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Uniformization of Bounded Domains and Rigidity of Bergman Metrics

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Uniformization of Bounded Domains and Rigidity of Bergman Metrics

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by

Ruoyi Wang

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I am grateful to my advisor, without whose help, I would not have been here.

To my parents for all the support.

ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy, Graduate Program in Mathematics
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Dr. Bun Wong, Chairperson

This dissertation investigates rigidity in several complex variables through the study of intrinsic metrics on bounded domains in $\mathbb{C}^n$. In particular, it focuses on geometric and analytic characterizations of the unit ball using the Carathéodory, Kobayashi, and Bergman metrics. These metrics are biholomorphically invariant and play a central role in understanding the complex geometry of domains independently of coordinate representations. The work is motivated by classical results of Lempert and Lu, as well as more recent developments concerning strongly pseudoconvex domains and curvature rigidity. The first main contribution concerns the Kähler structure of the Carathéodory metric on bounded strongly pseudoconvex domains. Lempert proved that on bounded convex domains the Carathéodory and Kobayashi metrics coincide, and an important consequence is that the Kobayashi metric is Kähler if and only if the domain is biholomorphic to the unit ball. More recently, Zimmer extended this rigidity phenomenon to strongly pseudoconvex domains by showing that if the Kobayashi metric is Kähler, then the universal cover of the domain is biholomorphic to the unit ball. In this dissertation, we establish an analogous character-

ization using the Carathéodory metric. Our approach builds upon the theory of complex geodesics developed by Lempert and later generalized by Bracci, Fornæss, and Wold. A key ingredient is the existence and behavior of complex geodesics near strongly pseudoconvex boundary points. By analyzing the relationship between extremal maps and intrinsic metrics, we prove that the Kähler condition for the Carathéodory metric imposes strong geometric rigidity on the domain. This result deepens the connection between intrinsic metrics and biholomorphic classification and provides a new perspective on metric rigidity phenomena in several complex variables. The second principal contribution generalizes Lu's classical uniformization theorem for the Bergman metric. Lu proved that if a bounded domain in $\mathbb{C}^n$ has a complete Bergman metric with constant negative holomorphic sectional curvature, then the domain must be biholomorphic to the unit ball. In this dissertation, we weaken the global curvature assumption and show that local constancy of the holomorphic sectional curvature is sufficient to obtain the same conclusion. Central to the proof is the theory of Bergman representative coordinates, which provides a local biholomorphic linearization of the domain and allows curvature information to be transferred effectively. By exploiting these coordinates and comparing the Bergman and Carathéodory metrics, we demonstrate that local geometric data can determine the global biholomorphic type of a domain. As an application, we extend observations of Cheung and Wong by relaxing convexity assumptions to strong pseudoconvexity in certain settings. In addition to these main results, the dissertation surveys foundational material on intrinsic metrics, curvature, and complex geodesics, and discusses broader connections to important open problems in complex geometry, including the Cheng–Yau conjecture and the Ramadanov conjecture. The results presented here suggest that comparisons among intrinsic metrics, especially through curvature and coincidence properties, may provide new tools for understanding rigidity and uniformization phenomena beyond the strongly pseudoconvex setting.

Contents

Chapter 1

Introduction

1.1 Overview

This thesis presents two main contributions to the field of several complex variables, encapsulated in Theorem 3.4.0.2 and Theorem 4.3.0.4. Theorem 3.4.0.2, which is the focus of Chapter 3, concerns the Kähler structure of intrinsic metrics on certain domains. Theorem 4.3.0.4, treated in Chapter 5, offers a generalization of Lu's classical uniformization theorem. In this introductory chapter, we provide the necessary background and discuss the significance of these results within the broader context of the field. We also outline the structure of the thesis.

We begin with a landmark result by Lempert [35], which illustrates the deep connection between intrinsic metrics on convex domains in $\mathbb{C}^n$.

Theorem 1.1.0.1 *For a bounded convex domain $\Omega \subset \mathbb{C}^n$, the Carathéodory metric and the Kobayashi metric coincide.*

An important consequence of Lempert's theorem is that the Kobayashi metric on Ω is *Kähler* if and only if Ω is biholomorphic to the unit ball $\mathbb{B}_n$. This result has had a substantial impact on the field of several complex variables, as it provides a powerful characterization of bounded domains through intrinsic geometric properties.

Intrinsic metrics—those that are biholomorphically invariant—serve as essential tools in complex geometry and analysis. They allow us to study complex manifolds in ways that are independent of specific coordinate representations. Several important intrinsic metrics exist in this setting, including the Carathéodory, Kobayashi, Bergman, and *Kähler* metrics. These metrics not only quantify complex-geometric structures but also encode significant curvature and convexity information about the domain.

Our first main result, Theorem 3.4.0.2, builds on recent work by Andrew Zimmer [21], who extended Lempert's ideas to the setting of strongly pseudoconvex domains. Zimmer established an analogue of Lempert's result, characterizing domains whose Kobayashi metric is Kähler and whose universal cover is biholomorphic to the unit ball. One key insight in his work—and in ours—is the role of complex geodesics near the boundary of such domains, particularly in light of the relation between the Kobayashi and Carathéodory metrics.

Lempert [36] originally showed that, for bounded strongly convex domains, extremal maps realizing the Carathéodory metric are complex geodesics. This result was later generalized to bounded strongly pseudoconvex domains by Bracci, Fornæss, and Wold [16]. Zimmer provided a more direct proof of the existence of complex geodesics near the boundary, which we revisit in Chapter 3 as Theorem 3.2.0.5. In this chapter, we also present classical theorems by Wong and Stanton (Theorems 3.3.0.3 and 3.3.0.2) that characterize the unit ball using the coincidence of the Carathéodory and Kobayashi metrics.

The second major contribution of this thesis is Theorem 4.3.0.4, which generalizes Lu's classical uniformization theorem concerning domains whose Bergman metric has constant negative holomorphic sectional curvature.

Theorem 1.1.0.2 (Lu's Uniformization Theorem) *Let Ω be a bounded domain in $\mathbb{C}^n$ with a complete Bergman metric B^2_Ω. If the holomorphic sectional curvature is equal to a negative constant $-c^2$, then Ω is biholomorphic to the unit ball $\mathbb{B}_n$ and $c^2 = \frac{2}{n+1}$.*

We refer to this foundational result as Lu's uniformization theorem. It has been generalized in various contexts and applied in several settings, including those discussed by Cheung and Wong [11]. A central tool in the proof of Lu's theorem is the Bergman representative coordinate system, which provides a local biholomorphic linearization of complex structures. In Chapter 4, we give a detailed exposition of this construction.

In our generalization (Theorem 4.3.0.4), we prove that the assumption of globally constant holomorphic sectional curvature can be weakened to a local condition. Specifically, we demonstrate that local constancy of curvature suffices to characterize the unit ball among bounded domains with a complete Bergman metric. This broadens the scope of Lu's original theorem and opens the door for further applications. Additional consequences and corollaries stemming from this result are presented in Chapter 4 as well.

The remainder of our work is structured as follows:

- **Chapter 2** provides an overview of several complex variables, with particular emphasis on intrinsic metrics and curvature. This chapter serves as background and contains no new results.

- **Chapter 3** presents Theorem 3.4.0.2. We begin with a discussion of relevant results and conjectures, including Zimmer's theorem 3.4.0.1, which inspires our work. The proof of Theorem 3.4.0.2 follows a similar strategy to Zimmer's but focuses on the Carathéodory metric rather than the Kobayashi metric. Several illustrative examples are provided.

- **Chapter 4** introduces Lu's original uniformization theorem 4.2.0.5 and provides a careful treatment of the Bergman representative coordinates, which are essential to the proof. We also present intermediate results comparing the Bergman and Carathéodory metrics.

- **Chapter 5** contains our proof of Theorem 4.3.0.4. We explain how our generalization is achieved using weaker curvature assumptions and discuss the implications of this broader characterization of the unit ball.

We now proceed with the preliminary material in Chapter 2.

Chapter 2

Background

2.1 Notation

We will denote the unit disk by

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

and the unit ball in $\mathbb{C}^n$ by

$$\mathbb{B}_n := \{z \in \mathbb{C}^n : |z| < 1\}.$$

Let $\Omega \subset \mathbb{C}^n$ be a domain and $f : \Omega \to \mathbb{C}^n$ be a holomorphic mapping $f(z) = (f_1(z), ..., f_n(z))$. We denote the differential map at p to be

$$df_p$$

and

$$||df_p||_{\mathbb{C}}$$

to be the determinant of the holomorphic Jacobian matrix $\frac{\partial(f_1,...,f_n)}{\partial(z_1,...,z_n)}$ at p.

On a complex manifold M, we have the holomorphic tangent bundle $T^{1,0}M$ with local frame $\{\frac{\partial}{\partial z_i}\}$ and the holomorphic cotangent bundle is then denoted to be $(T^{1,0}M)^*$. A section of $(T^{p,q}M)^* = \bigwedge^p (T^{1,0}M)^* \otimes \bigwedge^q \overline{(T^{0,1}M)^*}$ is then the $(p,q)-$form on the manifold M. We will denote

$$A^{p,q}(M)$$

to be the spaces of all $(p,q)-$forms. We then get the direct sum composition

$$A^r(M) = \bigoplus_{p+q=r} A^{p,q}M.$$

If E is a holomorphic vector bundle over complex manifold M, we will then denote

$$A^r(E)$$

to be be the space of all sections of $\bigwedge^p (T^{1,0}M)^* \otimes \bigwedge^q \overline{(T^{0,1}M)^*} \otimes E$, where $p+q=r$. We have the differentiation operator $d = \partial + \overline{\partial}$ and for any $(p,q)-$form ψ, $\partial\psi$ is a $(p+1,q)-$form and $\overline{\partial}\psi$ is a $(p,q+1)-$form.

For any complex valued function f in a local holomorphic coordinate, we will denote

$$\partial f = \sum_i^n \frac{\partial f}{\partial z_i} dz_i,$$

and

$$\overline{\partial} f = \sum_i^n \frac{\partial f}{\partial \overline{z}_i} dz_i.$$

2.2 Intrinsic and *Kähler* Metrics

We are mainly interested in the intrinsic metrics on complex manifolds for the property that they are invariant under biholomophism.

2.2.1 *Kähler* Metric

Let M be a complex manifold of dimension n with a Hermitian metric that assigns a Hermitian inner product h_p on the holomorphic vector bundle E of M for each $p \in \Omega$ such that h_p varies smoothly on p. Under a local frame $\{e_i\}$ of E, we may write $h_{i\bar{j}} = h(e_i, e_j)$ and $h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i d\overline{z_j}$ where $(z_1, ..., z_n)$ are local coordinates and $h = (h_{i\bar{j}})$ is an $n \times n$ matrix of smooth functions which is Hermitian symmetric and positively definite. If we take the real part $g = Re(h)$ of h, then g is a Riemannian metric on the underlying smooth manifold $M_{\mathbb{R}}$ of M. The imaginary part of h is important for the inherent complex structure: let $\omega_h = -\frac{1}{2} Im(h) = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \wedge d\overline{z_j}$ be the *Kähler* form of the metric h. It is a positive (1,1)-form independent of the choice of local coordinates. Let $\nabla : A^0(E) \to A^1(E)$ be the Hermitian connection of h that satisfies the Leibniz' rule:

$$\nabla(f\xi) = f\xi + df \otimes \xi, \forall f \in \mathcal{C}^\infty(M, \mathbb{C}), \forall \xi \in A^0(E).$$

It is the unique connection on the vector bundle E which is compatible with both h and the complex structure, i.e, for vector field $X \in E$ and $\xi, \zeta \in A^0(E)$

$$Xh(\xi, \zeta) = h(\nabla_X \xi, \zeta) + h(\xi, \nabla_{\overline{X}} \zeta),$$

and for the complexified tangent bundle $T_{\mathbb{C}}M = T^{1,0}M \bigoplus T^{0,1}M$ and splitting $\nabla = \nabla' + \nabla''$,

$$\nabla' = A^0(E) \to A^{1,0}(E)$$

$$\nabla'' = A^0(E) \to A^{0,1}(E).$$

In general, the Hermitian connection ∇ of metric h may not coincide with the Riemannian (Levi-Civita) connextion ∇^0 of the Riemannian metric $Re(h)$. The special case when those two connections agree is particularly important:

Definition 2.2.1.1 *A Hermitian metric h on M is called a Kähler metric if the and only if $\nabla = \nabla^0$.*

In other words, a *Kähler* metric on M is a Riemannian metric such that both the metric and the Riemannian connection are compatible with the complex structure J.

