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Sub. for pub. in Nuovo Cimento (Supplement)

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Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

STRANGE-PARTICLE DECAYS

Frank S. Crawford, Jr.

November 2, 1962

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STRANGE-PARTICLE DECAYS*

Frank S. Crawford, Jr.

Lawrence Radiation Laboratory
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INTRODUCTION

In these lectures we discuss the weak decays of strange particles. In particular, we discuss the present evidence concerning the $\Delta I = 1/2$ rule in the nonleptonic decays, and the $\Delta I = 1/2$ and $\Delta S = \Delta Q$ rules in the leptonic decays of strange particles.

We consider the hyperon decays

$$\Lambda \rightarrow N + \pi ,$$

$$\Sigma \rightarrow N + \pi ,$$

$$\text{and} \quad \Xi \rightarrow \Lambda + \pi ;$$

and the K-meson decays

$$K \rightarrow 2\pi ,$$

$$K \rightarrow 3\pi ,$$

$$\text{and} \quad K \rightarrow \pi + L + \nu, \text{ where}$$

L (lepton) stands for e or μ.

I will assume that the students are partly familiar with the material in Gell-Mann and Rosenfeld.[1] I will furthermore try to avoid repeating material given here at Varenna by Professor Rosenfeld.

---

*Lectures given in the course on "Elementary Particles," Enrico Fermi International School of Physics, July 23 through August 4, 1962, at Varenna, Como, Italy. (To be published by the Italian Physical Society in the Proceedings of the Varenna summer school.)

The five lectures are as follows:

I. Simple introductory examples illustrating invariance (or lack of invariance) with respect to S, I, I_3, P, T, and C in weak decays.

II. Review of the definition and measurement of the decay parameters α, β, and γ in hyperon decay.

III. $\Delta I = 1/2$ rule for the nonleptonic decays $K \rightarrow 2\pi$, $\Lambda \rightarrow N + \pi$, $\Xi \rightarrow \Lambda + \pi$, and $\Sigma \rightarrow N + \pi$.

IV. $K \rightarrow 3\pi$ and the $\Delta I = 1/2$ rule.

V. The $\Delta I = 1/2$ rule for leptonic K decays.

There will be no attempt to give complete references, especially to "well-known" results.

Lecture I. INTRODUCTORY EXAMPLES

We begin by considering the quantities S, I, I_3, P, T, and C. All of these (except T) are conserved in the strong interactions but not in the weak interactions. T (time-reversal) invariance is usually assumed to hold in both the strong and weak reactions. (There is no experimental evidence to the contrary.)

To illustrate nonconservation of (S, I, I_3) in weak (decay) interactions, consider $\Lambda \rightarrow p + \pi^-$. We have $(S = -1,\ I = 0,\ I_3 = 0)_\Lambda \rightarrow (0,\ 1/2 \text{ or } 3/2,\ -1/2)_{p\pi^-}$. Thus none of S, I, or I_3 is conserved. Notice that $|\Delta I_3| = 1/2$ but that $\Delta I = 1/2$ or $3/2$.

We now turn our attention briefly to P, T, and C, using a minimum of formalism.

In considering the meaning of P (parity) conservation (or nonconservation) we will use mirrors. The space inversion $x, y, z \rightarrow -x, -y, -z$ is (for example) equivalent to the reflection $x, y, z \rightarrow -x, y, z$, followed by a rotation R of 180° about the x axis, $x, y, z \rightarrow x, -y, -z$. Since R is assumed to have no observable consequences (i.e., the orientation of the system with respect to Andromeda, for instance, is assumed to be irrelevant), it is sufficient to consider only reflections in a mirror. The behavior of an axial vector (spin) or of a polar vector (linear momentum) upon reflection in a mirror is shown in Fig. 1.

To designate a spin we usually use $\Uparrow$ instead of . Sometimes we use if the spin is perpendicular to the paper.

We now consider, as an example of P conservation, the strong process $\pi^- + p \rightarrow \Lambda + K^0$. Suppose the target proton is unpolarized. Let the plane of the paper be the production plane. Consider the three production configurations of Fig. 2, which differ only as to the orientation of the spin of the Λ.

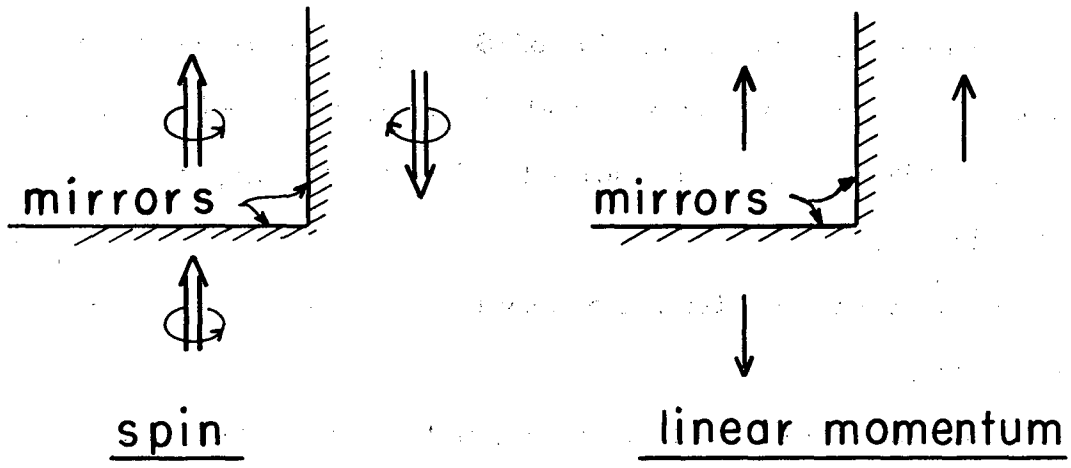

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Fig. 1

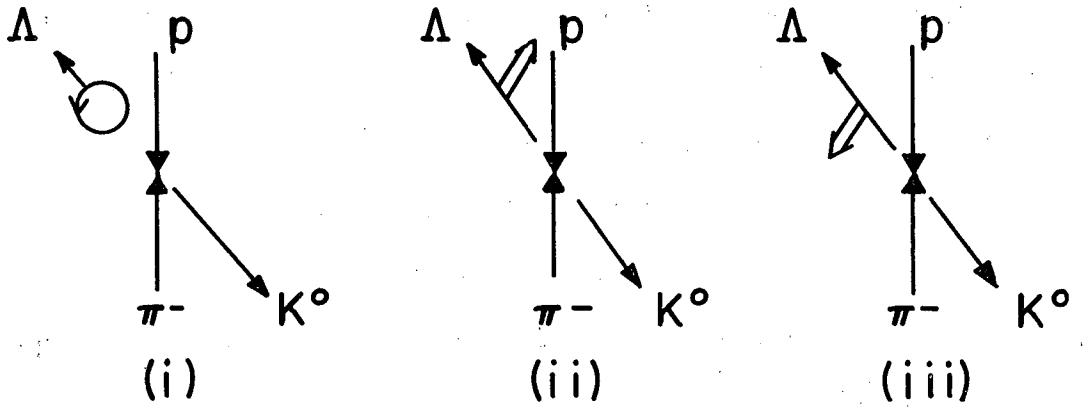

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Fig. 2

In case (i) the Λ spin is perpendicular to the production plane. In (ii) the Λ spin lies in the production plane. In (iii) the Λ spin is opposite to that in case (ii). If we view process (i) in a mirror held parallel to the production plane (plane of the paper) we "see" a process which we call (i'). A real process (i.e., with no mirror) that looks like (i') is also called (i'). Notice that, in our example, (i') happens to be indistinguishable from (i). Similarly the process (ii') looks like (iii), and (iii') looks like (ii). The following statements are all equivalent:

(a) "The process is invariant under reflection."

(b) "Parity is conserved in the process."

(c) "The process p and its reflected process p' occur with equal probability."

Thus if parity is conserved, processes (ii) and (ii')–that is (ii) and (iii)–occur with equal probability; therefore the Λ polarization components in the production plane must average to zero. Similarly (i) and (i') occur with equal probability. But these are the same process. Therefore a net polarization perpendicular to the production plane [as in (i)] is allowed (but not required). As a matter of fact, one finds experimentally that, in $\pi^- + p \rightarrow \Lambda + K^0$, the Λ's often have polarization of nearly 100% perpendicular to the production plane, but are never polarized in the production plane.[2]

Next consider the weak process $\Lambda \rightarrow p + \pi^-$. Consider the decay configurations (i) and (ii) of Fig. 3. Here we have suppressed the arrows corresponding to the vectors representing linear momentum. We represent a spin-zero pion by a dot, and a spin-1/2 particle by $\uparrow$, and think of the picture as a diagram in momentum space; the position x, y, z of the particle on the diagram gives its momentum p_x, p_y, p_z. (We will use this convention several times more in this lecture.)

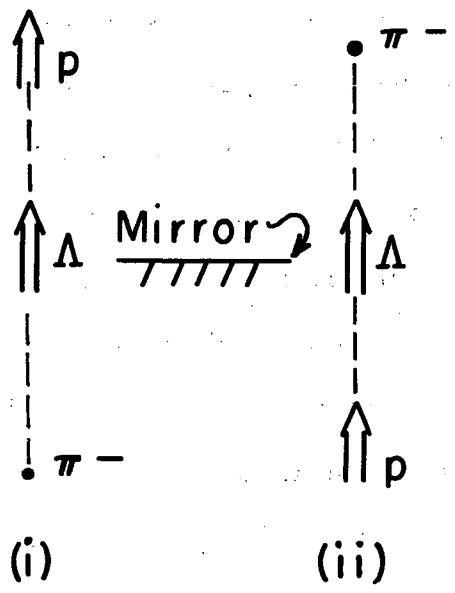

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Fig. 3

The decay (ii) is the reflection of the decay (i), for a mirror oriented as indicated. (Of course for any orientation of the mirror, (ii) is obtained by reflection of (i), followed by some rotation of the entire process. We have chosen the orientation of the mirror so as to preserve the Λ spin direction, and thus avoid an additional irrelevant rotation.) If P were conserved in the decay, then process (i) and its reflection (ii) would occur with equal probability. Thus there would be no decay asymmetry for a polarized source of Λ's—as many protons would be emitted parallel and antiparallel to the Λ polarization. The large "up-down" decay asymmetries (with respect to the production plane) that are observed experimentally show that P is not conserved in $\Lambda \rightarrow p + \pi^-$, and also in most of the other hyperon decays. The large asymmetries often observed correspond to nearly maximum parity nonconservation in the decay, and to Λ's strongly polarized in the production process. The decay asymmetry determines $\alpha_\Lambda p_\Lambda$, where α_Λ is the decay parameter, and p_Λ is the Λ polarization. That is, p_Λ = (number of + spins) minus (number of - spins) divided by the total number of Λ's. These quantities will be discussed in more detail in the second lecture.

Next we consider the consequences of T (time-reversal) invariance for hyperon decay. We will use the same type of pictures as before: diagrams in three-dimensional momentum space, with double-shafted arrows to represent spins. The application of T to a physical state leads to a new state related to the original state through reversal of all linear momenta and spins. Furthermore an outgoing wave becomes an incoming wave. (Think of a playback of a movie film in reverse.) An incoming wave does not correspond to an observable "final" state of free particles—the incoming particles must interact before one obtains an outgoing wave that can correspond to final free particles. Furthermore, consider a process in which an initial state i, say

a Λ, evolves into a final state f, say $p + \pi^-$. Then in the time-reversed picture the sense of evolution is reversed, and $p + \pi^-$ evolves into Λ. This process is of course unobservable by presently conceivable technique. However, in quantum mechanics, interchange of i and f in $\langle \psi_f | H | \psi_i \rangle \equiv m$ merely corresponds to complex conjugation, and thus does not affect $|m|^2$. Our pictures of course correspond to $|m|^2$. We therefore draw pictures in which the initial and final states are both present, with labels i and f, and include a step called "complex conjugation" (c.c.) which does not change the picture but interchanges i and f.

Consider an initial state that consists of a Λ at rest (and therefore at the origin in p_x, p_y, p_z space) with spin along +z. It evolves into a final state that is an outgoing proton with momentum along +x and spin along +y. This is picture (i), Fig. 4. (We have not chosen this configuration by accident, of course.) Now apply time reversal, T, to (i), to get (ii). Under T the Λ spin reverses, the decay proton spin and linear momentum reverse, and the outgoing proton wave becomes an incoming wave. The sense of evolution is reversed so Λ is final, f, instead of initial, i. Next apply c.c., to interchange i and f. Also perform a rotation R, of the entire process by 180° about the y axis, so that the Λ spin is again along +z. R and c.c. give (iii), and are assumed to have no observable consequences. Finally, let the incoming p-π^- wave scatter and become an outgoing wave, corresponding to an observable final state. Here, if we were using the formalism, we would obtain an s-matrix element factor. Instead we will merely give two extreme illustrations. One extreme is a "weak scattering" in the final (f) state, so weak in fact that "nothing happens," and the incoming wave becomes an outgoing wave with the same linear momenta and spins. This is picture (iv). In the other extreme example there is a strong spin-flip

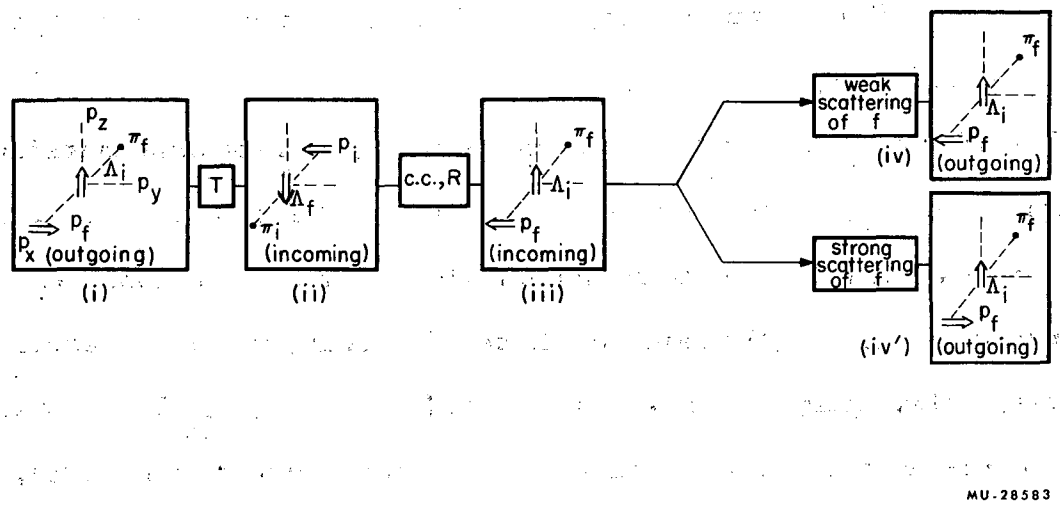
p_z
π_f
Λ_i
p_y
p_f
p_x (outgoing)
(i)
T
p_i
Λ_f
π_i
(incoming)
(ii)
c.c.,R
π_f
Λ_i
p_f
(incoming)
(iii)
weak scattering of f
(iv)
π_f
Λ_i
p_f
(outgoing)
strong scattering of f
(iv')
π_f
Λ_i
p_f
(outgoing)
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Fig. 4

scattering and the proton spin is reversed without deflection of the linear momentum, to give picture (iv').

The following statements are equivalent:

(a) "Time-reversal invariance holds in Λ-decay."

(b) "The decay corresponding to (iv) (for weak final-state scattering) or (iv') (for strong scattering) occurs with the same probability as that corresponding to (i)."

From the pictures we see that if the π^--p scattering is weak and if T invariance holds, then the Λ polarization corresponding to (i) is exactly canceled by the equally probable decay (iv), so that there is zero net polarization of the type (i). On the other hand, if the π-p scattering is strong, as in (iv'), a net polarization can be obtained. However, if the π-p scattering phase shifts are known (at the decay momentum) the effect of the scattering can be exactly taken into account, and one can still test T invariance. We need not write down the formulas, which are well known.[1]

The decay parameter corresponding to the Λ polarization shown in (i) is called β, with $-1 \leq \beta \leq 1$. We have $\beta = 0$ if T invariance holds and the f scattering is weak. This parameter will be discussed in the second lecture. It is clear from the discussion of Fig. 4 that one needs polarized Λ's in order to measure β_Λ.

There are two measurements of β for hyperon decays so far. Cronin and Overseth[2] find for $\Lambda \rightarrow p + \pi^-$ a value $\beta_\Lambda = 0.19 \pm 0.19$. This value is consistent with T invariance and the known π-p phase shifts. Another result is that of the U. C.-Berkeley-U. C. L. A. experiment. The experimenters find[2] for $\Xi^- \rightarrow \Lambda + \pi^-$ the preliminary result $\beta_{\Xi^-} = -0.68 \pm 0.27$. The experimental uncertainty is of course large, but the large value of β, if substantiated, probably indicates a strong Λ-π interaction. This should

not be surprising, since the Ξ mass is near that of the Y_1^* resonance.[2]

Lastly we consider C invariance. Again we use the example of Λ decay. Charge conjugation C applied to the process $\Lambda \rightarrow p + \pi^-$ gives the process $\bar{\Lambda} \rightarrow \bar{p} + \pi^+$. If C invariance holds then these two decays should occur with equal amplitudes for the same configuration of momentum and spins.

Insufficient experimental information is available for $\bar{\Lambda}$ decay. However, CPT invariance allows us to substitute PT for C. We can then consider the effect of PT invariance on Λ decay, since PT does not change Λ into $\bar{\Lambda}$. Our pictures will be similar to those used previously. We will prove that PT-invariance would, in the absence of final-state interactions, give zero for the "up-down" decay parameter α_Λ. We start with configuration (i), of Fig. 5, which implies a source of polarized Λ's. Application of T gives (ii), with reversed linear momenta and spins, with incoming p-π^-, and with i (initial) and f(final) reversed. Complex conjugation (c. c.) and reflection P in a <u>vertical</u> mirror (chosen to eliminate the need for a further rotation to orient the Λ spin) give (iii). The incoming f(final) wave scatters and becomes an outgoing f state. A weak f scattering ("nothing happens") is shown in (iv). A strong scattering, in which the π^- and p reverse their linear momenta (180° scattering) is shown in (iv'). The following statements are equivalent:

(a) "PT invariance is satisfied in $\Lambda \rightarrow p + \pi^-$."

(b) "Decay configuration (iv), for weak final state scattering [or (iv') for a particular strong scattering] has the same probability as configuration (i)."

We see that any up-down decay asymmetry implied by (i) is completely canceled by (iv), for weak scattering. Thus PT invariance (i.e., C

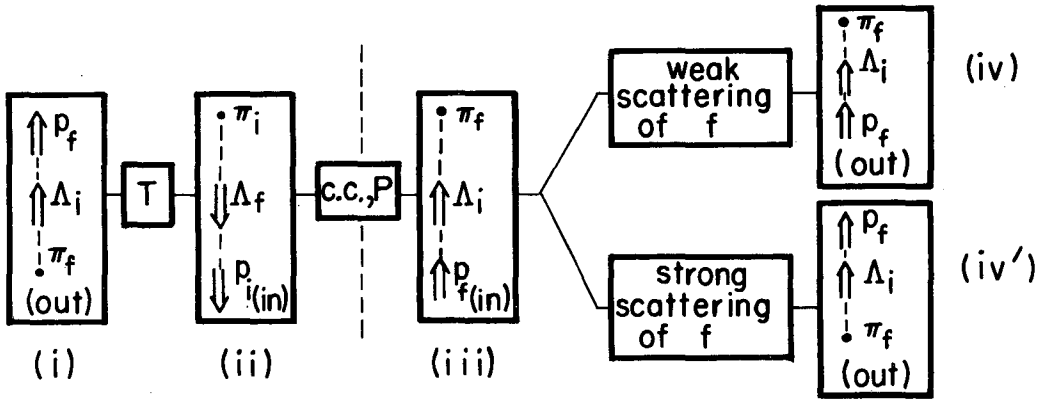

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Fig. 5

invariance) guarantees $\alpha_\Lambda = 0$, in the absence of π-p final-state interactions. At the momentum of Λ decay (100 MeV/c) the π-p scattering phase shifts are very small,[1] so that "weak scattering" holds. Experimentally the decay parameter α_Λ is large. We conclude that $\Lambda \rightarrow p + \pi^-$ does <u>not</u> satisfy PT invariance. This was first pointed out by R. Gatto.[3]

Lecture II. DECAY PARAMETERS

In this lecture we consider in detail the hyperon decay parameters α, β, and γ, and how they are measured. In every case we have a parent particle of spin 1/2 decaying into a daughter of spin 1/2 plus a pion (spin zero). The decays of interest are $\Lambda \to N + \pi$, $\Sigma \to N + \pi$, and $\Xi \to \Lambda + \pi$. Instead of speaking of "parent" and "daughter," we will for convenience take $\Lambda \to p + \pi^-$ as a model, most of the time.

Since the Λ has $J = 1/2$, the p-π^- system can only be in the state $S_{1/2}$ or $P_{1/2}$. Call S and P the corresponding amplitudes. Let ψ_+ describe the $\pi^- + p$ spin and space configuration for $(J, J_z) = (1/2, +1/2)$, and ψ_- that for $(J, J_z) = (1/2, -1/2)$. We can use the Clebsch-Gordan coefficients of Table I to construct ψ_+. We will represent the proton's spin state by

$\Uparrow \equiv \begin{pmatrix}1\\0\end{pmatrix} \equiv (1/2, +1/2)$ and

$\Downarrow \equiv \begin{pmatrix}0\\1\end{pmatrix} \equiv (1/2, -1/2)$. The orbital angular momentum state of $\pi^- + p$ is given by $Y_\ell^m(\theta, \phi)$, where θ and ϕ are the polar and azimuthal angles of emission of the proton with respect to the z axis (see Fig. 6).

The appropriate spherical harmonics are Y_0^0 for the $S_{1/2}$ state of π-p, and Y_1^1, Y_1^0, and Y_1^{-1} for $P_{1/2}$. We use

$$Y_0^0 = 1 , \tag{1a}$$

$$Y_1^{+1} = -\sqrt{3/2}\, \sin\theta\, e^{i\phi} , \tag{1b}$$

$$Y_1^0 = \sqrt{3}\, \cos\theta , \tag{1c}$$

$$Y_1^{-1} = \sqrt{3/2}\, \sin\theta\, e^{-i\phi} . \tag{1d}$$

That part of ψ_+ that corresponds to $S_{1/2}$ can be written down without using the table. It is just $S Y_0^0 \Uparrow = S \cdot 1 \cdot \Uparrow$. To obtain the $P_{1/2}$ part, we use Table I, which gives the composition of 1 (P wave) $\times$ 1/2 (spin).

