its covariant version $K^{\mu\nu}$, in terms of the lower-spin (longitudinal) components of $\mathcal{E}_{\mu_1 \dots \mu_s}$. This will be important when discussing FT3 which has physical lower-spin fields.

The coherent state in FT1 can be decomposed as

$$\mathcal{E}_{\mu_1 \dots \mu_s} = \mathcal{E}^{(s)}_{\mu_1 \dots \mu_s} + \left(u \mathcal{E}^{(s-1)}\right)_{\mu_1 \dots \mu_s} + \left(u^2 \mathcal{E}^{(s-2)}\right)_{\mu_1 \dots \mu_s} + \dots \,, \tag{2.2.44}$$

where the spin-$(s-k)$ component is represented by

$$\left(u^k \mathcal{E}^{(s-k)}\right)_{\mu_1 \dots \mu_s} = \binom{s}{k}^{1/2} u_{(\mu_1} \dots u_{\mu_k} \varepsilon_{\mu_{k+1}} \dots \varepsilon_{\mu_s)} \,. \tag{2.2.45}$$

The states with even k have positive norm and those with odd k have negative norm. We may now compute products involving these polarization tensors. We have

$$\mathcal{E}_1^{(s)} \cdot \left(u_2 \bar{\mathcal{E}}_2^{(s-1)}\right) = -\frac{\sqrt{s} (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1} \varepsilon_1 \cdot q}{m} + \mathcal{O}(q^2) \,. \tag{2.2.46}$$

At $\mathcal{O}(S^1)$, we plug (2.2.44) into (2.2.27) and find that

$$\begin{aligned} \mathcal{E}_1^{(s)} \cdot M^{\mu\nu} \cdot \left(u_2 \bar{\mathcal{E}}_2^{(s-1)}\right) &= i\sqrt{s} \, (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1} \, (u_2^\mu \varepsilon_1^\nu - u_2^\nu \varepsilon_1^\mu) + \mathcal{O}(q) \,, \\ \left(u_1 \mathcal{E}_1^{(s-1)}\right) \cdot M^{\mu\nu} \cdot \bar{\mathcal{E}}_2^{(s)} &= -i\sqrt{s} (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1} \, (u_1^\mu \bar{\varepsilon}_2^\nu - u_1^\nu \bar{\varepsilon}_2^\mu) + \mathcal{O}(q) \,, \end{aligned} \tag{2.2.47}$$

while all the other $\mathcal{E}_1^{(s)} \cdot M^{\mu\nu} \cdot \left(u_2^k \bar{\mathcal{E}}_2^{(s-k)}\right)$ vanish at the classical order. We note that contractions like $\left(u_1^k \mathcal{E}_1^{(s-k)}\right) \cdot M^{\mu\nu} \cdot \left(u_2^k \bar{\mathcal{E}}_2^{(s-k)}\right)$ and $\left(u_1^k \mathcal{E}_1^{(s-k)}\right) \cdot M^{\mu\nu} \cdot \left(u_2^k \bar{\mathcal{E}}_2^{(s-k-1)}\right)$ give identical result as (2.2.39),(2.2.47) in the limit $s \geqslant k$. They contribute an overall factor that can be absorbed in the normalization of the states. In this sense, we can identify the above combination with $K^{\mu\nu}$ or K^μ. More precisely, we have

$$-(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1} \Big[u_1^\mu \, (\varepsilon_1^\nu + \bar{\varepsilon}_2^\nu) - u_1^\nu \, (\varepsilon_1^\mu + \bar{\varepsilon}_2^\mu) \Big] \xrightarrow{\text{large } s} \frac{\mathcal{E}_1 \cdot \bar{\mathcal{E}}_2}{\sqrt{s}} K(p_1)^{\mu\nu} \,,$$

$$-(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1}\,(\varepsilon_1^\mu + \bar{\varepsilon}_2^\mu) \xrightarrow{\text{large } s} \frac{\mathcal{E}_1 \cdot \bar{\mathcal{E}}_2}{\sqrt{s}} K(p_1)^\mu \,, \tag{2.2.48}$$

Using these two relations in the product $\mathcal{E}_1 \cdot \bar{\mathcal{E}}_2$, we find that

$$\begin{aligned}\mathcal{E}_1 \cdot \bar{\mathcal{E}}_2 &= (\varepsilon_1 \cdot \bar{\varepsilon}_2)^s + \sqrt{s}(\varepsilon_1 \cdot u_2 + \bar{\varepsilon}_2 \cdot u_1)(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1} + \ldots \\ &= (\varepsilon_1 \cdot \bar{\varepsilon}_2)^s + \frac{q \cdot K}{m} \mathcal{E}_1 \cdot \bar{\mathcal{E}}_2 + \ldots \end{aligned} \tag{2.2.49}$$

where the first term comes from $\mathcal{E}_1^{(s)} \cdot \left(u_2 \bar{\mathcal{E}}_2^{(s-1)}\right) + \left(u_1 \mathcal{E}_1^{(s-1)}\right) \cdot \bar{\mathcal{E}}_2$ and the $\ldots$ contains the contraction between $\mathcal{E}^{(s)}$ and the states with spin less than $s-1$. Again, similar contractions between lower-spin states contribute an overall factor that can be normalized away. We thus get the relation between the transverse $\mathcal{E}_1^{(s)} \cdot \bar{\mathcal{E}}_2^{(s)} = (\varepsilon_1 \cdot \bar{\varepsilon}_2)^s$ and the full result $\mathcal{E}_1 \cdot \bar{\mathcal{E}}_2$ up to the first order in q and K,

$$(\varepsilon_1 \cdot \varepsilon_2)^s = \left(1 - \frac{q \cdot K}{m}\right)(\mathcal{E}_1 \cdot \bar{\mathcal{E}}_2) + \mathcal{O}(q^2, K^2)\,, \tag{2.2.50}$$

which is of course consistent with (2.2.29),(2.2.37). The relations (2.2.48) and (2.2.50) are used to extract the $\mathcal{O}(K^1)$ terms in FT3. More generally, the contractions

$$\mathcal{E}_1^{(s)} \cdot \left(u_2^k \bar{\mathcal{E}}_2^{(s-k)}\right) = \binom{s}{k}^{1/2} (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-k} \left(-\frac{q \cdot \varepsilon_1}{m}\right)^k \,, \tag{2.2.51}$$

$$\begin{aligned}\mathcal{E}_1^{(s)} \cdot \{M^{\mu_1\nu_1}, \ldots, M^{\mu_k\nu_k}\} \cdot \left(u_2^k \bar{\mathcal{E}}_2^{(s-k)}\right) &= \binom{s}{k}^{1/2} (k!)(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-k} \prod_{j=1}^{k} \left(2iu_2^{[\mu_i} \varepsilon_1^{\nu_i]}\right) \\ &\to s^{k/2} (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-k} \prod_{j=1}^{k} \left(2iu_2^{[\mu_i} \varepsilon_1^{\nu_i]}\right)\,, \end{aligned} \tag{2.2.52}$$

can be used to show that for the s to $s-k$ amplitudes,

$$(q \cdot K)^k \to s^{k/2} (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-k} \left(\frac{q \cdot \varepsilon_1}{m}\right)^k \,, \tag{2.2.53}$$

$$\mathcal{E}_1^{(s)} \cdot \{\underbrace{M, \ldots M}_{m}\} \cdot \left(u_2^k \bar{\mathcal{E}}_2^{(s-k)}\right) \sim K^k S^{m-k} \,, \tag{2.2.54}$$

which are necessary to identify the structures related to $K^k S^{m-k}$ in the amplitudes. Recall that the notation $u_2^k \bar{\mathcal{E}}_2^{(s-k)}$ includes a factor $s^{k/2}$, cf. Eq. (2.2.45). We leave for future work the detailed study of structures of higher orders in spin.

Apart from states created by the operators[11] $\mathcal{E}^{(s)} \cdot a_{(s)}^\dagger$ and $\mathcal{E}^{(s-1)} \cdot a_{(s-1)}^\dagger$ of the fields with definite spin, in FT3 we may also choose asymptotic states with indefinite spin, which are a normalized linear combinations of these definite-spin states (and, in general, also of lower-spin fields). In a quantum theory such a choice is disfavored as it breaks the little-group symmetry. In the classical theory, effectively with a single asymptotic

[11] We denote by $a_{(s)}^\dagger$ the creation operators of the field $\phi_{(s)}$ corresponding to the state labeled by the rest-frame polarization tensors in Eqs. (2.2.18) and (2.2.19).

state, this is not an issue. We therefore also evaluate amplitudes in FT3 with asymptotic states

$$|g\rangle = \frac{1}{\sqrt{2}}\left(\mathcal{E}^{(s)}\cdot a^\dagger_{(s)} + \mathcal{E}^{(s-1)}\cdot a^\dagger_{(s-1)}\right)|0\rangle\,. \tag{2.2.55}$$

Similar states have also been considered in Refs. [65, 236]. We will refer to these amplitudes as $\mathcal{A}^{\mathrm{FT3g}}$; in terms of definite-spin states they are

$$\mathcal{A}^{\mathrm{FT3g}} = \frac{1}{2}\left(\mathcal{A}^{\mathrm{FT3s}}_{s\to s} + \mathcal{A}^{\mathrm{FT3s}}_{s-1\to s-1} + \mathcal{A}^{\mathrm{FT3s}}_{s-1\to s} + \mathcal{A}^{\mathrm{FT3s}}_{s\to s-1}\right)\,. \tag{2.2.56}$$

There is no simple polarization tensor that can be assigned to the state $|g\rangle$; the closest analog of the sandwich of Lorentz generators and polarization tensors is the expectation value of the (field) generator of Lorentz transformations in the state $|g\rangle$. While does not have a simple interpretation in terms of the $\boldsymbol{S}$ and $\boldsymbol{K}$ vectors, the interaction (2.2.61) will supply the requisite factors of momenta for such an interpretation to be possible, cf. (2.2.48).

2.2.5 Nonminimal Lagrangian

We are primarily interested in amplitudes in the classical limit, where the spin s is taken to be large. We expect that the relevant interaction terms do not depend on a particular representation of the spin, and thus are Lorentz singlets constructed from covariant derivatives, photon field strengths, ϕ_s and Lorentz generators (2.2.17) in the same representation as ϕ_s. Moreover, we consider for the time being only those interactions that survive in the classical limit. The close relation in Eq. (2.2.27) between Lorentz generators and the spin tensor and the scaling of momenta in the classical limit imply that the number of derivatives on the photons must be equal to the number of Lorentz generators. Under these guidelines, we can write down the following nonminimal linear-in-$F_{\mu\nu}$ interactions up to two powers of spins,

$$\begin{aligned}(-1)^s\mathcal{L}_{\text{non-min}} &= QC_1F_{\mu\nu}\phi_sM^{\mu\nu}\bar{\phi}_s + \frac{QD_1}{m^2}F_{\mu\nu}(D_\rho\phi_sM^{\rho\mu}D^\nu\bar{\phi}_s + \text{c.c}) \\ &\quad - \frac{iQC_2}{2m^2}\partial_{(\mu}F_{\nu)\rho}(D^\rho\phi_s\mathbb{S}^\mu\mathbb{S}^\nu\bar{\phi}_s - \text{c.c}) - \frac{iQD_2}{2m^2}\partial_\mu F_{\nu\rho}(D_\alpha\phi_sM^{\alpha\mu}M^{\nu\rho}\bar{\phi}_s - \text{c.c})\,,\end{aligned} \tag{2.2.57}$$

where for later convenience we choose[12] to scale the Wilson coefficients by Q so that at each order amplitudes display overall powers of α, and the Pauli-Lubanski spin operator $\mathbb{S}^\mu$ is defined as

$$\mathbb{S}^\mu \equiv \frac{-i}{2m}\,\varepsilon^{\mu\nu\rho\sigma}M_{\rho\sigma}D_\nu\,. \tag{2.2.58}$$

We note that the C_i operators are the electrodynamics analogs of operators [140, 141, 145] of general relativity, and the D_i's are the electrodynamics analogs of the typical examples of "extra Wilson coefficients" of Ref. [101]. From the effective-field-theory point of view, we can write down another operator that contributes classically

[12] Neutral particles can also have nonminimal couplings analogous to those in Eq. (2.2.57). The corresponding Lagrangian is obtained by the double-scaling limit $Q \to 0$, $C_i, D_i \to \infty$ with fixed products QC_i and QD_i.

Field theory	Lagrangian	Amplitude	External state
FT1	$\mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{min}} + \mathcal{L}_{\text{non-min}}$	$\mathcal{A}^{\text{FT1s}}$ $\mathcal{A}^{\text{FT1g}}$	spin-s generic
FT2	$\mathcal{L}_{\text{EM}} + \mathcal{L}^{s}_{\text{min}} + \mathcal{L}_{\text{non-min}}$	$\mathcal{A}^{\text{FT2}}$	spin-s
FT3	$\mathcal{L}_{\text{EM}} + \mathcal{L}^{s,s-1}_{\text{min}} + \mathcal{L}^{s,s-1}_{\text{non-min}}$	$\mathcal{A}^{\text{FT3s}}$ $\mathcal{A}^{\text{FT3g}}$	spin-s indefinite spin

Table 2.1: Field-theory amplitudes, corresponding Lagrangians and external states. The Lagrangians are given in Eqs. (2.2.2), (2.2.7), (2.2.11), (2.2.57) and (2.2.61).

at the second order in spin,

$$\mathcal{L}_{D_{2b}} = \frac{iQD_{2b}}{2m^4}\partial_{(\mu}F_{\nu)\rho}(D_\lambda\phi_s M^{\lambda\mu}M^{\nu}{}_{\sigma}D^{(\sigma}D^{\rho)}\bar{\phi}_s - \text{c.c})\,. \tag{2.2.59}$$

While C_2 and D_2 give independent contribution to three-point amplitudes at $\mathcal{O}(S^2)$ and $\mathcal{O}(S^1K^1)$, see 2.3.1, the above D_{2b} operator gives independent contribution at $\mathcal{O}(K^2)$. Since the purpose of our current work is to understand the existence of extra Wilson coefficients, for simplicity we will not consider this operator further.

2.1 collects the Lagrangians of the four effective field theories that are our focus noting also the notation we use for their corresponding amplitudes. FT1 are described by the same Lagrangian, $\mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{min}} + \mathcal{L}_{\text{non-min}}$. To compute amplitudes in these two theories, we do not need to specify a particular value for s. As discussed in 2.2.3, the representations of the rotation group with spin $s-2k$ that are part of the field ϕ_s have positive norm and are therefore physical, while those with spin $s-(2k+1)$ have negative norm. In contrast, FT2 is described by the Lagrangian $\mathcal{L}_{\text{EM}} + \mathcal{L}^{s}_{\text{min}} + \mathcal{L}_{\text{non-min}}$, and contains only the physical spin-s degrees of freedom. When computing the amplitude $\mathcal{A}^{\text{FT1s}}$, we restrict the external states to be the physical spin-s states, which are transverse such that the resultant spin tensors satisfy the covariant SSC according to 2.2.3. Meanwhile, we keep the external states generic in $\mathcal{A}^{\text{FT1g}}$. As a result, the amplitude $\mathcal{A}^{\text{FT1g}}$ contains explicit SSC-violating terms compared to $\mathcal{A}^{\text{FT1s}}$.

We show in 2.3 that despite having the same physical spin-s external states, $\mathcal{A}^{\text{FT1s}}$ and $\mathcal{A}^{\text{FT2}}$ are different for four-point Compton scattering in the classical limit. In particular, the Compton amplitudes from FT2 depend only on C_1 and C_2 while $\mathcal{A}^{\text{FT1s}}$ also depend on D_1 and D_2, similar to the appearance of additional nontrivial Wilson coefficients in general relativity [59]. The differences between these amplitudes vanish for

$$C_1 = C_2 = 1 \qquad D_1 = D_2 = 0\,, \tag{2.2.60}$$

and reproduce the root-Kerr amplitudes of Ref. [193], so that the additional D_i operators do not contribute, in much the same way that additional operators do not contribute to the Kerr black hole. The similarity of the root-Kerr solution in electromagnetism and the Kerr solution in general relativity follows from the double copy.

These results indicate that additional lower-spin degrees of freedom are the origin of the extra Wilson coefficients. We consider an interpolation between FT1 and FT2 in 2.6 to understand the effect of the state projector. In FT1 these degrees of freedom have negative norm; a natural question is whether lower-spin states with positive norm have similar consequences. FT3 explores this question. With some foresight which is justified in 2.6, we choose the nonminimal interactions of ϕ_s and ϕ_{s-1} in (2.2.11), valid through the quadratic order in spin, to be

$$\begin{aligned}\mathcal{L}^{s,s-1}_{\text{non-min}} &= QC_1 F_{\mu\nu}\phi_s M^{\mu\nu}\bar{\phi}_s - \frac{2iQ\widetilde{C}_1\sqrt{s}}{m}F_{\mu\nu}\Big[(\phi_s)^{\mu}{}_{\alpha_2\ldots\alpha_s}D^{\nu}\bar{\phi}^{\alpha_2\ldots\alpha_s}_{s-1} - \text{c.c}\Big] \\ &- \frac{iQC_2}{2m^2}\partial_{(\mu}F_{\nu)\rho}(D^{\rho}\phi_s\mathbb{S}^{\mu}\mathbb{S}^{\nu}\bar{\phi}_s - \text{c.c}) - \frac{2iQ\widetilde{C}_2\sqrt{s}}{m}F_{\mu\nu}\Big[(\phi_s)^{\mu}{}_{\alpha_2\ldots\alpha_s}D^{\alpha_2}\bar{\phi}^{\nu\alpha_3\ldots\alpha_s}_{s-1} - \text{c.c}\Big]\,,\end{aligned} \tag{2.2.61}$$

and the Lagrangian of FT3 is given by the third line of Table 2.1. We shall see in 2.3 that the Wilson coefficients $\widetilde{C}_1$ and $\widetilde{C}_2$ appear at $\mathcal{O}(K^1)$ and $\mathcal{O}(S^1K^1)$ order of the Compton amplitudes respectively, and that there exists an effective map between the D_i and $\widetilde{C}_i$ coefficients. As discussed in 2.2.3, we need to include couplings between ϕ_s and ϕ_{s-2} to access the $\mathcal{O}(K^2)$ interactions, which we omit for simplicity.

Similar to gravity, operators describing tidal deformations under the influence of external fields are necessary to describe the electromagnetic interactions of generic spinning bodies. Simple counting of classical scaling indicates that in QED they first appear $\mathcal{O}(S^2)$. At this order in spin three independent operators are

$$\begin{aligned}(-1)^s\mathcal{L}_{F^2} &= \frac{Q^2E_1}{m^2}F_{\mu\nu}F_{\rho\sigma}\phi_s M^{\mu\nu}M^{\rho\sigma}\bar{\phi}_s + \frac{Q^2E_2}{m^2}F_{\mu\nu}F_{\rho}{}^{\mu}\phi_s M^{\nu\lambda}M_{\lambda}{}^{\rho}\bar{\phi}_s \\ &+ \frac{Q^2E_3}{m^4}F_{\mu\nu}F_{\rho\sigma}D^{\mu}\phi_s M^{\nu\lambda}M_{\lambda}{}^{\rho}D^{\sigma}\bar{\phi}_s + \mathcal{O}(M^3)\,.\end{aligned} \tag{2.2.62}$$

Including them we find that all E_i Wilson coefficients vanish for the root-Kerr states in much the same way as the D_i coefficients vanish for these states in FT1.

2.3 Scattering Amplitudes

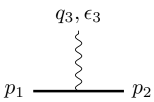
In this section we first compute the 1PL (tree) Compton amplitudes of the higher-spin effective Lagrangians introduced in the previous section and summarized in 2.1. We then use them as the basic building blocks of the $\mathcal{O}(\alpha^2)$ two-body amplitudes through generalized unitarity.[13] In addition, classical Compton amplitudes are also observables that can be directly compared with worldline computations along the lines of Ref. [247]. The comparison will be given in 2.4.

[13]For simplicity, we suppress a factor of Q in the three-point Compton amplitudes, and a factor of Q^2 in the four-point Compton amplitudes, where Q is the electric charge of the massive body.

2.3.1 Three-Point Amplitudes

We start with computing and comparing the three-point Compton amplitudes from the theories in 2.1. Assuming that all the momenta are outgoing, the Feynman rules for FT1 are given by

$$\mathcal{A}_3\left[\quad\right]$$



$$= (-1)^s \mathcal{E}_1 \cdot \mathbb{M}_3(p_1, p_2, q_3, \epsilon_3) \cdot \bar{\mathcal{E}}_2\,, \tag{2.3.1}$$

$$\begin{aligned}\mathbb{M}_3(p_1, p_2, q_3, \epsilon_3) &= 2\epsilon_3 \cdot p_1 \mathbb{1} - 2iC_1 M_{\mu\nu} q_3^\mu \epsilon_3^\nu - \frac{2iD_1}{m^2} \epsilon_3 \cdot p_1 M_{\mu\nu} p_1^\mu q_3^\nu \\ &+ \frac{C_2}{m^2} \epsilon_3 \cdot p_1 \{M^{\mu\nu}, M_\mu{}^\rho\} q_3^\nu q_3^\rho - \frac{C_2}{m^4} \epsilon_3 \cdot p_1 \{M_{\mu\nu}, M_{\rho\lambda}\} p_1^\mu q_3^\nu p_1^\rho q_3^\lambda \\ &+ \frac{2D_2}{m^2} \{M_{\mu\nu}, M_{\rho\lambda}\} p_1^\mu q_3^\nu q_3^\rho \epsilon_3^\lambda\,.\end{aligned}$$

The symmetric product between Lorentz generators is defined in (2.2.28).

In the classical limit, the massive spinning particles are described by the spin coherent states (2.2.30). We first consider FT1g with generic coherent states that do not satisfy the transversality. The expectation values of Lorentz generators are given by (2.2.27), which lead to the classical spin tensor $\mathsf{S}_{\mu\nu}$ that do not satisfy the covariant SSC. The three-point amplitude is

$$\begin{aligned}\mathcal{A}_3^{\text{FT1g}} &= (-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_2 \Bigg[2\epsilon_3 \cdot p_1 - 2iC_1 \mathsf{S}_{\mu\nu} q_3^\mu \epsilon_3^\nu + \frac{C_2}{m^2} \epsilon_3 \cdot p_1 \mathsf{S}_{\mu\nu} q_3^\nu \mathsf{S}^\mu{}_\lambda q_3^\lambda \\ &\qquad - \mathsf{S}_{\mu\nu} p_1^\mu q_3^\nu \left(\frac{2iD_1}{m^2} \epsilon_3 \cdot p_1 + \frac{C_2}{m^4} \epsilon_3 \cdot p_1 \mathsf{S}_{\lambda\sigma} p_1^\lambda q_3^\sigma - \frac{2D_2}{m^2} \mathsf{S}_{\lambda\sigma} q_3^\lambda \epsilon_3^\sigma\right)\Bigg] \\ &= 2(-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_2 \Bigg[\epsilon_3 \cdot p_1 - iC_1 S_{\mu\nu} q_3^\mu \epsilon_3^\nu - (C_1 - D_1) \epsilon_3 \cdot p_1 \frac{q \cdot K}{m} \\ &\qquad + \frac{C_2}{2m^2} \epsilon_3 \cdot p_1 S_{\mu\nu} q_3^\nu S^\mu{}_\lambda q_3^\lambda + iD_2 S_{\mu\nu} q_3^\mu \epsilon_3^\nu \frac{q \cdot K}{m} + D_2 \epsilon_3 \cdot p_1 \left(\frac{q \cdot K}{m}\right)^2\Bigg]\,,\end{aligned} \tag{2.3.2}$$

where in the second equal sign we have used (2.2.25) to expose the SSC preserving S-part and the SSC violating K-part in $\mathsf{S}^{\mu\nu}$. As expected, the extra Wilson coefficients D_i appear with the SSC-violating terms. If we further restrict the external states to be transverse, the K-part becomes subleading in the classical limit and thus drops out. This leads to the three-point amplitudes $\mathcal{A}_3^{\text{FT1s}}$ and $\mathcal{A}_3^{\text{FT2}}$

$$\mathcal{A}_3^{\text{FT1s}} = \mathcal{A}_3^{\text{FT2}} = 2(-\varepsilon_1 \cdot \bar{\varepsilon}_2)^s \left[\epsilon_3 \cdot p_1 - iC_1 S_{\mu\nu} q_3^\mu \epsilon_3^\nu + \frac{C_2}{2m^2} \epsilon_3 \cdot p_1 S_{\mu\nu} q_3^\nu S^\mu{}_\lambda q_3^\lambda\right]. \tag{2.3.3}$$

This amplitude only depends on the C_i Wilson coefficients. The fact that $\mathcal{A}_3^{\text{FT1s}}$ does not contain any additional Wilson coefficients is analogous to the three-point gravity amplitude of Ref. [59], which did not contain any additional Wilson coefficients either, connected to restricting the external states to traceless and transverse spin-s ones.

For FT3, we can similarly restrict the external states to be spin-s. The resulting amplitude $\mathcal{A}_3^{\text{FT3s}}$ is the

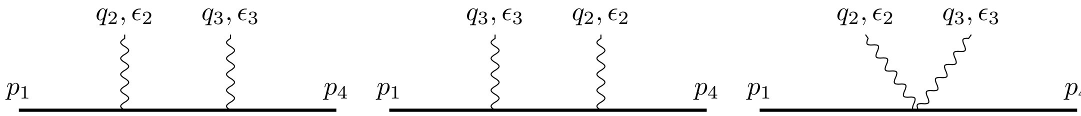


Figure 2.1: The three Feynman diagrams describing lowest-order Compton scattering.

same as (2.3.3), i.e.

$$\mathcal{A}_3^{\mathrm{FT1s}} = \mathcal{A}_3^{\mathrm{FT2}} = \mathcal{A}_3^{\mathrm{FT3s}} . \tag{2.3.4}$$

We may also choose the indefinite-spin states (2.2.55); the corresponding amplitude $\mathcal{A}_3^{\mathrm{FT3g}}$ receives contributions from both the spin-s and spin-$(s-1)$ external states,

$$\begin{aligned}\mathcal{A}_3^{\mathrm{FT3g}} &= \mathcal{A}_3^{\mathrm{FT3s}} + \frac{2i\sqrt{s}}{m}(-\varepsilon_1\cdot\bar{\varepsilon}_2)^{s-1}(\varepsilon_1\cdot q_3 + \bar{\varepsilon}_2\cdot q_3)\left[\widetilde{C}_1(\epsilon_3\cdot p_1) + \widetilde{C}_2(\mathcal{E}_1\cdot f_3\cdot\bar{\mathcal{E}}_4)\right] \\ &= 2(-1)^s\mathcal{E}_1\cdot\bar{\mathcal{E}}_2\left[\epsilon_3\cdot p_1 - iC_1 S_{\mu\nu}q_3^\mu\epsilon_3^\nu + \frac{C_2}{2m^2}\epsilon_3\cdot p_1 S_{\mu\nu}q_3^\nu S^\mu{}_\lambda q_3^\lambda\right. \\ &\qquad\left. + (i\widetilde{C}_1 - 1)\epsilon_3\cdot p_1\frac{q\cdot K}{m} + (iC_1 - \widetilde{C}_2)S_{\mu\nu}q_3^\mu\epsilon_3^\nu\frac{q\cdot K}{m}\right] ,\end{aligned} \tag{2.3.5}$$

where we have used (2.2.48),(2.2.50) to obtain the final expression in terms of the boost vector K. The appearance of the Wilson coefficients $\widetilde{C}_1$ and $\widetilde{C}_2$ associated with SSC violation. We find that, up to $\mathcal{O}(K^2)$ terms, the additional Wilson coefficients in FT1g and FT3g are related as

$$\mathcal{A}_3^{\mathrm{FT3g}} = \mathcal{A}_3^{\mathrm{FT1g}} \qquad \text{for} \quad i\widetilde{C}_1 = 1 - C_1 + D_1 \;\; \text{and} \;\; i\widetilde{C}_2 = D_2 - C_1 . \tag{2.3.6}$$

The extra factor of i in this map reflects the unphysical nature of the spin-$(s-1)$ states in FT1 vs. their physical nature in FT3. While this map is not an equivalence of Lagrangians, it provides a simple relation between the amplitudes of FT1 and FT3. A similar map also exists at $\mathcal{O}(K^2)$ if we include such interactions in both FT1 and FT3.

2.3.2 Four-Point Compton Amplitudes

At four points, the Compton amplitudes are given by the three Feynman diagrams in 2.1, with the relevant propagators and three- and four-point vertices derived from our field theories. While the Lagrangian of FT1 is independent of s and therefore general properties of Lorentz generators and coherent states are sufficient for amplitude calculations, the explicit dependence on s of FT2 and FT3 Lagrangians requires that for them we choose a particular representation. As noted in 2.2.3, we choose the (s, s) representation, for which the coherent states are given by (2.2.35) and the Lorentz generators are listed in (2.2.38). Specifically for four-point Compton amplitudes, the Feynman rules for FT2 and FT3 simplify considerably because every vertex has at least one on-shell massive particle represented by the symmetric, traceless and transverse polarization tensor. Thus, when deriving the three- and four-point vertex rules we can ignore all the interactions covered by the ellipsis in (2.2.7) because they only include traces or/and longitudinal modes of the external on-shell particle. For the same reason, we can also ignore all the (resummed) propagators that involve lower-spin auxiliary fields.

The spin-independent part of the Compton amplitude is common to FT1 though FT3,

$$\mathcal{A}^{\mathrm{FT1,2,3}}_{4,\text{ cl.}}\Big|_{S^0} = (-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_4 \frac{2(p_1 \cdot f_2 \cdot f_3 \cdot p_1)}{(p_1 \cdot q_2)^2}\,, \tag{2.3.7}$$

where $f_i^{\mu\nu} \equiv \varepsilon_i^\mu q_i^\nu - \varepsilon_i^\nu q_i^\mu$. For the (s,s) representation, we do not need to distinguish $\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4$ and $(\varepsilon_1 \cdot \bar{\varepsilon}_4)^s$ as their difference is higher order in S and K. Up to the overall factor $\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4$, Eq. (2.3.7) reproduces the classical limit of the scalar QED Compton amplitude given in, for example, Eq. (2.8) of Ref. [191].[14]

The Linear-in-Spin Compton Amplitudes

Consider now spin-dependent parts of amplitudes of the four field theories. For FT1, here and after we choose ϕ_s to be in the (s,s) representation to streamline the comparison with FT2 and FT3. We first consider $\mathcal{A}^{\mathrm{FT1s}}$ in which the external states are transverse. Evaluating the linear-in-spin part of the three Feynman diagrams in Fig. 2.1 with the propagators and vertices following from the Lagrangian of FT1 leads to

$$\begin{aligned}\mathcal{A}^{\mathrm{FT1s}}_{4,\text{ cl.}}\Big|_{S^1} = (-\varepsilon_1 \cdot \bar{\varepsilon}_4)^s S(p_1)_{\mu\nu} \bigg[&\frac{iC_1}{(p_1 \cdot q_2)^2} (f_2^{\mu\nu} q_{2\rho} f_3^{\rho\lambda} + f_3^{\mu\nu} q_{3\rho} f_2^{\rho\lambda}) p_{1\lambda} + \frac{2iC_1^2}{p_1 \cdot q_2} f_2^{\nu\rho} f_{3\rho}{}^{\mu} \\ &+ \frac{2iD_1(2C_1 - D_1 - 2)}{(p_1 \cdot q_2) m^2} p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda} \bigg]\,. \end{aligned} \tag{2.3.8}$$

The amplitude $\mathcal{A}^{\mathrm{FT1}}_{4,\text{ cl}}\Big|_{S^1}$ depends on both the C_1 and D_1 Wilson coefficients.[15] We note that D_1 appears only together with the combination $p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda}$. Repeating the calculation while relaxing the transversality on external states leads to the Compton amplitude,

$$\begin{aligned}\mathcal{A}^{\mathrm{FT1g}}_{4,\text{ cl.}}\Big|_{S^1} = (-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_4 \bigg\{ &\mathsf{S}(p_1)_{\mu\nu} \bigg[\frac{iC_1}{(p_1 \cdot q_2)^2} (f_2^{\mu\nu} q_{2\rho} f_3^{\rho\lambda} + f_3^{\mu\nu} q_{3\rho} f_2^{\rho\lambda}) p_{1\lambda} + \frac{2iC_1^2}{p_1 \cdot q_2} f_2^{\nu\rho} f_{3\rho}{}^{\mu} \\ &+ \frac{2iD_1(2C_1 - D_1 - 2)}{(p_1 \cdot q_2) m^2} p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda} \bigg] + \mathsf{S}(p_1)_{\mu\nu} p_1^\nu \bigg[\frac{2iD_1(C_1+1)}{(p_1 \cdot q_2) m^2} p_{1\rho} (f_3^{\rho\lambda} f_{2\lambda}{}^{\mu} - f_2^{\rho\lambda} f_{3\lambda}{}^{\mu}) \\ &+ \frac{2iD_1}{(p_1 \cdot q_2)^2 m^2} p_{1\rho} p_{1\lambda} (f_3^{\rho\mu} q_{3\sigma} f_2^{\sigma\lambda} + f_2^{\rho\mu} q_{2\sigma} f_3^{\sigma\lambda}) \bigg] \bigg\}\,. \end{aligned} \tag{2.3.9}$$

We note that the first term is formally identical to (2.3.8) for FT1 except for the replacement $S \to \mathsf{S}$, while the second term is proportional to the SSC condition $\mathsf{S}_{\mu\nu} p_1^\nu$.

Proceeding to FT2 of a single transverse spin-s field, we extract the classical $\mathcal{O}(S^1)$ Compton amplitude from explicit calculations for $s = 1, 2, 3$ using the Lagrangian given in the third row of 2.1. Unlike FT1, the amplitudes are now given in terms of explicit polarization vectors instead of spin tensors such that we need to

[14]In Ref. [191] higher order in q_i terms are also included since they are needed when feeding the Compton amplitudes into unitarity cuts for building higher PL two-body amplitudes.

[15]The gravitational analog of this amplitude is of $\mathcal{O}(S^2)$.

convert the former into the latter. Since the classical amplitudes in terms of spin tensors scale as $\mathcal{O}(q^0)$ and the spin tensors scale as $\mathcal{O}(q^{-1})$, in a fixed-spin calculation, the classical part of the $\mathcal{O}(S^1)$ amplitude is among the $\mathcal{O}(q)$ terms of the full quantum amplitude [59]. At this order, the massive polarization vectors appear in two structures, $(\varepsilon_1 \cdot \bar{\varepsilon}_4)^s$ and $(\varepsilon_1 \cdot \bar{\varepsilon}_4)^{s-1}\varepsilon_1^{[\mu}\bar{\varepsilon}_4^{\nu]}$. The terms proportional to $(\varepsilon_1 \cdot \bar{\varepsilon}_4)^s$ belong to the quantum spinless amplitude, which can be ignored here. We then use the relation (2.2.40) to convert the second structure to spin tensors. The final classical amplitude is obtained by extrapolating the finite-spin results to generic s and taking the $s \to \infty$ of that expression. At $\mathcal{O}(S^1)$, the amplitude after the replacement (2.2.40) is in fact independent of s, as we have explicitly checked for $s \leqslant 3$. After identifying the classical part, the final answer for the classical Compton amplitude is

$$\begin{aligned}\mathcal{A}_{4,\text{ cl}}^{\text{FT2}}\Big|_{S^1} = (-\varepsilon_1 \cdot \bar{\varepsilon}_4)^s S(p_1)_{\mu\nu} \Bigg[&\frac{iC_1}{(p_1 \cdot q_2)^2}(f_2^{\mu\nu} q_{2\rho} f_3^{\rho\lambda} + f_3^{\mu\nu} q_{3\rho} f_2^{\rho\lambda})p_{1\lambda} + \frac{2iC_1^2}{p_1 \cdot q_2} f_2^{\nu\rho} f_{3\rho}{}^{\mu} \\ &+ \frac{2i(C_1-1)^2}{(p_1 \cdot q_2)m^2} p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda}\Bigg]. \qquad (2.3.10)\end{aligned}$$

Notably, this amplitude is independent of D_1, and it can be obtained from (2.3.8) by setting D_1 to a special value,

$$\mathcal{A}_{4,\text{ cl}}^{\text{FT1s}}\Big|_{S^1} = \mathcal{A}_{4,\text{ cl}}^{\text{FT2}}\Big|_{S^1} \quad \text{for} \quad D_1 = C_1 - 1\,. \qquad (2.3.11)$$

In other words, for this special value of D_1, FT1 effectively propagates only the spin-s states. Moreover, the special value $C_1 = 1$ and $D_1 = 0$ reproduces the root-Kerr Compton amplitudes [193]. The appearance of additional Wilson coefficients in $\mathcal{A}^{\text{FT1s}}$ compared with $\mathcal{A}^{\text{FT2}}$ can be attributed to the additional propagating degrees of freedom.[16]

Finally, FT3 amplitudes also receive contributions from lower-spin states. We first restrict the lower spins to only appear in the intermediate states. Repeating the same steps as for FT2 we find that spin-$(s-1)$ intermediate states contribute as

$$\mathcal{A}_{4,\text{ cl}}^{\text{FT3s}}\Big|_{S^1} = \mathcal{A}_{4,\text{ cl}}^{\text{FT3}}\Big|_{S^1} + (-\varepsilon_1 \cdot \bar{\varepsilon}_4)^s S(p_1)_{\mu\nu}\left[\frac{2i\widetilde{C}_1^2}{(p_1 \cdot q_2)m^2} p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda}\right]. \qquad (2.3.12)$$

We note that the $\mathcal{O}(S^1)$ amplitude does not change if we include intermediate states with spin less than $s-1$. Comparing with (2.3.8), we find that the two amplitudes are formally related by the same map as (2.3.6),

$$\mathcal{A}_{4,\text{ cl.}}^{\text{FT3s}}\Big|_{S^1} = \mathcal{A}_{4,\text{ cl.}}^{\text{FT1s}}\Big|_{S^1} \quad \text{for} \quad i\widetilde{C}_1 = 1 - C_1 + D_1\,. \qquad (2.3.13)$$

Furthermore, this map persists even for amplitudes with external lower-spin states. To see this, we first rewrite (2.3.9) using (2.2.25),

$$\mathcal{A}_{4,\text{ cl.}}^{\text{FT1g}}\Big|_{S^1} = (-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_4 \Bigg\{ S(p_1)_{\mu\nu}\Bigg[\frac{iC_1}{(p_1 \cdot q_2)^2}(f_2^{\mu\nu} q_{2\rho} f_3^{\rho\lambda} + f_3^{\mu\nu} q_{3\rho} f_2^{\rho\lambda})p_{1\lambda} + \frac{2iC_1^2}{p_1 \cdot q_2} f_2^{\nu\rho} f_{3\rho}{}^{\mu}$$

[16] In Ref. [231], a similar computation was carried out for $s = 1$ and observed a similar effect. Their amplitudes are equivalent to our $D_1 = 0$ case.

$$+ \frac{2iD_1(2C_1 - D_1 - 2)}{(p_1 \cdot q_2)m^2} p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda} \Bigg] + \frac{2K(p_1)_\mu p_{1\nu}}{m} \Bigg[\frac{D_1 - C_1}{(p_1 \cdot q_2)^2} (q_2^\mu + q_3^\mu) f_2^{\nu\rho} f_{3\rho\lambda} p_1^\lambda$$
$$- \frac{C_1(1 - C_1 + D_1)}{p_1 \cdot q_2} (f_2^{\nu\rho} f_{3\rho}{}^\mu - f_3^{\nu\rho} f_{2\rho}{}^\mu) \Bigg] \Bigg\} . \tag{2.3.14}$$

We then find $\mathcal{A}_4^{\text{FT3g}}$, with external states in Eq. (2.2.55), from the Lagrangian of FT3. The momentum dependence of vertices is essential to express the contributions with spin-$(s-1)$ external states in terms of K^μ, using (2.2.48),(2.2.50). The result is

$$\mathcal{A}_4^{\text{FT3g}}\Big|_{S^1} = (-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_4 \Bigg\{ S(p_1)_{\mu\nu} \Bigg[\frac{iC_1}{(p_1 \cdot q_2)^2} (f_2^{\mu\nu} q_{2\rho} f_3^{\rho\lambda} + f_3^{\mu\nu} q_{3\rho} f_2^{\rho\lambda}) p_{1\lambda} + \frac{2iC_1^2}{p_1 \cdot q_2} f_2^{\nu\rho} f_{3\rho}{}^\mu$$
$$+ \frac{2i[(C_1 - 1)^2 + \widetilde{C}_1^2]}{(p_1 \cdot q_2)m^2} p_{1\rho} f_2^{\rho\mu} f_3^{\nu\lambda} p_{1\lambda} \Bigg] + \frac{2K(p_1)_\mu p_{1\nu}}{m} \Bigg[\frac{i\widetilde{C}_1 - 1}{(p_1 \cdot q_2)^2} (q_2^\mu + q_3^\mu) f_2^{\nu\rho} f_{3\rho\lambda} p_1^\lambda$$
$$- \frac{iC_1 \widetilde{C}_1}{p_1 \cdot q_2} (f_2^{\nu\rho} f_{3\rho}{}^\mu - f_3^{\nu\rho} f_{2\rho}{}^\mu) \Bigg] \Bigg\} . \tag{2.3.15}$$

Now comparing (2.3.14),(2.3.15), we find that

$$\mathcal{A}_{4,\text{ cl.}}^{\text{FT3g}}\Big|_{S^1} = \mathcal{A}_{4,\text{ cl.}}^{\text{FT1g}}\Big|_{S^1} \qquad \text{for} \quad i\widetilde{C}_1 = 1 - C_1 + D_1 . \tag{2.3.16}$$

The robustness of this map demonstrates that the terms tagged by the extra Wilson coefficients present in the amplitudes (and observables) of FT1 and FT3 carry new physical information compared to FT2.

The Quadratic-in-Spin Compton Amplitudes

Feynman-diagram calculations using the propagators and vertices of FT1 as well as properties of transverse coherent states show that, at $\mathcal{O}(S^2)$ the Compton amplitude depends on two distinct contractions of spin tensors:

$$\mathcal{A}_{4,\text{ cl.}}^{\text{FT1s}}\Big|_{S^2} = (-\varepsilon_1 \cdot \bar{\varepsilon}_4)^s \big(S(p_1)_{\mu\nu} S(p_1)_{\lambda\sigma} \mathcal{X}^{\mu\nu\lambda\sigma} + S(p_1)_{\mu\lambda} S(p_1)^\lambda{}_\nu \mathcal{X}^{\mu\nu} \big) . \tag{2.3.17}$$

Their kinematic coefficients are given by

$$\mathcal{X}^{\mu\nu\lambda\sigma} = \frac{C_1^2 (q_2 \cdot q_3)}{2(p_1 \cdot q_2)^2} f_2^{\mu\nu} f_3^{\lambda\sigma} + \frac{C_1 D_1 + D_2(C_1 - D_1 - 1)}{(p_1 \cdot q_2)m^2} p_{1\rho} (f_3^{\rho\mu} q_2^\nu f_2^{\lambda\sigma} - f_2^{\rho\mu} q_3^\nu f_3^{\lambda\sigma}) , \tag{2.3.18}$$

$$\mathcal{X}^{\mu\nu} = \frac{C_2}{m^2} \Bigg[\frac{p_{1\rho} p_{1\alpha} (f_2^{\rho\mu} q_2^\nu f_3^{\alpha\beta} q_{2\beta} + f_3^{\rho\mu} q_3^\nu f_2^{\alpha\beta} q_{3\beta})}{(p_1 \cdot q_2)^2} + \frac{p_{1\rho} (f_2^{\rho\sigma} f_{3\sigma}{}^\mu q_3^\nu - f_3^{\rho\sigma} f_{2\sigma}{}^\mu q_2^\nu)}{(p_1 \cdot q_2)}$$
$$+ \frac{2C_1\, p_{1\rho} (f_3^{\rho\mu} f_2^{\nu\sigma} q_{3\sigma} - f_2^{\rho\mu} f_3^{\nu\sigma} q_{2\sigma})}{(p_1 \cdot q_2)} + \frac{2(C_1 - 1) p_{1\rho} f_2^{\rho\mu} f_3^{\nu\sigma} p_{1\sigma}}{m^2} + 2C_1 f_{2\rho}{}^\mu f_3^{\nu\rho} \Bigg] . \tag{2.3.19}$$

The dependence on Wilson coefficients indicates that both terms originate from both $\mathcal{L}_{\text{min}}$ and $\mathcal{L}_{\text{non-min}}$, and the Lorentz algebra was used to reduce a product of three Lorentz generators to a sum of irreducible (symmetric) products. We also note that the D_1 and D_2 dependence only appear in $\mathcal{X}^{\mu\nu\lambda\sigma}$.

Choosing general asymptotic states instead of transverse ones leads to the amplitude $\mathcal{A}^{\text{FT1g}}$. Apart from the replacement $S \to \mathsf{S}$ in $\mathcal{A}_{4,\text{ cl}}^{\text{FT1s}}\Big|_{S^2}$, the amplitude contains terms proportional to the covariant SSC conditions:

$$\mathcal{A}_{4,\,\text{cl.}}^{\text{FT1g}}\Big|_{S^2} = (-1)^s \mathcal{E}_1 \cdot \bar{\mathcal{E}}_4 \Big[\mathsf{S}(p_1)_{\mu\nu}\mathsf{S}(p_1)_{\lambda\sigma}\mathcal{X}^{\mu\nu\lambda\sigma} + \mathsf{S}(p_1)_{\mu\lambda}\mathsf{S}(p_1)^{\lambda}{}_{\nu}\mathcal{X}^{\mu\nu}$$
$$+ \mathsf{S}(p_1)_{\mu\nu}p_1^\nu \mathsf{S}(p_1)_{\lambda\sigma}\mathcal{Y}^{\mu\lambda\sigma} + \mathsf{S}(p_1)_{\mu\nu}p_1^\nu \mathsf{S}(p_1)_{\lambda\sigma}p_1^\sigma \mathcal{Y}^{\mu\lambda} + \mathsf{S}(p_1)_{\mu\lambda}\mathsf{S}(p_1)^{\lambda}{}_{\nu}p_1^\nu \mathcal{Y}^{\mu}\Big], \tag{2.3.20}$$

where the additional kinematic coefficients are given by

$$\begin{aligned}
\mathcal{Y}^{\mu\lambda\sigma} = {}& \frac{C_1(D_1-D_2)}{(p_1\cdot q_2)m^2}(q_{2\rho}f_3^{\rho\mu}f_2^{\lambda\sigma} - q_{3\rho}f_2^{\rho\mu}f_3^{\lambda\sigma}) + \frac{C_1D_1(q_2\cdot q_3)}{(p_1\cdot q_2)^2m^2}p_{1\rho}(f_3^{\rho\mu}f_2^{\lambda\sigma} - f_2^{\rho\mu}f_3^{\lambda\sigma}) \\
&+ \frac{2C_1D_2}{(p_1\cdot q_2)m^2}(q_2+q_3)^\mu f_2^{\sigma\alpha}f_{3\alpha}{}^{\lambda} + \frac{2D_1^2}{(p_1\cdot q_2)m^4}p_{1\rho}p_{1\alpha}(f_2^{\rho\mu}f_3^{\alpha\lambda}q_2^\sigma - f_3^{\rho\mu}f_2^{\alpha\lambda}q_3^\sigma) \\
&+ \frac{2D_2}{(p_1\cdot q_2)^2m^2}p_{1\alpha}(q_2^\lambda f_2^{\sigma\mu}f_3^{\alpha\beta}q_{2\beta} + q_3^\lambda f_3^{\sigma\mu}f_2^{\alpha\beta}q_{3\beta}) \\
&+ \frac{2(C_2(C_1-D_1-1)-D_1D_2)}{(p_1\cdot q_2)m^4}(q_2+q_3)^\mu p_{1\rho}f_3^{\rho\lambda}f_2^{\sigma\alpha}p_{1\alpha} \\
&+ \frac{C_2(C_1-D_1-1)}{m^4}(f_2^{\mu\lambda}f_3^{\sigma\rho} + f_3^{\mu\lambda}f_2^{\sigma\rho})p_{1\rho} + \frac{D_1D_2}{m^4}p_{1\rho}(f_3^{\rho\mu}f_2^{\lambda\sigma} + f_2^{\rho\mu}f_3^{\lambda\sigma})\,,
\end{aligned} \tag{2.3.21}$$

$$\begin{aligned}
\mathcal{Y}^{\mu\lambda} = {}& \frac{2D_1^2}{(p_1\cdot q_2)m^4}p_{1\rho}(f_2^{\rho\mu}q_{2\alpha}f_3^{\alpha\lambda} - f_3^{\rho\mu}q_{3\alpha}f_2^{\alpha\lambda}) + \frac{2D_1^2(q_2\cdot q_3)}{(p_1\cdot q_2)^2m^4}p_{1\rho}f_2^{\rho\mu}p_{1\alpha}f_3^{\alpha\lambda} \\
&+ \frac{C_2}{(p_1\cdot q_2)^2m^4}p_{1\rho}(q_2^\mu f_2^{\rho\lambda}p_{1\alpha}f_3^{\alpha\beta}q_{2\beta} + q_3^\mu f_3^{\rho\lambda}p_{1\alpha}f_2^{\alpha\beta}q_{3\beta}) \\
&+ \frac{C_2-2D_1D_2}{(p_1\cdot q_2)m^4}p_{1\alpha}(q_3^\mu f_2^{\alpha\beta}f_{3\beta}{}^{\lambda} - q_2^\mu f_3^{\alpha\beta}f_{2\beta}{}^{\lambda}) + \frac{2C_1C_2}{(p_1\cdot q_2)m^4}p_{1\alpha}(q_2^\mu f_2^{\alpha\beta}f_{3\beta}{}^{\lambda} - q_3^\mu f_3^{\alpha\beta}f_{2\beta}{}^{\lambda}) \\
&- \frac{2C_1C_2}{m^4}f_2^{\mu\alpha}f_{3\alpha}{}^{\lambda} - \frac{2C_2D_1}{m^6}p_{1\rho}p_{1\alpha}(f_2^{\rho\mu}f_3^{\alpha\lambda} + \eta^{\mu\lambda}f_2^{\rho\beta}f_{3\beta}{}^{\alpha})\,,
\end{aligned} \tag{2.3.22}$$

$$\begin{aligned}
\mathcal{Y}^{\mu} = {}& \frac{2C_2D_1}{(p_1\cdot q_2)m^4}p_{1\rho}p_{1\alpha}(f_2^{\rho\mu}f_3^{\alpha\beta}q_{2\beta} - f_3^{\rho\mu}f_2^{\alpha\beta}q_{3\beta}) \\
&- \frac{C_2(C_1+D_1-1)}{m^4}p_{1\rho}(f_3^{\rho\alpha}f_{2\alpha}{}^{\mu} + f_2^{\rho\alpha}f_{3\alpha}{}^{\mu})\,.
\end{aligned} \tag{2.3.23}$$

We proceed next to the $\mathcal{O}(S^2)$ tree-level Compton amplitude of FT2. Repeating at this order the classical scaling argument we described at $\mathcal{O}(S^1)$ shows that, in a fixed-spin calculation, the classical tree-level Compton amplitude is contained in the $\mathcal{O}(q^2)$ terms of the quantum tree-level Compton amplitude. Thus, we extract these terms from explicit $s = 1, 2, 3$ calculations, extrapolate them to large spin and keep only the leading term. The massive polarization vectors now appear in four structures,

$$(\varepsilon_1\cdot\bar\varepsilon_4)^s\,, \qquad (\varepsilon_1\cdot\bar\varepsilon_4)^{s-1}\varepsilon_1^{[\mu}\bar\varepsilon_4^{\nu]}\,, \qquad (\varepsilon_1\cdot\bar\varepsilon_4)^{s-1}\varepsilon_1^{(\mu}\bar\varepsilon_4^{\nu)}\,, \qquad (\varepsilon_1\cdot\bar\varepsilon_4)^{s-2}\varepsilon_1^{[\mu}\bar\varepsilon_4^{\nu]}\varepsilon_1^{[\rho}\bar\varepsilon_4^{\lambda]}\,, \tag{2.3.24}$$

where the first two correspond to the quantum spinless and $\mathcal{O}(S^1)$ amplitudes that can be ignored here. We use the replacement (2.2.42) and (2.2.43) for the last two structures. It turns that the dependence on s is simple so that we can extrapolate it to obtain the general s dependence and take $s\to\infty$ limit. The kinematic coefficient of $S^{\mu\rho}S_\rho{}^\nu$ can be accessed by any $s \geqslant 1$; in the large s limit, it exactly reproduces the $\mathcal{X}^{\mu\nu}$ shown in (2.3.18). The structure $S^{\mu\nu}S^{\rho\lambda}$ appears for $s\geqslant 2$. A careful analysis with $s = 2$ and 3 gives identical results, so that we postulate that the coefficient of $S^{\mu\nu}S^{\rho\lambda}$ is independent of s. Thus we find that the tree-level Compton amplitude of FT2 is

$$\mathcal{A}^{\text{FT2}}_{4,\text{ cl.}}\Big|_{S^2} = (-\varepsilon_1\cdot\bar{\varepsilon}_4)^s \left(S_{\mu\nu}S_{\lambda\sigma}\widetilde{\mathcal{X}}^{\mu\nu\lambda\sigma} + S_{\mu\lambda}S^{\lambda}{}_{\nu}\mathcal{X}^{\mu\nu}\right)\,,$$
$$\widetilde{\mathcal{X}}^{\mu\nu\lambda\sigma} = \frac{C_1^2(q_2\cdot q_3)}{2(p_1\cdot q_2)^2}f_2^{\mu\nu}f_3^{\lambda\sigma} + \frac{C_1(C_1-1)}{(p_1\cdot q_2)m^2}p_{1\rho}(f_3^{\rho\mu}q_2^{\nu}f_2^{\lambda\sigma} - f_2^{\rho\mu}q_3^{\nu}f_3^{\lambda\sigma})\,, \tag{2.3.25}$$

where $\mathcal{X}^{\mu\nu}$ is defined in equation (2.3.19). These coefficients depend only on C_1 and C_2 and are independent of D_1 and D_2. We again observe the same pattern as in the linear-in-spin case,

$$\mathcal{A}^{\text{FT1s}}_{4,\text{ cl.}}\Big|_{S^2} = \mathcal{A}^{\text{FT2}}_{4,\text{ cl.}}\Big|_{S^2} \qquad \text{for} \quad D_1 = C_1 - 1\,. \tag{2.3.26}$$

We note that the special value of D_1 also removes the dependence on D_2.

Similar to $\mathcal{O}(S^1)$, the $\mathcal{O}(S^2)$ Compton amplitudes of FT3 receive contributions from lower-spin intermediate states. Keeping the external states transverse, we get

$$\mathcal{A}^{\text{FT3s}}_{4,\text{ cl}}\Big|_{S^2} = \mathcal{A}^{\text{FT2}}_{4,\text{ cl}}\Big|_{S^2} + (-\varepsilon_1\cdot\bar{\varepsilon}_4)^s S(p_1)_{\mu\nu}S(p_1)_{\lambda\sigma}\left[\frac{\widetilde{C}_1\widetilde{C}_2}{(p_1\cdot q_2)m^2}p_{1\rho}(f_3^{\rho\mu}q_2^{\nu}f_2^{\lambda\sigma} - f_2^{\rho\mu}q_3^{\nu}f_3^{\lambda\sigma})\right]. \tag{2.3.27}$$

Just like the previous cases, the same formal relations hold between the additional Wilson coefficients in FT1 and FT3. Indeed, comparing (2.3.27) and (2.3.17), it is easy to see that

$$\mathcal{A}^{\text{FT1s}}_{4,\text{ cl}}\Big|_{S^2} = \mathcal{A}^{\text{FT3s}}_{4,\text{ cl}}\Big|_{S^2} \qquad \text{for} \quad i\widetilde{C}_1 = 1 - C_1 + D_1 \;\text{ and }\; i\widetilde{C}_2 = D_2 - C_1\,, \tag{2.3.28}$$

which is identical to (2.3.6). This demonstrates that the relation between extra Wilson coefficients and extra propagating degrees of freedom holds also at $\mathcal{O}(S^2)$. A comparison between $\mathcal{A}^{\text{FT1g}}_{4,\text{ cl.}}$ and $\mathcal{A}^{\text{FT3g}}_{4,\text{ cl.}}$ at $\mathcal{O}(S^1K^1)$ and $\mathcal{O}(K^2)$ requires that we include in the Lagrangian of FT3 a spin-$(s-2)$ field ϕ_{s-2}, and additional operators that contribute independently at $\mathcal{O}(K^2)$, for example (2.2.59) for FT1. This is because the effect of $\mathcal{O}(K^2)$ operators show up at $\mathcal{O}(S^1K^1)$ in the four-point Compton amplitudes due to the commutator $[K^2, K] \sim SK$. Finally, we note that the spin-transition Compton amplitude $\mathcal{A}_4^{s\to s-1}$ under a fixed-spin calculation may superficially contain a super-classical contribution that does not cancel between the two matter channels. Consistency of the theory requires however that in the large-spin limit this term is subleading. We will assume that this cancellation holds as $s\to\infty$. It is nontrivial to carry out explicit calculations to demonstrate this, but would be worth investigating.

2.3.3 Two-Body Amplitudes

In previous subsections, we have explored and understood the effect of various types of interactions between higher-spin fields and photons on Compton amplitudes, and the number of Wilson coefficients necessary to describe such interactions. We found that, under suitable conditions like allowing spin magnitude change, this number is indeed larger than that required to describe the interactions of SSC preserving spins.

The rationale of this exercise is to eventually understand their effects on two-body observables, such as the momentum impulse and the spin kick. It was originally suggested in the context of gravity that a larger number of Wilson coefficients may be required to describe more general interactions [101]. We therefore proceed to expose the photon-mediated two-body amplitudes and, in later sections, the observables that follow from them as well as their comparison with a wordline perspective. We use the generalized unitarity method [206, 207, 209] to construct the relevant integrands, while taking advantage of the simplifications introduced in Ref. [248]. To reduce the encountered loop integrals to known ones we make use of integration by parts [212, 216] as implemented in FIRE [249, 250].

We use the momentum and mass variables

$$\begin{aligned} &\bar{m}_1 = m_1^2 - q^2/4\,, \qquad &&\bar{m}_2^2 = m_2^2 - q^2/4\,, \qquad &&y = \frac{\bar{p}_1 \cdot \bar{p}_2}{\bar{m}_1 \bar{m}_2}\,, \\ &\bar{p}_1 = p_1 + q/2 = -p_4 - q/2\,, \qquad &&\bar{p}_2 = p_2 - q/2 = -p_3 + q/2\,, && \end{aligned} \tag{2.3.29}$$

which are originally used for the expansion in the soft region of gravitational amplitudes in [33]. We primarily focus on FT1 because this is what we compare with a worldline theory. Unitarity guarantees that the two-body amplitudes of FT2 can be obtained from those of FT1 by setting $D_1 = C_1 - 1$ and imposing the covariant SSC, while the two-body amplitudes of FT3 can be obtained using the map between the extra Wilson coefficients proposed in the previous subsection.