Proposition 2.2.1.2 *Let (M, h) be a Hermitian manifold. Then the follwoing are equivalent:*

1. *h is a Kähler metric.*
2. *$\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$ for $1 \leq i, j, k \leq n$ under any local holomorphic coordinate system.*
3. *$d\omega_h = 0$.*
4. *For any $p \in M$, there exists a local holomorphic coordinates $(z_1, ..., z_n)$ in a neighborhood of p such that $h_{i\bar{j}}(p) = \delta_{ij}, dh_{i\bar{j}}(p) = 0$. Such a coordinate is said to be normal at p.*

Proof. Let $(z_1, ..., z_n)$ be a local holomorphic coordinate in a neighborhood $U \subset M$ and $dz = (dz_1, .., dz_n)$ be the vector of the natural frame. By Cartan, the torsion form of h

$$\begin{aligned}
\tau_i &= \sum_{j,l=1}^{n} (\partial h_{j\bar{l}} \cdot h^{\bar{l}i}) \wedge dz_j \\
&= \sum_{l=1}^{n} (\sum_{j,k=1}^{n} \frac{\partial h_{j\bar{l}}}{\partial z_k} dz_k \wedge dz_j) h^{\bar{l}i} \\
&= \sum_{l=1}^{n} h^{\bar{l}i} \sum_{1 \leq j \leq k \leq n} (\frac{\partial h_{k\bar{l}}}{\partial z_j} - \frac{\partial h_{j\bar{l}}}{\partial z_k}) dz_j \wedge dz_k.
\end{aligned}$$

So h is torsion free, i.e., a *Kähler* metric if and only if $\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$. By the definition of $\omega_h = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i \wedge d\bar{z}_j$, we see that $d\omega_h = \frac{i}{2}(\sum_{i,j,k=1}^{n} \frac{\partial h_{i,j}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j + \sum_{i,j,k=1}^{n} \frac{\partial h_{i,j}}{\partial \bar{z}_k} dz_i \wedge$

$d\bar{z}_j \wedge d\bar{z}_k) = 0$ if and only if $\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}$. Then we have shown that (1), (2) and (3) are equivalent.

If (4) holds, then the torsion tensor of h will vanish at any given point $p \in M$, so h is $K\ddot{a}hler$. Conversely, Fix a point $p \in M$ with holomorphic local coordinate $(z_1, ..., z_n)$. With out loss of generality, we can assume p is the origin and $h_{i\bar{j}}(p) = \delta_{ij}, 1 \leq i, j \leq n$ with change of coordinate. By (2), the constant matrix $A^j_{ik} = \frac{\partial h_{i\bar{j}}}{\partial z_k}(p)$ is a symmetric matrix for each $i \leq j \leq n$. Define a new holomorphic coordinate $(z'_1, ..., z'_n)$ by $z'_j = z_j + \frac{1}{2}\sum_{i,k=1}^n A^j_{ik} z_i z_k$. Then the matrix of the metric with the new coordinate is given by $h' = B^{-1}h(B^{-1})^*$ with $B_{ij} = \delta_{ij} + \sum_{k=1}^n A^j_{ik} z_K$. It follows that $(dh')(p) = 0$. ■

We see that having a $K\ddot{a}hler$ metric on M is the same as having a closed global (1,1)-form on M. If h is a $K\ddot{a}hler$ metric on M, than $h|_N$ is also a $K\ddot{a}hler$ metric, where $N \subset M$ is a complex submanifold because $d(\omega|_N) = (d\omega)|_N = 0$

Example 2.2.1.3 *The usual Euclidean metric on* $\mathbb{C}^n$ *has the Kähler form* $\omega = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$. *We see that it is a Kähler metric.*

Example 2.2.1.4 *Consider the unit ball* $\mathbb{B}_n$. *The (1,1)-form* $\omega = -\frac{i}{2}\partial\overline{\partial}\log(1-|Z|^2) = \frac{i}{2}\sum_{i,j=1}^n \frac{(1-|z|^2)\delta_{ij}+\bar{z}_i z_j}{(1-|z|^2)^2} dz_i \wedge d\bar{z}_j$ *is closed. It defines that Bergman metric on the unit ball.*

2.2.2 Kobayashi Metric and *Carathéodory* Metric

We will start with the construction of Kobayashi metric [33]: Let $\Omega \subset \mathbb{C}^n$ be a domain or a complex manifold and $\mathbb{D}$ be the unit disk. The Kobayashi infinitestemal pseudometric on a is a real valued function $K_\Omega : T\Omega \to \mathbb{R}^+ \cup \{0\}$ such that

$$K_\Omega(p, v) := inf\{|\lambda|, f \in Hol(\mathbb{D}, \Omega), f(0) = p, df|_0(\lambda) = v\},$$

where $|\lambda|$ is measured by *Poincaré* metric $ds^2 = \frac{2}{(1-|z|^2)^2}(dzd\bar{z})$ on the unit disk. The Kobayashi metric is not a *Kähler* metric and is a complex Finsler metric defined by 2.5.0.6. It is only semi-continuous in general.

Example 2.2.2.1 *Let $\Omega = \mathbb{D}$ be the unit disk. We will start with calculating the Kobayashi metric at the origin 0. Let $v = 1 + 0i \in T_0\Omega$ and*

$$f : \mathbb{D} \to \mathbb{D}$$

be holomorphic and satisfies $f(0) = 0$. By Schwartz's Lemma, we know that $|f'(0)| < 0$. Since the identity map id maps $\mathbb{D}$ to Ω with $id(0) = 0$ and $id'(0) = 1$, we can conclude that 1 is the extremal value and with the Poincaré *metric at the origin is normalized to be 2 in our construction, we have*

$$K_\Omega(0, v) = 1.$$

Now let v be any vector of length 1 $v = e^{i\theta}$ for some $0 \leq \theta < 2\pi$. Then we see that

$$K_\Omega(0, v) = K_\Omega(0, e^{i\theta} \cdot 1) = |e^{i\theta}| K_\Omega(0, 1) = 2.$$

Then for vector v of any length $v = re^{i\theta}$, we get that

$$K_\Omega(0, v) = K_\Omega(0, re^{i\theta}) = rK_\Omega(0, e^{i\theta}) = r = 2|v|.$$

Next for any arbitrary point $z \in \mathbb{D}$, consider the Möbius transformation

$$\psi(p) = \frac{p - z}{1 - \overline{z}p}$$

that maps z to 0. What's more, we get that

$$\psi'(z) = \left.\frac{(1 - \overline{z}p) \cdot 1 - (p - z) \cdot (-\overline{z})}{(1 - \overline{z}p)^2}\right|_{z=p} = \frac{1}{1 - |z|^2}.$$

Therefore, we calculate the Kobayashi metric for any vector v to be

$$K_\Omega(z,v) = K_\Omega(\psi(z), \psi'(z)v) = K_\Omega(0, \frac{v}{1-|z|^2}) = \frac{1}{1-|z|^2}K_\Omega(0,v) = \frac{2|v|}{1-|z|^2}$$

and we see that it coincides with the Poincaré metric on the unit disk.

One of the most important properties of the Kobayashi metric is that its distance decreasing:

Proposition 2.2.2.2 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be complex domains and $f : \Omega_1 \to \Omega_2$ be a holomorphic mapping. Then $K_{\Omega_2}(f(p), df_p(v)) \leq K_{\Omega_1}(p,v)$.*

Proof. Since $f \circ g \in Hol(\mathbb{D}, \Omega_2)$ for $g \in Hol(\mathbb{D}, \Omega_1)$, this property is a direct result from the definition. ■

Then we can see why the Kobayashi metric is preserved under biholomorphism,i.e.,

Corollary 2.2.2.3 *Every biholomrphic mapping $f : \Omega_1 \to \Omega_2$ is an isometry in Kobayashi metric.*

Proof. Applying the distance decreasing property on f andf^{-1}, we can see that $K_{\Omega_1}(p,v) = K_{\Omega_1}(f^{-1} \circ f(p), d(f^{-1} \circ f)_p(v)) \leq K_{\Omega_2}(f(p), df_p(v)) \leq K_{\Omega_1}(p,v)$. ■

Other properties to notice about the Kobayashi metric are that it is upper semi-continuous [47] and it is a complex Finsler metric that $K_\Omega(p,v) = 0$ if and only if $v = 0$ and $K_\Omega(z, sv) = |s|K_\Omega(z,v)$ for some $s \in \mathbb{C}$. This observation follows from the fact that if $f(p) = g(sp)$, then $f'(p) = sg'(sp)$. We then wish to define a distance based on he elements of $Hol(\mathbb{D}, \Omega)$.

Definition 2.2.2.4 *Let $\Omega \subset \mathbb{C}^n$ be open and $p, q \in \Omega$. The Kobayashi Pseudo-distance is defined by*

$$k_\Omega(p,q) := \inf \sum_{i}^{N} l_\Omega(p_i, p_{i-1}),$$

for any partition $p_0 = p, ..., p_N = q$, where $l_\Omega := \inf\{d_\mathbb{D}(a,b), f \in Hol(\mathbb{D},\Omega), f(a) = p, f(b) = q\}$and $d_\mathbb{D}$ denotes the Poincaré distance on unit disk. The distance function is given by

$$d_\mathbb{D}(0,z) = \int_0^1 |\alpha'(t)|dt \text{ for } \alpha(t) = tz.$$

Royden have proved that the Kobayashi distance $k_\Omega(p,q)$ is the integrated form of K_Ω [47].

Theorem 2.2.2.5 *Let $\gamma : [0,1] \to \Omega$ be any differentiable curves joining $p, q \in \Omega$ and $\Gamma_\Omega(p,q)$ be all the piece-wise $\mathcal{C}^1$ curve connecting p and q, then we have*

$$k_\Omega(p,q) = \inf_{\gamma \in \Gamma_\Omega(p,q)} \int_0^1 K_\Omega(\gamma(t), \gamma'(t))dt.$$

Combining Royden's result and Proposition 2.2.2.2, we can conclude the distance decreasing property for the Kobayashi distance:

Proposition 2.2.2.6 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be complex domains and $f : \Omega_1 \to \Omega_2$ be a holomorphic mapping. Then $k_{\Omega_2}(f(p), f(q)) \leq k_{\Omega_1}(p,q)$.*

It is easy to verify that $k_\Omega(p,q) \geq 0$, $k_\Omega(p,q) = k_\Omega(q,p)$, and the triangle inequality $k_\Omega(p,q) + k_\Omega(q,r) \geq k_\Omega(p,r)$. However, the Kobayashi distance is not smooth in general by the given example [33].

Example 2.2.2.7 *Consider the Kobayashi distance on* $\mathbb{D} \times \mathbb{D}$*. then* $k_{\mathbb{D}\times\mathbb{D}}((p,p'),(q,q')) = \max\{k_{\mathbb{D}}(p,q), k_{\mathbb{D}}(p',q')\}$*, which is not* $\mathcal{C}^1$*. To show the statement is true, we may assume without loss of generality that* $p = p' = 0$ *and* $k_{\mathbb{D}}(0,q) \geq k_{\mathbb{D}}(0,q')$*. Let* $f : \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ *be a holomorphic mapping* $f(z) = (z, \frac{q'}{q}z)$*. Since* k_Ω *is distance decreasing, then we have* $k_{\mathbb{D}\times\mathbb{D}}((0,0),(q,q')) = k_{\mathbb{D}\times\mathbb{D}}(f(0), f(q)) \leq k_{\mathbb{D}}(0,q)$ *and by the projection map we have* $k_{\mathbb{D}\times\mathbb{D}}((0,0),(q,q')) \geq k_{\mathbb{D}}(0,q), k_{\mathbb{D}\times\mathbb{D}}((0,0),(q,q')) \geq k_{\mathbb{D}}(0,q')$*. More generally, if* $\mathbb{D}^n$ *denotes the n-dimensional poly disk, then* $k_{\mathbb{D}^n}((p_1,...,p_k)) = \max\{k_{\mathbb{D}}(p_i,q_i), 1 \leq i \leq n\}$*.*

Other than the Kobayashi metric, the *Carathéodory* metric also rises from the holomorphic maps between the domain and the unit disk. We define the infinitesimal form of the *Carathéodory* metric on a complex domain $\Omega \subset \mathbb{C}^n$ by $c_\Omega : T\Omega \to \mathbb{R}^+ \cup \{0\}$,

$$C_\Omega(p,v) = \sup\{|\lambda| : f \in Hol(\Omega,\mathbb{D}), f(p) = 0, df|_p(v) = \lambda\}$$

and the *Carathéodory* distance $c_\Omega(p,q) = \sup\{d_{\mathbb{D}}(f(p), f(q)) : f \in Hol(\Omega,\mathbb{D})\}$ with respect to *Poincaré* metric. Since $\mathbb{D}$ is a homogeneous domain, it suffices take the supremum over the subfamily $\mathcal{F} = \{f \in Hol(\Omega,\mathbb{D}) | f(p) = 0\}$. Applying the Arzela-Ascoli Theorem and the following proposition, we see that $\mathcal{F}$ is compact and the supremum in our definitions are achieved by some $f \in \mathcal{F}$.