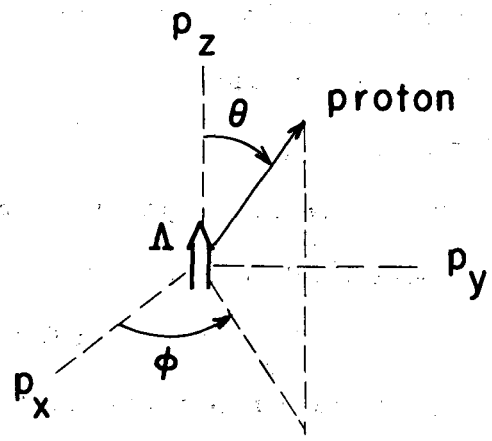

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Fig. 6

Looking in the column (1/2, +1/2) [because we want ψ_+] we find the decomposition

$$(1/2, +1/2) = \sqrt{2/3}\,(1,+1)(1/2,-1/2) - \sqrt{1/3}\,(1,0)(1/2,+1/2)\,.$$

Putting the $S_{1/2}$ and $P_{1/2}$ parts together, we have

$$\psi_+ = S\,Y_0^0 \Uparrow + P\,[\sqrt{2/3}\;Y_1^1 \Downarrow - \sqrt{1/3}\;Y_1^0 \Uparrow]\,.$$

Similarly, using the table, we get

$$\psi_- = S\,Y_0^0 \Downarrow + P\,[\sqrt{1/3}\;Y_1^0 \Downarrow - \sqrt{2/3}\;Y_1^{-1} \Uparrow]\,.$$

Using the spherical harmonics of Eq. (1), we have

$$\psi_+ = S \Uparrow + P\,[-\sin\theta\, e^{i\phi} \Downarrow - \cos\theta \Uparrow]$$

$$= (S - P\cos\theta) \Uparrow - P\sin\theta\, e^{i\phi} \Downarrow\,, \tag{2a}$$

$$\psi_- = S \Downarrow + P\,[\cos\theta \Downarrow - \sin\theta\, e^{-i\phi} \Uparrow]$$

$$= (S + P\cos\theta) \Downarrow - P\sin\theta\, e^{-i\phi} \Uparrow\,. \tag{2b}$$

The decay angular distribution for ψ_+ is given by $|\psi_+|^2 = \psi_+^* \psi_+$. We use the orthogonality of the spin functions, namely $\Uparrow^* \Uparrow \equiv (1,0)\begin{pmatrix}1\\0\end{pmatrix} = 1$, $\Downarrow^* \Downarrow \equiv (0,1)\begin{pmatrix}0\\1\end{pmatrix} = 1$, $\Uparrow^* \Downarrow = \Downarrow^* \Uparrow = 0$; so that

$$|\psi_+|^2 = |S - P\cos\theta|^2 + |-P\sin\theta|^2$$

$$= |S|^2 + |P|^2 - 2\,\mathrm{Re}\,S^* P\cos\theta\,.$$

$$|\psi_-|^2 = |S + P\cos\theta|^2 + |-P\sin\theta|^2$$

$$= |S|^2 + |P|^2 + 2\,\mathrm{Re}\,S^* P\cos\theta\,.$$

It is customary to define

$$\alpha \equiv \frac{2\,\mathrm{Re}\;S^* P}{|S|^2 + |P|^2}\,, \tag{3}$$

$$\beta \equiv \frac{2\,\mathrm{Im}\,S^* P}{|S|^2 + |P|^2}\,, \tag{4}$$

$$\gamma \equiv \frac{|S|^2 - |P|^2}{|S|^2 + |P|^2} . \tag{5}$$

(note that $\alpha^2 + \beta^2 + \gamma^2 = 1$.)

Then, from the above,

$$|\psi_+|^2 = [\,|S|^2 + |P|^2][1 - \alpha \cos\theta] , \tag{6}$$

$$|\psi_-|^2 = [\,|S|^2 + |P|^2][1 + \alpha \cos\theta] . \tag{7}$$

Now suppose a collection of Λ's is partially polarized, with a fraction f_+ in the state ψ_+, and a fraction f_- in the state ψ_-, with $f_+ + f_- = 1$. Then the weighted decay angular distribution is given by

$$|\psi|^2 = f_+ |\psi_+|^2 + f_- |\psi_-|^2$$

$$= [\,|S|^2 + |P|^2]\,\{(f_+ + f_-) - \alpha (f_+ - f_-) \cos\theta\} .$$

The polarization p of the collection of Λ's is defined to be

$$p_\Lambda = (f_+ - f_-)/(f_+ + f_-) , \tag{8}$$

with $-1 \leqslant p_\Lambda \leqslant +1$, so that

$$|\psi|^2 = [\,|S|^2 + |P|^2]\,\{1 - \alpha\, p \cos\theta\} . \tag{9}$$

The decay distribution for N decays is thus given by

$$dN = N\,[1 - \alpha p \cos\theta]\,\frac{d(\cos\theta)}{2} . \tag{10}$$

Notice that $\int_{-1}^{1} dN = N$, and

$$\int_{-1}^{1} \cos\theta \cdot dN = \frac{-N\alpha p}{3} , \text{ so that}$$

$$-\alpha p = \frac{3}{N} \int_{-1}^{1} \cos\theta\, dN \Rightarrow \frac{3\Sigma_i \cos\theta_i}{N} , \tag{11}$$

where the sum is over all the decays, and where the arrow means "corresponds, for large numbers, to". Equation (11) is often used by experimenters. An equivalent formula is $-\alpha p = 2(\text{up-down})/(\text{up+down})$.

We see that measurement of "up-down asymmetry" does not give α, but gives αp. Since the sign and magnitude of p are generally unknown, a measurement of αp gives a lower limit to $|\alpha|$. That is, $|\alpha| = |\alpha p|/|p| \geq |\alpha p|$.

In order to measure α directly one can measure the longitudinal polarization of the decay protons from an <u>unpolarized</u> collection of Λ's. This is easily seen as follows. First, consider only proton emission along the $\pm z$ axis. From Eqs. (6) and (7), with $\cos\theta = \pm 1$ we obtain the relative probabilities shown in Fig. 7. Notice that because of angular momentum conservation the proton spin direction must be the same as that of the Λ, for emission along the z axis (quantization axis), because the π-p orbital angular momentum can have no component along the proton's linear momentum, and therefore cannot flip the baryon spin. The definition of the longitudinal polarization of the proton, along its velocity $\hat{v}$ with respect to the Λ rest frame, is given by an expression analogous to Eq. (8). Using Fig. 7, we get, for the longitudinal polarization,

$$p(\text{long.}) = \frac{N_+ - N_-}{N_+ + N_-} = \frac{(1-\alpha)+(1-\alpha) - (1+\alpha) - (1+\alpha)}{(1-\alpha)+(1-\alpha)+(1+\alpha)+(1+\alpha)} = -\alpha_\Lambda,$$

where $N_\pm$ refer to $\pm\hat{v}$, and where we have used equal weights for ψ_+ and ψ_-. Since the Λ collection is unpolarized, all quantization directions are equivalent, so that we can always choose the z axis to be along the direction of emission of the proton and be assured that ψ_+ and ψ_- have equal populations. The above result therefore is general.

One still has the problem of <u>measuring</u> this longitudinal polarization of the daughter. In the case of $\Xi \to \Lambda + \pi$ one can measure the decay asymmetry

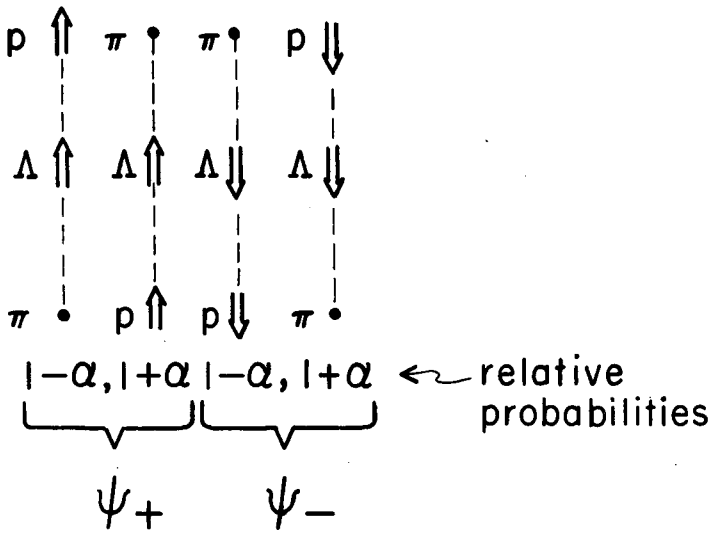

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Fig. 7

of the daughter Λ with respect to the direction of $\vec{v}_\Lambda - \vec{v}_\Xi$, and thus determine $\alpha_\Lambda p_\Lambda$ (longitudinal), using Eq. (11). But $p_\Lambda(\text{long.}) = -\alpha_\Xi$. Thus one measures $\alpha_\Lambda \alpha_\Xi$.[2]

In the case of $\Lambda \rightarrow p + \pi^-$ one can scatter the decay proton for instance from carbon to look for a scattering asymmetry, using a spark chamber.[2] Notice that if the Λ (unpolarized collection) decays at rest in the laboratory system, then the proton has a purely longitudinal polarization in the laboratory system (where the carbon scatterer is at rest). When this proton scatters from carbon (spin zero) there cannot be any "left-right" scattering asymmetry, merely from the symmetry of the initial p-carbon configuration. There also cannot be any front-back (0° versus 180°) scattering asymmetry that depends on the proton's longitudinal polarization. This follows from parity conservation in the strong p-carbon reaction. We can see this with our mirror. Suppose an incoming "spin-head-on" (as opposed to "tail-on") proton likes to scatter "strongly" (i.e., through 180°) from carbon. If the tail-on collision does <u>not</u> like to occur, we have a means of determining the polarization. However, the image of a head-on collision in a mirror held parallel to the proton velocity is a tail-on collision. By P conservation the two processes have the same probability. Thus head-on and tail-on protons <u>both</u> scatter strongly (or weakly), and we cannot distinguish the two polarizations (since parity <u>is</u> conserved in the strong reactions).

One gets around this by using fast Λ's that decay in flight. Then the decay protons, which have a polarization along $\vec{v}$ proton $- \vec{v}_\Lambda$, can have a component $\perp$ to $\vec{v}$ proton $- \vec{v}$ carbon. It is then possible to get azimuthal asymmetry in the scattering. This is illustrated in Fig. 8, which is our usual diagram in velocity space. We choose the carbon at rest. The Λ is shown without an arrow, since it is unpolarized. If $\vec{v}$ proton $- \vec{v}_\Lambda$ is along

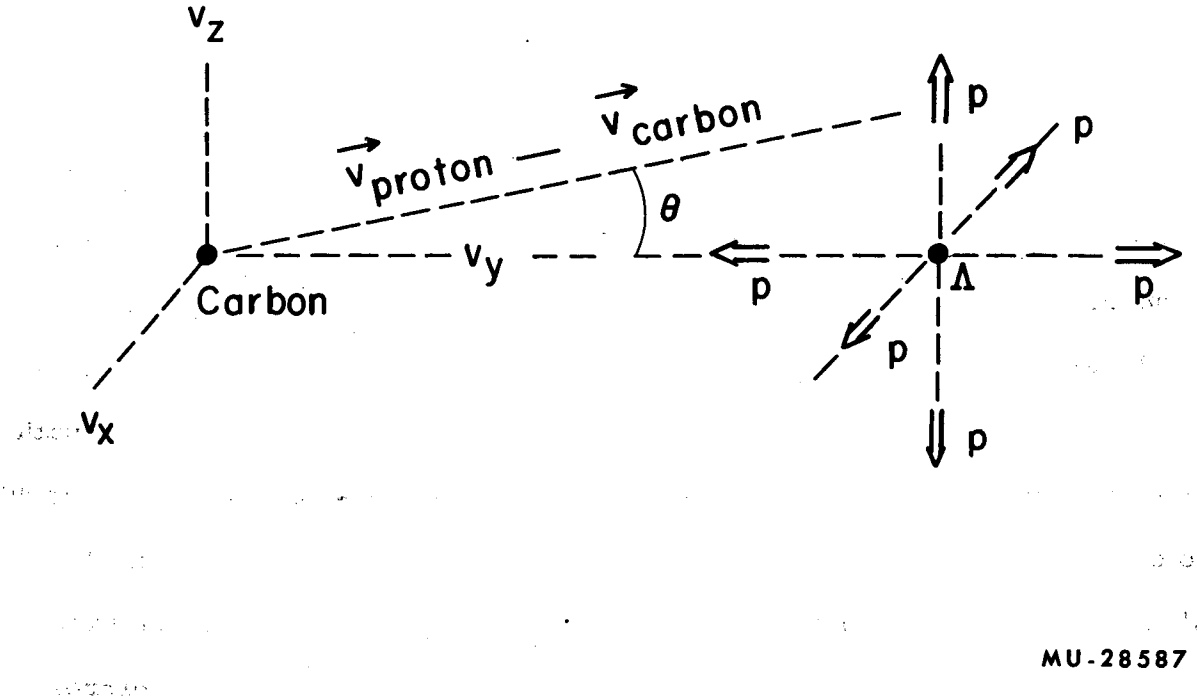
v_z
$\vec{v}_{proton}$
$\vec{v}_{carbon}$
θ
v_y
Carbon
v_x
p
p
p
p
p
p
Λ
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Fig. 8

$\pm\hat{x}$ or $\pm\hat{z}$ we see that the proton has a transverse polarization of approximately $-a_\Lambda \cos\theta$, where θ is as shown. If $\vec{v}_{\text{proton}}-\vec{v}_\Lambda$ is along $\vec{v}_\Lambda - \vec{v}_c$, i.e., along $\pm\hat{y}$, there is no transverse polarization. Thus about 4/6 of the decays are useful.

We turn now to the problem of measuring the decay parameter β. It was already mentioned in the first lecture that β is a measure of T invariance, and also it was shown that the proton polarization shown in Fig. 4 (i) (for a polarized Λ) must average to zero if T invariance holds and the scattering is weak (as it is in Λ decay). We will calculate the slightly more general proton polarization component shown in Fig. 9. We choose the Λ state ψ_+, i.e., 100% polarized Λ's along +z. (Our final answer can then be multiplied by p_Λ if $p_\Lambda \neq +1$). We choose the decay configuration with $\phi = 0$, as shown in Fig. 9. This simplifies the formulas and corresponds to an unessential rotation of the axes. We wish to calculate $\langle\sigma_y\rangle$. We have, for the state ψ_+,

$$\langle\sigma_y\rangle = \frac{\psi_+^* \sigma_y \psi_+}{\psi_+^* \psi_+} . \tag{12}$$

The denominator is given by Eq. (6). To calculate the numerator we use

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Uparrow \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Downarrow \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; ; \; \sigma_y \Uparrow = i \Downarrow, \quad \sigma_y \Downarrow = -i \Uparrow \; ;$$

$$\psi_+ = (S - P\cos\theta) \Uparrow - P\sin\theta \Downarrow, \quad \sigma_y \psi_+ = i\,[(S - P\cos\theta)\Downarrow + P\sin\theta \Uparrow];$$

$$\Uparrow^* \Uparrow = \Downarrow^* \Downarrow = 1, \quad \Uparrow^* \Downarrow = \Downarrow^* \Uparrow = 0; \; \psi_+^* \sigma_y \psi_+$$

$$= [(S - P\cos\theta)^* \Uparrow^* - P^* \sin\theta \Downarrow^*]\, i\,[(S - P\cos\theta)\Downarrow + P\sin\theta \Uparrow]$$

$$= i\,\{(S - P\cos\theta)^* P\sin\theta - P^* \sin\theta\,(S - P\cos\theta)\}$$

$$= i\,\{2i\,\mathrm{Im}\, S^* P \sin\theta\} \equiv -(|S|^2 + |P|^2)\,\beta\sin\theta .$$

Finally, then, for $\phi = 0$, and $p_\Lambda = 1$,

$$\langle\sigma_y\rangle = \frac{-\beta\sin\theta}{1 - a\cos\theta} . \tag{13}$$

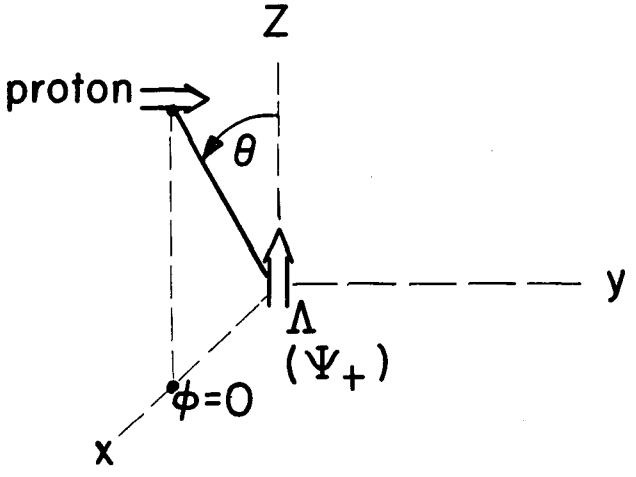

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Fig. 9

Clearly, for $p_\Lambda \neq 1$ we have, for $\phi = 0$,

$$\langle \sigma_y \rangle = \frac{-\beta p_\Lambda \sin\theta}{1 - \alpha p_\Lambda \cos\theta} . \tag{14}$$

It is clear that our choice of $\phi = 0$ was unessential, and Eq. (14) gives, more generally, the azimuthal or $\hat{\phi}$ component of polarization.

The case shown in Fig. 4 (i) has $\phi = 0$, $\theta = 90^\circ$, $p_\Lambda = 1$, so that $\langle \sigma_y \rangle = -\beta$. Since we had previously concluded that this polarization must vanish if T invariance holds, we see that β is a measure of lack of T invariance (for weak final-state interaction). If T-invariance holds, S and P are "relatively real," i.e., S/P is real.

The problem of measurement of $\langle \sigma_y \rangle$ (of the proton in Fig. 9) is illustrated in Fig. 10. We see that as far as transverse proton polarization is concerned, we could use Λ's at rest in the laboratory system and have four out of six "useful directions." However, we need polarized Λ's, and polarized Λ's are not produced at rest; furthermore, the proton would then have only a few MeV, and would not penetrate a scatterer of reasonable thickness. For fast Λ's we see that only 2/6 of the decays are useful—those with $\vec{v}_p - \vec{v}_\Lambda$ along $\pm\hat{y}$ in Fig. 10. Of course, in the decay $\Xi^- \to \Lambda + \pi^-$, "all four" directions of Λ emission in the production plane are useful.

One may ask: How can one in a single experiment measure α, using an unpolarized sample of parent hyperons, and β, using polarized parents? The answer is that, since the parent polarization must be perpendicular to the production plane, one obtains "effectively" unpolarized parents if one throws away information as to the orientation of the production plane. Crucial to this argument is the fact that, for a spin-1/2 parent, the decay distribution, Eq. (9), is <u>linear</u> in $p_\Lambda \cos\theta$, and so the term containing p_Λ averages to zero when we average over the distribution.

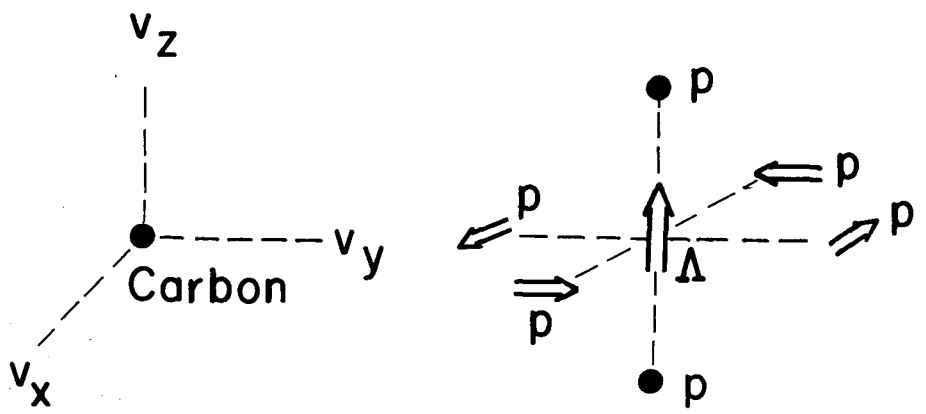

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Fig. 10

We turn now to the measurement of γ. From Eq. (3) we see that α is unchanged by the interchange of S and P. Thus if we know α we know the relative amounts of S and P, but don't know "which is which." That is, $|S|/|P| = 10/1$ and $|S|/|P| = 1/10$ are indistinguishable. From Eq. (5) we see that it is the <u>sign</u> of γ that tells us the correct ratio. (The magnitude of γ is already known once α and β are known, since $\gamma^2 = 1 - \alpha^2 - \beta^2$.)

To determine which is which (S or P) we consider first the limiting case of pure S-wave decay for a 100% polarized Λ, ψ_+. For pure S-wave there is no orbital angular momentum to flip the spin, and the proton polarization is the same as that of the Λ for all directions of emission. This also follows from Eq. (1) if we set $P = 0$ to get $\psi_+ = S \Uparrow$.

We next calculate $\langle\sigma_x\rangle$ and $\langle\sigma_z\rangle$ for the general case. We still set $\phi = 0$ for convenience. (Since we have already calculated $\langle\sigma_y\rangle$ in Eq. (13), we are at present interested only in $\langle\sigma_x\rangle$ and $\langle\sigma_z\rangle$.) We use $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_x \Uparrow = \Downarrow$; $\sigma_x \Downarrow = \Uparrow$; $\sigma_x \psi_+ = \sigma_x [(S - P\cos\theta) \Uparrow - P\sin\theta \Downarrow]$

$$= [(S - P\cos\theta) \Downarrow - P\sin\theta \Uparrow] .$$

$$\psi_+^* \sigma_x \psi_+ = [(S - P\cos\theta)^* \Uparrow^* - P^* \sin\theta \Downarrow^*][(S - P\cos\theta) \Downarrow - P\sin\theta \Uparrow]$$

$$= -(S - P\cos\theta)^* P\sin\theta - P^* \sin\theta (S - P\cos\theta)$$

$$= 2|P|^2 \sin\theta\cos\theta - 2\,\mathrm{Re}\, S^* P \sin\theta$$

$$= |P|^2 \sin 2\theta - (|S|^2 + |P|^2)\,\alpha \sin\theta. \tag{15}$$

Similarly, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_z \Uparrow = \Uparrow$, $\sigma_z \Downarrow = -\Downarrow$;

$$\sigma_z \psi_+ = \sigma_z [(S - P\cos\theta) \Uparrow - P\sin\theta \Downarrow]$$

$$= [(S - P\cos\theta) \Uparrow + P\sin\theta \Downarrow]$$

$$\psi_+^* \sigma \psi_+ = [(S - P\cos\theta)^* \Uparrow^* - P^* \sin\theta \Downarrow^*][(S - P\cos\theta) \Uparrow + P\sin\theta \Downarrow]$$

$$= |S - P\cos\theta|^2 - |P|^2 \sin^2\theta$$

$$= |S|^2 + |P|^2 (\cos^2\theta - \sin^2\theta) - 2\,\mathrm{Re}\, S^* P \cos\theta$$

$$= |S|^2 + |P|^2 \cos 2\theta - (|S|^2 + |P|^2)\,\alpha \cos\theta. \tag{16}$$

We can combine (16) and (15) into a vector $\vec{\sigma}$ in the xz plane. (We are not concerned with σ_y at the moment; we are not assuming $\langle \sigma_y \rangle = 0$.) We find

$$\begin{aligned} \psi_+^* \vec{\sigma} \ \psi_+ &= \psi_+^* \ [\sigma_z \ \hat{z} + \sigma_x \ \hat{x}] \psi_+ \\ &= |S|^2 \ \hat{z} + |P|^2 \ [(\cos 2\theta) \ \hat{z} + (\sin 2\theta) \ \hat{x}] \\ &\quad - (|S|^2 + |P|^2) \ \alpha [\cos\theta \ \hat{z} + \sin\theta \ \hat{x}] \ . \end{aligned} \tag{17}$$

But $\cos\theta \ \hat{z} + \sin\theta \ \hat{x} \equiv \hat{q}(\theta)$, where $\hat{q}$ is the unit vector along the proton momentum (in the Λ rest frame). And $(\cos 2\theta)\hat{z} + (\sin 2\theta)\hat{x} \equiv \hat{n}(2\theta)$, where $\hat{n}(2\theta)$ is a unit vector in the $\hat{z}\hat{q}$ plane, making an angle 2θ with $\hat{z}$.

Finally we obtain, from these definitions and Eqs. (16) and (6),

$$\langle \vec{\sigma} \rangle = \frac{|S|^2 \ \hat{z} + |P|^2 \ \hat{n}(2\theta) - (|S|^2 + |P|^2) \ \alpha \ \hat{q}(\theta)}{(|S|^2 + |P|^2) \ [1 - \alpha \cos\theta]} \ . \tag{18}$$

In addition there is a y component given by Eq. (13) or by (14). If we do not have $p = +1$ (pure ψ_+ state) we obtain, by a weighted average over ψ_+ and ψ_-, the final general result

$$\langle \vec{\sigma} \rangle = \frac{p[|S|^2 \ \hat{z} + |P|^2 \hat{n}(2\theta) - 2 \, \mathrm{Im} \, S^* P \sin\theta \ \hat{d}] - 2 \, \mathrm{Re} \, S^* P \hat{q}(\theta)}{(|S|^2 + |P|^2) \ [1 - \alpha p \cos\theta]} \ . \tag{19}$$

For a pure P wave, $|S| = 0$, $\alpha = 0$, $\beta = 0$ and we obtain, for the proton polarization

$$\langle \vec{\sigma} \rangle_{P\text{-wave}} = p \ \hat{n}(2\theta) \ . \tag{20}$$

Then the proton spin lies in the plane of the emission of the proton (and the Λ polarization z axis). Proton emission at angle θ gives proton spin at 2θ.