Tree Level

The structure of the two-body amplitude at tree-level and in the classical limit is

$$i\mathcal{M}_{4,\text{cl.}}^{\text{tree}} = (4\pi\alpha)(\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4)(\mathcal{E}_2 \cdot \bar{\mathcal{E}}_3)\left(\frac{d_{\perp}}{q^2}\right), \tag{2.3.30}$$

where q is the photon momentum, and the numerator $d_{\perp}$ is a function of momenta and spin tensors which scales as $d_{\perp} \sim q^0$ in the classical limit. Our results for FT1g through the quadratic order in spin are as follows:

$$\begin{aligned} d_{\perp}\Big|_{\text{spinless}} &= 4iy\bar{m}_1\bar{m}_2\,, \\ d_{\perp}\Big|^{\text{FT1g}}_{S_1^1 S_2^0} &= -4\bar{m}_2 \mathsf{S}_{1\mu\nu}\left(C_{1(1)}\bar{u}_2^{\mu} q^{\nu} - D_{1(1)} y \bar{u}_1^{\mu} q^{\nu}\right), \\ d_{\perp}\Big|^{\text{FT1g}}_{S_1^1 S_2^1} &= 4iC_{1(1)}C_{1(2)}\mathsf{S}_1^{\mu\nu} q_{\nu} \mathsf{S}_{2\mu\rho} q^{\rho} \\ &\quad - 4i\mathsf{S}_1^{\mu\nu} q_{\nu} \mathsf{S}_2^{\rho\lambda} q_{\lambda}\left(C_{1(1)}D_{1(2)}\bar{u}_{2\mu}\bar{u}_{2\rho} + C_{1(2)}D_{1(1)}\bar{u}_{1\mu}\bar{u}_{1\rho} - D_{1(1)}D_{1(2)} y \bar{u}_{1\mu}\bar{u}_{2\rho}\right), \\ d_{\perp}\Big|^{\text{FT1g}}_{S_1^2 S_2^0} &= \frac{2i\bar{m}_2}{\bar{m}_1}\left[yC_{2(1)}\mathsf{S}_1^{\mu\nu} q_{\nu} \mathsf{S}_{1\mu\rho} q^{\rho} - yC_{2(1)}(\mathsf{S}_1^{\mu\nu}\bar{u}_{1\mu} q_{\nu})^2 - 2D_{2(1)}\mathsf{S}_1^{\mu\nu}\bar{u}_{1\mu} q_{\nu} \mathsf{S}_1^{\rho\lambda}\bar{u}_{2\rho} q_{\lambda}\right], \end{aligned} \tag{2.3.31}$$

where the spinless case agrees with Ref. [246, 5, 191]. The notation $C_{i(j)}$ and $D_{i(j)}$ refers to the C_i and D_i coefficients associated with body j. If the external states are transverse, $(\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4)(\mathcal{E}_2 \cdot \bar{\mathcal{E}}_3) = (\varepsilon_1 \cdot \bar{\varepsilon}_4)^s(\varepsilon_2 \cdot \bar{\varepsilon}_3)^s$, then the spin tensor obeys the covariant SSC, such that,

$$\begin{aligned} d_{\perp}\Big|^{\text{FT1s}}_{S_1^1 S_2^0} &= d_{\perp}\Big|^{\text{FT2}}_{S_1^1 S_2^0} = d_{\perp}\Big|^{\text{FT3s}}_{S_1^1 S_2^0} = -4C_{1(1)}\bar{m}_2 S_1^{\mu\nu}\bar{u}_{2\mu}q^\nu \,, \\ d_{\perp}\Big|^{\text{FT1s}}_{S_1^1 S_2^1} &= d_{\perp}\Big|^{\text{FT2}}_{S_1^1 S_2^1} = d_{\perp}\Big|^{\text{FT3s}}_{S_1^1 S_2^1} = 4iC_{1(1)}C_{1(2)} S_1^{\mu\nu} q_\nu S_{2\mu\rho} q^\rho \,, \\ d_{\perp}\Big|^{\text{FT1s}}_{S_1^2 S_2^0} &= d_{\perp}\Big|^{\text{FT2}}_{S_1^2 S_2^0} = d_{\perp}\Big|^{\text{FT3s}}_{S_1^2 S_2^0} = \frac{2iC_{2(1)} y \bar{m}_2}{\bar{m}_1} S_1^{\mu\nu} q_\nu S_{1\mu\rho} q^\rho \,. \end{aligned} \tag{2.3.32}$$

The small velocity expansion of the first two expressions agrees with the results of Ref. [246]. They are related to (2.3.31) through the replacement $\mathsf{S}_i \to S_i$ and $S_{i\mu\nu}\bar{u}_i^\nu = 0$, which holds to all orders in spin at tree level. We note that, to first order on K^i, the amplitudes of FT3g can also be obtained from (2.3.31) through the Wilson coefficient map (2.3.6).

In 2.5 we compare observables from the amplitudes $\mathcal{A}^{\text{FT1g}}$ of FT1 and those from worldline calculations in the absence of an SSC. We find a perfect match both at $\mathcal{O}(\alpha)$, which follow from the amplitudes above, and at $\mathcal{O}(\alpha^2)$ which follow from the one-loop amplitudes we now summarize.

One Loop

While four-point Compton amplitudes are not relevant for the tree-level two-body scattering, they are an integral part of two-body scattering at one loop. The generalized unitarity method [206, 207, 209] provides a means to construct the classically-relevant parts of the latter in terms of the former. We should therefore expect that the precise intermediate states contributing to Compton amplitudes have observable consequences for the scattering of two matter particles. In particular, we note that intermediate states of spin different from the external spin can be projected out either by using only transverse spin-s fields *or* by choosing particular values for the extra Wilson coefficients, see (2.3.11). Since before loop integration, the part of the one-loop two-body amplitude that is relevant in the classical limit is literally the product of two Compton amplitudes summed over states, the latter observation must have hold at one loop as well. Thus, we may follow this strategy to compute the one-loop two-body amplitude of FT1g.

The complete one-loop amplitude exhibits classically-singular, classical and quantum terms. The former two are

$$i\mathcal{M}^{(1)}_{4,\text{cl.}} = (4\pi\alpha)^2\Big[C_{\text{box}}(I_{\square} + I_{\boxtimes}) + i\mathcal{M}_{\triangle+\triangledown}\Big]\,, \tag{2.3.33}$$

where the first one, given by the box and crossed-boxed integrals

$$I_{\square} = \int \frac{\mathrm{d}^d\ell}{(2\pi)^d}\frac{1}{\ell^2(\ell-q)^2(2\bar{p}_1\cdot\ell + i0)(-2\bar{p}_2\cdot\ell + i0)}$$

$$I_{\boxtimes} = \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^2 (\ell - q)^2 (2\bar{p}_1 \cdot \ell + i0)(2\bar{p}_2 \cdot \ell + i0)} \,, \tag{2.3.34}$$

is the classically-singular part, while the second term, containing the triangle integral

$$I_{\triangle} = \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \frac{1}{\ell^2 (\ell - q)^2 (2\bar{p}_1 \cdot \ell + i0)} \,, \tag{2.3.35}$$

is the classical part [96, 14].

The spin-independent part of the amplitude is

$$\begin{aligned} C_{\text{box}}\Big|_{\text{spinless}} &= -(\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4)(\mathcal{E}_2 \cdot \bar{\mathcal{E}}_3) \left(d_{\bot\!\!\!\!\top}\Big|_{\text{spinless}} \right)^2 \,, \\ i\mathcal{M}_{\triangle+\triangledown}\Big|_{\text{spinless}} &= (\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4)(\mathcal{E}_2 \cdot \bar{\mathcal{E}}_3) \frac{i(\bar{m}_1 + \bar{m}_2)}{4\sqrt{-q^2}} \,. \end{aligned} \tag{2.3.36}$$

As its tree-level counterpart, it agrees with Ref. [191, 251] and it is the same in all three field theories.

The linear in spin part of the classically singular term is

$$C_{\text{box}}\Big|^{\text{FT1g}}_{S_1^1 S_2^0} = -(\mathcal{E}_1 \cdot \bar{\mathcal{E}}_4)(\mathcal{E}_2 \cdot \bar{\mathcal{E}}_3) \left(d_{\bot\!\!\!\!\top}\Big|_{\text{spinless}} \times d_{\bot\!\!\!\!\top}\Big|^{\text{FT1g}}_{S_1^1 S_2^0} \right) \,, \tag{2.3.37}$$

As the spin-independent part (2.3.36), it is given by the product of tree-level amplitudes, in agreement with the expected exponential structure of the amplitude in the classical limit [14, 16, 30]. The corresponding expression at higher powers of the spins should be given by the IBP reduction of such products of trees summed over all the possible ways of distributing the spins in the two factors.

The classical part of the one-loop two-body amplitude can be organized in terms of the various possible contractions of spin tensors. As at tree level, we write explicitly the amplitude $\mathcal{A}^{\text{FT1g}}$ for FT1 and obtain the amplitudes in other theories via $\mathsf{S} \to S$ and other limits on Wilson coefficients. The structure of $i\mathcal{M}_{\triangle+\triangledown}\Big|_{S_1^{n_1} S_2^{n_2}}$ is

$$i\mathcal{M}_{\triangle+\triangledown}\Big|_{S_1^{n_1} S_2^{n_2}} = \frac{(\varepsilon_1 \cdot \bar{\varepsilon}_4)^s (\varepsilon_2 \cdot \bar{\varepsilon}_3)^s}{4\sqrt{-q^2}} \sum_i \alpha^{(n_1, n_2, i)} \mathcal{O}^{(n_1, n_2, i)} \,; \tag{2.3.38}$$

through second order in spin, the spin-tensor contractions are $\mathcal{O}^{(n_1,n_2,i)}$ are:

- Linear in spin:

$$\mathcal{O}^{(1,0,1)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{2\mu} q_\nu \,, \qquad \mathcal{O}^{(1,0,2)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} q_\nu \tag{2.3.39}$$

- Bilinear in spin:

$$\begin{aligned}
&\mathcal{O}^{(1,1,1)} = \mathsf{S}_1^{\mu\nu} q_\nu \mathsf{S}_{2\mu\rho} q^\rho\,, && \mathcal{O}^{(1,1,2)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{2\nu} \mathsf{S}_{2\mu\rho} \bar{u}_1^\rho\,, && \mathcal{O}^{(1,1,3)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{2\mu} q_\nu \mathsf{S}_2^{\lambda\sigma} \bar{u}_{1\lambda} q_\sigma\,,\\
&\mathcal{O}^{(1,1,4)} = \mathsf{S}_1^{\mu\nu} \mathsf{S}_{2\mu\nu}\,, && \mathcal{O}^{(1,1,5)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\nu} \mathsf{S}_{2\mu\rho} \bar{u}_1^\rho\,, && \mathcal{O}^{(1,1,6)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\nu} \mathsf{S}_{2\mu\rho} \bar{u}_2^\rho\,,\\
&\mathcal{O}^{(1,1,7)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{2\nu} \mathsf{S}_{2\mu\rho} \bar{u}_2^\rho\,, && \mathcal{O}^{(1,1,8)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} q_\nu \mathsf{S}_2^{\lambda\sigma} \bar{u}_{1\lambda} q_\sigma\,, && \mathcal{O}^{(1,1,9)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} q_\nu \mathsf{S}_2^{\lambda\sigma} \bar{u}_{2\lambda} q_\sigma\,,
\end{aligned}$$
$$\mathcal{O}^{(1,1,10)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{2\mu} q_\nu \mathsf{S}_2^{\lambda\sigma} \bar{u}_{2\lambda} q_\sigma\,, \qquad \mathcal{O}^{(1,1,11)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} \bar{u}_{2\nu} \mathsf{S}_2^{\lambda\sigma} \bar{u}_{1\lambda} \bar{u}_{2\sigma}\,, \tag{2.3.40}$$

- Quadratic in spin:

$$\begin{aligned}
&\mathcal{O}^{(2,0,1)} = \mathsf{S}_1^{\mu\nu} q_\nu \mathsf{S}_{1\mu\rho} q^\rho\,, && \mathcal{O}^{(2,0,2)} = (\mathsf{S}_1^{\mu\nu} \bar{u}_{2\mu} q_\nu)^2\,, && \mathcal{O}^{(2,0,3)} = \mathsf{S}_1^{\mu\nu} \mathsf{S}_{1\mu\nu}\,,\\
&\mathcal{O}^{(2,0,4)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\nu} \mathsf{S}_{1\mu\rho} \bar{u}_1^\rho\,, && \mathcal{O}^{(2,0,5)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\nu} \mathsf{S}_{1\mu\rho} \bar{u}_2^\rho\,, && \mathcal{O}^{(2,0,6)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{2\nu} \mathsf{S}_{1\mu\rho} \bar{u}_2^\rho\,,\\
&\mathcal{O}^{(2,0,7)} = (\mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} q_\nu)^2\,, && \mathcal{O}^{(2,0,8)} = \mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} q_\nu \mathsf{S}_1^{\lambda\sigma} \bar{u}_{2\lambda} q_\sigma\,, && \mathcal{O}^{(2,0,9)} = (\mathsf{S}_1^{\mu\nu} \bar{u}_{1\mu} \bar{u}_{2\nu})^2\,,
\end{aligned} \tag{2.3.41}$$

All contractions that contain the covariant SSC constraints, $\mathsf{S}_i^{\mu\nu} \bar{u}_{i\mu}$, vanish for $\mathcal{A}^{\rm FT1s}$, $\mathcal{A}^{\rm FT2}$ and $\mathcal{A}^{\rm FT3s}$. The coefficients of $\mathcal{O}^{(n_1,n_2,i)}$ at linear order in spin are:

$$\begin{aligned}
\alpha^{(1,0,1)} &= -\frac{y}{(y^2-1)\bar{m}_1}\Big[2C_{1(1)}\bar{m}_1 + (C_{1(1)}^2 - 2C_{1(1)}D_{1(1)} + D_{1(1)}^2 + 2D_{1(1)})\bar{m}_2\Big]\,,\\
\alpha^{(1,0,2)} &= \frac{1}{(y^2-1)\bar{m}_1}\Big\{\Big[(y^2+1)C_{1(1)} + (y^2-1)D_{1(1)}\Big]\bar{m}_1\\
&\qquad + \Big[C_{1(1)}^2 - (y^2+1)C_{1(1)}D_{1(1)} + y^2 D_{1(1)}^2 + (3y^2-1)D_{1(1)}\Big]\bar{m}_2\Big\}\,.
\end{aligned} \tag{2.3.42}$$

As we reduce $\mathcal{A}^{\rm FT1g}$ to $\mathcal{A}^{\rm FT1s}$, the coefficients of the surviving spin structures under the covariant SSC are unchanged. To obtain the amplitude for FT2 we further impose $D_1 = C_1 - 1$, which also makes the D_2 dependence vanish up to the quadratic order in spin as in the Compton amplitudes. Similarly, to obtain the amplitudes $\mathcal{A}^{\rm FT3s}$ we use the relations (2.3.6) to replace the coefficients D_1 and D_2 by $\widetilde{C}_1$ and $\widetilde{C}_2$ after imposing the covariant SSC. Last but not least, we can also obtain $\mathcal{A}^{\rm FT3g}$ up to linear order in K from $\mathcal{A}^{\rm FT1g}$ by simply using the relations (2.3.6).

2.4 Worldline Theories

What worldline theory can reproduce the field-theory results of the previous sections? In the field theories where multiple spin states propagate, the spin vectors magnitude is no longer conserved so to match this one needs to introduce additional degrees of freedom on the worldline. Because these additional degrees of freedom are constrained by the Lorentz generator algebra, the natural choice is to find these degrees of freedom in the spin tensor itself. In section 2.5, we construct a two-body Hamiltonian that explicitly exhibits these additional dynamical variables. In this section our task is to find a modified worldline that produces the same results as the field theory.

We start from a standard worldline construction [226] with the SSC corresponding to WL1, listed in section 2.1. We see that the results we obtain for this theory then match the field theory containing only a single massive quantum spin state [231, 179], which is related to the fact that both necessarily preserve the spin-vector magnitude. To match field-theory results when multiple quantum spin states are present, we introduce additional degrees of freedom on the worldline by releasing the SSC, corresponding to WL2. As in general relativity this has *no physical effect* at first order in the coupling [101], but starting at second order in the coupling, physical differences can appear; in general relativity physical effects start at cubic order in the spin tensor, but in electrodynamics this occurs at linear order.

Specifically, we compute the tree-level Compton amplitude to quadratic order in spin and probe-limit $\mathcal{O}(\alpha^2)$ two-body impulse and spin kicks to linear order in spin with a scalar source. We do so initially using WL1 with the covariant SSC imposed via a Lagrange multiplier. Then, we switch to WL2 by removing the Lagrange multiplier terms enforcing the SSC constraint. This Compton amplitude of the modified worldline formalism has the same spin tensor dependence as found in the classical limit of the amplitude $\mathcal{A}^{\mathrm{FT1g}}$ of field theory FT1 without a physical state projector limiting it to the states of a single quantum spin. We find that not only do the equations of motion consistently evolve all the degrees of freedom, but that it is possible to match the observables of the modified worldline with the field theory, with a direct correspondence between the Wilson coefficients of the two formalisms. The key consequence is that both have a larger number of independent Wilson coefficients than the conventional worldline approach in which the SSC is imposed. We emphasize that the match is rather nontrivial.

2.4.1 Worldline Action with Dynamical Mass Function

We begin with a brief review of the worldline formalism, following Ref. [226]. The worldline formalism seeks to describe the evolution of a body of matter in terms of its spacetime location and internal degrees of freedom. We refer to the spacetime location of the body “center” in coordinates as $z^\mu(\lambda)$ where λ is a real parameter which parameterizes the worldline, called the worldline time. For now we denote the internal degrees of freedom of the body as $\phi^a(\lambda)$ where a is an index running over all of those internal degrees of freedom. Below we take these degrees of freedom to track the orientation of the body but for now the particular structure of these degrees of freedom is not important. The body’s evolution is described by an action which is reparameterization invariant under monotonic redefinitions of the worldline time $\lambda' = \lambda'(\lambda)$. The reparametrization invariance can be imposed directly through the introduction of an einbein field $\mathsf{e}(\lambda)$. The einbein is defined to transform under reparameterizations as:

$$\mathsf{e}'(\lambda') = \frac{d\lambda}{d\lambda'}\,\mathsf{e}(\lambda)\,, \tag{2.4.1}$$

A generic reparameterization invariant action is then of the form:

$$S[\mathsf{e}, z, \phi] = \int_{-\infty}^{\infty} \mathcal{L}\left(z, \frac{\dot{z}}{\mathsf{e}}, \phi, \frac{\dot{\phi}}{\mathsf{e}}\right) \mathsf{e}\, d\lambda\,, \tag{2.4.2}$$

where dots indicate differentiation with respect to λ. Defining the conjugate momenta as usual:

$$p_\mu = -\frac{\partial(\mathcal{L}\mathsf{e})}{\partial \dot{z}^\mu}\,, \qquad\qquad \pi_a = -\frac{\partial(\mathcal{L}\mathsf{e})}{\partial \dot{\phi}^a}\,, \tag{2.4.3}$$

the Hamiltonian form of the action can be written as:

$$S[\mathsf{e}, z, p, \phi, \pi] = \int_{-\infty}^{\infty} \left(-\pi_a \dot{\phi}^a - p_\mu \dot{z}^\mu - \mathsf{e} H(z, p, \phi, \pi)\right) d\lambda\,, \tag{2.4.4}$$

and p, π, and H are reparameterization invariant. It is useful to introduce the notation:

$$|p| = \sqrt{p^\mu p_\mu}, \qquad \hat{p}^\mu = \frac{p^\mu}{|p|}\,. \tag{2.4.5}$$

For a free particle, the Hamiltonian $H = -|p| + m$ produces the geodesic equation of motion. In general, $H = -|p| + m + \delta H(z, p, \phi, \pi)$ for some function δH containing all additional couplings. The on-shell constraint imposed by the einbein's equation of motion is always $H = 0$, which then determines $|p| = m + \delta H$. So, it is useful to introduce the dynamical mass function $\mathcal{M}(z, \hat{p}, \phi, \pi)$ as the solution for $|p|$ imposed by the einbein equation of motion: $|p| = \mathcal{M}(z, \hat{p}, \phi, \pi)$. Then, we can take the Hamiltonian:

$$H(z, p, \phi, \pi) = -|p| + \mathcal{M}(z, \hat{p}, \phi, \pi). \tag{2.4.6}$$

Note that this is equivalent to taking $H = p^2 - \mathcal{M}^2$ as in [226], up to a redefinition of the Lagrange multiplier e.

In the context of electrodynamics it is possible to add the minimal coupling through the dynamical mass function but then the conjugate momentum of the body is not gauge invariant. Instead, by taking p_μ to be the kinetic momentum (the conjugate momentum plus QA_μ), we can have p_μ and consequently $\mathcal{M}$ be gauge invariant at the cost of shifting $p_\mu \dot{z}^\mu$ to $(p_\mu - QA_\mu)\dot{z}^\mu$. Thus, to couple the worldline particle to electromagnetism it is simplest to use the action:

$$S[\mathsf{e}, z, p, \phi, \pi] = \int_{-\infty}^{\infty} \left(-p_\mu \dot{z}^\mu + QA_\mu \dot{z}^\mu - \pi_a \dot{\phi}^a + \mathsf{e}\left(\sqrt{p^\mu p_\mu} - \mathcal{M}(z, \hat{p}, \phi, \pi)\right)\right) d\lambda\,, \tag{2.4.7}$$

with a gauge and reparameterization invariant Lorentz-scalar dynamical mass $\mathcal{M}$.

2.4.2 Worldline Theory with SSC

Worldline Spin Degrees of Freedom

The standard worldline formulation incorporates spin in a way reminiscent of rigid bodies in classical mechanics. For a moving body, there is some point defined as the "center" of that body, tracked by the worldline, which moves in spacetime and we assume that other points of the body move along with that center in "quasirigid" motion, as defined in Ref. [252], requiring that the internal structure is essentially unchanged.[17] The orientation of the body is tracked by a tetrad $e^{\mu}{}_{A}(\lambda)$ that represents the change of internal body displacements undergone during the motion with respect to some arbitrary default frame. Capital Latin indices are used for the body's internal Lorentz indices while lowercase Greek indices are used as spacetime indices. As usual, the tetrad satisfies:

$$e^{\mu}{}_{A} e^{\nu}{}_{B} \eta^{AB} = g^{\mu\nu}, \qquad g_{\mu\nu} e^{\mu}{}_{A} e^{\nu}{}_{B} = \eta_{AB}\,. \tag{2.4.8}$$

Internal body displacements are defined in the body's center of momentum frame, so that $\hat{p}^{\mu}$ is instantaneously taken as the time direction. Thus by definition we take:

$$e^{\mu}{}_{0} = \hat{p}^{\mu}\,. \tag{2.4.9}$$

Beyond this condition $e^{\mu}{}_{A}$ may be any tetrad satisfying Eq. (2.4.8). Any such tetrad can be decomposed into (1) a tetrad which is parallel transported along the worldine, then boosted by a standard boost so that its timelike element is boosted to $\hat{p}^{\mu}$, and (2) an arbitrary little-group element of $\hat{p}^{\mu}$. The three little-group parameters of $\hat{p}^{\mu}$ can then be taken to be the ϕ^{a} coordinates. The spin angular momentum of the body is the generator of Lorentz transformations of the body orientation about the body center, and so is given by:

$$\mathsf{S}_{\mu\nu} = -\pi_{a} \left.\frac{d\phi^{a}}{d\theta^{\mu\nu}}\right|_{\theta=0}, \tag{2.4.10}$$

with Lorentz transformation parameters $\theta^{\mu\nu}$. A short computation with the above definitions reveals that they enforce the covariant SSC:

$$\mathsf{S}_{\mu\nu} p^{\nu} = 0\,. \tag{2.4.11}$$

In addition:

$$-\frac{1}{2}\mathsf{S}_{\mu\nu}\Omega^{\mu\nu} = \pi_{a}\dot{\phi}^{a}\,, \tag{2.4.12}$$

where the angular velocity tensor $\Omega^{\mu\nu}$ is defined by:

$$\Omega^{\mu\nu} = \eta^{AB} e^{\mu}{}_{A} \frac{De^{\nu}{}_{B}}{D\lambda}\,. \tag{2.4.13}$$

Using the spin tensor and the arbitrariness of the default frame of the body, the action for a spinning body takes the form:

$$\begin{aligned} S[\mathsf{e}, \xi, \chi, z, p, e, \mathsf{S}] = \int_{-\infty}^{\infty} \Big(&-(p_{\mu} - QA_{\mu})\dot{z}^{\mu} + \frac{1}{2}\mathsf{S}_{\mu\nu}\Omega^{\mu\nu} + \mathsf{e}(|p| - \mathcal{M}(z, \hat{p}, \mathsf{S})) \\ &+ \xi_{\mu}\mathsf{S}^{\mu\nu}p_{\nu} + \chi_{\mu}(e^{\mu}{}_{0} - \hat{p}^{\mu})\Big)\, d\lambda\,. \end{aligned} \tag{2.4.14}$$

Lagrange multipliers ξ_{μ} and χ_{μ} enforce $\mathsf{S}^{\mu\nu}\hat{p}_{\nu} = 0$ and $e^{\mu}{}_{0} = \hat{p}^{\mu}$. This formulation of the action imposes the covariant SSC $\mathsf{S}_{\mu\nu}p^{\nu} = 0$, corresponding to the WL1 theory.

[17] More precisely, quasirigidity is the requirement that the multipole moments of the body's current density and stress tensor evolve only by translating along the worldline and Lorentz transforming according to the orientation tracking tetrad.

One can shift the definition of the worldline z^μ and in doing so one finds that the definition of the spin changes as does the constraint satisfied by the spin. Thus, one can change to a new SSC through a shift of the worldline. In this formalism, the ability to locally shift the definition of the worldline in this way may be thought of as a gauge transformation [229, 145, 230] and the Lagrange multipliers supplied to enforce the covariant SSC and ${e^\mu}_0 = \hat{p}^\mu$ correspond to a gauge fixing. Because the SSC removes the S_{0a} components of the spin tensor in the body's center of momentum frame, these timelike components are not physical degrees of freedom. (Even when an SSC other than the covariant SSC is considered, these timelike components are determined by the other degrees of freedom using the appropriate SSC.)

Equations of Motion with SSC

The variation of Eq. (2.4.14) in Minkowski space gives an electromagnetic version of the Mathisson–Papapetrou–Dixon (MPD) [253, 107, 108] equations:

$$\begin{aligned}
\dot{z}^\mu &= \hat{p}^\mu - \frac{1}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial\hat{p}_\mu} + \frac{\hat{p}_\sigma}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial\hat{p}_\sigma}\hat{p}^\mu + \mathsf{S}^{\mu\nu}\frac{\frac{\partial\mathcal{M}}{\partial z^\nu} - 2\frac{\partial\mathcal{M}}{\partial\mathsf{S}^{\nu\rho}}p^\rho - QF_{\nu\rho}\left(\hat{p}^\rho - \frac{1}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial\hat{p}_\rho} + \frac{\hat{p}_\sigma}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial\hat{p}_\sigma}\hat{p}^\rho\right)}{\mathcal{M}^2 - \frac{Q}{2}\mathsf{S}^{\alpha\beta}F_{\alpha\beta}}\,, \\
\dot{p}_\mu &= -QF_{\mu\nu}\dot{z}^\nu + \frac{\partial\mathcal{M}}{\partial z^\mu}\,, \\
\dot{\mathsf{S}}_{\mu\nu} &= p_\mu\dot{z}_\nu - p_\nu\dot{z}_\mu - 2{\mathsf{S}_\mu}^\rho\frac{\partial\mathcal{M}}{\partial\mathsf{S}^{\rho\nu}} + 2{\mathsf{S}_\nu}^\rho\frac{\partial\mathcal{M}}{\partial\mathsf{S}^{\rho\mu}} - \frac{\partial\mathcal{M}}{\partial\hat{p}^\mu}\hat{p}_\nu + \frac{\partial\mathcal{M}}{\partial\hat{p}^\nu}\hat{p}_\mu\,.
\end{aligned} \tag{2.4.15}$$

In varying to find these equations of motion we find that the equations of motion are consistent with simply taking $\chi_\mu = 0$ and so if $\hat{p}^\mu = {e^\mu}_0$ is imposed as an initial condition then never adding the χ_μ term to the action still preserves this condition for later times.

At linear order in spin the generic symmetry consistent dynamical mass function is:

$$\mathcal{M} = m - \frac{QC_1}{2m}\mathsf{S}^{\mu\nu}F_{\mu\nu}\,, \tag{2.4.16}$$

for constant free mass m and Wilson coefficient C_1. With this form of the dynamical mass function, the equations of motion to linear in spin order are:

$$\begin{aligned}
\dot{z}^\mu &= \left(1 + \frac{QC_1}{2m^2}\mathsf{S}^{\alpha\beta}F_{\alpha\beta}\right)\frac{p^\mu}{m} + \frac{Q(C_1-1)}{m^3}\mathsf{S}^{\mu\nu}F_{\nu\rho}p^\rho + \mathcal{O}(\mathsf{S}^2)\,, \\
\dot{p}_\mu &= -QF_{\mu\nu}\dot{z}^\nu - \frac{QC_1}{2m}\mathsf{S}^{\rho\sigma}\partial_\mu F_{\rho\sigma} + \mathcal{O}(\mathsf{S}^2)\,, \\
\dot{\mathsf{S}}_{\mu\nu} &= p_\mu\dot{z}_\nu - p_\nu\dot{z}_\mu + \frac{QC_1}{m}\left(\mathsf{S}_{\mu\rho}{F^\rho}_\nu - \mathsf{S}_{\nu\rho}{F^\rho}_\mu\right) + \mathcal{O}(\mathsf{S}^2)\,.
\end{aligned} \tag{2.4.17}$$

These linear in spin equations of motion depend only on a single Wilson coefficient following from the fact that with the SSC imposed, the only independent linear in spin operator is the one in equation (2.4.16). This is similar to the situation in general relativity where the SSC allows only a single independent Wilson coefficient at the linear in spin level [145]. The appearance of two Wilson coefficients in the field theory (cf. Eqs. (2.2.57), (2.3.8) and (2.3.9)) and one coefficient in the worldline with the SSC imposed is the analog of the similar appearance of a different number of Wilson coefficients in general relativity between the field-theory and worldline descriptions starting at the spin-squared level in the action [101].

2.4.3 Worldline Theory with no SSC

In Sect. 2.4.2 we reviewed that an SSC (and particularly the covariant SSC) is natural for the worldline formalism for quasirigid bodies. Here we consider a modified version of the worldline formalism in which we "remove" the SSC. This corresponds to our worldline theory WL2. It explicitly introduces additional physical degrees of freedom into the theory. Remarkably we find that this modified worldline theory cleanly matches the field-theory results of FT1g at 2PL $\mathcal{O}(\mathsf{S}^1)$, including its extra independent Wilson coefficients. This then allows us to interpret the appearance of extra Wilson coefficients purely on the worldline, tying them to additional dynamical degrees of freedom. A similar construction was described in Ref. [231]. We find that these extra degrees of freedom allow for the magnitude of the spin vector to change.

Removing the SSC

Consider the worldline action,

$$S[\mathsf{e}, \xi, \chi, z, p, e, \mathsf{S}] = \int_{-\infty}^{\infty} \left(-(p_\mu - QA_\mu)\dot{z}^\mu + \frac{1}{2}\mathsf{S}_{\mu\nu}\Omega^{\mu\nu} + \mathsf{e}(|p| - \mathcal{M}(z, \hat{p}, \mathsf{S})) \right) d\lambda \,, \tag{2.4.18}$$

which is identical to equation (2.4.14), except that the Lagrange multiplier terms that enforce the SSC are dropped. By not including these, the interdependence between the definition of the body center degrees of freedom (z, p) and the body orientation degrees of freedom (e, S) is removed. As already noted, in equation (2.4.14) the SSC implies that the S_{0a} components of the spin tensor are not independent physical degrees of freedom. In contrast, in equation (2.4.18) with no SSC imposed we are explicitly promoting these timelike components to be treated as physical. As we shall see, this does not lead to inconsistencies in the equations of motion, but instead adds dynamical degrees of freedom.

The variation of Eq. (2.4.18) with no SSC imposed results in equations of motion,

$$\begin{aligned}
\dot{z}^\mu &= \hat{p}^\mu - \frac{1}{\mathcal{M}}\frac{\partial \mathcal{M}}{\partial \hat{p}_\mu} + \frac{\hat{p}_\sigma}{\mathcal{M}}\frac{\partial \mathcal{M}}{\partial \hat{p}_\sigma}\hat{p}^\mu \,, \\
\dot{p}_\mu &= -qF_{\mu\nu}\dot{z}^\nu + \frac{\partial \mathcal{M}}{\partial z^\mu} \,, \\
\dot{\mathsf{S}}_{\mu\nu} &= p_\mu \dot{z}_\nu - p_\nu \dot{z}_\mu - 2\mathsf{S}_\mu{}^\rho \frac{\partial \mathcal{M}}{\partial \mathsf{S}^{\rho\nu}} + 2\mathsf{S}_\nu{}^\rho \frac{\partial \mathcal{M}}{\partial \mathsf{S}^{\rho\mu}} - \frac{\partial \mathcal{M}}{\partial \hat{p}^\mu}\hat{p}_\nu + \frac{\partial \mathcal{M}}{\partial \hat{p}^\nu}\hat{p}_\mu \,.
\end{aligned} \tag{2.4.19}$$

Comparing to Eq. (2.4.15) we see that only the equation of motion for the worldline trajectory z^μ differs from the case with the SSC imposed. Moreover, the absence of the SSC Lagrange multiplier term results in this equation being simpler.

At linear order in spin the generic symmetry consistent dynamical mass function is:

$$\mathcal{M} = m - \frac{QC_1}{2m}\mathsf{S}^{\mu\nu}F_{\mu\nu} - \frac{QD_1}{m}\hat{p}_\mu \mathsf{S}^{\mu\nu}F_{\nu\rho}\hat{p}^\rho\,, \tag{2.4.20}$$

for constants m, C_1, D_1. In this case, instead of the single Wilson coefficient C_1 we have the additional coefficient D_1 analogous to the appearance of a second coefficient in the field theory FT2. To give a physical meaning to D_1 it is useful to define the spin vector S^μ and mass moment vector K^μ:

$$S^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu \mathsf{S}_{\rho\sigma}\,, \qquad\qquad K^\mu = -\mathsf{S}^{\mu\nu}\hat{p}_\nu\,, \tag{2.4.21}$$

where $\epsilon_{0123} = +1$. The boost vector K^μ is precisely what is eliminated when the covariant SSC is imposed, or equivalently what is algebraically constrained when a different SSC is used. The complete information in the spin tensor is recovered from these two vectors by:

$$\mathsf{S}^{\mu\nu} = \hat{p}^\mu K^\nu - K^\mu \hat{p}^\nu + \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma\,. \tag{2.4.22}$$

Directly, K^μ is the generator of "intrinsic" Lorentz boosts (where by "intrinsic" we mean acting only on the internal degrees of freedom). As well, $-\frac{K^\mu}{|p|}$ can be interpreted as the displacement between the actual worldline $z^\mu(\lambda)$ being used and the worldline $z^\mu_{\mathrm{COM}}(\lambda)$ that would trace out the center of mass of the body. To see this, look at the total angular momentum $J^{\mu\nu}$:

$$J^{\mu\nu} = z^\mu p^\nu - p^\mu z^\nu + \mathsf{S}^{\mu\nu} = \left(z^\mu - \frac{K^\mu}{|p|}\right)p^\nu - p^\mu\left(z^\nu - \frac{K^\nu}{|p|}\right) + \epsilon^{\mu\nu\rho\sigma}\hat{p}_\rho S_\sigma\,. \tag{2.4.23}$$

We can see that if the definition of the worldline is shifted by $-\frac{K^\mu}{|p|}$ to a new worldline $z'^\mu = z^\mu - \frac{K^\mu}{|p|}$ then the resulting new spin tensor $\mathsf{S}'^{\mu\nu}$ would satisfy the covariant SSC. In the conventional worldline formalism this is considered as an allowed redefinition which should lead to a physically equivalent theory (in that whether the spin and coupling expansions are performed about one or the other should not affect observables). Here we do not require it to be so.

In the previous discussion, the bodies were treated as point-like. It is useful to remind ourselves of the meaning $\mathbf{K}$ in the context of classical extended bodies. A familiar analysis of the spin vector can allow further insight into the meaning of K^μ. Let $\mathbf{J}$ be the generator of rotations about the origin (not body centered) acting on a matter distribution with energy density $\mathcal{E}(\mathbf{x})$ and linear momentum density $\wp(\mathbf{x})$ in a region V of space,

$$\mathbf{J} = \int_V \mathbf{x}\times\wp(\mathbf{x})d^3\mathbf{x}. \tag{2.4.24}$$

When the center of the body is identified with the point $\mathbf{z}$, the orbital generator of Lorentz boosts is of course,

$$\mathbf{L} = \mathbf{z}\times\mathbf{p}. \tag{2.4.25}$$

Thus, the "intrinsic" generator of rotations (the spin) of the body is given by a familiar formula,

$$\mathbf{S} = \mathbf{J} - \mathbf{L} = \int_V (\mathbf{x} - \mathbf{z}) \times \wp(\mathbf{x}) d^3\mathbf{x}. \tag{2.4.26}$$

Now performing the same analysis for the generator of Lorentz boosts, let $\mathbf{K}_{\text{total}}$ be the generator of Lorentz boosts about the origin acting on the matter distribution,

$$\mathbf{K}_{\text{total}} = \int_V (t\wp(\mathbf{x}) - \mathbf{x}\mathcal{E}(\mathbf{x}))d^3\mathbf{x}. \tag{2.4.27}$$

The "orbital" generator of Lorentz boosts is then,

$$\mathbf{K}_{\text{orbital}} = \mathbf{p}t - \mathrm{E}\mathbf{z} \tag{2.4.28}$$

where E and $\mathbf{p}$ are the total energy and momentum of the body. Thus, the "intrinsic" generator of Lorentz boosts of the body is:

$$\mathbf{K} = \mathbf{K}_{\text{total}} - \mathbf{K}_{\text{orbital}} = \mathrm{E}\mathbf{z} - \int_V \mathbf{x}\mathcal{E}(\mathbf{x})d^3\mathbf{x} \tag{2.4.29}$$

Let $\mathbf{z}_{\text{COM}}$ be the center of momentum position of the body in the center of momentum frame ($\mathrm{E} = |p|$). Then, automatically:

$$\mathbf{z}_{\text{COM}} = \frac{1}{\mathrm{E}} \int_V \mathbf{x}\mathcal{E}(\mathbf{x})d^3\mathbf{x} \implies \mathbf{z}_{\text{COM}} = \mathbf{z} - \frac{\mathbf{K}}{|p|}. \tag{2.4.30}$$

This precisely establishes the interpretation of $-\frac{K^\mu}{|p|}$ as a displacement between the worldline around which the spin and coupling expansions are performed and the worldline which tracks the center of mass of the body.

Note that we use a different convention in this section compared to section 2.2. In particular, the worldline $\mathbf{K}$ and the field-theory $\boldsymbol{K}$ are related by an analytic continuation, $i\boldsymbol{K} \mapsto \mathbf{K}$ with both $\boldsymbol{K}$ and $\mathbf{K}$ being real, while the rest-frame spin vectors are simply equal, $\boldsymbol{S} \leftrightarrow \mathbf{S}$. We comment further in section 2.5.8 on the rationale behind this analytic continuation.

Writing K^μ as a spatial integral moment of the energy-momentum tensor as above identifies it as a mass dipole moment of the body about the worldine position z^μ. This identification can be made directly from Dixon's formalism [253]. For a body with a charge density proportional to its mass density then $-\frac{Q}{|p|}K^\mu$ would be the electric dipole moment of the body. However, for a generic object it is not necessarily the case that these densities are proportional and so we need not assume that the electric dipole moment is $-\frac{Q}{|p|}K^\mu$. In particular, in the body's center of momentum frame its energy is simply its dynamical mass function minus QA_0 and in that frame (2.4.20) becomes:

$$\mathcal{M} = m + \frac{Q(C_1 - D_1)}{m}\mathbf{E} \cdot \mathbf{K} - \frac{QC_1}{m}\mathbf{B} \cdot \mathbf{S} + \mathcal{O}(F^2)\,. \tag{2.4.31}$$

Thus the induced electric dipole moment $\mathbf{d}$ and magnetic dipole moment $\boldsymbol{\mu}$ relative to the worldline center $\mathbf{z}$:

$$\mathbf{d} = -\frac{Q(C_1 - D_1)}{m}\mathbf{K}, \qquad \boldsymbol{\mu} = \frac{QC_1}{m}\mathbf{S}\,. \tag{2.4.32}$$

Immediately, $2C_1$ is the gyromagnetic ratio of the body (which should take the value 1 for a classical distribution of mass and charge which are proportional). For a distribution in which mass and charge are proportional, $C_1 - D_1 = 1$. Here we consider the possibility that it takes a generic value different from 1. The value of $C_1 - D_1 = 1$ is explicitly required by a worldline formalism which is assumed to have worldline shift symmetry [229, 145, 230] because the definition of the electric dipole moment immediately implies a shift of the dipole moment by $-\frac{Q}{|p|}K^\mu$ whenever the worldline is shifted by $-\frac{K^\mu}{|p|}$. Thus, $C_1 - D_1 \neq 1$ breaks the worldline shift symmetry.

Of course, to have a proper description of extended bodies that fits into the WL2 framework one should understand the constraints on the energy and momentum distributions arising from the Lorentz algebra. It would also be very interesting to directly connect extended objects with appropriate distributions of energy and momentum to the extra Wilson coefficient of WL2.

Equations of Motion with no SSC

With the dynamical mass function (2.4.20) we find equations of motion in WL2 to linear order in spin:

$$\begin{aligned}
\dot{z}^\mu =& \frac{p^\mu}{m}\left(1 + \frac{QC_1}{2m^2}\mathsf{S}^{\rho\sigma}F_{\rho\sigma} - \frac{QD_1}{m^4}p_\nu \mathsf{S}^{\nu\rho}F_{\rho\sigma}p^\sigma\right) + \frac{QD_1}{m^3}p^\rho \mathsf{S}_{\rho\sigma}F^{\sigma\mu} + \frac{QD_1}{m^3}\mathsf{S}^{\mu\nu}F_{\nu\rho}p^\rho + \mathcal{O}(\mathsf{S}^2)\,, \\
\dot{p}_\mu =& -QF_{\mu\nu}\dot{z}^\nu - \frac{QC_1}{2m}\mathsf{S}^{\rho\sigma}\partial_\mu F_{\rho\sigma} - \frac{QD_1}{m^3}p_\rho \mathsf{S}^{\rho\sigma}\partial_\mu F_{\sigma\alpha}p^\alpha + \mathcal{O}(\mathsf{S}^2) \\
\dot{\mathsf{S}}^{\mu\nu} =& -\frac{QC_1}{m}\left(F^\mu{}_\rho \mathsf{S}^{\rho\nu} - F^\nu{}_\rho \mathsf{S}^{\rho\mu}\right) - \frac{QD_1}{m^3}\left(F^{\mu\rho}p_\rho \mathsf{S}^{\nu\sigma}p_\sigma - F^{\nu\rho}p_\rho \mathsf{S}^{\mu\sigma}p_\sigma\right) \\
&+ \frac{QD_1}{m^3}\left(p^\mu \mathsf{S}^{\nu\rho}F_{\rho\sigma}p^\sigma - p^\nu \mathsf{S}^{\mu\rho}F_{\rho\sigma}p^\sigma\right) + \mathcal{O}(\mathsf{S}^2).
\end{aligned} \tag{2.4.33}$$

If one begins the time evolution with initial conditions satisfying the covariant SSC and $C_1 - D_1 = 1$, the covariant SSC is preserved dynamically. $C_1 - D_1 \neq 1$ produces violations of the covariant SSC. In light of this, notice that if $C_1 - D_1$ is set to 1 in (2.4.33) and covariant SSC satisfying initial conditions are chosen, then the equations of motion (2.4.33) reduce to the equations of motion (2.4.17). Consequently, this modified worldline formalism is strictly more general than the conventional WL1 as it contains the WL1 as a special case when appropriate initial conditions and Wilson coefficient values are selected. In order to "turn on" the SSC and reduce to the conventional worldline formalism we can set D_1 to the special value $D_1 = C_1 - 1$ at any stage of calculation and use initial conditions satisfying the covariant SSC.

Worldline Compton Amplitude

Using the WL2 equations of motion we compute the classical Compton amplitude to order $\mathcal{O}(\alpha\mathsf{S}^2)$ for general values of C_1, D_1 The classical Compton is computed by computing the coefficient of the outgoing spherical electromagnetic wave produced by the response of the spinning body to in an incoming electromagnetic plane wave as in Appendix D of [5] or as is done for gravity in [247]. In particular, we consider an incoming plane wave vector potential in Lorenz gauge,

$$A^{\rm in}_\mu(X) = e^{ik\cdot x}\xi_\mu \tag{2.4.34}$$

and the response of a spinning particle to this potential using the equations of motion of WL2. The $\mathcal{O}(\alpha)$ perturbative solutions can be returned to the current,

$$\begin{aligned} J^\mu(X) &= \frac{\delta S}{\delta A_\mu} \\ &= \int_{-\infty}^{\infty}\left(Q\dot{z}^\mu\delta(X-Z) + \frac{\mathsf{e}Q}{m}\left(C_1\mathsf{S}^{\mu\nu} + D_1\left(\hat{p}^\mu\mathsf{S}^{\mu\rho}\hat{p}_\rho - \mathsf{S}^{\mu\rho}\hat{p}_\rho\hat{p}^\nu\right)\right)\partial_\nu\delta(X-Z)\right)d\lambda. \end{aligned} \tag{2.4.35}$$

Then, treating that current as a source we compute the perturbation of the vector potential. The large distance behavior of the perturbed vector potential allows one to read off the Compton amplitude $\mathcal{A}^{\mu\nu}$ by:

$$A^\mu(X) = e^{ik\cdot x}\xi^\mu + \frac{e^{ikr-i\omega t}}{4\pi r}\mathcal{A}^{\nu\mu}\xi_\nu + O(\frac{1}{r^2}). \tag{2.4.36}$$

The Compton amplitude can then be extracted directly from the current by using the Lorenz gauge solution to the wave equation at large distances. Doing so one finds:

$$\widetilde{J}^\mu = 2\pi\mathcal{A}^{\nu\mu}\xi_\nu \tag{2.4.37}$$

where $\widetilde{J}^\mu$ is the Fourier transform of the current evaluated at the outgoing photon momentum.

Using the current computed from the worldline equations of motion, the resulting classical Compton amplitude is found to fully agree with the $\mathcal{A}^{\rm FT1g}$ Compton amplitude in (2.3.7), (2.3.9), (2.3.20) (with the S^2 terms matching up to contact terms, which we did not explicitly include either on the field theory or in the worldline theory).

Worldline Impulses

For computing observables with these equations of motion we consider the probe limit of a spinning particle of mass m scattering off of a stationary scalar source. For simplicity, we consider only the probe limit; even so, the result is sufficiently complex to demonstrate a rather nontrivial comparison with the field-theory calculations. The source – a point charge moving with four-velocity u_2 – has vector potential,

$$A_\mu(x) = \frac{Qu_{2\mu}}{4\pi\sqrt{(x\cdot u_2)^2 - x\cdot x}}\,. \tag{2.4.38}$$

The solutions to the equations of motion of the probe in powers of $\alpha = Q^2/(4\pi)$ are of the form:

$$z^\mu(\lambda) = b^\mu + u_1^\mu\lambda + \alpha\delta z^\mu_{(1)}(\lambda) + \alpha^2\delta z^\mu_{(2)}(\lambda) + \mathcal{O}(\alpha^3) \tag{2.4.39}$$

$$p^\mu(\lambda) = mu_1^\mu + \alpha\delta p^\mu_{(1)}(\lambda) + \alpha^2\delta p^\mu_{(2)}(\lambda) + \mathcal{O}(\alpha^3) \tag{2.4.40}$$

$$\mathsf{S}^{\mu\nu}(\lambda) = \mathsf{S}_1^{\mu\nu} + \alpha\delta\mathsf{S}^{\mu\nu}_{(1)}(\lambda) + \alpha^2\delta\mathsf{S}^{\mu\nu}_{(2)} + \mathcal{O}(\alpha^3). \tag{2.4.41}$$

The impact parameter b^μ is defined to be transverse on the initial momentum, $b \cdot p_1 = 0$. The initial momentum mu_1^μ defines the initial four-velocity u_1^μ. All perturbations of p^μ and $\mathsf{S}^{\mu\nu}$ asymptotically vanish for $\lambda \to \pm\infty$ while the trajectory perturbations are logarithmically divergent with the worldline time due to the long range nature of the Coulomb potential. Due to this logarithmic divergence, in order to treat the $\mathcal{O}(\alpha^2)$ and higher solutions correctly, all the perturbations may be set to 0 at an initial cutoff time $\lambda = -T$. Impulse observables are then computed by taking the difference in observables at time T and $-T$ and at the end taking the limit $T \to \infty$. Equivalently, the perturbations may be given representations in terms of standard Feynman integrals and computed using dimensional regularization, such as in Ref. [26].

Computing the momentum impulse and spin kick to $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\mathsf{S}^1)$ in this way gives a perfect match to the corresponding observables obtained from $\mathcal{A}^{\mathrm{FT1g}}$ when the worldline Wilson coefficients C_1 and D_1 are identified with their field-theory counterparts, as detailed in Sec. 2.5.7 below. The results of WL1 can be recovered from the more general results of WL2 by setting the special value $D_1 = C_1 - 1$. To express the impulses, it is useful to define:

$$\gamma = u_1 \cdot u_2, \qquad v = \frac{\sqrt{\gamma^2 - 1}}{\gamma}, \tag{2.4.42}$$

$$\check{u}_1^\mu = u_1^\mu - \gamma u_2^\mu, \qquad \check{u}_2^\mu = u_2^\mu - \gamma u_1^\mu, \tag{2.4.43}$$

and to decompose the impulses according to:

$$\begin{aligned} \Delta p_1^\mu &= \alpha \Delta p^\mu_{1(\alpha^1)} + \alpha^2 \Delta p^\mu_{1(\alpha^2)} + \mathcal{O}(\mathsf{S}_1^2) + \mathcal{O}(\alpha^3), \\ \Delta \mathsf{S}_1^{\mu\nu} &= \alpha \Delta \mathsf{S}^{\mu\nu}_{1(\alpha^1)} + \alpha^2 \Delta \mathsf{S}^{\mu\nu}_{1(\alpha^2)} + \mathcal{O}(\mathsf{S}_1^2) + \mathcal{O}(\alpha^3). \end{aligned} \tag{2.4.44}$$

Then at order $\mathcal{O}(\alpha)$ and with the notation $|b| = \sqrt{-b^\mu b_\mu}$, we find the impulse

$$\begin{aligned} \Delta p^\mu_{1(\alpha^1)} =& \frac{2b^\mu}{v|b|^2} + \frac{2}{m_1 \gamma v |b|^2} \left(2 \frac{b^\mu b^\nu}{|b|^2} \mathsf{S}_{1\nu\rho} + \mathsf{S}_1{}^\mu{}_\rho \right) \left(D_1 \gamma u_1^\rho - C_1 u_2^\rho \right) \\ & - \frac{2 \mathsf{S}_{1\nu\rho} u_1^\nu u_2^\rho}{m_1 \gamma^3 v^3 |b|^2} \Big[(C_1 - D_1 \gamma^2) u_1^\mu + (D_1 - C_1) \gamma u_2^\mu \Big] \end{aligned} \tag{2.4.45}$$

and the spin kick

$$\Delta \mathsf{S}^{\mu\nu}_{1(\alpha^1)} = \frac{4}{m_1 \gamma v |b|^2} \left(\mathsf{S}_1{}^{[\mu}{}_\sigma \delta^{\nu]}{}_\rho b^\sigma - \mathsf{S}_1{}^{[\mu}{}_\rho b^{\nu]} \right) \left(D_1 \gamma u_1^\rho - C_1 u_2^\rho \right). \tag{2.4.46}$$

At order $\mathcal{O}(\alpha^2)$, we find the impulse,

$$\begin{aligned} \Delta p^\mu_{1(\alpha^2)} =& - \frac{\pi b^\mu}{2 m_1 \gamma v |b|^3} - \frac{2 \check{u}_1^\mu}{m_1 \gamma^2 v^4 |b|^2} \\ & + \pi \frac{3 b^\mu b^\nu \mathsf{S}_{1\nu\rho} + |b|^2 \mathsf{S}_1{}^\mu{}_\rho}{2 m_1^2 \gamma^3 v^3 |b|^5} \Big[(C_1^2 - C_1 D_1 - D_1) \check{u}_1^\rho + D_1 (C_1 - D_1 - 3) \gamma \check{u}_2^\rho \Big] \\ & + 2 \frac{C_1^2 - D_1 (2C_1 - D_1 - 2) \gamma^2}{m_1^2 \gamma^2 v^2 |b|^4} b^\nu \mathsf{S}_{1\nu\rho} \left[\eta^{\rho\mu} + \frac{\check{u}_1^\rho u_1^\mu + \check{u}_2^\rho u_2^\mu}{\gamma^2 v^2} \right] \end{aligned} \tag{2.4.47}$$

$$
\begin{aligned}
&- 4\frac{b^\nu \mathsf{S}_{1\nu\rho}}{m_1^2\gamma^6 v^6|b|^4}\Big[(D_1-C_1)\gamma^2\check{u}_2^\rho\check{u}_1^\mu + (D_1\gamma^2 - C_1)\check{u}_1^\rho\check{u}_2^\mu\Big] \\
&+ \pi\frac{\mathsf{S}_{1\nu\rho}\check{u}_1^\nu\check{u}_2^\rho}{2m_1^2\gamma^7v^7|b|^3}\Big\{\big[(C_1-D_1)^2+2D_1\big]\gamma\check{u}_1^\mu \\
&\qquad + \big[C_1^2 - C_1D_1 - D_1 - D_1(C_1-D_1-3)\gamma\big]\check{u}_2^\mu\Big\}
\end{aligned}
$$

and the spin kick,

$$
\begin{aligned}
\Delta\mathsf{S}^{\mu\nu}_{1(\alpha^2)} = {}& \frac{\pi}{m_1^2\gamma^3v^3|b|^3}\Big[(C_1^2 - C_1D_1 - D_1)\Big(b^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}\check{u}_1^\rho - \check{u}_1^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}b^\rho\Big) \\
&+ D_1(C_1-D_1-3)\gamma\Big(b^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}\check{u}_2^\rho - \check{u}_2^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}b^\rho\Big)\Big] \\
&+ \frac{4}{m_1^2\gamma^2v^2|b|^4}\Big\{2\Big(C_1 b^{[\mu}u_2^{\nu]} - D_1\gamma b^{[\mu}u_1^{\nu]}\Big)b^\rho\mathsf{S}_{1\rho\sigma}(C_1u_2^\sigma - D_1\gamma u_1^\sigma) \\
&- \big[C_1^2 - D_1(2C_1-D_1)\gamma^2\big]b^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}b^\rho\Big\} \\
&+ \frac{4}{m_1^2\gamma^6v^6|b|^2}\Big\{(C_1-D_1)^2\gamma^2\check{u}_1^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}\check{u}_1^\rho + (C_1-D_1\gamma^2)^2\check{u}_2^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}\check{u}_2^\rho \\
&+ (C_1-D_1\gamma^2)\gamma\Big[(C_1-D_1-1)\check{u}_2^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}\check{u}_1^\rho + (C_1-D_1+1)\check{u}_1^{[\mu}\mathsf{S}_{1\rho}{}^{\nu]}\check{u}_2^\rho\Big]\Big\}.
\end{aligned}
\tag{2.4.48}
$$

The nontrivial nature of the above results give us confidence that we have indeed identified a worldline model whose results match those of the field theory. It would of course be useful to carry out further comparisons to field theory, not only beyond the probe limit but also more importantly to higher orders in the spin, especially for the case of general relativity. Given the rather different setups, a direct proof that the field-theory and worldline descriptions will always yield equivalent results appears nontrivial.

2.5 Effective Hamiltonian Including Lower-Spin States

Refs. [59, 44, 61, 101] extend the spinless Hamiltonian of Ref. [14] to the case of spinning bodies. This corresponds to the two-body effective description EFT1, which is composed of the collection of all independent operators containing up to a given power of spin, each with arbitrary coefficients determined by matching to either field-theory or worldline results. Here we explicitly consider operators up to linear in spin. We also construct a second EFT Hamiltonian, referred to as EFT2, extending the degrees of freedom of $S^{\mu\nu}$ to include the intrinsic boost, K^μ. Interpreting the Hamiltonians as quantum operators allows us to obtain scattering amplitudes, which we then match to the quantum-field-theory amplitudes found in section 2.3.3. This determines the coefficients in the Hamiltonians. A suitable expectation value of the Hamiltonian operators are then reinterpreted as classical Hamiltonians. The corresponding equations of motion can be solved to give the impulse, spin and boost kick along a scattering trajectory, which we then compare to the corresponding observables obtained from the worldlines WL1 and WL2, described in section 2.4. We find that the extra Wilson coefficients that appear in WL2, FT1g and FT3g are naturally accounted for in EFT2. Finally, we find a compact eikonal formula [67, 254, 255, 70], extending the spin results of Ref. [59] to account for the appearance of the intrinsic boost operator, that matches the results obtained from the equations of motion and worldline. Eikonal representations are automatically compact because they encode the physical information in a single scalar function.

2.5.1 Hamiltonian 1: Solely Spinning Degrees of Freedom

We consider an effective description of the binary containing only spin degrees of freedom, which leads to equations of motion that preserve the magnitude of the spin vector. We refer to this effective description as EFT1. This is the same treatment as the one of Refs. [59, 61, 101] except that here we consider electrodynamics instead of general relativity. We briefly describe this Hamiltonian and then proceed with a more extensive description of a modified Hamiltonian which contains a boost operator and allows for spin-magnitude change.

EFT1 contains the usual spin-vector degrees of freedom, along with the usual commutation or Poisson-bracket relations for spin. In terms of the quantum-mechanical states that describe the bodies, this construction implies that we may take them to belong to a single irreducible representation of the rotation group. In particular, we choose the asymptotic scattering states to be spin coherent states [245], which are labeled by an integer s and a direction given by a unit vector $\hat{\boldsymbol{n}}$ [59], as in the field-theory discussion in Sec. 2.2. To build the most generic Hamiltonian that accommodates these spin degrees of freedom, we need only consider the spin operator $\hat{\mathbf{S}}$. This is in accordance with the classical description of these particles, where one describes such a spinning object in terms of the spin three-vector $\mathbf{S}$.[18]

For simplicity, here we limit the discussion to a Hamiltonian for one scalar and one spinning particle valid to linear order in spin. This center-of-mass (CoM) Hamiltonian is given by (see Ref. [59] for the corresponding one in general relativity),

$$\mathcal{H}_1 = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V^{(0)}(\boldsymbol{r}^2, \boldsymbol{p}^2) + V^{(1)}(\boldsymbol{r}^2, \boldsymbol{p}^2) \frac{\boldsymbol{L} \cdot \hat{\mathbf{S}}_1}{\boldsymbol{r}^2} \,, \tag{2.5.1}$$

where the potentials are

$$V^{(a)}(\boldsymbol{r}^2, \boldsymbol{p}^2) = \frac{\alpha}{|\boldsymbol{r}|} c_1^{(a)}(\boldsymbol{p}^2) + \left(\frac{\alpha}{|\boldsymbol{r}|}\right)^2 c_2^{(a)}(\boldsymbol{p}^2) + \mathcal{O}(\alpha^3) \,, \tag{2.5.2}$$

and we have taken the particle 1 to carry spin $\mathbf{S}_1$, with the binary system carrying angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. For these operators we have the commutation relations,

$$[\hat{S}_{1,i}, \hat{S}_{1,j}] = i\epsilon_{ijk}\hat{S}_{1,k} \,, \qquad [r_i, \hat{S}_{1,j}] = [p_i, \hat{S}_{1,j}] = 0 \,, \qquad [r_i, p_j] = i\delta_{ij} \,. \tag{2.5.3}$$

For any operator that is a function solely of $\boldsymbol{r}$, $\boldsymbol{p}$, $\hat{\mathbf{S}}_1$ the spin magnitude is preserved, since all such operators

[18] For compactness, for $\boldsymbol{r}$ and $\boldsymbol{p}$ we do not distinguish a quantum operator from the corresponding classical value by using a different symbol, as in Refs. [59, 61].

commute with the spin Casimir, i.e.,

$$[\hat{\mathbf{S}}_1^2, \mathcal{O}] = 0\,, \qquad\qquad \mathcal{O} = \{\boldsymbol{r},\, \boldsymbol{p},\, \hat{\mathbf{S}}_1\}\,. \tag{2.5.4}$$

Similarly, at the level of the classical equations of motion the above implies that the spin magnitude is a conserved quantity. Further details may be found in Ref. [59].

2.5.2 Hamiltonian 2: Inclusion of Boost Operator

In this subsection we expand the degrees of freedom so that we are able to properly describe the field theories and worldline theories that also contain additional degrees of freedom, allowing the spin magnitudes to vary by their interaction with the electromagnetic field. To this end, inspired by the worldline construction in section 2.4.3, we extend the above Hamiltonian to include the generator of intrinsic boosts $\hat{\mathbf{K}}$. We start by motivating this choice. We proceed to describe how does one build the most general two-body Hamiltonian out of the available operators for the problem at hand. Finally, as an explicity illustration, we build the Hamiltonian linear in the spin and boost of one of the particles.

In order to have a Hamiltonian whose amplitudes match those of FT1g and FT3 we are prompted to consider additional operators. The natural choice is operators built out of the vector $\mathbf{K}_1$, already encountered in Eq. (2.4.23). The operator $\hat{\mathbf{K}}_1$ should act on the intrinsic degrees of freedom of the body, hence it commutes with both $\boldsymbol{r}$ and $\boldsymbol{p}$. Accordingly, the commutation relations are,

$$[\hat{S}_{1,i}, \hat{K}_{1,j}] = i\epsilon_{ijk}\hat{K}_{1,k}\,, \qquad\qquad [r_i, \hat{K}_{1,j}] = [p_i, \hat{K}_{1,j}] = 0\,. \tag{2.5.5}$$

where the first relation simply implies that $\hat{\mathbf{K}}_1$ is a vector operator. To fully characterize the operator $\hat{\mathbf{K}}_1$ we need to specify the commutation relations with itself. Motivated by the connection to the worldline we take:

$$[\hat{K}_{1,i}, \hat{K}_{1,j}] = -i\epsilon_{ijk}\hat{S}_{1,k}\,, \tag{2.5.6}$$

which identifies $\hat{\mathbf{K}}_1$ with the generator of intrinsic boosts. The operator algebra is completed by the commutators familiar from the case without the $\hat{\mathbf{K}}_1$ operator given in Eq. (2.5.3).