Example 2.2.2.8 *A calculation nearly identical to example 2.2.2.1 shows that the Carathéodory metric coincides with the Kobayashi metric and Poincaré metric on the unit disk.*

Using similar argument for the Kobayashi metric, we can easily see that the *Carathéodory* metric (distance) is a complex Finsler metric and is distance decreasing:

Proposition 2.2.2.9 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be complex domains and $f : \Omega_1 \to \Omega_2$ be a holomorphic mapping. Then $C_{\Omega_2}(f(z), df_z(v)) \leq C_{\Omega_1}(z, v)$ and $c_{\Omega_2}(f(p), f(q)) \leq c_{\Omega_1}(p, q)$.*

However, unlike the Kobayashi metric, the *Carathéodory* metric behaves better in continuity and in general $c_\Omega(p, q) \neq \inf_{\gamma \in \Gamma_\Omega(p,q)} \int_0^1 C_\Omega(\gamma(t), \gamma'(t))dt$.

Theorem 2.2.2.10 *The Carathéodory metric C_Ω is continuous.*

Proof. Let $v_k \in T_{p_k}\Omega$ be a sequence of vectors converging to $v \in T_p\Omega$. As we have discussed, there exists $f \in Hol(\Omega, \mathbb{D})$ such that $C_\Omega(p, v) = |df_p(v)|$ and $f(p) = 0$. Then we have

$$C_\Omega(p, v) = |df_p(v)| = \lim |df_{p_k}(v_k)| \leq \lim C_\Omega(p_k, v_k).$$

Similarly, there exists $f_k \in Hol(\Omega, \mathbb{D})$ for each k such that $C_\Omega(p_k, v_k) = |(df_k)_{p_k}(v_k)|$ and $f_k(p_k) = 0$. Take a converging subsequence to get $\lim f_{k_n} = g$ so that

$$C_\Omega(p, v) \geq |dg_p(v)| = \lim |(df_{k_n})_{p_{k_n}}(v_{k_n})| = \lim C_\Omega(p_{k_n}, v_{k_n}) = \lim C_\Omega(p_k, v_k).$$

Hence, we have $C_\Omega(p, v) = \lim C_\Omega(p_k, v_K)$ and the *Carathéodory* metric C_Ω is then continuous. ■

Despite the differences between the Kobayashi Metric and the *Carathéodory* metric, these two intrinsic metrics are closely related in the sense that c_Ω is always bounded above by k_Ω and only in special circumstance they are equal to one another.

Proposition 2.2.2.11 *For complex domain $\Omega \subset \mathbb{C}^n$, we have $k_\Omega(p, q) \geq c_\Omega(p, q)$ [51].*

Proof. Let $p = p_0, p_1, ..., p_N = q$ be some partition and $\{f_i \in Hol(\mathbb{D}, \Omega)\}$ be holomorphic mappings such that $f_i(a_i) = p_{i-1}$ and $f_i(b_i) = p_i$ for $a_i, b_i \in \mathbb{D}$, as described in the definition of k_Ω. Take g to be any holomorphic map from Ω to $\mathbb{D}$. Notice that $g \circ f_i$ is holomrophic on $\mathbb{D}$. Then by the Schwarz lemma, the *Poincaré* distance satisfies

$$\sum_{i=1}^{N} d_{\mathbb{D}}(a_i, b_i) \geq \sum_{i=1}^{N} d_{\mathbb{D}}(g \circ f_i(a_i), g \circ f_i(b_i)) \geq d_{\mathbb{D}}(g \circ f_1(a_1), g \circ f_N(b_N)) = d_{\mathbb{D}}(f(p), f(q)),$$

where the second inequality comes from the triangle inequality. Hence, we have $k_\Omega(p,q) = \inf \sum_{i=1}^{N} d_{\mathbb{D}}(a_i, b_i) \geq \sup d_{\mathbb{D}}(f(p), f(q)) = c_\Omega(p,q)$. ■

Although we see that the Kobayashi metric is only upper semi-continuous in general, but the induced Kobayashi distance and the *Carathéodory* distance are continuous.

Proposition 2.2.2.12 *For a complex domain $\Omega \subset \mathbb{C}^n$, both c_Ω and k_Ω are continuous (as mappings from $\Omega \times \Omega$ to $\mathbb{R}$).*

Proof. Because the Kobayashi distance always dominates the *Carathéodory* distance, it suffices to show that k_Ω is continuous. Let $\{(p_n, q_n)\}$ be a sequence of points in $\Omega \times \Omega$ whcih converges to (p,q). From the triangle inequality, we obtain that

$$|k_\Omega(p_n, q_n) - k_\Omega(p,q)| \leq k_\Omega(p_n, p) + k_\Omega(q_n, q).$$

Then we desire to show that $k_\Omega(p_n, p) \to 0$ as $p_n \to p$. Let U be a neighborhood of p. Since $k_\Omega \leq k_U$ by the distance decreasing property, it suffices to show that $d_U(p_n, p) \to 0$ as $p_n \to p$ in U.

If p is a non-singular point of Ω, then we may assume that U is a polydisk and the conclusion follows from the example 2.2.2.7.

Then let p be a singular point, and assume that there exists a positive number δ such that $k_U(p_n, p) \geq \delta$ for all n. Let $\pi : \tilde{U} \to U$ be a resolution of the singularity and $\{q_n\}$ be a sequence in $\tilde{U}$ such that $\pi(q_n) = p_n$. Since π is proper, by taking a subsequence, we may assume that $\{q_n\}$ converges to a point $q \in \tilde{U}$. By continuity of π, we have that $\pi(q) = p$. Let V be a polydisk neighborhood of q in $\tilde{U}$. Since $\pi : V \to U$ is distance-decreasing and k_V is continuous by example 2.2.2.7, we have that

$$k_U(p_n, p) = k_U(\pi(q_n), \pi(q)) \leq k_V(q_n, q) \to 0,$$

as $n \to \infty$, which is a contradiction. ■

Proposition 2.2.2.13 *For the unit disk $\mathbb{D} \subset \mathbb{C}$, $k_{\mathbb{D}}$ and $c_{\mathbb{D}}$ coincide with the Poincaré distance $d_{\mathbb{D}}$.*

Proof. By the distance decreasing property of the *Poincaré* distance, we get that

$$\sum_i d_{\mathbb{D}}(0, z_i) \geq \sum_i d_{\mathbb{D}}(f_i(0), f_i(z_i)) \geq d_{\mathbb{D}}(p, q)$$

for holomorphic $f_i : \mathbb{D} \to \mathbb{D}$, which means that $k_{\mathbb{D}} \geq d_{\mathbb{D}}$ from the definition of the Kobayashi distance. On the other hand, if we take $f = id_{\mathbb{D}}$, then we get $d_{\mathbb{D}} \geq k_{\mathbb{D}}$. Hence, $k_{\mathbb{D}} = d_{\mathbb{D}}$. The argument for *Carathéodory* distance is the same. ■

The unit disk turns out to be a special case of bounded convex domains, where the Kobayashi Metric and the *Carathéodory* metric equal to one another, as the famous result proved by Lempert states [35]:

Theorem 2.2.2.14 *For a bounded convex domain $\Omega \subset \mathbb{C}^n$, the Carathéodory metric and the Kobayashi metric coincide.*

People investigated extensively on how the intrinsic metrics are related, especially equal up to scaling, on different complex manifolds. In general, this equality does not occur for arbitrary domains. We will discuss more in depth of this question in the following chapters.

2.2.3 Bergman Metric and *Szegö* Kernel

In the early twentieth century when the Banach space and Hilbert space were not defined yet, people commonly studied the complete, infinite-dimensional L^2 space. Stefan Bergman at the time conceived of the idea of the Bergman space of square-integrable holomorphic functions on the unit disk. Since the Bergman kernel behaves functorially under biholomorphism, it gives rise to the Bergman metric that is invariant under biholomorphism. Here we are following the construction in Krantz [34].

We start with the Bergman space on a bounded domain $\Omega \subset \mathbb{C}^n$:

$$A^2(\Omega) = \{f \text{ holomorphic on } \Omega : \textstyle\int_\Omega |f(z)|^2 dV(z)^{1//2} \equiv ||f||_{A^2(\Omega)} < \infty\}$$

where dV is the Lebesgue measure, and we equip the Bergman space with the inner product $\langle f, g\rangle = \int_\Omega f(z)\bar{g}(z)dV(z)$. The Bergman space is a separable Hilbert space and for each $z \in \Omega$ we see that $\Phi_z : f \to f(z)$ is a continuous linear functional on $A^2(\Omega)$. By the Riesz representation theorem, there exists some $\mathcal{K}_z \in A^2(\Omega)$ such that

$$f(z) = \Phi_z(f) = \langle f, \mathcal{K}_z\rangle.$$

We can then define the Bergman Kernel.

Definition 2.2.3.1 *For a bounded domain $\Omega \subset \mathbb{C}^n$, the Bergman kernel is the function $\mathcal{K}_\Omega(z,\zeta) \equiv \overline{\mathcal{K}_z(\zeta)} = \mathcal{K}(z,\zeta)$ with the reproducing property*

$$f(z) = \int_\Omega \mathcal{K}_\Omega(z,\zeta)f(\zeta)dv(\zeta), \forall f \in A^2(\Omega).$$

The Bergman kernel is conjugate symmetric: $\mathcal{K}(z,\zeta) = \overline{\mathcal{K}(\zeta, z)}$ and is uniquely determined by the properties that it is an element of $A^2(\Omega)$, is conjugate symmetric, and reproduces

$A^2(\Omega)$. Since $A^2(\Omega)$ is a separable Hilbert space, it has a countable, complete orthonormal basis $\{\phi_j\}_{j=1}^{\infty}$. For any choice of such basis, we get

$$\sum_{j=1}^{\infty} \phi_j(z)\overline{\phi_j(\zeta)} = \mathcal{K}(z,\zeta).$$

Note that the diagonal $\mathcal{K}(z,z)$ is strictly positive. To build towards the invariant *Kähler* metric, we first investigate the Bergman kernel behavior under biholomorphism.

Proposition 2.2.3.2 *Let Ω_1, Ω_2 be bounded domains in $\mathbb{C}^n$ and $f : \Omega_1 \to \Omega_2$ be biholomorphic. Then we get*

$$||df_z||_{\mathbb{C}}\mathcal{K}_{\Omega_2}(f(z), f(\zeta))\overline{||df_\zeta||_{\mathbb{C}}} = \mathcal{K}_{\Omega_1}(z,\zeta).$$

Proof. Let $\phi \in A(\Omega_1)$. Then by change of variable, we get

$$\begin{aligned}
&\int_{\Omega_1} ||df_z||_{\mathbb{C}}\mathcal{K}_{\Omega_2}(f(z), f(\zeta))\overline{||df_\zeta||}_{\mathbb{C}}\phi(\zeta)dV(\zeta)\\
&= \int_{\Omega_2} ||df_z||_{\mathbb{C}}\mathcal{K}_{\Omega_2}(f(z), f(\zeta'))\overline{||d(f\circ f^{-1})_{\zeta'}||}_{\mathbb{C}}\phi(f^{-1}(\zeta')||df^{-1}_{\zeta'}||_{\mathbb{R}}dV(\zeta')\\
&= ||df_z||_{\mathbb{C}}\int_{\Omega_2} \mathcal{K}_{\Omega_2}(f(z), \zeta')(||d(f\circ f^{-1})_{\zeta'}||^{-1}\phi(f^{-1}(\zeta')))dV(\zeta').
\end{aligned}$$

By change of variables, the expression $||d(f\circ f^{-1})_{\zeta'}||^{-1}\phi(f^{-1}(\zeta'))$ in the last equation is an element of $A^2(\Omega_2)$. So the reproducing property fo $\mathcal{K}_{\Omega_2}$ applies and the last line equals

$$= ||df_z||_{\mathbb{C}}(||df_z||_{\mathbb{C}}^{-1})\phi(f^{-1}\circ f(z)) = \phi(z).$$

By the uniqueness of the Bergman kernel, the proposition follows. ■

Definition 2.2.3.3 *For a bounded domain $\Omega \subset \mathbb{C}^n$, we define the Bergman metric B_Ω on Ω by*

$$g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\log\mathcal{K}(z,z), z\in\Omega.$$

This is a *Kähler* metric and is positive definite at each point $z \in \Omega$. Given a given tangent vector $v = (v_1, ..., v_n)$ at a point $p \in \Omega$, we have $B^2_\Omega(p, v) = \sum_{i,j} g_{ij}(p) v_i \bar{v}_j$. Consequently, for a $\mathcal{C}^1$ curve $\gamma : [0, 1] \to \Omega$, the length of the curve is given by $\int_0^1 \sum_{i,j} g_{i,j}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)}dt$ so that for any points $p, q \in \Omega$, the distance in Bergman metric is

$$b_\Omega(p, q) = \inf_{\gamma \in \Gamma_\Omega(p,q)} \int_0^1 \sum_{i,j} g_{i,j}(\gamma(t))\gamma_i'(t)\overline{\gamma_j'(t)}dt.$$