For pure S wave we have

$$\langle \vec{\sigma} \rangle_{S\text{-wave}} = p \ \hat{z} \ . \tag{21}$$

The two extremes are illustrated in Fig. 11.

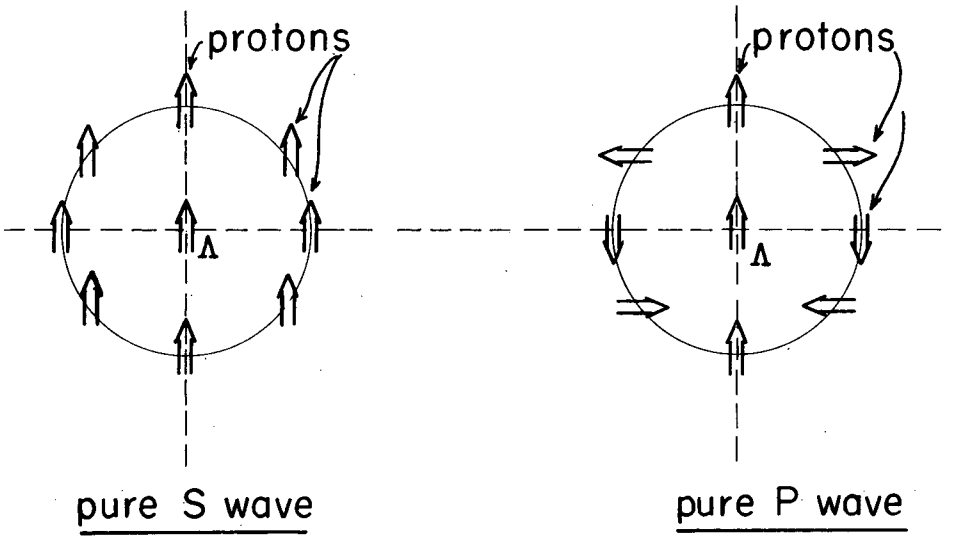

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Fig. 11

We can now put the proton polarization [Eqs. (19) and (14)] in a simpler form if we go to the unit vectors $\hat{q}$, $\hat{\theta}$, and $\hat{\phi}$ corresponding to spherical coordinates. See Fig. 12. Here $\hat{q}(\theta, \phi)$ is a unit vector in the direction of emission of the proton, relative to Λ rest frame, and $\hat{\theta}$ and $\hat{\phi}$ are unit vectors corresponding to increases in θ and ϕ. By inspection of Fig. 12, we see

$$\hat{z} = \hat{q} \cos \theta - \hat{\theta} \sin \theta,$$

$$\hat{n}(2\theta) = \hat{q} \cos \theta + \hat{\theta} \sin \theta.$$

Therefore for one part of the numerator of (19), we have

$$|S|^2 \hat{z} + |P|^2 \hat{n}(2\theta) = (|S|^2 + |P|^2)\, \hat{q} \cos \theta - (|S|^2 - |P|^2)\, \hat{\theta} \sin \theta$$
$$= (|S|^2 + |P|^2)[\hat{q} \cos \theta - \gamma\, \hat{\theta} \sin \theta] .$$

Accordingly from Eq. (19) we find the general expression for the daughter polarization $\langle \vec{\sigma} \rangle$ in terms of the daughter emission direction $\hat{q}$, the parent polarization $p\hat{z}$, and the decay parameters α, β, and γ:

$$\langle \vec{\sigma} \rangle = \frac{\hat{q}(p \cos \theta - \alpha) - p \sin \theta (\gamma\, \hat{\theta} + \beta\, \hat{\phi})}{(1 - \alpha p \cos \theta)} . \tag{22}$$

As checks, we see that for pure S wave we have $\alpha = \beta = 0$, $\gamma = +1$, and thus

$$\langle \vec{\sigma} \rangle = p[\hat{q} \cos \theta - \hat{\theta} \sin \theta] = p\hat{z} .$$

For pure P wave we have $\alpha = \beta = 0$, $\gamma = -1$, and find

$$\langle \vec{\sigma} \rangle = p[\hat{q} \cos \theta + \hat{\theta} \sin \theta] = p\hat{n}(2\theta).$$

The longitudinal polarization of the daughter along its direction of emission is given by

$$\langle \vec{\sigma} \rangle \cdot \hat{q} = \frac{p \cos \theta - \alpha}{1 - \alpha p \cos \theta} , \tag{23}$$

which reduces to $-\alpha$ if the Λ polarization p is zero.

The expression (22) has the advantage that the usually used parameters

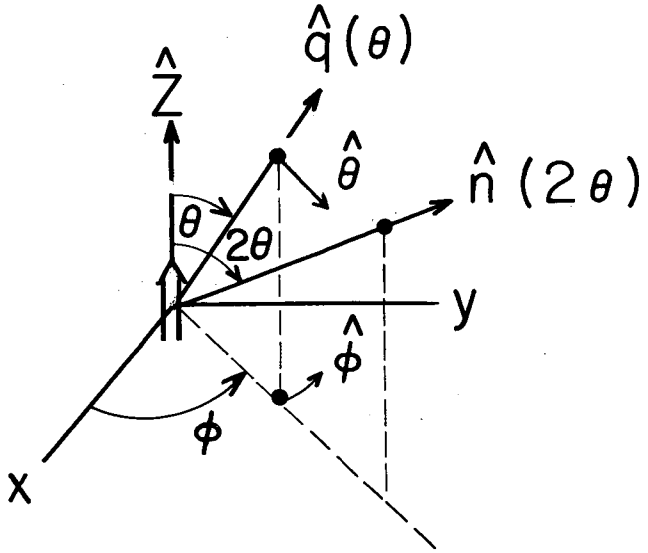

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Fig. 12

α, β, and γ appear explicitly, and the unit vectors are orthonormal. Expression (19) has perhaps the advantage that it is easier to see the separate effects of S and P wave.

From Eq. (22), by squaring and adding the three orthonormal components (and by using $\alpha^2 + \beta^2 + \gamma^2 = 1$), we find the square of the magnitude of the proton's polarization vector,

$$(\langle \vec{\sigma} \rangle)^2 = 1 - \frac{(1-p^2)(1-\alpha^2)}{(1-\alpha p \cos\theta)^2} \quad . \tag{24}$$

This means that if the Λ is 100% polarized ($p = \pm 1$), then for any given direction of emission of the proton, the proton polarization is 100%, in <u>some</u> direction [given by (22)]. On the other hand, if the Λ polarization is $p \neq 1$, the proton's polarization is not p but is given by Eq. (24).

Lecture III. $\Delta T = 1/2$ RULE FOR NONLEPTONIC DECAY

In this lecture we review the well-known[1] evidence that has led to the hypothesis of the $\Delta I = 1/2$ rule, for nonleptonic decays. We consider the decays $K \to 2\pi$, $\Lambda \to N + \pi$, $\Xi \to \Lambda + \pi$, and $\Sigma \to N + \pi$, and will make calculations illustrating spurion technique.[1]

1. The Decay $K \to 2\pi$

We consider the decays

$$K^+ \to \pi^+ + \pi^0 , \tag{25}$$

$$K^0 \to \pi^+ + \pi^- , \tag{26}$$

and

$$K^0 \to \pi^0 + \pi^0 , \tag{27}$$

Two pions can have total $I = 0$, 1, or 2. Let us use our Clebsch-Gordan Table V to construct the charge states for 2π with $I = 0$, 1, or 2. Reading from the table we get, using the notation $\psi(I, I_3)$,

$$\psi(0,0) = \sqrt{1/3}\,(1,+1)(1,-1) - \sqrt{1/3}\,(1,0)(1,0) + \sqrt{1/3}\,(1,-1)(1,1), \text{ or}$$

$$\psi(0,0) = \sqrt{1/3}\,\{\pi^+\pi^- - \pi^0\pi^0 + \pi^-\pi^+\} . \tag{28}$$

Similarly we read, by inspection of the table,

$$\psi(1,0) = \sqrt{1/2}\,\{\pi^+\pi^- - \pi^-\pi^+\} , \tag{29}$$

$$\psi(1,+1) = \sqrt{1/2}\,\{\pi^+\pi^0 - \pi^0\pi^+\} , \tag{30}$$

$$\psi(2,0) = \sqrt{1/6}\,\{\pi^+\pi^- + 2\pi^0\pi^0 + \pi^-\pi^+\} , \tag{31}$$

$$\psi(2,+1) = \sqrt{1/2}\,\{\pi^+\pi^0 + \pi^0\pi^+\} . \tag{32}$$

We do not need any other components in considering reactions (25), (26), and (27).

In this notation $\pi^+\pi^-$ means that pion #1 is π^+, #2 is π^-. The expressions $\pi^+\pi^-$ and $\pi^-\pi^+$ do not mean the same thing, since #1 and #2 may be distinguishable by position or by energy.

By inspection of Eqs. (28) through (32) we see that, upon interchanging the charge of #1 and #2, we have $\psi \rightarrow (-1)^I \psi$, for I = 0, 1, or 2.

Since the two pions are identical bosons, the total exchange of space (x), spin (σ), and charge (Q) must leave ψ unchanged. The pions have no spin, so we have

$$(x)(Q) = +1, \quad \text{i.e.,}$$
$$(-1)^{\ell} (-1)^I = +1 , \tag{33}$$

where ℓ is the relative angular momentum of the two pions. For K decay, the total spin J = 0, so $\ell = 0$, so $(-1)^I = +1$, so I = 0 or 2 only. I = 0 is excluded for K^+ by charge conservation, so K^+ can go to 2π only in the state $\psi(2, +1)$. However, K^0 can go either to $\psi(2, 0)$ or $\psi(0, 0)$.

In discussing K^0 we must distinguish between K_1^0 and K_2^0, which are even and odd, respectively under CP. For $\pi^0\pi^0$ or $\pi^+\pi^-$, CP has the same effect as interchanging the charge (when present) and space coordinates of the two particles. Therefore CP = +1 (identical bosons). Therefore K_1^0 can and K_2^0 cannot decay into $\pi^0\pi^0$ and $\pi^+\pi^-$.

Since K^+ and K^0 have I = 1/2, the change of $\vec{I}$ in the decay of $K^+ \rightarrow 2\pi$ is $|\Delta\vec{I}| \equiv \Delta I = 2 \pm 1/2 = 5/2$ or $3/2$. In $K_1^0 \rightarrow 2\pi$ we have $\Delta I = 2 \pm 1/2 = 5/2$ or $3/2$; or $0 + 1/2 = 1/2$. Thus $\Delta I = 1/2$ is available for $K_1^0 \rightarrow 2\pi$, but not for $K^+ \rightarrow 2\pi$. The rate $R(K^+ \rightarrow \pi^+ + \pi^0)$ is only about 1/600 of $R(K_1^0 \rightarrow 2\pi)$.[1] The most natural explanation is that there is a selection rule that nearly forbids decays with $\Delta I = 3/2$ or $5/2$ but allows those with $\Delta I = 1/2$. This proposed selection rule is called the $\Delta I = 1/2$ rule. If the $\Delta I = 1/2$ rule were strictly obeyed, $K^+ \rightarrow \pi^+ + \pi^0$ would be forbidden. F rthermore $K^0 \rightarrow 2\pi$ would go only to $\psi(0, 0)$. By Eq. (28) we see that in that case the branching ratio

$$B_1 \equiv \frac{R(K_1^0 \rightarrow \pi^0\pi^0)}{R(K_1^0 \rightarrow \pi^0\pi^0) + R(K_1^0 \rightarrow \pi^+ + \pi^-)} \equiv \frac{R(00)}{R(00) + R(+-)} \tag{34}$$

would have the value $B_1 = \frac{|-1|^2}{|1|^2 + |-1|^2 + |1|^2} = \frac{1}{3}$. This is close to what is in fact observed.[1]

Since $K^+ \to \pi^+ + \pi^0$ does after all exist, we next calculate the effect on the prediction of $B_1 = 1/3$, using spurion technique.[1]

The spurion $s(\Delta I, \Delta I_3)$ is introduced in order to keep track of the change of $\vec{I}$ in the decay. One can think of the spurion as carrying off $\Delta\vec{I}$, so that one now has $\vec{I}$ conservation in the decay. We illustrate by considering the K decay into 2π with $I = 2$. We assume that $\Delta I = 3/2$ occurs, but that there is no $\Delta I = 5/2$. We use Eqs. (31) and (32) to describe the 2π state. We have, using the notation (I, I_3),

$$K^+(1/2, +1/2) \to \psi(2, +1) + s(3/2, -1/2), \tag{35}$$

$$K^0(1/2, -1/2) \to \psi(2, 0) + s(3/2, -1/2). \tag{36}$$

Here we have chosen the $\Delta I = 3/2$ spurion, and have chosen $\Delta I_3 = -1/2$ for the spurion in order to "conserve" I_3. Notice also that because of the famous formula

$$\frac{Q}{e} = I_3 + \frac{S + B}{2}, \tag{37}$$

"conservation" of I_3 implies strangeness (S) conservation.

For convenience we now transpose the K and the spurion to opposite sides of the equation. To maintain conservation of I_3 we see, from Eq. (35), that we must reverse the sign of I_3 when we transpose. From (37) this means that S also must reverse its sign. Eqs. (35) and (36) can be combined as

$$K(1/2) \to \psi(2) + s(3/2, -1/2). \tag{38}$$

After transposing we get

$$s(3/2, +1/2) \to \psi(2) + \bar{K}(1/2), \tag{39}$$

which means

$$s(3/2, +1/2) \to \psi(2, +1) + K^-(1/2, -1/2), \tag{40}$$

$$\text{and } s(3/2, +1/2) \to \psi(2, 0) + \bar{K}^0(1/2, +1/2). \tag{41}$$

We now go to the Clebsch-Gordan Table III. According to Eq. (39) we want to compose $2\times 1/2$ so as to get $(3/2, +1/2)$. We therefore look under the column $(3/2, +1/2)$ to find

$$\begin{aligned}(3/2, +1/2) &= \sqrt{3/5}\,(2, +1)(1/2, -1/2) - \sqrt{2/5}\,(2, 0)(1/2, +1/2)\\ &= \sqrt{3/5}\,(2, +1)\,K^- - \sqrt{2/5}\,(2, 0)\,\bar{K}^0\\ &= \sqrt{3/5}\,K^-\{\sqrt{1/2}(\pi^+\pi^0 + \pi^0\pi^+)\} - \sqrt{2/5}\,\bar{K}^0\{\sqrt{1/6}(\pi^+\pi^- + 2\pi^0\pi^0 + \pi^-\pi^+)\}.\end{aligned} \tag{42}$$

Similarly we consider the $\Delta I = 1/2$ spurion. This cannot be obtained by composition of $2\times 1/2$, but only by $0\times 1/2$, so we have, analogous to (39),

$$s(1/2, +1/2) \rightarrow \bar{K}(1/2) + \psi(0), \tag{43}$$

$$\text{or}\quad (1/2, +1/2) = (1/2, +1/2)(0, 0)$$

$$= \bar{K}^0\sqrt{1/3}\,\{\pi^+\pi^- - \pi^0\pi^0 + \pi^-\pi^+\}. \tag{44}$$

Suppose now that both $\Delta I = 1/2$ and $3/2$ occur, with amplitudes a_1 and a_3, respectively. Then from (44) and (42), the total amplitude is

$$\psi = a_1(1/2, +1/2) + a_3(3/2, 1/2), \text{ or}$$

$$\begin{aligned}\psi = \bar{K}^0[(\pi^+\pi^- + \pi^-\pi^+)(\sqrt{1/3}\,a_1 - \sqrt{2/30}\,a_3) + \pi^0\pi^0(-\sqrt{1/3}\,a_1 - 2\sqrt{2/30}\,a_3)]\\ + K^-[(\pi^+\pi^0 + \pi^0\pi^+)\sqrt{3/10}\,a_3].\end{aligned} \tag{45}$$

We now remark that the overall relative phase between the $\bar{K}^0$ and K^- parts of ψ has no physical meaning. This is because charge conservation prevents "interference" between $\bar{K}^0$ and K^-. We can only compare intensities.

We next recall that it is K_1^0 decay we are interested in, not K^0 or $\bar{K}^0$. In Eq. (45), that part of ψ proportional to $\bar{K}^0(\pi^+\pi^- + \pi^-\pi^+)$, for example, represents (after transposing), the amplitude for $K^0 \rightarrow (\pi^+\pi^- + \pi^-\pi^+)$. Now,

$$K_1^0 = \frac{K^0 + \bar{K}^0}{\sqrt{2}}. \tag{46}$$

We have <u>chosen</u> the final state of 2π so that it corresponds to K_1^0 decay. Thus as far as decay into a state with CP = +1 (K_1^0 decay) is concerned, the K^0 and $\bar{K}^0$ behave in exactly the same way.

That is, they interfere constructively in just such a way as to give CP = +1. Therefore the amplitude for $\overline{K}^0 \to (\pi^+\pi^- + \pi^-\pi^+)$ is exactly the same as that for $K^0 \to (\pi^+\pi^- + \pi^-\pi^+)$. From Eq. (46), the amplitude for $K_1^0 \to (\pi^+\pi^- + \pi^-\pi^+)$ is just $(1+1)/\sqrt{2} = \sqrt{2}$ times that for $K^0 \to (\pi^+\pi^- + \pi^-\pi^+)$. Therefore when we write rates, we have

$$R(K_1^0 \to 2\pi) = (\sqrt{2})^2 \, R(K^0 \to 2\pi). \tag{47}$$

[Note: This argument is rephrased following Eq. (109).]

From Eq. (45) we now write the decay rates

$$R(K^+ \to \pi^+\pi^0) \equiv R(+0) = [(\sqrt{3/10})^2 + (\sqrt{3/10})^2]\,|a_3|^2 \text{ , i.e.,}$$

$$R(+0) = 3/5 \; |a_3|^2. \tag{48}$$

$$R(K_1^0 \to \pi^+\pi^-) \equiv R(+-) = 2\,(1^2 + 1^2)\,|\sqrt{1/3}\, a_1 - \sqrt{1/15}\, a_3|^2, \tag{49}$$

$$R(K_1^0 \to \pi^0\pi^0) \equiv R(00) = 2\,|-\sqrt{1/3}\, a_1 - 2\sqrt{1/15}\, a_3|^2. \tag{50}$$

Now choose units such that $a_1 \equiv 1$. Set $a_3 = |a_3| \exp i\,\delta$. Expand (49) and (50), neglecting quadratic terms in a_3, to get

$$R(+-) = 4/3 - 8/3 \; 1/\sqrt{5}\, |a_{31}| \cos\delta, \tag{51}$$

$$R(00) = 2/3 + 8/3 \; 1/\sqrt{5}\, |a_{31}| \cos\delta, \tag{52}$$

so that in these units

$$R_1 \equiv R(+-) + R(00) = 2. \tag{53}$$

$$R(+0) = 3/5 \; |a_3|^2 = 3/10 \; |a_3|^2 \, R_1 \, ,$$

so that

$$|a_3| = \sqrt{10/3} \; \sqrt{\frac{R(+0)}{R_1}} \; . \tag{54}$$

Putting in numbers,[1,2] $R_1 = 550\, R(+0)$, so

$$|a_3| = 0.078 \; |a_1| \, . \tag{55}$$

Now

$$B_1 \equiv \frac{R(00)}{R_1} = 1/3 + 4/3 \;\; 1/\sqrt{5} \cos\delta \; \frac{|a_3|}{|a_1|},$$

or finally,

$$B_1 = 0.333 + 0.047 \cos \delta .$$

For $-1 < \cos \delta < 1$ we get

$$0.29 \leq B_1 \leq 0.38 , \tag{56}$$

as the prediction of the $\Delta I = 1/2$ rule.

If there were no π-π interaction at 200 MeV/c in the S state, a_3 and a_1 would be relatively real, by T-invariance. Then $\cos \delta = \pm 1$.

Recent values of B_1 are[2]

0.260± 0.024, Anderson et al.,
0.294± 0.021, Chrétien, et al.,
0.329± 0.013, Brown, et al.

All are consistent with Eq. (56), although not completely with one another.

2. The Decay $\Lambda \rightarrow N + \pi$

Since I = 0 for the Λ, we can have $\Delta I = 1/2$ or $3/2$. We write the spurion reactions

$$s(1/2, -1/2) \rightarrow \overline{\Lambda}(0, 0)\, N(1/2)\, \pi(1). \tag{56}$$

The $\overline{\Lambda}(0, 0)$ contributes only a factor of unity, so, from Table I we find

$$(1/2, -1/2) = \overline{\Lambda}[\sqrt{1/3}\; n\pi^0 - \sqrt{2/3}\; p\pi^-] . \tag{57}$$

Similarly for $\Delta I = 3/2$ we have, by inspection Table I,

$$(3/2, -1/2) = \overline{\Lambda}[\sqrt{2/3}\; n\pi^0 + \sqrt{1/3}\; p\pi^-] . \tag{58}$$

The total amplitude is

$$\psi \equiv a_1\, (1/2, -1/2) + a_3\, (3/2, -1/2), \quad \text{or}$$

$$\psi = (\sqrt{1/3}\; a_1 + \sqrt{2/3}\, a_3)\, \overline{\Lambda} n \pi^0 + (-\sqrt{2/3}\; a_1 + \sqrt{1/3}\, a_3)\, \overline{\Lambda} p \pi^- . \tag{59}$$

The decay rates are therefore

$$R(n\pi^0) = 1/3\; |a_1 + \sqrt{2}\; a_3|^2 , \tag{60}$$

$$R(p\pi^-) = 1/3\; |\sqrt{2}\; a_1 - a_3|^2 . \tag{61}$$

We define the branching ratio

$$B_\Lambda \equiv \frac{R(p\pi^-)}{R_\Lambda \equiv R(p\pi^-) + R(n\pi^0)} , \tag{62}$$

and notice that if the $\Delta I = 1/2$ rule holds, i.e., if $a_3 = 0$, then $B_\Lambda = 2/3$.

If we expand (60) and (61), set $a_1 \equiv 1$, call $a_3 \equiv |a_3| e^{i\delta}$, and neglect $|a_3|^2$, we find

$$R(n\pi^0) = 1/3 + \frac{2\sqrt{2}}{3} |a_3| \cos\delta ,$$

$$R(p\pi^-) = 2/3 - \frac{2\sqrt{2}}{3} |a_3| \cos\delta ,$$

$$R_\Lambda = R(n\pi^0) + R(p\pi^-) = 1$$

$$B_\Lambda = \frac{R(p\pi^-)}{R_\Lambda} = 2/3 - \frac{2\sqrt{2}}{3} |a_3| \cos\delta ,$$

$$B_\Lambda = 0.660 - 0.95 |a_3| \cos\delta . \tag{63}$$

Here we know that the $N\pi$ scattering is weak, so we expect $\cos\delta = \pm 1$. Also, we have corrected $2/3 \to 0.660$, for phase space ($n\pi^0$ is lighter than $p\pi^-$).

A recent accurate value of B_Λ by Anderson et al.[2] gives $B_\Lambda = 0.685 \pm 0.017$. If we take $\cos\delta = \pm 1$ then we see that we must choose

$$\cos\delta = -1 , \quad \text{and}$$

$$\frac{|a_3|}{|a_1|} = 0.026 \pm 0.018 . \tag{64}$$

This result is of the same order as Eq. (55), for $K^+ \to 2\pi$. Of course the value of $|a_3|/|a_1|$ for Λ decay need have no relation to that for $K \to 2\pi$.