Alternatively, the introduction of $\hat{\mathbf{K}}_1$ may be motivated by the requirement that the spin magnitude should change under time evolution via the constructed Hamiltonian. In the quantum-mechanical language, this requires an operator that does not commute with $\hat{\mathbf{S}}_1^2$. It follows that it must also not commute with $\hat{\mathbf{S}}_1$, i.e. it must have tensor structure under intrinsic rotations. The simplest object that satisfies this criterion is a vector under intrinsic rotations that commutes with both $\boldsymbol{r}$ and $\boldsymbol{p}$. This reasoning leads to the introduction of an operator obeying the commutation relations (2.5.5), while (2.5.6) still needs to be motivated by the interpretation of $\hat{\mathbf{K}}_1$ as the boost generator. We indeed find that inclusion of $\hat{\mathbf{K}}_1$ leads to scattering amplitudes between states of different spin magnitude, similar to our field-theory constructions above. Furthermore, while in these scattering amplitudes the change in spin is minute, $s \to s-1$, the effect is resummed to a finite change via Hamilton's equations, as we see in section 2.5.5.

We proceed to construct the effective Hamiltonian. The first question is to find the complete set of terms that can appear. We constrain these based on symmetry considerations: the Hamiltonian is invariant under parity and time reversal (see, however, the discussion in section 2.5.8). To take advantage of these constraints we list how our operators transform under the action of these symmetries:

$$\begin{aligned} &\text{Parity:} && P^\dagger \boldsymbol{r} P = -\boldsymbol{r}\,, \quad P^\dagger \boldsymbol{p} P = -\boldsymbol{p}\,, \quad P^\dagger \hat{\mathbf{S}}_1 P = \hat{\mathbf{S}}_1\,, \quad P^\dagger \hat{\mathbf{K}}_1 P = -\hat{\mathbf{K}}_1\,, \\ &\text{Time Reversal:} && T^\dagger \boldsymbol{r} T = \boldsymbol{r}\,, \quad T^\dagger \boldsymbol{p} T = -\boldsymbol{p}\,, \quad T^\dagger \hat{\mathbf{S}}_1 T = -\hat{\mathbf{S}}_1\,, \quad T^\dagger \hat{\mathbf{K}}_1 T = \hat{\mathbf{K}}_1\,, \end{aligned} \tag{2.5.7}$$

see e.g. Sect. 2.6 of Ref. [256]. Furthermore, we construct terms that have classical scaling. The scaling of our operators in the classical limit is

$$\boldsymbol{r} \sim \frac{1}{\lambda}\boldsymbol{r}\,, \quad \boldsymbol{p} \sim \lambda^0 \boldsymbol{p}\,, \quad \hat{\mathbf{S}}_1 \sim \frac{1}{\lambda}\hat{\mathbf{S}}_1\,, \quad \hat{\mathbf{K}}_1 \sim \frac{1}{\lambda}\hat{\mathbf{K}}_1\,, \tag{2.5.8}$$

where λ is a small parameter that characterizes the classical limit (usually associated with $\hbar$).

Two additional properties that reduce the number of operators are on-shell conditions and Schouten identities. The former capture the freedom of field redefinitions in the quantum-mechanical context or the freedom of canonical transformations in the classical context. The latter stem from the fact that we work with more than three three-dimensional vectors, hence there must be linear relations among them. While these considerations are not important for the purposes of this chapter, they can significantly reduce the number of terms one needs to consider when looking at higher orders in spin and boost (see e.g. Ref. [101]).

Using the above one may systematically construct independent terms in the Hamiltonian. At linear order in spin and boost we have:

$$\mathbb{O}_1 = \frac{\boldsymbol{L}\cdot\hat{\mathbf{S}}_1}{\boldsymbol{r}^2}\,, \qquad \mathbb{O}_2 = \frac{\boldsymbol{r}\cdot\hat{\mathbf{K}}_1}{\boldsymbol{r}^2}\,. \tag{2.5.9}$$

The Hamiltonian valid to linear order in spin and boost is then,

$$\mathcal{H}_2 = \sqrt{\boldsymbol{p}^2+m_1^2} + \sqrt{\boldsymbol{p}^2+m_2^2} + V^{(0)}(\boldsymbol{r}^2,\boldsymbol{p}^2) + V^{(1)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\frac{\boldsymbol{L}\cdot\hat{\mathbf{S}}_1}{\boldsymbol{r}^2} + V^{(2)}(\boldsymbol{r}^2,\boldsymbol{p}^2)\frac{\boldsymbol{r}\cdot\hat{\mathbf{K}}_1}{\boldsymbol{r}^2}\,, \tag{2.5.10}$$

where we used the operators in Eq. (2.5.9) and the potential coefficients given in Eq. (2.5.2). The Hamiltonian has an additional operator containing $\hat{\mathbf{K}}_1$ compared to the one in Eq. (2.5.1).

2.5.3 Amplitudes from the Effective Hamiltonian

Having identified the general form of the Hamiltonian that can capture the classical physics of our field theories with additional degrees of freedom, Eq. (2.5.10), we proceed to determine its coefficient functions $V^{(i)}$. We follow closely Refs. [59, 61, 101] where analogous calculations were carried out for Hamiltonians depending only on the spin operator. As in that case, we consider scattering of spin-coherent states [245]. These states may be superpositions of fixed-spin-magnitude states or more general superpositions that involve states of different spin magnitude, similar to the field-theory construction in Eq. (2.2.44). For our purposes it is sufficient to consider incoming and outgoing states whose spin parts are identical. However, we note that it is possible to also consider different incoming and outgoing states. Since the incoming and outgoing states are taken to be the same, the amplitudes are expressed in terms of diagonal matrix elements of $\hat{\mathbf{S}}$ and $\hat{\mathbf{K}}$. A coherent state $|\boldsymbol{s}\rangle \equiv |s, \hat{\boldsymbol{n}}\rangle$ with fixed spin magnitude s and direction $\hat{\boldsymbol{n}}$ is the state of highest weight along the direction $\hat{\boldsymbol{n}}$. Similarly with the field-theory discussion in Sec. 2.2.2, for such a state we have,

$$\langle \boldsymbol{s}|\hat{\mathbf{S}}|\boldsymbol{s}\rangle = \mathbf{S} = s\,\hat{\boldsymbol{n}}\,, \quad \text{and} \quad \langle \boldsymbol{s}|\hat{\mathbf{K}}|\boldsymbol{s}\rangle = 0\,. \tag{2.5.11}$$

We build a generalized coherent state $|\Psi\rangle$ by superimposing states $|\boldsymbol{s}\rangle$ with different values of s, such that,

$$\langle \Psi|\hat{\mathbf{S}}|\Psi\rangle = \mathbf{S}\,, \quad \text{and} \quad \langle \Psi|\hat{\mathbf{K}}|\Psi\rangle = \mathbf{K}\,, \tag{2.5.12}$$

where on the right-hand side of the above equation we have the classical values of $\hat{\mathbf{S}}$ and $\hat{\mathbf{K}}$. These classical values depend on the details of the construction of $|\Psi\rangle$, but the exact dependence is not important for our purposes. Finally, these states are built such that they obey the property

$$\langle \Psi|\{\hat{S}_{i_1}\dots\hat{S}_{i_n}\}|\Psi\rangle = S_{i_1}\dots S_{i_n}\,, \quad \text{and} \quad \langle \Psi|\{\hat{K}_{i_1}\dots\hat{K}_{i_n}\}|\Psi\rangle = K_{i_1}\dots K_{i_n}\,, \tag{2.5.13}$$

up to terms that do not contribute in the classical limit, where the $\{\}$ brackets signify symmetrization and division by the number of terms (see also the discussion in section 2.2.2).

We may now proceed to compute the EFT amplitudes. For the details of such a computation we refer the reader to Refs. [59, 61], where corresponding computations are carried out for the purely-spin case. We give here the result for the amplitude obtained from $\mathcal{H}_2$. The corresponding amplitude from $\mathcal{H}_1$ follows by setting the coefficients of any operators containing $\hat{\mathbf{K}}$ to zero. The EFT amplitude may be organized as

$$\mathbb{M} = \mathbb{M}^{1\mathrm{PL}} + \mathbb{M}^{2\mathrm{PL}} + \dots\,, \tag{2.5.14}$$

where we have explicitly written the first and second PL contributions and the ellipsis denote higher PL orders. We have

$$\mathbb{M}^{1\mathrm{PL}} = \frac{4\pi\alpha}{\boldsymbol{q}^2}\left[a_1^{(0)} + a_1^{(1)}\boldsymbol{L}_q\cdot\mathbf{S}_1 + a_1^{(2)} i\boldsymbol{q}\cdot\mathbf{K}_1\right], \tag{2.5.15}$$

and

$$\begin{aligned}\mathbb{M}^{2\mathrm{PL}} &= \mathbb{M}^{2\mathrm{PL}}_{\triangle} + (4\pi\alpha)^2\, a_{\mathrm{iter}} \int \frac{d^{D-1}\boldsymbol{\ell}}{(2\pi)^{D-1}} \frac{2\xi E}{\boldsymbol{\ell}^2(\boldsymbol{\ell}+\boldsymbol{q})^2(\boldsymbol{\ell}^2+2\boldsymbol{p}\cdot\boldsymbol{\ell})}\,,\\ \mathbb{M}^{2\mathrm{PL}}_{\triangle} &= \frac{2\pi^2\alpha^2}{|\boldsymbol{q}|}\left[a_2^{(0)} + a_2^{(1)}\boldsymbol{L}_q\cdot\mathbf{S}_1 + a_2^{(2)} i\boldsymbol{q}\cdot\mathbf{K}_1\right],\end{aligned} \tag{2.5.16}$$

where the triangle subscript in $\mathbb{M}_{\triangle}^{\mathrm{2PL}}$ indicates that the origin of the contribution is an one-loop triangle integral. Here $\boldsymbol{p}$ and $\boldsymbol{p}-\boldsymbol{q}$ are the incoming and outgoing spatial momenta of particle 1 in the CoM frame respectively and $\boldsymbol{L}_q = i\boldsymbol{p}\times\boldsymbol{q}$. We also use

$$E = E_1 + E_2\,, \quad \text{and} \quad \xi = \frac{E_1 E_2}{E^2}\,, \tag{2.5.17}$$

where $E_{1,2}$ are the energies of particles 1 and 2, which are conserved in the CoM frame (see also Eq. (2.5.21)). The vectors $\mathbf{S}_1$ and $\mathbf{K}_1$ that appear in the above two equations are the classical values of the corresponding quantum operators. They depend on whether one chooses to scatter the $|\boldsymbol{s}\rangle$ or $|\Psi\rangle$ state as shown in Eqs. (2.5.11) and (2.5.12). The 1PL amplitude coefficients take the form

$$a_1^{(0)} = -c_1^{(0)}\,, \quad a_1^{(1)} = c_1^{(1)}\,, \quad a_1^{(2)} = -c_1^{(2)}\,, \tag{2.5.18}$$

while for the 2PL amplitude we have

$$\begin{aligned}
a_2^{(0)} &= -c_2^{(0)} + 2E\xi c_1^{(0)}\mathcal{D}c_1^{(0)} + \frac{(1-3\xi)\left(c_1^{(0)}\right)^2}{2E\xi}\,,\\
a_2^{(1)} &= \frac{c_2^{(1)}}{2} - E\xi c_1^{(1)}\mathcal{D}c_1^{(0)} - E\xi c_1^{(0)}\mathcal{D}c_1^{(1)} + \frac{(3\xi-1)c_1^{(0)}c_1^{(1)}}{2E\xi} + \frac{E\xi\left(\left(c_1^{(2)}\right)^2 - 2c_1^{(0)}c_1^{(1)}\right)}{2\boldsymbol{p}^2}\,,\\
a_2^{(2)} &= -\frac{c_2^{(2)}}{2} - \frac{1}{2}E\xi c_1^{(2)}\left(c_1^{(1)} - 2\mathcal{D}c_1^{(0)}\right) + E\xi c_1^{(0)}\mathcal{D}c_1^{(2)} + \frac{(1-3\xi)c_1^{(0)}c_1^{(2)}}{2E\xi}\,,
\end{aligned} \tag{2.5.19}$$

and

$$a_{\mathrm{iter}} = \left(c_1^{(0)}\right)^2 - c_1^{(0)}c_1^{(1)}\boldsymbol{L}_q\cdot\mathbf{S}_1 + c_1^{(0)}c_1^{(2)}i\boldsymbol{q}\cdot\mathbf{K}_1\,. \tag{2.5.20}$$

In the above we have used the shorthands $c_n^{(a)} \equiv c_n^{(a)}\left(\boldsymbol{p}^2\right)$ and $\mathcal{D} \equiv \frac{d}{d\boldsymbol{p}^2}$.

2.5.4 Hamiltonian Coefficients from Matching to Field Theory

We are now in position to determine the Hamiltonian coefficients that capture the same classical physics as the field theories discussed in section 2.2; we do so by matching the corresponding scattering amplitudes including their full mass dependence. We start by specializing the field-theory amplitudes to the CoM frame. We then match each field-theory construction to an appropriate Hamiltonian and we discuss our findings.

The CoM frame is defined by the kinematics

$$p_1 = -(E_1,\,\boldsymbol{p})\,, \qquad p_2 = -(E_2,\,-\boldsymbol{p})\,, \qquad q = (0,\,\boldsymbol{q})\,, \qquad \boldsymbol{p}\cdot\boldsymbol{q} = \boldsymbol{q}^2/2\,, \tag{2.5.21}$$

together with $q = p_2 + p_3$ and $p_1 + p_2 + p_3 + p_4 = 0$. To align with the field-theory construction, we express the barred variables defined in Eq. (2.3.29) in this frame,

$$\bar{p}_1 = -(E_1,\,\bar{\boldsymbol{p}})\,, \qquad \bar{p}_2 = -(E_2,\,-\bar{\boldsymbol{p}})\,, \qquad \bar{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{q}/2\,, \qquad \bar{\boldsymbol{p}}\cdot\boldsymbol{q} = 0\,. \tag{2.5.22}$$

For the asymptotic spin variables we have

$$\mathsf{S}_1^{\mu\nu} = \frac{1}{m_1}\left(\epsilon^{\mu\nu\rho\lambda}\bar{p}_{1\rho}S_{1\lambda} + \bar{p}_1^\mu K_1^\nu - \bar{p}_1^\nu K_1^\mu\right), \tag{2.5.23}$$

with

$$S_1^\mu = \left(\frac{\bar{\boldsymbol{p}}\cdot\mathbf{S}_1}{m_1}, \mathbf{S}_1 + \frac{\bar{\boldsymbol{p}}\cdot\mathbf{S}_1}{m_1(E_1+m_1)}\bar{\boldsymbol{p}}\right), \quad K_1^\mu = \left(\frac{\bar{\boldsymbol{p}}\cdot\mathbf{K}_1}{m_1}, \mathbf{K}_1 + \frac{\bar{\boldsymbol{p}}\cdot\mathbf{K}_1}{m_1(E_1+m_1)}\bar{\boldsymbol{p}}\right), \tag{2.5.24}$$

where $\mathbf{S}_1$ and $\mathbf{K}_1$ correspond to the values in the rest frame of particle 1. Finally, we may use Eq. (2.2.29) to express the wave-function products $\mathcal{E}_1\cdot\bar{\mathcal{E}}_4$ and $\mathcal{E}_2\cdot\bar{\mathcal{E}}_3$ as

$$\mathcal{E}_1\cdot\bar{\mathcal{E}}_4 = \exp\left[-\frac{\boldsymbol{L}_q\cdot\mathbf{S}_1}{m_1(E_1+m_1)}\right]\exp\left[\frac{i\boldsymbol{q}\cdot\mathbf{K}_1}{m_1}\right], \quad \text{and} \quad \mathcal{E}_2\cdot\bar{\mathcal{E}}_3 = 1\,, \tag{2.5.25}$$

up to terms that do not contribute to the classical limit. The second product in the above equation follows from the fact that we take the corresponding particle to be a scalar. Note that the $\mathbf{K}_1$ used here agrees with that from the worldline (2.4.21).

Using the above relations we express the field-theory amplitudes in terms of the same variables as the EFT ones. Then, we may match them and extract the Hamiltonian coefficients. In particular, we have

$$\mathbb{M}^{\text{1PL}} = \frac{\mathcal{M}_{4,\text{cl.}}^{\text{tree}}}{4E_1E_2}\,, \qquad \text{and} \qquad \mathbb{M}^{\text{2PL}} = \frac{\mathcal{M}_{4,\text{cl.}}^{(1)}}{4E_1E_2}\,. \tag{2.5.26}$$

We use the above equations to match to the field theories as follows:

$$\text{EFT1} \;\leftrightarrow\; \text{FT2}\,, \quad \text{and} \quad \text{EFT2} \;\leftrightarrow\; \text{FT1g}\,. \tag{2.5.27}$$

Our first EFT Hamiltonian EFT1 contains only operators that preserve the spin magnitude. Hence, it can describe the field theory that contains a single particle of spin s (FT2). Our second Hamiltonian allows for transitions between particles of different spin magnitude, and hence can describe a field theory that contains particles of different spin magnitude (FT1g). Regarding FT3, the amplitudes we have computed may be mapped to those of FT1g via appropriate relabeling. We expect this to be true for all amplitudes that may be computed in the two theories, in which case the same should be true for the Hamiltonian coefficients. Finally, FT1s may be thought of as a restriction of FT1g where we only allow for spin-s external states. We discuss the possible matching of FT1s to our two Hamiltonians separately.

For the 1PL matching of EFT2 to FT1g we find

$$c_1^{(0)} = \frac{m_1m_2\gamma}{4E_1E_2}\,, \quad c_1^{(1)} = \frac{m_1m_2\gamma - EC_1\,(m_1+E_1)}{4E_1E_2m_1\,(m_1+E_1)}\,, \quad c_1^{(2)} = \frac{m_2\gamma\,(-C_1+D_1+1)}{4E_1E_2}\,, \tag{2.5.28}$$

where γ is defined in Eq. (2.4.42). Importantly, we find $c_1^{(2)} = c_2^{(2)} = 0$ if $D_1 = C_1 - 1$, such that all $\mathbf{K}_1$ dependence in the Hamiltonian vanishes for this choice. For this reason, both the 1PL and 2PL coefficients related to the matching of EFT1 and FT2 follows from the above by setting $D_1 = C_1 - 1$, hence we do not report them separately.

We conclude this subsection by commenting on FT1s. Given that FT1s is defined as a collection of amplitudes that are a subset of the ones of FT1g, the most appropriate matching procedure is to extend to FT1g and follow the analysis given above to match to EFT2. Alternatively, one can also match EFT1 to FT1s as was carried out in Ref. [101] following similar steps. In this case, the effects of the lower-spin states propagating in the field-theory amplitude are captured by the vertices of the Hamiltonian. By examining the resulting Hamiltonian, we find that some of the coefficients (in particular $c_2^{(1)}(\boldsymbol{p}^2)$) admit only a Laurent series around $\boldsymbol{p}^2 = 0$. This is a familiar phenomenon in QFT where one integrates out a state that may go on-shell in the processes of interest, and, borrowing the terminology of that context, we refer to it as a non locality.[19] A non-local quantum description may be consistent as long as one always considers amplitudes with appropriate external states. However, we find that the observables computed from this Hamiltonian match the corresponding ones from WL1 or those from WL2 only for the choice $D_1 = C_1 - 1$, for which the non-locality vanishes.

2.5.5 Observables from the Equations of Motion

Having analyzed the implications of interpreting our Hamiltonians as quantum operators, we proceed to consider them as generating functions of the classical evolution of the system. In particular, given a classical Hamiltonian $\mathcal{H}_2(\boldsymbol{r}(t), \boldsymbol{p}(t), \mathbf{S}_1(t), \mathbf{K}_1(t))$ of the form (2.5.10), the classical time evolution of any quantity $f(\boldsymbol{r}(t), \boldsymbol{p}(t), \mathbf{S}_1(t), \mathbf{K}_1(t))$ is determined by $\dot{f} = \mathrm{d}f/\mathrm{d}t = \{f, \mathcal{H}_2\}$, where the classical Poisson brackets $\{f, g\}$ are given directly by the quantum-operator algebra of Eqs. (2.5.3), (2.5.5) and (2.5.6) with $\hat{f} \to f$ and $[\hat{f}, \hat{g}] \to i\{f, g\}$. This leads to the explicit equations of motion

$$\begin{aligned} \dot{\boldsymbol{r}} &= \frac{\partial \mathcal{H}_2}{\partial \boldsymbol{p}}, & \dot{\mathbf{S}}_1 &= \frac{\partial \mathcal{H}_2}{\partial \mathbf{S}_1} \times \mathbf{S}_1 + \frac{\partial \mathcal{H}_2}{\partial \mathbf{K}_1} \times \mathbf{K}_1 \,, \\ \dot{\boldsymbol{p}} &= -\frac{\partial \mathcal{H}_2}{\partial \boldsymbol{r}}, & \dot{\mathbf{K}}_1 &= \frac{\partial \mathcal{H}_2}{\partial \mathbf{S}_1} \times \mathbf{K}_1 - \frac{\partial \mathcal{H}_2}{\partial \mathbf{K}_1} \times \mathbf{S}_1 \,. \end{aligned} \tag{2.5.29}$$

The addition of $\mathbf{K}_1$ as a dynamical quantity changes basic properties of the equations. Specifically, the magnitude of the spin $\mathbf{S}_1$ is no longer conserved.

We solve the equations of motion order by order in α. Given that $\mathcal{H}_2 = E_1 + E_2 + \mathcal{O}(\alpha)$ at zeroth order in the coupling, with $E_{1,2} = \sqrt{m_{1,2}^2 + \boldsymbol{p}^2}$, we see that perturbative solutions to these equations take the form

$$\boldsymbol{r}(t) = \boldsymbol{b}^{(0)} + \frac{E_1 + E_2}{E_1 E_2} \boldsymbol{p}^{(0)} t + \alpha \boldsymbol{r}^{(1)}(t) + \alpha^2 \boldsymbol{r}^{(2)}(t) + \dots \,,$$

[19] We stress that not every Hamiltonian which contains some coefficient that does not admit a Taylor expansion around $\boldsymbol{p}^2 = 0$ is non local in the sense described here. Indeed, it is certainly possible to alter the Hamiltonian coefficients by performing a field redefinition in the quantum-mechanical context or a canonical transformation in the classical context, which may potentially remove such a behavior. In addition, when dealing with more than three three-dimensional vectors there exist Schouten identities that might cause the coefficients of the Hamiltonian to have apparent singularities in the $\boldsymbol{p}^2 \to 0$ limit.

$$
\begin{aligned}
\boldsymbol{p}(t) &= \boldsymbol{p}^{(0)} + \alpha \boldsymbol{p}^{(1)}(t) + \alpha^2 \boldsymbol{p}^{(2)}(t) + \dots\,, \\
\mathbf{S}_1(t) &= \mathbf{S}_1^{(0)} + \alpha \mathbf{S}_1^{(1)}(t) + \alpha^2 \mathbf{S}_1^{(2)}(t) + \dots\,, \\
\mathbf{K}_1(t) &= \mathbf{K}_1^{(0)} + \alpha \mathbf{K}_1^{(1)}(t) + \alpha^2 \mathbf{K}_1^{(2)}(t) + \dots\,,
\end{aligned}
\tag{2.5.30}
$$

where $\boldsymbol{b}^{(0)}$, $\boldsymbol{p}^{(0)}$, $\mathbf{S}_1^{(0)}$, and $\mathbf{K}_1^{(0)}$ are constants determined from the initial conditions. The constant $\boldsymbol{b}^{(0)}$ is the usual impact parameter for the scattering process. Substituting these expansions into the equations of motion, using the explicit Hamiltonian (2.5.10) and separating orders in α, we obtain integral expressions for

$$
\mathcal{O}^{(n)}(t) = \left\{ \boldsymbol{r}^{(n)}(t), \boldsymbol{p}^{(n)}(t), \mathbf{S}_1^{(n)}(t), \mathbf{K}_1^{(n)}(t) \right\}\,. \tag{2.5.31}
$$

These depend on lower-order solutions $\mathcal{O}^{(\tilde{n})}(t)$, with $0 \leq \tilde{n} < n$, as well as the Hamiltonian coefficients $c_n^{(a)}(\boldsymbol{p}^2)$ and their derivatives evaluated at $\boldsymbol{p}^2 = \left(\boldsymbol{p}^{(0)}\right)^2$. Working iteratively, we obtain explicit expressions for $\mathcal{O}^{(n)}(t)$ by performing simple one-dimensional integrals with respect to t. We choose the integration constants by enforcing $\mathcal{O}^{(n)}(t) \to \{0,0,0,0\}$ as $t \to -\infty$ for all $n \geq 1$, ensuring that $\boldsymbol{b}^{(0)}$, $\boldsymbol{p}^{(0)}$, $\mathbf{S}_1^{(0)}$, and $\mathbf{K}_1^{(0)}$ characterize the initial conditions. Without loss of generality, we can choose $\boldsymbol{b}^{(0)} \cdot \boldsymbol{p}^{(0)} = 0$ and identify $\boldsymbol{b}^{(0)}$ as the incoming impact parameter vector. In particular, we choose

$$
\boldsymbol{b}^{(0)} = (-b, 0, 0)\,, \quad \boldsymbol{p}^{(0)} = (0, 0, p_\infty)\,, \quad \mathbf{S}_1^{(0)} = (S_{1x}^{(0)}, S_{1y}^{(0)}, S_{1z}^{(0)})\,, \quad \mathbf{K}_1^{(0)} = (K_{1x}^{(0)}, K_{1y}^{(0)}, K_{1z}^{(0)})\,. \tag{2.5.32}
$$

Following the above procedure, we finally obtain $(\boldsymbol{p}, \mathbf{S}_1, \mathbf{K}_1)$ in the outgoing state from the limit $t \to +\infty$, given as functions of the incoming $\left\{\boldsymbol{b}^{(0)}, \boldsymbol{p}^{(0)}, \mathbf{S}_1^{(0)}, \mathbf{K}_1^{(0)}\right\}$.

As we emphasized, a key consequence of including $\mathbf{K}_1$ in the Hamiltonian is that the magnitude of $\mathbf{S}_1$ is not conserved under time evolution. Indeed, it is a straightforward consequence of the equations of motion that

$$
\frac{d}{dt}\left(\mathbf{S}_1^2 - \mathbf{K}_1^2\right) = 0\,, \tag{2.5.33}
$$

which reduces to the equation for spin-magnitude conservation only if $\mathbf{K}_1$ is constant throughout the trajectory, as would hold for a rigid object with no internal degrees of freedom other than the spin. Explicitly, solving the equations of motion we find that the spin magnitude does indeed change. We define the change of the spin and boost magnitude as

$$
\Delta \mathbf{S}_1^2 \equiv \mathbf{S}_1^2(t = \infty) - \mathbf{S}_1^2(t = -\infty)\,, \quad \Delta \mathbf{K}_1^2 \equiv \mathbf{K}_1^2(t = \infty) - \mathbf{K}_1^2(t = -\infty)\,. \tag{2.5.34}
$$

We have that through 1PL they are given by

$$
\Delta \mathbf{S}_1^2 = \Delta \mathbf{K}_1^2 = \frac{4\alpha E_1 E_2 \left(K_{1z}^{(0)} S_{1y}^{(0)} - K_{1y}^{(0)} S_{1z}^{(0)}\right) c_1^{(2)}(p_\infty^2)}{b\, p_\infty (E_1 + E_2)} + \mathcal{O}(\alpha^2)\,, \tag{2.5.35}
$$

in accordance with Eq. (2.5.33). Thus the spin magnitude is conserved to 1PL order if we choose the initial condition $\mathbf{K}_1^{(0)} = 0$. Similarly, the boost magnitude is also conserved if $\mathbf{S}_1^{(0)} = 0$. However, starting at 2PL order, this is no longer true. In particular,

$$\Delta \mathbf{S}_1^2 \Big|_{\mathbf{K}_1^{(0)} \to 0} = \Delta \mathbf{K}_1^2 \Big|_{\mathbf{K}_1^{(0)} \to 0} = \frac{4\alpha^2 E_1^2 E_2^2 \left(\left(S_{1y}^{(0)} \right)^2 + \left(S_{1z}^{(0)} \right)^2 \right) \left(c_1^{(2)}(p_\infty^2) \right)^2}{b^2\, p_\infty^2 (E_1 + E_2)^2} + \mathcal{O}(\alpha^3)\,. \tag{2.5.36}$$

As expected, the spin magnitude is conserved if we choose $D_1 = C_1 - 1$, as can be seen by combining the above equations with Eq. (2.5.28).

The above equations further imply that for an object with $D_1 \neq C_1 - 1$ the intrinsic boost, and hence the induced electric dipole moment (see Eq. (2.4.32)), is not a constant of motion. In particular, even if a body has $\mathbf{K}_1 = 0$ at some moment in time, time evolution induces non-zero values for $\mathbf{K}_1$. In other words, a body which satisfies the covariant SSC at the initial time violates it at later times.

It is interesting to ask whether we could instead remove $\mathbf{S}_1$ and have a system that is described only by $\mathbf{K}_1$. Up to 1PL order it is consistent to have $\mathbf{S}_1 = 0$ with $\mathbf{K}_1 \neq 0$, as can be seen in Eq. (2.5.35). However, at 2PL order we find

$$\Delta \mathbf{S}_1^2 \Big|_{\mathbf{S}_1^{(0)} \to 0} = \Delta \mathbf{K}_1^2 \Big|_{\mathbf{S}_1^{(0)} \to 0} = \frac{4\alpha^2 E_1^2 E_2^2 \left(\left(K_{1y}^{(0)} \right)^2 + \left(K_{1z}^{(0)} \right)^2 \right) \left(c_1^{(2)}(p_\infty^2) \right)^2}{b^2\, p_\infty^2 (E_1 + E_2)^2} + \mathcal{O}(\alpha^3)\,. \tag{2.5.37}$$

As for Eq. (2.5.36), this only vanishes for the special value $D_1 = C_1 - 1$. Hence, without the special choice, a non-rotating body starts spinning via the electromagnetic interaction if it starts with non-zero intrinsic boost $\mathbf{K}_1$.

The dynamics that we consider here are an extension of those that satisfy an SSC along their evolution. Indeed, at any step of the calculation one is free to set $D_1 = C_1 - 1$ and retrieve the evolution of an SSC-satisfying body. Such a restriction would remove all $\mathbf{K}_1$ dependence from the Hamiltonian as we mentioned below Eq. (2.5.28) and render $\mathbf{K}_1$ to be a constant of motion that does not affect the dynamics.

The complete results of solving the equations of motion through $\mathcal{O}(\alpha^2)$ as outlined above are quite lengthy. A much more compact way to represent the amplitude is through an eikonal formula, which we give below.

2.5.6 Observables from an Eikonal Formula

Analyzing the results of the perturbative integration of Hamilton's equations as described in the previous section, we find that the outgoing-state observables can be simply expressed in terms of derivatives of an eikonal phase, which is a scalar function of the incoming-state variables. This is motivated by the analagous eikonal formula found in Ref. [59] for the pure spin case, except now there are additional degrees of freedom from the intrinsic boost. At the order to which we are working here, the eikonal phase coincidences with a two-dimensional Fourier transform of the EFT amplitude.

For convenience we rename the incoming-state quantities, called $\left\{\boldsymbol{b}^{(0)}, \boldsymbol{p}^{(0)}, \mathbf{S}_1^{(0)}, \mathbf{K}_1^{(0)}\right\}$ above, now simply as $\{\boldsymbol{b}, \boldsymbol{p}, \mathbf{S}_1, \mathbf{K}_1\}$. Then we denote the outgoing-state observables by $\{\boldsymbol{p}+\Delta\boldsymbol{p}, \mathbf{S}_1+\Delta\mathbf{S}_1, \mathbf{K}_1+\Delta\mathbf{K}_1\}$.

We find empirically that the changes in the observables $\boldsymbol{p}$, $\mathbf{S}_1$, and $\mathbf{K}_1$ are given in terms of an eikonal phase $\chi(\boldsymbol{b}, \boldsymbol{p}, \mathbf{S}_1, \mathbf{K}_1)$ as follows: The impulse is given by

$$\Delta\boldsymbol{p} = \frac{\partial\chi}{\partial\boldsymbol{b}} + \frac{1}{2}\{\chi, \frac{\partial\chi}{\partial\boldsymbol{b}}\} + \mathcal{D}_L(\chi, \frac{\partial\chi}{\partial\boldsymbol{b}}) - \frac{1}{2}\frac{\partial}{\partial\boldsymbol{b}}\mathcal{D}_L(\chi,\chi) - \frac{\boldsymbol{p}}{2\boldsymbol{p}^2}\left(\frac{\partial\chi}{\partial\boldsymbol{b}}\right)^2 + \mathcal{O}\left(\chi^3\right) , \tag{2.5.38}$$

which simultaneously gives contributions orthogonal and along $\boldsymbol{p}$. In this formula $\boldsymbol{p}\cdot\boldsymbol{b} = 0$ so all the $\boldsymbol{b}$-derivatives are projected orthogonal to the incoming momentum $\boldsymbol{p}$. The spin and boost kicks are given by

$$\begin{aligned}
\Delta\mathbf{S}_1 &= \{\chi, \mathbf{S}_1\} + \frac{1}{2}\{\chi, \{\chi, \mathbf{S}_1\}\} + \mathcal{D}_L(\chi, \{\chi, \mathbf{S}_1\}) - \frac{1}{2}\{\mathcal{D}_L(\chi,\chi), \mathbf{S}_1\} + \mathcal{O}\left(\chi^3\right) , \\
\Delta\mathbf{K}_1 &= \{\chi, \mathbf{K}_1\} + \frac{1}{2}\{\chi, \{\chi, \mathbf{K}_1\}\} + \mathcal{D}_L(\chi, \{\chi, \mathbf{K}_1\}) - \frac{1}{2}\{\mathcal{D}_L(\chi,\chi), \mathbf{K}_1\} + \mathcal{O}\left(\chi^3\right) .
\end{aligned} \tag{2.5.39}$$

The brackets here are given by the Lorentz algebra,

$$\{S_{1i}, S_{1j}\} = \epsilon_{ijk}S_{1k} , \quad \{S_{1i}, K_{1j}\} = \epsilon_{ijk}K_{1k} , \quad \{K_{1i}, K_{1j}\} = -\epsilon_{ijk}S_{1k} , \tag{2.5.40}$$

with all others vanishing. We also define

$$\mathcal{D}_L(f,g) \equiv -\epsilon_{ijk}\left(S_{1i}\frac{\partial f}{\partial S_{1j}} + K_{1i}\frac{\partial f}{\partial K_{1j}}\right)\frac{\partial g}{\partial L_k} , \tag{2.5.41}$$

which is a K-dependent extension of the operator $\mathcal{D}_{SL}$ of Ref. [59]. The angular momentum $\boldsymbol{L}$ and the incoming impact parameter $\boldsymbol{b}$ are related by $\boldsymbol{L} = \boldsymbol{b}\times\boldsymbol{p}$ and $\boldsymbol{b} = \boldsymbol{p}\times\boldsymbol{L}/\boldsymbol{p}^2$, implying $(\partial/\partial L_i) = \epsilon_{ijk}(p_k/\boldsymbol{p}^2)(\partial/\partial b_j)$ in Eq. (2.5.41). An all-orders generalization may follow along the lines of Eq. (7.21) of Ref. [59] by including $\mathbf{K}$, although at higher orders in α the radial action may be more natural than the eikonal phase [30].

The appropriate eikonal function is proportional to the two-dimensional Fourier transform (from $\boldsymbol{q}$ space to $\boldsymbol{b}$ space) of the EFT amplitude as given in Eqs. (2.5.15) and (2.5.16), while keeping only the triangle contribution in Eq. (2.5.16) [59],

$$\chi = \frac{1}{4E|\boldsymbol{p}|}\int \frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{-i\boldsymbol{q}\cdot\boldsymbol{b}}(\mathbb{M}^{\text{1PL}} + \mathbb{M}_{\triangle}^{\text{2PL}}) + \mathcal{O}(\alpha^3) ; \tag{2.5.42}$$

the box contribution to the amplitude is effectively included in the exponentiation of the tree-level amplitude $\mathbb{M}^{\text{1PL}}$. Explicitly, we have

$$\begin{aligned}
\chi = \alpha\frac{\xi E}{|\boldsymbol{p}|}&\left[-a_1^{(0)}\log|\boldsymbol{b}|^2 - \frac{2a_1^{(1)}}{|\boldsymbol{b}|^2}\boldsymbol{b}\times\boldsymbol{p}\cdot\mathbf{S}_1 + \frac{2a_1^{(2)}}{|\boldsymbol{b}|^2}\boldsymbol{b}\cdot\mathbf{K}_1\right] \\
&+ \pi\alpha^2\frac{\xi E}{|\boldsymbol{p}|}\left[\frac{a_2^{(0)}}{|\boldsymbol{b}|} - \frac{a_2^{(1)}}{|\boldsymbol{b}|^3}\boldsymbol{b}\times\boldsymbol{p}\cdot\mathbf{S}_1 + \frac{a_2^{(2)}}{|\boldsymbol{b}|^3}\boldsymbol{b}\cdot\mathbf{K}_1\right] + \mathcal{O}(\alpha^3) ,
\end{aligned} \tag{2.5.43}$$

where the amplitude coefficients $a_n^{(m)}(\boldsymbol{p}^2)$ are given in terms of the Hamiltonian coefficients $c_n^{(m)}(\boldsymbol{p}^2)$ via the same relations (2.5.18) and (2.5.19) found from the EFT matching, here all evaluated at the incoming momentum $\boldsymbol{p}$. The above relations hold for general values of the Hamiltonian coefficients $c_n^{(m)}(\boldsymbol{p}^2)$.

2.5.7 Comparison to Observables from the Worldline Theory

Having in hand the observables $\Delta\boldsymbol{p}$, $\Delta\mathbf{S}_1$, and $\Delta\mathbf{K}_1$ obtained from Hamilton's equations resulting from an EFT matching to a QFT amplitude, we are in a position to ask how these compare to equivalent observables obtained from a worldline theory as in Sec. 2.4. We find that the observables of the spinning-probe worldline theory without an SSC match precisely onto those from the probe limit of FT1g via the transformations of variables detailed bellow — these are in one-to-one correspondence with the transformations used to relate the EFT amplitudes to the covariant forms of the field-theory amplitudes in section 2.5.4. As discussed in section 2.4.3, the probe limit provides a nontrivial check.

In the worldline theory, we considered a probe/test particle with mass m_1, initial momentum $p_1^\mu = m_1 u_1^\mu$, and initial spin tensor $\mathsf{S}_1^{\mu\nu}$, scattering off the field of a background Coulomb source with velocity u_2^μ. The changes Δp_1^μ and $\Delta\mathsf{S}_1^{\mu\nu}$ from the initial to the final state were expressed in terms of these quantities and the initial impact parameter b^μ.

Using three-dimensional vectors in the rest frame of the background source, we identify

$$u_2^\mu = (1,0,0,0)\,, \qquad\qquad p_1^\mu = m_1 u_1^\mu = (m_1\gamma, \boldsymbol{p})\,, \tag{2.5.44}$$

so $\boldsymbol{p}$ here is the spatial momentum of the probe in the background frame, with $\boldsymbol{p}^2 = m_1^2(\gamma^2-1)$, and $m_1\gamma$ is its energy, where $\gamma = u_1\cdot u_2$ is the relative Lorentz factor. For the spin tensor in the probe limit, just as in (2.5.23) and (2.5.24), we decompose it into components S_1^μ and K_1^μ in the probe's rest frame,

$$\mathsf{S}_1^{\mu\nu} = \epsilon^{\mu\nu\rho\lambda}u_{1\rho}S_{1\lambda} + u_1^\mu K_1^\nu - u_1^\nu K_1^\mu\,, \tag{2.5.45}$$

and we then relate these, respectively, to three-dimensional vectors $\mathbf{S}$ and $\mathbf{K}$ in the background frame by the standard boost taking u_2^μ into u_1^μ,

$$S_1^\mu = \left(\frac{\boldsymbol{p}\cdot\mathbf{S}_1}{m_1}, \mathbf{S}_1 + \frac{\boldsymbol{p}\cdot\mathbf{S}_1}{m_1^2(\gamma+1)}\boldsymbol{p}\right), \quad K_1^\mu = \left(\frac{\boldsymbol{p}\cdot\mathbf{K}_1}{m_1}, \mathbf{K}_1 + \frac{\boldsymbol{p}\cdot\mathbf{K}_1}{m_1^2(\gamma+1)}\boldsymbol{p}\right). \tag{2.5.46}$$

Note that for the complete translation of the observables, we must consider all of (2.5.44)–(2.5.46) applied to both the initial state quantities and to the final state quantities. Finally, for the impact parameter, we have $b^\mu = (0, \boldsymbol{b}_{\text{cov}})$, where this should be related to the vector $\boldsymbol{b}$ appearing in the solution of Hamilton's equations by

$$\boldsymbol{b} = \boldsymbol{b}_{\text{cov}} + \frac{\boldsymbol{p}\times\mathbf{S}_1}{m_1^2(\gamma+1)} + \frac{1}{m_1}\left(\mathbf{K}_1 - \frac{\boldsymbol{p}\cdot\mathbf{K}_1}{\boldsymbol{p}^2}\boldsymbol{p}\right), \tag{2.5.47}$$

which is the Fourier conjugate, under (2.5.42), of multiplication by the factor $\mathcal{E}_1\cdot\bar{\mathcal{E}}_4$ in Eq. (2.5.25), in the probe limit.

Taking the solutions for Δp_1^μ and $\Delta\mathsf{S}_1^{\mu\nu}$ from solving the worldline equations of motion, given in (2.4.44), and converting them into 3-vector forms using the translations given in the previous paragraph (again, being careful to apply (2.5.45) and (2.5.46) separately to both the initial and final states, using the initial and final momenta), we find expressions for $\Delta \boldsymbol{p}$, $\Delta \mathbf{S}_1$, and $\Delta \mathbf{K}_1$ which precisely match those coming from solving the equations of motion coming from the Hamiltonian matched to FT1g, given by (2.5.38) and (2.5.39) with (2.5.43), (2.5.28), and [257].

2.5.8 On the Reality of K

We conclude this section by commenting on the reality properties of $\mathbf{K}_1$. In the quantum theory $\hat{\mathbf{K}}_1$ is an antihermitian operator for any finite-dimensional representation[20] of the Lorentz group, which implies that its expectation value $\mathbf{K}_1$ in any such state is imaginary. On the other hand, if we allow for an infinite-dimensional representation, $\hat{\mathbf{K}}_1$ may be taken to be hermitian, which would result in $\mathbf{K}_1$ being real (see e.g. Sect. 10.3 of Ref. [258]).

We first consider the implications of choosing a finite-dimensional representation, given that these are the representations employed by our field-theory constructions. In this case, for the Hamiltonian to be a hermitian operator, we need the coefficients of all Hamiltonian terms that contain an odd number of factors of the boost operator to be imaginary. This is indeed so for FT3g, while for FT1g the coefficients are real. For the 1PL coefficients, this can be seen by combining Eqs. (2.3.13) and (2.5.28). In this way, the unphysical nature of the lower-spin states in FT1g results into a non-hermitian Hamiltonian. Interestingly, the hermitian Hamiltonian corresponding to FT3g breaks time-reversal symmetry, which can be seen by combining Eq. (2.5.7) with the fact that time-reversal is an antiunitary operator (see e.g. Sect. 2.6 of Ref. [256]).

Secondly, we examine the case of infinite-dimensional representations. For these, all Hamiltonian coefficients may be taken to be real. This implies that time-reversal symmetry is satisfied. Furthermore, this case meshes well with the classical interpretation of $\mathbf{K}_1$ as a mass moment, which implies that $\mathbf{K}_1$ is real.

While the above seem to suggest the use of a field theory for an infinite-dimensional representation, we do not attempt such a construction in the present chapter. Instead, we find that the analytical continuation below Eq. (2.4.30) is sufficient for our purposes. In particular, such an analytical continuation allows for the matching between our field-theory and worldline constructions, and also results in a hermitian and time-reversal-symmetric Hamiltonian. We defer further analysis of this issue to the future.

2.6 Wilson coefficients and propagating degrees of freedom

We have seen in the previous section that the Compton amplitudes computed in FT1s depend on additional Wilson coefficients compared to those of FT2 (see 2.1 for the Lagrangians for these field theories). In 2.4 we showed that the number of Wilson coefficients of FT2 matches the usual worldline formulation

[20] Here we refer to the size of the spin space available to the particle (e.g. the states $|1/2, \pm 1/2\rangle$ for a spin-1/2 particle), in other words the size of the little-group representation. In contrast, the complete Hilbert space of a particle is always infinite due to the momentum assuming continuous values.

WL1 with an SSC imposed. We also found a modified worldline theory, WL2, containing the same number of additional Wilson coefficients as found in FT1s, FT1g and FT3. Thus, additional Wilson coefficients (relative to e.g. FT2 or WL1) are a reflection of additional degrees of freedom in the short-distance theory. In FT1 some of these extra states are unphysical, having negative norm, see Sec. 2.2.4. In this section we elaborate on the rationale behind FT3, which may be thought of as a rewriting of FT1 such that all states have positive norm, and demonstrate that the same outcome—physically-relevant extra Wilson coefficients—can also result when all states have positive norm.

As in previous discussions of FT3, we focus on fields in the (s, s) representation. We begin by separating such a field into components with definite spin. While the external states of the amplitudes $\mathcal{A}^{\text{FT1s}}$ are transverse and thus spin s, the intermediate states may contain lower-spin components, some of which are unphysical. We use factorization and gauge invariance to study the exchanges of lower-spin particles in amplitudes with spin-s external states in FT1. We find that the map given in Eq. (2.3.13) yields the results of FT3s from those of FT1s; the imaginary unit in Eq. (2.3.13) is indicative of the negative-norm nature of the exchanged states of FT1. We also discuss from a general perspective the intermediate-state spins that can contribute in the classical limit and construct their contribution to the Compton amplitude. This analysis sets on firm footing the field content we chose for the Lagrangian of FT3. Because of the structure of the Lorentz generators in the (s, s) representation (2.2.38), the trace part of intermediate states can be projected out by simply choosing traceless external states, such as the coherent states in Eq. (2.2.30). We therefore focus on the consequences of transversality or lack thereof.

2.6.1 Resolution of the Identity and Amplitudes with Lower-Spin States

As reviewed earlier, a field in the representation (s, s) of the Lorentz group contains states of all spins between 0 and s. To develop a general picture of the interplay and couplings of these states it is useful to formally expose them in the Lagrangian of FT1. We use the resolution of the identity operator in this representation,

$$\delta^{\nu(s)}_{\mu(s)} = \sum_{n=0}^{s} \binom{s}{n} u_{(\mu_1} \dots u_{\mu_n} u^{(\nu_1} \dots u^{\nu_n} \mathscr{P}^{\nu_{n+1}\dots\nu_s)}_{\mu_{n+1}\dots\mu_s)} \,, \tag{2.6.1}$$

with the on-shell transverse projectors $\mathscr{P}^{\nu_1\nu_2\dots\nu_s}_{\mu_1\mu_2\dots\mu_s} = \Theta^{\nu_1}_{(\mu_1}\Theta^{\nu_2}_{\mu_2}\dots\Theta^{\nu_s}_{\mu_s)}$, which is the $j = 0$ term in the summation of (2.2.14) and the symmetrization follows the definition in footnote 7. For example, for the two-index and three-index-symmetric representations this becomes

$$\begin{aligned} \delta^{(\nu_1}_{\mu_1}\delta^{\nu_2)}_{\mu_2} &= \mathscr{P}^{\nu_1\nu_2}_{\mu_1\mu_2} + 2u_{(\mu_1}u^{(\nu_1}\mathscr{P}^{\nu_2)}_{\mu_2)} + u_{\mu_1}u_{\mu_2}u^{\nu_1}u^{\nu_2} \,, \\ \delta^{(\nu_1}_{\mu_1}\delta^{\nu_2}_{\mu_2}\delta^{\nu_3)}_{\mu_3} &= \mathscr{P}^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3} + 3u_{(\mu_1}u^{(\nu_1}\mathscr{P}^{\nu_2\nu_3)}_{\mu_2\mu_3)} + 3u_{(\mu_1}u_{\mu_2}u^{(\nu_1}u^{\nu_2}\mathscr{P}^{\nu_3)}_{\mu_3)} + u_{\mu_1}u_{\mu_2}u_{\mu_3}u^{\nu_1}u^{\nu_2}u^{\nu_3} \,. \end{aligned} \tag{2.6.2}$$

The projectors used here single out the longitudinal components of fields but not traces. We ignore trace states; while they are propagating, in four-point Compton amplitudes they can be projected out from all diagrams that do not include loops of higher-spin states by choosing traceless external states.

By inserting the resolution of the identity (2.6.1) into the nonminimal interaction $\mathcal{L}_{\text{non-min}}$ of FT1, we can expose and identify the couplings of all the definite-spin components of ϕ_s.[21] For example, in the $\mathcal{O}(S^1)$ interaction $F_{\mu\nu}\phi_s M^{\mu\nu}\bar{\phi}_s$, by using $u^\mu \to i\partial^\mu/m$, we get

$$\phi_s M^{\mu\nu}\bar{\phi}_s = \sum_{n=0}^{s}\frac{(-1)^n}{m^{2n}}\phi_s^{\rho_1\ldots\rho_s}(M^{\mu\nu})^{\mu_1\ldots\mu_s}_{\rho_1\ldots\rho_s}\partial_{(\mu_1}\ldots\partial_{\mu_n}\mathscr{P}^{\nu_{n+1}\ldots\nu_s}_{\mu_{n+1}\ldots\mu_s)}(\partial^n\bar{\phi}_s)_{\nu_{n+1}\ldots\nu_s} + \text{c.c}\,, \tag{2.6.3}$$

where $(\partial^n\bar{\phi}_s)_{\nu_{n+1}\ldots\nu_s} = \partial^{\nu_1}\ldots\partial^{\nu_n}\bar{\phi}_{s\nu_1\ldots\nu_s}$ is a field in the $(s-n, s-n)$ representation of the Lorentz group, and the projector $\mathscr{P}^{\nu_{n+1}\ldots\nu_s}_{\mu_{n+1}\ldots\mu_s}$ singles out its spin-$(s-n)$ component. In (2.6.3) each term in the summation is given by partial derivatives and is thus not invariant under the photon gauge transformation. We only use this equation as a guide to construct an effective field theory in which an s-index tensor nonminimally couples to an $(s-n)$-index tensor.

Schematically, we identify $\mathscr{P}^{\nu_{n+1}\ldots\nu_s}_{\mu_{n+1}\ldots\mu_s}(\partial^n\bar{\phi}_s)_{\nu_{n+1}\ldots\nu_s} \equiv (\phi_{s-n})_{\mu_{n+1}\ldots\mu_s}$ as an off-shell spin-$(s-n)$ field and assign to it the kinetic term is given by $\mathcal{L}^{s-n}_{\text{min}}$ defined in (2.2.7). We further replace all the remaining partial derivatives by their covariant version. For the coupling $F_{\mu\nu}\phi_s M^{\mu\nu}\bar{\phi}_s$ at the linear order in spin, this prescription leads to the following interaction between ϕ_s and ϕ_{s-1},

$$\frac{1}{m}F_{\mu\nu}\left[\phi_s^{\alpha_1\ldots\alpha_s}{M^{\mu\nu}}_{\alpha_1\ldots\alpha_s,\beta_1\ldots\beta_s}D^{(\beta_1}\,\bar{\phi}_{s-1}^{\beta_2\ldots\beta_s)} + \text{c.c}\right]. \tag{2.6.4}$$

This interaction agrees with the one included in FT3 for $\widetilde{C}_1 = \widetilde{C}_2$ (see (2.2.61)). If we further relax the requirement that the interaction has to be mediated by the Lorentz generator, we get one more gauge invariant structures and thus arrive exactly at (2.2.61).

Having identified the off-shell component fields that exist within the off-shell field ϕ_s, we may explore how does the amplitude change if we restrict both the on-shell and the off-shell states to $(2s+1)$ states of a spin-s particles. We study this by building the four-point Compton amplitude involving only massive spin-s degrees of freedom with on-shell methods. On general grounds, we should find $\mathcal{A}^{\text{FT2}}$; to carry out this calculation, we need to find products of spin-s polarization tensors and the projector, similarly to the products involving Lorentz generators we computed in 2.2.3. We then subtract it from the corresponding amplitude $\mathcal{A}^{\text{FT1s}}$ to obtain the contribution from the lower-spin degrees of freedom, i.e. the difference between $\mathcal{A}^{\text{FT1s}}$ and $\mathcal{A}^{\text{FT2}}$. In 2.3, the Compton amplitudes of FT2 are computed from fixed value of s and then extrapolated to the generic case. Here, we will keep s arbitrary, but only consider the linear order of spin; this will be sufficient to illustrate the main points of our discussion .

[21] The projectors may be replaced with their off-shell-transverse version, constructed from $(\eta_{\mu\nu} - p^\mu p^\nu/p^2)$. However, this yields a nonlocal Lagrangian. Moreover, transversality needs to be only an on-shell property, so using Eq. (2.6.1) is sufficient.

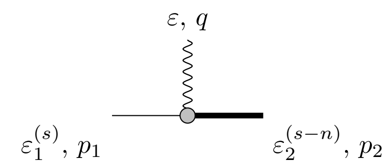


Figure 2.2: The three-point amplitude involving a massive spin-s particle (thin line), a massive spin-$(s-n)$ particle (thick line) and a photon (wiggly line).

We evaluate the products in question explicitly starting from low and fixed values of the spin, extrapolating to arbitrary s and then taking the classical limit. We find,

$$\mathcal{E}_1^{(s)} \cdot \mathcal{P}^{(s)}(p_1+q_2) \cdot \bar{\mathcal{E}}_4^{(s)} = \mathcal{E}_1^{(s)} \cdot \bar{\mathcal{E}}_4^{(s)} \left(1 + \frac{s\varepsilon_1 \cdot q_2 \bar{\varepsilon}_4 \cdot q_3}{\varepsilon_1 \cdot \bar{\varepsilon}_4 m^2} + \ldots\right), \tag{2.6.5}$$
$$\mathcal{E}_1^{(s)} \cdot \mathcal{P}^{(s)}(p_1+q_2) \cdot M^{\mu\nu} \cdot \bar{\mathcal{E}}_4^{(s)} = \mathcal{E}_1^{(s)} \cdot M^{\mu\nu} \cdot \bar{\mathcal{E}}_4^{(s)} + \frac{is(p_1^\mu \bar{\varepsilon}_4^\nu - p_1^\nu \bar{\varepsilon}_4^\mu)\varepsilon_1 \cdot q_2}{\varepsilon_1 \cdot \bar{\varepsilon}_4 \, m^2} \, \mathcal{E}_1^{(s)} \cdot \bar{\mathcal{E}}_4^{(s)} + \ldots\,,$$

where we used the on-shell conditions and transversality and q_2 and q_3 are the momenta of the Compton amplitude photons. We have omitted terms that do not contribute in the classical limit of the Compton amplitude at $\mathcal{O}(S^1)$.

Using Eqs. (2.6.5) it is straightforward to compute the pole part of the Compton amplitude. To complete the amplitude we construct an ansatz for the missing contact term and fix it by demanding gauge invariance for the two photon external lines. We find that the difference $\mathcal{A}^{\delta}_{4,\text{cl}}$ between the amplitude without the spin-s projector, $\mathcal{A}^{\text{FT1s}}_{4,\text{cl}}$, and the amplitude with the spin-s projector, which is indeed $\mathcal{A}^{\text{FT2}}$, is given by

$$\mathcal{A}^{\delta}_{4,\text{cl}} = \mathcal{A}^{\text{FT1s}}_{4,\text{cl}} - \mathcal{A}^{\text{FT2}}_{4,\text{cl}} = -(-1)^s \frac{\mathcal{E}_1^{(s)} \cdot \bar{\mathcal{E}}_4^{(s)}}{m^2} \frac{2i(1-C_1+D_1)^2}{p_1 \cdot q_2} p_1 \cdot f_2 \cdot S(p_1) \cdot f_3 \cdot p_1\,. \tag{2.6.6}$$

This is exactly the difference between (2.3.8) and (2.3.10). The sign difference compared to Eq. (2.3.12) reflects the negative norm of the spin-$(s-1)$ states that are part of ϕ_s compared to the positive norm of the analogous states in FT3. Eq. (2.6.6) also manifests that choosing $D_1 = C_1 - 1$ for $\mathcal{A}^{\text{FT1s}}$ is equivalent to consistently inserting the spin-s physical-state projector.

2.6.2 Lower-Spin States and their Scaling in the Classical Limit

Having identified the relevance of the lower-spin states for Compton amplitudes, we now proceed to examine the processes whose classical limit receives contributions from such states. While, as already noted, in FT1 such states have negative norm, we may either construct field theories such as FT3 in which their norm is positive so they are physical, or we may simply use maps such as (2.3.13) or (2.3.28) to modify the amplitudes of FT1 to agree with amplitudes with physical intermediate states.

We wish to characterize the classical scaling of the transitions from the spin-s to the spin-$(s-n)$ state via the emission of a photon. There are several distinct structures that can appear in such an amplitude, as

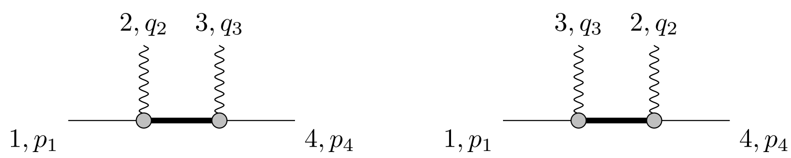


Figure 2.3: Representative diagram of the contribution of the spin-$(s-n)$ exchanges in the Compton amplitude. Legs 1 and 4 are massive spin-s particles, legs 2 and 3 are photons, and the intermediate thick line corresponds to the spin-$(s-n)$ particle for some $n > 0$.

illustrated for example in Eq. (2.3.1). For illustrative purposes we focus on the first term in that equation which arises from the covariant derivative in the quadratic Lagrangian $\mathcal{L}_{\rm min}$ of FT1; other interactions may be treated similarly with similar expected conclusions. We moreover interpret the lower-spin field as the longitudinal components of a higher-spin field, as discussed in Sec. 2.2.4. Thus, the three-point amplitude we consider here and illustrated in 2.2 is

$$\mathcal{A}_{3,\rm min}^{s\to s-n} = i\epsilon_3 \cdot p_1\, \mathcal{E}_1^{(s)} \cdot \left(u_2^n\, \bar{\mathcal{E}}_2^{(s-n)}\right) , \tag{2.6.7}$$

where all momenta are outgoing, the matter momenta are p_1 and p_2, $u_i = p_i/m$ and the photon momentum is q. Using the explicit form of the polarization tensors in Eq. (2.2.45), this three-point amplitude becomes

$$\mathcal{A}_{3,\rm min}^{s\to s-n} = i\epsilon_3 \cdot p_1\, (\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-n} \binom{s}{n}^{1/2} \left(\frac{q\cdot\varepsilon_1}{m}\right)^n , \tag{2.6.8}$$

where we used the on-shell conditions $p_2 = -p_1 - q$ and $\varepsilon_1 \cdot p_1 = 0$. For $n \ll s$ and $1 \ll s$ we may approximate $\binom{s}{n} \approx \frac{s^n}{n!}$. We may use the scaling of polarization tensors implied by their embedding in a nontransverse (s, s) representation of their Lorentz group to obtain the scaling of the transition amplitude. Together with (2.2.46),(2.2.47),(2.2.54), (2.6.8) implies that the transition three-point amplitude $\mathcal{A}_{3,\rm min}^{s\to s-n}$ depends on q and K as

$$\mathcal{A}_{3,\rm min}^{s\to s-n} \sim q^n K^n \sim q^0 . \tag{2.6.9}$$

Thus, the transition three-point amplitudes scale as q^0 in the classical limit, so they are classical.

We now discuss the contribution of three-point amplitudes to the residue of four-point amplitudes. Since in Eq. (2.6.9) the polarization tensors have already been used to generate the factors of K, the expression of the amplitudes that is useful for residue computation is Eq. (2.6.8) together with the fact that the sum of a product of spin-$(s-k)$ polarization tensors over all the physical states yields the projector onto the spin-$(s-k)$ states. It is then straightforward to see that the pole part of diagonal amplitudes, whose diagrams are illustrated in 2.3, is

$$\begin{aligned} \mathcal{A}_4^{s\to s}\Big|^{\rm exchange}_{\text{spin-}(s-n)} &= \sum_{(s-n)\text{ states}} \left[\frac{\mathcal{A}^{s\to s-n}_{3:p_1,q_2,P}\mathcal{A}^{s-n\to s}_{3:-P,q_3,p_4}}{2p_1\cdot q_2} + \frac{\mathcal{A}^{s\to s-n}_{3:p_1,q_3,P}\mathcal{A}^{s-n\to s}_{3:-P,q_2,p_4}}{2p_1\cdot q_3}\right] \\ &\sim \frac{s^n}{2p_1\cdot q_2}(\varepsilon_1\cdot\bar{\varepsilon}_4)^{s-n}\left[\left(\frac{q_2\cdot\varepsilon_1}{m}\right)^n\left(\frac{q_3\cdot\bar{\varepsilon}_4}{m}\right)^n - \left(\frac{q_3\cdot\varepsilon_1}{m}\right)^n\left(\frac{q_2\cdot\bar{\varepsilon}_4}{m}\right)^n\right] , \end{aligned} \tag{2.6.10}$$

where we assumed that the relevant higher-spin theory has standard factorization properties. The second term in (2.6.10) follows by interchanging q_2 and q_3. In the large s limit, (2.2.40),(2.2.42),(2.2.43) imply that

$$\left(\frac{q_2\cdot\varepsilon_1}{m}\frac{q_3\cdot\bar{\varepsilon}_4}{m}\right)^n \to \frac{(\varepsilon_1\cdot\bar{\varepsilon}_4)^n}{(2ms)^n}\left[iq_2\cdot S(p_1)\cdot q_3 - \frac{1}{ms}q_2\cdot S(p_1)\cdot S(p_1)\cdot q_3\right]^n$$
$$\to \frac{(\varepsilon_1\cdot\bar{\varepsilon}_4)^n}{(2ms)^n}\left[iq_2\cdot S(p_1)\cdot q_3\right]^n . \tag{2.6.11}$$

We observe that for any n the explicit factors of s cancel in Eq. (2.6.10), i.e. the various factors combine such that the only spin dependence is through $(\varepsilon_1\cdot\bar{\varepsilon}_4)^s$ and $S^{\mu\nu}$. However, since the square parenthesis in Eq. (2.6.11) scales as q^n and the propagator in Eq. (2.6.10) scales as $1/q$, only for $n=1$ the exchange term has a classical contribution. Heuristically, the existence of one matter propagator allows for transitions to spin states that differ from the external by one unit (i.e. $s\to s-1\to s$).