Proposition 2.2.3.4 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains and let $f : \Omega_1 \to \Omega_2$ be a biholomorphic mapping. Then f induces an isometry of Bergman metics*

$$B_{\Omega_1}(p, v) = B_{\Omega_2}(f(p), df_p(v)),$$

for all $p \in \Omega$ and $v \in \mathbb{C}^n$. Equivalently, f induces an isometry of Bergman distances such that

$$d_{\Omega_2}(f(p), f(q)) = d_{\Omega_1}(p, q).$$

Proof. From the definition, it suffices to check that

$$\sum g_{ij}^{\Omega_2}(f(z))(df_p(v_i))(df_p(v_j)) = \sum_{i,j} g_{ij}^{\Omega_1}(z) v_i \overline{v}_j,$$

for all $z \in \Omega$ and $v = (v_1, ..., v_n) \in \mathbb{C}^n$. By proposition 2.2.3.2, we get that

$$\begin{aligned} g_{ij}^{\Omega_1}(z) &= \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \mathcal{K}_{\Omega_1}(z, z) \\ &= \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \left(|df|^2 \mathcal{K}_{\Omega_2}(f(z), f(z))\right) \\ &= \frac{\partial^2}{\partial z_i \overline{z}_j} \log \left(\mathcal{K}_{\Omega_2}(f(z), f(z))\right) \end{aligned}$$

since $\log |df|^2$ is locally

$$\log df + \log d\overline{f} + C$$

hence is annihilated by the mixed second derivative. Notice that by definition

$$g_{ij}^{\Omega_2}(f(z)) = \frac{\partial^2}{\partial z_i \overline{z}_j} \log\left(\mathcal{K}_{\Omega_2}(f(z), f(z)\right)$$

and hence we obtain the desired result . ■

Bergman kernel can almost never be calculated explicitly unless the domain Ω has a great deal of symmetry, so that a useful orthonormal basis for $A^2(\Omega)$ can be determined. Now we restricted our attention to the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. The functions z^α are each in $A^2(\mathbb{B}_n)$ and are pair wise orthogonal by the symmetry of the ball. By the uniqueness of the power series expansion for an element of $A^{\mathbb{B}_n}$, the elements z^α form a complete orthonormal system on $\mathbb{B}_n$. By setting

$$\gamma_\alpha = \int_{\mathbb{B}_n} |z^\alpha| dV(z),$$

we see that $\{z^\alpha \backslash \sqrt{\gamma_\alpha}\}$ is a complete orthonormal system in $A^2(\mathbb{B}_n)$. Thus, it follows that

$$\mathcal{K}_{\mathbb{B}_n}(z, \zeta) = \sum_\alpha \frac{z^\alpha \overline{\zeta}^\alpha}{\gamma_\alpha}.$$

Since we have

$$N(k) = \int_{\mathbb{B}_n} |z_1|^{2k} dV(z) = \pi^n \frac{k!}{(k+n)!}$$

for non-negative integer k and $z = (z_1, ..., z_n) = (z_1, z')$, we obtain

$$\mathcal{K}_{\mathbb{B}_n}(z, r\bar{1}) = \sum_\alpha \frac{z^\alpha (r\bar{1}^\alpha)}{\gamma_\alpha} = \sum_{k=0}^{\infty} \frac{z_1^k r^k}{N(k)}$$

for $z \in \mathbb{B}_n$, $0 < r < 1$, and $\bar{1} = (1, 0, ..., 0)$.

Example 2.2.3.5 *Consider the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. Let $z = (z_1, ..., z_n) = r\tilde{z} \in \mathbb{B}_n$, where $r = |z|$ and $|\tilde{z}| = 1$. Fix some $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{B}_n$. Choose a unitary rotation ρ such that*

$\rho(\tilde{z}) = \bar{1} = (1, 0, ..., 0)$. *First of all, we can finish the calculation*

$$\begin{aligned}
\mathcal{K}_{\mathbb{B}_n}(z, r\bar{1}) &= \frac{1}{\pi^n} \sum_{k=0}^{\infty} (rz_1)^k \cdot \frac{(k+n)!}{k!} \\
&= \frac{n!}{\pi^n} \sum_{k=0}^{\infty} (rz_1)^k \begin{pmatrix} k+n \\ \\ n \end{pmatrix} \\
&= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - rz_1)^{n+1}}
\end{aligned}$$

Then using the above calculation, we can get the general case of the Bergman kernel on the unit ball $\mathbb{B}_n$.

$$\begin{aligned}
\mathcal{K}_{\mathbb{B}_n}(z, \zeta) = \mathcal{K}_{\mathbb{B}_n}(r\tilde{z}, \zeta) &= \mathcal{K}(r\rho^{-1}(\bar{1}), \zeta) \\
&= \mathcal{K}_{\mathbb{B}_n}(r\bar{1}, \rho(\zeta)) = \overline{\mathcal{K}_{\mathbb{B}_n}(\rho(\zeta), r\bar{1})} \\
&= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - r\overline{(\rho(\zeta)}_1)^{n+1}} \\
&= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - r\bar{1} \cdot \overline{(\rho(\zeta)})^{n+1}} \\
&= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - (r\rho^{-1}(\bar{1}) \cdot \overline{\zeta}))^{n+1}} \\
&= \frac{n!}{\pi^n} \cdot \frac{1}{(1 - z \cdot \overline{\zeta})^{n+1}}.
\end{aligned}$$

Then by direct computation we can get that the Bergman metric on the unit ball

$$\begin{aligned}
g_{ij}(z) &= \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \mathcal{K}_{\mathbb{B}_n}(z, z) \\
&= \frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log\left(\frac{n!}{\pi^n} \cdot \frac{1}{(1 - |z|^2)^{n+1}}\right) \\
&= \frac{n+1}{(1 - |z|^2)^2} ((1 - |z|^2)\delta_{ij} + \overline{z}_i z_j).
\end{aligned}$$

Then we see that the Bergman metric for the disk would be $g_{ij}(z) = \frac{2}{(1-|z|^2)^2}$ and this is exactly the *Poincaré* metric on the disk.

The *Szegö* kernel is similar to the Bergman kernel — they are both special cases of a general theory of Hilbert spaces with reproducing kernel. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with $\mathcal{C}^2$ boundary. Consider the set of functions $\{D(\Omega)\}$ that are continuous on $\bar{\Omega}$ and holomorphic on Ω. Let $H^2(\partial\Omega)$ be the space consisting of the closure in the $L^2(\partial\Omega, d\sigma)$ topology of the restrictions to the boundary $\partial\Omega$ of elements of $D(\Omega)$. Then $H^2(\partial\Omega)$ is a proper Hilbert subspace of $L^2(\partial\Omega)$. Each element $f \in H^2(\partial\Omega)$ has a natural holomorphic extension to Ω given by its Poisson integral Pf. It is a known fact that

$$\lim_{\epsilon \to 0^+} f(\zeta - \epsilon\nu_\zeta) = f(\zeta)$$

for almost every ζ in the boundary of Ω and ν_ζ is the unit outward normal to $\partial\Omega$ at the point ζ.

For each fixed $z \in \Omega$, the functional

$$\psi_z : f \to Pf(z)$$

is continuous. By the Riesz representation theorem, there exists unique Hilbert space representative $\tilde{k}_z(\zeta)$ such that $\psi_z(f) = \langle f, \tilde{k}_z(\zeta)\rangle$. Then we can define the *Szegö* kernel:

Definition 2.2.3.6 *The Szegö kernel is defined to be*

$$S(z,\zeta) = \overline{\tilde{k}_z(\zeta)},$$

for $z \in \Omega$ *and* $\zeta \in \partial\Omega$.

If $f \in H^2(\partial\Omega)$, then we have that

$$f(z) = \int_{\partial\Omega} S(z,\zeta) f(\zeta) d\sigma(\zeta),$$

for all $z \in \Omega$. Here $d\sigma$ is $(2n-1)-$dimensional Hausdorff measure on $\partial\Omega$.

Example 2.2.3.7 *Consider the unit ball $\mathbb{B}_n$ and $z \in \mathbb{B}_n$. Let ρ be the unique unitary rotation such that ρz is a multiple of the unit vector $\bar{1}$. Then we see that the Szegö kernel for the ball is*

$$\begin{aligned} S(z,\zeta) &= S(\rho^{-1}\bar{1},\zeta) = S(\bar{1},\rho\zeta) = \overline{S(\rho\zeta,\bar{1})} \\ &= \frac{(n-1)!}{2\pi^n}\frac{1}{(1-(\overline{\rho\zeta})\cdot\bar{1})^n} \\ &= \frac{(n-1)!}{2\pi^n}\frac{1}{(1-\bar{\zeta}\cdot(\rho^{-1}\bar{1}))^n} \\ &= \frac{(n-1)!}{2\pi^n}\frac{1}{(1-z\cdot\bar{\zeta})^n}. \end{aligned}$$

2.3 Domains of Holomorphy

Definition 2.3.0.1 *Let $\Omega \subset \mathbb{C}^n$ be a domain and a point $p \in \partial\Omega$. We say p is a singular point of $f \in Hol(\Omega)$if there exists no open set U containing p such that f can be extended to $\Omega \cup U$ holomorphically.*

Definition 2.3.0.2 *A domain $\Omega \subset \mathbb{C}^n$ is called a domain of holomorphy if there exists some function $f \in Hol(\Omega)$, which is singular simultaneously at all boundary points.*

The theory of domain of holomorphy present the deep phenomena of several complex variables as not being merely one-variable theory with multi-indices. In the theory of holomorphic functions of one complex variable, every open set is a domain of holomorphy. The situation in several variables is decidedly different. The fact that only some domains Ω can be enlarged to a larger domain Ω' such that every holomorphic function on Ω can be extended holomorphically to Ω' is a special to the theory of several complex variables.

Example 2.3.0.3 *Let $\mathbb{B}_n$ be the unit ball in $\mathbb{C}^n$. Consider the points with $||p|| = 1$ and without loss of generality, we can assume $p = (1, 0, ..., 0)$. Define the function*

$$f(x) = \frac{1}{1 - z_1},$$

with $z = (z_1, z_2, ..., z_n)$. Then f is holomorphic in the ball but f cannot be defined continuously at any point in the sphere, so f cannot be extended to any open set U containing some $p \in \partial\Omega$. Hence, we see that $\mathbb{B}_n$ is a domain of holomorphy.

The same trick can be applied to open sets in $\mathbb{C}$. However, the next example has no analogue in $\mathbb{C}$.

Example 2.3.0.4 *Consider the domain*

$$\Omega = \mathbb{B}_2(0, 1) \backslash \overline{\mathbb{B}_2(0, \frac{1}{2})} \subset \mathbb{C}^2.$$

The Hartogs phenomenon is the fact that every holomorphic f on Ω analytically continues to a holomorphic $\tilde{u}$ on $\mathbb{B}_2(0, 1)$. For this specific example, let f be holomorphic on Ω. For z_1 fixed, $|z_1| < 1$, write

$$f_{z_1}(z_2) = u(z_1, z_2) = \sum_{n=-\infty}^{\infty} a_n(z_1) z_2^n,$$

where the coefficients of the Laurent expansion are given by

$$a_n(z_1) = \frac{1}{2\pi i} \int_{|z_2| = \frac{3}{4}} \frac{f(z_1, \zeta)}{\zeta^{n+1}} d\zeta.$$

In particular, $a_n(z_1)$ depends holomorphically on z_1. But $a_n(z_1) = 0$ for $n < 0$ when $\frac{1}{2} < |z_1| < 1$. Therefore, $a_n(z_1) \equiv 0$ for $n < 0$. But then the power series expansion defines an analytic extension of f to all of $\mathbb{B}_2(0, 1)$.

Historically, people have dedicated to understand the domains of holomorphy in the study of several complex variables. The boundary of a domain is important to study the domains of holomorphy as the focus is understanding the extension of holomorphic functions to the boundary.

2.4 Pseudoconvexity

Convexity is a nice property. However, convexity is not usually preserved under biholomorphism, which is a direct result of the Riemann mapping theorem. Then we need to introduce pseudoconvexity, which is a reflection of the analytic complex structure of a bounded domain.

Definition 2.4.0.1 *Let $\Omega \subset \mathbb{C}^n$ be a domain. It is called pseudoconvex , or Hartogs pseudoconvex, if there exists a continuous plurisubharmoic exhaustion function $\phi : \Omega \to \mathbb{R}$.*

Will not be using this definition of pseudoconvexity and more details can be found in [34]. It is invariant under biholomorphism and a domain is Hartogs pseudoconvex if and only if it is a domain of holomorphy.