It is interesting to observe[4] that $B_\Lambda = 2/3$ is obtained not only for $a_3 = 0$, but also for $a_3 = -2\sqrt{2}\, a_1$. This can be seen by inspection of Eqs. (60) and (61). Let us examine this possibility more closely. So far we have discussed only branching ratios, as predicted by the $\Delta I = 1/2$ rule. That is, in terms of the decay amplitudes S and P we have been considering only $|S|^2 + |P|^2 = R$. The $\Delta I = 1/2$ rule predicts much more. For instance, Eq. (57) holds for <u>every</u> decay configuration, for the $\Delta I = 1/2$ decay, and thus holds for the S-wave and P-wave parts separately. That is, from (57),

and using the subscript 1 to stand for the $\Delta I = 1/2$ amplitude,

$$A_1(\Lambda \to p\pi^-) = -\sqrt{2}\, A_1(\Lambda \to n\pi^0)\,, \tag{65}$$

which means, for $\Delta I = 1/2$,

$$S_1(\Lambda \to p\pi^-) = -\sqrt{2}\, S_1(\Lambda \to n\pi^0)\,, \text{ and} \tag{66}$$

$$P_1(\Lambda \to p\pi^-) = -\sqrt{2}\, P_1(\Lambda \to n\pi^0)\,. \tag{67}$$

Then for the decay parameters α, β, and γ, defined in Eqs. (3), (4), (5), we see

$$\alpha_1(\Lambda \to p\pi^-) = \alpha_1(\Lambda \to n\pi^0)\,, \tag{68}$$

$$\beta_1(\Lambda \to p\pi^-) = \beta_1(\Lambda \to n\pi^0)\,, \tag{69}$$

$$\gamma_1(\Lambda \to p\pi^-) = \gamma_1(\Lambda \to n\pi^0)\,, \tag{70}$$

$$R_1(\Lambda \to p\pi^-) = 2R_1(\Lambda \to n\pi^0)\,. \tag{71}$$

Similarly if we had a <u>pure</u> $\Delta I = 3/2$ decay, then, from Eq. (58),

$$A_3(\Lambda \to p\pi^-) = (1/\sqrt{2})\, A_3(\Lambda \to n\pi^0)\,.$$

This again leads to <u>equality</u> of α_3, β_3, and γ_3, for $\Lambda \to p + \pi^-$ and $\Lambda \to n + \pi^0$, and a branching ratio $R_3(\Lambda \to p\pi^-) = 1/2\, R_3(\Lambda \to n\pi^0)$.

Suppose now that one has a mixture of $\Delta I = 3/2$ and $\Delta I = 1/2$. In general, the two decays ($\Delta I = 3/2$ and $1/2$) should have different S/P ratios. In that case, the S/P ratios for $\Lambda \to p\pi^-$ and $n\pi^0$ will <u>not</u> be the same, in general. We can see this in detail as follows. For $\Delta I = 1/2$ (designated by subscript 1), if we use π^- and π^0 to denote $\Lambda \to p + \pi^-$ and $n + \pi^0$, we have, from Eq. (65), for the S- and P-wave parts separately,

$$S_1(\pi^-) = -\sqrt{2}\, S_1(\pi^0) \tag{72}$$

and

$$P_1(\pi^-) = -\sqrt{2}\, P_1(\pi^0)\,. \tag{73}$$

For $\Delta I = 3/2$ (subscript 3) we have

$$S_3(\pi^-) = (1/\sqrt{2})\, S_3(\pi^0) \tag{74}$$

and

$$P_3(\pi^-) = (1/\sqrt{2})\, P_3(\pi^0)\,. \tag{75}$$

Now suppose $\Delta I = 1/2$ occurs with amplitude a_1, and $\Delta I = 3/2$ with amplitude a_3, then

$$\begin{aligned} S(\pi^-) &= a_1 S_1(\pi^-) + a_3 S_3(\pi^-) \\ &= -\sqrt{2}\, a_1 S_1(\pi^0) + (a_3/\sqrt{2})\, S_3(\pi^0)\,, \\ P(\pi^-) &= a_1 P_1(\pi^-) + a_3 P_3(\pi^-) \\ &= -\sqrt{2}\, a_1 P_1(\pi^0) + (a_3/\sqrt{2})\, P_3(\pi^0)\,, \\ S(\pi^0) &= a_1 S_1(\pi^0) + a_3 S_3(\pi^0)\,, \\ P(\pi^0) &= a_1 P_1(\pi^0) + a_3 P_3(\pi^0)\,. \end{aligned}$$

We see by inspection of these equations that if $a_3 = 0$ or if $a_1 = 0$, then $S(\pi^-)/P(\pi^-) = S(\pi^0)/P(\pi^0)$. The same is true if $S_1/P_1 = S_3/P_3$. In both cases α, β, and γ are the same for $p\pi^-$ and $n\pi^0$. The choice $a_3 = -2\sqrt{2}\, a_1$ gives $B_\Lambda = 2/3$. In general, if $S_1/P_1 \neq S_3/P_3$, then a value for a_3/a_1 that gives $B = 2/3$ leads to different values of α, β, and γ for $p\pi^-$ and $n\pi^0$.

It is thus important to check α, β, and γ for $\Lambda \to n\pi^0$. Cork et al.[5] have measured the up-down asymmetries for $\Lambda \to p\pi^-$ and $\Lambda \to n\pi^0$ "simultaneously", i.e., from Λ's produced in the same way. Therefore there is a single Λ polarization p_Λ. The decay asymmetries yield $\alpha(\pi^-)p_\Lambda$ and $\alpha(\pi^0)p_\Lambda$, and the ratio gives

$$\frac{\alpha(n\pi^0)}{\alpha(p\pi^-)} = +1.10 \pm 0.27\,,$$

in agreement with the $\Delta I = 1/2$ rule.

Block et al.[2] have measured $\gamma(n\pi^0)$ by an indirect method. The branching ratio for

$$\frac{{}_\Lambda He^4 \to (\pi^0 \text{ modes})}{{}_\Lambda He^4 \to (\pi^- \text{ modes})}$$

depends strongly on $S(\pi^0)/P(\pi^0)$. They find that S wave predominates. They find

$$\gamma(n\pi^0) = +0.78 \begin{smallmatrix} +0.22 \\ -0.42 \end{smallmatrix} .$$

This is to be compared with, for instance, the value obtained by Cronin and Overseth,[2]

$$\gamma(p\pi^-) = +0.78 \pm 0.04 .$$

Thus an "accidental" solution with $a_3 = -2\sqrt{2}\ a_1$, must (within the errors) also have the same S/P ratio for $\Delta I = 3/2$ and $1/2$, to agree with experiment. Such a double accident seems unlikely.

3. <u>The Decay $\Xi \to \Lambda + \pi$</u>

To find the prediction for $\Delta I = 1/2$ we write

$$s(1/2, +1/2) \to \overline{\Lambda}(0)\ \Xi(1/2)\ \overline{\pi}(1) ,$$

$$(1/2, +1/2) = \overline{\Lambda}\left[\sqrt{2/3}\,(1, +1)(1/2, -1/2) - \sqrt{1/3}\ (1, 0)(1/2, 1/2)\right]$$

$$= \overline{\Lambda}\left[\sqrt{2/3}\ \pi^+\Xi^- - \sqrt{1/3}\ \pi^0\Xi^0\right] ,$$

$\frac{\tau(\Xi^0) = 3 \pm .5}{\tau(\Xi^-) = 1.74 \pm .05} = 1.72 \pm .3$

which gives (transposing)

$$R(\Xi^- \to \Lambda + \pi^-) = 2\,R(\Xi^0 \to \Lambda + \pi^0) . \qquad (76)$$

The Ξ^- lifetime is about 1.2×10^{-10} sec.[2] The Ξ^0 lifetime is not yet known well enough to test Eq. (76).

4. <u>The Decay $\Sigma \to N + \pi$</u>

The final state $N + \pi$ can have $I = 1/2$ or $3/2$. The Σ has $I = 1$. Therefore we can have $\Delta I = 1 \times 1/2 = 3/2$ or $1/2$; or $1 \times 3/2 = 5/2$, $3/2$, or $1/2$. We assume, for simplicity, that $\Delta I = 5/2$ is absent, but include $\Delta I = 3/2$ as well as $1/2$.

We write $\Sigma \to N + \pi + s$. Transposing, we have $s \to \Sigma(\overline{N}\,\overline{\pi})$. From an example, say $\Sigma^- \to n + \pi^- + s$, we see that we have $\Delta I_3 = +1/2$ for the spurion s. There are four possible transition amplitudes, corresponding to $\Delta I = 1/2$

and 3/2, and $I(\bar{N}\bar{\pi}) = 1/2$ or 3/2. We write down the four charge states, using the notation

$$\psi(1,1) \equiv \psi(\Delta I = 1/2,\ I(\bar{N}\bar{\pi}) = 1/2)\ ,$$

$$\psi(1,3) \equiv \psi(\Delta I = 1/2,\ I(\bar{N}\bar{\pi}) = 3/2)\ ,$$

$$\psi(3,1) \equiv \psi(\Delta I = 3/2,\ I(\bar{N}\bar{\pi}) = 1/2)\ ,$$

$$\text{and } \psi(3,3) \equiv \psi(\Delta I = 3/2,\ I(\bar{N}\bar{\pi}) = 3/2)\ .$$

Correspondingly we define the four decay amplitudes $A(1,1)$, $A(1,3)$, $A(3,1)$ and $A(3,3)$, and have the superposition

$$\psi(\Sigma\bar{N}\bar{\pi}) = A(1,1)\psi(1,1) + A(1,3)\psi(1,3) + A(3,1)\psi(3,1) + A(3,3)\psi(3,3). \quad (77)$$

We now write down $\psi(1,1)$, etc., using Tables I and II, and recalling that $\Delta I_3 = +1/2$ in each case. To aid in reading the table we write an intermediate step, in a notation that is self-explanatory:

$$\begin{aligned}\psi(1,1) &= (\tfrac{1}{2}[1(\Sigma)\times\tfrac{1}{2}(\bar{N}\bar{\pi})];\ +\tfrac{1}{2})\\ &= \sqrt{2/3}\ \Sigma^+(\tfrac{1}{2},-\tfrac{1}{2}) - \sqrt{1/3}\ \Sigma^0(\tfrac{1}{2},+\tfrac{1}{2})\\ &= \sqrt{2/3}\ \Sigma^+(\sqrt{1/3}\ \pi^0\bar{p} - \sqrt{2/3}\ \pi^-\bar{n}) - \sqrt{1/3}\ \Sigma^0(\sqrt{2/3}\ \pi^+\bar{p} - \sqrt{1/3}\ \pi^0\bar{n}).\end{aligned} \quad (78)$$

$$\begin{aligned}\psi(1,3) &= (\tfrac{1}{2}[1(\Sigma)\times\tfrac{3}{2}(\bar{N}\bar{\pi})];\ +\tfrac{1}{2})\\ &= \sqrt{1/2}\ \Sigma^-(\tfrac{3}{2},+\tfrac{3}{2}) - \sqrt{1/3}\ \Sigma^0(\tfrac{3}{2},+\tfrac{1}{2}) + \sqrt{1/6}\ \Sigma^+(\tfrac{3}{2},-\tfrac{1}{2})\\ &= \sqrt{1/2}\ \Sigma^-\pi^+\bar{n} - \sqrt{1/3}\ \Sigma^0(\sqrt{1/3}\ \pi^+\bar{p} + \sqrt{2/3}\ \pi^0\bar{n})\\ &\qquad + \sqrt{1/6}\ \Sigma^+(\sqrt{2/3}\ \pi^0\bar{p} + \sqrt{1/3}\ \pi^-\bar{n}).\end{aligned} \quad (79)$$

$$\begin{aligned}\psi(3,1) &= (\tfrac{3}{2}[1(\Sigma)\times\tfrac{1}{2}(\bar{N}\bar{\pi});\ +\tfrac{1}{2})\\ &= \sqrt{1/3}\ \Sigma^+(\tfrac{1}{2},-\tfrac{1}{2}) + \sqrt{2/3}\ \Sigma^0(\tfrac{1}{2},+\tfrac{1}{2})\\ &= \sqrt{1/3}\ \Sigma^+(\sqrt{1/3}\ \pi^0\bar{p} - \sqrt{2/3}\ \pi^-\bar{n}) + \sqrt{2/3}\ \Sigma^0(\sqrt{2/3}\ \pi^+\bar{p} - \sqrt{1/3}\ \pi^0\bar{n}).\end{aligned} \quad (80)$$

$$\psi(3,3) = (\tfrac{3}{2}[1(\Sigma) \times \tfrac{3}{2}(\overline{N}\,\overline{\pi}); + \tfrac{1}{2})$$

$$= \sqrt{2/5}\,\Sigma^-(\tfrac{3}{2}, +\tfrac{3}{2}) + \sqrt{1/15}\;\Sigma^0(\tfrac{3}{2}, +\tfrac{1}{2}) - \sqrt{8/15}\;\Sigma^+(\tfrac{3}{2}, -\tfrac{1}{2})$$

$$= \sqrt{2/5}\;\Sigma^-\pi^+\overline{n} + \sqrt{1/15}\;\Sigma^0(\sqrt{1/3}\,\pi^+\overline{p} + \sqrt{2/3}\,\pi^0\,\overline{n})$$

$$- \sqrt{8/15}\;\Sigma^+(\sqrt{2/3}\,\pi^0\overline{p} + \sqrt{1/3}\;\pi^-\overline{n}). \qquad (81)$$

We could now write down the general superposition $\psi(\Sigma\overline{N}\,\overline{\pi})$ given by Eq. (77). However, since we are interested in the charge states rather than the I-spin states, we rewrite Eq. (77) as

$$\psi(\Sigma\overline{N}\,\overline{\pi}) \equiv A(\Sigma^+\pi^0\overline{p})\,\psi(\Sigma^+\pi^0\overline{p}) + A(\Sigma^+\pi^-\overline{n})\;\psi(\Sigma^+\pi^-\overline{n})$$

$$+ A(\Sigma^0\pi^+\overline{p})\,\psi(\Sigma^0\pi^+\overline{p}) + A(\Sigma^0\pi^0\overline{n})\psi(\Sigma^0\pi^0\overline{n})$$

$$+ A(\Sigma^-\pi^+\overline{n})\;\psi(\Sigma^-\pi^+\overline{n})\,. \qquad (82)$$

From Eqs. (77) through (82) we obtain the amplitudes

$$A(\Sigma^+\pi^0\overline{p}) = \frac{\sqrt{2}}{3}A(1,1) + \frac{1}{3}\;A(1,3) + \frac{1}{3}\;A(3,1) - \frac{4}{3}\sqrt{\frac{1}{5}}\;A(3,3), \qquad (83)$$

$$A(\Sigma^+\pi^-\overline{n}) = -\frac{2}{3}\;A(1,1) + \frac{\sqrt{2}}{6}\;A(1,3) - \frac{\sqrt{2}}{3}A(3,1) - \frac{2}{3}\sqrt{\frac{2}{5}}\;A(3,3), \qquad (84)$$

$$A(\Sigma^-\pi^+\overline{n}) = \frac{\sqrt{2}}{2}\;A(1,3) + \sqrt{\frac{2}{5}}\;A(3,3)\,, \qquad (85)$$

$$A(\Sigma^0\pi^+\overline{p}) = -\frac{\sqrt{2}}{3}\;A(1,1) - \frac{1}{3}\;A(1,3) + \frac{2}{3}A(3,1) + \frac{1}{3}\sqrt{\frac{1}{5}}\;A(3,3), \qquad (86)$$

$$A(\Sigma^0\pi^0\overline{n}) = \frac{1}{3}\;A(1,1) - \frac{\sqrt{2}}{3}\;A(1,3) - \frac{\sqrt{2}}{3}\;A(3,1) + \frac{1}{3}\sqrt{\frac{2}{5}}\;A(3,3). \qquad (87)$$

Unfortunately we cannot make use of Eqs. (86) and (87), since $\Sigma^0 \to N + \pi$ is unobservable because of the rapid death of the Σ^0 via the electromagnetic decay $\Sigma^0 \to \Lambda + \gamma$.

We are left with Eqs. (83), (84), and (85). These equations hold for either the S-wave or the P-wave parts of the decay amplitude. If we wished we could write the equations twice, once with new subscripts for S and once for P. In general the separate terms are complex numbers. However, if the final N-π interaction is small, then T invarinace demands that the separate terms all be real, except for an unimportant phase factor common to all terms. The N-π interaction is indeed negligible at the decay momentum.[1] We therefore take all the terms to be real. We now imagine Eq. (83) (for instance) written twice (once with subscripts for S wave, and once for P wave). We can imagine a two-dimensional S-P space, and think of the two equations (i.e., S and P) as equations involving the S and P components of vectors. We combine the components and write, for instance,

$$\underline{A}(1,1) \equiv A_S(1,1)\,\hat{S} + A_P(1,1)\,\hat{P}\ , \tag{88}$$

with $\hat{S}$ and $\hat{P}$ as unit vectors, and with similar expressions for $\underline{A}(1,3)$, $\underline{A}(3,1)$, and $\underline{A}(3,3)$. Since Eqs. (83), (84), and (85) hold for both the S and P components, they hold for the vectors. We can therefore imagine these equations rewritten, with the substitution of $\underline{A}(1,1)$ for $A(1,1)$, etc.

At first glance the right-hand sides of Eqs. (83), (84), and (85) seem to involve the four independent vectors $\underline{A}(1,1)$, $\underline{A}(1,3)$, $\underline{A}(3,1)$, and $\underline{A}(3,3)$. However, we observe that $\underline{A}(1,1)$ and $\underline{A}(3,1)$ occur only in the combination

$$\underline{A}(1,1;3,1) \equiv \underline{A}(1,1) + \sqrt{1/2}\ \underline{A}(3,1)\ , \tag{89}$$

so that we may rewrite Eqs. (83), (84), and (85) as follows:

$$\underline{A}^0 \equiv \underline{A}(\Sigma^+\pi^0\bar{p}) = \sqrt{\frac{2}{9}}\,\underline{A}(1,1;3,1) + \sqrt{\frac{1}{9}}\,\underline{A}(1,3) - \sqrt{\frac{16}{45}}\,\underline{A}(3,3), \tag{90}$$

$$\underline{A}^+ \equiv \underline{A}(\Sigma^+\pi^-\bar{n}) = -\frac{2}{3}\,\underline{A}(1,1;3,1) + \sqrt{\frac{1}{18}}\,\underline{A}(1,3) - \sqrt{\frac{8}{45}}\,\underline{A}(3,3), \tag{91}$$

$$\underline{A}^- \equiv \underline{A}(\Sigma^-\pi^+\bar{n}) = \sqrt{\frac{1}{2}}\,\underline{A}(1,3) + \sqrt{\frac{2}{5}}\,\underline{A}(3,3). \tag{92}$$

From Eqs. (90), (91), and (92) we form the linear combination

$$\sqrt{2}\,\underline{A}(\Sigma^+\pi^0\bar{p}) + \underline{A}(\Sigma^+\pi^-\bar{n}) - \underline{A}(\Sigma^-\pi^+\bar{n}) = -\sqrt{\frac{18}{5}}\,\underline{A}(3,3). \tag{93}$$

Using Eq. (93), we can make the following observations. (a) Suppose the $\Delta I = 1/2$ rule holds. Then $\underline{A}(3,1) = 0$ and $\underline{A}(3,3) = 0$. Since $\underline{A}(3,3) = 0$, Eq. (93) corresponds to a closed triangle in the S-P plane. This is the well-known triangle of Gell-Mann and Rosenfeld.[1] (b) Suppose we have $\underline{A}(3,1) \neq 0$, but $\underline{A}(3,3) = 0$. Since $\underline{A}(3,1) \neq 0$, the $\Delta I = 1/2$ rule does not hold. Nevertheless, according to Eq. (93) we obtain a closed triangle in the S-P plane. Thus if we find a closed triangle (experimentally) we cannot rule out $\Delta I = 3/2$. The linear combination of $\Delta I = 1/2$ and $3/2$ given by Eq. (89) cannot be resolved. (c) Suppose we have $\underline{A}(3,3) \neq 0$. Then Eq. (93) corresponds to a closed quadrangle instead of a triangle. Equation (93) can be used to determine $\underline{A}(3,3)$. Of course then the $\Delta I = 1/2$ rule does not hold exactly. (We already know this, from the decay $K^+ \to \pi^+\pi^0$.)

We turn now to our experimental knowledge of $\underline{A}(\Sigma^+\pi^0\bar{p})$, $\underline{A}(\Sigma^+\pi^-\bar{n})$, and $\underline{A}(\Sigma^-\pi^+\bar{n})$. From the partial decay rates we know[1,6] that

$$|\underline{A}(\Sigma^+\pi^0\bar{p})| \approx |\underline{A}(\Sigma^+\pi^-\bar{n})| \approx |\underline{A}(\Sigma^-\pi^+\bar{n})|. \tag{94}$$

Therefore, <u>if</u> $\underline{A}(3,3) = 0$, we see from Eq. (93) that the resulting triangle will be an isosceles right triangle with equal legs $|\underline{A}(\Sigma^+\pi^-\bar{n})|$ and

$|\underline{A}(\Sigma^-\pi^+\bar{n})|$, and hypotenuse $\sqrt{2}\,|\underline{A}(\Sigma^+\pi^0\bar{p})|$. The decay parameter α_0, corresponding to $\underline{A}(\Sigma^+\pi^0\bar{p})$--i.e., to $\Sigma^+ \to p + \pi^0$--has been determined by Beall, et al.[7] by measuring the scattering asymmetry of the decay proton. Their result is $\alpha_0 = +0.78^{+0.08}_{-0.09}$. The other decay parameters, β_0 and γ_0, are not known. We assume $\beta_0 = 0$ (T invariance and small N-π interaction). The decay parameter α_+, corresponding to $\underline{A}(\Sigma^+\pi^-\bar{n})$, --i.e., to $\Sigma^+ \to n + \pi^+$-- has been measured by Cork et al.;[5] they measured in a single experiment the up-down asymmetry for $\Sigma^+ \to n + \pi^+$, to obtain $\alpha_+ p_{\Sigma^+}$, and for $\Sigma^+ \to p + \pi^0$, to obtain $\alpha_0 p_{\Sigma^+}$. The ratio gives α_+/α_0, and the known value of α_0 gives α_+. They find $\alpha_+ = +0.03 \pm 0.08$. In our present notation, $\alpha = 2A_S A_P/(A_S^2 + A_P^2)$, so that $\alpha_+ \cong 0$ means that $\underline{A}(\Sigma^+\pi^-\bar{n})$ is oriented along either the $\hat{S}$ axis or the $\hat{P}$ axis. Until γ_+ is measured we cannot choose between these alternative possibilities. The decay parameter α_-, corresponding to $\underline{A}(\Sigma^-\pi^+\bar{n})$--i.e., to $\Sigma^- \to n + \pi^-$--has been measured by Tripp, Watson, and Ferro-Luzzi,[8] who obtain $\alpha_- = +0.16 \pm 0.21$, and by Nussbaum et al.,[2] who obtain $\alpha_- = +0.04 \pm 0.23$. Therefore $\underline{A}(\Sigma^-\pi^+\bar{n})$ is oriented (approximately) either along $\hat{S}$ or along $\hat{P}$. There is as yet no knowledge of γ_-, so that either alternative is possible. If $\underline{A}(3,3) = 0$, then according to Eqs. (93) and (94) and the results $\alpha_+ \approx 0$ and $\alpha_- \approx 0$, we have the two possibilities

$$\underline{A}(\Sigma^+\pi^-\bar{n}) \approx -\hat{S}\,, \tag{95a}$$

$$\underline{A}(\Sigma^-\pi^+\bar{n}) \approx \hat{P}\,, \tag{95b}$$

$$\underline{A}(\Sigma^+\pi^0\bar{p}) \approx \frac{\hat{S}+\hat{P}}{\sqrt{2}}\,, \tag{95c}$$

or, instead,

$$\underline{A}(\Sigma^+\pi^-\bar{n}) \approx -\hat{P}\,, \tag{96a}$$

$$\underline{A}(\Sigma^-\pi^+\bar{n}) \approx \hat{S}\,, \tag{96b}$$

$$\underline{A}(\Sigma^+\pi^0\bar{p}) \approx \frac{\hat{P}+\hat{S}}{\sqrt{2}} \tag{96c}$$

We have chosen units such that $|\underline{A}(\Sigma^- \pi^+ \bar{n})| \equiv 1$. Either solution could of course be multiplied by -1, or by exp $i\alpha$, with no physical consequences.