A similar argument reveals the contribution of transition three-point amplitudes to off-diagonal $s\to s-m$ two-photon amplitudes. It is intuitive that intermediate spin-$(s-n)$ states can contribute if $0\le n\le m$. For $n>m$, we find that the existence of one matter propagator in the four-point amplitude allows for the state $n=m+1$ to also contribute. Indeed, factorization together with Eq. (2.6.8) imply that

$$\mathcal{A}_4^{s\to s-m}\Big|^{\text{exchange}}_{\text{spin-}(s-n)} = \sum_{(s-n)\text{ states}}\left[\frac{\mathcal{A}^{s\to s-n}_{3:p_1,q_2,P}\mathcal{A}^{s-n\to s-m}_{3:-P,q_3,p_4}}{2p_1\cdot q_2} + \frac{\mathcal{A}^{s\to s-n}_{3:p_1,q_3,P}\mathcal{A}^{s-n\to s-m}_{3:-P,q_2,p_4}}{2p_1\cdot q_3}\right]$$
$$\sim \frac{s^{n-m/2}}{2p_1\cdot q_2}(\varepsilon_1\cdot\bar{\varepsilon}_4)^{s-n}\left[\left(\frac{q_2\cdot\varepsilon_1}{m}\right)^n\left(\frac{q_3\cdot\bar{\varepsilon}_4}{m}\right)^{n-m} - \left(\frac{q_3\cdot\varepsilon_1}{m}\right)^n\left(\frac{q_2\cdot\bar{\varepsilon}_4}{m}\right)^{n-m}\right], \tag{2.6.12}$$

where we assumed that $n\ge m$ and that, as before, the relevant higher-spin theory has standard factorization properties.[22] Eqs. (2.2.40), (2.2.42) and (2.2.43) imply that, as in the diagonal amplitude, the factors with an equal number of ε_1 and $\bar{\varepsilon}_4$ can be effectively written in the large s limit as

$$\left(\frac{q_2\cdot\varepsilon_1}{m}\frac{q_3\cdot\bar{\varepsilon}_4}{m}\right)^{n-m} \to \frac{(\varepsilon_1\cdot\bar{\varepsilon}_4)^{n-m}}{(2ms)^{n-m}}\left[iq_2\cdot S(p_1)\cdot q_3 - \frac{1}{ms}q_2\cdot S(p_1)\cdot S(p_1)\cdot q_3\right]^{n-m}, \tag{2.6.13}$$

and similarly for $q_2\leftrightarrow q_3$. Eqs. (2.2.52) and (2.2.53) can be used to write the remaining unbalanced dependence on ε_1 and $\bar{\varepsilon}_4$ as a linear combination of K and S vectors of degree larger or equal to m, which contains at least one term with m factors of K and has an overall factor of $s^{-m/2}$ (see for e.g. Eq. (2.2.53)). The overall factors of s cancel out, as in the case of diagonal amplitudes. The terms with m factors of K and one power of S if $n>m$ and the terms with m factors of K and no power of S if $n=m$ exhibit classical scaling. In other words, suppressing factors of q and S, we have $\mathcal{A}_4^{s\to s-m}\sim K^m$ as its three-point counterpart in Eq. (2.6.9). This dependence prompted us to restrict our analysis of FT3 to a single power of K or an arbitrary power of

[22] An expression analogous with Eq. (2.6.12) can be written for $m>n$. Both in that expression and in Eq. (2.6.12) the sum over intermediate states yields an on-shell transverse projector. Transversality of external states implies, however, that the momentum-dependent terms in that projector are subleading in the classical limit, which justifies why no projector is included in Eq. (2.6.12).

S and no K, i.e. $m \leq 1$, since these are the terms we may probe by considering a spin-s and a spin-$(s-1)$ field. It would be interesting to extend FT3 with further lower-spin fields and access nonlinear dependence on S and K. Consistency of the theory should lead to the cancellation of possible superclassical terms.

The arguments above can be repeated to analyze the possible intermediate states that can contribute to higher-point tree-level amplitudes. For example, starting with Eq. (2.6.9), the two-pole part of a diagonal $s \to s$ three-photon amplitude can receive contributions from suitable combinations of intermediate states of spin different from s. Further contributions from single-pole terms depend on the scaling of four-point contact terms; for example, if it is the same as for the three-point amplitude, $\mathcal{A}_4^{s\to s-n} \sim (q \cdot K/m)^n$, then such four-point amplitudes contribute to single-pole terms of the five-point amplitude. Such higher-point amplitudes are some of the ingredients of higher-PL spin-dependent calculations, so it would be interesting to investigate them further.

2.6.3 Lower-Spin States in the Compton Amplitude

With the information we acquired from the analysis of the soft-region scaling of amplitudes with states of different large spin we may construct Compton amplitudes using a standard on-shell approach: We start with three-point amplitudes with the appropriate scaling and use them to construct the $O(S)$ exchange part of the Compton amplitude. We then fix the contact terms by demanding gauge invariance and that their dimension is the same as that of contact terms arising from the Lagrangians of FT1, FT2 and FT3.

The three-point amplitude is shown diagrammatically in Fig. 2.2. With $n = 1$ and all-outgoing momenta, its expression that follows from a Lagrangian such as that of FT1 is

$$\mathcal{A}_3^{s\to s-1} = (-1)^s \mathcal{E}_1^{(s)} \cdot \mathbb{M}_3(p_1, p_2, q_3, \epsilon_3) \cdot \left(u_2 \bar{\mathcal{E}}_2^{(s-1)}\right) , \tag{2.6.14}$$

where $\mathbb{M}_3$ is given in Eq. (2.3.1). Using Eqs. (2.2.46) and (2.2.47), the linear-in-S or K part of the three-point amplitude to leading order in s can be written as

$$\mathcal{A}_3^{s\to s-1} = (-1)^s \frac{2\sqrt{s}(C_1 - D_1 - 1)(p_1 \cdot \epsilon_3)(\varepsilon_1 \cdot q)(\varepsilon_1 \cdot \bar{\varepsilon}_2)^{s-1}}{m} + \dots . \tag{2.6.15}$$

where the ellipsis stands for terms of higher order in s and q.

Next, we sew together two of these three-point amplitudes to obtain the residues of the two matter-exchange poles of the Compton amplitude corresponding to the two diagrams in Fig. 2.3. Focusing solely on the spin-$(s-1)$ exchange, we have

$$\begin{aligned}\text{Res}\Big(\mathcal{A}_{4,\text{cl}}\Big|_{\text{spin-}(s-1)}\Big)\Big|_{2p_1\cdot q_2=0} &= (-1)^s \frac{4s(C_1 - D_1 - 1)^2(p_1 \cdot \epsilon_2)(\varepsilon_1 \cdot q_2)(p_4 \cdot \epsilon_3)(\bar{\varepsilon}_4 \cdot q_3)}{m^2} \\ &\times \sum_{\text{phys. } \varepsilon_\ell \text{ states}} (\varepsilon_1 \cdot \bar{\varepsilon}_\ell)^{s-1}(\varepsilon_\ell \cdot \bar{\varepsilon}_4)^{s-1} ,\end{aligned} \tag{2.6.16}$$

where ϵ_2 and ϵ_3 are photon polarization vectors. The physical state sum is evaluated using

$$\sum_{\text{phys. } \varepsilon_\ell \text{ states}} (\varepsilon_1 \cdot \bar{\varepsilon}_\ell)^{s-1} (\varepsilon_\ell \cdot \bar{\varepsilon}_4)^{s-1} = \varepsilon_{1\mu_1} \dots \varepsilon_{1\mu_{s-1}} \left(\mathcal{P}^{(s-1)}(\ell)\right)^{\mu(s-1)}_{\nu(s-1)} \bar{\varepsilon}_4^{\nu_1} \dots \bar{\varepsilon}_4^{\nu_{s-1}} , \tag{2.6.17}$$

and Eq. (2.6.5). Given that the residue scales as q, we may replace the projector with the identity, as all the other terms are subleading in small q. The residue becomes

$$\begin{aligned} &\text{Res}\Big(\mathcal{A}_{4,\text{cl}}\Big|_{\text{spin-}(s-1)}\Big)\Big|_{2p_1\cdot q_2=0} = \\ &\qquad (-1)^s \frac{4s(C_1 - D_1 - 1)^2 (p_1 \cdot \epsilon_2)(\varepsilon_1 \cdot q_2)(p_4 \cdot \epsilon_3)(\bar{\varepsilon}_4 \cdot q_3)}{m^2} (\varepsilon_1 \cdot \bar{\varepsilon}_4)^{s-1} . \end{aligned} \tag{2.6.18}$$

To complete the amplitude we need to add the other exchange channel, with a pole at $p_1 \cdot q_3 = 0$, and to find the contact term so that the result is invariant under photon gauge transformations, $\varepsilon_i \to \varepsilon_i + \lambda q_i$ with $i = 2$ and separately $i = 3$. Allowing for at most two powers of momenta in the contact term, its effect is only the replacements

$$(p_1 \cdot \epsilon_2)(\varepsilon_1 \cdot q_2) \to p_1^\mu f_{2,\mu\nu} \varepsilon_1^\nu , \qquad (p_4 \cdot \epsilon_3)(\bar{\varepsilon}_4 \cdot q_3) \to p_4^\alpha f_{3,\alpha\beta} \bar{\varepsilon}_4^\beta , \tag{2.6.19}$$

with $f_{i,\mu\nu}$ defined below Eq. (2.3.7). Thus, the classical Compton amplitude of two spin-s particles due to an intermediate spin-$(s-1)$ exchange is

$$\mathcal{A}_{4,\text{cl}}^{s\to s}\Big|_{\text{spin-}(s-1)} = (-1)^s \frac{(\varepsilon_1 \cdot \bar{\varepsilon}_4)^{s-1}}{m^2} \frac{4s(C_1 - D_1 - 1)^2}{2p_1 \cdot q_2} p_1 \cdot f_2 \cdot \varepsilon_1 \, p_4 \cdot f_3 \cdot \bar{\varepsilon}_4 + (2 \leftrightarrow 3) . \tag{2.6.20}$$

Finally, replacing the polarization vectors ε_1 and $\bar{\varepsilon}_4$ in terms of the spin tensor as in Eq. (2.2.40) and keeping only the classical terms leads to

$$\mathcal{A}_{4,\text{cl}}^{s\to s}\Big|_{\text{spin-}(s-1)} = \mathcal{A}_{4,\text{cl}}^{\delta} = \mathcal{A}_{4,\text{cl}}^{\text{FT1s}} - \mathcal{A}_{4,\text{cl}}^{\text{FT2}} , \tag{2.6.21}$$

where for the second equality we used Eq. (2.6.6). Thus, we explicitly identify the difference between $\mathcal{A}_{4,\text{cl}}^{\text{FT1s}}$ and $\mathcal{A}_{4,\text{cl}}^{\text{FT2}}$ as due to the propagation of an intermediate $(s-1)$-spin state.

2.7 Discussion and Conclusion

In this chapter we addressed a puzzle regarding the description and dynamical evolution of spinning bodies in Lorentz invariant theories, with an eye towards applications to the two-body problem in general relativity. Their gravitational or electromagnetic interactions are described via an effective field theory of point particles in terms of a set of higher-dimension operators each with a free Wilson coefficient. Ref. [101] found that the amplitudes-based framework of Ref. [59] leads to additional independent Wilson coefficients in observables compared to the usual worldline description. These additional Wilson coefficients appear to vanish identically for black holes, but seem to contribute to scattering observables for more general spinning objects starting at the second order in Newton's constant and at cubic order in the spin.

To identify the origin and the physics described by the extra Wilson coefficients we analyzed the simpler case of electromagnetic interactions of charged spinning bodies. This theory is inherently simpler than general relativity because it has no photon self-interactions and more importantly the analogous effects are already present at linear order in spin. We constructed several such electromagnetic field theories: one with two physical propagating higher-spin fields, another with multiple physical and unphysical propagating higher-spin states packaged in a single higher-spin field, and finally one with a single quantum spin. When available, we also considered several possible classical asymptotic states. In the classical limit we found that simple maps connect the amplitudes of the various cases and reached the conclusion that the presence of states beyond those of a spin-s particle leads to additional Wilson coefficients. These Wilson coefficients govern transitions between states of different spin which in turn lead to changes in the magnitude of the classical spin vector even for conservative dynamics. While the magnitude of the spin vector can change in theories with additional propagating states, the magnitude of the spin tensor is conserved.

We found that these results have an interpretation in a more conventional worldline framework and exposed it by analyzing two distinct worldline theories. The first one corresponds to the standard construction [140, 226] where a spin supplementary condition is imposed. The second theory relaxes this constraint, introducing additional degrees of freedom. As for field theories with transitions between states with different spin, the dynamics of this theory allows for changes in the magnitude of the spin vector along classical trajectories.

While the results of all of our field theories can be obtained as limits of results of these two worldline theories, we did not find a worldline theory that reproduces observables obtained from $\mathcal{A}^{\mathrm{FT1s}}$ whose asymptotic states are limited to a single quantum spin. It would be interesting to pursue the construction of such a theory; to this end it may be profitable to interpret $\mathcal{A}^{\mathrm{FT1s}}$ as a sequence of absorption amplitudes and match them with a worldline theory with additional non-asymptotic states, along the lines of Ref. [234]. Another interesting direction would be to generalize FT3, which was constructed using spin s and $(s-1)$ states, to include spin $(s-k)$ state with $k \geq 2$, in order to describe interactions beyond the spin-orbit case.

We evaluated tree-level Compton amplitudes to provide a direct comparison between the various field and worldline theories. We carried out this comparison to second order in the spin tensor. Field theories restricted to propagate only the states of a spin-s particle preserve the magnitude of the classical spin vector, and the results match those of the worldline with a spin supplementary condition imposed, compatible with Refs. [231, 179]. In contrast, if states of different spin propagate and transitions are allowed between them, the field-theory Compton amplitudes contain additional Wilson coefficients and match those of the worldline with no spin supplementary condition. The results of the theory with propagating states of a single spin-s particle are reproduced for special values of the Wilson coefficients; thus, for these values, the SSC condition is effectively imposed (albeit not actively), and the spin gauge symmetry is restored. This holds true both for the field theory where some of the additional spin states were negative norm [59] and for the alternative construction with all positive-norm states.

To establish a closer connection between the extra degrees of freedom present in the various field-theory descriptions of spinning bodies and classical observables, we constructed a pair of two-body Hamiltonians where the obtained amplitudes match the field-theory amplitudes [59]. The first of these Hamiltonians is the standard two-body one including the standard spin-orbit terms. The second incorporates the mass moment as a new (boost) degree of freedom, and is the one that can match both the field theories with transitions between states of different spins and the worldline with no spin supplementary condition imposed. We carried out detailed comparisons of the impulse, and spin and mass-moment kicks through $\mathcal{O}(\alpha^2 S)$ between the predictions of these two-body Hamiltonians and the corresponding worldline approaches and found agreement to this order. It would be interesting to generalize our field theory with two propagating fields to contain multiple propagating fields and in this way verify the connection to the worldline through $\mathcal{O}(\alpha^2 S^{k\geq 2})$.

We also succeeded in finding a compact way to express scattering observables via an eikonal formula. The spin eikonal formula of Ref. [59] provides a direct connection between amplitudes and scattering observables and bypasses explicit use of the Hamiltonian. We found a generalization of this formula, which is valid through $\mathcal{O}(\alpha^2 S)$, compactly contains the intricate results of Hamilton's equations for scattering observables and includes extra degrees of freedom (in the form of the rest-frame boost vector) and all Wilson coefficients. It would be interesting to extent this comparison to higher powers of the spin and boost vectors. While this eikonal formula was not derived from first principles, its existence strongly suggests that a first-principles derivation should exist.

Our primary conclusion is that, whether using a four-dimensional field-theory or a worldline description of spinning bodies, the extra Wilson coefficients are directly associated with additional propagating degrees of freedom. These extra coefficients induce a dynamical change in the magnitude of the rest-frame spin vector even for conservative dynamics. This change in spin magnitude is necessarily associated with a change in the mass moment, which in turn induces a change in the electric dipole moment. It would be very interesting to identify physical systems where these additional degrees of freedom lead to observable effects whether in electrodynamics or general relativity.

We expect that carrying out similar field theory, worldline and effective two-body Hamiltonian constructions and comparisons for general relativity should be straightforward. We look forward to studying the phenomena described here in detail for the case of general relativity where they were originally observed.

Chapter 3

Dynamical Implications of the Kerr Multipole Moments for Spinning Black Holes

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Previously the linearized stress tensor of a stationary Kerr black hole has been used to determine some of the values of gravitational couplings for a spinning black hole to linear order in the Riemann tensor in the action (worldline or quantum field theory). In particular, the couplings on operators containing derivative structures of the form $(S \cdot \nabla)^n$ acting on the Riemann tensor were fixed, with S^μ the spin vector of the black hole. In this chapter we find that the Kerr solution determines all of the multipole moments in the sense of Dixon of a stationary spinning black hole and that these multipole moments determine all linear in R couplings. For example, additional couplings beyond the previously mentioned are fixed on operators containing derivative structures of the form $S^{2n}(p \cdot \nabla)^{2n}$ acting on the Riemann tensor with p^μ the momentum vector of the black hole. These additional operators do not contribute to the three-point amplitude, and so do not contribute to the linearized stress tensor for a stationary black hole. However, we find that they do contribute to the Compton amplitude. Additionally, we derive formal expressions for the electromagnetic and gravitational Compton amplitudes of generic spinning bodies to all orders in spin in the worldline formalism and evaluated expressions for these amplitudes to $\mathcal{O}(S^3)$ in electromagnetism and $\mathcal{O}(S^5)$ in gravity.

3.1 Introduction

3.1.1 General Overview

The observation of gravitational waves by the LIGO/Virgo collaboration [102, 103] began a new era of gravitational physics, with implications for astronomy, cosmology, and possibly particle physics. Physical black holes and neutron stars generically carry significant spin angular momentum which affects their dynamics during mergers in binary systems and the gravitational wave signals they emit. These spin effects will play an increasingly important role in signal analysis as gravitational wave detectors become more sensitive [104, 105, 106] and also lead to rich theoretical structure for generic bodies, but especially so for black holes.

The study of the dynamics of generic spinning bodies in general relativity has a long history [107, 108, 109, 110, 253, 259, 260]. Multiple successful field theoretic and worldline based approaches exist for the study of spinning bodies in both the post-Newtonian (PN) approximation [111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 51, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163] and the post-Minkowskian (PM) approximation [164, 165, 6, 57, 56, 52, 78, 21, 22, 58, 53, 54, 55, 59, 61, 60, 65, 62, 166, 167, 168, 169, 66, 170, 171, 172, 173, 174, 101, 168, 175, 176, 177, 178, 179, 180, 181, 182, 183, 184, 2, 261, 262]. The electromagnetic [187, 188, 189, 5, 190, 191, 192] and non-abelian gauge theory [7, 1] cases are very similar in structure to gravity and can be used to develop helpful insights for the harder gravitational problem.

In both the field theoretic and worldline approaches when considering only the minimal Poincaré degrees of freedom, the interaction of the body with gravity is characterized by a tower of effective field theory operators in the action, each carrying a Wilson coefficient, some number of powers of the spin of the body, and some number of powers of the Riemann tensor and its derivatives. For generic bodies these Wilson coefficients take arbitrary values. We will specialize our interest in this chapter exclusively to spinning black holes. This restriction in principle determines the values of all such Wilson coefficients. However, presently the values for these Wilson coefficients on linear and quadratic in Riemann tensor are only partly known.

The coefficients for operators of the form $(S \cdot \nabla)^n R_{....}$ were fixed in Ref. [145]. Such operators are precisely those which contribute to the three point amplitude. There are possible operators which are linear in the Riemann tensor but not of this form, such as those of the form $S^{2n}(p \cdot \nabla)^{2n} R_{....}$, whose Wilson coefficients cannot be determined from the three point amplitude. Using the equations of motion in the action, one can see that such operators contribute at order R^2 for scattering processes. We find that for a black hole the coefficients for all operators which are linear in the Riemann curvature can be fixed by matching against the multipole moments of the Kerr solution. Several proposals [173, 101, 179, 180] have appeared in the literature to fix the coefficients on quadratic in Riemann operators based on a shift-symmetry principle which is already true of the linear in Riemann results. Refs. [185, 186] find that the Compton amplitude derived by solving the Teukolsky equation agrees with the shift-symmetry principle through $\mathcal{O}(S^4)$ but that tension with the shift-symmetry begins at $\mathcal{O}(S^5)$ (though the results from the Teukolsky equation involve a subtle analytic continuation between the black-hole and naked-singularity regimes). The couplings we find based on consideration of multipole moments can be made consistent with spin-exponentiation, shift-symmetry, or the Teukolsky results through $\mathcal{O}(S^5)$. As well, they can be made simultaneously consistent with spin-exponentiation and the Teukolsky results through $\mathcal{O}(S^5)$. (Beginning at $\mathcal{O}(S^5)$ one helicity combination develops a spurious pole in the the spin-exponentiated amplitude; when we say that we can match spin-exponentiation at $\mathcal{O}(S^5)$ or beyond we only mean that we match to the helicity combination without a spurious pole after $\mathcal{O}(S^4)$.)

3.1.2 Summary of Method and Results

In Dixon's landmark papers Ref. [259, 260] on the worldline formalism, among other results, he proves that there is a unique way to define the multipole moments of the current density or stress tensor for an extended body in general relativity so that those multipole moments can be made into a generating function for the current density/stress tensor in the usual way and so that those multipole moments are fully reduced (i.e. contain no interdependencies between moments of different orders). The definitions of these moments are highly nontrivial and only coincide with the "naive" moments (from integrating powers of displacement against the current density/stress tensor over a spatial slice) for a body in uniform motion in flat spacetime. From the Kerr solution, following Israel's analysis [263], we compute the stress tensor which acts as its source (in the maximally causally extended spacetime) and from that source we compute the multipole moments of a spinning black hole using Dixon's definitions of the multipole moments. Those multipole moments can then be used to determine a stress tensor, which in turn can be used to determine an action for the spinning black hole, up to couplings to operators which are quadratic in the Riemann tensor. The action we find is put in dynamical mass function form in (3.6.1) and the specific dynamical mass function we find for spinning black holes is given in (3.7.43).

The dynamical mass function we find contains all of the equivalent black hole couplings identified in Ref. [145], which can be found by comparison to the three-point amplitude (the stationary stress tensor), as well as many new terms. Dixon's formalism specifies the unique way to lift those naive moments to proper multipole moments, and that lifting fixes the additional couplings in (3.7.43) relative to Ref. [145]. This lifting is only able to fix linear in Riemann couplings in the action because no information about higher order in Riemann operators is contained in the stationary moments. We find that the dynamical mass function in (3.7.43) can be made consistent with the spin-exponentiation proposed by Ref. [53] and shift-symmetry proposed by Refs. [173, 101, 179, 180] through $\mathcal{O}(S^4)$ for the appropriate choice of quadratic in Riemann couplings. As well, it can be made consistent with any one of the three principles of spin-exponentiation, shift-symmetry, or the Teukolsky equation results found in Ref [185] through $\mathcal{O}(S^5)$ and made consistent with any pair of them except the combination of shift-symmetry and the Teukolsky equation, which are incompatible. To facilitate this analysis, we find a formal expression for the gravitational Compton amplitude for a generic spinning body to all orders in spin and explicitly compute that amplitude in terms of all possible

Wilson coefficients in the action through $\mathcal{O}(S^5)$. We find that there is one linearly independent structure in the amplitude at $\mathcal{O}(S^2)$, with one more appearing at $\mathcal{O}(S^3)$, seven more at $\mathcal{O}(S^4)$, and eleven more at $\mathcal{O}(S^5)$.

In order to understand these gravitational results, it is instructive to first follow all of the same steps of analysis for a $\sqrt{\text{Kerr}}$ particle in electromagnetism. In section 3.2 we review the worldline formalism with a dynamical mass function for electromagnetism. In section 3.3 we review the basics of Dixon's theory of multipole moments and specialize his results to the current-density in flat spacetime. In section 3.4 we use Dixon's formalism to compute the multipole moments of a $\sqrt{\text{Kerr}}$ particle and from those moments we compute the necessary dynamical mass function for such a particle, up to corrections which are quadratic in the field strength. Our electromagnetic analysis culminates in section 3.5 in which we compute the electromagnetic Compton amplitude for a generic spinning body to all orders in spin in terms of its dynamical mass function. We then specialize that all orders result to cubic order in spin by enumerating all possible operators in the action and study the spin-exponentiation and shift-symmetry properties of the resultant amplitude. We find that for electromagnetism, it is possible to simultaneously demand spin-exponentiation, shift-symmetry, and consistency with the $\sqrt{\text{Kerr}}$ multipole moment based dynamical mass function through $\mathcal{O}(S^3)$.

In the second half of the chapter, we perform the same analysis for gravity. We begin in 3.6 by reviewing the worldline formalism with a dynamical mass function in general relativity. In section 3.7 we use Dixon's formalism to compute the multipole moments of a Kerr particle and from those moments we compute the necessary dynamical mass function for such a particle, up to corrections which are quadratic in the Riemann tensor. Then in 3.8 we derive a formal expression for the gravitational Compton amplitude for a generic spinning body to all orders in spin, which is unfortunately much more complex than the corresponding electromagnetic formula. We then specialize that all orders result to quintic order in spin by enumerating all possible operators in the action and study the requirements imposed on that amplitude by spin-exponentiation, shift-symmetry, and matches to the Teukolsky equation. We find that for gravity, the multipole moment based dynamical mass function is consistent with the combination of spin-exponentiation and the Teukolsky equation at $\mathcal{O}(S^5)$ and that requiring these fixes all available coefficients in the dynamical mass function at this order.

3.1.3 Notation

We call the spacetime manifold $\mathbb{T}$. Beginning alphabet Greek letter indices $\alpha, \beta, \gamma, \delta, \ldots$ are used for spacetime indices at a generic event $X \in \mathbb{T}$. The tangent space to $\mathbb{T}$ at X is written $T_X(\mathbb{T})$. Late alphabet Greek letter indices $\mu, \nu, \rho, \kappa, \ldots$ are used for spacetime indices at a particular event of interest $Z \in \mathbb{T}$. Indices are symmetrized using parentheses and antisymmetrized with brackets, both with the typical symmetry factors:

$$M^{(\alpha\beta)} = \frac{M^{\alpha\beta} + M^{\beta\alpha}}{2}, \qquad M^{[\alpha\beta]} = \frac{M^{\alpha\beta} - M^{\beta\alpha}}{2}. \tag{3.1.1}$$

For a generic vector v^α, we define:

$$|v| = \sqrt{|g_{\alpha\beta} v^\alpha v^\beta|}, \qquad \hat{v}^\alpha = \frac{v^\alpha}{|v|}. \tag{3.1.2}$$

Let $\zeta(s, Z, v)$ be a geodesic with affine parameter s so that $\zeta^\mu(0, Z, v) = z^\mu$ and $\frac{d\zeta^\mu}{ds}(0, Z, v) = v^\mu$. Then, the exponential map is defined by the event:

$$\exp_Z(v) = \zeta(1, Z, v). \tag{3.1.3}$$

Consider the geodesic $\zeta(s)$ with affine parameter s so that $\zeta^\mu(0) = z^\mu$ and $\zeta^\alpha(1) = x^\alpha$ (for Z and X sufficiently close for one only such geodesic to exist). Then, Synge's worldfunction $\sigma(Z, X)$ is defined by:

$$\sigma(Z, X) = \frac{1}{2} \int_0^1 g_{\alpha\beta}(\zeta) \frac{d\zeta^\alpha}{ds} \frac{d\zeta^\beta}{ds} ds. \tag{3.1.4}$$

Instead viewing σ as a functional or the path, under variation with respect to the path ζ, we find:

$$\delta\sigma = \frac{d\zeta_\alpha}{ds}(1)\delta x^\alpha - \frac{d\zeta_\mu}{ds}(0)\delta z^\mu. \tag{3.1.5}$$

We place indices on σ to indicate covariant derivatives with $\alpha, \beta, ...$ indices for x and $\mu, \nu, ...$ indices for z:

$$\sigma_\alpha = \nabla_\alpha \sigma = \frac{\partial\sigma}{\partial x^\alpha} = \frac{d\zeta_\alpha}{ds}(1), \qquad \sigma_\mu = \nabla_\mu \sigma = \frac{\partial\sigma}{\partial z^\mu} = -\frac{d\zeta_\mu}{ds}(0). \tag{3.1.6}$$

If more that two indices of the same type were to be placed on σ the order would be important due to the noncommutativity of covariant derivatives, however we will have no need to do this. As well, we introduce the inverse matrix σ^{-1}:

$$\sigma^{-\alpha}{}_\mu \sigma^\mu{}_\beta = \delta^\alpha_\beta. \tag{3.1.7}$$

We will describe the motion of the spinning body so that the worldline $z^\mu(\lambda)$ tracks the center of momentum of the body with λ the worldline time parameter. $u^\mu(\lambda)$ will be a smooth one parameter family of future oriented timelike unit vectors which later will be specialized to be $\hat{p}^\mu(\lambda)$, the unit vector in the direction of the linear momentum $p^\mu(\lambda)$ of the body. We let $\Sigma(\lambda)$ be the Cauchy slice formed by shooting out geodesics based at $z^\mu(\lambda)$ which are orthogonal to $u^\mu(\lambda)$. Explicitly:

$$\Sigma(\lambda) = \{X \in \mathbb{T} : u^\mu(\lambda)\sigma_\mu(Z(\lambda), X) = 0\}. \tag{3.1.8}$$

Let $\tau(X)$ be the value of λ so that $X \in \Sigma(\lambda)$. Let $w_1^\alpha(X)$ be any vector field satisfying:

$$\lambda = \tau(X) \implies \lambda + \delta\lambda = \tau(\exp_X(w_1 \delta\lambda)) + \mathcal{O}(\delta\lambda^2). \tag{3.1.9}$$

That is, if each point of $\Sigma(\lambda)$ is displaced by $w_1^\alpha \delta\lambda$ then it produces a point in $\Sigma(\lambda + \delta\lambda)$ for sufficiently small $\delta\lambda$. Automatically:

$$w_1^\alpha \nabla_\alpha \tau = 1. \tag{3.1.10}$$

Let D^4x be the invariant spacetime volume measure:

$$D^4x = \sqrt{-\det g}\, d^4x. \tag{3.1.11}$$

Let $d\Sigma_\alpha$ be the future oriented invariant volume measure on Σ. For any scalar function $f(X)$ then:

$$\int_{\mathbb{T}} f(X) D^4x = \int_{-\infty}^{\infty} \int_{\Sigma(\lambda)} f(X) w_1^\alpha d\Sigma_\alpha d\lambda. \tag{3.1.12}$$

Because λ is an arbitrary parameter for the worldline, it is useful to introduce the einbein $\mathsf{e}(\lambda)$ to help manage reparameterization invariance manifestly. The einbein is an arbitrary function defined so that under a smooth monotone increasing reparameterization $\lambda' = \lambda'(\lambda)$:

$$\mathsf{e}'(\lambda') = \frac{d\lambda}{d\lambda'} \mathsf{e}(\lambda). \tag{3.1.13}$$

The reparameterization invariant worldline measure $D\lambda$ is then defined by:

$$D\lambda = \mathsf{e}(\lambda) d\lambda. \tag{3.1.14}$$

We also define the reparameterization invariant version of the vector field $w_1^\alpha(X)$:

$$w^\alpha(X) = \frac{w_1^\alpha(X)}{\mathsf{e}(\tau(X))}. \tag{3.1.15}$$

Then, we have:

$$\int_{\mathbb{T}} f(X) d\mathbb{T} = \int_{-\infty}^{\infty} \int_{\Sigma(\lambda)} f(X) w^\alpha d\Sigma_\alpha D\lambda. \tag{3.1.16}$$

Selection of a worldline parameter then amounts to choosing $\mathsf{e}(\lambda)$ as an arbitrary function.

3.2 Electromagnetic MPD Equations

We begin with an analysis of the $\sqrt{\text{Kerr}}$ electromagnetic particle in flat spacetime to develop a road-map for the gravitational analysis which follows. The motion of a generic spinning body under the influence of electromagnetism in Minkowski space is described by the electromagnetic flat space Mathisson Papapetrou Dixon (MPD) equations [260]. It is well established [260, 252, 264, 58, 226, 261] that the electromagnetic MPD equations can be derived from a variational principle through an action $\mathcal{S}$ of the form:

$$\mathcal{S}[z, p, \Lambda, S, \alpha, \beta] = \int_{-\infty}^{\infty} \left(p_\mu \dot{z}^\mu + q A_\mu \dot{z}^\mu + \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^\mu S^\nu \Omega^{\rho\sigma} - \frac{\alpha}{2}(p^2 + \mathcal{M}^2) + \beta p \cdot S \right) d\lambda. \tag{3.2.1}$$

In this action, $z^\mu(\lambda)$ is the center of momentum worldline of the body, its conjugate is the linear momentum carried by the body $p_\mu(\lambda)$, q is the charge of the body, A_μ is the vector potential, $u^\mu = \hat{p}^\mu$, $\alpha = \frac{\mathsf{e}}{\mathcal{M}}$ is a Lagrange multiplier which reinforces reparameterization invariance, and S^μ is the spin vector of the body. The spin vector is defined so that it becomes the angular momentum vector of the body in its center of momentum frame. Automatically, then $S \cdot p = 0$ and so β is a Lagrange multiplier included to enforce this constraint. Also appearing is $\Lambda^\mu{}_A(\lambda)$, a tetrad tracking the orientation of the body, which satisfies:

$$\eta^{\mu\nu} = \Lambda^\mu{}_A \Lambda^\nu{}_B \eta^{AB}, \qquad \eta_{AB} = \eta_{\mu\nu}\Lambda^\mu{}_A\Lambda^\nu{}_B. \tag{3.2.2}$$

Without loss of generality, we take:

$$\Lambda^\mu{}_0(\lambda) = u^\mu(\lambda), \qquad \Lambda^\mu{}_3(\lambda) = \hat{S}^\mu(\lambda). \tag{3.2.3}$$

(If these are set as initial conditions, they are maintained dynamically automatically.) Capital Latin indices $A, B, C, D, ...$ are always used for Lorentz indices in the default frame of the body so that $\Lambda^\mu{}_A$ represents the Lorentz transformation the body has undergone in its motion relative to an arbitrary default orientation. The angular velocity tensor of the body is defined by:

$$\Omega^{\mu\nu} = \eta^{AB}\Lambda^\mu{}_A \frac{d\Lambda^\nu{}_B}{d\lambda}. \tag{3.2.4}$$

Finally, $\mathcal{M}(z, u, S)$, called the dynamical mass function of the body, encodes the free mass of the body and all of its nonminimal couplings to electromagnetism. In particular, it takes the form:

$$\mathcal{M}^2(z, u, S) = m^2 + \mathcal{O}(q\mathcal{F}) \tag{3.2.5}$$

where m is the mass of the body in vacuum.

For variations of the action it is useful to define the antisymmetric tensor:

$$\delta\theta^{\mu\nu} = \eta^{AB}\Lambda^\mu{}_A \delta\Lambda^\nu{}_B. \tag{3.2.6}$$

Then, the variation of the above action gives:

$$\begin{aligned}
\delta\mathcal{S} = \int_{-\infty}^{\infty} \Bigg(&\delta z^\mu \left(-\dot{p}_\mu + q\mathcal{F}_{\mu\nu}\dot{z}^\nu - \mathsf{e}\frac{\partial\mathcal{M}}{\partial z^\mu}\right) \\
&+\delta p_\mu \left(\dot{z}^\mu - \mathsf{e}u^\mu - \frac{\mathsf{e}}{|p|}\frac{\partial\mathcal{M}}{\partial u^\nu}(\eta^{\mu\nu} + u^\mu u^\nu) + \beta S^\mu + \frac{1}{2|p|}(\delta^\mu_\alpha + u^\mu u_\alpha)\epsilon^{\alpha\nu\rho\sigma}S_\nu\Omega_{\rho\sigma}\right) \\
&+\frac{1}{2}\delta\theta^{\rho\sigma}\left(-\frac{d}{d\lambda}\left(\epsilon_{\mu\nu\rho\sigma}u^\mu S^\nu\right) + \epsilon_{\mu\nu\rho\alpha}u^\mu S^\nu \Omega^\alpha{}_\sigma - \epsilon_{\mu\nu\sigma\alpha}u^\mu S^\nu\Omega^\alpha{}_\rho\right) \\
&+\delta S^\mu\left(-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}u^\nu\Omega^{\rho\sigma} - \mathsf{e}\frac{\partial\mathcal{M}}{\partial S^\mu} + \beta p_\mu\right) - \frac{\delta\alpha}{2}\left(p^2 + \mathcal{M}^2\right) + \delta\beta p\cdot S\Bigg)\, d\lambda.
\end{aligned} \tag{3.2.7}$$

Using the δS^μ variation to solve for the angular velocity tensor, one can then determine the value of β. That value of β can then be used to simplify the spin and trajectory equations of motion. Explicitly, these give:

$$\Omega^{\mu\nu} = \dot{u}^\mu u^\nu - u^\mu \dot{u}^\nu + \mathsf{e}\epsilon^{\mu\nu\rho\sigma}u_\rho\frac{\partial\mathcal{M}}{\partial S^\sigma} \tag{3.2.8}$$

$$\beta = -\frac{\mathsf{e}}{\mathcal{M}}u^\mu\frac{\partial\mathcal{M}}{\partial S^\mu} \tag{3.2.9}$$

$$\dot{S}^\mu = u^\mu \dot{u}\cdot S + \mathsf{e}\epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\frac{\partial\mathcal{M}}{\partial S^\sigma} \tag{3.2.10}$$

$$\dot{z}^\mu = \mathsf{e}u^\mu + \frac{\mathsf{e}}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u_\mu} + \frac{\mathsf{e}}{\mathcal{M}}u^\mu u^\nu\frac{\partial\mathcal{M}}{\partial u^\nu} + \frac{\mathsf{e}}{\mathcal{M}}S^\mu u^\nu\frac{\partial\mathcal{M}}{\partial S^\nu} + \frac{1}{\mathcal{M}^2}\epsilon^{\mu\nu\rho\sigma}S_\nu u_\rho\dot{p}_\sigma. \tag{3.2.11}$$

In order to determine the trajectory evolution explicitly we must insert the momentum equation of motion into (3.2.11). To simplify, it is useful to introduce the dual field strength:

$$ {}^\star\mathcal{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\rho\sigma} \implies \mathcal{F}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}{}^\star\mathcal{F}^{\rho\sigma}. \tag{3.2.12} $$

Simplifying finally gives the electromagnetic MPD equations of motion for the spinning body:

$$ \begin{aligned} \left(1 - \frac{q}{\mathcal{M}^2}{}^\star\mathcal{F}^{\alpha\beta}u_\alpha S_\beta\right)\frac{\dot{z}^\mu}{\mathsf{e}} &= u^\mu + \frac{1}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u_\mu} + u^\mu\frac{u^\nu}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial u^\nu} + S^\mu\frac{u^\nu}{\mathcal{M}}\frac{\partial\mathcal{M}}{\partial S^\nu} + \frac{1}{\mathcal{M}^2}\epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\frac{\partial\mathcal{M}}{\partial z^\sigma} \\ &\quad + \frac{q}{\mathcal{M}^2}{}^\star\mathcal{F}^{\mu\nu}S_\nu + \frac{q}{\mathcal{M}^3}\left(S^\rho\frac{\partial\mathcal{M}}{\partial u^\rho} + S^2u^\rho\frac{\partial\mathcal{M}}{\partial S^\rho}\right){}^\star\mathcal{F}^{\mu\nu}u_\nu \end{aligned} \tag{3.2.13} $$

$$ \dot{p}_\mu = q\mathcal{F}_{\mu\nu}\dot{z}^\nu - \mathsf{e}\frac{\partial\mathcal{M}}{\partial z^\mu} \tag{3.2.14} $$

$$ \dot{S}^\mu = u^\mu\dot{u}\cdot S + \mathsf{e}\epsilon^{\mu\nu\rho\sigma}u_\nu S_\rho\frac{\partial\mathcal{M}}{\partial S^\sigma}. \tag{3.2.15} $$

For solving these equations of motion we will always choose λ so that $\mathsf{e} = 1$.

To understand how the dynamical mass function relates to the multipole moments of the body, we will need to study the current produced by our action. Define the $\mathcal{Q}_n$ moments:

$$ \mathcal{Q}_n^{\rho_1\ldots\rho_n\mu\nu} = \mathcal{Q}_n^{(\rho_1\ldots\rho_n)[\mu\nu]} = \frac{\partial\mathcal{M}}{\partial\partial^n_{\rho_1\ldots\rho_n}\mathcal{F}_{\mu\nu}} \tag{3.2.16} $$

Then, our action produces a formal distributional expression for J^μ:

$$ J^\mu(X) = \frac{\delta\mathcal{S}}{\partial A_\mu} = \int_{-\infty}^{\infty}\left(q\dot{z}^\mu\delta(X-Z) - 2\mathsf{e}\sum_{n=0}^{\infty}(-1)^n\mathcal{Q}_n^{\rho_1\ldots\rho_n\mu\nu}\partial^{n+1}_{\rho_1\ldots\rho_n\nu}\delta(X-Z)\right)d\lambda \tag{3.2.17} $$

3.3 Dixon's Multipole Moments

In this section we summarize some of the ingredients and results of Dixon's definition of multipole moments [259]. The multipole moments of the current density (in the case of electromagnetism) and of the energy-momentum tensor (in the case of gravity) directly enter the equations of motion of the body and can be computed from the stationary fields produced by the body when isolated. We find that these multipole moments determine all linear in $\mathcal{F}$ (for electromagnetism) or linear in R (for gravity) operators in the dynamical mass function. The necessary matching is similar to the three-point amplitude matching performed in Ref. [58] in the worldline or in Ref. [59] in the field theory. However, using Dixon's multipole construction we are able to extract more physical information from the stationary $\sqrt{\text{Kerr}}$ or Kerr solutions than is contained in the three-point amplitude, which allows the determination of an increased number of Wilson coefficients.

Following Dixon's discussion, we first consider how multipole moments are defined for a generic scalar field, then a generic tensor field. Then, we see how the general multipole moments of a vector field are constrained in complicated ways if that vector field satisfies the continuity equation. This leads to Dixon's definition of the reduced multipole moments of a conserved vector field which we apply to the current density. In this section, in all but the final subsection, we keep the spacetime generic in anticipation of applying our analysis to gravity. In the final subsection, we specialize to Minkowski space in preparation for electromagnetic calculations.

3.3.1 Moments of a scalar field

Using the exponential map we can take functions on $\mathbb{T}$ and turn them into functions on $T_z(\mathbb{T})$. For a scalar field $\phi(X)$ ($X \in \mathbb{T}$) we may define the function $\phi'(Z,v)$ ($v \in T_Z(\mathbb{T})$) by:

$$\phi'(Z,v) = \phi(\exp_Z(v)). \tag{3.3.1}$$

Because $\phi'(Z,v)$ is a function on the flat tangent space $T_Z(\mathbb{T})$, it is simple to define the Fourier transform $\widetilde{\phi}(Z,k)$ and inverse Fourier transform:

$$\widetilde{\phi}(Z,k) = \int e^{-ik\cdot v}\phi'(Z,v)\frac{D^4v}{(2\pi)^2}, \qquad \phi'(Z,v) = \int e^{ik\cdot v}\widetilde{\phi}(Z,k)\frac{D^4k}{(2\pi)^2} \tag{3.3.2}$$

where the invariant tangent space measure and Fourier space measure are defined as:

$$D^4v = \sqrt{-\det g(Z)}d^4v, \qquad D^4k = \frac{d^4k}{\sqrt{-\det g(Z)}}. \tag{3.3.3}$$

Using that $v^\mu = -\sigma^\mu(Z,X)$ we can express the inverse Fourier transform for our original scalar function as:

$$\phi(X) = \int e^{-ik_\mu\sigma^\mu(Z,X)}\widetilde{\phi}(Z,k)\frac{D^4k}{(2\pi)^2} \tag{3.3.4}$$

It immediately follows that for scalar functions $\phi(X)$ and $\psi(X)$ and their associated $\phi'(Z,v), \psi'(Z,v)$:

$$\int_{\Sigma(\lambda)}\psi^*(X)\phi(X)w^\alpha d\Sigma_\alpha = \int\sum_{n=0}^{\infty}\frac{i^n}{n!}k_{\mu_1}...k_{\mu_n}\widetilde{\psi}^*(Z,k)\int_\Sigma\sigma^{\mu_1}...\sigma^{\mu_n}\phi(X)w^\alpha d\Sigma_\alpha\frac{D^4k}{(2\pi)^2}. \tag{3.3.5}$$

Define then the moments $F_n^{\mu_1\cdots\mu_n}$ and the moment generating function F associated to the scalar function ϕ:

$$F_n^{\mu_1\cdots\mu_n}[\phi](\lambda) = \int_\Sigma(-\sigma^{\mu_1})...(-\sigma^{\mu_n})\phi(X)w^\alpha d\Sigma_\alpha \tag{3.3.6}$$

$$F[\phi](\lambda,k) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}F_n^{\mu_1\cdots\mu_n}k_{\mu_1}...k_{\mu_n}. \tag{3.3.7}$$

Automatically the moments of ϕ satisfy the conditions:

$$F_n^{\mu_1\cdots\mu_n} = F_n^{(\mu_1\cdots\mu_n)}, \qquad u_{\mu_1}F_n^{\mu_1\cdots\mu_n} = 0 \tag{3.3.8}$$

where for this last condition it is useful to remember equation (3.1.8). The moment generating function determines how $\phi(X)$ behaves against test functions and thus determines $\phi(X)$ completely according to:

$$\int_{\Sigma(\lambda)}\psi^*(X)\phi(X)w^\alpha d\Sigma_\alpha = \int\widetilde{\psi}^*(Z,k)F(\lambda,k)\frac{D^4k}{(2\pi)^2}. \tag{3.3.9}$$

3.3.2 Moments of a general tensor field

Suppose we now consider a tensor field $\phi^{\alpha_1...\alpha_m}{}_{\beta_1...\beta_n}(X)$. Using Synge's world function and the exponential map we can translate this to a tensor function on the tangent space at z by:

$$\phi'^{\mu_1...\mu_m}{}_{\nu_1...\nu_n}(Z,v) = (-\sigma^{\mu_1}{}_{\alpha_1})...(-\sigma^{\mu_m}{}_{\alpha_m})(-\sigma^{-\beta_1}{}_{\nu_1})...(-\sigma^{-\beta_n}{}_{\nu_n})\phi^{\alpha_1...\alpha_m}{}_{\beta_1...\beta_n}(\exp_Z(v)). \tag{3.3.10}$$

The Fourier transform and its inverse may now be defined as:

$$\widetilde{\phi}^{\mu_1...\mu_m}{}_{\nu_1...\nu_n}(Z,k) = \int \frac{e^{-ik\cdot v}}{(2\pi)^2}\phi'^{\mu_1...\mu_m}{}_{\nu_1...\nu_n}(Z,v)D^4v \tag{3.3.11}$$

$$\phi'^{\mu_1...\mu_m}{}_{\nu_1...\nu_n}(Z,v) = \int \frac{e^{ik\cdot v}}{(2\pi)^2}\widetilde{\phi}^{\mu_1...\mu_m}{}_{\nu_1...\nu_n}(Z,k)D^4k \tag{3.3.12}$$

For a given ϕ, suppose we define the lower rank tensor φ by contracting two indices:

$$\varphi^{\alpha_1...\alpha_{m-1}}{}_{\beta_1...\beta_{n-1}} = \phi^{\alpha_1...\alpha_{m-1}\alpha_m}{}_{\beta_1...\beta_{n-1}\alpha_m} \implies \varphi'^{\mu_1...\mu_{m-1}}{}_{\nu_1...\nu_{n-1}} = \phi'^{\mu_1...\mu_{m-1}\mu_m}{}_{\nu_1...\nu_{n-1}\mu_m}. \tag{3.3.13}$$

This property only holds because the upper index bitensor propagator in (3.3.10) $(-\sigma^{\mu}{}_{\alpha})$ is the matrix inverse of the lower index bitensor propagator $(-\sigma^{-\beta}{}_{\nu})$. Alternatively, suppose we consider the moments of a vector field ϕ_β which is itself the gradient of a scalar field φ:

$$\phi_\beta = \nabla_\beta\varphi \implies \widetilde{\phi}_\nu = ik_\nu\widetilde{\varphi}. \tag{3.3.14}$$

This property only holds if the lower index bitensor propagator is $-\sigma^{-\beta}{}_{\nu}$. Therefore while one could have imagined other bitensor propagators to use for transporting the components of ϕ, such as parallel transport, the choice in (3.3.10) is unique in satisfying both (3.3.13) and (3.3.14). Unfortunately, the expression given in (3.3.10) has the property that translation to a tensor function on tangent space does not commute with raising/lowering indices of the original tensor field. This causes no real inconvenience for our calculations but one should be aware that $\phi'^{\mu} \neq g^{\mu\nu}(Z)\phi'_\nu$ if ϕ'^{μ} is defined as above from ϕ^μ and ϕ'_μ is defined as above from ϕ_μ.

For a generic ϕ, the moments and moment generating function of ϕ are defined by:

$$F_N^{\mu_1...\mu_N\rho_1...\rho_m}{}_{\nu_1...\nu_n}[\phi](\lambda) = \int_\Sigma (-\sigma^{\mu_1})....(-\sigma^{\mu_N})\phi'^{\rho_1...\rho_m}{}_{\nu_1...\nu_n}w^\alpha d\Sigma_\alpha \tag{3.3.15}$$

$$F^{\rho_1...\rho_m}{}_{\nu_1...\nu_n}[\phi](\lambda,k) = \sum_{N=0}^{\infty}\frac{(-i)^N}{N!}F_N^{\mu_1...\mu_N\rho_1...\rho_m}{}_{\nu_1...\nu_n}k_{\mu_1}...k_{\mu_N}. \tag{3.3.16}$$

Automatically the moments of ϕ satisfy the conditions:

$$F_N^{\mu_1...\mu_N\rho_1...\rho_m}{}_{\nu_1...\nu_n} = F_N^{(\mu_1...\mu_N)\rho_1...\rho_m}{}_{\nu_1...\nu_n}, \qquad u_{\mu_1}F_N^{\mu_1...\mu_N\rho_1...\rho_m}{}_{\nu_1...\nu_n} = 0. \tag{3.3.17}$$

As well, ϕ's behavior against test functions is determined by:

$$\int_\Sigma \psi^{*\beta_1...\beta_n}{}_{\alpha_1...\alpha_m}\phi^{\alpha_1...\alpha_m}{}_{\beta_1...\beta_n}w^\gamma d\Sigma_\gamma = \int \widetilde{\psi}^{*\nu_1...\nu_n}{}_{\mu_1...\mu_m}F^{\mu_1...\mu_m}{}_{\nu_1...\nu_n}(\lambda,k)\frac{D^4k}{(2\pi)^2}. \tag{3.3.18}$$

Thus, just as in the scalar case the moment generating function determines the original tensor field.

3.3.3 Moments of a conserved vector field

If we consider a vector field $\phi^\alpha(X)$ with moments as defined above then a brief calculation reveals:

$$\int_\mathbb{T} \psi^*\nabla_\alpha\phi^\alpha d\mathbb{T} = -\int_{-\infty}^{\infty}\int \widetilde{\psi}^*(z,k)\sum_{n=1}^{\infty}\frac{(-i)^n}{(n-1)!}k_{\mu_1}...k_{\mu_n}F_{n-1}^{(\mu_1...\mu_n)}\frac{D^4k}{(2\pi)^2}D\lambda \tag{3.3.19}$$

If ϕ^α is a conserved vector field so that $\nabla_\alpha\phi^\alpha = 0$, then the left hand side is 0 for all ψ. One may then wish based on this to conclude that $F_{n-1}^{(\mu_1...\mu_n)}$ is 0 for each $n \geq 1$. However, this is not a valid deduction. $\widetilde{\psi}^*(Z,k)$ at each fixed λ determines the function $\psi(X)$ through the inverse Fourier transform and so $\widetilde{\psi}^*(Z(\lambda_1),k)$ and $\widetilde{\psi}^*(Z(\lambda_2),k)$ for $\lambda_1 \neq \lambda_2$ are not independent of each other. Because of this, one cannot conclude that the integrand above at each λ must be individually 0 as they may conspire to cancel at different λ for arbitrary $\psi^*(X)$. Consequently, it is difficult to conclude anything explicit about the moments of ϕ^α from the condition $\nabla_\alpha\phi^\alpha = 0$.

Because of the mentioned difficulty, Dixon helpfully introduced an alternate set of reduced moments for a conserved vector field. For us the vector field of interest will always be J^α. The goal of this reduced set of moments is precisely to produce a set which does not have the entanglements of the naive moments for a conserved vector field. Due to the absence of entanglements between the moments, when using Dixon's moments the it is valid to equate integrands moment-by-moment. To arrive at Dixon's reduced multipole moments, which for a conserved vector field are written $m_n^{\lambda_1...\lambda_n\mu}$, we first need to introduce a few building blocks. Define:

$$\Theta_n^{\kappa\lambda}(Z,X) = (n+1)\int_0^1 \sigma^\kappa{}_\alpha(Z,\zeta(t))\sigma^{\alpha\lambda}(Z,\zeta(t))t^n dt \tag{3.3.20}$$

$$\mathfrak{q}_n^{\lambda_1...\lambda_n\mu\nu} = (-1)^n\int_\Sigma \sigma^{\lambda_1}...\sigma^{\lambda_n}\Theta_{n-1}^{\mu\nu}J^\alpha d\Sigma_\alpha \qquad (n\geq 1) \tag{3.3.21}$$

$$\mathfrak{j}_n^{\lambda_1...\lambda_n\mu} = (-1)^n\int_\Sigma \sigma^{\lambda_1}...\sigma^{\lambda_n}\sigma^\mu{}_\alpha J^\alpha w^\beta d\Sigma_\beta \tag{3.3.22}$$

$$Q_n^{\lambda_1...\lambda_n\mu\nu} = \mathfrak{j}_{n+1}^{\lambda_1...\lambda_n[\mu\nu]} + \frac{1}{n+1}\mathfrak{q}_{n+1}^{\lambda_1...\lambda_n[\mu\nu]\kappa}\frac{\dot{z}^\kappa}{\mathsf{e}} \tag{3.3.23}$$

$$m_n^{\lambda_1...\lambda_n\mu} = \frac{2n}{n+1}Q_{n-1}^{(\lambda_1...\lambda_n)\mu} \qquad (n\geq 1). \tag{3.3.24}$$

The $m_n^{\lambda_1...\lambda_n\mu}$ moments (called the reduced moments) will be the actual moments of interest. The other quantities defined are useful intermediate pieces for calculation. The reduced moments automatically satisfy:

$$m_n^{\lambda_1...\lambda_n\mu} = m_n^{(\lambda_1...\lambda_n)\mu}, \qquad m_n^{(\lambda_1...\lambda_n\mu)} = 0, \tag{3.3.25}$$

$$u_{\lambda_1} m_n^{\lambda_1 \dots \lambda_{n-1}[\lambda_n \mu]} = 0 \qquad (n \geq 2). \tag{3.3.26}$$

Dixon finds that beyond these conditions, the reduced moments are not restricted by the conservation of J^α and that they are independent of each other for different values of n. It is useful to define the 0^{th} moment:

$$m_0^\mu = q \frac{\dot{z}^\mu}{\mathsf{e}} \tag{3.3.27}$$

where q is the total charge of the body defined by:

$$q = \int_\Sigma J^\alpha d\Sigma_\alpha. \tag{3.3.28}$$

Due to the conservation of J^α, q is independent of λ. Dixon finds also that the reduced moments are independent of this 0^{th} moment. Then, define the reduced moment generating function:

$$M^\mu(\lambda, k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} m_n^{\lambda_1 \dots \lambda_n \mu} k_{\lambda_1} \dots k_{\lambda_n}. \tag{3.3.29}$$

Like the naive moments, the reduced moment generating functions determine the behavior of J^α against test functions. In particular, for an arbitrary vector field $A_\alpha(X)$:

$$\int_\Sigma A_\alpha^*(X) J^\alpha(X) w^\beta d\Sigma_\beta = \int \widetilde{A}_\mu^*(Z, k) M^\mu(\lambda, k) \frac{D^4 k}{(2\pi)^2}. \tag{3.3.30}$$

The moment generating function automatically satisfies:

$$M^\mu k_\mu = \frac{q}{\mathsf{e}} \dot{z} \cdot k. \tag{3.3.31}$$

This implies that for the gradient of a scalar function, using the definition of the reduced moment generating function:

$$\int_\Sigma J^\alpha \nabla_\alpha f w^\beta d\Sigma_\beta = \frac{q}{\mathsf{e}} \frac{d}{d\lambda} f(Z). \tag{3.3.32}$$

For f which decay sufficiently quickly for no boundary term to be necessary under integration by parts we immediately have:

$$0 = \int_{\mathbb{T}} f \nabla_\alpha J^\alpha d\mathbb{T} = -q \; f(Z)|_{\lambda \to -\infty}^{\lambda \to \infty} \tag{3.3.33}$$

which is true without placing any restrictions on the reduced moments of J^α beyond m_0^μ. Dixon proved [259] that these reduced moments are the unique set of moments which are independent of each other for different n, have only m_0^μ restricted by the conservation law, and satisfy the index symmetry conditions in equations (3.3.25) and (3.3.26).

Through (3.3.30), the current density is determined in terms of the reduced multipole moments. Explicitly comparing that behavior against test functions to (3.2.17) and using crucially that the reduced multipole moments are unique and contain no interdependencies, we can identify:

$$m_n^{\rho_1 \dots \rho_n \mu} = -2\, n! \mathcal{Q}_{n-1}^{(\rho_1 \dots \rho_n)\mu} \tag{3.3.34}$$

which gives the reduced multipole moments from the couplings in the action. Alternatively, this can be nicely inverted using the index symmetry conditions of both quantities to find:

$$\mathcal{Q}_n^{\rho_1 \ldots \rho_n \mu\nu} = -\frac{1}{n!}\frac{1}{n+2} m_{n+1}^{\rho_1 \ldots \rho_n [\mu\nu]}. \tag{3.3.35}$$

This allows the direct determination of the coupling of the body to the field strength in the action from its exact reduced multipole moments.