There is another form of pseudoconvexity called the Levi pseudoconvexity. We will use this concept of pseudoconvex here. For Levi pseudoconvexity, we need to start with the idea of a defining function:

Definition 2.4.0.2 *Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^1$ boundary. We say $\rho : \mathbb{C}^n \to \mathbb{R}$ is a defining function if ρ satisfies that for all $p \in \partial\Omega$ with $V \ni p$ open we have*

- $\rho(z) > 0 \ \forall z \in V \cap \Omega^c$

- $\rho(z) < 0 \ \forall z \in V \cap \Omega$
- $\rho(z) = 0 \ \forall z \in V \cap \partial\Omega$, *and* $\nabla\rho \neq 0$.

The fact that the gradient of ρ at $\partial\Omega$ is nonvanishing means that $\partial\Omega$ is a $\mathcal{C}^1$ embedded submanifold of $\mathbb{C}^n$. If ρ can be chosen to be $\mathcal{C}^k$ smooth for some $1 \leq k \leq \infty$, then we say that Ω is a $\mathcal{C}^k$ domain.

Definition 2.4.0.3 *A domain $\Omega \subset \mathbb{C}^n$ with $\mathcal{C}^2$ boundary is said to be strongly (Levi) pseudoconvex if it has a defining function ρ such that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k > 0$$

for all $p \in \partial\Omega$ and all $w \in \mathbb{C}^n$ satisfying $\sum_{j=1}^{n} (\partial\rho/\partial z_j)(p) w_j = 0$. The expression on the left side is call the Levi form.

The point p is a point of strong pseudoconvexity if the Levi form at p is positive definite for some choice of defining function and the same set of vectors w, whereas p is a point of weak speudoconvex if the Levi form is positive semidefinite.

Proposition 2.4.0.4 *If $\Omega \subset \mathbb{C}^n$ is strongly speudoconvex, then Ω has a defining function ρ that*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq C|w|^2,$$

for some $C > 0$, and for all $p \in \partial\Omega$, $w \in \mathbb{C}^n$.

The pseudoconvexity if a inherent property of the complex domain so we see that it is independent of the choice of the defining function. We have a couple a notions of pseudoconvexity and we see that they are inherently the same by the following theorem.

Theorem 2.4.0.5 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Then the following are equivalent:*

1. *Ω is a domain of holomorphy.*

2. *Ω is holomorphically convex.*

3. *Ω is Hartogs pseudoconvex.*

4. *Ω is Levi pseudoconvex.*

The detailed proof can be found in [34]. The difficult part of Theorem 2.4.0.5 is to prove that a pseudoconvex domain is a domain of holomorphy for either notion of pseudoconvexity. This is widly-known as the Levi problem raised by Levi in 1910's. It was proved by Oka [44] and the converse direction is not hard to show.

Example 2.4.0.6 *Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^4 < 1\}$. Let $\rho(z) = |z_1|^2 + |z_2|^4 - 1$ be a defining function for the domain and we have that the Levi form*

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k = |w_1|^2 + 4|z_2|^2 |w_2|^2 \geq 0.$$

Thus, we see that $p \in \partial\Omega$ is strongly pseudoconvex except at points where $|z_2|^2 = 0$ and tangent vector $|w_1|^2 = 0$. The domain is weakly pseudoconvex at the exceptional points.

For a strongly pseudoconvex point $p \in \partial\Omega$, since the second derivatives of any $\mathcal{C}^2$ defining function ρ of Ω is continuous, all points of $\partial\Omega$ sufficiently close to p are also strongly pseudoconvex.

Lemma 2.4.0.7 *Let $\Omega \subset \mathbb{C}^n$ be a domain with $\mathcal{C}^2$ boundary. Letp $\in \partial\Omega$ be a strongly pseudoconvex point. There exists a neighborhood $U \subset \mathbb{C}^n$ containing p and a biholomorphic coordinate change ψ on U such that $\psi(U \cap \partial\Omega)$ is strongly convex.*

Proof. By proposition 2.4.0.4, we may choose a defining function ρ for Ω such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq C|w|^2,$$

for some $C > 0$, and for all $p \in \partial\Omega$, $w \in \mathbb{C}^n$. Without loss of generality, we may assume that $p = 0$ and $(1, 0, ..., 0)$ is the unit outward normal vector to $\partial\Omega$ at p. The second order Taylor expansion of ρ about p is given by

$$\begin{aligned}
\rho(w) &= \rho(0) + \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) \cdot w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) w_j w_k \\
&\quad + \sum_{j=1}^{n} \frac{\partial \rho}{\partial \bar{z}_j}(p) \cdot \bar{w}_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial \bar{z}_j \partial \bar{z}_k}(p) \bar{w}_j \bar{w}_k + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k + o(|w|^2) \\
&= 2Re(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) \cdot w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) w_j w_k) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k + o(|w|^2).
\end{aligned}$$

Let $w'_1 = \psi_1(w) = w_1 + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(p) w_j w_k$, $w'_2 = \psi(w) = w_2$, ..., and $w'_n = \psi(w) = w_n$ be a change of coordinates. It is well-defined for sufficiently small w in some neighborhood U'. Then the Taylor expansion about the new coordinates is given by

$$\rho(w') = 2Re(w'_1) + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w'_j \bar{w}'_k + o(|w'|^2).$$

Thus the real Hessian at p of ρ in the coordinates w' is the Levi form which is positive definite. Therefore, $\psi(U' \cap \Omega)$ is strictly convex at $\psi(p)$. By the continuity of the second derivatives of ρ, we have that $\psi(U' \cap \Omega)$ is strictly convex in a neighborhood V containing $\psi(p)$. Let $U \subseteq U'$ be an open set so that $\psi(U \cap \Omega) \subseteq V$ is strictly convex. ■

2.5 Curvature

Let $h = \sum_{i,j=1}^{n} h_{i\bar{j}} dz_i d\overline{z_j}$ be a Hermitian metric on a complex manifold M with tangent bundle TM. Then h has a canonical Hermitian connection ∇. Under a local frame $\{\frac{\partial}{\partial z_i}\}_1^n$ of TM in a neighborhood of $p \in M$, we may denote

$$\nabla \frac{\partial}{\partial z_i} = \sum_{j=1}^{n} \theta_{ij} \otimes \frac{\partial}{\partial z_j}, \nabla^2 \frac{\partial}{\partial z_i} = \sum_{j=1}^{n} \Theta_{ij} \otimes \frac{\partial}{\partial z_j},$$

where the $r \times r$ matrices $\theta = (\theta_{ij})$ of 1-forms and $\Theta = (\Theta_{ij})$ of 2-forms are called the connection matrix and curvature matrix respectively. We get that $\theta_{ij} = \sum_{k=1}^{n} h^{kj} \partial h_{ik}$, where $\sum h^{ij} h_{ik} = \delta_{ik}$.

For holomorphic tangent vectors $X = \sum x_i \frac{\partial}{\partial z_i}$ and $Y = \sum y_i \frac{\partial}{\partial z_i}$ in TM, define the curvature tensor to be the $(1,1)$-form $\Theta_{X\bar{Y}}$ by

$$\Theta_{X\bar{Y}} = \sum_{i,j,k=1}^{n} \Theta_{ik} h_{k\bar{j}} x_i \bar{y}_j.$$

For holomorphic tangent vector $Z, W \in TM$, the curvature is a covariant 4-tensor denoted by $R_{X\bar{Y}Z\bar{W}} = \Theta_{Z\bar{W}}(X, \bar{Y})$. We say $R_{X\bar{X}Y\bar{Y}}$ the curvature of TM in the direction of X and Y. Naturally, we will say that it is positively curved if and only if $R_{X\bar{Y}Z\bar{W}} > 0$ for any $p \in M$ and any non-zero $X, Y \in T_pM$.

Definition 2.5.0.1 *The normalized curvature in the direction of X, Z*

$$B(X, Z) = \frac{R_{X\bar{X}Z\bar{Z}}}{|X|^2 |Z|^2}$$

is called the bisectional curvature in the direction of X and Z.

Definition 2.5.0.2 *The holomorphic sectional curvature in the direction of X is defined to be*

$$H(X) = B(X, X) = \frac{R_{X\bar{X}X\bar{X}}}{|X|^4}.$$

The curvature form is then give by $\Theta_{kl} = \sum_{i,j=1}^{n} R_{kij}^{l} dz_i \wedge d\bar{z}_j$, where

$$R_{kij}^{l} = \sum_{m=1}^{n} (\frac{\partial h_{km}}{\partial z_i} \frac{\partial h^{ml}}{\partial \bar{z}_j} - h^{ml} \frac{\partial^2 h_{km}}{\partial z_i \partial \bar{z}_j}). \tag{2.1}$$

Let $R_{klij} = \sum_{m=1}^{n} h_{ml} R_{kij}^{m}$ be the 4-fold covariant tensor field and then we get

$$R_{klij} = -\frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} + \sum_{m,h=1}^{n} h^{hm} \frac{\partial h_{kh}}{\partial z_i} \frac{\partial h_{ml}}{\partial \bar{z}_j}. \tag{2.2}$$

Let $X = \sum_{i=1}^{n} x_i \frac{\partial}{\partial z_i}(p)$ be a unit vector at $p \in M$. Then the holomorphic sectional curvature of X is given by

$$H(X) = \sum_{k,l,i,j=1}^{n} R_{klij}(p) x_k \bar{x}_l x_i \bar{x}_j. \tag{2.3}$$

A certain special case will frequently be encountered: let $\{\frac{\partial}{\partial z_i}, ..., \frac{\partial}{\partial z_n}\}$ are orthonormal at $p \in M$ so that $g_{ij}(p) = \delta_{ij}$. Let t be the unit vector $\{\frac{\partial}{\partial z_n}(p)\}$, we obatin from (2.3) that

$$H(t) = R_{nnnn} = (-\frac{\partial^2 g_{nn}}{\partial z_n \partial \overline{z_n}} + \sum_{i=1}^{n} |\frac{\partial g_{ni}}{\partial z_n}|^2)(p).$$

We now recall the following well-known factor about the decreasing property of holomorphic sectional curvature on submanifolds.

Lemma 2.5.0.3 *Let M' be a complex submanifold of the Hermitian manifold M. Then the holomorphic sectional curvature of M' does not exceed that of M. More precisely, let G be the Hermitian metric of M and let G' be the induced Hermitian metric of M'. If t is a unit vector tangent to M', then $H(G', t) \leq H(G, t)$*

Proof. Let t be tangent to M' at p and we may assume local coordinates $\{z_1, ..., z_n\}$ so that locally $M' = \{z_1 = ... = z_s = 0\}$ for $s < n$ and $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\}$ are orthonormal at p, where $t = \frac{\partial}{\partial z_n}(p)$. Then we have that the local coordinates $\{\frac{\partial}{\partial z_{s+1}}, ..., \frac{\partial}{\partial z_n}\}$ restricted to M' give local coordinates of M' at p and they are orthonormal at p. We see that the induced Hermitian metric $G' = \sum_{\alpha,\beta=s+1}^{n} g_{\alpha\beta} dz_\alpha d\overline{z}_\beta$. From the above discussion, we can conclude that

$$H(G', t) = (-\frac{\partial^2 g_{nn}}{\partial z_n \partial \overline{z_n}} + \sum_{\alpha=s+1}^{n} |\frac{\partial g_{n\alpha}}{\partial z_n}|^2)(p)$$
$$H(G, t) = (-\frac{\partial^2 g_{nn}}{\partial z_n \partial \overline{z_n}} + \sum_{i=1}^{n} |\frac{\partial g_{ni}}{\partial z_n}|^2)(p).$$

Thus, combining the two equations gives that

$$H(G', t) = H(G, t) - \sum_{\alpha=1}^{s} |\frac{\partial g_{n\alpha}}{\partial z_n}|^2(p) \leq H(G, t).$$

■

Example 2.5.0.4 *Consider the Kähler metric h on the unit ball*

$$h_{i\bar{j}} = \frac{1}{1 - 2|z|^2}\delta_{ij} + \frac{1}{(1 - 2|z|^2)^2}\bar{z}_i z_j,$$

under a standard coordinates $(z_1, ..., z_n)$. The curvature form under the natural frame is given by

$$\Theta_{ij} = -(\sum_{k,l=1}^{n} h_{k\bar{l}} dz_k \wedge d\bar{z}_l)\delta_{ij} + \sum_{l=1}^{n} h_{i\bar{j}} d\bar{z}_l \wedge dz_j.$$

Then we get

$$R_{ijkl} = -(h_{i\bar{j}} h_{k\bar{l}} + h_{k\bar{j}} h_{i\bar{l}}),$$

and therefore the holomorphic sectional curvature of h in any direction X is constant for $H(X) = -2$.