Solutions (95) and (96) demand $\alpha_0 \approx +1$. This is in disagreement with the measured value,[7] $\alpha_0 = +.78^{+.08}_{-.09}$. In Fig. 13 we reproduce the diagram from Tripp et al.[8] (Their notation $\underline{N}_+$, $\underline{N}_-$, and $\underline{N}_0$ corresponds to our $\underline{A}(\Sigma^+ \pi^- \bar{n})$, $\underline{A}(\Sigma^- \pi^+ \bar{n})$, and $\underline{A}(\Sigma^+ \pi^0 \bar{p})$. Their sign convention for α is opposite to ours.) The two possibilities for $\underline{N}_0$ correspond to the two possibilities $\gamma_0 > 0$, and $\gamma_0 < 0$, i.e., $|S|/|P| > 1$ and < 1. From the diagram and Eq. (93) we find

$$\frac{|\underline{A}(3,3)|}{|\underline{A}(\Sigma^- \pi^+ \bar{n})|} \approx 0.23 \pm 0.09, \text{ or } 0.30 \pm 0.10, \tag{97}$$

where the errors are only estimated from the diagram, and where the first possibility corresponds to $\gamma_0/\gamma_+ > 0$ and the second to $\gamma_0/\gamma_+ < 0$. In the diagram it is implicitly assumed that $\gamma_+/\gamma_- < 0$; i.e., that if $\Sigma^+ \to n + \pi^+$ goes by S wave, then $\Sigma^- \to n + \pi^-$ goes by P wave, and vice versa. In other words it is assumed that the violation of the $\Delta I = 1/2$ rule is small. Of course, if the $\Delta I = 1/2$ rule does not hold, one can have $\gamma_+/\gamma_- > 0$; that is, both decays can go by S wave or both by P wave. We must have some reservations until γ_+, γ_-, and γ_0 are measured.

It is perhaps worth remarking that even if experiments finally tell us that, for example, $\Sigma^+ \to n + \pi^+$ is pure P wave, $\Sigma^- \to n + \pi^-$ is pure S wave, and $\alpha_0 = +1.0$ (instead of 0.78), then we still will not be able to rule out a large violation of the $\Delta I = 1/2$ rule. For instance if in the example of Eq. (95a) we replaced $-\hat{S}$ by $\hat{S}$, but left (95b) and (95c) as they are, we would obtain $\sqrt{18/5}\ \underline{A}(3,3) = 2\hat{S}$, instead of zero, as is seen from Eq. (93). This type of ambiguity, and also the ambiguity corresponding to Eq. (89), is not "inherent" but, as we see from Eqs. (86) and (87), could be resolved if it were possible to measure the rates for $\Sigma^0 \to p + \pi^-$ and $\Sigma^0 \to n + \pi^0$.

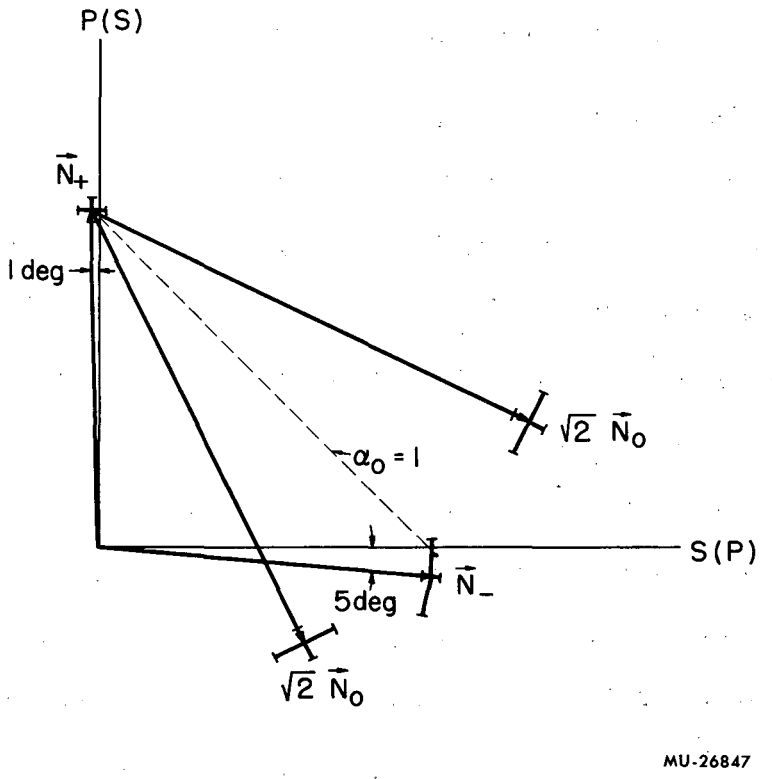
P(S)
$\vec{N}_+$
1 deg
$\sqrt{2}\ \vec{N}_o$
$\alpha_o = 1$
S(P)
$\vec{N}_-$
5deg
$\sqrt{2}\ \vec{N}_o$
MU-26847

Fig. 13

Lecture IV. $K \to 3\pi$ AND THE $\Delta I = 1/2$ RULE

In this lecture we consider the decays

$$K^+ \to \pi^+ + \pi^+ + \pi^- \equiv (++-) ,$$
$$K^+ \to \pi^+ + \pi^0 + \pi^0 \equiv (+00) ,$$
$$K_2^0 \to \pi^+ + \pi^- + \pi^0 \equiv (+-0) ,$$
$$\text{and } K_2^0 \to \pi^0 + \pi^0 + \pi^0 \equiv (000) .$$

change state

The final (3π) state can have $I = 0, 1, 2$ or 3. For K^+ decay we have $Q = +1$, so $I_3 = +1$; therefore $I = 0$ is excluded. Thus for $K^+ \to 3\pi$ one has the possibilities $\Delta I = 1 \pm 1/2 = 3/2$ or $1/2$, $2 \pm 1/2 = 5/2$ or $3/2$, and $3 \pm 1/2 = 7/2$ or $5/2$. For $K^0 \to 3\pi$ we shall see that K_1^0 goes to $I = 0$ or 2, and K_2^0 goes to $I = 1$ or 3. We consider only K_2^0 decay. Thus for $K_2^0 \to 3\pi$ one has the possibilities $\Delta I = 1 \pm 1/2 = 3/2$ or $1/2$, and $3 \pm 1/2 = 7/2$ or $5/2$.

Consider now the states $\pi^+\pi^-\pi^0$ and $\pi^0\pi^0\pi^0$. Let $\underline{L}$ be the angular momentum of pion #1 relative to the c.m. of #2 and #3, and let $\underline{\ell}$ be the angular momentum of #2 and #3 in their c.m. Then $\underline{J} = \underline{L} + \underline{\ell}$. But $\underline{J} = 0$, since the spin of the K is zero. Therefore $|\underline{L}| = |\underline{\ell}|$. Therefore $P = (-1)^3(-1)^L(-1)^\ell = -1$. Thus for $K_1^0 \to 3\pi$, for which $CP = +1$, we have $C = -1$, and for $K_2^0 \to 3\pi$, we have $CP = -1$, and $C = +1$. (Here we are assuming CP invariance in the decay.) Since $3\pi^0$ obviously has $C = +1$, we see that $K_2^0 \to 3\pi^0$ is allowed, and $K_1^0 \to 3\pi^0$ is forbidden.

We next wish to show that for 3π, $I = 0$ and 2 have $C = -1$, and $I = 1$ and 3 have $C = +1$. There are several ways to show this. The easiest is to assume the theorem proven by Professor Rosenfeld in his accompanying lectures, namely

$$C = G(-1)^I .$$

For 3π we have $G = -1$, so we have $C = (-1)^{I+1}$.

Another way is to use the Clebsch-Gordan tables by brute force, as follows. The notation is (I, I_3).

I = 0

We have from Table V,

$$
\begin{aligned}
(0,0)_{3\pi} &= 1(\pi)\times 1(2\pi) \\
&= \sqrt{1/3}\,\pi^+(1,-1)_{2\pi} - \sqrt{1/3}\,\pi^0(1,0)_{2\pi} + \sqrt{1/3}\,\pi^-(1,+1)_{2\pi} \\
&= \sqrt{1/3}\,\pi^+(\sqrt{1/2}\,\pi^0\pi^- - \sqrt{1/2}\,\pi^-\pi^0) \\
&\quad - \sqrt{1/3}\,\pi^0(\sqrt{1/2}\,\pi^+\pi^- - \sqrt{1/2}\,\pi^-\pi^+) \\
&\quad + \sqrt{1/3}\,\pi^-(\sqrt{1/2}\,\pi^+\pi^0 - \sqrt{1/2}\,\pi^0\pi^+)\,. \qquad (98)
\end{aligned}
$$

Under C, we have $\pi^0 \to \pi^0$, $\pi^+ \to \pi^-$, and $\pi^- \to \pi^+$, and by inspection of Eq. (98) we see that $(0,0)_{3\pi} \to -(0,0)_{3\pi}$. That is, $C = -1$.

I = 1

Here there are three possibilities. We have,

first,
$$
\begin{aligned}
(1,0)_{3\pi} &= 1(\pi)\times 0(2\pi) \qquad (99a) \\
&= \pi^0(0,0)_{2\pi} = \pi^0(\sqrt{1/3}\,\pi^+\pi^- - \sqrt{1/3}\,\pi^0\pi^0 + \sqrt{1/3}\,\pi^-\pi^+),
\end{aligned}
$$

second,
$$
\begin{aligned}
(1,0)_{3\pi} &= 1(\pi)\times 1(2\pi) \qquad (99b) \\
&= \sqrt{1/2}\,\pi^+(1,-1)_{2\pi} - \sqrt{1/2}\,\pi^-(1,+1)_{2\pi} \\
&= \sqrt{1/2}\,\pi^+(\sqrt{1/2}\,\pi^0\pi^- - \sqrt{1/2}\,\pi^-\pi^0) \\
&\quad - \sqrt{1/2}\,\pi^-(\sqrt{1/2}\,\pi^+\pi^0 - \sqrt{1/2}\,\pi^0\pi^+)\,,
\end{aligned}
$$

and lastly,
$$
\begin{aligned}
(1,0)_{3\pi} &= 1(\pi)\times 2(2\pi) \qquad (99c) \\
&= \sqrt{3/10}\,\pi^+(2,-1)_{2\pi} - \sqrt{2/5}\,\pi^0(2,0)_{2\pi} + \sqrt{3/10}\,\pi^-(2,+1)_{2\pi} \\
&= \sqrt{3/10}\,\pi^+(\sqrt{1/2}\,\pi^0\pi^- + \sqrt{1/2}\,\pi^-\pi^0) \\
&\quad - \sqrt{2/5}\,\pi^0(\sqrt{1/6}\,\pi^+\pi^- + \sqrt{2/3}\,\pi^0\pi^0 + \sqrt{1/6}\,\pi^-\pi^+) \\
&\quad + \sqrt{3/10}\,\pi^-(\sqrt{1/2}\,\pi^+\pi^0 + \sqrt{1/2}\,\pi^0\pi^+)\,.
\end{aligned}
$$

We see that in all three cases, $C = +1$.

I = 2

There are two possibilities,

first,

$$
\begin{aligned}
(2,0)_{3\pi} &= 1(\pi) \times 1(2\pi) \qquad (100a)\\
&= \sqrt{1/6}\ \pi^+(1,-1)_{2\pi} + \sqrt{2/3}\ \pi^0(1,0)_{2\pi} + \sqrt{1/6}\ \pi^-(1,+1)_{2\pi}\\
&= \sqrt{1/6}\ \pi^+(\sqrt{1/2}\ \pi^0\pi^- - \sqrt{1/2}\ \pi^-\pi^0)\\
&\quad + \sqrt{2/3}\ \pi^0(\sqrt{1/2}\ \pi^+\pi^- - \sqrt{1/2}\ \pi^-\pi^+)\\
&\quad + \sqrt{1/6}\ \pi^-(\sqrt{1/2}\ \pi^+\pi^0 - \sqrt{1/2}\ \pi^0\pi^+)\ ,
\end{aligned}
$$

and second,

$$
\begin{aligned}
(2,0)_{3\pi} &= 1(\pi) \times 2(2\pi) \qquad (100b)\\
&= \sqrt{1/2}\ \pi^-(2,+1)_{2\pi} - \sqrt{1/2}\ \pi^+(2,-1)_{2\pi}\\
&= \sqrt{1/2}\ \pi^-(\sqrt{1/2}\ \pi^+\pi^0 + \sqrt{1/2}\ \pi^0\pi^+)\\
&\quad - \sqrt{1/2}\ \pi^+(\sqrt{1/2}\ \pi^0\pi^- + \sqrt{1/2}\ \pi^-\pi^0)\ .
\end{aligned}
$$

In both (100a) and (100b) we have $C = -1$.

I = 3

We have

$$
\begin{aligned}
(3,0)_{3\pi} &= 1(\pi) \times 2(2\pi) \qquad (101)\\
&= \sqrt{1/5}\ \pi^+(2,-1)_{2\pi} + \sqrt{3/5}\ \pi^0(2,0)_{2\pi} + \sqrt{1/5}\ \pi^-(2,+1)_{2\pi}\\
&= \sqrt{1/5}\ \pi^+(\sqrt{1/2}\ \pi^0\pi^- + \sqrt{1/2}\ \pi^-\pi^0)\\
&\quad + \sqrt{3/5}\ \pi^0(\sqrt{1/6}\ \pi^+\pi^- + \sqrt{2/3}\ \pi^0\pi^0 + \sqrt{1/6}\ \pi^-\pi^+)\\
&\quad + \sqrt{1/5}\ \pi^-(\sqrt{1/2}\ \pi^+\pi^0 + \sqrt{1/2}\ \pi^0\pi^+)\ ,
\end{aligned}
$$

for which $C = +1$.

In Eqs. (98) through (101) we have exhibited the seven possible charge states for $\pi^+\pi^-\pi^0$ and $\pi^0\pi^0\pi^0$ and seen by inspection that we have $C = (-1)^{I+1}$. We notice as a check that $3\pi^0$ occurs only for I = 1 or 3, i.e., for C = +1.

We consider at first only the predictions of the $\Delta I = 1/2$ rule. Then for $K^+ \to 3\pi$ and $K_2^0 \to 3\pi$ we can have only the 3π states $(I, I_3) = (1, +1)$, and $(1, 0)$, respectively. There are three independent 3π states with I = 1, as we saw from the combinations $1(3\pi) = 1(\pi) \times 0(2\pi)$, $1(\pi) \times 1(2\pi)$, or $1(\pi) \times 2(2\pi)$. We could use the Clebsch-Gordan table to construct these states, as was done for the (1, 0) states in Eqs. (99a, b, c). However, it is more convenient to use another approach. (The functions we obtain for (1, 0) are linear combinations of those found in Eqs. (99).)

We have the three individual pion wave amplitudes $\underline{\pi}_1$, $\underline{\pi}_2$, and $\underline{\pi}_3$, each of which transforms like a vector (I = 1) in I-spin space. We want to form a probability amplitude for 3π. This must be trilinear in $\underline{\pi}_1$, $\underline{\pi}_2$, and $\underline{\pi}_3$. We want that combination that transforms like a vector in I-spin space. There are three such combinations, which are, most simply, $\underline{A} \equiv \underline{\pi}_1 (\underline{\pi}_2 \cdot \underline{\pi}_3)$, $\underline{B} \equiv \underline{\pi}_2 (\underline{\pi}_3 \cdot \underline{\pi}_1)$, and $\underline{C} \equiv \underline{\pi}_3 (\underline{\pi}_1 \cdot \underline{\pi}_2)$. The most general vector is then

$$\underline{V} = A\,\underline{A} + B\,\underline{B} + C\,\underline{C}\ , \tag{102}$$

where A, B, and C are complex numbers.

The meaning of, for instance, $\underline{\pi}_2 \cdot \underline{\pi}_3$, can be expressed in two ways (which unfortunately differ by a factor of -1). We can use the Clebsch-Gordan table to find that combination of $\underline{\pi}_2$ and $\underline{\pi}_3$ that transforms like a scalar. That is

$$\begin{aligned}(\underline{\pi}_2 \cdot \underline{\pi}_3) &= (0, 0)_{2\pi} = 1(\pi_2) \times 1(\pi_3) \\ &= (\sqrt{1/3}\ \pi_2^+ \pi_3^- - \sqrt{1/3}\ \pi_2^0 \pi_3^0 + \sqrt{1/3}\ \pi_2^- \pi_3^+ .\end{aligned} \tag{103}$$

Or, instead, we can use the spherical harmonics in Eq. 1 (with the addition of a normalization factor) and write

$$\underline{\pi}_2 \cdot \underline{\pi}_3 = \pi_{x_2}\pi_{x_3} + \pi_{y_2}\pi_{y_3} + \pi_{z_2}\pi_{z_3}$$

$$= \left(\frac{\pi_x + i\pi_y}{\sqrt{2}}\right)_2 \left(\frac{\pi_x - i\pi_y}{\sqrt{2}}\right)_3 + \left(\frac{\pi_x - i\pi_y}{\sqrt{2}}\right)_2 \left(\frac{\pi_x + i\pi_y}{\sqrt{2}}\right)_3 + \pi_{z_2}\pi_{z_3}$$

$$= -Y_1^1(2)\, Y_1^{-1}(3) - Y_1^{-1}(2)\, Y_1^1(3) + Y_1^0(2)\, Y_1^0(3)$$

$$= -\pi_2^+ \pi_3^- - \pi_2^- \pi_3^+ + \pi_2^0 \pi_3^0 ,$$

which is the same as (103) except for a common factor. We use Eq. (103).

We can take x, y, and z components of the vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$; or we can take +, -, and 0 "components," since these are just linear combinations of the x, y, and z components. Thus

$$A^+ = \pi_1^+ (\underline{\pi}_2 \cdot \underline{\pi}_3) = \pi_1^+ \frac{(\pi_2^+ \pi_3^- + \pi_2^- \pi_3^+ - \pi_2^0 \pi_3^0)}{\sqrt{3}} , \tag{104}$$

$$A^0 = \pi_1^0 (\underline{\pi}_2 \cdot \underline{\pi}_3) = \pi_1^0 \frac{(\pi_2^+ \pi_3^- + \pi_2^- \pi_3^+ - \pi_2^0 \pi_3^0)}{\sqrt{3}} . \tag{105}$$

Instead of $\underline{A}$, $\underline{B}$, and $\underline{C}$ we could take as our independent states the linear combinations

$$\underline{S} \equiv \underline{A} + \underline{B} + \underline{C} = \underline{\pi}_1 (\underline{\pi}_2 \cdot \underline{\pi}_3) + \underline{\pi}_2 (\pi_3 \cdot \underline{\pi}_1) + \underline{\pi}_3 (\underline{\pi}_1 \cdot \underline{\pi}_2), \tag{106}$$

$$\underline{M}_1 \equiv \underline{B} - \underline{C} = \underline{\pi}_1 \times (\underline{\pi}_2 \times \underline{\pi}_3) ,$$

and $\underline{M}_2 \equiv -\underline{A} + \underline{B} = (\underline{\pi}_1 \times \underline{\pi}_2) \times \underline{\pi}_3$.

The combination $\underline{S}$ is completely symmetric with respect to interchange of any two pions. The functions $\underline{M}_1$ and $\underline{M}_2$ have mixed symmetries. (For instance $\underline{M}_1$ is antisymmetric under interchange of #2 and #3 but has no other symmetry.)

We return to the general expression, Eq. (102). We first write out the expression completely. Then we rearrange the terms so that π_1, π_2, π_3 always occur in order. We can then drop the subscripts 1, 2, and 3. For instance, $\pi_2^+ \pi_1^- \pi_3^0 = \pi_1^- \pi_2^+ \pi_3^0 \equiv \pi^-\pi^+\pi^0 \equiv (-+0)$. Thus we have

$$\begin{aligned}
\underline{V} &\equiv A\,\underline{A} + B\,\underline{B} + C\,\underline{C} \\
&= A\,\underline{\pi}_1(\underline{\pi}_2 \cdot \underline{\pi}_3) + B\,\underline{\pi}_2(\underline{\pi}_3 \cdot \underline{\pi}_1) + C\,\underline{\pi}_3(\underline{\pi}_1 \cdot \underline{\pi}_2) \\
&= \frac{A}{\sqrt{3}}\,\underline{\pi}\,(\pi^+\pi^- + \pi^-\pi^+ - \pi^0\pi^0) \\
&\quad + \frac{B}{\sqrt{3}}\,(\pi^+\underline{\pi}\,\pi^- + \pi^-\underline{\pi}\,\pi^+ - \pi^0\underline{\pi}\,\pi^0) \\
&\quad + \frac{C}{\sqrt{3}}\,(\pi^+\pi^-\underline{\pi} + \pi^-\pi^+\underline{\pi} - \pi^0\pi^0\underline{\pi})\,.
\end{aligned}$$

Taking components, we find

$$\begin{aligned}
(1,+1)_{3\pi} \equiv V^+ &= \sqrt{1/3}\,\{A[(++-) + (+-+) - (+00)] \\
&\qquad + B[(++-) + (-++) - (0+0)] \\
&\qquad + C[(+-+) + (-++) - (00+)]\} \\
&= \sqrt{1/3}\,\{(A+B)(++-) + (B+C)(-++) + (C+A)(+-+) \\
&\qquad - A(+00) - B(0+0) - C(00+)\}, \qquad (107)
\end{aligned}$$

$$\begin{aligned}
(1,0)_{3\pi} \equiv V^0 &= \sqrt{1/3}\,\{A[(0+-) + (0-+) - (000)] \\
&\qquad + B[(+0-) + (-0+) - (000)] \\
&\qquad + C[(+-0) + (-+0) - (000)]\} \\
&= \sqrt{1/3}\,\{A[(0+-) + (0-+)] + B[(+0-) + (-0+] + C[(+-0) + (-+0)] \\
&\qquad - (A+B+C)(000)\}\,. \qquad (108)
\end{aligned}$$

We now turn to the predictions of the $\Delta I = 1/2$ rule. We have, using our usual notation (I, I_3), and the spurion $s(\Delta I, \Delta I_3)$,

$$K^+ \to (3\pi) + s \ ,$$

i.e., $(1/2, +1/2) \to (1, +1) + s(1/2, -1/2)$,

and $$K^0 \to (3\pi) + s \ ,$$

i.e., $(1/2, -1/2) \to (1, 0) + s(1/2, -1/2)$.

Or, transposing both s and K, we have, from Table I,

$$\begin{aligned} s(1/2, +1/2) &= 1/2\,(\bar{K}) \times 1(3\pi) \\ &= \sqrt{2/3}\,(1/2, -1/2)(1, +1) - \sqrt{1/3}\,(1/2, +1/2)(1, 0) \\ &= \sqrt{2/3}\ K^- V^+ - \sqrt{1/3}\ \bar{K}^0 V^0 \ , \end{aligned} \tag{109}$$

where V^+ and V^0 are given by (107) and (108).

In the term $\bar{K}^0 V^0$ we have contributions like $\bar{K}^0(0+-)$. This represents, after transposing, $K^0 \to \pi_1^0 + \pi_2^+ + \pi_3^-$. We are actually interested in $K_2^0 \to 3\pi$, rather than in $K^0 \to 3\pi$. Because of the relation $K^0 = (K_1^0 + K_2^0)/\sqrt{2}$, a pure K^0 beam is, in terms of intensities, half K_1^0 and half K_2^0. Only the K_2^0 half of the beam contributes to $K^0 \to 3\pi$ in the $I = 1$ state. Therefore a pure K_2^0 beam would give twice the decay rate of a pure K^0 beam, into $I = 1$. In terms of amplitudes we should therefore multiply the K^0 decay amplitude by $\pm\sqrt{2}$ to get the K_2^0 decay amplitude. (The choice of sign is arbitrary, since charge conservation prevents interference between K_2^0 and K^+ decay; the relative phase of K_2^0 and K^+ has no physical consequence.)