3.3.4 Moments of the current density in Minkowski space

We now specialize our calculations to Minkowski space. We can represent an arbitrary element v^μ of $T_z(\mathbb{T})$ by a vector $Y = y^A \vec{\iota}_A = y^A \Lambda^\mu{}_A \vec{e}_\mu$ where $\vec{\iota}_A$ are local Minkowskian basis vectors ($\vec{\iota}_A \cdot \vec{\iota}_B = \eta_{AB}$) and $\vec{e}_\mu$ are coordinate basis vectors. Thus:

$$v^\mu = \Lambda^\mu{}_A y^A. \tag{3.3.36}$$

Just as in section 3.2 we continue to always choose the tetrad so that $\Lambda^\mu{}_0 = u^\mu$. In flat space:

$$\sigma(Z,X) = \frac{1}{2}(x-z)^2, \qquad \sigma^\mu = -(x^\mu - z^\mu), \qquad \sigma^\mu{}_\alpha = -\delta^\mu_\alpha \implies \Theta_n^{\mu\nu} = \eta^{\mu\nu} \tag{3.3.37}$$

We always use lowercase beginning Latin alphabet indices $a, b, c, \ldots$ for values $1, 2, 3$ on the tangent space. Then, we have:

$$X \in \Sigma(\lambda) \implies \exists\, y^a \in \mathbb{R}^3 \;:\; x^\mu = z^\mu + \Lambda^\mu{}_a y^a. \tag{3.3.38}$$

Using the y^a coordinates and the definition of $\tau(X)$ we may identify explicit flat space expressions for w^α and $d\Sigma_\alpha$:

$$\lambda = \tau(z + \lambda u) \implies 1 = (\dot{z}^\alpha + \dot{\Lambda}^\alpha{}_a y^a)\nabla_\alpha \tau \tag{3.3.39}$$

$$w^\alpha = \frac{\dot{z}^\alpha + \dot{\Lambda}^\alpha{}_a y^a}{\mathsf{e}}, \qquad d\Sigma_\alpha = -u_\alpha d^3 y. \tag{3.3.40}$$

With these the $\mathfrak{q}_n$ and $\mathfrak{j}_n$ moments become:

$$\mathfrak{q}_n^{\lambda_1 \ldots \lambda_n \mu\nu} = \eta^{\mu\nu} \Lambda^{\lambda_1}{}_{A_1} \ldots \Lambda^{\lambda_n}{}_{A_n} \int_\Sigma y^{A_1} \ldots y^{A_n} (-J \cdot u) d^3 y \tag{3.3.41}$$

$$\mathfrak{j}_n^{\lambda_1 \ldots \lambda_n \mu} = \Lambda^{\lambda_1}{}_{A_1} \ldots \Lambda^{\lambda_n}{}_{A_n} \Lambda^\mu{}_B \int_\Sigma y^{A_1} \ldots y^{A_n} \Lambda_\nu{}^B J^\nu \frac{-u \cdot \dot{z} - u \cdot \dot{\Lambda} \cdot y}{\mathsf{e}} d^3 y. \tag{3.3.42}$$

Using the explicit formula for the worldline velocity $\dot{z}^\mu$ in (3.2.13), $u \cdot \dot{z} = -\mathsf{e}$. As well, due to (3.2.8), $\dot{\Lambda} = \mathcal{O}(\mathcal{F})$. Therefore:

$$\mathfrak{j}_n^{\lambda_1 \ldots \lambda_n \mu} = \Lambda^{\lambda_1}{}_{A_1} \ldots \Lambda^{\lambda_n}{}_{A_n} \Lambda^\mu{}_B \int_\Sigma y^{A_1} \ldots y^{A_n} \Lambda_\nu{}^B J^\nu d^3 y + \mathcal{O}(\mathcal{F}). \tag{3.3.43}$$

It is useful to define one last class of moments:

$$K_n^{a_1 \ldots a_n B} = \int_\Sigma y^{a_1} \ldots y^{a_n} \Lambda_\nu{}^B J^\nu d^3 y \tag{3.3.44}$$

$$\implies \mathfrak{q}_n^{\lambda_1 \ldots \lambda_n \mu\nu} = \eta^{\mu\nu} \Lambda^{\lambda_1}{}_{a_1} \ldots \Lambda^{\lambda_n}{}_{a_n} K_n^{a_1 \ldots a_n 0} \tag{3.3.45}$$

$$\implies \mathfrak{j}_n^{\lambda_1 \ldots \lambda_n \mu} = \Lambda^{\lambda_1}{}_{a_1} \ldots \Lambda^{\lambda_n}{}_{a_n} \Lambda^\mu{}_B K_n^{a_1 \ldots a_n B} + \mathcal{O}(\mathcal{F}). \tag{3.3.46}$$

For a stationary body in its center of momentum frame, the K_n moments coincide with the naive moments of the stationary current density. By following the chain of definitions from (3.3.20) to (3.3.24) these K_n moments determine the full reduced moments m_n. Therefore, computing the naive moments of a stationary body allows the determination of the (coupling independent part of the) full set of reduced moments.

3.4 Root-Kerr Multipole Moments

In this section we determine the linear in $\mathcal{F}$ couplings in the dynamical mass function to all orders in spin for a $\sqrt{\text{Kerr}}$ particle. We begin by reviewing how the vector potential which defines the $\sqrt{\text{Kerr}}$ particle arises from the Kerr-Newman solution and computing some basic mechanical properties of the $\sqrt{\text{Kerr}}$ fields. Then, we use the $\sqrt{\text{Kerr}}$ fields to determine the charge and current densities which are the source of the $\sqrt{\text{Kerr}}$ solution using analysis which parallels the calculation of the source of the Kerr metric performed in Ref. [263]. From these charge and current densities we are then able to determine the K_n moments for $\sqrt{\text{Kerr}}$ and consequently determine the m_n reduced multipole moments up to corrections of $\mathcal{O}(\mathcal{F})$. Comparing these reduced moments to those produced by the current density produced by (3.2.17) fully determines all linear in $\mathcal{F}$ terms in the dynamical mass function for a $\sqrt{\text{Kerr}}$ particle.

3.4.1 From Kerr-Newman to Root-Kerr

The Kerr-Newman solution for a stationary charged spinning black hole is defined by the line element and vector potential:

$$
\begin{aligned}
-d\tau^2 = & -\frac{r^2 - 2Gmr + a^2 + \frac{q^2 G}{4\pi}}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2\theta}{r^2 + a^2\cos^2\theta}((r^2+a^2)d\varphi - adt)^2 \\
& + (r^2 + a^2\cos^2\theta)\left(\frac{dr^2}{r^2 - 2Gmr + a^2 + \frac{q^2 G}{4\pi}} + d\theta^2\right) && (3.4.1) \\
A_\mu dx^\mu = & \ \frac{q}{4\pi}\frac{-rdt + ar\sin^2\theta d\varphi}{r^2 + a^2\cos^2\theta}. && (3.4.2)
\end{aligned}
$$

The $G \to 0$ limit of the Kerr-Newman metric produces the line element:

$$
-d\tau^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\varphi^2. \quad (3.4.3)
$$

This is just the Minkowski line element in a particular choice of coordinates. The associated spatial metric $g^{(3)}_{ab}$ implied by this line element is the natural metric of oblate-spheroidal coordinates, related to Cartesian coordinates by:

$$
x = \sqrt{r^2 + a^2}\sin\theta\cos\varphi, \qquad y = \sqrt{r^2 + a^2}\sin\theta\sin\varphi, \qquad z = r\cos\theta \quad (3.4.4)
$$

and with corresponding coordinate basis vectors:

$$\vec{e}_r = \frac{\partial \vec{x}}{\partial r}, \qquad \vec{e}_\theta = \frac{\partial \vec{x}}{\partial \theta}, \qquad \vec{e}_\varphi = \frac{\partial \vec{x}}{\partial \varphi}, \qquad g^{(3)}_{ab} = \vec{e}_a \cdot \vec{e}_b. \tag{3.4.5}$$

The vector potential of this system defines the vector potential of a stationary $\sqrt{\text{Kerr}}$ particle:

$$\phi = -A_0 = \frac{q}{4\pi} \frac{r}{r^2 + a^2 \cos^2\theta} \tag{3.4.6}$$

$$\vec{A} = g^{(3)ab} A_a \vec{e}_b = \frac{q}{4\pi} \frac{ar}{(r^2+a^2)(r^2+a^2\cos^2\theta)} \vec{e}_\varphi. \tag{3.4.7}$$

These potentials the produce the electric and magnetic fields:

$$\vec{E} = \frac{q}{4\pi} \frac{(r^2+a^2)(r^2-a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^3} \vec{e}_r - \frac{q}{4\pi} \frac{2ra^2 \sin\theta\cos\theta}{(r^2+a^2\cos^2\theta)^3} \vec{e}_\theta \tag{3.4.8}$$

$$\vec{B} = \frac{q}{4\pi} \frac{2ar(r^2+a^2)\cos\theta}{(r^2+a^2\cos^2\theta)^3} \vec{e}_r + \frac{q}{4\pi} \frac{a(r^2-a^2\cos^2\theta)\sin\theta}{(r^2+a^2\cos^2\theta)^3} \vec{e}_\theta. \tag{3.4.9}$$

Away from any singularities, this magnetic field satisfies $\nabla \times \vec{B} = 0$ and so can be expressed in terms of a magnetic scalar potential $\vec{B} = -\nabla\psi$. In general coordinates in Minkowski space, the condition that the magnetic field (away from any current sources) be given by both a magnetic scalar potential and vector potential is:

$$-\sqrt{\det g^{(3)}} g^{(3)ab} \frac{\partial \psi}{\partial x^b} = \epsilon^{abc} \frac{\partial A_c}{\partial x^b}. \tag{3.4.10}$$

Using the stationary $\sqrt{\text{Kerr}}$ vector potential, one may use this to explicitly compute the associated magnetic scalar potential. Doing so gives:

$$\psi = \frac{q}{4\pi} \frac{a\cos\theta}{r^2+a^2\cos^2\theta}. \tag{3.4.11}$$

Defining $\vec{a} = a\vec{e}_z$, in oblate-spheroidal coordinates we have the identity:

$$|\vec{x} - i\vec{a}| = r - ia\cos\theta. \tag{3.4.12}$$

Consequently, the stationary $\sqrt{\text{Kerr}}$ particle is equivalently defined [265] by producing the electric and magnetic scalar potentials or electric and magnetic fields given by:

$$\phi + i\psi = \frac{q}{4\pi|\vec{x}-i\vec{a}|} \implies \vec{E} + i\vec{B} = -\nabla\left(\frac{q}{4\pi|\vec{x}-i\vec{a}|}\right). \tag{3.4.13}$$

3.4.2 Mechanical Properties of the Stationary Root-Kerr Solution

Consider a surface of constant $r = R$ for integration. If we take $R \to \infty$ this becomes the spatial boundary. The surface area element is:

$$d\vec{\mathbb{A}} = (r^2+a^2)\sin\theta \vec{e}_r d\theta d\varphi. \tag{3.4.14}$$

Then, by Gauss' law the total charge of the $\sqrt{\text{Kerr}}$ particle is given by:

$$\int_{\mathbb{S}} \rho d^3\vec{x} = \int_{\partial\mathbb{S}} \vec{E} \cdot d\vec{\mathbb{A}} = \lim_{R\to\infty} \int_0^\pi \int_0^{2\pi} \frac{q}{4\pi} \frac{R^2 - a^2\cos^2\theta}{(R^2+a^2\cos^2\theta)^2} (R^2+a^2)\sin\theta d\varphi d\theta = q \tag{3.4.15}$$

confirming our consistent use of the symbol q for the total charge of the distribution.

The magnetic dipole moment of a charge distribution is by definition:

$$\vec{\mu} = \frac{1}{2}\int_{\mathbb{S}} \vec{x} \times \vec{J} d^3\vec{x}. \tag{3.4.16}$$

Because the distribution is stationary, rewriting $\vec{J} = \nabla \times \vec{B}$ and integrating by parts sufficiently many times this is equivalently:

$$\vec{\mu} = -\frac{3}{2}\int_{\partial\mathbb{S}} \vec{A} \times d\vec{\mathbb{A}}. \tag{3.4.17}$$

With the same surface of constant $r = R$:

$$\vec{\mu} = -\frac{3}{2}\lim_{R\to\infty}\int_0^{\pi}\int_0^{2\pi} \frac{qa}{4\pi}\frac{R}{R^2 + a^2\cos^2\theta^2}\vec{e}_{\varphi} \times \vec{e}_r \sin\theta d\varphi d\theta = q\vec{a}. \tag{3.4.18}$$

So, $\vec{a}$ is the magnetic dipole moment per unit charge.

Like for a point charge, the total energy in the electromagnetic field for this system is divergent. To regulate this instead of integrating all the way down to $r = 0$ for the total energy, we integrate down to a cutoff $r = \varepsilon$. Then, the total energy is:

$$\mathrm{E} = \int_{\mathbb{S}} \frac{|\vec{E}|^2 + |\vec{B}|^2}{2} d^3\vec{x} = \frac{q^2}{16\pi\varepsilon}\left(1 + \frac{a}{\varepsilon}\arctan\left(\frac{a}{\varepsilon}\right) + \frac{\varepsilon}{a}\arctan\left(\frac{a}{\varepsilon}\right)\right). \tag{3.4.19}$$

The Poynting vector for the particle is:

$$\vec{\wp} = \vec{E} \times \vec{B} = \frac{q^2}{16\pi^2}\frac{a\vec{e}_{\varphi}}{(r^2 + a^2\cos^2\theta)^3}. \tag{3.4.20}$$

This leads to the total linear momentum:

$$\vec{p} = \int_{\mathbb{S}} \vec{\wp} d^3\vec{x} = 0. \tag{3.4.21}$$

So, the stationary $\sqrt{\text{Kerr}}$ fields given are in fact in the center of momentum frame. Consequently, the energy found before is the invariant mass of the distribution:

$$m = \frac{q^2}{16\pi\varepsilon}\left(1 + \frac{a}{\varepsilon}\arctan\left(\frac{a}{\varepsilon}\right) + \frac{\varepsilon}{a}\arctan\left(\frac{a}{\varepsilon}\right)\right). \tag{3.4.22}$$

The Poynting vector we found leads to the total angular momentum (which because we are in the center of momentum frame is the spin angular momentum):

$$\vec{S} = \int_{\mathbb{S}} \vec{x} \times \vec{\wp} d^3\vec{x} = \frac{q^2}{16\pi\varepsilon}\left(1 + \frac{a}{\varepsilon}\arctan\left(\frac{a}{\varepsilon}\right) + \frac{2\varepsilon}{a}\arctan\left(\frac{a}{\varepsilon}\right) - \frac{\varepsilon^2}{a^2} + \frac{\varepsilon^3}{a^3}\arctan\left(\frac{a}{\varepsilon}\right)\right)\vec{a}. \tag{3.4.23}$$

While both the spin and the mass of the solution are divergent in ε, their ratio has a finite ε to 0 limit. Therefore, the physical value of the ratio of the spin to the mass is:

$$\frac{\vec{S}}{m} = \vec{a}. \tag{3.4.24}$$

Immediately it follows that for this distribution:

$$\vec{\mu} = \frac{q}{m}\vec{S}. \tag{3.4.25}$$

Thus, the $\sqrt{\text{Kerr}}$ particle has a gyromagnetic ratio of exactly 2.

3.4.3 Charge and Current density

The electric and magnetic fields as well as scalar and vector potential given diverge whenever $r^2 + a^2\cos^2\theta = 0$ and nowhere else. For $r > 0$ the potentials and their derivatives are continuous and so there are no surface charges or currents for $r > 0$. Because the divergence of the right hand side in (3.4.8) is 0 away from any poles, there is no charge density for $r > 0$. Because the curl of the right hand side in (3.4.9) is 0 away from any poles, there is no current density for $r > 0$. Consequently, the source for the $\sqrt{\text{Kerr}}$ particle must only have support for $r = 0$. For $r = 0$ the oblate spheroidal coordinates reduce to the disk in the xy plane with center at the origin and radius a. In order to approach this disk from above we must take $r \to 0$ with $\theta < \frac{\pi}{2}$ and in order to approach from below we must take $r \to 0$ with $\theta > \frac{\pi}{2}$. It is useful to define the coordinate $\chi = \theta$ for $z > 0$ and $\chi = \pi - \theta$ for $z < 0$. This way, two points approaching the disk, one from above and one from below, will limit to the same point in space when their limiting values of χ and φ coincide. The disk is then parameterized by the coordinates χ and φ by:

$$x = a\sin\chi\cos\varphi, \qquad y = a\sin\chi\sin\varphi, \qquad z = 0. \tag{3.4.26}$$

We can also use the cylindrical radial coordinate $\mathscr{r} = \sqrt{x^2+y^2} = a\sin\chi$.

The surface charge density inside the disk is given by:

$$\sigma_{\text{disk}} = \vec{e}_z \cdot (\vec{E}(0,\chi,\varphi) - \vec{E}(0,\pi-\chi,\varphi)) = -\frac{qa}{2\pi(a^2-\mathscr{r}^2)^{\frac{3}{2}}}. \tag{3.4.27}$$

We can integrate this from $\chi = 0$ to $\chi = \frac{\pi}{2} - \varepsilon$ to get the total charge inside the disk:

$$Q_{\text{disk}} = \int_{\text{disk}} \sigma_{\text{disk}} d\mathbb{A} = q\left(1 - \frac{1}{\sin\varepsilon}\right). \tag{3.4.28}$$

The surface charge density cannot be extended all the way to the ring at $\chi = \frac{\pi}{2}$ as it has a nonintegrable divergence. We know from before that the total charge of the distribution is q and so there must be a charged ring at $\chi = \frac{\pi}{2}$ so that the total charge between this ring and the disk is q. This ring is precisely the ring on which $r^2 + a^2\cos^2\theta = 0$ and so where the potentials and fields are divergent. The charge density on the ring must be azimuthally symmetric because the fields are. So:

$$q = Q_{\text{disk}} + Q_{\text{ring}} = q - \frac{q}{\sin\varepsilon} + \int_0^{2\pi} \lambda_{\text{ring}} a d\varphi \implies \lambda_{\text{ring}} = \frac{q}{2\pi a\sin\varepsilon}. \tag{3.4.29}$$

Thus the total charge density of the stationary $\sqrt{\text{Kerr}}$ particle is:

$$\rho = -\frac{qa}{2\pi(a^2-\mathscr{r}^2)^{\frac{3}{2}}}\delta(z)\vartheta(a\cos\varepsilon - \mathscr{r}) + \frac{q}{2\pi a\sin\varepsilon}\delta(z)\delta(\mathscr{r} - a). \tag{3.4.30}$$

The surface current density inside the disk is:

$$\vec{K}_{\text{disk}} = \vec{e}_z \times (\vec{B}(0,\chi,\varphi) - \vec{B}(0,\pi-\chi,\varphi)) = \sigma_{\text{disk}}\frac{r}{a}\vec{\iota}_\varphi \tag{3.4.31}$$

where $\vec{\iota}_\varphi$ is the unit vector in the direction of $\vec{e}_\varphi$. This is precisely the surface current density produced by rigidly rotating the surface charge density of the disk with angular velocity $\vec{\omega}$ given by:

$$\vec{\omega} = \frac{1}{a}\vec{e}_z. \tag{3.4.32}$$

Rotating the ring with this same angular velocity produces a current in the ring of:

$$\vec{I}_{\text{ring}} = \lambda_{\text{ring}}\vec{\iota}_\varphi = \frac{q}{2\pi a \sin\varepsilon}\vec{\iota}_\varphi. \tag{3.4.33}$$

These currents produce a magnetic dipole moment:

$$\vec{\mu} = \frac{1}{2}\int_{\text{disk}} \vec{x}\times\vec{K}_{\text{disk}} d\mathbb{A} + \frac{1}{2}\int_{\text{ring}} \vec{x}\times\vec{I}_{\text{ring}} ds = q\vec{a}\left(1 - \frac{1}{2}\sin\varepsilon\right). \tag{3.4.34}$$

In the $\varepsilon \to 0$ limit this produces exactly the magnetic dipole moment we found before. Therefore, the current density for the stationary $\sqrt{\text{Kerr}}$ distribution is:

$$\vec{J} = \rho\frac{\vec{a}\times\vec{x}}{a^2} \tag{3.4.35}$$

and the stationary $\sqrt{\text{Kerr}}$ particle is exactly a charged disk and ring with the charge distribution given by equation (3.4.30) and rotating with the angular velocity $\frac{1}{a}$ about the central axis of the disk. These charge and current densities are precisely analogous to the mass density and energy-momentum tensor found in Ref. [263] for a stationary Kerr black hole.

3.4.4 Stationary Multipole Moments

We can now compute the K_n moments explicitly for a stationary $\sqrt{\text{Kerr}}$ particle:

$$K_n^{a_1...a_n0} = \int_{\mathbb{S}} x^{a_1}...x^{a_n}\rho(\vec{x})d^3\vec{x}, \qquad K_n^{a_1...a_nb} = \int_{\mathbb{S}} x^{a_1}...x^{a_n}J^b(\vec{x})d^3\vec{x}. \tag{3.4.36}$$

By using (3.4.35), we find immediately:

$$K_n^{a_1...a_nb} = \frac{1}{a^2}\epsilon^b{}_{cd}a^c K_{n+1}^{da_1...a_n0} \tag{3.4.37}$$

and so only the multipole moments of ρ need to be computed in order to determine the moments of the full distribution. Further, due to the full symmetrization of $K_n^{a_1...a_n0}$, for an arbitrary 3-vector k_a:

$$K_n^{a_1...a_n0} = \frac{1}{n!}\frac{\partial^n}{\partial k_{a_1}...\partial k_{a_n}}\left(K_n^{b_1...b_n0}k_{b_1}...k_{b_n}\right). \tag{3.4.38}$$

The charge density of the $\sqrt{\text{Kerr}}$ particle is localized to the xy plane and is rotationally symmetric about the z axis. So, it can be written as:

$$\rho(\vec{x}) = \sigma(r)\delta(z) \tag{3.4.39}$$

where in particular for $\sqrt{\text{Kerr}}$:

$$\sigma(\mathfrak{r}) = -\frac{qa}{2\pi(a^2-\mathfrak{r}^2)^{\frac{3}{2}}}\vartheta(a\cos\varepsilon - \mathfrak{r}) + \frac{q}{2\pi a\sin\varepsilon}\delta(\mathfrak{r}-a). \tag{3.4.40}$$

Then:

$$K_n^{a_1\ldots a_n 0}k_{a_1}\ldots k_{a_n} = (k_x^2+k_y^2)^{\frac{n}{2}}\int_0^{2\pi}\cos^n\varphi d\varphi\int_0^\infty \sigma(\mathfrak{r})\mathfrak{r}^{n+1}d\mathfrak{r}. \tag{3.4.41}$$

If n is odd the azimuthal integral gives 0, so we only need to consider even n. For even n the azimuthal integral is:

$$\int_0^{2\pi}\cos^{2n}\varphi d\varphi = 2\pi\frac{(2n)!}{4^n n!^2}. \tag{3.4.42}$$

With the integration variable $x = \frac{1}{a}\sqrt{a^2-\mathfrak{r}^2}$ the radial integral becomes:

$$\int_0^\infty \sigma(\mathfrak{r})\mathfrak{r}^{2n+1}d\mathfrak{r} = \frac{qa^{2n}}{2\pi}\left(1+\int_{\sin\varepsilon}^1 \frac{1-(1-x^2)^n}{x^2}dx\right). \tag{3.4.43}$$

Here it is safe to take the $\varepsilon\to 0$ limit to give:

$$\int_0^\infty \sigma(\mathfrak{r})\mathfrak{r}^{2n+1}d\mathfrak{r} = \frac{qa^{2n}}{2\pi}\frac{4^n n!^2}{(2n)!}. \tag{3.4.44}$$

Thus the nonzero K_n moments become:

$$K_{2n}^{a_1\ldots a_{2n}0}v_{a_1}\ldots v_{a_{2n}} = q|\vec{a}\times\vec{v}|^{2n} \tag{3.4.45}$$

$$K_{2n+1}^{a_1\ldots a_{2n+1}b}v_{a_1}\ldots v_{a_{2n+1}} = q|\vec{a}\times\vec{v}|^{2n}(\vec{a}\times\vec{v})^b. \tag{3.4.46}$$

3.4.5 Dynamical Multipole Moments

Now we return the K_n moments to the definition of the reduced multipole moments. For a generic body allowed to respond to external fields, the multipole moments will in general depend on those external fields and so the general moments in external fields cannot be fully determined from the stationary moments. It is unclear at this time what is the appropriate response for a $\sqrt{\text{Kerr}}$ body to external fields. So, for an exact dynamical $\sqrt{\text{Kerr}}$ body:

$$K_n^{a_1\ldots a_n b} = K_{n\ \text{stat}}^{a_1\ldots a_n b} + \mathcal{O}(\mathcal{F}) \tag{3.4.47}$$

where $K_{n\ \text{stat}}^{a_1\ldots a_n b}$ are the K_n moments we found for a stationary $\sqrt{\text{Kerr}}$ particle. To express the reduced moments it is useful to define the projector $\perp^\mu{}_\nu$ which projects 4-vectors into the plane of the disk:

$$\perp^\mu{}_\nu = \delta^\mu_\nu + u^\mu u_\nu - \hat{a}^\mu\hat{a}_\nu. \tag{3.4.48}$$

We also write $k_\perp^\mu$ as a shorthand for $k_\perp^\mu = \perp^\mu{}_\nu k^\nu$ and $k_\perp$ without an index as a shorthand for $\sqrt{|k_\mu \perp^\mu{}_\nu k^\nu|}$. Using the definition of the reduced multipole moments for arbitrary 4-vectors k_μ and v_μ we find:

$$k_{\lambda_1}\ldots k_{\lambda_{2n}}v_\mu m_{2n}^{\lambda_1\ldots\lambda_{2n}\mu} = qa^{2n}k_\perp^{2n}u\cdot v - qa^{2n}k_\perp^{2n-2}(u\cdot k)(k_\perp\cdot v) + \mathcal{O}(\mathcal{F}) \tag{3.4.49}$$

$$k_{\lambda_1}\ldots k_{\lambda_{2n+1}}v_\mu m_{2n+1}^{\lambda_1\ldots\lambda_{2n+1}\mu} = qa^{2n}k_\perp^{2n}\epsilon_{\mu\nu\rho\sigma}u^\mu a^\nu k^\rho v^\sigma + \mathcal{O}(\mathcal{F}). \tag{3.4.50}$$

Returning these to (3.3.35), we find:

$$\mathcal{Q}_{2n}^{\rho_1\ldots\rho_{2n}\mu\nu}\partial^{2n}_{\rho_1\ldots\rho_{2n}}\mathcal{F}_{\mu\nu} = \frac{q}{(2n+1)!}a^{2n}\partial_\perp^{2n\star}\mathcal{F}_{\mu\nu}a^\mu u^\nu + \mathcal{O}(\mathcal{F}^2) \tag{3.4.51}$$

$$\mathcal{Q}_{2n+1}^{\rho_1\ldots\rho_{2n+1}\mu\nu}\partial^{2n+1}_{\rho_1\ldots\rho_{2n+1}}\mathcal{F}_{\mu\nu} = -\frac{q}{(2n+2)!}a^{2n+2}\partial_\perp^{2n}\partial_\perp^\mu\mathcal{F}_{\mu\nu}u^\nu + \mathcal{O}(\mathcal{F}^2). \tag{3.4.52}$$

With the definition of $\mathcal{Q}_n$, these produce the dynamical mass function:

$$\mathcal{M} = m + q\frac{\sinh(a\partial_\perp)}{a\partial_\perp}{}^\star\mathcal{F}_{\mu\nu}a^\mu u^\nu + q\frac{1-\cosh(a\partial_\perp)}{\partial_\perp^2}\partial_\perp^\mu\mathcal{F}_{\mu\nu}u^\nu + \mathcal{O}(\mathcal{F}^2). \tag{3.4.53}$$

There is no subtlety in defining the square root or inverse of the differential operator here because once the trigonometric functions are series expanded only positive even powers of $\partial_\perp$ survive. We will see later that for $\sqrt{\text{Kerr}}$ particles, at least at low orders in spin, the squared dynamical mass function $\mathcal{M}^2$ may be a simpler object when $\mathcal{O}(\mathcal{F})^2$ operators are considered. So, going forward we will express the dynamical mass function as:

$$\mathcal{M}^2 = m^2 + 2mq\frac{\sinh(a\partial_\perp)}{a\partial_\perp}{}^\star\mathcal{F}_{\mu\nu}a^\mu u^\nu + 2mq\frac{1-\cosh(a\partial_\perp)}{\partial_\perp^2}\partial_\perp^\mu\mathcal{F}_{\mu\nu}u^\nu + \mathcal{O}(\mathcal{F}^2). \tag{3.4.54}$$

This $\sqrt{\text{Kerr}}$ dynamical mass function is our primary result for electromagnetism. When acting on a field strength which in the neighborhood of the body is a vacuum solution ($\partial_\nu\mathcal{F}^{\mu\nu}=0$) of Maxwell's equations, this reduces to:

$$\mathcal{M}^2 = m^2 + 2mq\frac{\sin(a\Delta)}{a\Delta}{}^\star\mathcal{F}_{\mu\nu}a^\mu u^\nu + 2mq\frac{1-\cos(a\Delta)}{(a\Delta)^2}(a\cdot\partial)\mathcal{F}_{\mu\nu}a^\mu u^\nu + \mathcal{O}(\mathcal{F}^2) \tag{3.4.55}$$

where the Δ differential operator is defined by:

$$a\Delta = \sqrt{(a\cdot\partial)^2 - a^2(u\cdot\partial)^2}. \tag{3.4.56}$$

Again there is no subtlety in defining the square root or inverse of the differential operator. In general (3.4.55) and (3.4.54) are not equivalent, however we show that for computing Compton amplitudes they are interchangeable.

The dynamical mass function in (3.4.55) very nearly matches the couplings in Ref. [145] (if the analysis done there for gravity is done for electromagnetism). In particular, phrased as a dynamical mass function, the couplings in Ref. [145] determine:

$$\mathcal{M}^2 = m^2 + 2mq\frac{\sin(a\cdot\partial)}{a\cdot\partial}{}^\star\mathcal{F}_{\mu\nu}a^\mu u^\nu + 2mq\frac{1-\cos(a\cdot\partial)}{a\cdot\partial}\mathcal{F}_{\mu\nu}a^\mu u^\nu + \mathcal{O}(\mathcal{F}^2) \tag{3.4.57}$$

Equation (3.4.55) becomes this result with the replacement:

$$a\Delta \to a\cdot\partial. \tag{3.4.58}$$

The $(u\cdot\partial)$ terms present in $a\Delta$ do not contribute to the three point amplitude and so $a\Delta$ and $a\cdot\partial$ are indistinguishable in a three point matching calculation. For this reason, the analyses of Refs. [58, 59] were insensitive to the couplings on terms of the form $S^{2n}(u\cdot\partial)^{2n}\mathcal{F}_{\ldots}$ The first of such terms is present in Ref. [79]

with an undetermined Wilson coefficient. Similarly, in Ref. [145] $(u \cdot \partial)$ are not considered because if one uses the equations of motion in the action, they can be shuffled into order $\mathcal{F}^2$ terms. One can see this quickly on the lowest order such term, for example:

$$\mathsf{e}a^2(u\cdot\partial)^{2\star}\mathcal{F}_{\mu\nu}a^\mu u^\nu = a^2 a^\mu u^\nu u^\rho \dot{z}^\sigma \partial_\sigma \partial_\rho {}^\star\mathcal{F}_{\mu\nu} + \mathcal{O}(\mathcal{F}^2) \tag{3.4.59}$$

$$= \frac{d}{d\lambda}\left(a^2 a^\mu u^\nu u^\rho \partial_\rho {}^\star\mathcal{F}_{\mu\nu}\right) - \frac{d}{d\lambda}(a^2 a^\mu u^\nu u^\rho)\partial_\rho {}^\star\mathcal{F}_{\mu\nu} + \mathcal{O}(\mathcal{F}^2) \tag{3.4.60}$$

$$= \frac{d}{d\lambda}\left(a^2 a^\mu u^\nu u^\rho \partial_\rho {}^\star\mathcal{F}_{\mu\nu}\right) + \mathcal{O}(\mathcal{F}^2). \tag{3.4.61}$$

For three-point amplitudes, (3.4.54), (3.4.55), and (3.4.57) are interchangeable. For Compton amplitudes, (3.4.54) and (3.4.55) are interchangeable but distinct from (3.4.57). The worldline evolution of all three are distinct. Importantly, by the definition of Dixon's moments (3.3.30) holds for any test function vector field $A_\alpha(X)$, not only the physically relevant vector potential and that it holds without using the solution to the equations of motion. We have shown that for a $\sqrt{\text{Kerr}}$ particle the only dynamical mass function for which (3.3.30) holds without using the electromagnetic MPD equations is (3.4.54). If one shuffles away $(u \cdot \partial)$ operators in favor of $\mathcal{F}^2$ so that the linear in $\mathcal{F}$ dynamical mass function is given by (3.4.57), then (3.3.30) will not be true identically. In this sense, the advantage of (3.4.54) is that it uniquely provides at each λ the physically correct multipole moments for $\sqrt{\text{Kerr}}$.

3.5 Electromagnetic Compton Amplitude

In this section we formally compute the $\sqrt{\text{Kerr}}$ Compton amplitude to all orders in spin up to contact terms. Those contact terms are determined by $\mathcal{F}^2$ operators in the dynamical mass function which the multipole analysis is insensitive to. We begin by computing a formal expression for the all orders in spin Compton amplitude for a generic charged spinning body in terms of the dynamical mass function. Next we consider that generic Compton amplitude explicitly to order S^3. At $\mathcal{O}(S^1)$ there is a single Wilson coefficient in the action and it can be determined by matching the $\mathcal{O}(S^1)$ three-point amplitude or Compton amplitude. At $\mathcal{O}(S^2)$ there are 5 new Wilson coefficients in the action. One of them can be determined by matching the $\mathcal{O}(S^2)$ three-point amplitude while the other 4 can be determined uniquely by matching the $\mathcal{O}(S^2)$ Compton amplitude. At $\mathcal{O}(S^3)$ there are 8 new Wilson coefficients in the action. One of them can be determined by matching the $\mathcal{O}(S^3)$ three-point amplitude while the other 7 appear only as contact terms in the Compton amplitude. Among those 7, there are only 6 linearly independent structures appearing in the $\mathcal{O}(S^3)$ Compton amplitude, and so there is one linear combination of Wilson coefficients that the Compton amplitude is independent of at this order in spin.

Once we have the results for a generic body through $\mathcal{O}(S^3)$, we specialize our interest to $\sqrt{\text{Kerr}}$. Requiring the exponentiation of spin structure as found in Ref. [53] through $\mathcal{O}(S^2)$ for the helicity-preserving amplitude (which develops a spurious pole starting at $\mathcal{O}(S^3)$) and through $\mathcal{O}(S^3)$ for the helicity-reversing amplitude (which has no such spurious pole) fixes all Wilson coefficients through $\mathcal{O}(S^2)$ and 4 of the 8 new operators at $\mathcal{O}(S^3)$. Alternatively, requiring the helicity-preserving amplitude to have the shift-symmetry described in Refs. [173, 101, 179, 180] through $\mathcal{O}(S^3)$ fixes 3 of the 8 new operators at $\mathcal{O}(S^3)$. The spin-exponentiation and shift-symmetry are consistent with each other, and together fix 6 of the 8 new operators at $\mathcal{O}(S^3)$ (as they share one redundant condition). The dynamical mass function (3.4.55) determines 2 Wilson coefficients at $\mathcal{O}(S^3)$, one of which is fixed by the three-point amplitude and the other of which is independent of and consistent with both spin-exponentiation and shift-symmetry. Together then Dixon's multipole moments, spin-exponentiation, and shift-symmetry fix 7 of the 8 Wilson coefficients at $\mathcal{O}(S^3)$, which is the maximum amount possible by using the Compton amplitude (due to the presence of a linearly independent combination of Wilson coefficients that the amplitude is independent of).

3.5.1 Formal Classical Compton

We consider Compton scattering of an incoming photon with polarization $\mathcal{E}_1$ and momentum k_1 off of a massive spinning charged body with initial momentum mv ($v \cdot v = -1$) and initial spin s to an outgoing photon with polarization $\mathcal{E}_2$ and momentum k_2 and perturbed massive body. For a plane wave vector potential with strength ϵ:

$$A_\mu(X) = \epsilon \mathcal{E}_\mu e^{ik\cdot x}, \qquad \mathcal{F}_{\mu\nu}(X) = \epsilon f_{\mu\nu} e^{ik\cdot x}, \qquad f_{\mu\nu} = ik_\mu \mathcal{E}_\nu - ik_\nu \mathcal{E}_\mu. \tag{3.5.1}$$

Because the tree level Compton amplitude is $\mathcal{O}(q^2)$, it depends only on the $\mathcal{O}(q)$ and $\mathcal{O}(q^2)$ pieces of the dynamical mass function. Consequently, we will only be concerned with operators in $\mathcal{M}$ which are linear or quadratic in $\mathcal{F}$ and so we consider an $\mathcal{M}$ of the form:

$$\mathcal{M}^2(z,u,S) = m^2 + q\delta\mathcal{M}_1^2(z,u,S) + q^2\delta\mathcal{M}_2^2(z,u,S) + \mathcal{O}(q^3\mathcal{F}^3) \tag{3.5.2}$$

where $\delta\mathcal{M}_1^2$ is of the form:

$$\delta\mathcal{M}_1^2(z,u,S) = \sum_{n=0}^{\infty} T_n^{\rho_1\cdots\rho_n\mu\nu}(u,S)\partial^n_{\rho_1\ldots\rho_n}\mathcal{F}_{\mu\nu}(z) \tag{3.5.3}$$

for some functions $T_n^{\rho_1\cdots\rho_n\mu\nu}(u,S)$ satisfying:

$$T_n^{\rho_1\cdots\rho_n\mu\nu} = T_n^{(\rho_1\cdots\rho_n)[\mu\nu]} \tag{3.5.4}$$

and $\delta\mathcal{M}_2^2$ is of the form:

$$\delta\mathcal{M}_2^2(z,u,S) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} V_{nl}^{\rho_1\ldots\rho_n\mu\nu|\kappa_1\ldots\kappa_l\tau\omega}(u,S)\partial^n_{\rho_1\ldots\rho_n}\mathcal{F}_{\mu\nu}(z)\partial^l_{\kappa_1\ldots\kappa_l}\mathcal{F}_{\tau\omega}(z) \tag{3.5.5}$$

for some functions $V_{nl}^{\cdots}(u,S)$ satisfying:

$$V_{nl}^{\rho_1\ldots\rho_n\mu\nu|\kappa_1\ldots\kappa_l\tau\omega} = V_{nl}^{(\rho_1\ldots\rho_n)[\mu\nu]|(\kappa_1\ldots\kappa_l)[\tau\omega]} \tag{3.5.6}$$

$$V_{nl}^{\rho_1\ldots\rho_n\mu\nu|\kappa_1\ldots\kappa_l\tau\omega} = V_{ln}^{\kappa_1\ldots\kappa_l\tau\omega|\rho_1\ldots\rho_n\mu\nu} \tag{3.5.7}$$

Maxwell's equations together with the flat space electromagnetic MPD equations with the described initial conditions will produce solutions of the form:

$$z^\mu(\lambda) = v^\mu\lambda + q\epsilon\delta z^\mu(\lambda) + \mathcal{O}(q^2) \tag{3.5.8}$$

$$p_\mu(\lambda) = mv_\mu + q\epsilon\delta p_\mu(\lambda) + \mathcal{O}(q^2) \tag{3.5.9}$$

$$S^\mu(\lambda) = s^\mu + q\epsilon\delta S^\mu(\lambda) + \mathcal{O}(q^2) \tag{3.5.10}$$

$$J^\mu(X) = qJ^\mu_{\text{stat}}(X) + q^2\epsilon\delta J(X) + \mathcal{O}(q^3) \tag{3.5.11}$$

$$A_\mu(X) = \epsilon\mathcal{E}_{1\mu}e^{ik\cdot x} + qA_\mu^{\text{stat}}(X) + q^2\epsilon\delta A_\mu(X) + \mathcal{O}(q^3) \tag{3.5.12}$$

where qJ^μ_{stat} is the current density produced by the stationary spinning body in the absence of the incoming photon ($\epsilon \to 0$) and qA_μ^{stat} is the (Lorenz gauge) vector potential produced by that current density. The equation of motion perturbations will be oscillatory from solving the electromagnetic MPD equations. In particular, their solutions take the form:

$$\delta z^\mu = \delta\widetilde{z}^\mu e^{ik_1\cdot v\lambda}, \qquad \delta u^\mu = \delta\widetilde{u}^\mu e^{ik_1\cdot v\lambda}, \qquad \delta S^\mu = \delta\widetilde{S}^\mu e^{ik_1\cdot v\lambda} \tag{3.5.13}$$

for constant vectors $\delta\widetilde{z}, \delta\widetilde{u}, \delta\widetilde{S}$. ($\delta p$ is determined from δu and $\delta\mathcal{M}_1^2$ because $p^\mu = \mathcal{M}u^\mu$.)

The $\mathcal{O}(q^2\epsilon)$ piece of the vector potential, δA_μ, determines the linear in ϵ outgoing electromagnetic field and thus determines the tree level Compton amplitude. From the Lorenz-gauge Maxwell equation, δA_μ satisfies:

$$-\partial^2\delta A^\mu = \delta J^\mu \tag{3.5.14}$$

and limits to 0 in the asymptotic past (when the appropriate $i\varepsilon$ is used for the oscillatory solutions). Therefore, it is determined by the delayed Green's function solution to the wave equation and given by:

$$\delta A^\mu(X) = \int_{\mathbb{S}} \frac{\delta J^\mu(t - |\vec{x} - \vec{x}'|, \vec{x}')}{4\pi|\vec{x} - \vec{x}'|} d^3\vec{x}' \tag{3.5.15}$$

where:

$$t = -v\cdot x, \qquad \vec{x}^\mu = x^\mu + v^\mu v\cdot x. \tag{3.5.16}$$

Thus, δA is determined by δJ. The current perturbation is determined by the equation of motion perturbations $\delta z, \delta p, \delta S$. From (3.2.16) we may identify:

$$\mathcal{Q}_n^{\rho_1\ldots\rho_n\mu\nu} = \frac{q}{2m}\left(1 - \frac{q\delta\mathcal{M}_1^2}{2m^2}\right)T_n^{\rho_1\ldots\rho_n\mu\nu} + \frac{q^2}{m}\sum_{l=0}^\infty V_{nl}^{\rho_1\ldots\rho_n\mu\nu|\kappa_1\ldots\kappa_l\tau\omega}\partial^l_{\kappa_1\ldots\kappa_l}\mathcal{F}_{\tau\omega} + \mathcal{O}(\mathcal{F}^2). \tag{3.5.17}$$

Define the Fourier transform of the current:

$$\widetilde{J}^{\mu}(k) = \int \frac{e^{-ik\cdot x}}{(2\pi)^2} J^{\mu}(X) d^4X. \tag{3.5.18}$$

In terms of the $\mathcal{Q}_n$ the exact Fourier transform of $J^{\mu}(X)$ is:

$$\widetilde{J}^{\mu}(k_2) = \int_{-\infty}^{\infty} \left(q\dot{z}^{\mu} + 2\sum_{n=0}^{\infty} (-i)^{n+1} \mathcal{Q}_n^{\rho_1\ldots\rho_n\mu\nu} k_{2\rho_1}\ldots k_{2\rho_n} k_{2\nu} \right) \frac{e^{-ik_2\cdot z}}{(2\pi)^2} d\lambda \tag{3.5.19}$$

For a plane wave vector potential it is useful to introduce the function:

$$\mathcal{N}(u, S, k, \mathcal{E}) = \sum_{n=0}^{\infty} \frac{i^n}{2m} T_n^{\rho_1\ldots\rho_n\mu\nu}(u, S) k_{\rho_1}\ldots k_{\rho_n} f_{\mu\nu} \tag{3.5.20}$$

so that if we evaluate $\delta\mathcal{M}_1^2$ on a plane wave vector potential:

$$\delta\mathcal{M}_1^2\big|_{\text{plane wave}} = 2m\epsilon\mathcal{N}(u, S, k, \mathcal{E}) e^{ik\cdot z}. \tag{3.5.21}$$

Similarly, for a pair of plane waves it is useful to introduce the function:

$$\mathcal{P}(u, S, k, \mathcal{E}, k', \mathcal{E}') = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{i^{l-n}}{m} V_{nl}^{\rho_1\ldots\rho_n\mu\nu|\kappa_1\ldots\kappa_l\tau\omega}(u, S) k'_{\rho_1}\ldots k'_{\rho_n} f'^{*}_{\mu\nu} k_{\kappa_1}\ldots k_{\kappa_l} f_{\tau\omega}. \tag{3.5.22}$$

As a shorthand, we write:

$$\mathcal{N}_1 = \mathcal{N}(v, s, k_1, \mathcal{E}_1), \qquad \mathcal{N}_2 = \mathcal{N}(v, s, k_2, \mathcal{E}_2), \qquad \mathcal{P}_{12} = \mathcal{P}(v, s, k_1, \mathcal{E}_1, k_2, \mathcal{E}_2). \tag{3.5.23}$$

Expanding the definition of the $\mathcal{Q}_n$ moments and returning the result to the Fourier transform of the current produces:

$$\delta\widetilde{J}^{\mu}(k_2) = H^{\mu}(k_2)\frac{\delta(k_1\cdot v - k_2\cdot v)}{2\pi} \tag{3.5.24}$$

where:

$$H^{\mu}(k_2) = -ik_2\cdot\delta\widetilde{z}v^{\mu} + ik_2\cdot v\delta\widetilde{z}^{\mu} + 2\sum_{n=0}^{\infty}(-i)^{n+1}\delta\widetilde{\mathcal{Q}}_n^{\rho_1\ldots\rho_n\mu\nu} k_{2\rho_1}\ldots k_{2\rho_n} k_{2\nu} \tag{3.5.25}$$

$$\begin{aligned}\delta\widetilde{\mathcal{Q}}_n^{\rho_1\ldots\rho_n\mu\nu} = &\frac{1}{2m}\left(-\frac{\mathcal{N}_1}{m} + \delta\widetilde{u}^{\sigma}\frac{\partial}{\partial v^{\sigma}} + \delta\widetilde{S}^{\sigma}\frac{\partial}{\partial s^{\sigma}}\right) T_n^{\rho_1\ldots\rho_n\mu\nu}(v, s) \\ &+ \sum_{l=0}^{\infty}\frac{i^l}{m} V_{nl}^{\rho_1\ldots\rho_n\mu\nu|\kappa_1\ldots\kappa_l\tau\omega} k_{1\kappa_1}\ldots k_{1\kappa_l} f_{1\tau\omega}.\end{aligned} \tag{3.5.26}$$

Returning our current perturbation to δA produces:

$$\delta A^{\mu} = \frac{e^{i\omega(r-t)}}{4\pi r} H^{\mu}(k_2) + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{3.5.27}$$

where

$$r = \sqrt{x^2 + (x\cdot v)^2}, \qquad n^{\mu} = v^{\mu} + \frac{x^{\mu} + (x\cdot v)v^{\mu}}{r}, \qquad \omega = -k_1\cdot v, \qquad k_2^{\mu} = \omega n^{\nu}. \tag{3.5.28}$$

Therefore, the canonically normalized Compton amplitude is:

$$\mathcal{A}_{\text{canonical}} = \mathcal{E}^{*}_{2\mu} H^{\mu}(k_2). \tag{3.5.29}$$

The covariant (Feynman) normalized Compton amplitude can be obtained by multiplying by the usual factor of $\sqrt{(2\mathrm{E}_1)(2\mathrm{E}_2)}$. Because the calculation is performed in the classical limit and in a frame in which the initial body is at rest, this normalization factor simply becomes $2m$. Thus:

$$\mathcal{A} = 2m\mathcal{E}^*_{2\mu}H^\mu(k_2). \tag{3.5.30}$$

Expanding the electromagnetic MPD equations to linear order in ϵ produces the solutions:

$$\delta\widetilde{u}^\mu = \frac{-if_1^{\mu\nu}v_\nu}{mk_1\cdot v} - \frac{\mathcal{N}_1}{m}\left(v^\mu + \frac{k_1^\mu}{k_1\cdot v}\right) \tag{3.5.31}$$

$$\delta\widetilde{S}^\mu = v^\mu\delta\widetilde{u}\cdot s - \frac{i}{k_1\cdot v}\epsilon^{\mu\nu\rho\sigma}v_\nu s_\rho\frac{\partial\mathcal{N}_1}{\partial s^\sigma} \tag{3.5.32}$$

$$\begin{aligned}\delta\widetilde{z}^\mu = &\frac{-i}{k_1\cdot v}\left(\delta\widetilde{u}^\mu + \frac{\delta^\mu_\alpha + v^\mu v_\alpha}{m^2}{}^\star f_1^{\alpha\beta}s_\beta + \frac{\eta^{\mu\nu}+v^\mu v^\nu}{m}\frac{\partial\mathcal{N}_1}{\partial v^\nu}\right.\\ &\left.+\frac{s^\mu}{m}v^\nu\frac{\partial\mathcal{N}_1}{\partial s^\nu} + \frac{i}{m^2}\mathcal{N}_1\epsilon^{\mu\nu\rho\sigma}v_\nu s_\rho k_{1\sigma}\right).\end{aligned} \tag{3.5.33}$$

Simplifying the Compton amplitude allows it to be expressed in terms of these solutions as:

$$\mathcal{A} = 2m\left(f^*_{2\mu\nu}\delta\widetilde{z}^\mu v^\nu + ik_2\cdot\delta\widetilde{z}\mathcal{N}_2^* - \delta\widetilde{u}^\sigma\frac{\partial\mathcal{N}_2^*}{\partial v^\sigma} - \delta\widetilde{S}^\sigma\frac{\partial\mathcal{N}_2^*}{\partial s^\sigma} + \frac{\mathcal{N}_1\mathcal{N}_2^*}{m} - \mathcal{P}_{12}\right). \tag{3.5.34}$$

This gives the formal tree-level electromagnetic Compton amplitude for an arbitrary dynamical mass function.

Using either the dynamical mass function in (3.4.54) or in (3.4.55) determines the $\mathcal{N}$ function to be:

$$\begin{aligned}\mathcal{N}(u,S,k,\mathcal{E}) = &\frac{\sin\left(\frac{1}{m}\sqrt{S^2(u\cdot k)^2-(S\cdot k)^2}\right)}{\sqrt{S^2(u\cdot k)^2-(S\cdot k)^2}}{}^\star f_{\mu\nu}S^\mu u^\nu\\ &+ik\cdot S\frac{1-\cos\left(\frac{1}{m}\sqrt{S^2(u\cdot k)^2-(S\cdot k)^2}\right)}{S^2(u\cdot k)^2-(S\cdot k)^2}f_{\mu\nu}S^\mu u^\nu\end{aligned} \tag{3.5.35}$$

however the multipole analysis is unable to determine the $\mathcal{P}$ function. Thus, this determines the Compton amplitude through equations (3.5.31)-(3.5.34) up to contact terms. Because the electromagnetic Compton amplitude only depends on the linear in $\mathcal{F}$ dynamical mass function through the $\mathcal{N}$ function and (3.4.54) and (3.4.55) determine the same $\mathcal{N}$ function, they are interchangeable for Compton amplitudes.

3.5.2 Compton Amplitude through Spin Cubed

In this subsection we explicitly compute the Compton amplitude for a generic dynamical mass function through order $\mathcal{O}(S^3)$. Requiring a match to the spin-exponentiated result for a $\sqrt{\text{Kerr}}$ particle fixes all Wilson coefficients on $\mathcal{F}^1$ and $\mathcal{F}^2$ operators through $\mathcal{O}(S^2)$. The implied values of the Wilson coefficients on $\mathcal{F}^1$ operators match those determined by (3.4.55). For the helicity-conserving Compton amplitude, the spin-exponentiation cannot be continued past $\mathcal{O}(S^2)$ due to spurious poles. However, the helicity-reversing Compton amplitude has a perfectly healthy exponentiation at $\mathcal{O}(S^3)$. The helicity-conserving amplitude can instead be required to satisfy the shift symmetry at $\mathcal{O}(S^3)$ (which is automatic for lower orders which satisfy spin-exponentiation). We find that at $\mathcal{O}(S^3)$ requiring spin-exponentiation for the helicity-reversing Compton and shift symmetry for the helicity-conserving Compton are consistent with each other and consistent with (3.4.55). However, we find that these three requirements still leave one remaining free parameter in the $\mathcal{O}(\mathcal{F}^2 S^3)$ piece of the dynamical mass function.

We consider only effects in the dynamical mass function which introduce no additional length scales beyond $\frac{S}{m}$. Consequently, we only consider terms for which the number of powers of spin equals the number of derivatives on the vector potential(s). As well, we only consider terms which are parity symmetric and not proportional to the field equations (no $\partial_\nu \mathcal{F}^{\mu\nu}$ or $\partial^2 \mathcal{F}^{\mu\nu}$ terms). (Terms proportional to $\partial_\nu \mathcal{F}^{\mu\nu}$ or $\partial^2 \mathcal{F}_{\mu\nu}$ do not contribute to the electromagnetic Compton amplitude because they evaluate to 0 in the $\mathcal{N}$ function.) The most general such $\delta\mathcal{M}_1^2$ to order S^3 is:

$$\begin{aligned}\delta\mathcal{M}_1^2 = {} & 2C_1 {}^\star\mathcal{F}_{\mu\nu} S^\mu u^\nu + \frac{C_2}{m}(S\cdot\partial)\mathcal{F}_{\mu\nu}S^\mu u^\nu \\ & - \frac{C_3}{3m^2}(S\cdot\partial)^{2\star}\mathcal{F}_{\mu\nu}S^\mu u^\nu + \frac{E_3}{3m^2}S^2(u\cdot\partial)^{2\star}\mathcal{F}_{\mu\nu}S^\mu u^\nu + \mathcal{O}(S^4)\end{aligned} \tag{3.5.36}$$

for some Wilson coefficients C_1, C_2, C_3, E_3. The most general such $\delta\mathcal{M}_2^2$ to order S^3 is:

$$\begin{aligned}\delta\mathcal{M}_2^2 = {} & \frac{D_{2a}}{m^2}(\mathcal{F}_{\mu\nu}S^\mu u^\nu)^2 + \frac{D_{2b}}{m^2}({}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu)^2 + \frac{D_{2c}}{m^2}S^2\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{D_{2d}}{m^2}S^2 u^\mu \mathcal{F}_{\mu\nu}\mathcal{F}^{\nu\rho}u_\rho \\ & + \frac{D_{3a}}{m^3}{}^\star\mathcal{F}_{\mu\nu}S^\mu u^\nu (S\cdot\partial)\mathcal{F}_{\rho\sigma}S^\rho u^\sigma + \frac{D_{3b}}{m^3}S^{2\star}\mathcal{F}^{\mu\nu}u_\mu \partial_\nu \mathcal{F}_{\rho\sigma}S^\rho u^\sigma \\ & + \frac{D_{3c}}{m^3}S^{2\star}\mathcal{F}^{\mu\nu}u_\mu (S\cdot\partial)\mathcal{F}_{\nu\rho}u^\rho + \frac{D_{3d}}{m^3}S^{2\star}\mathcal{F}^{\mu\nu}S_\mu(u\cdot\partial)\mathcal{F}_{\nu\rho}u^\rho \\ & + \frac{D_{3e}}{m^3}{}^\star\mathcal{F}^{\mu\nu}S_\mu(S\cdot\partial)\mathcal{F}_{\nu\rho}S^\rho + \frac{D_{3f}}{m^3}S^{2\star}\mathcal{F}^{\mu\nu}(S\cdot\partial)\mathcal{F}_{\mu\nu} + \mathcal{O}(S^4)\end{aligned} \tag{3.5.37}$$

for some Wilson coefficients D_{2a}, D_{2b}, D_{2c}, D_{2d} for quadratic-in-spin terms and D_{3a}, D_{3b}, D_{3c}, D_{3d}, D_{3e}, D_{3f} for cubic in spin. These lead to the $\mathcal{N}$ and $\mathcal{P}$ functions:

$$\mathcal{N} = \frac{C_1}{m}{}^\star f_{\mu\nu}S^\mu u^\nu + \frac{iC_2}{2m^2}k\cdot S f_{\mu\nu}S^\mu u^\nu + \frac{C_3}{6m^3}(k\cdot S)^{2\star}f_{\mu\nu}S^\mu u^\nu - \frac{E_3}{6m^3}S^2(k\cdot u)^{2\star}f_{\mu\nu}S^\mu u^\nu, \tag{3.5.38}$$

$$\begin{aligned}\mathcal{P} = {} & \frac{D_{2a}}{m^3}f'^*_{\mu\nu}S^\mu u^\nu f_{\rho\sigma}S^\rho u^\sigma + \frac{D_{2b}}{m^3}{}^\star f'^*_{\mu\nu}S^\mu u^{\nu\star} f_{\rho\sigma}S^\rho u^\sigma + \frac{D_{2c}}{m^3}S^2 f'^*_{\mu\nu}f^{\mu\nu} + \frac{D_{2d}}{m^3}S^2 u^\mu f'^*_{\mu\nu}f^{\nu\rho}u_\rho \\ & + \frac{iD_{3a}}{2m^4}{}^\star f'^*_{\mu\nu}S^\mu u^\nu (S\cdot k)f_{\rho\sigma}S^\rho u^\sigma - \frac{iD_{3a}}{2m^4}{}^\star f_{\mu\nu}S^\mu u^\nu (S\cdot k')f'^*_{\rho\sigma}S^\rho u^\sigma \\ & + \frac{iD_{3b}}{2m^4}S^{2\star}f'^*_{\mu\nu}u^\mu k^\nu f_{\rho\sigma}S^\rho u^\sigma - \frac{iD_{3b}}{2m^4}S^{2\star}f_{\mu\nu}u^\mu k'^\nu f'^*_{\rho\sigma}S^\rho u^\sigma \\ & + \frac{iD_{3c}}{2m^4}S^{2\star}f'^{*\mu\nu}u_\mu (S\cdot k)f_{\nu\rho}u^\rho - \frac{iD_{3c}}{2m^4}S^{2\star}f^{\mu\nu}u_\mu(S\cdot k')f'^*_{\nu\rho}u^\rho \\ & + \frac{iD_{3d}}{2m^4}S^{2\star}f'^{*\mu\nu}S_\mu(u\cdot k)f_{\nu\rho}u^\rho - \frac{iD_{3d}}{2m^4}S^{2\star}f^{\mu\nu}S_\mu(u\cdot k')f'^*_{\nu\rho}u^\rho \\ & + \frac{iD_{3e}}{2m^4}{}^\star f'^{*\mu\nu}S_\mu(S\cdot k)f_{\nu\rho}S^\rho - \frac{iD_{3e}}{2m^4}{}^\star f^{\mu\nu}S_\mu(S\cdot k')f'^*_{\nu\rho}S^\rho \\ & + \frac{iD_{3f}}{2m^4}S^{2\star}f'^{*\mu\nu}(S\cdot k)f_{\mu\nu} - \frac{iD_{3f}}{2m^4}S^{2\star}f^{\mu\nu}(S\cdot k')f'^*_{\mu\nu}.\end{aligned} \tag{3.5.39}$$

We find it advantageous to express our results for the Compton amplitude in a basis of definite-helicity/circularly polarized/(anti-)self-dual states for the incoming and outgoing electromagnetic waves, or "photons," while manifesting special covariance and gauge invariance (ultimately). For concreteness (initially), we can work in a particular Lorentz frame, associated to inertial Cartesian coordinates $x^\mu = (t, x, y, z)$, such that the charged massive spinning particle's initial velocity v and the two photon wavevectors, k_1 incoming and k_2 outgoing (both future-pointing), are given by

$$v^\mu = (1, 0, 0, 0), \qquad k_1^\mu = \omega(1, 0, 0, 1), \qquad k_2^\mu = \omega(1, \sin\theta, 0, \cos\theta). \tag{3.5.40}$$

Then θ is the photon scattering angle in the z-x-plane, and $\omega = -v \cdot k_1 = -v \cdot k_2$ is the waves' angular frequency. The "momentum transfer" (per $\hbar$) $q = k_2 - k_1$ squares to

$$q^2 = (k_2 - k_1)^2 = -2k_1 \cdot k_2 = 2\omega^2(1 - \cos\theta) = 4\omega^2 \sin^2\frac{\theta}{2}, \tag{3.5.41}$$

vanishing at forward scattering, $\theta = 0$. In choosing a particular basis of definite-helicity (complex null) polarization vectors, $\mathcal{E}_{1\pm}$ incoming, $\mathcal{E}_{2\pm}$ outgoing, with $k_n \cdot \mathcal{E}_{n\pm} = 0 = \mathcal{E}_{n\pm}^2$, it is natural to fix the gauge freedom $\mathcal{E}_n \to \mathcal{E}_n + \alpha k_n$ by imposing $v \cdot \mathcal{E} = 0$. Up to little group transformations ($\mathcal{E} \to e^{2i\varphi}\mathcal{E}$), this determines

$$\mathcal{E}_{1\sigma_1}^\mu = \frac{1}{\sqrt{2}}\left(0, 1, i\sigma_1, 0\right), \qquad \mathcal{E}_{2\sigma_2}^\mu = \frac{1}{\sqrt{2}}\left(0, \cos\theta, i\sigma_2, -\sin\theta\right), \tag{3.5.42}$$

for helicities $\sigma_1 = \pm 1$ and $\sigma_2 = \pm 1$, with complex conjugates $\mathcal{E}_{n\pm}^{*\mu} = \mathcal{E}_{n\mp}^\mu$, normalized as $\mathcal{E}_{n\sigma_n} \cdot \mathcal{E}_{n\sigma_n}^* = 1$. This frame (3.5.40) and polarization basis (or gauge) (3.5.42) are just as in [247] and in [185]. The spinless Compton amplitude is given simply by the contraction of the ingoing and conjugate-outgoing polarization vectors (only) in this $v \cdot \mathcal{E} = 0$ gauge:

$$\mathcal{A}_{\sigma_1\sigma_2}^{(0)} = -2\mathcal{E}_{1\sigma_1} \cdot \mathcal{E}_{2\sigma_2}^* = -\sigma_1\sigma_2 - \cos\theta. \tag{3.5.43}$$

In analyzing the helicity-preserving amplitudes $\mathcal{A}_{\pm\pm} \propto \mathcal{E}_{1\pm}^\mu \mathcal{E}_{2\pm}^{*\nu}$, it is useful to define as in [22][1] a complex null vector w orthogonal to both k_1 and k_2. The conditions $w^2 = k_1 \cdot w = k_2 \cdot w = 0$ determine w up to an overall normalization, which we fix by setting $v \cdot w = -\omega$, and up to a binary choice of branch ($w \propto |1\rangle[2|$ or $w \propto |2\rangle[1|$) to be correlated with the photons' helicities. For our $++$ (or $--$) case, the appropriate w is given by

$$w^\mu = \frac{\omega}{4\omega^2 - q^2}\Big(2\omega(k_1^\mu + k_2^\mu) - q^2 v^\mu - 2i\epsilon^\mu{}_{\nu\rho\sigma} v^\nu k_1^\rho k_2^\sigma\Big) \tag{3.5.44}$$

(or the complex conjugate $w^{*\mu}$). In the frame of (3.5.40),

$$w^\mu = \omega\Big(1, \tan\frac{\theta}{2}, i\tan\frac{\theta}{2}, 1\Big), \tag{3.5.45}$$

noting $4\omega^2 - q^2 = 4\omega^2\cos^2\frac{\theta}{2}$ along with (3.5.41). The complex null direction $\propto w$ provides an alternative

[1] In [22] as in most Amplitudes literature, our *helicity-preserving* amplitudes $\mathcal{A}_{\pm\pm} \propto \mathcal{E}_{1+}^\mu \mathcal{E}_{2+}^{\pm\nu}$ are called "opposite-helicity amplitudes $\mathsf{A}_{\pm\mp}$", while our *helicity-reversing* amplitudes $\mathcal{A}_{\pm\mp} \propto \mathcal{E}_{1\pm}^\mu \mathcal{E}_{2\mp}^{*\nu}$ are called "same-helicity amplitudes $\mathsf{A}_{\pm\pm}$", due to differing conventions (essentially $k_n \leftrightarrow -k_n$ entailing $\sigma_n \leftrightarrow -\sigma_n$).

