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Von Neumann's Informal Hidden-Variable Argument

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Von Neumann was convinced that the randomness observed in quantum mechanical systems is inherent in them, and not due to an ignorance of additional random variables unspecified in the quantum formalism. His formal "proof" of the nonexistence of these hidden-variables,[1] however, relied on overly restrictive assumptions concerning their nature, and thus must be considered unacceptable.[2]

For historical perspective, Wigner[3] has recently described in this journal the informal argument which motivated Von Neumann to his conviction. Also presented is Schrödinger's objection to his reasoning, but in a manner which misleads the reader into concluding that the objection is untenable. It is the purpose of the present note to show that Schrödinger's objection is valid, and that Von Neumann's motivating agrument is also unacceptable.[4]

Von Neumann's Informal Argument

Von Neumann's argument[5] concerns successive measurements of different spin components of a spin - 1/2 particle, with the assumption that the result is determined by a hidden-variable (or set of variables).

It may be stated briefly as follows:

1. A single measurement yielding a given sign will restrict the range of values which the hidden-variable(s) had before the measurement.

2. The restriction will be present after the conclusion of the measurement, otherwise successive measurements of the same component would not yield the same result.

3. A subsequent measurement operation of a different spin component will further restrict this range;

and

4. A sufficiently large number of these operations will allow the production of a state for which the spin components have a definite sign in all directions.

5. The resultant state will violate the predictions of the quantum theory, and no such violations have been observed.

Schrödinger's Objection

Schrödinger objected to Von Neumann's reasoning. He suggested that a later measurement, while further restricting the range of the hidden-variable(s) may restore a range blocked by an earlier measurement. He thus felt that such a restoration allowed the predictions of quantum mechanics for a spin-1/2 particle to be achieved by a hidden-variable theory.

Von Neumann and Wigner counter with two assertions.[5] First they claim that such a restoration "...presupposed the existence of hidden-variables in the apparatus used for the measurement." Second they assert that the existence of these hidden-variables still allows the generation of a state with well defined spin components in all directions. Thus they believe that they have refuted Schrödinger's objection.

In this note both of these assertions are demonstrated to be false. A trivial counter-example is provided which accomplishes Schrödinger's restoration without requiring the existence of hidden-variables in the apparatus. The model is capable of duplicating the predictions of quantum mechanics for an arbitrary series of spin component measurements of a spin-1/2 particle. Obviously then since the existence of hidden-variables in the apparatus is unnecessary for the measurement operation, the second assertion in likewise untrue, as the apparatus may choose simply to ignore their existence.

Counter-Example

Consider an ensemble of spin-1/2 particles which are polarized along the direction $\underset{\sim}{p}$. The polarization direction is characteristic of, and carried by every member of this ensemble. Assume that each member of the ensemble also hasa hidden-variable which is the unit vector $\underset{\sim}{\lambda}$, and that $\underset{\sim}{\lambda}$ has initially a uniform probability distribution over the hemisphere $\underset{\sim}{\lambda} \cdot \underset{\sim}{p} \geq 0$.

Next consider an apparatus which measures the spin component along the unit vector $\underset{\sim}{a}$. The action of the apparatus is two fold. First it must be sensitive to the information conveyed to it by the particle (in this case $\underset{\sim}{\lambda}$ and $\underset{\sim}{p}$), and from this information determine a binary result $A(\underset{\sim}{a}, \underset{\sim}{p}, \underset{\sim}{\lambda}) = \pm 1$. Second it must prepare the state for future measurements, without the use of any additional random variables intrinsic to the apparatus.

Construction of a model for the first part of this operation is straight forward, and has already been done by Bell[6,7]. Define

$$\Theta \equiv \cos^{-1} (\underset{\sim}{a} \cdot \underset{\sim}{p}),$$

and construct a new vector $\underset{\sim}{a}'$ in the plane of $\underset{\sim}{a}$ and $\underset{\sim}{p}$, defined so that

$$\Theta' \equiv \cos^{-1} (\underset{\sim}{a}' \cdot \underset{\sim}{p}) = \frac{\pi}{2} (1 - \cos\Theta)$$

as shown in Figure 1.

Now specify the result of the measurement to be

$$A(\underset{\sim}{a}, \underset{\sim}{p}, \underset{\sim}{\lambda}) = \text{sign}(\underset{\sim}{\lambda} \cdot \underset{\sim}{a}').$$

Averaging over $\underset{\sim}{\lambda}$ yields the expectation value

$$\langle m_{\underset{\sim}{p}} = +1/2 \mid \underset{\sim}{\sigma} \cdot \underset{\sim}{a} \mid m_{\underset{\sim}{p}} = +1/2 \rangle = 1 - \frac{2\Theta'}{\pi} = \cos\Theta$$

in agreement with the predictions of quantum mechanics.