Finally we can write the decay rates, remembering that, for instance, (++-) and (-++) are distinguishable and do not interfere. After including a factor of 2 for K_2^0 decay, as discussed above, we have, from Eqs. (109), (107), and (108),

$$R(K^+ \to ++-) = 2/3 \cdot 1/3 \cdot \{|A + B|^2 + |B + C|^2 + |C + A|^2\} \ , \tag{110}$$

$$R(K^+ \to +00) = 2/3 \cdot 1/3 \cdot \{|A|^2 + |B|^2 + |C|^2\}, \tag{111}$$

$$R(K_2^0 \to +-0) = 2 \cdot 1/3 \cdot 1/3 \{2|A|^2 + 2|B|^2 + 2|C|^2\}, \tag{112}$$

$$R(K_2^0 \to 000) = 2 \cdot 1/3 \cdot 1/3 \{|A + B + C|^2\}. \tag{113}$$

These equations contain the predictions of the $\Delta I = 1/2$ rule. We can think of A, B, and C as functions of the momenta in the decay. Then the equations refer to a given configuration. (We consider only the rates and not the spectra. See Rosenfeld's notes for spectral considerations.)

From Eqs. (110) through (113) we find

$$R(K_2^0 \to +-0) = 2\,R(K^+ \to +00), \tag{114}$$

and $$R(K^+ \to ++-) - R(K^+ \to +00) = R(K_2^0 \to 000). \tag{115}$$

Equations (114) and (115) hold for <u>any</u> choice of A, B, and C; in other words for any admixture of the symmetric I = 1 state <u>S</u>, given by A = B = C, and the mixed symmetry states $\underline{M}_1$ and $\underline{M}_2$. These two equations give the best tests for the $\Delta I = 1/2$ rule.[9]

The symmetric I = 1 state <u>S</u> plays a dominant role, empirically, as we shall see. We therefore write down the predictions for this state. Taking A = B = C in Eqs. (110) and (111), and then in Eqs. (112) and (113), we obtain

$$R(K^+ \to ++-) = 4\,R(K^+ \to +00), \tag{116}$$

$$R(K_2^0 \to 000) = 3/2\,R(K_2^0 \to +-0). \tag{117}$$

Notice that if the $\Delta I = 1/2$ rule holds then I = 1 is the only allowed 3π state. However, I = 1 <u>can</u> be reached through either $\Delta I = 1/2$ or $\Delta I = 3/2$. Equations (116) and (117) hold only for the symmetric I = 1 state. We will find they are well satisfied experimentally; but of course this has not much bearing on the $\Delta I = 1/2$ rule, since $\Delta I = 3/2$ can reach this state. On the

other hand, Eqs. (114) and (115), which relate the charged and neutral decays, depend directly on the $\Delta I = 1/2$ rule through the spurion equation (109), and do not hold when $\Delta I = 3/2$ is present.

Before giving the predictions when $\Delta I = 3/2$ is included, we turn to the experiments. We include phase-space factors and will indicate their insertion by a double-stemmed arrow, $\Longrightarrow$. From (116) we have, for the state $\underline{S}$,

$$\frac{R(K^+ \to +00)}{R(K^+ \to ++-)} = 0.25 \Longrightarrow 0.311 \ . \tag{118}$$

Recent experimental values are summarized in Ref. 10, and average to 0.298 ± 0.025. The agreement with (118) is excellent. We conclude that the symmetric $I = 1$ state ($\underline{S}$) is important.

From (117) we expect, for $\underline{S}$,

$$\frac{R(K_2^0 \to 000)}{R(K_2^0 \to +-0)} = 1.5 \Longrightarrow 1.82 \ . \tag{119}$$

Results from Dubnya presented at Geneva (1962) by Anikina et al.[2] give $R(K_2^0 \to 000)/R(K_2^0 \to \text{all charged}) = 0.38 \pm 0.07$. Luers et al.[11] have obtained

$$R(K_2^0 \to +-0)/R(K_2^0 \to \text{all charged}) = 0.134 \pm 0.018 \ . \tag{120}$$

Combining these two results, we obtain $R(K_2^0 \to 000)/R(K_2^0 \to +-0) = 2.83 \pm 0.52$. This result is in only fair agreement with Eq. (119). On the otherhand, the disagreement amounts to only two standard deviations, and does not shake our faith in the dominance of the state $\underline{S}$.

We now turn to the prediction (114) of the $\Delta I = 1/2$ rule. Alexander et al.[10] have measured an absolute decay rate for $K_2^0 \to \pi^\pm + L^\mp + \nu$. When this is combined with the branching ratio (120) of Luers et al., they find

$$R(K_2^0 \to +-0) = (1.44 \pm 0.43) \times 10^6 \ \text{sec}^{-1}. \tag{121}$$

This is to be compared with[12]

$$2R(K^+ \to +00) = (2.78 \pm 0.22) \times 10^6 \text{ sec}^{-1}. \tag{122}$$

The agreement with (114) is very poor. Alexander et al. quote 100/1 statistical odds against agreement.

We are thus motivated to look at the more complicated formulas that result when the $\Delta I = 1/2$ rule does not hold. We must also ask whether it is reasonable to expect that the presence of $\Delta I = 3/2$ could preserve the beautiful agreement of (116) with experiment and still give the expected disagreement with (114). The point here of course is that once we allow $\Delta I = 3/2$ then we must allow $I = 2$ (for 3π) in $K^+ \to 3\pi$, as well as $I = 1$, and (116) should presumably not hold.

We now give up the $\Delta I = 1/2$ rule and allow $\Delta I = 3/2$. We still omit $\Delta I = 5/2$ and $7/2$. (They will be included later!)

With $\Delta I = 1/2$ and $3/2$ we can reach $I = 1$ and 2 in $K^+ \to 3\pi$, and $I = 1$ for $K_2^0 \to 3\pi$. (CP invariance rules out $I = 2$ for $K_2^0 \to 3\pi$.) The relation between K^+ and K_2^0 decay for $\Delta I = 3/2$ going to $I = 1$ is given by a spurion equation similar to Eq. (109), namely, from Table I,

$$s(3/2, +1/2) = \sqrt{1/3}\ K^- V^+ + \sqrt{2/3}\ \bar{K}^0 V^0 . \tag{123}$$

In the arbitrary constants A, B, and C in V^+ and V^0 we use subscript 1 for $\Delta I = 1/2$ and subscript 3 for $\Delta I = 3/2$. We have, then,

$$\underline{V}_1 = A_1 \underline{A} + B_1 \underline{B} + C_1 \underline{C} , \tag{124}$$

$$\underline{V}_3 = A_3 \underline{A} + B_3 \underline{B} + C_3 \underline{C} . \tag{125}$$

Thus for the $I = 1$ part of the wave function we have, using (109) with (124), and (123) with (125),

$$
\begin{aligned}
\psi(I=1) &= s(1/2,1/2) + s(3/2,1/2) \\
&= K^- [\sqrt{2/3}\, V_1^+ + \sqrt{1/3}\, V_3^+] \\
&\quad + \bar{K}^0 [-\sqrt{1/3}\, V_1^0 + \sqrt{2/3}\, V_3^0] \\
&= \sqrt{1/3}\, K^- \{(\sqrt{2}\, A_1 + A_3)\, A^+ + (\sqrt{2}\, B_1 + B_3)\, B^+ + (\sqrt{2}\, C_1 + C_3)\, C^+\} \\
&\quad + \sqrt{1/3}\, \bar{K}^0 \{(-A_1 + \sqrt{2}\, A_3)\, A^0 + (-B_1 + \sqrt{2}\, B_3)\, B^0 + (-C_1 + \sqrt{2}\, C_3) C^0\}.
\end{aligned}
$$

But

$$
\begin{aligned}
A^+ &= \sqrt{1/3}\ [(\pi^+)\, \pi^+\pi^- + (\pi^+)\, \pi^-\pi^+ - (\pi^+)\, \pi^0\pi^0] , \\
B^+ &= \sqrt{1/3}\ [\pi^+(\pi^+)\, \pi^- + \pi^-(\pi^+)\, \pi^+ - \pi^0(\pi^+)\, \pi^0] , \\
C^+ &= \sqrt{1/3}\, [\pi^+\pi^-\, (\pi^+) + \pi^-\pi^+(\pi^+) - \pi^0\pi^0\, (\pi^+)] , \\
A^0 &= \sqrt{1/3}\, [(\pi^0)\, \pi^+\pi^- + (\pi^0)\, \pi^-\pi^+ - (\pi^0)\, \pi^0\pi^0] , \\
B^0 &= \sqrt{1/3}\, [\pi^+(\pi^0)\, \pi^- + \pi^-\, (\pi^0)\, \pi^+ - \pi^0(\pi^0)\, \pi^0] , \\
C^0 &= \sqrt{1/3}\, [\pi^+\pi^-(\pi^0) + \pi^-\pi^+\, (\pi^0) - \pi^0\pi^0(\pi^0)] .
\end{aligned}
$$

Combining these, we find

$$
\begin{aligned}
\psi(I=1) = 1/3\, K^- \{ & (\sqrt{2}\, A_1 + A_3 + \sqrt{2}\, B_1 + B_3)(++-) \\
& + (\sqrt{2}\, B_1 + B_3 + \sqrt{2}\, C_1 + C_3)(-++) \\
& + (\sqrt{2}\, C_1 + C_3 + \sqrt{2}\, A_1 + A_3)(+-+) \\
& - (\sqrt{2}\, A_1 + A_3)(+00) \\
& - (\sqrt{2}\, B_1 + B_3)(0+0) \\
& - (\sqrt{2}\, C_1 + C_3)(00+)\} \\
+ 1/3\, \bar{K}^0 \{ & (-A_1 + \sqrt{2}\, A_3)\, [(0+-) + (0-+)] \\
& + (-B_1 + \sqrt{2}\, B_3)[(+0-) + (-0+)] \\
& + (-C_1 + \sqrt{2}\, C_3)[(+-0) + (-+0)] \\
& + (A_1 - \sqrt{2}\, A_3 + B_1 - \sqrt{2}\, B_3 + C_1 - \sqrt{2}\, C_3)(000)\} . \qquad (126)
\end{aligned}
$$

We still need the I = 2 wave function for 3π, for $K^+ \to 3\pi$. There are two possibilities, which we label with subscripts D and E. We have, from Table V,

$$\begin{aligned}
\psi_D(2,+1) &\equiv 1(\pi)\times 1(2\pi) \\
&= \sqrt{1/2}\,\pi^+(1,0)_{2\pi} + \sqrt{1/2}\,\pi^0(1,+1)_{2\pi} \\
&= \sqrt{1/2}\,\pi^+(\sqrt{1/2}\,\pi^+\pi^- - \sqrt{1/2}\,\pi^-\pi^+) \\
&\quad + \sqrt{1/2}\,\pi^0(\sqrt{1/2}\,\pi^+\pi^0 - \sqrt{1/2}\,\pi^0\pi^+) \\
&= 1/2\,[(++-) - (+-+) + (0+0) - (00+)]\,, \qquad (127)
\end{aligned}$$

and from Table VI (and Table V)

$$\begin{aligned}
\psi_E(2,+1) &\equiv 1(\pi)\times 2(2\pi) \\
&= \sqrt{1/3}\,\pi^-(2,+2)_{2\pi} + \sqrt{1/6}\,\pi^0(2,+1_{2\pi} - \sqrt{1/2}\,\pi^+(2,0)_{2\pi} \\
&= \sqrt{1/3}\,\pi^-\pi^+\pi^+ + \sqrt{1/6}\,\pi^0[\sqrt{1/2}\,\pi^+\pi^0 + \sqrt{1/2}\,\pi^0\pi^+] \\
&\quad - \sqrt{1/2}\,\pi^+[\sqrt{1/6}\,\pi^+\pi^- + \sqrt{2/3}\,\pi^0\pi^0 + \sqrt{1/6}\,\pi^-\pi^+] \\
&= \sqrt{1/3}\,\{(-++) - 1/2\,[(++-) + (+-+)] + 1/2\,[(0+0) + (00+)] - (+00)\}. \\
&\qquad (128)
\end{aligned}$$

For the general case (for I = 2) we have the superposition

$$\psi(2,+1) = D\psi_D + E\psi_E, \tag{129}$$

where D and E are complex numbers, and ψ_D and ψ_E are given by (127) and (128).

The corresponding spurion equation is not really needed, since we have only K^+ decay into I = 2, and thus no coefficients relating K^+ and K_2^0 decay. However, for uniformity of notation we include the spurion. We have, from Table III,

$$\begin{aligned} s(3/2,+1/2) &= 1/2\,(\bar{K})\times 2(3\pi) \\ &= \sqrt{3/5}\,K^-(2,+1)_{3\pi} - \sqrt{2/5}\,\bar{K}^0(2,0)_{3\pi}. \end{aligned} \tag{130}$$

The term involving $\bar{K}^0$ corresponds to K_1^0 decay and is of no interest to us here. Omitting this term, and using (129), (127), and (128), we have

$$\begin{aligned} \psi(I=2) &= \sqrt{3/5}\,K^-\,\{D\,\psi_D + E\psi_E\} \\ &= \sqrt{3/5}\,K^-\,\{1/2\,D - \sqrt{1/12}\,E)\,[++-) - (00+)] \\ &\qquad - (1/2\,D + \sqrt{1/12}\,E)\,[+-+) - (0+0)] \\ &\qquad + \sqrt{1/3}\,E\,[(-++) - (+00)]\}\,. \end{aligned} \tag{131}$$

Finally we combine (131) and (126) to write

$$\psi = \psi(I=1) + \psi(I=2).$$

We can now pick out the coefficients for (++-), etc., and write the intensities. From (131), (126), and the above equation, and including the usual factor of 2 for K_2^0 decay, we have

$$\begin{aligned} R(K^+ \to ++-) = &\left|\tfrac{1}{3}\,[\sqrt{2}\,(A_1+B_1) + (A_3+B_3)] + \tfrac{1}{2}\sqrt{1/5}\,(\sqrt{3}\,D-E)\right|^2 \\ &+ \left|\tfrac{1}{3}\,[\sqrt{2}\,(B_1+C_1) + (B_3+C_3)] + \sqrt{1/5}\;E\right|^2 \\ &+ \left|\tfrac{1}{3}\,[\sqrt{2}\,(C_1+A_1) + (C_3+A_3)] - \tfrac{1}{2}\sqrt{1/5}\,(\sqrt{3}\,D+E)\right|^2, \end{aligned} \tag{132}$$

$$R(K^+ \to +00) = \left|\frac{1}{3}\left[-\sqrt{2}\,A_1 - A_3 - \sqrt{1/5}\,E\right.\right|^2$$
$$+ \left|\frac{1}{3}\left[-\sqrt{2}\,B_1 - B_3 + \frac{1}{2}\sqrt{1/5}\,(\sqrt{3}\,D + E)\right|^2\right.$$
$$+ \left|\frac{1}{3}\left[-\sqrt{2}\,C_1 - C_3 - \frac{1}{2}\sqrt{1/5}\,(\sqrt{3}\,D - E)\right|^2\right., \qquad (133)$$

$$R(K_2^0 \to +-0) = 2\left\{\left|\frac{1}{3}(-A_1 + \sqrt{2}\,A_3)\right|^2 (1^2 + 1^2)\right.$$
$$+ \left|\frac{1}{3}(-B_1 + \sqrt{2}\,B_3)\right|^2 (1^2 + 1^2)$$
$$\left. + \left|\frac{1}{3}(-C_1 + \sqrt{2}\,C_3)\right|^2 (1^2 + 1^2)\right\}, \qquad (134)$$

and

$$R(K_2^0 \to 000) = 2\left\{\left|\frac{1}{3}[A_1 + B_1 + C_1 - \sqrt{2}\,(A_3 + B_3 + C_3)]\right|^2\right\}. \qquad (135)$$

Equations (132) through (135) are completely general for $\Delta I = 1/2$ and $3/2$. As a check we see that if we turn off the $\Delta I = 3/2$ decay, i.e., set $O = A_3 = B_3 = C_3 = D = E$, we then get back our original equations (110) through (113).

In order to simplify the equations, we now make two assumptions. (We will later be able to verify that these were good assumptions.)

<u>Assumption I.</u> Assume that $\Delta I = 1/2$ dominates. That is, neglect quadratic terms in A_3, B_3, C_3, D, and E, but keep linear terms in these quantities.

Assumption II. Assume that the dominant terms-that is $\Delta I = 1/2$-go completely to the symmetric $I = 1$ state, but that $\Delta I = 3/2$ is completely free in this respect.

Assumption II is motivated by the good agreement of (116) with experiment. According to assumption II we have

$$A_1 = B_1 = C_1 \,. \tag{136}$$

We choose units so that

$$A_1 \equiv 1 \,.$$

Next, expand Eqs. (132) through (135), dropping the quadratic terms according to assumption I. It is easy to see by inspection of (132) and (133) that if $A_1 = B_1 = C_1 = 1$, then the linear terms in D and E cancel identically, and in addition the linear terms in A_3, B_3, and C_3 occur only in the combination

$$A_3 + B_3 + C_3 \equiv 3a_3 . \tag{137}$$

We thus find (neglecting quadratic terms),

$$R(K^+ \to ++-) = 1/9\,[24 + 24\sqrt{2}\ \mathrm{Re}\ a_3] \,, \tag{138}$$

$$R(K^+ \to +00) = 1/9\,[\ 6 + \ 6\sqrt{2}\ \mathrm{Re}\ a_3] \,, \tag{139}$$

$$R(K_2^0 \to +-0) = 1/9\,[12 - 24\sqrt{2}\ \mathrm{Re}\ a_3] \,, \tag{140}$$

and

$$R(K_2^0 \to 000) = 1/9\,[18 - 36\sqrt{2}\ \mathrm{Re}\ a_3] \,. \tag{141}$$

We see that we have $R(K^+ \to ++-) = 4R(K^+ \to +00)$, and $R(K_2^0 \to 000) = 3/2\ R(K_2^0 \to +-0)$; i.e., Eqs. (116) and (117) still hold! However, Eq. (114) does not hold. We thus see that there is no incompatibility between the good agreement of experiment with (116), and the poor agreement of the experimental results (121) and (122) with (114), provided $\Delta I = 3/2$ is present.

We must still verify that the $\Delta I = 3/2$ terms are small, to justify our neglect of quadratic terms. From Eqs. (139) and (140) we have

$$\sqrt{2}\,\mathrm{Re}\,\alpha_3 = \frac{2R(K^+ \to +00) - R(K_2^0 \to +-0)}{4\,R(K^+ \to +00) + R(K_2^0 \to +-0)} ,$$

$$\to \frac{2\,R(K^+ \to +00) - 0.97\,R(K_2^0 \to +-0)}{4\,R(K^+ \to +00) + 0.97\,R(K_2^0 \to +-0)} . \qquad (142)$$

Putting in the experimental values from (121) and (122) we find

$$\mathrm{Re}\,\alpha_3 = +0.136 \pm 0.053 . \qquad (143)$$

This is to be compared with $A_1 \equiv 1$.

We conclude from (143) that the neglect of quadratic terms is justified. Furthermore, we see that the amount of $\Delta I = 3/2$ needed to satisfy the experiments is small. In fact, by comparision of Eq. (143) with Eq. (55) we see that the ratio of the amplitude for $\Delta I = 3/2$ to that for 1/2 that is required in $K \to 3\pi$ is about the "same" as that required in $K \to 2\pi$ to explain the existence of $K^+ \to \pi^+ + \pi^0$. Thus the ratio of the experimental results (121) and (122) is not actually in disagreement with the $\Delta I = 1/2$ rule, but rather is "expected," from the well-known inexactness of the rule.

We now turn to the question of the possible presence of $\Delta I = 5/2$ and 7/2. The $\Delta I = 5/2$ decay can lead to 3π states with $I = 2$ or $I = 3$. The $I = 2$ state cannot be reached by K_2^0 but only by K_1^0. Therefore the $\Delta I = 5/2$ spurion equation relating K^+ and K_0 [analogous to the $\Delta I = 3/2$ equation (130)] is of no interest for $I = 2$. We need only the K^+ amplitude. Aside from normalization, we get the same answer as when we considered $I = 2$ in $K^+ \to 3\pi$ via $\Delta I = 3/2$. There we found that if the $I = 2$ amplitude is small (compared to the $I = 1$ amplitude from $\Delta I = 1/2$), so that quadratic terms are negligible,

and if the $\Delta I = 1/2$ amplitude goes to the symmetric $I = 1$ state, then the counting rates are not affected. This conclusion still holds. (This means that $I = 2$ final states are difficult to detect.)