(+)-helicity polarization direction for the incoming k_1 photon, as well as an alternative conjugate (+)-helicity polarization direction for the outgoing k_2 photon. We see that we can recover the normalized orthogonal-to-v polarization vectors $\mathcal{E}_{1+}$ and $\mathcal{E}^*_{2+}$ of (3.5.42) from w via $\mathcal{E}_n \to \mathcal{E}_n + \alpha k_n$ shifts and rescalings:

$$\mathcal{E}^\mu_{1+} = \frac{w^\mu - k_1^\mu}{\sqrt{2}\omega \tan\frac{\theta}{2}}, \qquad \mathcal{E}^{*\mu}_{2+} = \frac{k_2^\mu - w^\mu}{\sqrt{2}\omega \tan\frac{\theta}{2}}, \tag{3.5.46}$$

As in [179] (modulo conventions), let us define the vectors

$$\check{k}_1^\mu = k_1^\mu - w^\mu, \qquad \check{k}_2^\mu = k_2^\mu - w^\mu, \tag{3.5.47}$$

proportional to those in (3.5.46), which, along with w as in (3.5.44), provide a relatively compact way to express the ++ Compton amplitude.

The complex field-strength amplitudes $f^\pm_{\mu\nu} = 2ik_{[\mu}\mathcal{E}^\pm_{\nu]}$ are invariant under $\mathcal{E} \to \mathcal{E} + \alpha k$, they transform under the little groups like the $\mathcal{E}$s, and they are self-dual (or anti-self-dual),

$${}^\star f^\pm_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda} f^\pm_{\kappa\lambda} = \pm i f^\pm_{\mu\nu}, \tag{3.5.48}$$

for states of helicity +1 (or −1), while the complex conjugates are reversed: ${}^\star f^{\pm *}_{\mu\nu} = \mp i f^{\pm *}_{\mu\nu}$. For one way to see this, we can construct $f^+_{1\mu\nu}$ from the $\mathcal{E}^+_{1\mu}$ of (3.5.42), using (3.5.46) with (3.5.44),

$$\begin{aligned} f^+_{1\mu\nu} &= 2ik_{1[\mu}\mathcal{E}^+_{1\nu]} = \frac{\sqrt{2}i}{\omega\tan\frac{\theta}{2}} k_{1[\mu} w_{\nu]} \\ &= \frac{\sqrt{2}i}{\omega^2 \sin\theta} k_{1[\mu}\Big(v_{\nu]}(k_1\cdot k_2) + \omega k_{2\nu]} - i\epsilon_{\nu]\rho\alpha\beta} v^\rho k_1^\alpha k_2^\beta\Big) \\ &= \frac{\sqrt{2}i}{\omega^2 \sin\theta}\Big(\delta_{[\mu}{}^\alpha \delta_{\nu]}{}^\beta - \frac{i}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}\Big) k_{1\alpha}\big(v_\beta(k_1\cdot k_2) + \omega k_{2\beta}\big) \\ &= \frac{\sqrt{2}i}{\omega^2\sin\theta} 2\,{}^+\mathcal{G}_{\mu\nu}{}^{\alpha\beta} k_{1\alpha} 2v_{[\beta} k_{2\gamma]} k_1^\gamma, \end{aligned} \tag{3.5.49}$$

where we have used $0 = 5k_{1[\mu}\epsilon_{\nu\rho\alpha\beta]}v^\rho k_1^\alpha k_2^\beta = 2k_{1[\mu}\epsilon_{\nu]\rho\alpha\beta}v^\rho k_1^\alpha k_2^\beta - \epsilon_{\mu\nu\alpha\beta}k_1^\alpha(v^\beta(k_1\cdot k_2) + \omega k_2^\beta) = 2k_{1[\mu}\epsilon_{\nu]vk_1k_2} - \epsilon_{\mu\nu\alpha\beta}k_1^\alpha\, 2v^{[\beta}k_2^{\gamma]}k_{1\gamma}$ and $2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \sin\theta$, and where we recognize the (anti-)self-dual ((A)SD) projector,

$${}^\pm\mathcal{G}_{\mu\nu}{}^{\kappa\lambda} = \frac{1}{2}\delta_{[\mu}{}^\kappa\delta_{\nu]}{}^\lambda \mp \frac{i}{4}\epsilon_{\mu\nu}{}^{\kappa\lambda} = {}^\pm\mathcal{G}_{\mu\nu}{}^{\alpha\beta}\,{}^\pm\mathcal{G}_{\alpha\beta}{}^{\kappa\lambda} = \mp i\,{}^{\star\pm}\mathcal{G}_{\mu\nu}{}^{\kappa\lambda}, \tag{3.5.50}$$

which maps a 2-form (or any tensor) $A_{\mu\nu}$ onto its (A)SD part ${}^\pm A_{\mu\nu}$:

$${}^\pm A_{\mu\nu} = {}^\pm\mathcal{G}_{\mu\nu}{}^{\kappa\lambda}A_{\kappa\lambda} = \frac{1}{2}\big(A_{[\mu\nu]} \mp i\,{}^\star A_{[\mu\nu]}\big), \qquad {}^{\star\pm}A_{\mu\nu} = \pm i^\pm A_{\mu\nu}. \tag{3.5.51}$$

Note the useful identity

$${}^\pm\mathcal{G}_{\mu\nu}{}^{(\alpha}{}_\rho\,{}^\pm\mathcal{G}_{\kappa\lambda}{}^{\beta)\rho} = \frac{1}{4}\,{}^\pm\mathcal{G}_{\mu\nu\kappa\lambda}g^{\alpha\beta}, \tag{3.5.52}$$

or ${}^\pm\mathcal{G}_{\mu\nu(\alpha}{}^\gamma\,{}^\pm A_{\beta)\gamma} = \frac{1}{4}{}^\pm A_{\mu\nu}g_{\alpha\beta}$ and thus ${}^\pm\mathcal{G}_{\mu\nu}{}^{\rho\sigma}p_\sigma\,{}^\pm A_{\rho\tau}p^\tau = \frac{1}{4}p^2\,{}^\pm A_{\mu\nu}$, following from the double Levi-Civita identity $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} = -24\delta_\alpha{}^{[\mu}\delta_\beta{}^\nu\delta_\gamma{}^\rho\delta_\delta{}^{\sigma]}$ and $\delta_\omega{}^{[\lambda}\delta_\alpha{}^\mu\delta_\beta{}^\nu\delta_\gamma{}^\rho\delta_\delta{}^{\sigma]} = 0$. Collecting (3.5.49) along with its conjugate outgoing versions, also following from (3.5.44) and (3.5.46), we have

$$\begin{aligned}
f^{+}_{1\mu\nu} &= +2ik_{1[\mu}\mathcal{E}^{+}_{1\nu]} = \frac{4\sqrt{2}i}{\omega^2\sin\theta}\,{}^{+}\mathcal{G}_{\mu\nu}{}^{\alpha\beta}k_{1\alpha}v_{[\beta}k_{2\gamma]}k_1^{\gamma} = \frac{\sqrt{2}i}{\omega\tan\frac{\theta}{2}}k_{1[\mu}w_{\nu]}\,,\\
f^{+*}_{2\mu\nu} &= -2ik_{2[\mu}\mathcal{E}^{+*}_{2\nu]} = \frac{4\sqrt{2}i}{\omega^2\sin\theta}\,{}^{-}\mathcal{G}_{\mu\nu}{}^{\alpha\beta}k_{2\alpha}v_{[\beta}k_{1\gamma]}k_2^{\gamma} = \frac{\sqrt{2}i}{\omega\tan\frac{\theta}{2}}k_{2[\mu}w_{\nu]}\,,\\
f^{-*}_{2\mu\nu} &= -2ik_{2[\mu}\mathcal{E}^{-*}_{2\nu]} = \frac{4\sqrt{2}i}{\omega^2\sin\theta}\,{}^{+}\mathcal{G}_{\mu\nu}{}^{\alpha\beta}k_{2\alpha}v_{[\beta}k_{1\gamma]}k_2^{\gamma} = \frac{\sqrt{2}i}{\omega\tan\frac{\theta}{2}}k_{2[\mu}w^{*}_{\nu]}\,,
\end{aligned}\tag{3.5.53}$$

noting $w^{*\mu} = -w^{\mu} + \omega\dfrac{2\omega(k_1+k_2)^{\mu} - q^2v^{\mu}}{4\omega^2 - q^2}$ and

$$\omega^2\sin\theta = \sqrt{4\omega^4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{q^2}{4}(4\omega^2-q^2) = -\epsilon^{\mu}{}_{vk_1k_2}\epsilon_{\mu}{}^{vk_1k_2}} \propto [12]\langle 12\rangle[2|v|1\rangle[1|v|2\rangle. \tag{3.5.54}$$

In simplifying the helicity-basis Compton amplitudes $\mathcal{A}_{+\pm} \propto f^{+}_{1\mu\nu}f^{\pm*}_{2\alpha\beta}$, the (A)SD properties ${}^{\star}f^{+}_{1\mu\nu} = +if_{1\mu\nu}$ and ${}^{\star}f^{\pm*}_{2\mu\nu} = \mp if^{\pm*}_{2\mu\nu}$ can be used immediately within (before differentiation of) the $\mathcal{N}(f)$ and $\mathcal{P}(f,f')$ functions in (3.5.38). These functions completely determine the amplitudes via (3.5.31)–(3.5.34). They can finally be evaluated directly in terms of the complex null $w^{\mu}(k_1,k_2,v)$ by using the extreme equalities of (3.5.53), noting e.g. $\epsilon_{vak_1w} = i\omega\check{k}_1\cdot a$, $\epsilon_{vak_2w} = -i\omega\check{k}_2\cdot a$, $\epsilon_{vk_1k_2w} = i\omega q^2/2$ following from (3.5.44), recalling $\check{k}_1 = k_1 - w$ and $\check{k}_2 = k_2 - w$; they otherwise depend only on k_1^{μ}, k_2^{μ}, v^{μ} and the initial spin $s^{\mu} = ma^{\mu}$.

For the ++ amplitudes $\mathcal{A}^{(n)}_{++} \propto f_1^{+}f_2^{+*}a^n$ at nth order in spin, we find

$$\mathcal{A}^{(0)}_{++} = -\frac{4\omega^2-q^2}{2\omega^2}, \tag{3.5.55a}$$

$$\mathcal{A}^{(1)}_{++} = -\frac{4\omega^2-q^2}{2\omega^2}\Big\{C_1(\check{k}_1+\check{k}_2)\cdot a - (C_1-1)^2 w\cdot a\Big\}, \tag{3.5.55b}$$

$$\begin{aligned}
\mathcal{A}^{(2)}_{++} = -\frac{4\omega^2-q^2}{2\omega^2}\bigg\{&\frac{C_2}{2}[(\check{k}_1+\check{k}_2)\cdot a]^2 - \Big((C_1-1)C_2 + D_{2d}\Big)(w\cdot a)^2\\
&+ (C_1-1)(C_1-C_2)\,w\cdot a\,(\check{k}_1+\check{k}_2)\cdot a\\
&+ \bigg[\Big((C_1-1)(2C_1-C_2) + 2D_{2d} - D_{2a} - D_{2b}\Big)\frac{2\omega^2}{q^2} + C_1^2 - C_2\bigg]\check{k}_1\cdot a\,\check{k}_2\cdot a\bigg\},
\end{aligned}\tag{3.5.55c}$$

$$\begin{aligned}
\mathcal{A}^{(3)}_{++} = -\frac{4\omega^2-q^2}{2\omega^2}\bigg\{&\frac{C_3}{6}[(\check{k}_1+\check{k}_2)\cdot a]^3 + \hat{\mathcal{A}}^{(3)(C_2-1,C_1-1)}_{++}\\
&+ \left(\frac{4}{3} + 2F_{\mathrm{i}} - \frac{F_{\mathrm{ii}}}{4} + \frac{F_{\mathrm{iii}}}{2}\right)(w\cdot a)^3 + \left(F_{\mathrm{i}} - \frac{C_3-1}{6}\right)(w\cdot a)^2(\check{k}_1+\check{k}_2)\cdot a\\
&+ \bigg[F_{\mathrm{ii}}\frac{\omega^2}{q^2}w\cdot a + \left(F_{\mathrm{iii}}\frac{\omega^2}{q^2} - \frac{C_3-1}{2}\right)(\check{k}_1+\check{k}_2)\cdot a\bigg]\check{k}_1\cdot a\,\check{k}_2\cdot a\bigg\},
\end{aligned}\tag{3.5.55d}$$

where

$$F_{\mathrm{i}} = \frac{E_3-1}{6} - \frac{D_{3b}+D_{3c}+D_{3e}}{2}, \qquad F_{\mathrm{ii}} = 3 - 2D_{3a} + 4D_{3c} + 4D_{3d},$$

$$F_{\text{iii}} = -1 - \frac{2}{3}E_3 - D_{3a} + 2D_{3b} + 2D_{3c}, \tag{3.5.55e}$$

and $\hat{\mathcal{A}}_{++}^{(3)(C_2-1,C_1-1)}$ vanishes when $C_2 = C_1 = 1$ (which for $\sqrt{\text{Kerr}}$ are fixed by lower-order-in-spin pieces of the Compton amplitude, or by the three-point amplitude). Here we have used

$$\omega^2 a^2 = (w\cdot a)^2 - \frac{4\omega^2}{q^2}\check{k}_1\cdot a\,\check{k}_2\cdot a, \tag{3.5.56}$$

resulting from $0 = (u^{[\lambda}k_1^\mu k_2^\nu w^\rho a^{\sigma]})^2$, to eliminate a^2 in favor of $(w\cdot a)^2$. This leads to the relatively compact expressions (3.5.55) —— paralleling the parametrization (3.8.85) of the black hole–graviton ++ Compton amplitude at fifth order in spin as formulated in [185] —— but it makes the amplitude appear to have (in addition to the physical poles at $v\cdot k_1 = v\cdot k_2 = -\omega = 0$) unphysical poles: firstly, explicitly, at $q^2 = 4\omega^2\sin^2\frac{\theta}{2} = 0$ — at forward scattering, $\theta = 0$ — the would-be physical pole corresponding to an internal photon of momentum $q = k_2 - k_1$ going on shell, but on which the residue must be zero because the three-photon amplitude vanishes; and secondly, hidden inside w (and $\check{k}_1$ and $\check{k}_2$) in (3.5.44), at $4\omega^2 - q^2 = 4\omega^2\cos^2\frac{\theta}{2} = 0$ — at back-scattering, $\theta = \pi$ — "the spurious pole." However, no unphysical poles are actually present, for arbitrary values of the Wilson coefficients, as can be made manifest by using (3.5.56) to eliminate factors of $(w\cdot a)^2$ in favor of a^2.

The helicity-preserving amplitude $\mathcal{A}_{++}$ is well expressed in terms of the spin component $w\cdot a$ along the complex null $w^\mu(k_1,k_2,u)$ of (3.5.44) because of its symmetry $w \leftrightarrow w^*$ under $k_1 \leftrightarrow k_2$ [with $-2\omega = (k_1+k_2)\cdot v$].

Turning to the helicity-reversing amplitude $\mathcal{A}_{+-}$, the appropriate symmetry is reflected by a vector $x^\mu(k_1,k_2,v)$ with $x \leftrightarrow -x$ under $k_1 \leftrightarrow k_2$ (modulo any component along v). An apt choice is $x(k_1,k_2,v)\cdot a \propto w(k_2,-k_1,v)\cdot a$:

$$\begin{aligned}
w\cdot a &= \Big(\omega(k_1+k_2)\cdot a + i\epsilon_{\mu\nu\rho\sigma}v^\mu k_1^\nu k_2^\rho a^\sigma\Big)\frac{2\omega}{4\omega^2-q^2}\,,\\
x\cdot a &= \Big(\omega(k_2-k_1)\cdot a + i\epsilon_{\mu\nu\rho\sigma}v^\mu k_1^\nu k_2^\rho a^\sigma\Big)\frac{2\omega}{q^2} = w\cdot a + \frac{4\omega^2}{q^2}(k_1-w)\cdot a\\
&= \frac{2\omega}{q^2}\Big(\omega\, q\cdot a + ik_1\times k_2\cdot a\Big)\,,
\end{aligned} \tag{3.5.57}$$

coinciding with "$w_O\cdot a$" from [247] or $\propto$ "$w_{++}\cdot a$" from [179], with $x\cdot a \propto \langle 1|av|1\rangle$, in contrast to $w\cdot a \propto [2|a|1\rangle$. The identity (3.5.56) for $(w\cdot a)^2$ translates into

$$\begin{aligned}
\frac{q^2}{4\omega^2}\Big((x\cdot a)^2 - \omega^2a^2\Big) &= \frac{-q^2}{4\omega^2-q^2}(k_1\cdot a - x\cdot a)(k_2\cdot a + x\cdot a)\\
&= (k_1\cdot a)(k_2\cdot a) - (x\cdot a)(q\cdot a) - \omega^2a^2 = (aya)
\end{aligned} \tag{3.5.58}$$

for $(x\cdot a)^2$ and defines a convenient quadratic (aya) in the spin. With this, we find

$$\begin{aligned}
\mathcal{A}_{+-} = \frac{q^2}{2\omega^2}\Bigg\{&1 - C_1\,q\cdot a + (C_1^2-1)x\cdot a\\
&+ \frac{C_2}{2}(q\cdot a)^2 - (C_1-1)(C_1+C_2)k_1\cdot a\,k_2\cdot a
\end{aligned} \tag{3.5.59}$$

$$
\begin{aligned}
&+\left[\Big(D_{2b}-D_{2a}-(C_1-1)(2C_1+C_2)\Big)\frac{2\omega^2}{q^2}+(C_2-1)C_1\right](aya)\\
&-\Big(4D_{2c}+D_{2d}+(C_1-1)C_1\Big)\omega^2a^2\\
-&\frac{C_3}{6}(q\cdot a)^3+(C_3-1)x\cdot a\,k_1\cdot a\,k_2\cdot a+\hat{\mathcal{A}}_{+-}^{(3)(C_2-1,C_1-1)}\\
&-\left((1+D_{3a})\frac{\omega^2}{q^2}+\frac{C_3-1}{2}\right)(aya)q\cdot a-\left(\frac{3}{4}+E_3+D_{3b}+D_{3d}\right)\omega^2a^2x\cdot a\\
&-\left(\frac{1+E_3}{6}+\frac{D_{3b}+D_{3c}-D_{3e}}{2}+2D_{3f}+\frac{C_3-1}{6}\right)\omega^2a^2q\cdot a+\mathcal{O}(a^4)\Bigg\},
\end{aligned}
$$

where again $\hat{\mathcal{A}}_{+-}^{(3)(C_2-1,C_1-1)}$ vanishes when $C_2=C_1=1$.

At linear order in spin, C_1 is determined by the $\mathcal{O}(S^1)$ amplitude. At $\mathcal{O}(S^2)$, all 5 new operators, $C_2, D_{2a,b,c,d}$ contribute linearly independent structures to the amplitude. At $\mathcal{O}(S^3)$ there are 8 new operators $C_3, D_{3a,b,c,d,e,f}, E_3$ but only 7 linearly independent structures in the Compton. In particular, the Compton amplitude is independent of the value of the linear combination:

$$
Z=-6E_3+D_{3f}+2C_1D_{3b}-2(1+C_1)D_{3c}+4D_{3d} \tag{3.5.60}
$$

These amplitudes produce the spin-exponentiation of Ref. [53] through order $\mathcal{O}(S^2)$ if and only if:

$$
C_1=C_2=1, \qquad D_{2a}=D_{2b}=D_{2c}=D_{2d}=0. \tag{3.5.61}
$$

For these values of the Wilson coefficients, we recover the spin-exponentiated amplitudes:

$$
\mathcal{A}_{++}=\mathcal{A}_{++}^{(0)}\exp\big(a\cdot(\check{k}_1+\check{k}_2)\big)+\mathcal{O}(S^3) \tag{3.5.62}
$$

$$
\mathcal{A}_{+-}=\mathcal{A}_{+-}^{(0)}\exp\big(a\cdot(k_1-k_2)\big)+\mathcal{O}(S^3) \tag{3.5.63}
$$

For the equal helicity amplitude the spin-exponentiation cannot continue past $\mathcal{O}(S^2)$ due to the spurious pole in $\cos(\theta/2)$. There is no such trouble for the opposite helicity amplitude. If one demands the continuance of the spin-exponentiation for the opposite helicity amplitude through $\mathcal{O}(S^3)$, it fixes C_3 and 3 of the $D_{3a,b,c,d,e,f}$ coefficients. In particular it determines:

$$
\begin{aligned}
C_3=1, \qquad D_{3a}=-1, \qquad D_{3d}=-D_{3b}-E_3-\frac{3}{4},\\
D_{3f}=-\frac{1}{12}-\frac{E_3}{12}-\frac{D_{3b}}{4}-\frac{D_{3c}}{4}+\frac{D_{3e}}{4}.
\end{aligned} \tag{3.5.64}
$$

The value $C_3=1$ is consistent with (3.4.55). Equation (3.4.55) fixes the value of $E_3=1$ which is possible while continuing the spin-exponentiation but not demanded by it.

It is also interesting to study the shift symmetry condition identified in Refs. [173, 101, 179, 180]. In order for the same helicity Compton amplitude to maintain shift symmetry through $\mathcal{O}(S^3)$ according to the criteria of Ref. [173], some of the C_3, E_3, $D_{3a,b,c,d,e,f}$ are fixed. In particular:

$$C_3 = 1, \qquad D_{3c} = \frac{E_3}{3} + \frac{1 + D_{3a}}{2} - D_{3b}, \qquad D_{3d} = D_{3b} - \frac{E_3}{3} - \frac{5}{4}. \tag{3.5.65}$$

Thus, at this order the shift symmetry is also consistent with (3.4.55).

The conditions necessary to maintain opposite helicity spin-exponentiation, shift symmetry, and match (3.4.55) are consistent with each other at $\mathcal{O}(S^3)$ and lead to the combined set of conditions:

$$\begin{gathered} C_3 = 1, \qquad E_3 = 1, \qquad D_{3a} = -1, \qquad D_{3b} = -\frac{1}{12} \\ D_{3c} = \frac{5}{12}, \qquad D_{3d} = -\frac{5}{3}, \qquad D_{3f} = \frac{D_{3e} - 1}{4}. \end{gathered} \tag{3.5.66}$$

Thus, at $\mathcal{O}(S^3)$ there is a one parameter family of dynamical mass functions (as D_{3e} is fully undetermined) satisfying all of these constraints.

Following the decomposition of Ref. [173], the same helicity amplitude in terms of D_{3e} through $\mathcal{O}(S^3)$ may be written as:

$$\mathcal{A}_{++} = e^{a\cdot(k_1+k_2)} \sum_{n=0}^{3} \frac{\bar{I}_n}{n!} + \mathcal{O}(S^4) \tag{3.5.67}$$

with:

$$\begin{aligned} \bar{I}_0 &= -2\cos^2\frac{\theta}{2}, \qquad & \bar{I}_1 &= -2a\cdot w\bar{I}_0, \\ \bar{I}_2 &= (2a\cdot w)^2\,\bar{I}_0, \qquad & \bar{I}_3 &= -(3D_{3e}+1)(a\cdot w)^2 a\cdot(k_1+k_2)\bar{I}_0. \end{aligned} \tag{3.5.68}$$

Thus we can identify $c_0^{(3)}$ in equation (3.9b) of Ref. [173] with $-2 - 6D_{3e}$.

We can also compare to the recent work of [266] on $\sqrt{\text{Kerr}}$ amplitudes from higher-spin gauge interactions. To that end, following the lead of [266], instead of using (3.5.56) to eliminate $a^2\omega^2$ leaving $(w\cdot a)^2$ as we did in (3.5.55), we can express the $++$ amplitude in terms of both $(w\cdot a)^2$ and $\omega^2 a^2$ while eliminating q^2/ω^2 using (3.5.56). Defining $k_\pm = k_2 \pm k_1$,

$$\begin{gathered} k_+ \;=\; k_1 + k_2\,, \qquad \check{k}_1 + \check{k}_2 \;=\; k_+ - 2w\,, \\ q \;=\; k_- \;=\; k_2 - k_1 \;=\; \check{k}_2 - \check{k}_1\,, \qquad [(k_+ - 2w)\cdot a]^2 - (q\cdot a)^2 = 4\,\check{k}_1\cdot a\,\check{k}_2\cdot a\,, \end{gathered} \tag{3.5.69}$$

and replacing ω^2/q^2 with $[(w\cdot a)^2 - \omega^2 a^2]/(4\,\check{k}_1\cdot a\,\check{k}_2\cdot a)$, our amplitude (3.5.55) becomes

$$\begin{aligned} \frac{\mathcal{A}_{++}}{\mathcal{A}_{++}^{(0)}} &= 1 + C_1(k_+ - 2w)\cdot a - (C_1 - 1)^2 w\cdot a \\ &\quad + \frac{1}{2}[(k_+ - 2w)\cdot a]^2\frac{C_1^2 + C_2}{2} + \frac{C_2 - C_1^2}{4}(q\cdot a)^2 \\ &\qquad + (C_1 - 1)\Big[(C_1 - C_2)k_+\cdot a\,w\cdot a + \frac{C_2 - 2C_1}{2}\Big(\omega^2 a^2 + (w\cdot a)^2\Big)\Big] \\ &\quad - \frac{D_{2a} + D_{2b}}{2}(w\cdot a)^2 + \frac{D_{2a} + D_{2b} - 2D_{2d}}{2}\omega^2 a^2 \\ &\quad + \frac{1}{6}(k_+ - 2w)\cdot a\Big[(k_+ - 4w)\cdot a\,k_+\cdot a\frac{3 + C_3}{4} + 3\frac{C_3 - 1}{4}(q\cdot a)^2\Big] + \hat{\mathcal{A}}_{++}^{(3)(C_2-1,C_1-1)} \end{aligned}$$

$$+\frac{1-D_{3a}-2D_{3e}}{4}(w\cdot a)^2k_+\cdot a+\frac{-15+12(D_{3b}-D_{3d})-4E_3}{12}\omega^2a^2w\cdot a$$
$$+\frac{3+3D_{3a}-6(D_{3b}+D_{3c})+2E_3}{12}\omega^2a^2k_+\cdot a+\mathcal{O}(a^4). \tag{3.5.70}$$

Similarly, using (3.5.59),

$$\begin{aligned}\frac{\mathcal{A}_{+-}}{\mathcal{A}_{+-}^{(0)}} &= 1-C_1\,q\cdot a+(C_1^2-1)x\cdot a\\ &+\frac{1}{2}(q\cdot a)^2\frac{C_1^2+C_2}{2}+\frac{C_2-C_1^2}{4}(k_+\cdot a)^2-(C_2-1)C_1\,q\cdot a\,x\cdot a\\ &+\frac{1}{2}\Big((1-C_1)(C_2+2C_1)-D_{2a}+D_{2b}\Big)(x\cdot a)^2\\ &+\frac{1}{2}\Big(2C_1-C_2-C_1C_2+D_{2a}-D_{2b}-8D_{2c}-2D_{2d}\Big)\omega^2a^2\\ &-\frac{1}{6}(q\cdot a)^3\frac{3+C_3}{4}+\frac{C_3-1}{4}\Big[(q\cdot a)^2x\cdot a-\frac{1}{2}(k_+\cdot a)^2(q-2x)\cdot a+\frac{4}{3}\omega^2a^2q\cdot a\Big]\\ &-\frac{1+D_{3a}}{4}(x\cdot a)^2q\cdot a-\frac{3+4(D_{3b}+D_{3d}+E_3)}{4}\omega^2a^2x\cdot a+\hat{\mathcal{A}}_{+-}^{(3)(C_2-1,C_1-1)}\\ &+\frac{1+3D_{3a}-6(D_{3b}+D_{3c}-D_{3e})-24D_{3f}-2E_3}{12}\omega^2a^2q\cdot a+\mathcal{O}(a^4).\end{aligned} \tag{3.5.71}$$

With $C_1=C_2=C_3=1$,

$$\begin{aligned}\frac{\mathcal{A}_{++}}{\mathcal{A}_{++}^{(0)}} &= 1+(k_+-2w)\cdot a+\frac{1}{2}[(k_+-2w)\cdot a]^2+\frac{D_{2a}+D_{2b}}{2}\big[\omega^2a^2-(w\cdot a)^2\big]-D_{2d}\,\omega^2a^2\\ &+\frac{1}{6}(k_+-4w)\cdot a\,(k_+-2w)\cdot a\,k_+\cdot a+\frac{1-D_{3a}-2D_{3e}}{4}(w\cdot a)^2k_+\cdot a\\ &+\frac{-15+12(D_{3b}-D_{3d})-4E_3}{12}\omega^2a^2w\cdot a+\frac{3+3D_{3a}-6(D_{3b}+D_{3c})+2E_3}{12}\omega^2a^2k_+\cdot a\\ &+\mathcal{O}(a^4),\end{aligned} \tag{3.5.72}$$

and

$$\begin{aligned}\frac{\mathcal{A}_{+-}}{\mathcal{A}_{+-}^{(0)}} &= 1-q\cdot a+\frac{1}{2}(q\cdot a)^2+\frac{D_{2a}-D_{2b}}{2}\big[\omega^2a^2-(x\cdot a)^2\big]+(4D_{2c}+D_{2d})\omega^2a^2\\ &-\frac{1}{6}(q\cdot a)^3-\frac{1+D_{3a}}{4}(x\cdot a)^2q\cdot a-\frac{3+4(D_{3b}+D_{3d}+E_3)}{4}\omega^2a^2x\cdot a\\ &+\frac{1+3D_{3a}-6(D_{3b}+D_{3c}-D_{3e})-24D_{3f}-2E_3}{12}\omega^2a^2q\cdot a+\mathcal{O}(a^4).\end{aligned} \tag{3.5.73}$$

This matches (6.76) of [266],

$$\begin{aligned}\frac{\mathcal{A}_{++}}{\mathcal{A}_{++}^{(0)}} &= 1+(k_+-2w)\cdot a+\frac{1}{2}[(k_+-2w)\cdot a]^2+2\delta\big[\omega^2a^2-(w\cdot a)^2\big]\\ &+\frac{1}{6}(k_+-2w)\cdot a\Big[(k_+-4w)\cdot a\,k_+\cdot a+4\omega^2a^2\Big]+\frac{4}{3}\delta\big[\omega^2a^2-(w\cdot a)^2\big]k_+\cdot a+\mathcal{O}(a^4),\end{aligned} \tag{3.5.74}$$

and $\mathcal{A}_{+-}/\mathcal{A}_{+-}^{(0)}=e^{-q\cdot a}+\mathcal{O}(a^4)$ if

$$D_{2a}=D_{2b}=2\delta,\qquad D_{2c}=D_{2d}=0, \tag{3.5.75}$$

$$D_{3a} = -1, \qquad D_{3b} = -\frac{5+4E_3}{12}, \qquad D_{3c} = \frac{-11+8E_3-32\delta}{12},$$
$$D_{3d} = -\frac{1+2E_3}{3}, \qquad D_{3e} = 1 + \frac{8}{3}\delta, \qquad D_{3f} = \frac{3-E_3+8\delta}{6}.$$

3.6 Gravitational MPD Equations

We now turn our attention to gravity. We find that the line of analysis is directly analogous to that of electromagnetism and the resulting dynamical mass function is very similar. The motion of a generic spinning body in general relativity is described by the MPD equations [107, 108, 110, 253, 259, 260]. It is well established [260, 252, 264, 58, 230, 226, 267, 145] that the MPD equations can be derived from a variational principle through an action $\mathcal{S}$ of the form:

$$\mathcal{S}[z,p,\Lambda,S,\alpha,\beta] = \int_{-\infty}^{\infty} \left(p_\mu \dot{z}^\mu + \frac{1}{2}\widetilde{\epsilon}_{\mu\nu\rho\sigma} u^\mu S^\nu \Omega^{\rho\sigma} - \frac{\alpha}{2}(p^2 + \mathcal{M}^2) + \beta p \cdot S \right) d\lambda \tag{3.6.1}$$

where the pseudotensor Levi-Civita symbol is defined by:

$$\widetilde{\epsilon}_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}\sqrt{-\det g} \tag{3.6.2}$$

in terms of the purely numerical antisymmetric Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ which has $\epsilon_{0123} = 1$. In curved spacetime, the $\Lambda^\mu{}_A(\lambda)$ tetrad satisfies:

$$g^{\mu\nu}(Z) = \Lambda^\mu{}_A \Lambda^\nu{}_B \eta^{AB}, \qquad \eta_{AB} = g_{\mu\nu}(Z)\Lambda^\mu{}_A\Lambda^\nu{}_B. \tag{3.6.3}$$

Just as in the case of electromagnetism, we take:

$$\Lambda^\mu{}_0(\lambda) = u^\mu(\lambda), \qquad \Lambda^\mu{}_3(\lambda) = \hat{S}^\mu(\lambda). \tag{3.6.4}$$

The angular velocity tensor of the body is defined by:

$$\Omega^{\mu\nu} = \eta^{AB}\Lambda^\mu{}_A \frac{D\Lambda^\nu{}_B}{D\lambda} \tag{3.6.5}$$

where $\frac{D}{D\lambda}$ indicates covariant λ differentiation:

$$\frac{D\Lambda^\mu{}_A}{D\lambda} = \frac{d\Lambda^\mu{}_A}{d\lambda} + \Gamma^\mu{}_{\rho\sigma}\dot{z}^\rho\Lambda^\sigma{}_A. \tag{3.6.6}$$

The dynamical mass function $\mathcal{M}(z,u,S)$ now encodes the mass of the body and all of its nonminimal couplings to gravity and in particular takes the form:

$$\mathcal{M}^2(z,u,S) = m^2 + \mathcal{O}(R) \tag{3.6.7}$$

where m is the mass of the body in vacuum and $R_{\mu\nu\rho\sigma}$ is the Riemann tensor.

For variations of the action, it is useful to define the covariant variations:

$$\Delta p_\mu = \delta p_\mu - \Gamma^\rho{}_{\sigma\mu} p_\rho \delta z^\sigma \tag{3.6.8}$$

$$\Delta S^{\mu} = \delta S^{\mu} + \Gamma^{\mu}{}_{\rho\sigma} S^{\rho} \delta z^{\sigma} \tag{3.6.9}$$

$$\Delta \Lambda^{\mu}{}_{A} = \delta \Lambda^{\mu}{}_{A} + \Gamma^{\mu}{}_{\rho\sigma} \delta z^{\rho} \Lambda^{\sigma}{}_{A} \tag{3.6.10}$$

and the antisymmetric tensor:

$$\Delta \theta^{\mu\nu} = \eta^{AB} \Lambda^{[\mu}{}_{A} \Delta \Lambda^{\nu]}{}_{B}. \tag{3.6.11}$$

This definition leads to the identity:

$$\Delta \Omega^{\mu\nu} = \frac{D}{D\lambda} \Delta \theta^{\mu\nu} + \Omega^{\mu}{}_{\rho} \Delta \theta^{\rho\nu} - \Omega^{\nu}{}_{\rho} \Delta \theta^{\rho\mu} + R^{\mu\nu}{}_{\rho\sigma} \dot{z}^{\rho} \delta z^{\sigma}. \tag{3.6.12}$$

Then, the variation of the above action gives:

$$\begin{aligned}
\delta \mathcal{S} = \int_{-\infty}^{\infty} \Bigg(& \delta z^{\mu} \left(-\frac{D p_{\mu}}{D\lambda} - R^{\star}_{\mu\nu\rho\sigma} \dot{z}^{\nu} u^{\rho} S^{\sigma} - \mathsf{e} \nabla_{\mu} \mathcal{M} \right) \\
& + \Delta p_{\mu} \Bigg(\dot{z}^{\mu} - \mathsf{e} u^{\mu} - \frac{\mathsf{e}}{|p|} \frac{\partial \mathcal{M}}{\partial u^{\nu}} (g^{\mu\nu} + u^{\mu} u^{\nu}) + \beta S^{\mu} \\
& \qquad + \frac{1}{2|p|} \widetilde{\epsilon}^{\mu\nu\rho\sigma} S_{\nu} \Omega_{\rho\sigma} + \frac{u^{\mu}}{2|p|} \widetilde{\epsilon}_{\alpha\beta\rho\sigma} u^{\alpha} S^{\beta} \Omega^{\rho\sigma} \Bigg) \\
& + \frac{1}{2} \Delta \theta^{\rho\sigma} \left(-\frac{D}{D\lambda} \left(\widetilde{\epsilon}_{\mu\nu\rho\sigma} u^{\mu} S^{\nu} \right) + \widetilde{\epsilon}_{\mu\nu\rho\alpha} u^{\mu} S^{\nu} \Omega^{\alpha}{}_{\sigma} - \widetilde{\epsilon}_{\mu\nu\sigma\alpha} u^{\mu} S^{\nu} \Omega^{\alpha}{}_{\rho} \right) \\
& + \Delta S^{\mu} \left(-\frac{1}{2} \widetilde{\epsilon}_{\mu\nu\rho\sigma} u^{\nu} \Omega^{\rho\sigma} - \mathsf{e} \frac{\partial \mathcal{M}}{\partial S^{\mu}} + \beta p_{\mu} \right) - \frac{\delta\alpha}{2} \left(p^2 + \mathcal{M}^2 \right) + \delta\beta\, p \cdot S \Bigg) d\lambda
\end{aligned} \tag{3.6.13}$$

where the right-dual of the Riemann tensor is defined by:

$$R^{\star}_{\mu\nu\rho\sigma} = \frac{1}{2} \widetilde{\epsilon}_{\rho\sigma}{}^{\alpha\beta} R_{\mu\nu\alpha\beta}. \tag{3.6.14}$$

Using the δS^{μ} variation to solve for the angular velocity tensor, one can then determine the value of β. That value of β can then be used to simplify the spin and trajectory equations of motion. Explicitly, these give:

$$\Omega^{\mu\nu} = \frac{D u^{\mu}}{D\lambda} u^{\nu} - u^{\mu} \frac{D u^{\nu}}{D\lambda} + \mathsf{e} \widetilde{\epsilon}^{\mu\nu\rho\sigma} u_{\rho} \frac{\partial \mathcal{M}}{\partial S^{\sigma}} \tag{3.6.15}$$

$$\beta = -\frac{\mathsf{e}}{\mathcal{M}} u^{\mu} \frac{\partial \mathcal{M}}{\partial S^{\mu}} \tag{3.6.16}$$

$$\frac{D S^{\mu}}{D\lambda} = u^{\mu} \frac{D u^{\nu}}{D\lambda} S_{\nu} + \mathsf{e} \epsilon^{\mu\nu\rho\sigma} u_{\nu} S_{\rho} \frac{\partial \mathcal{M}}{\partial S^{\sigma}} \tag{3.6.17}$$

$$\dot{z}^{\mu} = \mathsf{e} u^{\mu} + \frac{\mathsf{e}}{\mathcal{M}} (g^{\mu\nu} + u^{\mu} u^{\nu}) \frac{\partial \mathcal{M}}{\partial u^{\nu}} + \frac{\mathsf{e}}{\mathcal{M}} S^{\mu} u^{\nu} \frac{\partial \mathcal{M}}{\partial S^{\nu}} + \frac{1}{\mathcal{M}^2} \widetilde{\epsilon}^{\mu\nu\rho\sigma} S_{\nu} u_{\rho} \frac{D p_{\sigma}}{D\lambda}. \tag{3.6.18}$$

In order to determine the trajectory evolution explicitly we must insert the momentum equation of motion into (3.6.18). To simplify, it is useful to introduce the two sided dual Riemann tensor:

$${}^{\star}R^{\star}_{\mu\nu\rho\sigma} = \frac{1}{2} \widetilde{\epsilon}_{\mu\nu\alpha\beta} R^{\star\alpha\beta}{}_{\rho\sigma} \tag{3.6.19}$$

Simplifying finally gives the gravitational MPD equations of motion for the spinning body:

$$\begin{aligned}
\left(1 + \frac{{}^{\star}R^{\star}_{uSuS}}{\mathcal{M}^2} \right) \frac{\dot{z}^{\mu}}{\mathsf{e}} = u^{\mu} + \frac{g^{\mu\nu} + u^{\mu} u^{\nu}}{\mathcal{M}} \frac{\partial \mathcal{M}}{\partial u^{\nu}} + S^{\mu} \frac{u^{\nu}}{\mathcal{M}} \frac{\partial \mathcal{M}}{\partial S^{\nu}} + \frac{1}{\mathcal{M}^2} \widetilde{\epsilon}^{\mu\nu\rho\sigma} u_{\nu} S_{\rho} \nabla_{\sigma} \mathcal{M} \\
- \frac{{}^{\star}R^{\star\mu SuS}}{\mathcal{M}^2} - \frac{1}{\mathcal{M}^3} \left(S^{\nu} \frac{\partial \mathcal{M}}{\partial u^{\nu}} + S^2 u^{\nu} \frac{\partial \mathcal{M}}{\partial S^{\nu}} \right) {}^{\star}R^{\star\mu uuS}
\end{aligned} \tag{3.6.20}$$

$$\frac{Dp_\mu}{D\lambda} = -R^\star_{\mu\nu\rho\sigma}\dot{z}^\nu u^\rho S^\sigma - \mathsf{e}\nabla_\mu \mathcal{M} \tag{3.6.21}$$

$$\frac{DS^\mu}{D\lambda} = u^\mu S_\nu \frac{Du^\nu}{D\lambda} + \mathsf{e}\widetilde{\epsilon}^{\mu\nu\rho\sigma}u_\nu S_\rho \frac{\partial \mathcal{M}}{\partial S^\sigma}. \tag{3.6.22}$$

Vectors such as u and S are used as indices to indicate contractions with them (${}^\star R^{\star\mu uuS} = {}^\star R^{\star\mu\nu\rho\sigma}u_\nu u_\rho S_\sigma$). For solving these equations of motion we will always choose λ so that $\mathsf{e} = 1$.

To understand how the dynamical mass function relates to the multipole moments of the body, we will need to study the energy-momentum tensor produced by our action. The energy-momentum tensor will be given by:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-\det g}}\frac{\delta \mathcal{S}}{\delta g^{\mu\nu}}. \tag{3.6.23}$$

In varying the metric, we have $\delta z^\mu = 0$, $\delta p_\mu = 0$, $\delta S^\mu = 0$, and $\Delta\theta^{\mu\nu} = 0$. $\delta \Lambda^\mu{}_A$ cannot be made 0 as its variation is related to the metric variation on the worldline. It is useful to introduce the DeWitt index shuffling operator [226] $\hat{G}^\alpha{}_\beta$, which acts on tensors according to:

$$\begin{aligned} \hat{G}^\alpha{}_\beta F^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} =& \delta^{\mu_1}_\beta F^{\alpha\mu_2\ldots\mu_m}{}_{\nu_1\ldots\nu_n} + \ldots + \delta^{\mu_m}_\beta F^{\mu_1\ldots\mu_{m-1}\alpha}{}_{\nu_1\ldots\nu_n} \\ &- \delta^\alpha_{\nu_1} F^{\mu_1\ldots\mu_m}{}_{\beta\nu_2\ldots\nu_n} - \ldots - \delta^\alpha_{\nu_n} F^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_{n-1}\beta} \end{aligned} \tag{3.6.24}$$

so that:

$$\nabla_\rho F^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \partial_\rho F^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} + \Gamma^\beta{}_{\alpha\rho}\hat{G}^\alpha{}_\beta F^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}. \tag{3.6.25}$$

For expressing the energy-momentum tensor simply it is important to use the scalarity of the dynamical mass function. In particular, requiring it to be a scalar function of u_μ, S^μ, $g_{\mu\nu}$, and symmetric covariant derivatives of $R_{\mu\nu\rho\sigma}$ implies it is invariant under a small diffeomorphism. This is only true if:

$$\frac{\partial \mathcal{M}}{\partial g^{\alpha\beta}} = \frac{1}{2}\frac{\partial \mathcal{M}}{\partial u^\alpha}u_\beta - \frac{1}{2}S_\alpha \frac{\partial \mathcal{M}}{\partial S^\beta} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{\partial \mathcal{M}}{\partial \nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma}}\hat{G}_{\alpha\beta}\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma}. \tag{3.6.26}$$

With these ingredients, the variation of the action with respect to the metric becomes:

$$\begin{aligned} \delta\mathcal{S} = \int_{-\infty}^{\infty} \Bigg(& -\frac{1}{2}p_\mu \dot{z}_\nu \delta g^{\mu\nu} + \frac{1}{2}\widetilde{\epsilon}_{\mu\nu\rho\sigma}u^\mu S^\nu \dot{z}_\alpha \nabla^\sigma \delta g^{\rho\alpha} \\ & + \mathsf{e}\sum_{n=0}^{\infty}\frac{\partial \mathcal{M}}{\partial \nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma}}\left(\frac{1}{2}\delta g^{\alpha\beta}\hat{G}_{\alpha\beta}\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma} - \delta(\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma})\right)\Bigg) d\lambda \end{aligned} \tag{3.6.27}$$

For computing variations of derivatives of the Riemann curvature, a strategy from Ref. [268] is helpful. Consider a tensor field $F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}$ with the same index symmetries as $\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma}$ which decays to 0 sufficiently quickly at infinity for no surface terms to be necessary upon the relevant integrations by parts we will perform. Then, a short calculation gives:

$$\begin{aligned} &\int_{\mathbb{T}} F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta(\nabla^n_{\lambda_1\ldots\lambda_n}R_{\mu\nu\rho\sigma})D^4x \\ &\quad = \int_{\mathbb{T}}\left(F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta\Gamma^\beta{}_{\alpha\lambda_n}\hat{G}^\alpha{}_\beta\nabla^{n-1}_{\lambda_1\ldots\lambda_{n-1}}R_{\mu\nu\rho\sigma} - \nabla_{\lambda_n}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta(\nabla^{n-1}_{\lambda_1\ldots\lambda_{n-1}}R_{\mu\nu\rho\sigma})\right)D^4x. \end{aligned} \tag{3.6.28}$$

The final term is now in the same form as the initial variational problem, but of a lower rank. Applying this formula iteratively allows all derivatives on the Riemann tensor to eventually be pushed past the variation, resulting in:

$$\int_{\mathbb{T}} F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta(\nabla^n_{\lambda_1\ldots\lambda_n}R_{\mu\nu\rho\sigma})D^4x$$
$$= \int_{\mathbb{T}} \left(\sum_{k=0}^{n-1}(-1)^k\nabla^k_{\lambda_n\ldots\lambda_{n-k+1}}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta\Gamma^{\beta}{}_{\alpha\lambda_{n-k}}\hat{G}^{\alpha}{}_{\beta}\nabla^{n-k-1}_{\lambda_1\ldots\lambda_{n-k-1}}R_{\mu\nu\rho\sigma}\right.$$
$$\left.+(-1)^n\nabla^n_{\lambda_1\ldots\lambda_n}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta R_{\mu\nu\rho\sigma}\right) D^4x. \tag{3.6.29}$$

Now using:

$$\delta\Gamma^{\rho}{}_{\mu\nu} = \frac{1}{2}g_{\mu\alpha}g_{\nu\beta}\nabla^{\rho}\delta g^{\alpha\beta} - \frac{1}{2}g_{\mu\sigma}\nabla_{\nu}\delta g^{\rho\sigma} - \frac{1}{2}g_{\nu\sigma}\nabla_{\mu}\delta g^{\rho\sigma} \tag{3.6.30}$$

$$\delta R^{\rho}{}_{\mu\sigma\nu} = \nabla_{\sigma}\delta\Gamma^{\rho}{}_{\nu\mu} - \nabla_{\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} \tag{3.6.31}$$

we are able to arrive at:

$$\int F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta(\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma})D^4x$$
$$= \int \left(\sum_{k=0}^{n-1}(-1)^k\delta g^{\alpha\beta}g_{\alpha\lambda_{n-k}}\nabla^{\gamma}\nabla^k_{(\lambda_{n-k+1}\ldots\lambda_n)}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\hat{G}_{[\gamma\beta]}(\nabla^{n-k-1}_{(\lambda_1\ldots\lambda_{n-k-1})}R_{\mu\nu\rho\sigma})\right.$$
$$+\sum_{k=0}^{n-1}(-1)^k\delta g^{\alpha\beta}g_{\alpha\lambda_{n-k}}\nabla^k_{(\lambda_{n-k+1}\ldots\lambda_n)}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}g^{\gamma\tau}\hat{G}_{[\tau\beta]}(\nabla_{\gamma}\nabla^{n-k-1}_{(\lambda_1\ldots\lambda_{n-k-1})}R_{\mu\nu\rho\sigma})$$
$$+\sum_{k=0}^{n-1}2(-1)^k\delta g^{\alpha\beta}g_{\alpha\lambda_{n-k}}\nabla^k_{(\lambda_{n-k+1}\ldots\lambda_n)}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\nabla_{\beta}\nabla^{n-k-1}_{(\lambda_1\ldots\lambda_{n-k-1})}R_{\mu\nu\rho\sigma}$$
$$+(-1)^n\delta g^{\alpha\beta}\nabla^n_{(\lambda_1\ldots\lambda_n)}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}g_{\alpha\mu}R_{\beta\nu\rho\sigma} + 2(-1)^n\nabla^2_{(\mu\nu)}\nabla^n_{(\lambda_1\ldots\lambda_n)}F^{\lambda_1\ldots\lambda_n\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\delta g^{\alpha\beta}$$
$$\left.+\frac{1}{2}\delta g^{\alpha\beta}F^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\hat{G}_{\alpha\beta}(\nabla^n_{\lambda_1\ldots\lambda_n}R_{\mu\nu\rho\sigma})\right) D^4x. \tag{3.6.32}$$

Define the gravitational $\mathcal{Q}_n$ moments:

$$\mathcal{Q}_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma} = \frac{\partial\mathcal{M}}{\partial\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\mu\nu\rho\sigma}}. \tag{3.6.33}$$

As well, define the scalar Dirac delta distribution:

$$\delta(X,Z) = \frac{\delta(x-z)}{\sqrt{-\det g}} \tag{3.6.34}$$

and the Φ_n fields:

$$\Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}(X) = \int_{-\infty}^{\infty}\mathcal{Q}_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}\delta(X,Z)D\lambda. \tag{3.6.35}$$

Then, formally as a distributional expression the energy-momentum tensor becomes:

$$T_{\alpha\beta} = \int_{-\infty}^{\infty}\left(\dot{z}_{(\alpha}p_{\beta)}\delta(X,Z) + \nabla^{\gamma}(\dot{z}_{(\alpha}\widetilde{\epsilon}_{\beta)\gamma\rho\sigma}u^{\rho}S^{\sigma}\delta(X,Z))\right)d\lambda + \sum_{n=0}^{\infty}\Psi^{(n)}_{\alpha\beta} \tag{3.6.36}$$

with:

$$\Psi^{(n)}_{\alpha\beta} = \sum_{k=0}^{n-1}(-1)^k \nabla^k_{\lambda_{n-k+1}\ldots\lambda_n}\Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma} g_{\lambda_{n-k}(\alpha}\nabla_{\beta)}\nabla^{n-k-1}_{\lambda_1\ldots\lambda_{n-k-1}}R_{\mu\nu\rho\sigma}$$
$$+\sum_{k=0}^{n-1}2(-1)^k\nabla_\gamma\left(\nabla^k_{\lambda_{n-k+1}\ldots\lambda_n}\Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}g_{\lambda_{n-k}(\alpha}g_{\beta)\delta}\hat{G}^{[\gamma\delta]}(\nabla^{n-k-1}_{\lambda_1\ldots\lambda_{n-k-1}}R_{\mu\nu\rho\sigma})\right)$$
$$+2(-1)^n\nabla^n_{\lambda_1\ldots\lambda_n}\Phi_n^{\lambda_1\ldots\lambda_n\mu\nu\rho\sigma}g_{\mu(\alpha}R_{\beta)\nu\rho\sigma}+4(-1)^n\nabla^2_{\mu\nu}\nabla^n_{\lambda_1\ldots\lambda_n}\Phi_n^{\lambda_1\ldots\lambda_n\mu}{}_{(\alpha}{}^{\nu}{}_{\beta)}. \tag{3.6.37}$$

3.7 Kerr Multipole Moments

In this section we perform the analysis of sections 3.3 and 3.4 but for the Kerr metric in gravity instead of the $\sqrt{\text{Kerr}}$ solution in electromagnetism. We begin by describing Dixon's definition of the multipoles of the energy-momentum tensor. Then, following analysis done by Israel in Ref. [263] we identify the energy-momentum tensor which acts as the source of the Kerr metric in the causally maximal extension of the Kerr spacetime.

3.7.1 Moments of the Energy Momentum Tensor

For precisely the same reasons that cause the naive moments of the current density to be interdependent due to the continuity equation, the naive moments of the energy-momentum tensor are interdependent due to its covariant conservation. Define the quantities:

$$\Theta_n^{\kappa\lambda\mu\nu}(Z,X) = (n-1)\int_0^1 \sigma^{\kappa\alpha}\sigma^{(\mu}{}_{\alpha}\sigma^{\nu)}{}_{\beta}\sigma^{\lambda\beta}t^{n-2}dt, \quad (n\geq 2) \tag{3.7.1}$$

$$\mathfrak{p}_n^{\kappa_1\ldots\kappa_n\lambda\mu\nu} = 2(-1)^n\int_\Sigma \sigma^{\kappa_1}\ldots\sigma^{\kappa_n}\Theta_n^{\rho\nu\lambda\mu}(-\sigma^{-1}_{\alpha\rho})T^{\alpha\beta}d\Sigma_\beta, \quad (n\geq 2) \tag{3.7.2}$$

$$\mathfrak{t}_n^{\kappa_1\ldots\kappa_n\lambda\mu} = (-1)^n\int_\Sigma \sigma^{\kappa_1}\ldots\sigma^{\kappa_n}\sigma^{\lambda}{}_{\alpha}\sigma^{\mu}{}_{\beta}T^{\alpha\beta}w^\gamma d\Sigma_\gamma, \quad (n\geq 2) \tag{3.7.3}$$

$$\mathcal{J}_n^{\kappa_1\ldots\kappa_n\lambda\mu\nu\rho} = \mathfrak{t}_{n+2}^{\kappa_1\ldots\kappa_n[\lambda[^{\prime}\nu\mu]\rho]^{\prime}} + \frac{1}{n+1}\mathfrak{p}_{n+2}^{\kappa_1\ldots\kappa_n[\lambda[^{\prime}\nu\mu]\rho]^{\prime}\tau}\frac{\dot{z}_\tau}{\mathsf{e}} \tag{3.7.4}$$

$$I_n^{\lambda_1\ldots\lambda_n\mu\nu} = \frac{4(n-1)}{n+1}\mathcal{J}_{n-2}^{(\lambda_1\ldots\lambda_{n-1}|\mu|\lambda_n)\nu}, \quad (n\geq 2). \tag{3.7.5}$$

These definitions are precisely analogous to equations (3.3.20) through (3.3.24) for electromagnetism, in precisely the same order. The I_n moments will serve as the interdependence-free reduced multipole moments of the energy-momentum tensor. The other quantities defined are useful intermediate pieces for calculation. For the necessary index symmetrizations, we use the notation that [] brackets antisymmetrize together and [′]′ brackets antrisymmetrize together but that the two ignore each other. For example:

$$I_2^{[\lambda[^{\prime}\nu\mu]\rho]^{\prime}} = \frac{1}{4}\left(I_2^{\lambda\nu\mu\rho}-I_2^{\mu\nu\lambda\rho}-I_2^{\lambda\rho\mu\nu}+I_2^{\mu\rho\lambda\nu}\right). \tag{3.7.6}$$

As well, the inclusion of vertical bars $|\mu|$ around indices in the midst of an (anti)symmetrization indicates that those indices should be skipped over when performing the (anti)symmetrization. For example:

$$I_1^{(\lambda|\mu|\nu)} = \frac{1}{2}(I_1^{\lambda\mu\nu} + I_1^{\nu\mu\lambda}). \tag{3.7.7}$$

Dixon's reduced moments automatically satisfy:

$$I_n^{\lambda_1...\lambda_n\mu\nu} = I_n^{(\lambda_1...\lambda_n)(\mu\nu)}, \qquad I_n^{(\lambda_1...\lambda_n\mu)\nu} = 0, \tag{3.7.8}$$

$$u_{\lambda_1} I_n^{\lambda_1...\lambda_{n-2}[\lambda_{n-1}[' \lambda_n\mu]\nu]'} = 0, \qquad (n \geq 3). \tag{3.7.9}$$

Dixon finds that beyond these conditions, the reduced moments are not restricted by the covariant conservation of $T^{\alpha\beta}$ and that they are independent of each other for different values of n. It is useful to define the 0^{th} and 1^{st} moments:

$$I_0^{\lambda\mu} = \frac{p^{(\lambda}\dot{z}^{\mu)}}{\mathsf{e}}, \qquad I_1^{\kappa\lambda\mu} = \frac{S^{\kappa(\lambda}\dot{z}^{\mu)}}{\mathsf{e}} \tag{3.7.10}$$

where p^λ is the total linear momentum of the body defined by:

$$p^\kappa = \int_\Sigma (-\sigma^{-1}_{\alpha\lambda})\sigma^{\lambda\kappa} T^{\alpha\beta} d\Sigma_\beta \tag{3.7.11}$$

and $S^{\lambda\mu}$ is the total spin tensor of the body defined by:

$$S^{\kappa\lambda} = 2\int_\Sigma \sigma^{[\kappa}(\sigma^{-1})^{\lambda]}{}_\alpha T^{\alpha\beta} d\Sigma_\beta. \tag{3.7.12}$$

We will always choose our definition of the worldline and Cauchy slicing so that $p^\mu(\lambda)$ is orthogonal to all tangent vectors to $\Sigma(\lambda)$. With this choice we can require:

$$p^\mu = |p|u^\mu, \qquad S^{\mu\nu}p_\nu = 0 \tag{3.7.13}$$

and thus we can introduce the spin vector S^μ defined so that:

$$S^\mu = -\frac{1}{2}\tilde{\epsilon}^{\mu\nu\rho\sigma}u_\nu S_{\rho\sigma} \implies S_{\mu\nu} = \tilde{\epsilon}_{\mu\nu\rho\sigma}u^\rho S^\sigma. \tag{3.7.14}$$

The conservation of $T^{\alpha\beta}$ determines the time evolution of p^μ and S^μ (through the MPD equations) but determines nothing about the time evolution of the higher moments. Dixon finds also that the reduced moments are independent of the 0^{th} and 1^{st} moments. Then, define the reduced moment generating function:

$$I^{\mu\nu}(\lambda, k) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} I_n^{\lambda_1...\lambda_n\mu\nu} k_{\lambda_1}...k_{\lambda_n}. \tag{3.7.15}$$

Like the naive moments, the reduced moment generating functions determine the behavior of $T^{\alpha\beta}$ against test functions. In particular, for an arbitrary symmetric tensor field $h_{\alpha\beta}(X)$:

$$\int_\Sigma h^*_{\alpha\beta}(X)T^{\alpha\beta}(X)w^\gamma d\Sigma_\gamma = \int \widetilde{h}^*_{\mu\nu}(Z,k)I^{\mu\nu}(\lambda,k)\frac{D^4k}{(2\pi)^2}. \tag{3.7.16}$$

The moment generating function automatically satisfies:

$$I^{\lambda\mu}k_\lambda = I_0^{\lambda\mu}k_\lambda - ik_\kappa k_\lambda I_1^{\kappa\lambda\mu}. \tag{3.7.17}$$

Dixon proved [259] that these reduced moments are the unique set of moments which are independent of each other for different n, have only I_0 and I_1 restricted by the conservation law, and satisfy the index symmetry conditions in equations (3.7.8) and (3.7.9).