Example 2.5.0.5 *For the Bergman metric* $g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \mathcal{K}(z,z)$ *on the unit ball* $z \in \mathbb{B}_n$*, we can calculate the holomorphic sectional curvature directly. Since the unit ball equipped with the Bergman metric is homogeneous, it suffices to check at 0. With direct computation,*

$$g_{ij} = \frac{n+1}{(1-|z|^2)^2}((1-|z|^2)\delta_{ij} + \overline{z}_i z_j.$$

At $z = 0$, $g_{ij}(0) = (n+1)\delta_{ij}$ *and* $\frac{\partial g_{ij}}{\partial z_k}(0) = 0$, $\frac{\partial g_{ij}}{\partial \bar{z}_l}(0) = 0$. *So that* $\frac{\partial g_{ij}}{\partial z_k \partial \bar{z}_l}(0) = (n+1)(\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il})$. *We get that*

$$R_{ijkl}|_0 = (n+1)(\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il}) = \frac{1}{n+1}(g_{ij}g_{kl} + g_{kj}g_{il})|_0 = -\frac{2}{n+1}.$$

The holomorphic sectional curvature is then $-\frac{2}{n+1}$.

Definition 2.5.0.6 *Let* M *be a complex manifold. Then* $F : TM \to \mathbb{R} \cup \{0\}$ *is a hermitian Finsler metric if*

1. $F(z, sv) = |s| \cdot F(z, v)$ *for* $s \in \mathbb{C}$ *and* $v \in TM$.

2. $F(z, v) = 0$ *if anf only if* $v = 0$ *for any* $z \in M$.

Suppose F is a $\mathcal{C}^2$ hermitian Finsler metric on a complex one dimensional manifold, which is a *Kähler* manifolds. Then the holomorphic sectional curvature is given by

$$H = -\frac{1}{F}\frac{\partial^2 \log F}{\partial z \partial \overline{z}}.$$

The holomorphic sectional curvature of a $\mathcal{C}^2$ hermitian Finsler metric F at (z, v) is defined to be

$$H_F(z, v) = \sup\{H_{F|_{M_z(v)}}(z, v)\},$$

where we denote $M_z(v)$ to be the complex one dimensional submanifold at the point $z \in M$ and whose tangent space at z is spanned by $\{v, Jv\}$. Then $F|_{M_z(v)}$ is the restriction of the metric F to the submanifold $M_z(v)$.

Any two simply-connected complete *Kähler* manifolds with constant holomorphic sectional curvature c are holomorphically isometric to each other. Thus, we have the following uniformization theorem characterizing the covering space of complete *Kähler* manifold of constant sectional curvature [26] [30].

Theorem 2.5.0.7 *For a complete Kähler manifold of constant sectional curvature, its universal cover is holomorphically isometric to* $\mathbb{B}_n(< 0), \mathbb{C}^n(= 0)$ *or* $\mathbb{CP}^n(> 0)$.

We also have this well-know fact in differential geometry:

Lemma 2.5.0.8 *For any Hermitian metric on complex manifold, if it has non-zero constant holomorphic curvature, then the metric is Kähler.*

The curvature of invariant metrics on strongly pseudoconvex domains is of particular interests. Klembeck [32] showed that near the boundary of strongly pseudoconvex domains, the curvature Bergman metrics approaches that of the Bergman metric on the unit ball, which is not unexpected since near the boundary, the metrics are comparable to that of the unit ball.

Theorem 2.5.0.9 *Let* $\Omega \subset \mathbb{C}^n$ *be a strongly pseudoconvex domain with smooth boundary and* $\delta_\Omega(z)$ *be the distance from* $z \in \Omega$ *to* $\partial\Omega$. *Then near the boundary of* Ω,

1. *The sectional curvature of the Bergman metric approaches the curvature tensor of constant holomrophic sectional curvature of* $-\frac{2}{n+1}$.

2. *The sectional curvature of the Kähler metric* $g_{ij} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j}(-\log \delta_\Omega(z))$ *approaches the curvature tensor of constant holomrophic sectional curvature of -1.*

Chapter 3

The Intrinsic Metrics on Strongly Pseudoconvex Domain

3.1 Conjectures

Strong pseudoconvexity captures how closely a domain resembles the unit ball near its boundary. In many respects, the unit ball serves as the canonical model of a strongly pseudoconvex domain in $\mathbb{C}^n$, and it plays a central role in the study of several complex variables.

In 1979, Cheng [8] conjectured that if the Bergman metric on a bounded strongly pseudoconvex domain is Kähler–Einstein, then the domain must be biholomorphic to the unit ball. Later, Yau [58] formulated a broader question: can we classify domains of holomorphy whose Bergman metrics are Kähler–Einstein?

Substantial progress has been made on this conjecture. Fu and Wong [20] confirmed Cheng's conjecture in the case where the domain is simply connected and two-dimensional. More recently, in 2020, Huang and Xiao [28] settled the conjecture for all bounded strongly pseudoconvex domains with $\mathcal{C}^\infty$ boundary in dimension $n \geq 2$.

The Cheng–Yau conjecture, now fully resolved under smooth boundary assumptions, has inspired a wide range of studies into the relationship between intrinsic metrics and the complex geometry of domains. One recurring theme is the uniqueness and coincidence of intrinsic metrics: the Bergman metric, Kähler–Einstein metric, Kobayashi metric, and Carathéodory metric all coincide (up to a constant) on the unit ball. This naturally raises a broader question: on which domains do some or all of these intrinsic metrics coincide up to scaling?

On bounded convex domains, Lempert [35] proved that the Kobayashi and Carathéodory metrics coincide, and that the Kobayashi metric is Kähler if and only if the domain is biholomorphic to the unit ball. Beyond convex domains, Wong [55] showed that for simply connected domains, if the Kobayashi and Carathéodory metrics coincide and are Kähler, then the domain is biholomorphic to the unit ball. Stanton [52] later generalized this result: if the Kobayashi metric and the *Carathéodory* metric coincide at a point and either one of the metric is s smooth Hermitian metric, then the domain is biholomorphic to a ball. Graham [29] showed that when approaching the boundary of a strongly pseudoconvex domian, the Kobayashi metric and Carathéodory metrics have a finite limit and later Klembeck [32] obtained Graham's precise estimation on the boundary behavior of Kobayashi metric.

The primary motivation for the current chapter stems from the work of Gaussier and Zimmer [21], who provided a characterization of domains whose universal covering spaces are biholomorphic to the unit ball, under the assumption that the Kobayashi metric is Kähler. Inspired by their approach, we extend the analysis by examining the Carathéodory metric instead. Our goal is to investigate whether similar characterizations hold when assuming the Carathéodory metric is Kähler, thereby offering a new perspective on rigidity phenomena in several complex variables.

3.2 Complex Geodesic near the Boundary of Domains

For a domain $\Omega \subset \mathbb{C}^n$ with a point $z \in \Omega$ and a non-zero vector $v \in T_z\Omega = \mathbb{C}^n$, define a holomorphic map $f_{z,v} : \mathbb{D} \to \Omega$ to be extremal with respect to (z, v) if $f_{z,v}(0) = z$, $f'(0) = \lambda v$ for some $\lambda > 0$ and for any holomorphic $g : \mathbb{D} \to \Omega$ with $g(0) = z, g'(0) = \tilde{\lambda} v$, we have $|\tilde{\lambda}| < \lambda$. We say a subset $U \subset \Omega$ is a holomorphic retract of Ω if there exists a holomorphic map $r : \Omega \to \Omega$ such that $r(\Omega) = U$ and $r(z) = z$ for all $z \in U$.

Definition 3.2.0.1 *A holomorphic map $\psi : \mathbb{D} \to \Omega$ is a complex geodesic with respect to the Kobayashi metric (or Carathéodory metric) if*

$$k_\Omega(\psi(z_1), \psi(z_2)) = k_\mathbb{D}(z_1, z_2), c_\Omega(\psi(z_1), \psi(z_2)) = c_\mathbb{D}(z_1, z_2).$$

for all $z_1, z_2 \in \mathbb{D}$. If $z, w \in \psi(\mathbb{D})$ are two distinct points, then we refer to ψ as complex geodesic joining z and w.

For a bounded strongly convex domain Ω with $\mathcal{C}^3$ boundary, Lempert [35] [36] proved that the unique extremal map $f_{z,v}$ is a complex geodesic and the extremal disk $f_{z,v}(\mathbb{D})$ is s holomorphic retract of Ω. Another deep observation of Lempert states that for a complex geodesic $\psi : \mathbb{D} \to D$ of a bounded strongly convex domain of $\mathcal{C}^2$ boundary, there exists a holomorphic map $\rho : D \to \mathbb{D}$ such that $\rho \circ \psi = id_{\mathbb{D}}$. Then a direct result is that the Kobayashi metric agrees with *Carathéodory* metric on a strongly convex domain. However, this is no longer true for pseudoconvex domain. As we discussed before, the strongly pseudoconvex domain behaves like a ball locally on the boundary and Bracci, Fornass, and Wold [16] showed that near the boundary of a pseudoconex domain, it continues to hold to hold to some extent.

For a bounded domain $\Omega \subset \mathbb{C}^n$, let $\delta_\Omega(z) = \min\{||z-p|| : p \in \partial\Omega\}$ be the distance from z to the boundary of domain Ω and if $\delta_\Omega(z)$ is small enough, then there exists a unique point $z' \in \partial\Omega$ such that $\delta_\Omega(z) = ||z - z'||$. For any vector $v = T_z\Omega = \mathbb{C}^n$, we can decompose it into the normal direction and tangential direction to the boundary, i.e., $v = v_N + v_T$ with $v_T \in T^{\mathbb{C}}_{z'}\partial\Omega = T_{z'}\partial\Omega \cap iT_{z'}\partial\Omega$. Then similar results to Lempert's theorem were proven [16]:

Theorem 3.2.0.2 *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with $\mathcal{C}^k$ boundary with $k \geq 3$. Then there exists some $\epsilon > 0$ such that for any $z \in \Omega$ with $\delta_\Omega(z) < \epsilon$ and non-zero vector $v \in T_z\Omega$ with $||v_N|| < \epsilon||v_T||$ the following holds:*

1. *The extremal map $f_{z,v}$ is unique.*

2. *The extremal map $f_{z,v}$ extends to a $\mathcal{C}^{k-2}$ map on $\overline{\mathbb{D}}$ and $f_{z,v}$ embeds $\overline{\mathbb{D}}$ into $\overline{\Omega}$ with $f(\partial\mathbb{D}) \subset \partial\Omega$.*

3. *The corresponding extremal disk $f_{z,v}(\mathbb{D})$ is a holomorphic retract of Ω.*

4. *The extremal map $f_{z,v}$ is a complex geodesic.*

5. *For any $z, w \in \Omega$ such that $\max\{\delta_\Omega(z), \delta_\Omega(w), ||z - w||\} < \epsilon$ and $||(w - z)_N|| <$*

$\epsilon||(w-z)_T||$, there exists a complex geodesic f joining z and w, which is unique up to pre-composition with automorphism of $\mathbb{D}$. Such a complex geodesic is an extremal map with respect to $(f(\zeta), f'(\zeta))$ for all $\zeta \in \mathbb{D}$ and its image is a holomorphic retract of Ω.

In particular, the above theorem shows that for a strongly pseudoconvex domain, the Kobayashi metric and *Carathéodory* metric coincide in the almost tangent direction near that boundary, and the Kobayashi distance between z and w equals the *Carathéodory* distance.

In order to present the proof of theorem 3.2.0.2, we need the following two statements by Huang [27] and Fornass [31] [19] of embedding complex geodesic and pseudoconvex domain into bounded convex domains near the boundary.

Proposition 3.2.0.3 *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with $\mathcal{C}^3$ boundary. Let $p \in \partial D$ and U be an open neighborhood of p.*

1. *Then there exists an open neghborhood V of p and $\epsilon > 0$ such hat for every $z \in V$ and for all $v \in \mathbb{C}^n \backslash \{0\}$ with $||v_N|| < \epsilon||v_T||$, the complex geodesic $\psi : \mathbb{D} \to D$ such that $\psi(0) = p$ and $\psi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}$ satisfies $\psi(\mathbb{D}) \subset U$.*

2. *There exists an open neighborhood $V \subset U$ of p and $\epsilon > 0$ such that for all $z, w \in V$ with $||(z-w)_N|| < \epsilon||(z-w)_T||$ the complex geodesic $\psi : \mathbb{D} \to D$ joining z and w is contained in U*

Proposition 3.2.0.4 *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of $\mathcal{C}^k$ boundary with $k \geq 2$ and let $p \in \partial\Omega$. Then there exits a bounded strongly convex domain D with*

$\mathcal{C}^k$ boundary and a biholomorphism $\phi : \Omega \to \mathbb{C}^n$ such that

1. *ϕ extends as a $\mathcal{C}^k$ diffeomorphism on $\overline{\Omega}$.*

2. *$\phi(\Omega) \subset D$.*

3. *There exists an open neighborhood U of $\phi(p)$ such that $U \cap \phi(\Omega) = U \cap D$.*

From the above two proposition, we can see that the basic idea is to embed neighborhood of strongly pseudoconvex point into strongly convex domain and us the nice property of convex domains. We will then present the proof of theorem 3.2.0.2 following the demonstration in [16].