The preparation of the new state for a subsequent measurement must now be done. We shall consider the case of a measurement apparatus that passes only particles for which the result of the measurement is $A = +1$. All of these have $\underset{\sim}{\lambda}$ within the intersection of the two hemispheres $\underset{\sim}{\lambda} \cdot \underset{\sim}{p} \geq 0$ and $\underset{\sim}{\lambda} \cdot \underset{\sim}{a}' \geq 0$. Define ϕ to be the azimuthal angle of $\underset{\sim}{\lambda}$ referenced from $\underset{\sim}{p}$ about the

$\underset{\sim}{p} \times \underset{\sim}{a}$ axis (see Figure 1), and prescribe that the measurement apparatus rotate $\underset{\sim}{\lambda}$ about the $\underset{\sim}{p} \times \underset{\sim}{a}$ axis, keeping the polar angle fixed, to a new azimuthal angle given by

$$\phi' = \frac{\phi}{1 - \Theta'/\pi} + \Theta - \frac{\Theta'}{2(1 - \Theta'/\pi)} .$$

By doing so the initial phase space is mapped on to the hemisphere $\underset{\sim}{\lambda} \cdot \underset{\sim}{a} \geq 0$. Finally prescribe that the apparatus define $\underset{\sim}{p}' = \underset{\sim}{a}$ as the new polarization direction after the measurement operation.

The above deterministic proceedure assures that the distribution of $\underset{\sim}{\lambda}$ after the measurement will be uniform over the hemisphere $\underset{\sim}{\lambda} \cdot \underset{\sim}{p}' \geq 0$. Thus the new hidden-variable distribution will be identical to that before the measurement, only rotated to the new orientation in the direction of $\underset{\sim}{p}' = \underset{\sim}{a}$.

A second measurement following a similar set of prescriptions for the direction $\underset{\sim}{b}$ will then yield the expectation value

$$< m_{\underset{\sim}{p}'} = +1/2 \;|\; \underset{\sim}{\sigma} \cdot \underset{\sim}{b} \;|\; m_{\underset{\sim}{p}'} = +1/2 > = \underset{\sim}{b} \cdot \underset{\sim}{p}' .$$

again in agreement with the predictions of quantum mechanics. Nowhere in our example is there any need of external (apparatus) hidden-variables.

Conclusions

The above trivial example serves to demonstrate that a hidden-variable theory is capable of yielding the predictions of quantum mechanics for an arbitrary series of measurements of different spin components of a spin-1/2 particle. Thus Von Neumann's informal argument is also invalid, as well as his formal one.

Von Neumann's original intention was to show that for all systems, the quantum mechanical predictions are substantially different from those of a hidden-variable theory. This apparently is not so in general. However, Bell's analysis of the peculiar case of a two spin-1/2 particle system[4, 6] shows that in this special case the quantum theory and any local hidden-variable theory will yield observably different predictions. Unfortunately, these predictions are as yet experimentally untested.[8]

Bell's theorem then should be regarded not as a supplement to Von Neumann's old ideas, but as a basis for new and important experimental predictions. It is hoped that this note will correct earlier misleading statements, and put these arguments in their proper historical perspective.

References

* This work supported in part by the U.S. Atomic Energy Commission.

1. J. Von Neumann, Mathematische Grundlagen der Quantenmechanik (Springer-Verlag, Berlin 1932) [English translation: Mathematical Foundations of Quantum Mechanics, (Princeton University Press, Princeton, New Jersey 1955)]
2. J. S. Bell, Rev. Mod. Phys. 38, 447 (1966), D. Bohm and J. Bub, Rev. Mod. Phys. 38, 453 (1966)
3. E. P. Winger, Amer. J. Phys. 38, 1005 (1970)
4. An important additional criticism of Wigner's paper is also warranted with regard to his conclusions concerning Bell's theorem. He suggests that Bell's inequality is generally applicable to the theory of quantum mechanical measurements.

   To this end he makes two claims. First he states that Bell's reasoning can be applied to the states of any two component systems, in particular the composite states of object plus apparatus in an idealized quantum mechanical measurement. Second he claims that Bell's use of the singlet state of two spin-1/2 particles was for demonstration purposes only, and otherwise irrelevant to the argument.

   However, both of these claims can be shown to be incorrect. It is readily apparent that the locality assumption, inherent in Bell's argument, while emminently reasonable for the two spin case, is unjustifiable for any quantum object plus measuring apparatus. The object and apparatus will generally occupy the same position during the measurement operation, and thus may be free to simultaneously exchange information. Hence the locality assumption is highly unreasonable in this application.

   Moreover, a careful analysis will show that the two spin-1/2 particle system is in fact crucial to Bell's argument. In addition to the locality assumption's ready applicability, there is the more important consideration

References (cont.)

that the quantum mechanical predictions for this system violate the inequality! Indeed many composite systems which do satisfy the locality condition do not do this (e.g. in angular correlation experiments). Thus quantum mechanics and a local hidden-variable theory need not yield different predictions for such systems.