Through (3.7.16), the energy-momentum tensor is determined in terms of the reduced multipole moments. Explicitly comparing that behavior against test functions to (3.6.36) and using crucially that the reduced multipole moments are unique and contain no interdependencies, we can identify:

$$I_n^{\rho_1\ldots\rho_n\mu\nu} = 4n!\mathcal{Q}_{n-2}^{(\rho_1\ldots\rho_{n-1}|\mu|\rho_n)\nu} + \mathcal{O}(R) \tag{3.7.18}$$

which gives the reduced multipole moments from the couplings in the action. Alternatively, this can be nicely inverted using the index symmetry conditions of both quantities to find:

$$\mathcal{Q}_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu} = \frac{n+1}{(n+3)!}I_{n+2}^{\lambda_1\ldots\lambda_n[\rho['\sigma\mu]\nu]'} + \mathcal{O}(R) \tag{3.7.19}$$

This allows the direct determination of the coupling of the body to the Riemann tensor in the action from its reduced multipole moments.

3.7.2 Source of the Kerr Metric

Here we summarize the analysis of Ref. [263] to identify the energy-momentum tensor which produces the Kerr metric. The Kerr metric has no intrinsic singularities away from $r = 0$ and everywhere away from $r = 0$ it is a solution to the vacuum Einstein equations. Therefore, the source of the Kerr metric can only have support on the surface $r = 0$. This surface is a disk and we introduce the same coordinate $\chi = \theta$ above the disk and $\chi = \pi - \theta$ below the disk as we did for $\sqrt{\text{Kerr}}$ so that t, χ, φ provide an intrinsic coordinate system. Restricting the Kerr solution to the surface $r = 0$ produces the flat metric γ_{ij} on the disk:

$$\begin{aligned} &\gamma_{tt} = -1, && \gamma_{t\chi} = 0, && \gamma_{t\varphi} = 0, \\ &\gamma_{\chi\chi} = a^2\cos^2\chi, && \gamma_{\chi\varphi} = 0, && \gamma_{\varphi\varphi} = a^2\sin^2\chi. \end{aligned} \tag{3.7.20}$$

The extrinsic curvature K_{ij} of the disk, when approached from above toward $r = 0$, is determined by the first r derivative of the Kerr solution and produces:

$$\begin{aligned} &K_{tt} = \frac{Gm}{a^2\cos^3\chi}, && K_{t\chi} = 0, && K_{t\varphi} = -\frac{Gm}{a}\frac{\sin^2\chi}{\cos^3\chi}, \\ &K_{\chi\chi} = 0, && K_{\chi\varphi} = 0, && K_{\varphi\varphi} = Gm\frac{\sin^4\chi}{\cos^3\chi}. \end{aligned} \tag{3.7.21}$$

The extrinsic curvature tensor of the disk when approached from below is simply the negative of the extrinsic curvature when approached from above. We may now use Israel's junction conditions to determine the surface energy-momentum tensor on the disk $\mathrm{S}_{ij}^{\text{disk}}$:

$$\mathrm{S}_{ij}^{\text{disk}} = -\frac{1}{8\pi G}\left(K_{ij} - K\gamma_{ij}\right)\Big|_{\text{below}}^{\text{above}}. \tag{3.7.22}$$

The resulting surface energy-momentum tensor is:

$$\mathrm{S}_{ij}^{\text{disk}} = \frac{\sigma_{\text{disk}}}{2}(\zeta_i\zeta_j + \xi_i\xi_j) \tag{3.7.23}$$

where:

$$\sigma_{\text{disk}} = -\frac{m}{2\pi a^2\cos^3\chi} \tag{3.7.24}$$

$$\zeta_t = 0, \qquad \zeta_\chi = a\cos^2\chi, \qquad \zeta_\varphi = 0 \tag{3.7.25}$$

$$\xi_t = -\sin\chi, \qquad \xi_\chi = 0, \qquad \xi_\varphi = a\sin\chi. \tag{3.7.26}$$

As a distribution the disk energy-momentum tensor is:

$$T_{\mu\nu}^{\text{disk}} = \frac{\sigma_{\text{disk}}}{2}(\zeta_\mu\zeta_\nu + \xi_\mu\xi_\nu)\delta(r\cos\chi). \tag{3.7.27}$$

Taking the worldline which passes through the center of the disk as the worldline of the metric, we can use equations (3.7.11) and (3.7.12) to compute the total momentum and spin of the Kerr solution. With only the given surface energy-momentum tensor, the resultant linear momentum and spin are not mu^μ and $m\epsilon_{\mu\nu\rho\sigma}u^\rho a^\sigma$. Instead, the integrals diverge as $\chi \to \frac{\pi}{2}$ in precisely the same way as occurred for $\sqrt{\text{Kerr}}$. In order to produce the correct total momentum and spin it is necessary to have a linear energy-momentum tensor density on the ring singularity at $r = 0, \chi = \frac{\pi}{2}$. In particular, the necessary effective mass density is:

$$\rho = -\frac{m}{2\pi a^2\cos^4\chi}\delta(r)\vartheta\left(\frac{\pi}{2} - \varepsilon - \chi\right) + \frac{m}{2\pi a^2\sin\varepsilon\cos^2\chi}\delta(r)\delta\left(\chi - \frac{\pi}{2}\right) \tag{3.7.28}$$

which is the same density as the $\sqrt{\text{Kerr}}$ solution. The resulting energy-momentum tensor is:

$$T_{\mu\nu} = \frac{\rho}{2}(\zeta_\mu\zeta_\nu + \xi_\mu\xi_\nu). \tag{3.7.29}$$

With this energy-momentum tensor using (3.7.11) and (3.7.12) we have precisely:

$$p^\mu = mu^\mu, \qquad S^\mu = ma^\mu. \tag{3.7.30}$$

3.7.3 Stationary Multipole Moments of Kerr

We now consider the Minkowski space limit of Dixon's moments for the energy-momentum tensor. Using the same y^a coordinates as before, we find:

$$\mathfrak{p}_n^{\kappa_1...\kappa_n\lambda\mu\nu} = 2\Lambda^{\kappa_1}{}_{A_1}...\Lambda^{\kappa_n}{}_{A_n}\eta^{\nu(\mu}\int_\Sigma(-T^{\lambda)\rho}u_\rho)y^{A_1}...y^{A_n}d^3y + \mathcal{O}(R) \tag{3.7.31}$$

$$\mathfrak{t}_n^{\kappa_1...\kappa_n\lambda\mu} = \Lambda^{\kappa_1}{}_{A_1}...\Lambda^{\kappa_n}{}_{A_n}\int_\Sigma y^{A_1}...y^{A_n}T^{\lambda\mu}d^3y + \mathcal{O}(R). \tag{3.7.32}$$

Just like with the current density, we now define the naive K moments:

$$K_n^{a_1...a_nBC} = \int_\Sigma y^{a_1}...y^{a_n}\Lambda_\alpha{}^B\Lambda_\beta{}^C T^{\alpha\beta}d^3y. \tag{3.7.33}$$

In terms of these moments, we have:

$$\mathfrak{p}_n^{\kappa_1...\kappa_n\lambda\mu\nu} = 2\Lambda^{\kappa_1}{}_{a_1}...\Lambda^{\kappa_n}{}_{a_n}\eta^{\nu(\mu}\Lambda^{\lambda)}{}_B K_n^{a_1...a_nB0} + \mathcal{O}(R) \tag{3.7.34}$$

$$\mathfrak{t}_n^{\kappa_1...\kappa_n\lambda\mu} = \Lambda^{\kappa_1}{}_{a_1}...\Lambda^{\kappa_n}{}_{a_n}\Lambda^\lambda{}_B\Lambda^\mu{}_C K_n^{a_1...a_nBC} + \mathcal{O}(R). \tag{3.7.35}$$

For arbitrary vectors on the internal Lorentz indices, k^A and v^A we have:

$$K_n^{a_1...a_nBC}k_{a_1}...k_{a_n}v_Bv_C = \int_\Sigma(x^ak_a)^n\frac{\sigma(\mathfrak{r})}{2}\delta(z)((\zeta^Av_A)^2 + (\xi^Av_A)^2)d^3x. \tag{3.7.36}$$

Following the same integrations as for $\sqrt{\text{Kerr}}$ allows these moments to be computed with no additional complications giving:

$$K_{2n}^{a_1...a_{2n}BC}k_{a_1}...k_{a_{2n}}v_Bv_C = m\frac{n+1}{2n+1}|\vec{k}\times\vec{a}|^{2n}v_0^2 + m\frac{n}{2n+1}|\vec{k}\times\vec{a}|^{2n-2}(\vec{a}\cdot(\vec{k}\times\vec{v}))^2 \tag{3.7.37}$$

$$K_{2n+1}^{a_1...a_{2n+1}BC}k_{a_1}...k_{a_{2n+1}}v_Bv_C = mv_0|\vec{a}\times\vec{k}|^{2n}\vec{a}\cdot(\vec{k}\times\vec{v}). \tag{3.7.38}$$

3.7.4 Dynamical Multipole Moments of Kerr

Now that we have the stationary moments of the Kerr solution, we can use these to compute the dynamical moments of a spinning black hole, up to corrections of order of the Riemann tensor, exactly analogously to the calculation for $\sqrt{\text{Kerr}}$. We can compute the reduced moments of the energy-momentum tensor by returning the stationary results through the chain of definitions defining $I_n^{\rho_1\cdots\rho_n\mu\nu}$. Using the same notation as for electromagnetism, we find:

$$\begin{aligned} I_{2n}^{\rho_1...\rho_{2n}\mu\nu}k_{\rho_1}...k_{\rho_{2n}}v_\mu v_\nu = \frac{n+1}{2n+1}ma^{2n}k_\perp^{2n-4}\bigg(&k_\perp^4v_0^2 - 2k_\perp^2(k_\perp\cdot v)k_0v_0 + \frac{k_\perp^2v_\perp^2}{2n-1}k_0^2 \\ &+2\frac{n-1}{2n-1}(k_\perp\cdot v)^2k_0^2 + \frac{n}{n+1}k_\perp^2\left(\epsilon_{\mu\nu\rho\sigma}u^\mu\hat{a}^\nu k^\rho v^\sigma\right)^2\bigg) + \mathcal{O}(R) \end{aligned} \tag{3.7.39}$$

$$I_{2n+1}^{\rho_1...\rho_{2n+1}\mu\nu}k_{\rho_1}...k_{\rho_{2n+1}}v_\mu v_\nu = ma^{2n+1}k_\perp^{2n-2}\epsilon_{\mu\nu\rho\sigma}u^\mu\hat{a}^\nu k^\rho v^\sigma(k_\perp^2v_0 - (k_\perp\cdot v)k_0) + \mathcal{O}(R). \tag{3.7.40}$$

By returning these to (3.7.19), we find that:

$$\mathcal{Q}_{2n}^{\lambda_1\ldots\lambda_{2n}\rho\mu\sigma\nu}\nabla^{2n}_{(\lambda_1\ldots\lambda_{2n})}R_{\rho\mu\sigma\nu} = \frac{ma^{2n+2}\nabla_\perp^{2n-2}}{(2n+3)!}\Big((n+2)\nabla_\perp^2 \perp^{\rho\sigma} R_{\rho u\sigma u} + \frac{n+1}{2}\nabla_\perp^2 \perp^{\rho\sigma}\perp^{\mu\nu} R_{\rho\mu\sigma\nu} - \frac{n(n+2)}{2n+1}(u\cdot\nabla)^2 \perp^{\rho\sigma}\perp^{\mu\nu} R_{\rho\mu\sigma\nu}\Big) \tag{3.7.41}$$

$$\mathcal{Q}_{2n+1}^{\lambda_1\ldots\lambda_{2n+1}\rho\mu\sigma\nu}\nabla^{2n+1}_{(\lambda_1\ldots\lambda_{2n+1})}R_{\rho\mu\sigma\nu} = \frac{ma^{2n+2}}{(2n+3)!}\nabla_\perp^{2n}R^\star_{\nabla_\perp uua} \tag{3.7.42}$$

(up to terms which are quadratic in the Riemann tensor). $\nabla_\perp$ is consistent with our use of the $\perp$ symbol: $\nabla_\perp^\rho = \perp^{\rho\sigma}\nabla_\sigma$ and is used as an index to indicate contraction just as with u and a. Now returning these to the dynamical mass function produces:

$$\mathcal{M}^2 = m^2 + 2m^2a^2\mathscr{F}_1(a\nabla_\perp)\perp^{\rho\sigma} R_{\rho u\sigma u} + 2m^2a^2\mathscr{F}_2(a\nabla_\perp)\perp^{\rho\sigma}\perp^{\mu\nu} R_{\rho\mu\sigma\nu} + 2m^2a^4\mathscr{F}_3(a\nabla_\perp)(u\cdot\nabla)^2\perp^{\rho\sigma}\perp^{\mu\nu} R_{\rho\mu\sigma\nu} + 2m^2a^2\mathscr{F}_4(a\nabla_\perp)R^\star_{\nabla_\perp uua} + \mathcal{O}(R^2) \tag{3.7.43}$$

where:

$$\mathscr{F}_1(x) = \frac{\cosh x}{2x^2} + \frac{\sinh x}{2x^3} - \frac{1}{x^2} \tag{3.7.44}$$

$$\mathscr{F}_2(x) = \frac{\cosh x}{4x^2} - \frac{\sinh x}{4x^3} \tag{3.7.45}$$

$$\mathscr{F}_3(x) = \frac{1}{x^4} - \frac{5}{8}\frac{\cosh x}{x^4} - \frac{3}{8}\frac{\sinh x}{x^5} + \frac{3}{8x^3}\int_0^x \frac{\sinh t}{t}dt \tag{3.7.46}$$

$$\mathscr{F}_4(x) = \frac{\sinh x - x}{x^3}. \tag{3.7.47}$$

The dynamical mass function in (3.7.43) is our principal result for spinning black holes, analogous to (3.4.54). If we neglect $u\cdot\nabla$ terms and consider only contributions to the dynamical mass function which are nonzero for a local vacuum solution of Einstein's equations ($R_{\mu\nu}=0$), then (3.7.43) simplifies to:

$$\mathcal{M}^2 = m^2 - 2m^2\frac{1-\cos(a\cdot\nabla)}{(a\cdot\nabla)^2}R_{uaua} + 2m^2\frac{(a\cdot\nabla)-\sin(a\cdot\nabla)}{(a\cdot\nabla)^2}R^\star_{uaua} + \mathcal{O}(R^2) \tag{3.7.48}$$

which are the equivalent couplings of Ref. [145]. While (3.7.43) and (3.7.48) produce the same three point amplitude (and so the same stationary energy-momentum tensor), they do not produce the same Compton amplitudes. Only (3.7.43) satisfies (3.7.16) as an off-shell statement for black holes. In this way, (3.7.43) uniquely captures the physical multipole moments for a spinning black hole independent of its motion.

3.8 Gravitational Compton Amplitude

3.8.1 Formal Classical Compton

To write Einstein's equations explicitly for the metric perturbation, it is useful to introduce a shorthand for the volume form Jacobian:

$$V = \frac{1}{\sqrt{-\det g}}. \tag{3.8.1}$$

and to define the inverse metric tensor density $\mathfrak{g}^{\mu\nu}$:

$$\mathfrak{g}^{\mu\nu} = g^{\mu\nu}\sqrt{-\det g}. \tag{3.8.2}$$

Then, derivatives of the metric can be expressed as:

$$\partial_\rho g^{\mu\nu} = V\left(\partial_\rho \mathfrak{g}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\partial_\rho \mathfrak{g}^{\alpha\beta}\right), \qquad \partial_\rho V = -\frac{V^2}{2}g_{\mu\nu}\partial_\rho \mathfrak{g}^{\mu\nu}. \tag{3.8.3}$$

With these definitions, we find that:

$$\begin{aligned}
\frac{2}{V^2}\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right) = {} & \mathfrak{g}^{\rho\sigma}\partial^2_{\rho\sigma}\mathfrak{g}^{\mu\nu} + \mathfrak{g}^{\mu\nu}\partial^2_{\rho\sigma}\mathfrak{g}^{\rho\sigma} - \mathfrak{g}^{\mu\sigma}\partial^2_{\rho\sigma}\mathfrak{g}^{\rho\nu} - \mathfrak{g}^{\nu\sigma}\partial^2_{\rho\sigma}\mathfrak{g}^{\rho\mu} \\
& - \partial_\sigma \mathfrak{g}^{\mu\rho}\partial_\rho \mathfrak{g}^{\nu\sigma} + \partial_\rho \mathfrak{g}^{\rho\sigma}\partial_\sigma \mathfrak{g}^{\mu\nu} - g_{\alpha\beta}g^{\rho\sigma}\partial_\rho \mathfrak{g}^{\mu\alpha}\partial_\sigma \mathfrak{g}^{\nu\beta} \\
& + g^{\mu\beta}g_{\rho\alpha}\partial_\sigma \mathfrak{g}^{\nu\alpha}\partial_\beta \mathfrak{g}^{\rho\sigma} + g^{\nu\beta}g_{\rho\alpha}\partial_\sigma \mathfrak{g}^{\mu\alpha}\partial_\beta \mathfrak{g}^{\rho\sigma} - \frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\partial_\sigma \mathfrak{g}^{\alpha\beta}\partial_\beta \mathfrak{g}^{\rho\sigma} \\
& - \frac{1}{8}(2g^{\mu\tau}g^{\nu\omega} - g^{\mu\nu}g^{\tau\omega})(2g_{\alpha\rho}g_{\beta\sigma} - g_{\alpha\beta}g_{\rho\sigma})\partial_\tau \mathfrak{g}^{\alpha\beta}\partial_\omega \mathfrak{g}^{\rho\sigma}.
\end{aligned} \tag{3.8.4}$$

This expression is true in any coordinates. Going forward we will only use de Donder gauge, defined so that the coordinates are harmonic functions when viewed as scalars:

$$\nabla^2 x^\mu = -g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta} = V\partial_\nu \mathfrak{g}^{\mu\nu} \overset{!}{=} 0. \tag{3.8.5}$$

Using this gauge, Einstein's equations can be written exactly in Landau-Lifshitz form as:

$$\begin{aligned}
-\mathfrak{g}^{\rho\sigma}\partial^2_{\rho\sigma}\mathfrak{g}^{\mu\nu} = {} & -16\pi G T^{\mu\nu}|\det \mathfrak{g}| - \partial_\sigma \mathfrak{g}^{\mu\rho}\partial_\rho \mathfrak{g}^{\nu\sigma} - g_{\alpha\beta}g^{\rho\sigma}\partial_\rho \mathfrak{g}^{\mu\alpha}\partial_\sigma \mathfrak{g}^{\nu\beta} \\
& + g^{\mu\beta}g_{\rho\alpha}\partial_\sigma \mathfrak{g}^{\nu\alpha}\partial_\beta \mathfrak{g}^{\rho\sigma} + g^{\nu\beta}g_{\rho\alpha}\partial_\sigma \mathfrak{g}^{\mu\alpha}\partial_\beta \mathfrak{g}^{\rho\sigma} - \frac{1}{2}g^{\mu\nu}g_{\alpha\rho}\partial_\sigma \mathfrak{g}^{\alpha\beta}\partial_\beta \mathfrak{g}^{\rho\sigma} \\
& - \frac{1}{8}(2g^{\mu\tau}g^{\nu\omega} - g^{\mu\nu}g^{\tau\omega})(2g_{\alpha\rho}g_{\beta\sigma} - g_{\alpha\beta}g_{\rho\sigma})\partial_\tau \mathfrak{g}^{\alpha\beta}\partial_\omega \mathfrak{g}^{\rho\sigma}
\end{aligned} \tag{3.8.6}$$

For studying gravitational waves we perturb about Minkowski space:

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}, \qquad \kappa = \sqrt{32\pi G}. \tag{3.8.7}$$

For this choice of coupling constant, the Einstein-Hilbert action is canonically normalized as a functional of the perturbation $h^{\mu\nu}$. When considering perturbations of Minkowski space, we raise and lower indices by using the Minkowski metric. In terms of $h^{\mu\nu}$, de Donder gauge is the requirement:

$$\partial_\nu h^{\mu\nu} = 0. \tag{3.8.8}$$

In terms of $h^{\mu\nu}$, Einstein's equations are exactly:

$$\begin{aligned}
-\partial^2 h^{\mu\nu} = \kappa \Bigg(& -\frac{T^{\mu\nu}}{2}|\det \mathfrak{g}| + h^{\rho\sigma}\partial^2_{\rho\sigma}h^{\mu\nu} - \partial_\sigma h^{\mu\rho}\partial_\rho h^{\nu\sigma} - g_{\alpha\beta}g^{\rho\sigma}\partial_\rho h^{\mu\alpha}\partial_\sigma h^{\nu\beta} \\
& + g^{\mu\beta}g_{\rho\alpha}\partial_\sigma h^{\nu\alpha}\partial_\beta h^{\rho\sigma} + g^{\nu\beta}g_{\rho\alpha}\partial_\sigma h^{\mu\alpha}\partial_\beta h^{\rho\sigma} - \frac{1}{2}g^{\mu\nu}g_{\alpha\rho}\partial_\sigma h^{\alpha\beta}\partial_\beta h^{\rho\sigma} \\
& - \frac{1}{8}(2g^{\mu\tau}g^{\nu\omega} - g^{\mu\nu}g^{\tau\omega})(2g_{\alpha\rho}g_{\beta\sigma} - g_{\alpha\beta}g_{\rho\sigma})\partial_\tau h^{\alpha\beta}\partial_\omega h^{\rho\sigma}\Bigg)
\end{aligned} \tag{3.8.9}$$

With κ acting as the coupling constant, $h^{\mu\nu}$ and $T^{\mu\nu}$ will have solutions in powers of κ:

$$\begin{aligned}
h^{\mu\nu} &= h^{\mu\nu}_{(0)} + \kappa h^{\mu\nu}_{(1)} + \kappa^2 h^{\mu\nu}_{(2)} + \mathcal{O}(\kappa^3) \\
T^{\mu\nu} &= T^{\mu\nu}_{(0)} + \kappa T^{\mu\nu}_{(1)} + \mathcal{O}(\kappa^2)
\end{aligned} \tag{3.8.10}$$

The κ^0 piece then satisfies the homogeneous wave equation from Einstein's equations:

$$\partial^2 h^{\mu\nu}_{(0)} = 0. \tag{3.8.11}$$

For Compton scattering, we will consider the incoming gravitational field to be a plane wave:

$$h^{\mu\nu}_{(0)} = \mathcal{E}^{\mu\nu}_1 e^{ik_1\cdot x} \tag{3.8.12}$$

for some polarization tensor $\mathcal{E}^{\mu\nu}_1$. We will further gauge fix within de Donder gauge so that $\mathcal{E}^{\mu\nu}_1$ is traceless, transverse to $k_{1\mu}$ and orthogonal to a vector v^μ. Using the helicity polarization vectors from electromagnetism, the most general such tensor may be written as a linear combination:

$$\mathcal{E}^{\mu\nu}_1 = c_+\mathcal{E}^\mu_{1+}\mathcal{E}^\nu_{1+} + c_-\mathcal{E}^\mu_{1-}\mathcal{E}^\nu_{1-}. \tag{3.8.13}$$

Therefore for considering Compton scattering in the helicity basis we will take the incoming plane wave to be of the form:

$$h^{\mu\nu}_{(0)} = \epsilon\mathcal{E}^\mu_1\mathcal{E}^\nu_1 e^{ik_1\cdot x}. \tag{3.8.14}$$

From Einstein's equations, the κ^1 piece $h^{\mu\nu}_{(1)}$ then satisfies:

$$\begin{aligned}
-\partial^2 h^{\mu\nu}_{(1)} = & -\frac{T^{\mu\nu}_{(0)}}{2} + h^{\rho\sigma}_{(0)}\partial^2_{\rho\sigma}h^{\mu\nu}_{(0)} - \partial_\sigma h^{\mu\rho}_{(0)}\partial_\rho h^{\nu\sigma}_{(0)} - \eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho h^{\mu\alpha}_{(0)}\partial_\sigma h^{\nu\beta}_{(0)} \\
& + \eta^{\mu\beta}\eta_{\rho\alpha}\partial_\sigma h^{\nu\alpha}_{(0)}\partial_\beta h^{\rho\sigma}_{(0)} + \eta^{\nu\beta}\eta_{\rho\alpha}\partial_\sigma h^{\mu\alpha}_{(0)}\partial_\beta h^{\rho\sigma}_{(0)} - \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\rho}\partial_\sigma h^{\alpha\beta}_{(0)}\partial_\beta h^{\rho\sigma}_{(0)} \\
& - \frac{1}{8}(2\eta^{\mu\tau}\eta^{\nu\omega} - \eta^{\mu\nu}\eta^{\tau\omega})(2\eta_{\alpha\rho}\eta_{\beta\sigma} - \eta_{\alpha\beta}\eta_{\rho\sigma})\partial_\tau h^{\alpha\beta}_{(0)}\partial_\omega h^{\rho\sigma}_{(0)}.
\end{aligned} \tag{3.8.15}$$

For Compton scattering we are only concerned with the response of the system to linear order in the incoming field strength ϵ and so it is useful to define $h^{\mu\nu}_{\text{stat}}$ as the stationary response of the metric perturbation to the unperturbed energy-momentum tensor. In the equation of motion for $h^{\mu\nu}_{(1)}$, all of the contributions from $h^{\mu\nu}_{(0)}$

are of order ϵ^2 and so:
$$h^{\mu\nu}_{(1)} = h^{\mu\nu}_{\text{stat}} + \mathcal{O}(\epsilon^2), \qquad -\partial^2 h^{\mu\nu}_{\text{stat}} = -\frac{T^{\mu\nu}_{(0)}}{2}. \tag{3.8.16}$$

The κ^1 correction to the metric perturbation can be decomposted into a statioanry piece from iterating corrections from $h^{\mu\nu}_{\text{stat}}$, which is ϵ independent, a piece which is linear in ϵ, and pieces which are of at least $\mathcal{O}(\epsilon^2)$:
$$h^{\mu\nu}_{(2)} = h^{\mu\nu}_{\text{stat},2} + \epsilon\delta h^{\mu\nu} + \mathcal{O}(\epsilon^2). \tag{3.8.17}$$

Because the lowest order stationary correction to the actual metric $g_{\mu\nu}$ is $\mathcal{O}(\kappa^2)$, the κ^1 correction to the energy-momentum tensor has no ϵ^0 piece and so may be written:
$$T^{\mu\nu}_{(1)} = \epsilon\delta T^{\mu\nu} + \mathcal{O}(\epsilon^2). \tag{3.8.18}$$

The quantity $\delta h^{\mu\nu}$ thus encodes the linearized response of gravity to the interaction of an incoming plane wave with the massive body and so determines the Compton amplitude. Returning these definitions to Einstein's equations, we find that $\delta h^{\mu\nu}$ satisfies:
$$\begin{aligned}
-\epsilon\partial^2\delta h^{\mu\nu} = \; & -\frac{\epsilon}{2}\delta T^{\mu\nu} + h^{\rho\sigma}_{\text{stat}}\partial^2_{\rho\sigma}h^{\mu\nu}_{(0)} + h^{\rho\sigma}_{(0)}\partial^2_{\rho\sigma}h^{\mu\nu}_{\text{stat}} - \partial_\sigma h^{\mu\rho}_{\text{stat}}\partial_\rho h^{\nu\sigma}_{(0)} - \partial_\sigma h^{\mu\rho}_{(0)}\partial_\rho h^{\nu\sigma}_{\text{stat}} \\
& - \partial^\rho h^{\mu\sigma}_{\text{stat}}\partial_\rho h^{\nu}_{(0)\sigma} - \partial^\rho h^{\mu\sigma}_{(0)}\partial_\rho h^{\nu}_{\text{stat}\sigma} + \partial^\mu h^{\rho\sigma}_{\text{stat}}\partial_\rho h^{\nu}_{(0)\sigma} + \partial^\mu h^{\rho\sigma}_{(0)}\partial_\rho h^{\nu}_{\text{stat}\sigma} \\
& + \partial^\nu h^{\rho\sigma}_{\text{stat}}\partial_\rho h^{\mu}_{(0)\sigma} + \partial^\nu h^{\rho\sigma}_{(0)}\partial_\rho h^{\mu}_{\text{stat}\sigma} - \eta^{\mu\nu}\partial_\lambda h^{\rho\sigma}_{\text{stat}}\partial_\rho h^{\lambda}_{(0)\sigma} \\
& - \frac{1}{4}\left(\eta^{\mu\tau}\eta^{\nu\omega} + \eta^{\nu\tau}\eta^{\mu\omega} - \eta^{\mu\nu}\eta^{\tau\omega}\right)\left(2\eta_{\alpha\rho}\eta_{\beta\sigma} - \eta_{\alpha\beta}\eta_{\rho\sigma}\right)\partial_\tau h^{\alpha\beta}_{\text{stat}}\partial_\omega h^{\rho\sigma}_{(0)}.
\end{aligned} \tag{3.8.19}$$

Define the source fluctuation $\delta\tau^{\mu\nu}$ so that:
$$-\partial^2\delta h^{\mu\nu} = -\frac{\delta\tau^{\mu\nu}}{2}. \tag{3.8.20}$$

Because the tree level Compton is $\mathcal{O}(\kappa^2)$, it only depends on linear and quadratic in R terms in the dynamical mass function. Consequently, we will consider the dynamical mass function to be of the form:
$$\mathcal{M}^2 = m^2 + \delta\mathcal{M}_1^2 + \delta\mathcal{M}_2^2 + \mathcal{O}(R^3) \tag{3.8.21}$$

where $\delta\mathcal{M}_1^2$ is of the form:
$$\delta\mathcal{M}_1^2 = \sum_{n=0}^{\infty} T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}(g,u,S)\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\rho\mu\sigma\nu} \tag{3.8.22}$$

for some functions $T_n^{\cdots}$ with the same index symmetries as $\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\rho\mu\sigma\nu}$ and where $\delta\mathcal{M}_2^2$ is of the form:
$$\delta\mathcal{M}_2^2 = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} V_{nl}^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu|\kappa_1\ldots\kappa_l\gamma\alpha\delta\beta}(g,u,S)\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\rho\mu\sigma\nu}\nabla^l_{(\kappa_1\ldots\kappa_l)}R_{\gamma\alpha\delta\beta} \tag{3.8.23}$$

for some functions $V_{nl}^{\cdots}$ with the same index symmetries as $\nabla^n_{(\lambda_1\ldots\lambda_n)}R_{\rho\mu\sigma\nu}\nabla^l_{(\kappa_1\ldots\kappa_l)}R_{\gamma\alpha\delta\beta}$.

Einstein's equations together with the MPD equations with the described initial conditions will produce solutions of the form:

$$z^\mu(\lambda) = v^\mu \lambda + \kappa\epsilon\delta z^\mu(\lambda) + \mathcal{O}(\kappa^2) \tag{3.8.24}$$

$$u_\mu(\lambda) = v_\mu + \kappa\epsilon\delta u_\mu(\lambda) + \mathcal{O}(\kappa^2) \tag{3.8.25}$$

$$S^\mu(\lambda) = s^\mu + \kappa\epsilon\delta S^\mu(\lambda) + \mathcal{O}(\kappa^2) \tag{3.8.26}$$

$$T^{\mu\nu}(X) = T^{\mu\nu}_{(0)}(X) + \kappa\epsilon\delta T^{\mu\nu} + \mathcal{O}(\kappa^2, \epsilon^2) \tag{3.8.27}$$

$$h^{\mu\nu}(X) = \epsilon\mathcal{E}_1^\mu\mathcal{E}_1^\nu e^{ik_1\cdot x} + \kappa h^{\mu\nu}_{\text{stat}}(X) + \kappa^2 h^{\mu\nu}_{\text{stat},2}(X) + \kappa^2\epsilon\delta h^{\mu\nu}(X) + \mathcal{O}(\kappa^3, \epsilon^2). \tag{3.8.28}$$

The equation of motion perturbations will be oscillatory from solving the MPD equations. In particular, their solutions take the form:

$$\delta z^\mu = \delta\widetilde{z}^\mu e^{ik_1\cdot v\lambda}, \qquad \delta u_\mu = \delta\widetilde{u}_\mu e^{ik_1\cdot v\lambda}, \qquad \delta S^\mu = \delta\widetilde{S}^\mu e^{ik_1\cdot v\lambda} \tag{3.8.29}$$

for constant vectors $\delta\widetilde{z}, \delta\widetilde{u}, \delta\widetilde{S}$.

It is useful to define the function $\mathcal{N}(u, S, k, \mathcal{E})$:

$$\mathcal{N}(u, S, k, \mathcal{E}) = -\sum_{n=0}^{\infty}\frac{i^n}{m}T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}(\eta, u, S)k_{\lambda_1}\ldots k_{\lambda_n}\mathcal{E}_\rho k_\mu\mathcal{E}_\sigma k_\nu \tag{3.8.30}$$

so that for a plane-wave:

$$\delta\mathcal{M}_1^2\big|_{\text{plane-wave}} = 2m\kappa\epsilon\mathcal{N}(u, S, k, \mathcal{E})e^{ik\cdot z} + \mathcal{O}(\kappa^2). \tag{3.8.31}$$

Similarly, we define the function $\mathcal{P}(u, S, k, \mathcal{E}, k', \mathcal{E}')$:

$$\mathcal{P}(u, S, k, \mathcal{E}, k', \mathcal{E}') = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{4i^{l-n}}{m}V_{nl}^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu|\kappa_1\ldots\kappa_l\gamma\alpha\delta\beta}k'_{\lambda_1}\ldots k'_{\lambda_n}k'_\rho\mathcal{E}'^*_\mu k'_\sigma\mathcal{E}'^*_\nu k_{\kappa_1}\ldots k_{\kappa_l}k_\gamma\mathcal{E}_\alpha k_\delta\mathcal{E}_\beta \tag{3.8.32}$$

and adopt the shorthand:

$$\mathcal{N}_1 = \mathcal{N}(v, s, k_1, \mathcal{E}_1), \qquad \mathcal{N}_2 = \mathcal{N}(v, s, k_2, \mathcal{E}_2), \qquad \mathcal{P}_{12} = \mathcal{P}(v, s, k_1, \mathcal{E}_1, k_2, \mathcal{E}_2). \tag{3.8.33}$$

Using the delayed Green's function solution to the wave equation to solve Einstein's equations for $\delta h^{\mu\nu}$, we find:

$$\delta h^{\mu\nu}(X) = -\int_{\mathbb{S}}\frac{\delta\tau^{\mu\nu}(t - |\vec{x} - \vec{x}'|, \vec{x}')}{8\pi|\vec{x} - \vec{x}'|}d^3\vec{x}' \tag{3.8.34}$$

where t and $\vec{x}$ retain their definitions from electromagnetism. The source perturbation is determined by the equation of motion perturbations $\delta z, \delta u, \delta S$. Define the flat-space Fourier transform of the energy-momentum tensor:

$$\widetilde{T}^{\mu\nu}(k) = \int\frac{e^{-ik\cdot x}}{(2\pi)^2}T^{\mu\nu}(X)d^4X. \tag{3.8.35}$$

From (3.6.36) we can identify that the Fourier transform of the stationary stress tensor of the unperturbed body is:

$$\widetilde{T}^{\alpha\beta}_{(0)}(k) = M^{\alpha\beta}(k)\frac{\delta(k\cdot v)}{2\pi} \tag{3.8.36}$$

where:

$$M^{\alpha\beta}(k) = mv^\alpha v^\beta + iv^{(\alpha}\epsilon^{\beta)\gamma\rho\sigma}k_\gamma v_\rho s_\sigma - \frac{2}{m}\sum_{n=0}^{\infty}(-i)^n T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}(\eta,v,s)k_{\lambda_1}\ldots k_{\lambda_n}k_\mu k_\nu. \tag{3.8.37}$$

The resulting Fourier transform of the stationary metric perturbation is:

$$\widetilde{h}^{\alpha\beta}_{\text{stat}}(k) = -\frac{M^{\alpha\beta}(k)}{2k^2}\frac{\delta(k\cdot v)}{2\pi}. \tag{3.8.38}$$

In terms of this solution, (3.8.19) produces:

$$\begin{aligned}
\delta\widetilde{\tau}^{\mu\nu}(k_2) = {} & \delta\widetilde{T}^{\mu\nu}(k_2) + 2\mathcal{E}_1^\mu\mathcal{E}_1^\nu(\widetilde{h}^{\rho\sigma}_{\text{stat}}k_{1\rho}k_{1\sigma}) + 2(\mathcal{E}_1\cdot k_2)^2\widetilde{h}^{\mu\nu}_{\text{stat}} - 4(\mathcal{E}_1\cdot k_2)\mathcal{E}_1^{(\mu}\widetilde{h}^{\nu)\rho}_{\text{stat}}k_{1\rho} \\
& - 4(k_1\cdot k_2)\mathcal{E}_1^{(\mu}\widetilde{h}^{\nu)\rho}_{\text{stat}}\mathcal{E}_{1\rho} + 4\mathcal{E}_1^{(\mu}(k_2-k_1)^{\nu)}(\widetilde{h}^{\rho\sigma}_{\text{stat}}k_{1\rho}\mathcal{E}_{1\sigma}) + 4(\mathcal{E}_1\cdot k_2)k_1^{(\mu}\widetilde{h}^{\nu)\rho}_{\text{stat}}\mathcal{E}_{1\rho} \\
& - 2\eta^{\mu\nu}(\mathcal{E}_1\cdot k_2)\widetilde{h}^{\rho\sigma}_{\text{stat}}k_{1\rho}\mathcal{E}_{1\sigma} - 2k_1^{(\mu}(k_2-k_1)^{\nu)}\widetilde{h}^{\alpha\beta}_{\text{stat}}\mathcal{E}_{1\alpha}\mathcal{E}_{1\beta} + \eta^{\mu\nu}k_1\cdot k_2\widetilde{h}^{\alpha\beta}_{\text{stat}}\mathcal{E}_{1\alpha}\mathcal{E}_{1\beta}
\end{aligned} \tag{3.8.39}$$

where everywhere it appears in the above equation, $\widetilde{h}^{\rho\sigma}_{\text{stat}}$ is evaluated at $k_2 - k_1$.

From the definition of $\mathcal{Q}_n$, we find:

$$\mathcal{Q}_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu} = \left(1 - \frac{\delta\mathcal{M}_1^2}{2m^2}\right)\frac{T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}}{2m} + \frac{1}{m}\sum_{l=0}^{\infty}V_{nl}^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu|\kappa_1\ldots\kappa_l\gamma\alpha\delta\beta}\nabla^l_{(\kappa_1\ldots\kappa_l)}R_{\gamma\alpha\delta\beta} + \mathcal{O}(R^2). \tag{3.8.40}$$

In terms of the solutions to the equations of motion:

$$\mathcal{Q}_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu} = \frac{T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}(\eta,v,s)}{2m} + \kappa\epsilon\delta\widetilde{\mathcal{Q}}_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}e^{ik_1\cdot v\lambda} + \mathcal{O}(\kappa^2) \tag{3.8.41}$$

where:

$$\begin{aligned}
\delta\widetilde{\mathcal{Q}}_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu} = {} & \frac{1}{4m}\mathcal{E}_{1\alpha}\mathcal{E}_{1\beta}\hat{G}^{\alpha\beta}T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu} + \frac{1}{2m}\frac{\partial T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}}{\partial v^\alpha}\delta\widetilde{u}^\alpha \\
& + \frac{1}{2m}\frac{\partial T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}}{\partial s^\alpha}\left(\delta\widetilde{S}^\alpha - \frac{1}{2}s\cdot\mathcal{E}_1\mathcal{E}_1^\alpha\right) - \frac{\mathcal{N}_1}{2m^2}T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu} \\
& - \frac{2}{m}\sum_{l=0}^{\infty}i^l V_{nl}^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu|\kappa_1\ldots\kappa_l\gamma\alpha\delta\beta}k_{1\kappa_1}\ldots k_{1\kappa_l}k_{1\gamma}k_{1\delta}\mathcal{E}_{1\alpha}\mathcal{E}_{1\beta}.
\end{aligned} \tag{3.8.42}$$

Returning these to $\delta\tau$, we find that in Fourier space it takes the form:

$$\delta\widetilde{\tau}^{\mu\nu}(k_2) = H^{\mu\nu}(k_2)\frac{\delta(k_2\cdot v - k_1\cdot v)}{2\pi} \tag{3.8.43}$$

and that at large distances, recycling the notation from electromagnetism. The metric perturbation becomes:

$$\delta h^{\mu\nu}(X) = -\frac{e^{i\omega(r-t)}}{8\pi r}H^{\mu\nu}(k_2) + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{3.8.44}$$

Therefore, the covariantly normalized Compton amplitude is given by:

$$\mathcal{A} = -m\mathcal{E}^*_{2\mu}\mathcal{E}^*_{2\nu}H^{\mu\nu}(k_2). \tag{3.8.45}$$

The solutions to the equations of motion are:

$$\delta\widetilde{u}^\mu = \frac{i}{2m}\mathcal{E}_1^\mu\epsilon^{vsk_1\mathcal{E}_1} - \frac{\mathcal{N}_1}{m}\left(v^\mu + \frac{k_1^\mu}{k_1\cdot v}\right) \tag{3.8.46}$$

$$\delta\widetilde{S}^{\mu} = \frac{1}{2}(\mathcal{E}_1 \cdot s)\mathcal{E}_1^{\mu} + v^{\mu} s \cdot \delta\widetilde{u} - \frac{i}{k_1 \cdot v}\epsilon^{\mu\nu\rho\sigma} v_{\nu} s_{\rho} \frac{\partial \mathcal{N}_1}{\partial s^{\sigma}} \tag{3.8.47}$$

$$\begin{aligned}\delta\widetilde{z}^{\mu} = \frac{-i}{k_1 \cdot v}\Bigg(&\delta\widetilde{u}^{\mu} + \frac{\epsilon^{vsk_1\mathcal{E}_1}}{2m^2}\left(\delta^{\mu}_{\nu} + v^{\mu}v_{\nu}\right)\epsilon^{\nu s k_1 \mathcal{E}_1} + \frac{\eta^{\mu\nu} + v^{\mu}v^{\nu}}{m}\frac{\partial \mathcal{N}_1}{\partial v^{\nu}} \\ &+ \frac{s^{\mu}v^{\nu}}{m}\frac{\partial \mathcal{N}_1}{\partial s^{\nu}} + \frac{i}{m^2}\mathcal{N}_1 \epsilon^{\mu\nu\rho\sigma} v_{\nu} s_{\rho} k_{1\sigma}\Bigg).\end{aligned} \tag{3.8.48}$$

Expressed in terms of these solutions, the gravitational Compton amplitude to all orders in spin is:

$$\begin{aligned}\mathcal{A} = 2m\Bigg(&\frac{1}{2}(k_2 \cdot v)(\mathcal{E}_2^* \cdot \delta\widetilde{z})\epsilon^{\mathcal{E}_2^* k_2 vs} - \frac{i}{4}(k_2 \cdot v)(\mathcal{E}_1 \cdot \mathcal{E}_2^*)\epsilon^{\mathcal{E}_1 \mathcal{E}_2^* vs} \\ &+ i(k_2 \cdot \delta\widetilde{z})\mathcal{N}_2^* - \delta\widetilde{u}^{\alpha}\frac{\partial \mathcal{N}_2^*}{\partial v^{\alpha}} - \left(\delta\widetilde{S}^{\alpha} - \frac{1}{2}(s \cdot \mathcal{E}_1)\mathcal{E}_1^{\alpha}\right)\frac{\partial \mathcal{N}_2^*}{\partial s^{\alpha}} + \frac{\mathcal{N}_1\mathcal{N}_2^*}{m} - \mathcal{P}_{12}\Bigg) \\ &- \frac{m(\mathcal{E}_1 \cdot \mathcal{E}_2^*)^2}{2(k_1 \cdot k_2)}(M_{\mu\nu}k_1^{\mu}k_1^{\nu}) - \frac{m(\mathcal{E}_1 \cdot k_2)^2}{2(k_1 \cdot k_2)}(M_{\mu\nu}\mathcal{E}_2^{*\mu}\mathcal{E}_2^{*\nu}) + \frac{m(\mathcal{E}_1 \cdot k_2)(\mathcal{E}_1 \cdot \mathcal{E}_2^*)}{(k_1 \cdot k_2)}(M_{\mu\nu}k_1^{\mu}\mathcal{E}_2^{*\nu}) \\ &+ m(\mathcal{E}_1 \cdot \mathcal{E}_2^*)(M_{\mu\nu}\mathcal{E}_1^{\mu}\mathcal{E}_2^{*\nu}) + \frac{m(\mathcal{E}_1 \cdot \mathcal{E}_2^*)(k_1 \cdot \mathcal{E}_2^*)}{(k_1 \cdot k_2)}(M_{\mu\nu}k_1^{\mu}\mathcal{E}_1^{\nu}) \\ &- \frac{m(\mathcal{E}_1 \cdot k_2)(\mathcal{E}_2^* \cdot k_1)}{(k_1 \cdot k_2)}(M_{\mu\nu}\mathcal{E}_1^{\mu}\mathcal{E}_2^{*\nu}) - \frac{m(\mathcal{E}_2^* \cdot k_1)^2}{2(k_1 \cdot k_2)}(M_{\mu\nu}\mathcal{E}_1^{\mu}\mathcal{E}_1^{\nu}) + \sum_{n=0}^{\infty}\delta\mathcal{A}_n\end{aligned} \tag{3.8.49}$$

where:

$$\begin{aligned}\delta\mathcal{A}_n = &-\frac{1}{2}\sum_{l=0}^{n-1} i^n(-1)^l(k_1 \cdot \mathcal{E}_2^*)T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}k_{1\lambda_1}\ldots k_{1\lambda_{n-l-1}}\mathcal{E}^*_{2\lambda_{n-l}}q_{\lambda_{n-l+1}}\ldots q_{\lambda_n}\delta\widetilde{R}_{\rho\mu\sigma\nu} \\ &+ \sum_{l=0}^{n-1} i^n(-1)^l T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}\mathcal{E}_2^{*[\tau}k_2^{\omega]}\hat{G}_{\tau\omega}(k_{1\lambda_1}\ldots k_{1\lambda_{n-l-1}}\delta\widetilde{R}_{\rho\mu\sigma\nu})\mathcal{E}^*_{2\lambda_{n-l}}q_{\lambda_{n-l+1}}\ldots q_{\lambda_n} \\ &- (-i)^n T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu}q_{\lambda_1}\ldots q_{\lambda_n}\mathcal{E}^*_{2\rho}\mathcal{E}_2^{*\alpha}\delta\widetilde{R}_{\alpha\mu\sigma\nu} \\ &+ 2(-1)^n\sum_{l=0}^{n-1} i^{n-1}\mathcal{E}^*_{2\alpha}\mathcal{E}^*_{2\beta}k_{2\mu}k_{2\nu}k_{2\lambda_{n-l+1}}\ldots k_{2\lambda_n}\delta\widetilde{\Gamma}^{\sigma}{}_{\rho\lambda_{n-l}}\hat{G}^{\rho}{}_{\sigma}(T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}q_{\lambda_1}\ldots q_{\lambda_{n-l-1}}) \\ &- 2i(-i)^n\mathcal{E}^*_{2\alpha}\mathcal{E}^*_{2\beta}k_{2\mu}\delta\widetilde{\Gamma}^{\sigma}{}_{\rho\nu}\hat{G}^{\rho}{}_{\sigma}(T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}q_{\lambda_1}\ldots q_{\lambda_n}) \\ &- 2i(-i)^n\mathcal{E}^*_{2\alpha}\mathcal{E}^*_{2\beta}\delta\widetilde{\Gamma}^{\sigma}{}_{\rho\mu}\hat{G}^{\rho}{}_{\sigma}(T_n^{\lambda_1\ldots\lambda_n\mu\alpha\nu\beta}q_{\nu}q_{\lambda_1}\ldots q_{\lambda_n}) \\ &+ (-i)^n k_{2\lambda_1}\ldots k_{2\lambda_n}k_{2\rho}\mathcal{E}^*_{2\mu}k_{2\sigma}\mathcal{E}^*_{2\nu}\mathcal{E}_1^{\alpha}\mathcal{E}_1^{\beta}\hat{G}_{\alpha\beta}(T_n^{\lambda_1\ldots\lambda_n\rho\mu\sigma\nu})\end{aligned} \tag{3.8.50}$$

and where:

$$q^{\mu} = k_2^{\mu} - k_1^{\mu} \tag{3.8.51}$$

$$\delta\widetilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{i}{2}(k_1^{\rho}\mathcal{E}_{1\mu}\mathcal{E}_{1\nu} - \mathcal{E}_1^{\rho}k_{1\mu}\mathcal{E}_{1\nu} - \mathcal{E}_1^{\rho}k_{1\nu}\mathcal{E}_{1\mu}) \tag{3.8.52}$$

$$\delta\widetilde{R}_{\rho\mu\sigma\nu} = -\frac{1}{2}(k_{1\rho}\mathcal{E}_{1\mu} - k_{1\mu}\mathcal{E}_{1\rho})(k_{1\sigma}\mathcal{E}_{1\nu} - k_{1\nu}\mathcal{E}_{1\sigma}). \tag{3.8.53}$$

3.8.2 Compton Amplitude through Spin to the Fifth

Similarly to electromagnetism, we consider only operators which have the same number of powers of spin as they do number of derivatives acting on the metric, so as not to include any length scales beyond

$\frac{S}{m}$. For enumerating linear in Riemann operators, it is not safe to use Einstein's equations to simplify the possible terms when considering the Compton amplitude as the linear in Riemann piece of the dynamical mass function does not contribute solely in a perfectly on-shell way to the Compton amplitude in the way the linear in field strength piece did for electromagnetism (only through the $\mathcal{N}$ function). We find explicitly that some Ricci operators make independent contributions to the Compton amplitude. Through $\mathcal{O}(S^5)$, the most general possible dynamical mass function with only length scale $\frac{S}{m}$ is:

$$\delta\mathcal{M}_1^2 = \delta\mathcal{M}_{1S^2}^2 + \delta\mathcal{M}_{1S^3}^2 + \delta\mathcal{M}_{1S^4}^2 + \delta\mathcal{M}_{1S^5}^2 + \mathcal{O}(S^6) \tag{3.8.54}$$

$$\delta\mathcal{M}_{1S^2}^2 = -C_2 R_{uSuS} - \frac{E_{2a}}{3} R_{SS} + E_{2b} S^2 R_{uu} + \frac{E_{2c}}{6} S^2 R, \tag{3.8.55}$$

$$\delta\mathcal{M}_{1S^3}^2 = \frac{C_3}{3m}(S\cdot\nabla)R^\star_{uSuS} + \frac{E_3}{3m} S^2 \nabla^\rho R^\star_{\rho uuS}, \tag{3.8.56}$$

$$\begin{aligned}\delta\mathcal{M}_{1S^4}^2 = {} & \frac{C_4}{12m^2}(S\cdot\nabla)^2 R_{uSuS} - \frac{E_{4a}}{20m^2} S^2 (u\cdot\nabla)^2 R_{uSuS} - \frac{E_{4b}}{12m^2} S^2 \nabla^2 R_{uSuS} \\ & + \frac{E_{4c}}{30m^2}(S\cdot\nabla)^2 R_{SS} - \frac{E_{4d}}{30m^2} S^2 (u\cdot\nabla)^2 R_{SS} - \frac{E_{4e}}{30m^2} S^2 \nabla^2 R_{SS} \\ & - \frac{E_{4f}}{12m^2} S^2 (S\cdot\nabla)^2 R_{uu} + \frac{E_{4g}}{20m^2} S^4 (u\cdot\nabla)^2 R_{uu} + \frac{E_{4h}}{12m^2} S^4 \nabla^2 R_{uu} \\ & - \frac{E_{4i}}{60m^2} S^2 (S\cdot\nabla)^2 R - \frac{E_{4j}}{60m^2} S^4 (u\cdot\nabla)^2 R + \frac{E_{4k}}{60m^2} S^4 \nabla^2 R,\end{aligned} \tag{3.8.57}$$

$$\begin{aligned}\delta\mathcal{M}_{1S^5}^2 = {} & -\frac{C_5}{60m^3}(S\cdot\nabla)^3 R^\star_{uSuS} + \frac{E_{5a}}{60m^3} S^2 (u\cdot\nabla)^2 (S\cdot\nabla) R^\star_{uSuS} + \frac{E_{5b}}{60m^3}(S\cdot\nabla)\nabla^2 R^\star_{uSuS} \\ & - \frac{E_{5c}}{60m^3} S^2 (S\cdot\nabla)^2 \nabla^\rho R^\star_{\rho uuS} + \frac{E_{5d}}{60m^3} S^4 (u\cdot\nabla)^2 \nabla^\rho R^\star_{\rho uuS} + \frac{E_{5e}}{60m^3} S^4 \nabla^2 \nabla^\rho R^\star_{\rho uuS}.\end{aligned} \tag{3.8.58}$$

Expanding (3.7.43) through order S^5 we find that it implies that all of these Wilson coefficients should be 1 except E_{4d} and E_{4j} which should both be 0:

$$\begin{aligned} C_2 &= C_3 = C_4 = C_5 = E_{2a} = E_{2b} = E_{2c} = E_3 \\ &= E_{4a} = E_{4b} = E_{4c} = E_{4e} = E_{4f} = E_{4g} = E_{4h} \\ &= E_{4i} = E_{4k} = E_{5a} = E_{5b} = E_{5c} = E_{5d} = E_{5e} = 1 \\ E_{4d} &= E_{4j} = 0. \end{aligned} \tag{3.8.59}$$

While it is not valid to use Einstein's equations to simplify the linear in Riemann piece of the dynamical mass function when computing the Compton amplitude, because the quadratic in Riemann piece of the dynamical mass function only contributes to the Compton amplitude by being evaluated fully on-shell (through the $\mathcal{P}$ function) we are perfectly safe to only consider terms within it which are nonzero for vacuum solutions of Einstein's equations. If we assume vacuum, $R^\rho{}_{\mu\rho\nu} = 0$, then the left and right duals of the Riemann tensor are equal, ${}^\star R = R^\star$, ${}^\star R^\star = -R$, and its (A)SD part can be obtained by projecting with

(3.5.50) from either side, and it commutes through to the other,

$$
{}^{\pm}R_{\rho\mu\sigma\nu} = \frac{1}{2}(R \mp i\,{}^{\star}R)_{\rho\mu\sigma\nu} = {}^{\pm}\mathcal{G}_{\rho\mu}{}^{\alpha\beta}R_{\alpha\beta\sigma\nu} = R_{\rho\mu\gamma\delta}\,{}^{\pm}\mathcal{G}^{\gamma\delta}{}_{\sigma\nu} = {}^{\pm}\mathcal{G}_{\rho\mu}{}^{\alpha\beta}R_{\alpha\beta\gamma\delta}\,{}^{\pm}\mathcal{G}^{\gamma\delta}{}_{\sigma\nu}, \tag{3.8.60}
$$

satisfying ${}^{\star}\,{}^{\pm}R = {}^{\pm}R^{\star} = \pm i\,{}^{\pm}R$ and ${}^{\pm}R^{*} = {}^{\mp}R$. Defining the quadrupolar gravito-electric and -magnetic "tidal" curvature tensors with respect to a unit timelike direction u,

$$
E_{\mu\nu} \mp iB_{\mu\nu} = (R_{\rho\mu\sigma\nu} \mp i\,{}^{\star}R_{\rho\mu\sigma\nu})u^{\rho}u^{\sigma} = 2\,{}^{\pm}R_{\rho\mu\sigma\nu}u^{\rho}u^{\sigma}, \tag{3.8.61}
$$

we see that the identities (3.5.50)–(3.5.52) allow us to reconstruct $R_{\rho\mu\sigma\nu}$ from its components $E_{\mu\nu}$ and $B_{\mu\nu}$ (and u_{ρ}), as the real part of

$$
2\,{}^{\pm}R_{\rho\mu\sigma\nu} = (R \mp i\,{}^{\star}R)_{\rho\mu\sigma\nu} = 16\,{}^{\pm}\mathcal{G}_{\rho\mu}{}^{\gamma\alpha}\,{}^{\pm}\mathcal{G}_{\sigma\nu}{}^{\delta\beta}(E_{\alpha\beta} \mp iB_{\alpha\beta})u_{\gamma}u_{\delta}. \tag{3.8.62}
$$

The tidal tensors $E_{\mu\nu}$ and $B_{\mu\nu}$ are symmetric and trace-free (STF) in vacuum, and orthogonal to u^{μ}, forming irreducible representations of the SO(3) rotation little group of u^{μ}. Moving on to the first derivative $\nabla_{\lambda}R_{\rho\mu\sigma\nu}$, the irreducible pieces w.r.t. u^{μ} are the fully STF-($\perp u$) octupolar tidal tensors

$$
E_{\lambda\mu\nu} \mp iB_{\lambda\mu\nu} = 2\nabla_{\kappa}\,{}^{\pm}R^{\rho}{}_{(\lambda}{}^{\sigma}{}_{\mu}\big(\delta_{\nu)}{}^{\kappa} + u_{\nu)}u^{\kappa}\big)u_{\rho}u_{\sigma}, \tag{3.8.63}
$$

and the 'time derivatives' of the quadrupolar curvature components,

$$
\dot{E}_{\mu\nu} \mp i\dot{B}_{\mu\nu} = 2u^{\lambda}\nabla_{\lambda}\,{}^{\pm}R_{\rho\mu\sigma\nu}u^{\rho}u^{\sigma}. \tag{3.8.64}
$$

In terms of the tidal tensors, the most general dynamical mass function through S^5 which is quadratic in the Riemann tensor and contains only pieces which are nonzero in vacuum is:

$$
\delta\mathcal{M}_2^2 = \delta\mathcal{M}_{2S^4}^2 + \delta\mathcal{M}_{2S^5}^2 + \mathcal{O}(S^6), \tag{3.8.65}
$$

$$
\begin{aligned}
\delta\mathcal{M}_{2S^4}^2 = {}& \frac{D_{4a}}{m^2}(E_{SS})^2 + \frac{D_{4b}}{m^2}S^2E^{\mu}{}_{S}E_{\mu S} + \frac{D_{4c}}{m^2}S^4E^{\mu\nu}E_{\mu\nu} \\
&+ \frac{D_{4d}}{m^2}(B_{SS})^2 + \frac{D_{4e}}{m^2}S^2B^{\mu}{}_{S}B_{\mu S} + \frac{D_{4f}}{m^2}S^4B^{\mu\nu}B_{\mu\nu},
\end{aligned} \tag{3.8.66}
$$

$$
\begin{aligned}
\delta\mathcal{M}_{2S^5}^2 = {}& \frac{D_{5a}}{m^3}B_{SS}E_{SSS} + \frac{D_{5b}}{m^3}S^2B^{\mu}{}_{S}E_{\mu SS} + \frac{D_{5c}}{m^3}S^4B^{\mu\nu}E_{\mu\nu S} \\
&+ \frac{D_{5d}}{m^3}E_{SS}B_{SSS} + \frac{D_{5e}}{m^3}S^2E^{\mu}{}_{S}B_{\mu SS} + \frac{D_{5f}}{m^3}S^4E^{\mu\nu}B_{\mu\nu S} \\
&+ \left(\frac{D_{5g}}{m^3}E_{S\mu}\dot{E}_{S\nu} + \frac{D_{5h}}{m^3}S^2E^{\lambda}{}_{\mu}\dot{E}_{\lambda\nu} + \frac{D_{5i}}{m^3}B_{S\mu}\dot{B}_{S\nu} + \frac{D_{5j}}{m^3}S^2B^{\lambda}{}_{\mu}\dot{B}_{\lambda\nu}\right)\epsilon^{\mu\nu}{}_{\rho\sigma}u^{\rho}S^{\sigma}S^2.
\end{aligned} \tag{3.8.67}
$$

The spinless and linear in spin pieces of the amplitudes are universal (independent of the values of any Wilson coefficients). At quadratic order in spin, there are 4 Wilson coefficients in the dynamical mass function, C_2, $E_{2a,b,c}$. The amplitude for a given body at $\mathcal{O}(S^2)$ determines C_2 and is independent of $E_{2a,b,c}$. In fact, the amplitude is independent of $E_{2a,b,c}$ to all orders in spin. At cubic order in spin, there are 2 Wilson coefficients in the dynamical mass function, C_3 and E_3. The amplitude for a given body at $\mathcal{O}(S^3)$

determines C_3 and is independent of E_3. In fact, the amplitude is independent of E_3 to all orders in spin. At quartic order in spin, there are 18 Wilson coefficients in the dynamical mass function, C_4, $D_{4a,b,c,d,e,f}$, $E_{4a,b,c,d,e,f,g,h,i,j,k}$. The amplitude for a given body at $\mathcal{O}(S^4)$ is independent of $E_{4c,d,e,f,g,h,i,j,k}$ and remains so at all orders in spin. Of the 9 remaining coefficients C_4, $D_{4a,b,c,d,e,f}$, $E_{4a,b}$, only 7 linear combinations can be determined from the $\mathcal{O}(S^4)$ Compton amplitude. In particular, the S^4 amplitude is independent of the value of E_{4a} (though the S^5 amplitude does depend on E_{4a}) and of the linear combination:

$$Z_4 = E_{4b} + \frac{D_{4b}}{2} - \frac{D_{4c}}{6} - \frac{D_{4e}}{2} + \frac{D_{4f}}{6}. \tag{3.8.68}$$

We find agreement with the Compton amplitude computed by Ben-Shahar in Ref [262] to quartic order in spin.