The $\Delta I = 7/2$ decay can lead only to the 3π state with $I = 3$. Thus we need consider only $I = 3$, from $\Delta I = 5/2$ and $7/2$ transitions. Both K_2^0 and K^+ can go to $I = 3$, so that the spurion relations are important. These are given by Table IV. We find

$$s(5/2, +1/2) = 1/2\,(\overline{K}) \times 3(3\pi)$$

$$= \sqrt{4/7}\; K^-(3,1)_{3\pi} - \sqrt{3/7}\; \overline{K}^0(3,0)_{3\pi}, \qquad (144)$$

and

$$s(7/2, +1/2) = 1/2\,(\overline{K}) \times 3(3\pi)$$

$$= \sqrt{3/7}\; K^-(3,1)_{3\pi} + \sqrt{4/7}\; \overline{K}^0(3,0)_{3\pi}\,. \qquad (145)$$

We find the 3π states in the usual way. There is only one state with $I = 3$, given by $1(\pi) \times 2(2\pi)$. From Table VI (and Table V),

$$(3,+1)_{3\pi} = \sqrt{1/15}\;\pi^-(2,2)_{2\pi} + \sqrt{8/15}\;\pi^0(2,1)_{2\pi} + \sqrt{6/15}\;\pi^+(2,0)_{2\pi}$$

$$= \sqrt{1/15}\;\pi^-\pi^+\pi^+ + \sqrt{8/15}\;\pi^0\{\sqrt{1/2}\;\pi^+\pi^0 + \sqrt{1/2}\;\pi^0\pi^+\}$$

$$+ \sqrt{6/15}\;\pi^+\{\sqrt{1/6}\;\pi^+\pi^- + \sqrt{2/3}\;\pi^0\pi^0 + \sqrt{1/6}\;\pi^-\pi^+\}$$

$$= \sqrt{1/15}\;\{(-++) + (+-+) + (++-)\}$$

$$+ \sqrt{4/15}\;\{(+00) + (0+0) + (00+)\}\;; \qquad (146)$$

$$
\begin{aligned}
(3,0)_{3\pi} &= \sqrt{1/5}\,\pi^+ (2,-1)_{2\pi} + \sqrt{3/5}\,\pi^0 (2,0)_{2\pi} + \sqrt{1/5}\,\pi^- (2,1)_{2\pi} \\
&= \sqrt{1/5}\,\pi^+ \{\sqrt{1/2}\,\pi^0\pi^- + \sqrt{1/2}\,\pi^-\pi^0\} \\
&\quad + \sqrt{3/5}\,\pi^0 \{\sqrt{1/6}\,\pi^+\pi^- + \sqrt{4/6}\,\pi^0\pi^0 + \sqrt{1/6}\,\pi^-\pi^+\} \\
&\quad + \sqrt{3/5}\,\pi^- \{\sqrt{1/2}\,\pi^+\pi^0 + \sqrt{1/2}\,\pi^0\pi^+\} \\
&= \sqrt{1/10}\,\{(+0-) + (+-0) + (0+-) + (0-+) + (-+0) + (-0+)\} \\
&\quad + \sqrt{4/10}\,(000). \qquad (147)
\end{aligned}
$$

We now associate the complex numbers F_5 and G_7 with $s(5/2, 1/2)$ and $s(7/2, 1/2)$, and write, for ψ, (omitting $\Delta I = 3/2$),

$$
\begin{aligned}
\psi &= s(1/2, 1/2) + F_5\, s(5/2, 1/2) + G_7\, s(7/2, 1/2) \\
&= \sqrt{2/3}\, K^- V_1^+ - \sqrt{1/3}\, \bar{K}^0 V_1^0 \\
&\quad + F_5 \{\sqrt{4/7}\, K^- (3,1)_{3\pi} - \sqrt{3/7}\, \bar{K}^0 (3,0)_{3\pi}\} \\
&\quad + G_7 \{\sqrt{3/7}\, K^- (3,1)_{3\pi} + \sqrt{4/7}\, \bar{K}^0 (3,0)_{3\pi}\}\,.
\end{aligned}
$$

For V_1^+ and V_1^0 we use Eqs. (107) and (108), with $A = B = C = 1$ (assumption that symmetric $I = 1$ dominates for $\Delta I = 1/2$). Using Eqs. (146) and (147) and collecting common terms, we have

$$
\begin{aligned}
\psi = K^- \{ & (\tfrac{2\sqrt{2}}{3} + \sqrt{\tfrac{4}{105}}\, F_5 + \sqrt{\tfrac{3}{105}}\, G_7)\,[(++-) + (+-+) + (-++)] \\
& + (-\tfrac{\sqrt{2}}{3} + 2\sqrt{\tfrac{4}{105}}\, F_5 + 2\sqrt{\tfrac{3}{105}}\, G_7)\,[(+00) + (0+0) + (00+)]\} \\
& - \bar{K}^0 \{(\tfrac{1}{3} + \sqrt{\tfrac{3}{70}}\, F_5 - \sqrt{\tfrac{4}{70}}\, G_7)\,[(0+-) + (0-+) + (+0-) + (-0+) + (+-0) + (-+0)] \\
& + (-1 + 2\sqrt{\tfrac{3}{70}}\, F_5 - 2\sqrt{\tfrac{4}{70}}\, G_7)\,(000)\}\,. \qquad (148)
\end{aligned}
$$

Finally we can write down the counting rates, keeping only linear terms in F_5 and G_5. We now include the terms due to $\Delta I = 3/2$, as appearing in Eqs. (138) through (141). We thus obtain from (148) the relative counting rates

$$R(K^+ \to ++-) = 1/9\,[24 + 24\sqrt{2}\ \mathrm{Re}\ \alpha_3 + 36\sqrt{2}\ \mathrm{Re}\ (\sqrt{\tfrac{4}{105}}\ F_5 + \sqrt{\tfrac{3}{105}}\ G_7)], \quad (149)$$

$$R(K^+ \to +00) = 1/9\,[\,6 + 6\sqrt{2}\ \mathrm{Re}\ \alpha_3 - 36\sqrt{2}\ \mathrm{Re}(\sqrt{\tfrac{4}{105}}\ F_5 + \sqrt{\tfrac{3}{105}}\ G_7)]\,, \quad (150)$$

$$R(K_2^0 \to +-0) = 1/9\,[12 - 24\sqrt{2}\ \mathrm{Re}\ \alpha_3 + 72\ \mathrm{Re}\ (\sqrt{\tfrac{3}{70}}\ F_5 - \sqrt{\tfrac{4}{70}}\ G_7)]\,, \quad (151)$$

and

$$R(K_2^0 \to 000) = 1/9\,[18 - 36\sqrt{2}\ \mathrm{Re}\ \alpha_3 - 72\ \mathrm{Re}\ (\sqrt{\tfrac{3}{70}}\ F_5 - \sqrt{\tfrac{4}{70}}\ G_7)]\,. \quad (152)$$

By inspection of (149) and (150) we see that for the ratio $R(K^+ \to ++-)/R(K^+ \to +00)$ to equal 4, in agreement with experiment [following Eq. (118)], we need, within rather small experimental errors,

$$\mathrm{Re}(\sqrt{\tfrac{4}{105}}\ F_5 + \sqrt{\tfrac{3}{105}}\ G_7) = 0. \quad (153)$$

The most reasonable conclusion is that

$$F_5 = 0, \text{ and } G_7 = 0. \quad (154)$$

The unlikely possibility that the result (153) is due to an accidental cancelation, i.e.,

$$\sqrt{3}\ G_7 = -\sqrt{4}\ F_5\,, \quad (155)$$

can be checked by measurement of the ratio $R(K_2^0 \to 000)/R(K_2^0 \to +-0)$. According to (151) and (152) this ratio should be 3/2 (plus phase-space corrections) if (154) holds, but not if (155) holds. The present experimental results are consistent with (154) but are not accurate enough to rule out (155). [See the discussion following Eq. (119).]

In summary, the evidence from $K \to 3\pi$ branching ratios indicates

(a) Dominance of $\Delta I = 1/2$.

(b) Dominance of the symmetric $I = 1$ state.

(c) Roughly that amount of $\Delta I = 3/2$ going to $I = 1$ expected from $K^+ \to \pi^+\pi^0$.

(d) Negligible amounts of $\Delta I = 5/2$ and $7/2$ going to $I = 3$.

(e) Possibly small amounts of $I = 2$ in K^+ decay from $\Delta I = 3/2$ and $5/2$. These could be present to, say, 30% in the amplitude (relative to $\Delta I = 1/2$) and still be undetectable via K^+ branching ratios, since they give no effect in linear approximation, and the quadratic terms should give effects of $< 10\%$. (If $I = 2$ is present, the $\Delta I = 3/2$ and $5/2$ contributions can be separated only by comparing $K_1^0 \to 3\pi$ with $K^+ \to 3\pi$.)

Lecture V. THE $\Delta I = 1/2$ RULE FOR LEPTONIC K-DECAYS

We begin with a summary of the decays we have studied so far, and also the strangeness-conserving decays, by means of a Puppi diagram, which is constructed as follows. A decay, for instance $n \to p e^- \bar{\nu}$, is written in transposed form, $p\bar{n} \to e^+ \nu$. Then $p\bar{n}$ and $e^+\nu$ are called "vertices." Similarly, $\mu^+ \to e^+ \nu\bar{\nu}$ becomes $\mu^+\nu \to e^+\nu$, $\mu^- + p \to n + \nu$ becomes $p\bar{n} \to \mu + \nu$, $\Lambda \to pe^-\bar{\nu}$ becomes $p\bar{\Lambda} \to e^+\nu$. (We do not need to distinguish between ν_e and ν_μ.) A given vertex is characterized by its quantum numbers for the strongly interacting particles. Thus the $p\bar{n}$ vertex has the same quantum numbers as π^+, and $p\bar{\Lambda}$ the same as K^+. We therefore call these the π^+ and K^+ vertices. Transitions are assumed to occur between any pair of vertices. (With each vertex we may associate a "current." Then transitions between two vertices are due to interaction between the two currents.)

Until recently the four vertices $e^+\nu$, $\mu^+\nu$, π^+, and K^+ seemed sufficient to summarize all known decays. One had a Puppi tetrahedron. [In addition one has the charge-conjugate diagram.] In our discussion we will need two additional vertices. Since a Puppi hexagon may become unwieldy, we use a "Puppi Table." For each vertex we give the total charge Q, strangeness S, isotopic spin I, and its third component I_3, for the <u>strongly interacting particles only</u>. Thus Q = 0 for the $e^+\nu$ vertex (and so are S and I) since there are no strongly interacting particles. By this convention, Q is not conserved in $\pi^+ \to \mu + \nu$, although of course the total charge is conserved. For each vertex the baryon number is zero, and Q, I_3, and S are related through the famous formula

$$Q = I_3 + \frac{S}{2} . \tag{156}$$

The two additional vertices, needed in our later discussion, will be named the (3/2, 1/2) and (3/2, 3/2) vertices, after their (I, I_3) values. The table follows.

Vertex	Q	S	I	I_3	Particles
$e^+\nu$	0	0	0	0	$e^+\nu$
$\mu^+\nu$	0	0	0	0	$\mu^+\nu$
π^+	1	0	1	+1	π^+, $p\bar{n}$, $\Sigma^+\bar{\Lambda}$, $\sqrt{\frac{1}{2}}(\pi^+\pi^0 - \pi^0\pi^+)$, ...
K^+	1	+1	$\frac{1}{2}$	$+\frac{1}{2}$	K^+, $p\bar{\Lambda}$, $\sqrt{\frac{2}{3}}\, n\bar{\Sigma}^- - \sqrt{\frac{1}{3}}\, p\bar{\Sigma}^0$, $\sqrt{\frac{2}{3}} K^0\pi^+ - \sqrt{\frac{1}{3}} K^+\pi^0$, ...
$(\frac{3}{2}, +\frac{1}{2})$	1	+1	$\frac{3}{2}$	$+\frac{1}{2}$	$\sqrt{\frac{1}{3}}\, n\bar{\Sigma}^- + \sqrt{\frac{2}{3}}\, p\bar{\Sigma}^0$, $\sqrt{\frac{1}{3}}\, K^0\pi^+ + \sqrt{\frac{2}{3}}\, K^+\pi^0$, ...
$(\frac{3}{2}, +\frac{3}{2})$	1	-1	$\frac{3}{2}$	$+\frac{3}{2}$	$\bar{n}\Sigma^+$, $\bar{K}^0\pi^+$, ...

Puppi Table

The first three vertices take care of neutron β decay, π decay, μ decay, and μ capture, and (for example) predict $\Sigma^+ \to \Lambda e^+\nu$. (An example of this decay has recently been reported by Block et al.[2])

The K^+ vertex is certainly present, since $K^+ \to \mu^+\nu$ occurs. Transitions between the K^+ and π^+ vertices can give only $\Delta I = 1/2$ and 3/2. (We have seen that both $\Delta I = 1/2$ and 3/2 are present but that $\Delta I = 1/2$ dominates, in non-leptonic decays of strange particles.) If either of the two I = 3/2 vertices is present, transitions to the π^+ vertex can occur with $\Delta I = 1/2$, 3/2, or 5/2.

Transitions between the (3/2, 3/2) vertex, which has S = -1, and the K^+ vertex (S = +1) can lead to decays with $\Delta S = 2$. For instance $\Xi^- \to n\pi^-$ can take place via

$$K^+ = \Xi^- \Lambda \to \bar{K}^0 \pi^+ = (3/2, 3/2),$$

or transposing,

$$\Xi^- \to \Lambda K^0 \pi^- = n\pi^-. \tag{157}$$

(Here the "equals" sign represents a strong reaction that conserves the quantum numbers of the vertex, and the arrow represents the weak reaction.) Since the decay $\Xi \to n\pi$ has not yet been observed, there has been good reason to assume the (3/2, 3/2) vertex to be absent.

Another argument (by Okun) against the existence of the (3/2, 3/2) vertex is provided by the smallness of the observed $K_1^0 - K_2^0$ mass difference. The transition $(3/2, 3/2) \leftrightarrow (1/2, 1/2)$ allows $\bar{K}^0\pi^+ = (3/2, 3/2) \to (1/2, 1/2) = K^0\pi^+$ in first order; i.e., $K^0 \leftrightarrow \bar{K}^0$ "rapidly." Since we have $K^0 = \frac{1}{\sqrt{2}}(K_1^0 + K_2^0)$ and $\bar{K}^0 = \frac{1}{\sqrt{2}}(K_1^0 - K_2^0)$, rapid transitions $K^0 \leftrightarrow \bar{K}^0$ would correspond to rapid change of the relative phase of K_1^0 and K_2^0, i.e. to rapid time variation in $\exp i(E_1 - E_2)t/\hbar$, and thus to a large mass difference $m_1 - m_2$. If the (3/2, 3/2) vertex is absent, $K^0 \leftrightarrow \bar{K}^0$ can only proceed in second order, via $K^0 \leftrightarrow \pi^+\pi^- \leftrightarrow \bar{K}^0$, leading to a "small" $K_1^0 - K_2^0$ mass difference, as seems to be observed.

The "$\Delta S/\Delta Q = +1$" rule, for leptonic decays of strange particles follows from the exclusion of the (3/2, 3/2) vertex. We see from the Puppi table that $\Delta S/\Delta Q = -1/-1 = +1$ for the leptonic decays $(1/2, 1/2) \to L^+\nu$ (L^+ means e^+ or μ^+), and for $(3/2, 1/2) \to L^+\nu$, but we have $\Delta S/\Delta Q = +1/-1 = -1$ for $(3/2, 3/2) \to L^+\nu$.

We now turn to the three-body leptonic decays $K \to \pi L \nu$. We have the three possibilities

$$K^+ \to \pi^0 L^+ \nu \quad , \tag{158}$$

$$K^0 \to \pi^- L^+ \nu \quad , \tag{159}$$

$$K^0 \to \pi^+ L^- \bar{\nu} \quad , \tag{160}$$

and the three reactions obtained from these by charge conjugation. The only possibilities are $\Delta I = 1/2$ or $3/2$ (for the strongly interacting particles, always).

We can transpose all particles to the left side of the equations, and add a spurion s to the right side, to conserve I and I_3 (as well as Q and S). Reactions (158) and (159) have $I_3 = +1/2$ for the spurion, so that the spurion can have I = 1/2 or 3/2. Using Table I we find the amplitudes

$$s(1/2, +1/2) = \bar{L}^+\bar{\nu}\ \{\sqrt{\tfrac{2}{3}}\ K^0\pi^+ - \sqrt{\tfrac{1}{3}}K^+\pi^0\} \tag{161}$$

and

$$s(3/2, +1/2) = \bar{L}^+\bar{\nu}\ \{\sqrt{\tfrac{1}{3}}\ K^0\pi^+ + \sqrt{\tfrac{2}{3}}\ K^+\pi^0\}\ . \tag{162}$$

(These correspond to the K^+ and (3/2, 1/2) vertices in the Puppi table.) Reaction (160) has $I_3 = +3/2$ for the spurion, so that the spurion must have I = 3/2. We then have

$$s(3/2, +3/2) = \bar{L}^-\nu\ K^0\pi^-\ . \tag{163}$$

corresponding to the (3/2, 3/2) vertex in the Puppi table. We define the complex numbers a_{11}, a_{31}, and a_{33} corresponding to s(1/2, 1/2), s(3/2, 1/2), and s(3/2, 3/2), and write

$$\begin{aligned}\psi \equiv\ & a_{11}\ s(1/2, 1/2) + a_{31}\ s(3/2, 3/2) + a_{33}\ s(3/2, 3/2)\\ =\ & \bar{L}^+\bar{\nu}\ [\sqrt{2/3}\ a_{11} + \sqrt{1/3}\ a_{31}]\ K^0\pi^+\\ &+ \bar{L}^+\bar{\nu}\ [-\sqrt{1/3}\ a_{11} + \sqrt{2/3}\ a_{31}]\ K^+\pi^0\\ &+ \bar{L}^-\nu\ a_{33}\ K^0\pi^-\ .\end{aligned} \tag{164}$$

Thus we have the transition amplitudes

$$a(K^+ \to \pi^0 L^+\nu) = -\sqrt{1/3}\ a_{11} + \sqrt{2/3}\ a_{31} \equiv a_+\ , \tag{165}$$

$$a(K^0 \to \pi^- L^+\nu) = \sqrt{2/3}\ a_{11} + \sqrt{1/3}\ a_{31} \equiv a\ , \tag{166}$$

and

$$a(K^0 \to \pi^+ L^-\nu) = a_{33} \equiv \bar{a}\ . \tag{167}$$

(The amplitude a_{33} corresponds to $\Delta S = -\Delta Q$.) Under the assumption of CP invariance we have

$$a(\bar{K}^0 \to \pi^+ L^-\bar{\nu}) = a(K^0 \to \pi^- L^+\nu) \equiv a \tag{168}$$

and

$$a(\bar{K}^0 \to \pi^- L^+\nu) = a(K^0 \to \pi^+ L^-\bar{\nu}) \equiv \bar{a}\ . \tag{169}$$

[Equations (168) and (169) are not completely obvious. See remarks following Eq. (176)]. Therefore for K_1^0 and K_2^0, since $K_{1,2}^0 = \frac{1}{\sqrt{2}}(K^0 \pm \bar{K}^0)$, we have

$$a(K_1^0 \to \pi^- L^+ \nu) = \frac{1}{\sqrt{2}}[a(K^0 \to \pi^- L^+ \nu) + a(\bar{K}^0 \to \pi^- L^+ \nu)]$$

$$= \frac{1}{\sqrt{2}}(a + \bar{a}), \tag{170}$$

$$a(K_1^0 \to \pi^+ L^- \nu) = \frac{1}{\sqrt{2}}[a(K^0 \to \pi^+ L^- \nu) + a(\bar{K}^0 \to \pi^+ L^- \nu)]$$

$$= \frac{1}{\sqrt{2}}[\bar{a} + a], \tag{171}$$

and similarly

$$a(K_2^0 \to \pi^- L^+ \nu) = \frac{1}{\sqrt{2}}(a - \bar{a}), \tag{172}$$

$$a(K_2^0 \to \pi^+ L^- \nu) = \frac{1}{\sqrt{2}}(\bar{a} - a). \tag{173}$$

Thus we have the rates

$$R(K_1^0 \to \pi^- L^+ \nu) = R(K_1^0 \to \pi^+ L^- \nu) = 1/2\ \left|\sqrt{\tfrac{2}{3}}\ a_{11} + \sqrt{\tfrac{1}{3}}\ a_{31} + a_{33}\right|^2, \tag{174}$$

$$R(K_2^0 \to \pi^- L^+ \nu) = R(K_2^0 \to \pi^+ L^- \nu) = 1/2\ \left|\sqrt{\tfrac{2}{3}}\ a_{11} + \sqrt{\tfrac{1}{3}}\ a_{31} - a_{33}\right|^2, \tag{175}$$

and

$$R(K^+ \to \pi^0 L^+ \nu) = \left|-\sqrt{\tfrac{1}{3}}\ a_{11} + \sqrt{\tfrac{2}{3}}\ a_{31}\right|^2. \tag{176}$$

Before examining the predictions of Eqs. (174), (175), and (176) we make some parenthetical remarks. First, time-reversal invariance requires that a_{11}, a_{31}, and a_{33} be all real, except for a common phase factor. (Final-state interactions are negligible here.)

Second, in Eqs. (168) and (169) we wish to invoke CP invariance, <u>not</u> C invariance. In order to have interfering amplitudes we must have exactly the same configuration of charges, momenta, and spins. In the K rest frame the configuration can be specified by giving the linear momenta,

$\vec{p}_i$ ($i = \pi, L, \nu$) and spins $\underline{\sigma}_i$ ($i = L, \nu$). Under P the spin is unchanged, but the $\vec{p}_i$ are reversed. Then we should write that from CP invariance,

$$a \equiv a(K^0 \to \pi^- L^+ \nu; \vec{p}_i; \vec{\sigma}_i) = a(\bar{K}^0 \to \pi^+ L^- \bar{\nu}; -\vec{p}_i; \vec{\sigma}_i),$$

and

$$\bar{a} \equiv a(\bar{K}^0 \to \pi^- L^+ \nu; \vec{p}_i; \vec{\sigma}_i) = a(K^0 \to \pi^+ L^- \bar{\nu}; -\vec{p}_i; \vec{\sigma}_i).$$

Then

$$a(K_1^0 \to \pi^- L^+ \nu; \vec{p}_i; \vec{\sigma}_i) = \frac{1}{\sqrt{2}}(a + \bar{a}),$$

$$a(K_2^0 \to \pi^- L^+ \nu; \vec{p}_i; \vec{\sigma}_i) = \frac{1}{\sqrt{2}}(a - \bar{a}),$$

$$a(K_1^0 \to \pi^+ L^- \nu; -\vec{p}_i; \vec{\sigma}_i) = \frac{1}{\sqrt{2}}(\bar{a} + a),$$

$$a(K_2^0 \to \pi^+ L^- \nu; -\vec{p}_i; \vec{\sigma}_i) = \frac{1}{\sqrt{2}}(\bar{a} - a).$$

Finally, then, Eqs. (174) and (175) should read

$$R(K_1^0 \to \pi^- L^+ \nu; \vec{p}_i; \vec{\sigma}_i) = R(K_1^0 \to \pi^+ L^- \bar{\nu}; -\vec{p}_i; \vec{\sigma}_i) = \frac{1}{2}|a + \bar{a}|^2,$$

and

$$R(K_2^0 \to \pi^- L^+ \nu; p_i; \vec{\sigma}_i) = R(K_2^0 \to \pi^+ L^- \bar{\nu}; -p_i; \vec{\sigma}_i) = \frac{1}{2}|a + \bar{a}|^2.$$

These equations should actually be modified once more. Since $\vec{p}_\pi + \vec{p}_L + \vec{p}_\nu = 0$, $\vec{p}_\pi$, $\vec{p}_L$, and $\vec{p}_\nu$ cannot form a pseudoscalar, and the entire configuration $-\vec{p}_\pi$, $-\vec{p}_L$, $-\vec{p}_\nu$ can be rotated until it coincides with $\vec{p}_\pi$, $\vec{p}_L$, $\vec{p}_\nu$. (This is allowed since the K spin is zero.) In this rotation the spins are also reversed. Thus we have $R(K_1^0 \to \pi^- L^+ \nu; \vec{p}_i; \vec{\sigma}_i) = R(K_1^0 \to \pi^+ L^- \bar{\nu}; \vec{p}_i; -\vec{\sigma}_i)$ and similarly for K_2^0 decay. Thus the <u>spectra</u> for $K_2^0 \to \pi^- L^+ \nu$ and $K_2^0 \to \pi^+ L^- \bar{\nu}$ are the same, and our use of (168) and (169) is justified, as long as we do not measure spins.

Next we consider the predictions of Eqs. (174), (175), and (176). We first sum over both signs of charge and let

$$R(K_1^0 \to \pi^- L^+ \nu) + R(K_1^0 \to \pi^+ L^- \nu) \equiv \Gamma_1, \tag{177}$$

$$R(K_2^0 \to \pi^- L^+ \nu) + R(K_2^0 \to \pi^+ L^- \nu) \equiv \Gamma_2, \tag{178}$$

and

$$R(K^+ \to \pi^0 L^+ \nu) \equiv \Gamma_+. \tag{179}$$

The predictions become

$$\Gamma_1 = |\sqrt{2/3}\ a_{11} + \sqrt{1/3}\ a_{31} + a_{33}|^2 \equiv |a + \bar{a}|^2, \tag{180}$$

$$\Gamma_2 = |\sqrt{2/3}\ a_{11} + \sqrt{1/3}\ a_{31} - a_{33}|^2 \equiv |a - \bar{a}|^2, \tag{181}$$

$$\Gamma_+ = |-\sqrt{1/3}\ a_{11} + \sqrt{2/3}\ a_{31}|^2 \equiv |a_+|^2 . \tag{182}$$

The predictions for some special cases follow:

1. <u>Pure $\Delta I = 1/2$ Rule.</u> (Includes $\Delta S = +\Delta Q$ rule.)

We have $a_{11} \neq 0$, $a_{31} = a_{33} = 0$. Then

$$\Gamma_1 = \Gamma_2 = 2\Gamma_+ . \tag{183}$$

2. <u>$\Delta S = +\Delta Q$ Rule.</u> (Without $\Delta I = 1/2$ rule.)

We have $a_{33} = 0$, $a_{11} \neq 0$, $a_{31} \neq 0$. Then

$$\Gamma_1 = \Gamma_2 \neq 2\Gamma_+ . \tag{184}$$

3. <u>No $\Delta I = 1/2$ Rule.</u> (For three-body decay.)

By this we mean $a_{11} = 0$; $a_{31} \neq 0$, $a_{33} \neq 0$. At first sight we might expect that the existence of $K^+ \to \mu + \nu$ would guarantee $a_{11} \neq 0$, since we can write $(K^+\pi^0)_{I=1/2} = K^+ \to \mu + \nu$, where the "equals" sign corresponds to a strong reaction. But conservation of angular momentum and parity forbids the strong reaction in this case. (Of course there are other possibilities.) Thus we should not assume, a priori, that $a_{11} \neq 0$, for three-body decay.