Proof of Theorem 3.2.0.2. By proposition 3.2.0.4, we can assume without loss of generality that there exists a bounded strongly convex domain D with $\mathcal{C}^3$ boundary such that $\Omega \subset D$ and for a point $p \in \partial\Omega$ there exists an open neighborhood U of p such that $\Omega \cap U = D \cap U$.

Let $z \in \Omega$ and non-zero vector $v \in T_z\Omega$. Since D is convex, by Lempert's theorem, we know that there exists a unique extremal map $f_{z,v}$ for D with respect to (z, v). By proposition 3.2.0.3, there exists an open neighborhood V of p and some $\epsilon_0 > 0$ such that $f_{z,v}(\mathbb{D}) \subset U$ for all $z \in V$ and $||v_N|| < \epsilon_0||v_T||$. By the distance decreasing property of the Kobayashi metric, we see that $K_D(z, v) \leq K_\Omega(z, v)$. Then it follows that $f_{z,v}$ is an extremal map for Ω as well. It is also an unique extremal map for Ω since $\Omega \subset D$ and if not, D would have two different extremal maps with respect to (z, v).

By Lempert's theorem, the extremal disk $f_{z,v}(\mathbb{D})$ is a holomorphic retract of C and since $\Omega \subset D, \Omega \cap U = D \cap U$, we see that $f_{z,v}(\mathbb{D})$ is also a holomorphic retract of Ω

as well. It follows that $f_{z,v}$ is a complex geodesic of Ω. By the compactness of $\overline{\Omega}$, we can choose a uniform $\epsilon > 0$ for all $p \in \partial\Omega$ and 1,2,3,4 holds.

By proposition 3.2.0.3, there exists an open neighborhood $V \subset U$ of p and $\epsilon_0 > 0$ such that for all $z, w \in V$ with $||(z-w)_N|| < \epsilon_0||(z-w)_T||$ the complex geodesic $fF\psi : \mathbb{D} \to D$ joining z and w is contained in U. Since $U \cap D = U \cap \Omega$, the image of such map $f : \mathbb{D} \to \Omega$ is contained in Ω. By the distance decreasing property of the Kobayashi distance, we have that $k_D(z,w) \leq k_\Omega(z,w)$ and $k_\Omega(z,w) \leq k_\mathbb{D}(z', w')$ for $f(z') = z$ and $f(w') = w$ and thus we get $k_\Omega(z,w) = k_\mathbb{D}(z', w')$. It follows that f is a complex geodesic for Ω as well. By Lempert's theorem, such geodesic f is unique up to precompostion with automorphism of $\mathbb{D}$ and it is an extremal map with respect to $(f(\zeta), f'(\zeta))$ for all $\zeta \in \mathbb{D}$ and its image is a holomorphic retract of D. Then by the same argument and choosing a proper ϵ, the geodesic f satisfies the same properties with respect to Ω, hence proving 5. ■

It is a direct consequence from theorem 3.2.0.2 that for bounded strongly pseudoconvex domain with $\mathcal{C}^3$ boundary, there exists an open set of non-zero tangent vectors at where the Kobayashi and *Carathéodory* metrics agree near the boundary. For the purpose of building towards , We need to improve the boundary condition to $\mathcal{C}^2$ boundary. The existence of complex geodesic in strongly pseudocovex domains was first established by Lempert [36], whose original analysis demonstrated that since the estimation of existence depend only on the normal curvature, the results naturally extends to $\mathcal{C}^2$ case by the Montel's theorem. We will present a direct proof of the existence of such open set by Gaussier and Zimmer[21]:

Theorem 3.2.0.5 *Suppose* $\Omega \subset \mathbb{C}^n, n \geq 2$, *is a bounded strongly pseudoconvex domain* *with $\mathcal{C}^2$ boundary. Then there exists $\epsilon > 0$ such that if $z \in \Omega$ and $\delta_\Omega(z) < \epsilon$, then $E_\Omega(z)$ has non-empty interior, where $E_\Omega \subset \mathbb{C}^n = T_z\Omega$ denotes the set of non-zero vectors $v \in \mathbb{C}^d$ where there exist a complex geodesic $\psi : \mathbb{D} \to \Omega$ with $\psi(0) = z$ and $\psi'(0) = \lambda v$.*

Proof. Fix $p \in \partial\Omega$. It suffices to show that for a neighborhood U_p of p, if $z \in \Omega \cap U_p$, then $E_\Omega(z)$ has non-empty interior.

By proposition 3.2.0.4, we can find a bounded strongly convex domain $D \subset \mathbb{C}^d$ with $\mathcal{C}^2$ boundary and holomorphic embedding $\phi : \Omega \to \mathbb{C}^d$ such that ϕ extends to $\mathcal{C}^2$ embedding on $\bar{\Omega}$, $\phi(\Omega) \subset D$ and there exists open neighborhood V of $\phi(p)$ such that $V \cap \Phi(\Omega) = V \cap D$. Then by proposition 3.2.0.3, there exists $\epsilon > 0$ such that for $\phi(z) = \hat{z} \in D$ with $||\phi(z) - \phi(p)|| < \epsilon$ and $v \in T_{\hat{z}}D$ in the tangential direction, the complex geodesic $\psi_{\hat{z},v} : \mathbb{D} \to D$ with $\psi_{\hat{z},v}(0) = \hat{z}, \psi'_{\hat{z},v}(0) = \lambda v$ for some $\lambda \in \mathbb{C}$ satisfies that $\psi_{\hat{z},v}(\mathbb{D}) \subset V$. Let $\mathcal{O}_p$ be a neighborhood of p such that $||\Phi(z) - \Phi(p)|| < \epsilon$ for all $z \in \mathcal{O}_p \cap \bar{\Omega}$. For a fixed $\phi(z) = \hat{z}$, let $\hat{E} \subset \mathbb{C}^d$ denote the set of non-zero vectors $v \in T_{\hat{z}}D$ such that the complex geodesic $\psi_{\hat{z},v} : \mathbb{D} \to D$ satisfies $\psi_{\hat{z},v}(\mathbb{D}) \subset V$. By Lempert's theorem, $\psi_{\hat{z},v}$ extends to $\bar{\psi}_{\hat{z},v} : \bar{\mathbb{D}} \to \bar{D}$ and that $\hat{E} \neq \emptyset$. If $\hat{z}_n \to \hat{z}$ in D and non-zero vectors $v_n \to v$ in TD, then the complex geodesic $\bar{\psi}_{\hat{z}_n,v_n} \to \bar{\psi}_{\hat{z},v}$ converges uniformly on $\bar{\mathbb{D}}$. Let non-zero vectors $v_n \to v$ in TD and $v \in \hat{E}$, then we have $\bar{\psi}_{\hat{z},v_n}(\mathbb{D}) \subset V$ for all v_n. It follows that $v_n \in \hat{E}$ and $\hat{E}$ is open. Consider the map $\phi : \Omega \to D$ and the induced map on the tangent space: $d\phi_z : T_z\Omega \to T_{\phi(z)}\phi(\Omega) \subset T_{\hat{z}}D$ with inverse $d(\phi)_z^{-1} : T_{\hat{z}}D \to T_z\Omega$.

Since $\hat{E}$ is non-empty and open in $T_{\hat{z}}D$, then $d(\phi)_z^{-1}(\hat{E})$ is non-empty and open since $d\phi_z$ is continuous. Now we are left to show that $d(\phi)_z^{-1}(\hat{E}) \subset E_\Omega(z)$, which implies $E_\Omega(z)$ has non-empty interior. Let $v \in \hat{E}$ be fixed. We have $\psi_{\hat{z},v}(\mathbb{D}) \subset V \cap D = V \cap$

$\Phi(\Omega) \subset \Phi(\Omega)$, so $\psi := \phi^{-1} \circ \psi_{\hat{z},v} : \mathbb{D} \to \Omega$ is well-defined and holomorphic. There exists some $\hat{\rho}$ such that $\hat{\rho} \circ \psi_{\hat{z},z} = id_{\mathbb{D}}$ so that $\rho = \hat{\rho} \circ \phi : \Omega \to \mathbb{D}$ is holomorphic and $\rho \circ \psi = id_{\mathbb{D}}$. By the decreasing property of Kobayashi pseudo-distance, $k_{\mathbb{D}}(x_1, x_2) = k_{\mathbb{D}}(\rho \circ \psi(x_1), \rho \circ \psi(x_2)) \le k_\Omega(\psi(x_1), \psi(x_2)) \le k_{\mathbb{D}}(x_1, x_2)$, so $k_\Omega(\psi(x_1), \psi(x_2)) = k_{\mathbb{D}}(x_1, x_2)$ and ψ is a complex geodesic. Finally, since $d(\phi)_z^{-1} \lambda v = d(\phi)_z^{-1} \psi'_{\hat{z},v}(0)$ and $\psi(0) = \phi^{-1} \circ \psi_{\hat{z},v}(0) = z$, $\psi'(0) = (\phi^{-1})'(\psi_{\hat{z},v}(0)) \cdot \psi'_{\hat{z},v}(0) = \frac{1}{\phi'(z)} \cdot \psi'_{\hat{z},v}(0)$, then we see that $d(\phi)_z^{-1} \psi'_{\hat{z},v}(0) \in E_\Omega(z)$ and $\hat{E} \subset E_\Omega(z)$. ■

The existence of complex geodesic near the boundary of strongly pseudoconvex domain is fundamental for our analysis of the main result, as we manage to extend the open set of Kobayashi metric and *Carathéodory* metric agreeing to the whole domain under some conditions.

3.3 Holomorphic Sectional Curvature and Intrinsic Metrics

From the uniformization theorem of complex manifolds with constant holomorphic sectional curvature, we see that the holomorphic sectional curvature play an important role in characterizations of complex manifolds. We will first start with two Theorems by Wong [55] on the holomorphic sectional curvature of Kobayashi metric and *Carathéodory* metric:

Theorem 3.3.0.1 *For a complex manifold M,*

1. *if M is* Carathéodory*-hyperbolic and the* Carathéodory *metric C_M is of class $\mathcal{C}^2$, then the holomorphic sectional curvature is less than or equal to -1.*

2. *if M is Kobayashi-hyperbolic and the Kobayashi metric K_M is os class $\mathcal{C}^2$, then the holomorphic sectional curvature is greater than or equal to -1.*

Remark: A complex manifold is said to be Kobayashi hyperbolic (*Carathéodory* hyperbolic) if the Kobayashi metric $K_M(p,v)$ (*Carathéodory* metric $C_M(p,v)$) is locally bounded by a strictly positive constant as $v \neq 0$ varies over the tangent boundle T_pM. The complex hyperbolic space is a Kähler manifold, and it is characterized by being the only simply connected Kähler manifold whose holomorphic sectional curvature is a negative constant [33].

With this observation and using the fact that any simply connected complete *Kähler* manifold with constant negative holomorphic sectional curvature is biholomorphic to the ball, we can then get the following characterization of the unit ball [55]:

Theorem 3.3.0.2 *For a simply connected complete hyperbolic complex domain Ω with the Carathéodory metric equal to the Kobayashi metric on Ω, assumed to be smooth hermitian metrics, then Ω is biholomorphic to the unit ball.*

Later, Stanton [52] proved the following result, which generalizes Theorem 3.3.0.2:

Theorem 3.3.0.3 *For a connected complete hyperbolic complex manifold M, if $K_M = C_M$ at a point of M and one of these metrics is hermitian of class $\mathcal{C}^\infty$, then M is biholomorphically equivalent to the unit ball.*

The above two theorems provide us with characterization of the unit ball from the intrinsic metrics perspective and play an essential role in our analysis on intrinsic metrics on bounded domains in the following sections. We seek to continue investigate the conditions where different intrinsic metrics coincide.

In the theory of functions of several complex variables, we have the famous Hartogs' extension Theorem [24]:

Theorem 3.3.0.4 *Let M be a Stein manifold with dimension $n \geq 2$ and $K \subset M$ be a compact subset. If $M \backslash K$ is connected, then any holomorphic function $f : M \backslash K \to \mathbb{C}$ extends to a holomorphic function $\bar{f}$ on all of M, where $f = \bar{f}|_{M \backslash K}$.*

Remark: A Stein manifold is a closed complex submanifold in some $\mathbb{C}^n$.

We would like to raise attention to a theorem by Gaussier and Zimmer, which proves a metric analogue of the Hartogs' extension Theorem where the holomorphic function is replaced by a locally symmetric Hermitian metric. The key importance of this extension theorem is that with further analysis on the existence of complex geodesic as discussed in the previous section, we are able to extend the open set of Kobayashi metric and *Carathéodory* metric agreeing. We will state the theorem here and proof can be found in [21].