At quintic order in spin, there are 16 Wilson coefficients in the dynamical mass function, C_5, D_{5a-j}, $E_{5a,b,c,d,e}$. As well, E_{4a} contributes to the S^5 amplitude. Of these 17 coefficients, 11 linearly combinations can be determined from the Compton amplitude. The amplitude is independent of the values of the 6 linear combinations:

$$Z_{5a} = E_{5a} + \frac{D_{5e}}{180} + \frac{D_{5g}}{60} - \frac{D_{5i}}{90} \tag{3.8.69}$$

$$Z_{5b} = E_{5b} - \frac{4}{15}D_{5b} + \frac{D_{5c}}{10} - \frac{4}{15}D_{5e} + \frac{D_{5f}}{10} + \frac{D_{5h}}{30} - \frac{D_{5j}}{30} \tag{3.8.70}$$

$$Z_{5c} = E_{5c} - \frac{D_{5b}}{12} + \frac{D_{5c}}{36} - \frac{17}{180}D_{5e} + \frac{D_{5f}}{30} - \frac{D_{5g}}{180} + \frac{D_{5h}}{180} - \frac{D_{5i}}{180} \tag{3.8.71}$$

$$Z_{5d} = E_{5d} - \frac{D_{5f}}{180} - \frac{D_{5h}}{90} + \frac{D_{5j}}{180} \tag{3.8.72}$$

$$Z_{5e} = E_{5e} + \frac{D_{5c}}{60} + \frac{D_{5f}}{60} + \frac{D_{5h}}{20} - \frac{D_{5j}}{20} \tag{3.8.73}$$

$$Z_{5f} = E_{4a} - \frac{D_{5b}}{20} - \frac{C_2}{10}D_{5g} + \frac{3}{20}D_{5i}. \tag{3.8.74}$$

There are precisely as many Z combinations which the Compton amplitude is independent of as there are E coefficients in the amplitude at this order, so if the amplitude is fully determined by some matching conditions those conditions can be expressed so that all of the D coefficients are parameterized by the matched values and the undetermined E coefficients. If (3.8.59) is used, all of these E coefficients are set to 1, which then fully determines the D coefficients and hence the dynamical mass function through this order. Because of these null Z combinations, the values of the any values of E coefficients can be made consistent with any matching conditions on the Compton amplitude.

In order to match the spin-exponentiated amplitude of Ref. [53] through $\mathcal{O}(S^4)$, we must have:

$$\begin{gathered} C_2 = C_3 = C_4 = 1, \qquad D_{4a} = D_{4d} = 0, \\ D_{4b} = \frac{E_{4b}}{2}, \qquad D_{4c} = -\frac{E_{4b}}{6}, \qquad D_{4e} = -\frac{E_{4b}}{2}, \qquad D_{4f} = \frac{E_{4b}}{6}. \end{gathered} \tag{3.8.75}$$

These conditions are consistent with (3.8.59). For the opposite-helicity amplitude, it is possible to continue the spin-exponentiation through S^5. Doing so requires:

$$\begin{aligned}
C_5 &= 1,\\
D_{5d} &= \frac{1}{6} - D_{5a},\\
D_{5e} &= -\frac{1}{15} - D_{5b} - \frac{E_{4a}}{20} + \frac{E_{5a}}{180} - \frac{8}{15}E_{5b} - \frac{8}{45}E_{5c},\\
D_{5f} &= -D_{5c} + \frac{E_{5b}}{5} + \frac{11}{180}E_{5c} - \frac{E_{5d}}{180} + \frac{E_{5e}}{30},\\
D_{5i} &= \frac{2}{9} + D_{5g} + \frac{E_{4a}}{4} - \frac{E_{5a}}{36}\\
D_{5j} &= D_{5h} - \frac{E_{5b}}{15} - \frac{E_{5c}}{180} + \frac{E_{5d}}{60} - \frac{E_{5e}}{10}
\end{aligned} \tag{3.8.76}$$

which are completely consistent with (3.8.59) (which simply sets $E_{4a} = E_{5a,b,c,d,e} = 1$ in these expressions). Therefore, if the dynamical mass function in (3.7.43) is used, opposite-helicity spin exponentiation can be maintained at $\mathcal{O}(S^5)$.

It is also interesting to see what is required by shift-symmetry. In order for the same-helicity Compton amplitude to have shift-symmetry through $\mathcal{O}(S^4)$ we find that it requires:

$$C_4 = 1, \qquad D_{4e} = -D_{4a} - D_{4b} - D_{4d}, \qquad D_{4f} = \frac{D_{4a}}{4} - D_{4c} + \frac{D_{4d}}{4} \tag{3.8.77}$$

which are consistent with both (3.8.59) and (3.8.75). It is possible to demand shift-symmetry at S^5 as well. Doing so requires:

$$\begin{aligned}
C_5 &= 1,\\
D_{5e} &= \frac{4}{5} + D_{5a} + D_{5b} - D_{5d} + \frac{E_{4a}}{20} + \frac{E_{5a}}{180} - \frac{E_{5c}}{90},\\
D_{5f} &= -\frac{43}{120} - \frac{D_{5a}}{4} + D_{5c} + \frac{D_{5d}}{4} + \frac{E_{5c}}{180} - \frac{E_{5d}}{180},\\
D_{5i} &= \frac{5}{18} - D_{5g} + \frac{E_{4a}}{20} + \frac{E_{5a}}{180} - \frac{E_{5c}}{90}\\
D_{5j} &= -D_{5h} + \frac{E_{5c}}{180} - \frac{E_{5d}}{180}
\end{aligned} \tag{3.8.78}$$

which are completely consistent with (3.8.59). Therefore, if the dynamical mass function in (3.7.43) is used, shift-symmetry will may be continued at $\mathcal{O}(S^5)$. The spin-exponentiation conditions and shift-symmetry conditions can be demanded simultaneously through $\mathcal{O}(S^5)$.

A final interesting case of comparison is to results from the Teukolsky equation. Following the analysis of Ref [185], a match to the analytically continued results of the Teukolsky equation depends on the combinations:

$$\begin{aligned}
c_2^{(0)} &= D_{5f} - D_{5c} + \frac{E_{5d} - E_{5c}}{180}, \qquad & c_3^{(0)} &= -2D_{5h} - 2D_{5j} + \frac{E_{5c} - E_{5d}}{90},\\
c_2^{(1)} &= \frac{1}{20} + 2c_2^{(0)} + \frac{D_{5e} - D_{5b}}{2} - \frac{9E_{4a} + E_{5a} - 2E_{5c}}{360},
\end{aligned}$$

$$c_3^{(1)} = \frac{1}{90} + 2c_3^{(0)} - D_{5g} - D_{5i} + \frac{9E_{4a} + E_{5a} - 2E_{5c}}{180},$$
$$c_2^{(2)} = -\frac{11}{120} + \frac{D_{5d} - D_{5a}}{4} + c_2^{(1)} - c_2^{(0)}, \qquad c_3^{(2)} = \frac{4}{15} + c_3^{(1)} - c_3^{(0)}. \tag{3.8.79}$$

Matching Teukolsky at $\mathcal{O}(S^5)$ requires:

$$c_2^{(0)} = c_2^{(1)} = c_2^{(2)} = 0, \qquad c_3^{(0)} = \frac{64}{15}\alpha, \qquad c_3^{(1)} = \frac{16}{3}\alpha, \qquad c_3^{(2)} = \frac{4}{15}(1+4\alpha), \tag{3.8.80}$$

where $\alpha = 1$ if contributions from analytically continued digamma functions are to be kept or $\alpha = 0$ if such contributions are to be dropped. Matching to Teukolsky is inconsistent with shift-symmetry but consistent with continuing spin-exponentiation for the same helicity amplitude and with (3.7.43). The combination of Teukolsky, spin-exponentiation, and (3.8.59) are consistent with each other and fully determine the D type Wilson coefficients to be:

$$\begin{aligned}
&D_{5a} = -\frac{1}{10}, && D_{5b} = -\frac{23}{60}, && D_{5c} = \frac{13}{90}, && D_{5d} = \frac{4}{15},\\
&D_{5e} = -\frac{79}{180}, && D_{5f} = \frac{13}{90}, && D_{5g} = -\frac{7}{36} + \frac{8}{5}\alpha, && D_{5h} = \frac{7}{90} - \frac{16}{15}\alpha,\\
& && D_{5i} = \frac{1}{4} + \frac{8}{5}\alpha, && D_{5j} = -\frac{7}{90} - \frac{16}{15}\alpha.
\end{aligned} \tag{3.8.81}$$

For expressing the full amplitudes, recall $\check{k}_1 = k_1 - w$, $\check{k}_2 = k_2 - w$. Then, for the helicity-preserving amplitude we find:

$$\begin{aligned}
\mathcal{A}_{++} = \frac{(4\omega^2 - q^2)^2}{16q^2\omega^2}\Big\{ & 1 + \check{k}_1 \cdot a + \check{k}_2 \cdot a \\
& + \frac{1}{2}(\check{k}_1 \cdot a + \check{k}_2 \cdot a)^2 + \frac{C_2 - 1}{2}\Big((\check{k}_1 \cdot a)^2 + (\check{k}_2 \cdot a)^2\Big) \\
& + \frac{1}{6}(\check{k}_1 \cdot a + \check{k}_2 \cdot a)^3 + \frac{C_3 - 1}{2}\Big((\check{k}_1 \cdot a)^3 + (\check{k}_2 \cdot a)^3\Big) \\
& \quad + \frac{C_2 - 1}{2}(k_1 + k_2 - 2C_2 w) \cdot a\, \check{k}_1 \cdot a\, \check{k}_2 \cdot a \\
& + \hat{\mathcal{A}}_{++}^{(4)} + \hat{\mathcal{A}}_{++}^{(5)} + \mathcal{O}(a^6)\Big\},
\end{aligned} \tag{3.8.82}$$

with

$$\begin{aligned}
\hat{\mathcal{A}}_{++}^{(4)} = & \frac{1}{24}(\check{k}_1 \cdot a + \check{k}_2 \cdot a)^4 + \frac{C_4 - 1}{24}\Big((\check{k}_1 \cdot a)^4 + (\check{k}_2 \cdot a)^4\Big) \\
& \quad + \frac{C_3 - 1}{6}\Big((\check{k}_1 \cdot a)^2 + (\check{k}_2 \cdot a)^2\Big)\check{k}_1 \cdot a\, \check{k}_2 \cdot a + \hat{\mathcal{A}}_{++}^{(4)(C_2-1)} \\
& - (D_{4a} + D_{4d})\frac{2\omega^2}{q^2}(\check{k}_1 \cdot a)^2(\check{k}_2 \cdot a)^2 \\
& + (D_{4b} + D_{4e})\omega^2 a^2 \check{k}_1 \cdot a\, \check{k}_2 \cdot a - \frac{D_{4c} + D_{4f}}{2}q^2\omega^2 a^4,
\end{aligned} \tag{3.8.83}$$

noting

$$\omega^2 a^2 = (w \cdot a)^2 - \frac{4\omega^2}{q^2}\check{k}_1 \cdot a\, \check{k}_2 \cdot a, \tag{3.8.84}$$

and with

$$\hat{\mathcal{A}}^{(5)}_{++} = \frac{1}{120}(\check{k}_1 \cdot a + \check{k}_2 \cdot a)^5 + \hat{\mathcal{A}}^{(5)(C_n-1)}_{++}$$
$$+ \Big(c_2^{(0)}(k_1+k_2)\cdot a + c_3^{(0)} w\cdot a\Big)(w\cdot a)^4 \frac{q^2}{4\omega^2}$$
$$- \Big(c_2^{(1)}(k_1+k_2)\cdot a + c_3^{(1)} w\cdot a\Big)(w\cdot a)^2 \check{k}_1\cdot a\, \check{k}_2\cdot a$$
$$+ \Big(c_2^{(2)}(k_1+k_2)\cdot a + c_3^{(2)} w\cdot a\Big)\frac{4\omega^2}{q^2}(\check{k}_1\cdot a)^2(\check{k}_2\cdot a)^2, \tag{3.8.85}$$

using the previous definitions of the $c_i^{(j)}$ coefficients in (3.8.79), and where

$$\hat{\mathcal{A}}^{(5)(C_n-1)}_{++} = \frac{C_5-1}{120}\Big[(\check{k}_1\cdot a)^5 + (\check{k}_2\cdot a)^5 - (w\cdot a)^2(\check{k}_1\cdot a + \check{k}_2\cdot a)\check{k}_1\cdot a\,\check{k}_2\cdot a\Big] \tag{3.8.86}$$
$$+ \frac{C_4-1}{24}\Big[(\check{k}_1\cdot a)^3 + (\check{k}_2\cdot a)^3 - (w\cdot a)^2(\check{k}_1\cdot a + \check{k}_2\cdot a)\Big]\check{k}_1\cdot a\,\check{k}_2\cdot a + \hat{\mathcal{A}}^{(5)(C_3-1,C_2-1)}_{++},$$

with $\hat{\mathcal{A}}^{(4)(C_2-1)}_{++}$ vanishing when $C_2 = 1$ and $\hat{\mathcal{A}}^{(5)(C_3-1,C_2-1)}_{++}$ vanishing when $C_3 = C_2 = 1$. Because the C coefficients are all determined by the three-point amplitude, for black holes they are all known to take the value 1.

For the helicity reversing amplitude, it is useful to recall:

$$(aya) := (k_1\cdot a)(k_2\cdot a) - (x\cdot a)(q\cdot a) - \omega^2 a^2$$
$$= \frac{-q^2}{4\omega^2 - q^2}(k_1\cdot a - x\cdot a)(k_2\cdot a + x\cdot a) = \frac{q^2}{4\omega^2}\Big((x\cdot a)^2 - \omega^2 a^2\Big). \tag{3.8.87}$$

With this, we find:

$$\mathcal{A}_{+-} = \frac{q^2}{16\omega^2}\Bigg\{1 - q\cdot a + \frac{(q\cdot a)^2}{2}C_2 + (C_2-1)(aya)$$
$$- \frac{(q\cdot a)^3}{6}C_3 + (aya)\Big((1 - C_2 - C_2^2 + C_3)x + \frac{C_2 - C_3}{2}q\Big)\cdot a$$
$$+ \hat{\mathcal{A}}^{(4)}_{+-} + \hat{\mathcal{A}}^{(5)}_{+-} + \mathcal{O}(a^6)\Bigg\}, \tag{3.8.88}$$

with

$$\hat{\mathcal{A}}^{(4)}_{+-} = \frac{(q\cdot a)^4}{24}C_4 + (aya)^2\frac{3 - 4C_3 + C_4}{12} + \frac{C_4 - C_3}{6}(aya)q\cdot a(q - 2x)\cdot a$$
$$+ (C_2-1)(aya)\left[\frac{C_3 + 3C_2}{3}\Big(2(aya)\frac{\omega^2}{q^2} + \omega^2 a^2\Big) + \frac{C_3 + C_2}{2}q\cdot a\, x\cdot a\right]$$
$$+ (aya)^2\frac{\omega^2}{q^2}2(D_{4d} - D_{4a}) + q^2\omega^2 a^4\left(\frac{D_{4f} - D_{4c}}{2} - \frac{E_{4b}}{6}\right)$$
$$+ (aya)\omega^2 a^2(D_{4b} - D_{4e} - E_{4b}), \tag{3.8.89}$$

and

$$\hat{\mathcal{A}}^{(5)}_{+-} = -\frac{(q\cdot a)^5}{120}C_5 + \frac{C_4 - C_5}{24}(aya)(q\cdot a)^2(q - 2x)\cdot a + \frac{1 - C_4}{12}(aya)^2 q\cdot a + \hat{\mathcal{A}}^{(5)(C_3-1,C_2-1)}_{+-}$$

$$
\begin{aligned}
&\quad + \frac{5C_4 - C_5 - 4}{120}(aya)\left[5(aya)\left((q-2x)\cdot a - 8\frac{\omega^2}{q^2} q\cdot a\right) + \omega^2 a^2 (9q+10x)\cdot a\right] \\
&+ q\cdot a\,(aya)^2 \frac{\omega^2}{q^2}\left(\frac{1}{6} - (D_{5a}+D_{5d})\right) \\
&+ q\cdot a\,(aya)\omega^2 a^2 \left(\frac{1}{30} + \frac{D_{5b}+D_{5e}}{2} + \frac{4}{15}E_{5b} + \frac{4E_{5c}}{45} + \frac{9E_{4a}-E_{5a}}{360}\right) \\
&+ x\cdot a\,(aya)\omega^2 a^2 \left(\frac{2}{9} + D_{5g} - D_{5i} + \frac{9E_{4a}-E_{5a}}{36}\right) \\
&+ q\cdot a\, q^2\omega^2 a^4 \left(-\frac{D_{5c}+D_{5f}}{4} + \frac{E_{5b}}{20} + \frac{11E_{5c}}{720} + \frac{6E_{5e}-E_{5d}}{720}\right) \\
&+ x\cdot a\, q^2\omega^2 a^4 \left(\frac{D_{5j}-D_{5h}}{2} + \frac{E_{5b}}{30} + \frac{E_{5c}}{360} + \frac{6E_{5e}-E_{5d}}{120}\right),
\end{aligned}
\tag{3.8.90}
$$

where similarly $\hat{\mathcal{A}}_{+-}^{(5)(C_3-1,C_2-1)}$ vanishes when $C_3 = C_2 = 1$.

3.9 Conclusion

Using the dynamical mass function worldline formalism, we derived formal expressions for the electromagnetic/gravitational Compton amplitudes of a generic spinning body to all orders in spin, with precise parameterized expressions in terms of Wilson coefficients for the amplitudes to order S^3 in electromagnetism and S^5 in gravity for bodies which match the $\sqrt{\text{Kerr}}$ /Kerr three-point amplitude. In electromagnetism we found 1 Wilson coefficient and 1 independent structure in the Compton at S^1, 5 new Wilson coefficients and 5 independent structures at S^2, and 8 new Wilson coefficients but only 7 independent structures at S^3. In gravity we found 4 Wilson coefficients and 1 independent structure in the Compton at S^2, 2 new Wilson coefficients and 1 independent structure at S^3, 18 new Wilson coefficients and 7 independent structures at S^4, and 16 new Wilson coefficients and 11 independent structures at S^5. As well, one of the Wilson coefficients on an S^4 operator in gravity (E_{4a}) does not contribute to S^4 piece of the Compton but does contribute to the S^5 piece (whereas the other operator contributions which do not contribute at the order they are introduced actually do not contribute at all through S^5).

Dixon's multipole moment formalism provides additional physical constraints on the dynamical mass function beyond those required by naive multipole moments which are determined by the three-point amplitude. Many of the additional terms induced by Dixon's formalism (the operators with E coefficients), especially at low orders in spin, happen to not affect the Compton amplitude. However, at sufficiently high orders in spin (S^5 and beyond in gravity) at least some of these additional terms do contribute to the Compton amplitude. Through $\mathcal{O}(S^5)$ these combinations are linearly redundant in the Compton amplitude to contributions from Riemann squared operators. It would be interesting to see if these additional coefficients begin to contribute in linearly independent ways in the Compton at higher orders in spin or in higher-point processes, such as the five-point amplitude (with three graviton lines).

In electromagnetism, using Dixon's multipole moments for $\sqrt{\text{Kerr}}$ determines the dynamical mass function to be given by (3.4.54) with no room for additional operators which are linear in the field strength. Because the stationary $\sqrt{\text{Kerr}}$ solution only determines its multipole moments up to corrections which are linear in the field strength, they only determine the couplings in the action up to corrections which are quadratic in the field strength. Without some additional physical principle which specifies how the multipole moments of the $\sqrt{\text{Kerr}}$ particle deform in the presence of a background field (which would determine its electromagnetic susceptibility tensors), it is not possible to determine the quadratic in field strength couplings using the multipole moment formalism. The couplings in (3.4.54) contain all of the couplings of (3.4.57) (in that the coefficients of operators which are present in both agree). The story is very similar in gravity. Dixon's multipole moment formalism applied to the Kerr solution determines the dynamical mass function to be given by (3.7.43) with no room for additional operators which are linear in the Riemann tensor. Because the stationary Kerr solution only determines its multipole moments up to corrections which are linear in the Riemann tensor, they only determine the couplings in the action up to corrections which are quaratic in the Riemann tensor. Without some further knowledge of the gravitational susceptibility tensors of a spinning black hole, it is not possible to determine the quadratic in Riemann couplings using the multipole moment formalism. The couplings in (3.7.43) contain all of the couplings of (3.7.48).

It is uncertain what precise physical principles determine the correction couplings in the action for a spinning black hole in general. Through $\mathcal{O}(S^4)$, spin-exponentiation, shift-symmetry, the results of the Teukolsky equation, and the results from the multipole moment formalism can all be maintained simultaneously by appropriately choosing the values of Wilson coefficients for quadratic in Riemann tensor operators. However, beginning at $\mathcal{O}(S^5)$ these different principles cannot all be maintained. Spin-exponentiation is only possible to maintain for one of the two independent helicity combinations and is consistent with shift-symmetry at $\mathcal{O}(S^5)$. However, shift-symmetry and the Teukolsky results are inconsistent with each other at $\mathcal{O}(S^5)$. The couplings fixed by the multipole moment formalism are the unique values for the couplings so that the stress tensor they produce behaves correctly against test functions (meaning satisfies (3.7.16) with the multipole moments of the Kerr solution used to form the generating function on the right hand side). In this way, only those couplings produce the Kerr multipole moments (up to corrections which are linear in the Riemann tensor) for a black hole which is in arbitrary nonuniform motion. Such couplings are consistent with maintaining a match to spin-exponentiation and the Teukolsky equation at $\mathcal{O}(S^5)$.

Chapter 4

Spin Magnitude Change in Orbital Evolution in General Relativity

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We show that physical scattering observables for compact spinning objects in general relativity can depend on additional degrees of freedom in the spin tensor beyond those described by the spin vector alone. The impulse, spin kick and leading order waveforms exhibit such a nontrivial dependence. A clear signal of this additional structure is the change of the magnitude of the spin vector under conservative Hamiltonian evolution, similar to our previous studies in electrodynamics. These additional degrees of freedom describe dynamical mass multipoles of compact objects and decouple for black holes.

4.1 Introduction

The detection of gravitational waves by the LIGO/Virgo collaboration [102, 103] has opened a new era in astronomy, cosmology, and perhaps even particle physics. As the sensitivity of gravitational-wave detectors will continue to improve [104, 105, 106], the spin of compact astrophysical objects will become increasingly important for signal identification and interpretation. Spin introduces rich three-dimensional dynamics due to the angular momentum exchange between the objects and their orbital motion. For further details, see reviews [269, 225, 227, 241] and references therein.

The study of spinning compact objects brings to the forefront interesting and subtle theoretical questions regarding the number of physical degrees of freedom and independent effective operators describing their interactions and their Wilson coefficients [101, 2]. Here, we discuss these issues along the lines of Ref. [2], which dealt with the case of electrodynamics.

In traditional worldline approaches (see e.g. [117, 229, 145, 60]) as well as in WQFT [166], a spin supplementary condition (SSC) [228] identifies the three physical spin degrees of freedom and restricts their interactions [1]. While a consequence of the SSC is the invariance of the spin magnitude under conservative time evolution, rotation and Lorentz invariance generally guarantee only the conservation of the magnitude of the total angular momentum.

Ref. [2] studied the effective operators and associated Wilson coefficients [101] in the simpler case of electrodynamics. It does so from the amplitudes-based field theory, worldline, and two-body Hamiltonian perspectives by relaxing the SSC, and found all approaches consistent. Additional Wilson coefficients vis-à-vis worldline approaches reflect the presence of additional dynamical degrees of freedom. In the field theory approach, this is connected to the description of the classical spin as a combination of quantum spins with allowed transitions. To match the field theory results, the worldline formulation of Ref. [226] was modified by relaxing the SSC, identifying the additional degrees of freedom denoted below by K^μ with components of the spin tensor

$$\mathsf{S}^{\mu\nu} = \frac{1}{m}\epsilon^{\mu\nu\rho\sigma}p_\rho S_\sigma + \frac{i}{m}(p^\mu K^\nu - p^\nu K^\mu)\,. \tag{4.1.1}$$

Here S^μ is the usual spin vector and K^μ can be interpreted as a mass dipole. They satisfy $p^\mu S_\mu = p^\mu K_\mu = 0$. Finally, the two-body Hamiltonian that reproduces the field theory and worldline results with the extra degrees of freedom necessarily should include both S and K [2].

Similar considerations apply just as well to gravity. Indeed, the self-consistency and physical inequivalence of the worldline approach with a dynamical mass dipole (i.e., with no SSC imposed) was understood a while ago [270, 271, 272] in this context.

Here, we show that the conclusions of Ref. [2] for electrodynamics carry over to gravity, including that the spin magnitude can evolve under conservative Hamiltonian dynamics, contrary to the usual approaches. While the linear-in-spin (dipole) interactions are not altered by the presence of K in accordance with the principles of general relativity, physical observables such as scattering angles and waveforms reflect the presence of additional degrees of freedom at $\mathcal{O}(G\mathsf{S}^2)$ and $\mathcal{O}(G^2\mathsf{S}^2)$, where S is the spin tensor of one of the objects [2]. Interestingly, starting at $\mathcal{O}(G^2S^3)$, artificially excluding the extra degrees of freedom in the external states

[1]The SSC was interpreted in Ref. [229, 145, 230] in terms of a spin-gauge symmetry encoding the freedom to locally shift the worldline in the ambient space.

[2]This contrasts with electrodynamics, where the first effect is linear in the spin tensor, and reflects the lack of universality of electromagnetic interactions at linear order in spin.

does not remove the additional Wilson coefficients [101]. Moreover, for special values of Wilson coefficients the extra degrees of freedom decouple from the spin and position dynamics, indicating the emergence of a local symmetry, possibly related to the spin gauge symmetry [145]. This interpretation is compatibile with the Kerr stress tensor [58], the results of Teukolsky equations [247, 185, 186] and with earlier results for black hole scattering through $\mathcal{O}(G^2S^4)$ [59, 60, 61, 167, 173, 101][3].

4.2 Field Theory

We begin by outlining our amplitude-based field theory approach of Refs. [59, 61, 101, 2], which describes a spinning body by a symmetric and traceless tensor field ϕ_s. In the local frame defined by the vierbein e^a_μ, the minimal interaction with Einstein gravity is,

$$\mathcal{L}_0 = -\frac{1}{2}\phi_s(\nabla^2 + m^2)\phi_s + \frac{H}{8}R_{abcd}\phi_s M^{ab}M^{cd}\phi_s\,, \tag{4.2.1}$$

where M^{ab} is the Lorentz generator acting in the space of ϕ_s. For generic bodies, such as neutron stars, we can write down additional interactions starting at $\mathcal{O}(M^2)$,

$$\begin{aligned}\mathcal{L}_{\text{non-min}} =\ & -\frac{C_2}{2m^2}R_{af_1bf_2}\nabla^a\phi_s\mathbb{S}^{(f_1}\mathbb{S}^{f_2)}\nabla^b\phi_s + \frac{D_2}{2m^2}R_{abcd}\nabla_i\phi_s\{M^{ai}M^{cd}\}\nabla^b\phi_s \\ & + \frac{E_2 - 2D_2}{2m^4}R_{abcd}\nabla^{(a}\nabla^{i)}\phi_s\{M^{b_i}M^{d_j}\}\nabla^{(c}\nabla^{j)}\phi_s\,.\end{aligned} \tag{4.2.2}$$

where $\mathbb{S}^a = -i\epsilon^{abcd}M_{bc}\nabla_d/(2m)$. The C_2 term corresponds to a standard interaction included on the worldline. It is the unique interaction at this order compatible with an internal SO(3) symmetry in the body-fixed frame and an imposed worldline spin gauge symmetry [145]. In contrast, in the field-theory formalism, it is natural to include all interactions that can be relevant to classical physics. The Wilson coefficients (H, C_2, D_2, E_2) are not independent, *allowing us to fix* $H = 1$. Note that we introduce a certain mixing of E_2 and D_2 to align the spin structures in amplitudes and observables.

Analogous interactions are not included in the usual worldline formulations because the SSC, which effectively removes three dynamical degrees of freedom from the spin tensor, sets them to zero. As discussed in some detail for the case of electrodynamics [2] through quadratic order in spin, demanding that only states of fixed definite spin propagates in the field theory formulation enforces the conservation of the magnitude of the spin vector during the interactions and is equivalent to enforcing the SSC. In contrast, when states of different spins propagate, transitions between them are allowed, the additional interactions in Eq. (4.2.2) contribute to physical observables. While negative norm states can appear in the analog of Eq. (4.2.2), using a more involved Lagrangian with only positive norm states does not change the conclusion that additional

[3]Through $\mathcal{O}(S^4)$ Refs. [247, 185, 186] find that the black hole Compton amplitude obtained via the Teukolsky equation agrees with these previous results, but beyond this, the situation is less clear. See Refs. [273, 274] for recent discussions on reconciling the approaches based on scattering amplitudes and the Teukolsky equation.

Wilson coefficients contribute in the classical limit if transitions between different spin states are allowed. Furthermore, physical observables obtained in the two cases are related by a simple mapping of parameters. The same conclusions hold for gravity. Since the appearance of negative norm states does not change the classical limit, we find it more practical to use Eq. (4.2.2).

We define the classical spin tensor S^{ab} as the expectation value of a Lorentz generator M^{ab} in the boosted spin-coherent states of the massive spinning particles (see Eq. (2.27) of Ref. [2]). Due to the presence of degrees of freedom beyond those of a single fixed spin and of the ensuing absence of an SSC, the product of massive polarization tensors also depends on the mass dipole K [2],

$$\mathcal{E}_1 \cdot \mathcal{E}_2 = \exp\left[\frac{q\cdot K}{m}\right] \mathcal{E}_1^{(s)} \cdot \mathcal{E}_2^{(s)} + \mathcal{O}(q)\,, \tag{4.2.3}$$

where $q = p_2 - p_1$ and $\mathcal{E}^{(s)}$ is the transverse traceless component of $\mathcal{E}$, corresponding to the coherent state of a fixed spin s. The product $\mathcal{E}_1^{(s)} \cdot \mathcal{E}_2^{(s)}$ only depends on the spin vector. It provides the transition between the covariant and canonical impact parameter [59] and between the covariant spin variable used in field-theory amplitudes and the canonical spin used in the effective Hamiltonian.

We construct the three-point and four-point Compton amplitudes, schematically shown in Fig. 4.1, using the Feynman rules from the Lagrangian $\mathcal{L}_0 + \mathcal{L}_{\text{non-min}}$. We separate the mass dipole K from the spin tensor using Eq. (4.1.1) and expose in the classical amplitudes both the K-dependence coming from both the interactions and from Eq. (4.2.3), but suppress the spin vector dependent factor $\mathcal{E}_1^{(s)} \cdot \mathcal{E}_2^{(s)}$. With this understanding, the classical Compton amplitudes up to $\mathcal{O}(\mathsf{S}^2)$ are

$$\begin{aligned}
\mathcal{M}_{\mathrm{C}}^{3\mathrm{pt}} =& -(\epsilon_1\cdot p)^2 + \frac{(\epsilon_1\cdot p)\tilde{f}_1(p,S)}{m} - \frac{(1+C_2)(\epsilon_1\cdot p)^2(k_1\cdot S)^2}{2m^2} \\
&- \frac{D_2(k_1\cdot K)(\epsilon_1\cdot p)\tilde{f}_1(p,S)}{m^2} - \frac{E_2(k_1\cdot K)^2(\epsilon_1\cdot p)^2}{2m^2}\,, \\
\mathcal{M}_{\mathrm{C}}^{4\mathrm{pt}} =& \frac{4}{\hat{s}t\hat{u}}\left[\alpha^2 - \alpha\,\mathcal{O}_{(1)} + \frac{1}{2}\mathcal{O}_{(1)}^2 + C_2\,\alpha\,\mathcal{O}_{(2)} + D_2\,\alpha\left(\mathcal{O}_{(1)}\frac{(k_1+k_2)\cdot K}{m} - \mathcal{K}_{(1,1)}\right) + E_2\left(\alpha\,\mathcal{O}_{(2)}\Big|_{S\to K}\right)\right]\,,
\end{aligned} \tag{4.2.4}$$

where $\alpha = p\cdot f_1\cdot f_2\cdot p$, and $f_i^{\mu\nu} = k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu$ is the linearized field strength. The kinematic variables here are defined as $\hat{s} = 2p\cdot k_1$, $t = 2k_1\cdot k_2$ and $\hat{u} = 2p\cdot k_2$. The operators $\mathcal{O}$ and $\mathcal{K}$ are given by

$$\begin{aligned}
\mathcal{O}_{(1)} &= \frac{1}{m}\Big[f_2(p,k_1)\tilde{f}_1(p,S) + \frac{\hat{s}}{2}\,\tilde{f}_{12}(p,S) + (1\leftrightarrow 2)\Big]\,, \\
\mathcal{O}_{(2)} &= \frac{1}{2m^2}\Big[tf_1(p,S)f_2(p,S) + \alpha(k_1\cdot S + k_2\cdot S)^2\Big]\,, \\
\mathcal{K}_{(1,1)} &= \frac{t}{2m^2}\Big[f_2(p,K)\tilde{f}_1(p,S) + f_1(p,K)\tilde{f}_2(p,S)\Big]\,,
\end{aligned} \tag{4.2.5}$$

where $f_i(a,b) = f_i^{\mu\nu}a_\mu b_\nu$, and $(\tilde{f}_i^{\mu\nu}, \tilde{f}_{12}^{\mu\nu})$ are respectively the Hodge duals [4] of $(f_i^{\mu\nu}, f_1^{\mu\rho}f_{2,\rho}{}^{\nu})$.

This result reveals interesting correlations between the additional degrees of freedom in ϕ_s and Wilson

[4]We define the Hodge dual as $\tilde{f}^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}f_{\alpha\beta}$

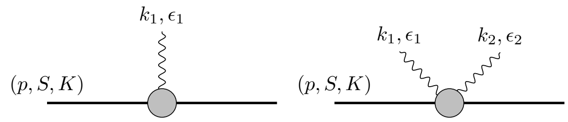


Figure 4.1: Three-point and four-point Compton amplitudes

coefficients. Firstly, the appearance of K is due to $\mathcal{L}_0$ not enforcing transversality on ϕ_s. However, the spin-orbit terms are unchanged and there are no linear-in-K contributions at this order, in line with the absence of interactions of mass dipoles in general relativity. At $\mathcal{O}(\mathsf{S}^2)$ level, K appears in the form of SK and K^2. Note that the K^2 dependence is identical to the S^2 dependence with Wilson coefficient C_2. The Compton amplitude (4.2.4) is independent of K if $D_2 = E_2 = 0$, as the mass dipoles originating from interactions exactly cancel the contribution of the exponential factor in Eq. (4.2.3). With these choices, the results agree with the conventional formalism for neutron stars or other compact astrophysical objects considered in Ref. [145]. Finally, the Kerr black hole corresponds to setting $C_2 = 0$, in close analogy with the root-Kerr solution in electrodynamics [2].

Starting from the Compton amplitudes in Fig. 4.1, two-body scattering amplitudes at $\mathcal{O}(G^2)$ can be found using generalized unitarity [206, 207, 248]. We construct tree-level and one-loop four-point two-body amplitudes, from which we may obtain the classical two-body effective Hamiltonian and observables through a matching process [14, 59].

4.3 Worldline

The standard approach to spinning particles using worldline formalisms, an SSC, e.g., $p_\mu \mathsf{S}^{\mu\nu} = 0$, is imposed via a Lagrange multiplier, which has the effect of eliminating K^μ in Eq. (4.1.1) as a dynamical degree of freedom. This implies that the magnitude of the spin vector, $S^\mu S_\mu$, is conserved and it eliminates various operators and their associated Wilson coefficients. A basic result of the field-theory analysis above and in Refs. [101, 2] is that even for conservative systems, $S^\mu S_\mu$ can change under conservative time evolution and that additional Wilson coefficients appear compared to the standard worldline framework. In Ref. [2], these features matched to a worldline description of electrodynamics with no imposed SSC, rediscovering the observation of Refs. [270, 271, 272] that the worldline with no SSC is distinct and consistent. Here we construct the analogous gravitational worldline theory and show that the Compton amplitudes match those of the abovementioned field theory.

To this end, we follow the same steps as for electrodynamics, which in turn is based on the dynamical mass formalism of Ref. [226], except that the SSC-imposing terms are dropped. In this approach, a spinning body is described by a timelike worldline $z^\mu(\lambda)$, its conjugate momentum $p_\mu(\lambda)$, a body tetrad $\Lambda^A{}_\mu(\lambda)$, and the body spin tensor $\mathsf{S}^{\mu\nu}(\lambda)$, where λ is a parameter describing the position along the worldline. The Lagrangian is

$$L = -p_\mu \dot{z}^\mu + \frac{1}{2}\mathsf{S}^{\mu\nu}\Lambda_{A\mu}\frac{D\Lambda^A{}_\nu}{D\lambda} + \frac{\xi}{2}(p^2 - \mathrm{M}^2) \tag{4.3.1}$$

where $\xi(\lambda)$ is a Lagrange multiplier that enforces the on-shell constraint $p^2 = \mathrm{M}^2$ and $\mathrm{M}(z, p, \mathsf{S})$ is the dynamical mass function of the body, which contains the body's free mass and all its non-minimal couplings to gravity. The stationary variation of the corresponding action describes the body dynamics. In terms of the 4-velocity $u^\mu = p^\mu/\sqrt{p^2}$ and to second order in the vectors S^μ and K^μ, the dynamical mass function can be written as:

$$\mathrm{M}^2 = m^2 + \frac{1-C_2}{4}R_{uSuS} + \frac{1-D_2}{2}R_{uSuK} + \frac{1-E_2}{4}R_{uKuK}\,,$$

with $R_{uSuK} = R_{\mu\nu\rho\sigma}u^\mu S^\nu u^\rho K^\sigma$. This worldline Lagrangian is equivalent to the one of Ref. [226] except that no Lagrange multiplier term that imposes an SSC is included.

The classical Compton is computed [247] as the coefficient of the outgoing spherical wave produced by the response of the spinning body to in an incoming plane wave. The metric perturbation,

$$h^{\mu\nu} = e^{ik\cdot x}\varepsilon^{\mu\nu} + \frac{e^{i(kr-\omega t)}}{4\pi r}\mathcal{M}_{\mathrm{C}}^{\mu\nu,\rho\sigma}\varepsilon_{\rho\sigma} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{4.3.2}$$

and the Compton amplitude $\mathcal{M}_{\mathrm{C}}^{\mu\nu,\rho\sigma}$ is extracted directly from the stress tensor $T_{\mu\nu} = \frac{\delta}{\delta h^{\mu\nu}}\int d\lambda L$ using the solution to the wave equation at large distances in de Donder gauge. With the Wilson coefficients assigned as in Eq. (4.3.2), the result exactly matches Eq. (4.2.4), pointing to the equivalence of this worldline theory and the field theory described above.

As discussed, the description of spinning particles in the presence of the covariant SSC can be obtained from the unconstrained one by simply setting $D_2 = 0 = E_2$. Fur such special values of the Wilson coefficients, initial data satisfying the covariant SSC (i.e. with $K^\mu = 0$) is preserved under time evolution, similarly to the case of electrodynamics [2]. This feature implies that, from a practical standpoint, it is convenient not to impose the SSC but instead to make appropriate choices of Wilson coefficients that reduce the system to the one without mass dipole K.

4.4 Scattering waveform at leading order

To explore the physical relevance of the mass dipole K and of the additional Wilson coefficients, we study their effect on the scattering waveform. To this end, we assume that we are given a waveform signal that can be fitted by a $K = 0$ system and study whether it is possible to accurately describe the same signal by turning on K and readjusting Wilson coefficients. Since our spin-dependent amplitudes are to a fixed (second) order in spin, we carry out this comparison at each order separately.

Since K does not enter at linear order, S should remain fixed as K is turned on. Thus, we may only adjust the Wilson coefficients. We will find that, to the order we are working and unless $D_2 = 0$, the waveform at a fixed observation angle can discern a nonvanishing mass dipole. The leading order waveform for the scattering Kerr black hole off a spinless body was computed in [275, 276, 277]. We have verified that our results at $K = 0$ exactly agrees with Ref. [275].

The metric perturbation at infinity is given by

$$\begin{aligned} h^{\infty}_{\mu\nu} &= \frac{1}{\kappa} \lim_{|\boldsymbol{x}|\to\infty} 4\pi|\boldsymbol{x}|(g_{\mu\nu} - \eta_{\mu\nu}) \\ &= \kappa M \left[\frac{\kappa^2 M}{|b|} \, \hat{h}^{(1)}_{\mu\nu} + \mathcal{O}(\kappa^4) \right] , \end{aligned} \tag{4.4.1}$$

where $M = m_1 + m_2$ and $\kappa^2 = 32\pi G$. Ref. [8] established the connection between $h^{\infty}_{\mu\nu}$ and scattering amplitudes. At leading order, the waveform $W \equiv \varepsilon^{\mu\nu} h^{\infty}_{\mu\nu}$ takes the simple form

$$\begin{aligned} W^{\mathrm{LO}}(t) &= \frac{\kappa^3 M^2}{|b|} \varepsilon^{\mu\nu} \hat{h}^{(1)}_{\mu\nu} \\ &= -2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \int d\mu \, \mathcal{M}_5^{\mathrm{tree}} e^{-iq_1\cdot b} e^{-i\omega t} , \end{aligned} \tag{4.4.2}$$

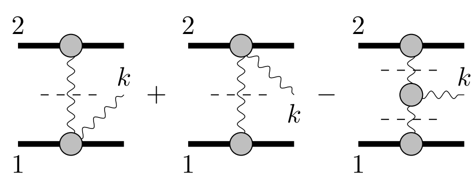
where $\mathcal{M}_5^{\mathrm{tree}} \equiv \mathcal{M}_5^{\mathrm{tree}}(p_1, p_2, q_1, k, \varepsilon)$ is the classical five-point amplitude with an outgoing graviton of momentum k and physical polarization ε. The measure $d\mu$ is

$$d\mu = \frac{d^4 q_1}{(2\pi)^4} \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}\big(2p_2 \cdot (k - q_1)\big) , \tag{4.4.3}$$

with $\hat{\delta}(x) = 2\pi\delta(x)$ and $\hat{\delta}^d(x) = (2\pi)^d \delta^d(x)$, and q_1 is the momenta lost by particle p_1.

Using generalized unitarity, the classical tree-level five-point amplitude necessary for W^{LO} can be computed by sewing together three- and four-point Compton amplitudes. Schematically, it is given by

$$\mathcal{M}_5 \sim$$



$, \qquad (4.4.4)$

where the dashed lines indicate on-shell conditions and summation over physical states. We ignore the local terms as they do not contribute in the classical regime.

The gravitational-wave memory, which is given by the soft limit of this amplitude, provides clues into the structure of the complete waveform. Interestingly, explicit calculation reveals that it follows the same pattern as for the scattering of spinless particles, i.e. [278, 279] that in the frequency domain, $k^\mu = \omega n^\mu$,

$$\varepsilon^{\mu\nu} h^{\infty}_{\mu\nu}(\omega) = \frac{i}{\omega} \sum_{i=1}^{4} \frac{\eta_i (\varepsilon \cdot p_i)^2}{n \cdot p_i} + \mathcal{O}(\omega^0) , \tag{4.4.5}$$

where $\eta_{1,2} = 1$, $\eta_{3,4} = -1$ and $\varepsilon^{\mu\nu} = \varepsilon^\mu \varepsilon^\nu$ for on-shell gravitons. Momentum conservation requires $p_4 = p_1 + \Delta p$ and $p_3 = p_2 - \Delta p$, where Δp is the leading order covariant impulse.

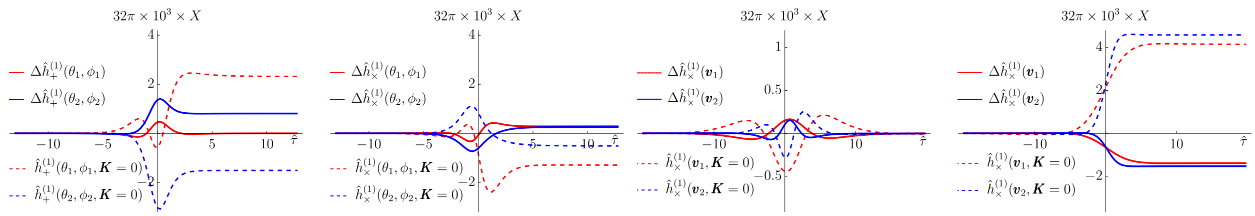


Figure 4.2: Plots showing the waveforms for $\boldsymbol{S} = (\cos\pi/4, \sin\pi/4, 0)$, $C_2 = 1$, denoted by $h_+^{(1)}(\ldots, \boldsymbol{K} = 0)$, and their difference from a $\boldsymbol{K}$-dependent waveform as follows. *Left two figures:* the $+$ and $\times$ polarizations observed at angles $(\theta_1, \phi_1) = (7,4)\pi/10$, $(\theta_2, \phi_2) = (1,4)\pi/10$ for particles with COM velocity $\boldsymbol{v} = (0,0,1/5)$ and impact parameter $\boldsymbol{b}_{\rm cov} = (5,0,0)$. We choose $\boldsymbol{K} = \boldsymbol{S}$, $C_2 = 1/3$, $D_2 \simeq 0.286$ so that the memory difference for the $+$ polarization at observation angle (θ_2, ϕ_1) vanishes, $\Delta\hat{h}_+^{(1)}(\theta_1, \phi_1) = 0$. *Right two figures:* dashed lines: the $+$ and $\times$ polarizations observed at $(\theta, \phi) = (5,10)\pi/10$ for COM velocities $\boldsymbol{v}_1 = (0,0,1/14)$, $\boldsymbol{v}_2 = (0,0,1/7)$ impact parameter $\boldsymbol{b}_{\rm cov} = (5,0,0)$; solid lines: the difference from $\boldsymbol{K} = \boldsymbol{S}$, $C_2 = 1/3$, $D_2 \simeq 0.431$; D_2 is chosen so that the memory difference for the $+$ polarization and velocity $\boldsymbol{v}_1$ vanishes, $\Delta\hat{h}_+^{(1)}(\boldsymbol{v}_1) = 0$. The three-vector $\boldsymbol{S}$ and $\boldsymbol{K}$ are related to the four-vector S^μ and K^μ as $S^\mu = (0, \boldsymbol{S})$, $K^\mu = (0, \boldsymbol{K})$. We note that the spin S^μ here is the covariant spin, and the impact parameter $b^\mu_{\rm cov} = (0, \boldsymbol{b}_{\rm cov})$ is defined accordingly [59].

It is not difficult to see that the third term in Eq. (4.4.4) is subleading as $k^\mu \to 0$, and thus the soft limit of the classical five-point tree amplitude is determined by the soft limit of the four-point tree-level Compton amplitude. As discussed below Eq. (4.2.5), in this amplitude, the terms bilinear in the spin vector originating from Wilson coefficient C_2 are the same (up to $S \to K$) as the terms bilinear in K originating from E_2. We may, therefore, expect a similar property for the entire five-point amplitude, which indeed turns out to be the case:

$$\mathcal{M}_5\Big|_{E_2 K^2} = \mathcal{M}_5\Big|_{C_2 S^2}^{S\to K, C_2 \to E_2}. \tag{4.4.6}$$

Namely, if we turn on K^μ parallel ($\|$) to S^μ, we can compensate the effect by adjusting C_2. Thus, for special systems with $D_2 = 0$, such mass dipoles are degenerate with the spin. In the following we consider only $D_2 \neq 0$ and $K^\mu \parallel S^\mu$. To streamline the comparison, we keep $E_2 = 0$ and interpret $E_2 \neq 0$ as a change in C_2.

Exploring the memory contribution of the $\mathcal{O}(SK)$ terms, we find that it is possible to choose the Wilson coefficients C_2 (or equivalently E_2) and D_2 so that K is not distinguishable *at late times* in a fixed observation direction specified by the polar angles (θ, ϕ) and for a center-of-mass (COM) velocity $\boldsymbol{v}$ in one, say $+$, polarization. At finite times, however, and for the second polarization at all times, the difference between waveforms with and without K is nontrivial, as illustrated at fixed velocity and several observation angles in the left two plots of Fig. 4.2. Depending on the observation angle, we note that the difference can be as large as about 50% of the K-independent waveform.

A similar conclusion can be drawn by inspecting the waveform at a fixed observation angle as a function of the relative velocity of the particles in the COM. Examples are shown in the right two plots in Fig. 4.2. We choose to align the gravitational wave memory for the $+$ polarization and for the lower of the two velocities.

$\mathbf{1}$	$\left(\left(\boldsymbol{r}\times\boldsymbol{p}\right)\cdot\mathbf{S}\right)/\boldsymbol{r}^2$	$\left(\boldsymbol{r}\cdot\mathbf{K}\right)/\boldsymbol{r}^2$
$\left(\boldsymbol{r}\cdot\mathbf{S}\right)^2/\boldsymbol{r}^4$	$\left(\boldsymbol{r}\cdot\mathbf{K}\right)\left(\left(\boldsymbol{r}\times\boldsymbol{p}\right)\cdot\mathbf{S}\right)/\boldsymbol{r}^4$	$\left(\boldsymbol{r}\cdot\mathbf{K}\right)^2/\boldsymbol{r}^4$
$\mathbf{S}^2/\boldsymbol{r}^2$	$\left(\mathbf{K}\cdot\left(\boldsymbol{p}\times\mathbf{S}\right)\right)/\boldsymbol{r}^2$	$\mathbf{K}^2/\boldsymbol{r}^2$
$\left(\boldsymbol{p}\cdot\mathbf{S}\right)^2/\boldsymbol{r}^2$	$\left(\boldsymbol{r}\cdot\mathbf{S}\right)\left(\left(\boldsymbol{r}\times\mathbf{K}\right)\cdot\boldsymbol{p}\right)/\boldsymbol{r}^4$	$\left(\boldsymbol{p}\cdot\mathbf{K}\right)^2/\boldsymbol{r}^2$

Table 4.1: The operators up to terms quadratic in $\mathbf{S}$ and $\mathbf{K}$ appearing in the Hamiltonian for a single spinning body.

Interestingly, the maximum waveform difference for the + polarization appears to exhibit a very weak velocity dependence. The variation of the width of the curves can be traced to a similar variation in the waveform itself, which is due to the fact that, at fixed impact parameter, lower-velocity particles experience a longer period of stronger acceleration, leading to a broader waveform. As before, the differences shown in the two right plots of Fig. 4.2 can be as large as 50% of the K-independent waveform.

4.5 Effective Hamiltonian

Using the two-body elastic amplitudes obtained in our field-theory analysis, we match to a Hamiltonian that describes the conservative evolution of the two bodies in general relativity. This Hamiltonian can serve as an input to EOB models for gravitational waveforms, connecting theoretical computations and experimental observations (see e.g. Refs. [280, 281, 282, 283, 164]).

The steps to construct a two-body Hamiltonian from our amplitudes are described in detail in our earlier paper [2], which builds on effective Hamiltonians used for the non-spinning [14] and spinning cases without $\mathbf{K}$ [59]. We start with an ansatz for the most general Hamiltonian with a set of to-be-determined coefficients of spin structures that include the $\mathbf{S}$ and $\mathbf{K}$ degrees of freedom and determine these coefficients by matching the two-to-two scattering amplitudes from this Hamiltonian and the ones obtained from field theory calculations.

A one-loop consistency is that all IR divergences of the amplitude (i.e. the coefficients of box integrals) are completely determined in terms of tree-level Hamiltonian coefficients.

We organize our Hamiltonian in terms of spin structures Σ_a and corresponding coefficients $c_n^a(\boldsymbol{p}^2)$ as

$$\begin{aligned}\mathcal{H} &= \sqrt{\boldsymbol{p}^2+m_1^2}+\sqrt{\boldsymbol{p}^2+m_2^2}+V\,,\\ V &= \sum_a V^a\Sigma_a\,,\quad V^a=\sum_{n=1}^{\infty}\left(\frac{G}{|\boldsymbol{r}|}\right)^n c_n^a(\boldsymbol{p}^2)\,,\end{aligned}\tag{4.5.1}$$

where m_1 and m_2 are the masses of the two objects, $\boldsymbol{r}$ and $\boldsymbol{p}$ is the relative distance and the relative momentum in the COM, respectively. We give the spin structures for the case of one spinning body in Table 4.1. For choices of Wilson coefficients that match the worldline with SSC, the resulting scattering angles reproduce in the overlap with those of Refs. [78, 21, 22, 167, 163].

The earlier two-body Hamiltonian in Ref. [101] was constructed with $\mathbf{K}=0$ and cannot be used to match the field theory or worldline where $\mathbf{K}$ is nonvanishing since the Hamiltonian must contain all the dynamical degrees of freedom.

An important prediction of general relativity is that the mass dipole of compact objects does not partake on its own in gravitational interactions. A non-trivial consistency check is that the introduction of $\mathbf{K}$ does not contradict this basic result. Indeed, our Hamiltonian allows multiple interactions that are linear in the mass dipole of one of the objects, e.g.

$$\Sigma_{(1,3)} = \frac{\boldsymbol{r}\cdot\mathbf{K}_1}{\boldsymbol{r}^2}\,,\ \Sigma_{(2,13)} = \frac{(\boldsymbol{r}\cdot\mathbf{K}_1)\left((\boldsymbol{r}\times\boldsymbol{p})\cdot\mathbf{S}_2\right)}{\boldsymbol{r}^4}\,. \tag{4.5.2}$$

We expect this property to hold to all orders in the spin and G, as the vanishing of these coefficients may be traced back to the fact that at linear order in the spin tensor, the only possible gravitational coupling is the minimal one. Since there is no freedom in adjusting the minimal coupling with additional Wilson coefficients, at linear order in the spin the gravitational interactions do not differentiate between black holes, conventional neutron stars and generic astrophysical objects considered here. In this way, the vanishing of the gravitational mass-dipole interaction in our formalism follows from general coordinate invariance.

Having obtained a Hamiltonian that captures the conservative evolution of our binary in general relativity, we demonstrate that including the additional degrees of freedom leads to spin-magnitude change. Indeed, by solving Hamilton's equations, we find at $\mathcal{O}(G)$:

$$\begin{aligned}\Delta\mathbf{S}_1^2 = {} & GD_2\frac{8\sigma m_2\left(m_1^2+2\sigma m_1 m_2+m_2^2\right)\left(K_{1z}^{(0)}\left(S_{1x}^{(0)}-S_{1y}^{(0)}\right)\left(S_{1x}^{(0)}+S_{1y}^{(0)}\right)-S_{1z}^{(0)}\left(K_{1x}^{(0)}S_{1x}^{(0)}-K_{1y}^{(0)}S_{1y}^{(0)}\right)\right)}{|\boldsymbol{b}|^2\left(\sqrt{p_\infty^2+m_1^2}+\sqrt{p_\infty^2+m_2^2}\right)^2 m_1} \\ & + GE_2\frac{4\left(2\sigma^2-1\right)m_2^2\left(m_1^2+2\sigma m_1 m_2+m_2^2\right)\left(K_{1y}^{(0)}K_{1z}^{(0)}S_{1x}^{(0)}+K_{1x}^{(0)}K_{1z}^{(0)}S_{1y}^{(0)}-2K_{1x}^{(0)}K_{1y}^{(0)}S_{1z}^{(0)}\right)}{|\boldsymbol{b}|^2\left(\sqrt{p_\infty^2+m_1^2}+\sqrt{p_\infty^2+m_2^2}\right)^3 p_\infty}\,,\end{aligned} \tag{4.5.3}$$

where $\mathbf{S}_1^{(0)}$ and $\mathbf{K}_1^{(0)}$ are respectively the initial value of the spin and mass dipole. The impact parameter $\boldsymbol{b}$ points in the x direction and the incoming momenta are $p_1 = (\sqrt{p_\infty^2+m_1^2}, 0, 0, p_\infty)$ and $p_2 = (\sqrt{p_\infty^2+m_2^2}, 0, 0, -p_\infty)$, with $\sigma = \frac{p_1\cdot p_2}{m_1 m_2}$ (see Ref. [2] for more details). As noted there, while neither $\mathbf{S}_1^2$ nor $\mathbf{K}_1^2$ are conserved, the difference $\mathbf{S}_1^2 - \mathbf{K}_1^2$ is, so we have $\Delta\mathbf{K}_1^2 = \Delta\mathbf{S}_1^2$.

4.6 Eikonal Phase

Remarkably, the spin-dependent scattering observables can be encoded in a single scalar function—the eikonal phase [59, 284, 285], which to $\mathcal{O}(G^2)$ is given by the two-dimensional Fourier transform (from $\boldsymbol{q}$ space to $\boldsymbol{b}$ space) of the classical part of the EFT amplitude. Ref. [2] gave a generalization of the construction of [59], including the effects of K and applied it to electrodynamics. Here we explicitly confirm that this construction gives the correct results through $\mathcal{O}(G^2\mathsf{S}^2)$. From Ref. [2] we have

$$\Delta \boldsymbol{p} = \frac{\partial \chi}{\partial \boldsymbol{b}} + \frac{1}{2}\{\chi, \frac{\partial \chi}{\partial \boldsymbol{b}}\} + \mathcal{D}_L(\chi, \frac{\partial \chi}{\partial \boldsymbol{b}}) - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{b}}\mathcal{D}_L(\chi,\chi) - \frac{\boldsymbol{p}}{2\boldsymbol{p}^2}\left(\frac{\partial \chi}{\partial \boldsymbol{b}}\right)^2 + \mathcal{O}\left(\chi^3\right), \tag{4.6.1}$$

which simultaneously gives contributions orthogonal and along $\boldsymbol{p}$. Here, $\boldsymbol{p}\cdot\boldsymbol{b} = 0$, so all the $\boldsymbol{b}$-derivatives are projected orthogonal to the incoming momentum $\boldsymbol{p}$. The spin and mass dipole change are

$$\Delta\mathbb{O} = \{\chi, \mathbb{O}\} + \frac{1}{2}\big\{\chi, \{\chi, \mathbb{O}\}\big\} + \mathcal{D}_L\big(\chi, \{\chi, \mathbb{O}\}\big) - \frac{1}{2}\big\{\mathcal{D}_L(\chi,\chi), \mathbb{O}\big\}, \tag{4.6.2}$$

where $\mathbb{O} = (\mathbf{S}, \mathbf{K})$ and

$$\mathcal{D}_L(f,g) \equiv -\epsilon_{ijk}\left(S_{1i}\frac{\partial f}{\partial S_{1j}} + K_{1i}\frac{\partial f}{\partial K_{1j}}\right)\frac{\partial g}{\partial L_k}. \tag{4.6.3}$$

with an obvious generalization when both particles are spinning. The Lorentz algebra gives the brackets involving $\mathbf{S}$ and $\mathbf{K}$ (see Eq. (5.40) of Ref. [2]).

We have explicitly checked that Eqs. (4.6.1) and (4.6.2) indeed reproduce the impulse, spin kick and mass dipole change given by Hamilton's equations.

4.7 Conclusions and Outlook

The body-fixed tetrad has six degrees of freedom. Demanding the minimal EFT description of the body causes only three of them to be dynamical and requires an SSC to fix the other three [286]. The spin gauge symmetry then guarantees independence of this constraint. Relaxing minimality removes the need of an SSC and promotes the additional tetrad components to dynamical variables in the form of a mass dipole.

We demonstrated that the traditional description of gravitating spinning compact objects can be naturally extended to include such additional degrees of freedom describing properties of their mass distribution. The mass dipole K yields to new interaction terms with associated free Wilson coefficients in the field-theory, worldline, and two-body-Hamiltonian descriptions, which are all in physical agreement. This mirrors the conclusions of Ref. [2] obtained in electrodynamics. The distinct dynamics and self-consistency of worldline theories with no SSC imposed were also appreciated a while ago [270, 271, 272]. An unconstrained description of spinning particles is technically simpler and, if desired, the SSC can be imposed by choosing Wilson coefficients leading to the cancellation of all K-dependent contributions. Notably, the known Wilson coefficients for fixed-spin compact objects have this property.

An important feature is that in agreement with the principles of general relativity, interactions linear in the mass dipole do not affect physical observables. We explicitly demonstrated in general relativity that multilinears in the mass dipole and spin and their associated Wilson coefficients not only affect the impulse and spin kick at $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$ but also directly modify the waveform in ways that cannot be replicated by merely readjusting the Wilson coefficients in the absence of a mass dipole. We also explicitly showed that a K-dependent generalization [2] of the spinning eikonal formula [59] yields the correct impulse, spin and boost kick. It would be interesting to understand whether this agreement continues to higher orders.

While we have demonstrated that a mass dipole affects physical observables, it remains an interesting question whether this could lead to measurable effects for any compact astrophysical bodies in our Universe [5]. The Wilson coefficients governing their interactions could be determined by matching them onto suitable models or comparing theoretical waveforms to numerical relativity simulations or gravitational-wave data. While we focused here on the scattering regime, the absence of the tail effect at these low orders makes it straightforward to use the Hamiltonian we derived in the bound regime. It would be very interesting to contrast the $K \neq 0$ and $K = 0$ bound-orbit waveforms. Defining black holes in purely field-theoretic terms remains an interesting problem. Here, we observed, to low orders, that a suitable criterion is the decoupling of the mass dipole. An all-order proof is desirable.

[5] See Refs. [287, 240] for a discussion of the hierarchical three-body system in which the inner body may be interpreted as exhibiting a mass dipole.

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