5. The form of Von Neumann's informal argument referred to in this article is that as related by Wigner in reference 3.

6. J. S. Bell, Physics 1, 195 (1965)

7. Indeed models have been published earlier which duplicate the predictions of quantum mechanics for a succession of spin component measurements. Notable of these efforts was that of D. Bohm and J. Bub, reference 2. Their model was not used in the present discussion for two reasons. First it includes a dynamical description of the wave function collapse. The associated finite collapse time may yield predictions at variance with quantum theory. Second the elements of the theory are given no physical significance. A more geometrically based model is presented here for sake of clarity and ease of visualization.

8. J. F. Clauser, Bul. Am. Phys. Soc. 14, 578 (1969); J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Letters 23, 880 (1969).

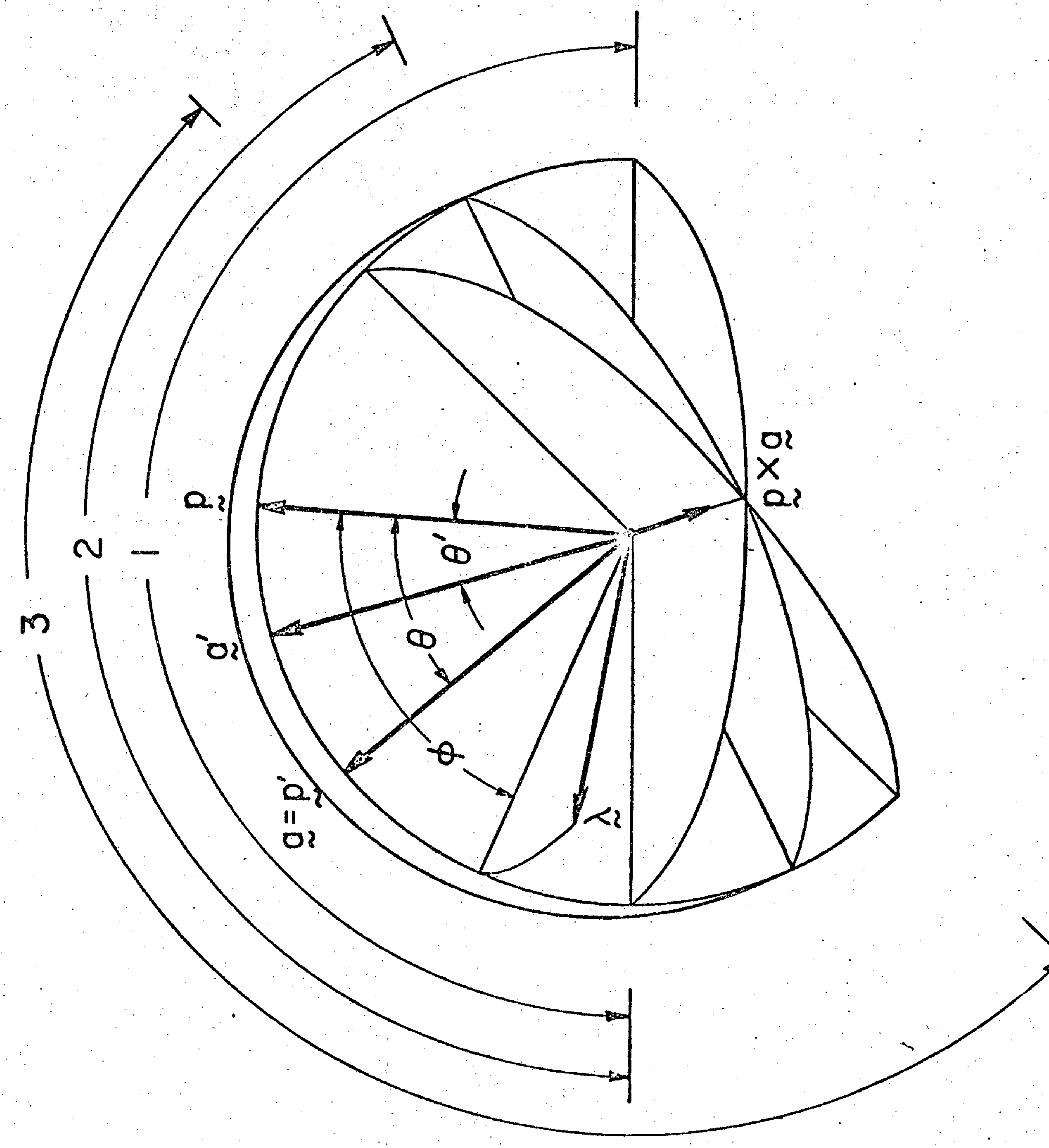
Figure Caption

Figure 1: Hidden variable phase space :

(1) Initial domain of λ .

(2) Portion of initial domain for which $A = +1$

(3) Final domain of λ .


3
2
1
p
q'
q = p'
θ'
θ
φ
λ
p × q

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