The No $\Delta I = 1/2$ rule is easily seen to lead to a quadratic relation between the counting rates, namely

$$(\Gamma_1 - \Gamma_2)^2 = 4\ \Gamma_+(\Gamma_1 + \Gamma_2 - \Gamma_+). \tag{185}$$

If we let

$$x \equiv \Gamma_1/\Gamma_2 \tag{186}$$

and

$$y \equiv \Gamma_+/\Gamma_2 , \tag{187}$$

then (185) becomes

$$x = 1 + 2y \pm \sqrt{1 + 8y} . \tag{188}$$

4. No (3/2, 1/2) Rule.

We mean $a_{31} = 0$, $a_{11} \neq 0$, $a_{33} \neq 0$. Then we have the quadratic relation

$$(\Gamma_1 - \Gamma_2)^2 = 16\, \Gamma_+(\Gamma_1 + \Gamma_2 - 4\Gamma_+)\,, \tag{189}$$

which is equivalent to

$$x = 1 + 8y \pm 2\sqrt{8y}\ . \tag{190}$$

5. Takeda Rule.

The intermediate-boson scheme of Takeda[12] allows all of a_{11}, a_{31}, and a_{33} to be nonzero, but imposes the constraint

$$a_{33} = \sqrt{3}\ \ a_{31}\ . \tag{191}$$

[Eq. (191) is equivalent to Eq. (48) of Takeda's paper. However, Takeda's Eq. (48) has a typographical error--the factor $(1/3)^{-1/2}$ should be replaced by $(1/3)^{+1/2}$. (Private communication from G. Takeda.)]. If we insert formula (191) into Eq. (181) and compare the result with Eqs. (182) and (180) we find the predictions

$$\Gamma_1 \neq \Gamma_2 = 2\Gamma_+\ . \tag{192}$$

Remarkably, one of the predictions--namely $\Gamma_2 = 2\Gamma_+$--coincides with a prediction of the pure $\Delta I = 1/2$ rule. [See Eq. (183).]

We now turn to the experiments. The K^+ rates are obtained by combining branching ratios from emulsions and bubble chambers, and the K^+ lifetime from counter experiments.[13] The combined rates for $K^+ \to e^+ \pi^0 \nu$ and $\mu^+ \pi^0 \nu$ give

$$\Gamma_+ (e^+, \mu^+) = (8.25 \pm 0.59) \times 10^6\ \mathrm{sec}^{-1}. \tag{193}$$

The rates for K_1^0 and K_2^0 are obtained as follows. Suppose one has a number N of K^0 produced at time t = 0, by means of a reaction like

$$K^+ + n \to K^0 + p \tag{194}$$

or

$$\pi^- + p \to K^0 + \Lambda\ . \tag{195}$$

At t = 0 we have, for $\psi(t)$, the wave function in the rest system of the neutral K-meson,

$$\psi(0) = |K^0\rangle = \frac{|K_1^0\rangle + |K_2^0\rangle}{\sqrt{2}} .$$

For $t > 0$ we must include the oscillating time-dependent factor $\exp(-iE_1 t/\hbar) \equiv \exp(-im_1 t)$, and the decay factor $\exp(-\lambda_1 t/2)$, in the K_1^0 amplitude, and a similar factor for K_2^0, to get

$$|\psi(t) = \frac{1}{\sqrt{2}} |K_1^0 \quad \exp(-im_1 t - \lambda_1 t/2) + \frac{1}{\sqrt{2}} |K_2^0 \quad \exp(-im_2 t - \lambda_2 t/2).$$

We now calculate the time-dependent amplitude for decay into $\pi^- L^+ \nu$ and $\pi^+ L^- \nu$, using Eqs. (170) to (173), to obtain

$$a(\pi^- L^+ \nu) = \langle K_1^0 | \psi(t) \rangle \; a(K_1^0 \to L^+) + \langle K_2^0 | \psi(t) \rangle \; a(K_2^0 \to L^+)$$

$$= \frac{\exp(-im_1 t - \lambda_1 t/2)}{\sqrt{2}} \frac{(a + \bar{a})}{\sqrt{2}} + \frac{\exp(-im_2 t - \lambda_2 t/2)}{\sqrt{2}} \frac{(a - \bar{a})}{\sqrt{2}} .$$

Similarly,

$$a(\pi^+ L^- \bar{\nu}) = \frac{\exp[-im_1 t - \lambda_1 t/2]}{\sqrt{2}} \frac{(\bar{a} + a)}{\sqrt{2}} + \frac{\exp[-im_2 t - \lambda_2 t/2]}{\sqrt{2}} \frac{(\bar{a} - a)}{\sqrt{2}} .$$

The decay rate is given by the absolute square, so that the two decay rates (corresponding to a single K^0 at t = 0) are

$$R(L^\pm) = 1/4 \; \{ |a + \bar{a}|^2 \exp(-\lambda_1 t) + |a - \bar{a}|^2 \exp(-\lambda_2 t)$$
$$\pm 2(|a|^2 - |\bar{a}|^2) \exp[-(\lambda_1 + \lambda_2)t/2] \cos \Delta m t \}, \quad (196)$$

where the + and - signs in the cross term go with L^+ and L^-, respectively. In the cross term we have set equal to zero a term proportional to $\sin(\Delta m t) \, \mathrm{Im}\, \bar{a}^* a$. Time-reversal invariance requires a and $\bar{a}$ to have a common phase factor, so that $\bar{a}^* a$ is real, and $\mathrm{Im}\, \bar{a}^* a$ vanishes.

We see from Eq. (196) that at t = 0,

$$R(L^\pm) = 1/4 \; \{ |a + \bar{a}|^2 + |a - \bar{a}|^2 \pm 2(|a|^2 - |\bar{a}|^2) \} , \quad (197)$$

i.e., $R(L^+) = |a|^2$, $R(L^-) = |\bar{a}|^2$.

Thus the ratio $R(L^-)/R(L^+)$ at $t = 0$ gives the ratio $|\bar{a}|^2/|a|^2$.

If one adds the rates for L^+ and L^-, the cross term in Eq. (194) cancels, and one obtains

$$R(L^+) + R(L^-) = 1/2\ |a+\bar{a}|^2 \exp(-\lambda_1 t) + 1/2\ |a-\bar{a}|^2 \exp(-\lambda_2 t)$$
$$= 1/2\ \Gamma_1 \exp(-\lambda_1 t) + 1/2\ \Gamma_2 \exp(-\lambda_2 t). \qquad (198)$$

Thus one can obtain Γ_1 and Γ_2 by studying the time dependence of $R(L^+)+R(L^-)$, without any knowledge of $m_1 - m_2 \equiv \Delta m$. In this case, however, the result is unchanged under the interchange of a and $\bar{a}$, as is evident from Eq. (198).

Ely et al.,[14] using K^0 produced by K^+ in propane through reaction (191), have studied the time dependence of decays into $\pi^- e^+ \nu$ and $\pi^+ e^- \nu$, using both Eqs. (196) and (198). They find, in disagreement with Eqs. (183) or (184),

$$\frac{\Gamma_1(e^\pm)}{\Gamma_2(e^\pm)} = 11.9^{+7.5}_{-5.6}. \qquad (199)$$

This is in bad agreement with the prediction of the $\Delta S = +\Delta Q$ rule. (They are not able to find the absolute rate for Γ_1 or Γ_2, since their sample is highly selected, so no comparison can be made with Γ_+.)

Alexander et al.,[10] using K^0 produced via reaction (195) in the 72-inch hydrogen chamber have studied the time dependence of $(\pi^\pm e^\mp \nu)+(\pi^\pm \mu^\mp \nu)$. No separation of charges was made, so that Eq. (198) was used. Combining the decays into $e^\pm$ and $\mu^\pm$, they find

$$\frac{\Gamma_1(e^\pm, \mu^\pm)}{\Gamma_2(e^\pm, \mu^\pm)} = 6.6^{+6.0}_{-4.0}. \qquad (200)$$

They also measure the absolute K_2^0 rates, and find

$$\Gamma_2(e^\pm, \mu^\pm) = (9.31 \pm 2.49) \times 10^6\ \text{sec}^{-1}. \qquad (201)$$

This is accomplished by using decays with sufficiently long K^0 flight time to insure that the K_1^0 have completely decayed ($\tau_1 \approx 0.9 \times 10^{-10}$ sec) but the K_2^0 have not ($\tau_2 \approx 7 \times 10^{-8}$ sec.). Comparing (201) with (193) we see that the prediction, $\Gamma_2 = 2\Gamma_+$, of the $\Delta I = 1/2$ rule, or of the Takeda rule, is not satisfied.

Crawford et al.[15] used K^0 produced via reaction (195) in the 10-inch hydrogen chamber. They found

$$\frac{\Gamma_1(e^\pm, \mu^\pm)}{\Gamma_2(e^\pm, \mu^\pm)} = 3.5^{+3.9}_{-2.7} \quad . \tag{202}$$

The chamber was too small to get rid of K_1^0 by attenuation in time, so that to measure $\Gamma_2(e^\pm, \mu^\pm)$ they had to assume a value for Γ_1/Γ_2. They assumed $\Gamma_1 = \Gamma_2$ [this is not in disagreement with (202)] and found

$$\Gamma_2(e^\pm, \mu^\pm) = 20.4^{+7.2}_{-5.6} \times 10^6 \text{ sec}^{-1}, \tag{203}$$

if $\Gamma_1 = \Gamma_2$. If instead one assumes $\Gamma_1/\Gamma_2 = 9$, [this is taken as a compromise between (199) and (200)] one obtains from the same experiment

$$\Gamma_2(e^\pm, \mu^\pm) = (8.5 \pm 2.8) \times 10^6 \text{ sec}^{-1}. \tag{204}$$

This agrees well with the result (201) of Alexander et al. (whose result does not depend on Γ_1/Γ_2), and poorly with the prediction of the $\Delta I = 1/2$ rule, or the Takeda rule.

Let us next see whether the "No $\Delta I = 1/2$ rule" can be ruled out. We want to test Eq. (188). The K_2^0 experiment of Alexander et al.,[10] combined with the K^+ results of other experiments,[13] gives, from (201) and (193),

$$\begin{aligned} y &= \Gamma_+(e^+, \mu^+) / \Gamma_2(e^\pm, \mu^\pm) \\ &= (8.25 \pm 0.59)/(9.3 \pm 2.49) = 0.89 \pm 0.24 \ . \end{aligned} \tag{205}$$

We insert this into (188) to <u>predict</u> (if $a_{11} = 0$),

$$\begin{aligned} x &= 1 + 2(.89) \pm \sqrt{1 + 8(.89)} \\ &= 2.78 \pm 2.85 \\ &= (5.63 \pm 0.83) \text{ or } (0 \pm 0.15) \ , \end{aligned} \tag{206}$$

where we have included the statistical errors in the last step. The prediction (206) of the "No $\Delta I = 1/2$ rule" is to be compared with Eq. (200), the value obtained by Alexander et al., namely $x = 6.6 \pm 5$. We see that the No $\Delta I = 1/2$ rule (for three-body leptonic decays) cannot be ruled out by the present experimental data.

We next test the "No (3/2, 1/2) rule," through its prediction (190), which becomes, according to (205),

$$\begin{aligned} x &= 1 + 8(0.89) \pm 2\sqrt{8(0.89)} \\ &= 8.1 \pm 5.3 \\ &= (13.4 \pm 2.7) \text{ or } (2.8 \pm 1.2), \end{aligned} \tag{207}$$

where experimental errors are only included in the final step. Neither of these predictions can be said to be in strong disagreement with the experimental result (200).

It is clear that more data are needed, to find the relative amounts of a_{11}, a_{31}, and a_{33}, and to see whether universality holds between e and μ.

Lastly, we must remark that part of our discussion has been oversimplified. Equations (180), (181), and (182) should be interpreted as giving the counting rates for a <u>specified configuration</u> of all the momenta and spins. Then a_{11}, a_{31}, and a_{33} are not constants, but are complicated functions of the configuration variables, the function depending on the dynamics of the decay. The comparisons of experiment with the predictions of the "pure $\Delta I = 1/2$ rule" [Eq. (183)], the "$\Delta S = \Delta Q$ rule" [Eq. (184)], and the "Takeda rule" [Eq. (192)] are not affected by the fact that we have suppressed information on the spectra, since these predictions are such that they refer both to a given configuration, and to the total decay rates, and in fact to the sum over e and μ decays. However, predictions (188) and (190), of the "No $\Delta I = 1/2$ rule" and the "No (3/2, 1/2) rule", while they do hold for a <u>given configuration</u>,

are not applicable to the total decay rates, nor to a sum of e and μ modes. The reason is that these (quadratic) predictions do not involve simple ratios. Therefore the "predictions" (206) and (207), and subsequent comparison with the experimental result (200), would make sense only if the form factors involved in a_{11}, a_{31}, and a_{33} were all the same function of the configuration and furthermore were the same for e and μ decay. Thus, only to the extent that the spectra correspond to phase space alone—and to the extent that we neglect the μ-e mass difference!—can the comparison of (206) and (207) with (200) be justified.

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TABLE I

$1 \times \frac{1}{2}$

	$(\frac{3}{2}, \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2})$	$(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, -\frac{3}{2})$
$(1, 1)(\frac{1}{2}, \frac{1}{2})$	1			
$(1, 1)(\frac{1}{2}, -\frac{1}{2})$		$\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$		
$(1, 0)(\frac{1}{2}, \frac{1}{2})$		$\sqrt{\frac{2}{3}}$ $-\sqrt{\frac{1}{3}}$		
$(1, 0)(\frac{1}{2}, -\frac{1}{2})$			$\sqrt{\frac{2}{3}}$ $\sqrt{\frac{1}{3}}$	
$(1, -1)(\frac{1}{2}, \frac{1}{2})$			$\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{2}{3}}$	
$(1, -1)(\frac{1}{2}, -\frac{1}{2})$				1

TABLE II

$\frac{3}{2} \times 1$

	$(\frac{5}{2}, \frac{5}{2})$	$(\frac{5}{2}, \frac{3}{2})(\frac{3}{2}, \frac{3}{2})$	$(\frac{5}{2}, \frac{1}{2})(\frac{3}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2})$	$(\frac{5}{2}, -\frac{1}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, -\frac{1}{2})$	$(\frac{5}{2}, -\frac{3}{2})(\frac{3}{2}, -\frac{3}{2})$	$(\frac{5}{2}, -\frac{5}{2})$
$(\frac{3}{2}, \frac{3}{2})(1, 1)$	1					
$(\frac{3}{2}, \frac{3}{2})(1, 0)$		$\sqrt{\frac{2}{5}}$ $\sqrt{\frac{3}{5}}$				
$(\frac{3}{2}, \frac{1}{2})(1, 1)$		$\sqrt{\frac{3}{5}}$ $-\sqrt{\frac{2}{5}}$				
$(\frac{3}{2}, \frac{3}{2})(1, -1)$			$\sqrt{\frac{1}{10}}$ $\sqrt{\frac{2}{5}}$ $\sqrt{\frac{1}{2}}$			
$(\frac{3}{2}, \frac{1}{2})(1, 0)$			$\sqrt{\frac{3}{5}}$ $\sqrt{\frac{1}{15}}$ $-\sqrt{\frac{1}{3}}$			
$(\frac{3}{2}, -\frac{1}{2})(1, 1)$			$\sqrt{\frac{3}{10}}$ $-\sqrt{\frac{8}{15}}$ $\sqrt{\frac{1}{6}}$			
$(\frac{3}{2}, \frac{1}{2})(1, -1)$				$\sqrt{\frac{3}{10}}$ $\sqrt{\frac{8}{15}}$ $\sqrt{\frac{1}{6}}$		
$(\frac{3}{2}, -\frac{1}{2})(1, 0)$				$\sqrt{\frac{3}{5}}$ $-\sqrt{\frac{1}{15}}$ $-\sqrt{\frac{1}{3}}$		
$(\frac{2}{3}, -\frac{3}{2})(1, 1)$				$\sqrt{\frac{1}{10}}$ $-\sqrt{\frac{2}{5}}$ $\sqrt{\frac{1}{2}}$		
$(\frac{3}{2}, -\frac{1}{2})(1, -1)$					$\sqrt{\frac{3}{5}}$ $\sqrt{\frac{2}{5}}$	
$(\frac{3}{2}, -\frac{3}{2})(1, 0)$					$\sqrt{\frac{2}{5}}$ $-\sqrt{\frac{3}{5}}$	
$(\frac{3}{2}, -\frac{3}{2})(1, -1)$						1

TABLE III

$2 \times \frac{1}{2}$

	$(\frac{5}{2}, \frac{5}{2})$	$(\frac{5}{2}, \frac{3}{2})$ $(\frac{3}{2}, \frac{3}{2})$	$(\frac{5}{2}, \frac{1}{2})$ $(\frac{3}{2}, \frac{1}{2})$	$(\frac{5}{2}, -\frac{1}{2})$ $(\frac{3}{2}, -\frac{1}{2})$	$(\frac{5}{2}, -\frac{3}{2})$ $(\frac{3}{2}, -\frac{3}{2})$	$(\frac{5}{2}, -\frac{5}{2})$
$(2, 2)(\frac{1}{2}, \frac{1}{2})$	1					
$(2, 2)(\frac{1}{2}, -\frac{1}{2})$		$\sqrt{\frac{1}{5}}$ $\sqrt{\frac{4}{5}}$				
$(2, 1)(\frac{1}{2}, \frac{1}{2})$		$\sqrt{\frac{4}{5}}$ $-\sqrt{\frac{1}{5}}$				
$(2, 1)(\frac{1}{2}, -\frac{1}{2})$			$\sqrt{\frac{2}{5}}$ $\sqrt{\frac{3}{5}}$			
$(2, 0)(\frac{1}{2}, \frac{1}{2})$			$\sqrt{\frac{3}{5}}$ $-\sqrt{\frac{2}{5}}$			
$(2, 0)(\frac{1}{2}, -\frac{1}{2})$				$\sqrt{\frac{3}{5}}$ $\sqrt{\frac{2}{5}}$		
$(2, -1)(\frac{1}{2}, \frac{1}{2})$				$\sqrt{\frac{2}{5}}$ $-\sqrt{\frac{3}{5}}$		
$(2, -1)(\frac{1}{2}, -\frac{1}{2})$					$\sqrt{\frac{4}{5}}$ $\sqrt{\frac{1}{5}}$	
$(2, -2)(\frac{1}{2}, \frac{1}{2})$					$\sqrt{\frac{1}{5}}$ $-\sqrt{\frac{4}{5}}$	
$(2, -2)(\frac{1}{2}, \frac{1}{2})$						1

TABLE IV

$3 \times \frac{1}{2}$

	$(\frac{7}{2},\frac{7}{2})$	$(\frac{7}{2},\frac{5}{2})$ $(\frac{5}{2},\frac{5}{2})$	$(\frac{7}{2},\frac{3}{2})$ $(\frac{5}{2},\frac{3}{2})$	$(\frac{7}{2},\frac{1}{2})$ $(\frac{5}{2},\frac{1}{2})$	$(\frac{7}{2},-\frac{1}{2})(\frac{5}{2},-\frac{1}{2})$	$(\frac{7}{2},-\frac{3}{2})(\frac{5}{2},-\frac{3}{2})$	$(\frac{7}{2},-\frac{5}{2})(\frac{5}{2},-\frac{5}{2})$	$(\frac{7}{2},-\frac{7}{2})$
$(3,3)(\frac{1}{2},\frac{1}{2})$	1							
$(3,3)(\frac{1}{2},-\frac{1}{2})$		$\sqrt{\frac{1}{7}}$ $\sqrt{\frac{6}{7}}$						
$(3,2)(\frac{1}{2},\frac{1}{2})$		$\sqrt{\frac{6}{7}}$ $-\sqrt{\frac{1}{7}}$						
$(3,2)(\frac{1}{2},-\frac{1}{2})$			$\sqrt{\frac{2}{7}}$ $\sqrt{\frac{5}{7}}$					
$(3,1)(\frac{1}{2},\frac{1}{2})$			$\sqrt{\frac{5}{7}}$ $-\sqrt{\frac{2}{7}}$					
$(3,1)(\frac{1}{2},-\frac{1}{2})$				$\sqrt{\frac{3}{7}}$ $\sqrt{\frac{4}{7}}$				
$(3,0)(\frac{1}{2},\frac{1}{2})$				$\sqrt{\frac{4}{7}}$ $-\sqrt{\frac{3}{7}}$				
$(3,0)(\frac{1}{2},-\frac{1}{2})$					$\sqrt{\frac{4}{7}}$ $\sqrt{\frac{3}{7}}$			
$(3,-1)(\frac{1}{2},\frac{1}{2})$					$\sqrt{\frac{3}{7}}$ $-\sqrt{\frac{4}{7}}$			
$(3,-1)(\frac{1}{2},-\frac{1}{2})$						$\sqrt{\frac{5}{7}}$ $\sqrt{\frac{2}{7}}$		
$(3,-2)(\frac{1}{2},\frac{1}{2})$						$\sqrt{\frac{2}{7}}$ $-\sqrt{\frac{5}{7}}$		
$(3,-2)(\frac{1}{2},-\frac{1}{2})$							$\sqrt{\frac{6}{7}}$ $\sqrt{\frac{1}{7}}$	
$(3,-3)(\frac{1}{2},\frac{1}{2})$							$\sqrt{\frac{1}{7}}$ $-\sqrt{\frac{6}{7}}$	
$(3,3)(\frac{1}{2},-\frac{1}{2})$								1

TABLE V

1×1

	(2, 2)	(2, 1)(1, 1)	(2, 0)(1, 0)(0, 0)	(2, -1)(1, -1)	(2, -2)
(1, 1)(1, 1)	1				
(1, 1)(1, 0) (1, 0)(1, 1)		$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$			
(1, 1)(1, -1) (1, 0)(1, 0) (1, -1)(1, 1)			$\sqrt{\frac{1}{6}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$ 0 $-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{3}}$		
(1, 0)(1, -1) (1, -1)(1, 0)				$\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{2}}$ $-\sqrt{\frac{1}{2}}$	
(1, -1)(1, -1)					1

TABLE VI

2×1

	(3, 3)	(3, 2)(2, 2)	(3, 1)(2, 1)(1, 1)	(3, 0)(2, 0)(1, 0)	(3, -1)(2, -1)(1, -1)	(3, -2)(2, -2)	(3, -3)
(2, 2)(1, 1)	1						
(2, 2)(1, 0)		$\sqrt{\frac{1}{3}}$ $\sqrt{\frac{2}{3}}$					
(2, 1)(1, 1)		$\sqrt{\frac{2}{3}}$ $-\sqrt{\frac{1}{3}}$					
(2, 2)(1, -1)			$\sqrt{\frac{1}{15}}$ $\sqrt{\frac{1}{3}}$ $\sqrt{\frac{3}{5}}$				
(2, 1)(1, 0)			$\sqrt{\frac{8}{15}}$ $\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{3}{10}}$				
(2, 0)(1, 1)			$\sqrt{\frac{6}{15}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{10}}$				
(2, 1)(1, -1)				$\sqrt{\frac{1}{5}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$			
(2, 0)(1, 0)				$\sqrt{\frac{3}{5}}$ 0 $-\sqrt{\frac{2}{5}}$			
(2, -1)(1, 1)				$\sqrt{\frac{1}{5}}$ $-\sqrt{\frac{1}{2}}$ $\sqrt{\frac{3}{10}}$			
(2, 0)(1, -1)					$\sqrt{\frac{6}{15}}$ $\sqrt{\frac{1}{2}}$ $\sqrt{\frac{1}{10}}$		
(2, -1)(1, 0)					$\sqrt{\frac{8}{15}}$ $-\sqrt{\frac{1}{6}}$ $-\sqrt{\frac{3}{10}}$		
(2, -2)(1, 1)					$\sqrt{\frac{1}{15}}$ $-\sqrt{\frac{1}{3}}$ $\sqrt{\frac{3}{5}}$		
(2, -1)(1, -1)						$\sqrt{\frac{2}{3}}$ $\sqrt{\frac{1}{3}}$	
(2, -2)(1, 0)						$\sqrt{\frac{1}{3}}$ $-\sqrt{\frac{2}{3}}$	
(2, -2)(1, -1)							1

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