Theorem 3.3.0.5 *Let M be a Stein manifold with complex dimension $n \geq 2$ and $K \subset M$ is a compact subset where $X \backslash K$ is connected. For a Hermitian metric h_0 on $X \backslash K$ complete at infinity, if h_0 is a Hermitian locally symmetric metric, then there exists a complete Hermitian locally symmetric metric h on M such that $h = h_0$ on $M \backslash K$.*

The technical definitions of locally symmetric space and locally symmetric metrics are given as follow:

Definition 3.3.0.6 *A Riemannian manifold (M, h) is called a locally symmetric space if for every $p \in M$ there is an open neighborhood U_p of p and a local isometry $s_p : U_p \to M$ such that $s_p(p) = p$ and $d(s_p)_p = -id$. If each s_p ia an isometry defined on all of M and M is connected, then M is a global symmetric space. The map s_p is called a geodesic symmetry at p.*

Definition 3.3.0.7 *A Riemannian metric h on complex manifold M is hermitian (locally) symmetric metric is h is Hermitian, (M, h) is a (locally) symmetric space and each geodesic symmetry s_p is holomorphic.*

With the above theorem and using the fact that a *Kähler* metric with constant holomorphic sectional curvature is locally symmetric, Gaussier and Zimmer [21] then proved the following corollary. Building on the uniformization theorem of manifolds with constant holomorphic sectional curvature 2.5.0.7, this corollary established the nice relation between local holomorphic sectional curvature conditions and global structure for complete *Kähler* metrics, which shows that constant negative holomorphic sectional curvature on the complement of a compact set ensures the universal cover is biholomorphic to the unit ball. A similar result for simply connected non-positively curved *Kähler* manifolds were proved by by Seshadri and Verma [50] previously. We will use corollary 3.3.0.8 as a crucial step in the proof of our main theorem.

Corollary 3.3.0.8 *Let M be a Stein manifold with complex dimension $n \geq 2$ and h is a complete Kähler metric on M. If there exists a compact set $K \subset M$ such that h has constant negative holomorphic sectional curvature on $M \backslash K$, then the universal cover of M is biholomorphic to the unit ball.*

Proof. Suppose M is a Stein manifold with dimension greater than 2, $K \subset M$ is compact and h_0 is a complete *Kähler* metric on M with constant holomorphic sectional curvature on $M \backslash K$. Since Stein manifolds are one ended [23], i.e., if $K \subset M$ is a compact set, then there exists a compact set $K' \subset M$ such that $K \subset K'$ and $M \backslash K'$ is connected, we can assume that $M \backslash K$ is connected. Fix $z \in M \backslash K$ and let $\exp_z : T_zM \to M$ be the exponential map at z associate to the *Kähler* metric h_0. Then fix a ball $\mathbb{B}_n(z,r)$ centered at z with $\mathbb{B}_n(z,r) \subset M \backslash K$ and define

$$s_z := \exp_z \circ (-id_{T_z M}) \circ \exp_z^{-1} : \mathbb{B}_n(z,r) \to \mathbb{B}_n(z,r).$$

Then the map s_z is a diffeomorphism with $s_z(z) = z$. Since the holomorphic sectional curvature of h_0 is constant on $M \backslash K$, by the Cartan-Ambrose-Hicks Theorem [23], s_z is a holomorphic local isometry. Since $d(\exp_z)_0 = id$, we have that $d(s_z)_z = -id_{T_z M}$, so h_0 is locally symmetric at z. Then by Theorem 3.3.0.5, there exists a complete locally symmetric metric h on M such that $h = h_0$ on $M \backslash K$. Since the universal cover of M is globally symmetric, hence homogeneous, and g has constant negative holomorphic sectional curvature on $M \backslash K$, we then see that g has constant negative holomorphic sectional curvature on M. By Theorem 2.5.0.7, the universal cover of M is biholomorphic to the unit ball. ∎

With this characterization of *Kähler* manifolds universally covered by the unit ball, we then can continue to investigate the conditions of intrinsic metrics being *Kähler* and characterize the domains with other intrinsic metrics.

3.4 *Kähler* Condition of ***Carathéodory*** Metric

Building up from the discussion in previous sections, we'd like first to state this interesting theorem by Gaussier and Zimmer [21], which characterized domains whose universal cover are biholomorphic to a ball through the *Kähler* Condition of the Kobayashi metric. So far, we have mentioned many results on characterizations of the unit ball and now we shift our attention to the cases where intrinsic metrics coincides.

Theorem 3.4.0.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of dimension $n \geq 2$ with $\mathcal{C}^2$ boundary. Then the following are equivalent:*

1. *The Kobayashi metric on Ω is Kähler on Ω is Kähler;*

2. *The Kobayashi metric on Ω is Kähler with constant holomorphic sectional curvature;*

3. *The universal covering of Ω is biholomorphic to the unit ball.*

Motivated by the result of Gaussier and Zimmer, we established the following parallel results for *Carathéodory* metric that provides characterization of domains biholomorphic to the unit ball. However, our theorem is different than the above one mainly because we extend the characterization to domains directly biholomorphic to the unit ball instead of just the universal cover being biholomorphic to the unit ball. We will state the theorem as follow:

Theorem 3.4.0.2 *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain of dimension $n \geq 2$ with $\mathcal{C}^2$ boundary. Then the following are equivalent:*

1. *The* Carathéodory *metric on Ω is Kähler.;*

2. *The* Carathéodory *metric on Ω is Kähler with constant holomorphic sectional curvature;*

3. *Ω is biholomorphic to the unit ball.*

4. *The* Carathéodory *metric on Ω is a scalar multiple of the Bergman metric.*

5. *The* Carathéodory *metric on Ω is Kähler on $U \cap \Omega$, where U is an open set that contains the boundary of Ω.*

Proof. It is well known that the The *Carathéodory* metric on the unit ball is *Kähler* and (2) implies (1) and (5) by definition. Then we only need to show the non-trivial part that (1) implies (3). By theorem 3.2.0.5, we know that there exists some complex geodesic near the boundary of Ω, i.e., there exist $\epsilon > 0$ such that if $z \in \Omega$ and $\delta_\Omega(z) < \epsilon$, then there is a open set in $E_\Omega(z)$ where the *Carathéodory* metric and Kobayashi metric agree. Let $\psi : \mathbb{D} \to \Omega$ be the complex geodesic with $\psi(0) = z$ and $\psi'(0) = \lambda v$ and $\rho \circ \psi = id$. By the non-decreasing property of the Kobayashi distance and *Carathéodory* distance, we have that

$$k_\Omega(\psi(\zeta), \psi(\eta)) \leq k_\mathbb{D}(\zeta, \eta) = k_\mathbb{D}((\rho \circ \psi)(\zeta), (\rho \circ \psi)(\eta)) \leq k_\Omega(\psi(\zeta), \psi(\eta)),$$

and similarly

$$c_\Omega(\psi(\zeta), \psi(\eta)) \leq c_\mathbb{D}(\zeta, \eta) = c_\mathbb{D}((\rho \circ \psi)(\zeta), (\rho \circ \psi)(\eta)) \leq c_\Omega(\psi(\zeta), \psi(\eta)),$$

for every $\zeta, \eta \in \mathbb{D}$. Then $\psi : (\mathbb{D}, K_\mathbb{D}) \to (\Omega, K_\Omega)$ is an isometric embedding and the holomorphic sectional curvature $H(z; v) = -1$. For a fixed z, since the map $v \in T_z\Omega \to H(z; v)$ is algebraic, we obtain that $H(z; v) = -1$ for all non-zero $v \in T_z\Omega$. Let $K := \{z \in \Omega | \delta_\Omega(z) \geq \epsilon\}$ to be a compact set. Then the holomorphic sectional curvature of C_Ω equal -1 on $\Omega \backslash K$ and C_Ω is a Hermitian locally symmetric metric by the fact that a *Kähler* metric with constant holomorphic sections curvature is locally symmetric. By the work of Graham [29], the *Carathéodory* metric C_Ω is complete on Ω. Then by Corollary 3.3.0.8, there exists a complete *Kähler* metric h on Ω with constant negative holomorphic sectional curvature such that $h = C_\Omega$ on $\Omega \backslash K$ and the universal cover of Ω is biholomorphic to the unit ball. By Theorem 3.4.0.1, the Kobayashi metric is also a complete *Kähler* metric of constant negative holomorphic sectional curvature on Ω. Let $\pi : \mathbb{B}^n \to \Omega$ be the holomorphic covering map. The pullback metric $\pi^*(C_\Omega)$ has constant negative holomorphic curvature on $\tilde{p} \in \mathbb{B}_n$ such that $\pi(\tilde{p}) = p$ for $p \in \Omega \backslash \Sigma$. For any point $q \in \Sigma$, $p \in \Omega \backslash \Sigma$ such that $\pi(\tilde{p}) = p, \pi(\tilde{q}) = q$, we can find some automorphism f on $\mathbb{B}_n$ such that $f(\tilde{q}) = \tilde{p}$ since $\mathbb{B}_n$ is a symmetric space and therefore $(f \circ \pi)^*(C_\Omega)$ has the same negative holomorphic sectional curvature on $\tilde{q}$ as well. Since f is an isometry, then we see that the pullback $\pi^*(C_\Omega)$ has constant negative holomorphic curvature in $\mathbb{B}_n$, which is equal to the holomorphic sectional curvature of Kobayashi metric in $\mathbb{B}_n$, since there is there is only one unique metric on the unit ball. Hence, we have $\pi^*(K_\Omega) = \pi^*(C_\Omega)$ on the unit ball with holomorphic sectional curvature equaling -1, both K_Ω and C_Ω are complete Kähler metrics of the same constant negative holomorphic sectional curvature on Ω. Then by Theorem 3.3.0.3, we can conclude that Ω is biholomorphic to the unit ball. ■

For complex domains of dimension $n = 1$, the same conclusion holds. To discuss the 1-dim case, we first need to introduce the following Theorem by Suita [43]:

Theorem 3.4.0.3 *For a bounded domain $\Omega \subset \mathbb{C}$, the Carathéodory metric is real analytic.*

Proof. Let $\{\Omega_n\}$ be a canonical exhaustion of Ω such that the boundary of Ω_n consists of a finite number of analytic curves. Let C_{Ω_n} be the *Carathéodory* metric for each Ω_n. By the decreasing property of *Carathéodory* metric, we see that $\{C_{\Omega_n}\}$ is decreasing and tends to C_Ω. Then there exists extremal functions f_n such that $f_n' = C_{\Omega_n}$ and $f_0' = C_\Omega$. It is shown in [25] that these extremal functions are unique and in [3] that the function $\log C_\Omega$ is subharmonic.

In every Ω_n there exists the *Szegö* kernel $S_n(z, \zeta)$ and its adjoint kernel $l_n(z, \zeta) =$

$\frac{1}{\pi(z-\zeta)^2} + \frac{2\partial^2 S_n(z,\zeta)}{\pi\partial z\partial\zeta}$ [5]. The *Szegö* kernel $S_n(z,\zeta)$ is Hermitian and analytic with respect to z and ζ. It is shown in [5] that

$$f_n(z) = \frac{S_n(z,\zeta)}{l_n(z,\zeta)} \text{ with } C_{\Omega_n}(\zeta) = 2\pi S_n(\zeta,\zeta).$$

and

$$|S_n(z,\zeta)|^2 \leq |S_n(z,z)||S_n(\zeta,\zeta)|.$$

Thus $S_n(z,\zeta)$ is uniformly bounded on every compact subset and hence forms a normal family of analytic functions of two variables z,ζ. We want to show that $\{S_n(z,\zeta)\}$ converges to $S(z,\zeta)$ uniformly on every compact subset of Ω.

Suppose there exists two limits $S(z,\zeta)$ and $S'(z,\zeta)$. We may assume $0 \in \Omega$. Then we see that $S(z,\zeta) - S'(z,\zeta)$ has an expansion in a polydisk $\{|z| < r\} \times \{|\zeta| < r\}$ such that

$$S(z,\zeta) - S'(z,\zeta) = \sum_{\alpha,\beta=0}^{\infty} a_{\alpha\beta} z_\alpha \overline{\zeta_\beta}.$$

with $a_{\alpha\beta} = \bar{a}_{\beta\alpha}$. Since $S(z,z) = S'(z,z) = C_\Omega(z)\backslash 2\pi$, by setting $z = |z|e^{i\theta}$, we have that $a_{00} = 0$ and by induction we see that $\sum_{\alpha+\beta=n} a_{\alpha\beta} e^{i(\alpha-\beta)\theta} = 0$. Hence, $a_{\alpha\beta} = 0$ and $S(z,\zeta)$ coincides with $S'(z,\zeta)$. Then we see that $\{f_n(z)\}$ converges to $f_0(z)$ uniformly on every compact subset of Ω and that C_Ω is real analytic. ■

Then it is easy to see that if the *Carathéodory* metric is complete with holomorphic sectional curvature equal to -1, then the domain is biholomorphic to the disk.

We will then present couple of examples related to the above theorem to better understand the *Kähler* condition for Kobayashi metric as well as *Carathéodory* metric.

Example 3.4.0.4 *By the work of Lempert [36], the* Carathéodory *metric and Kobayashi metric coincide on convex domains. The theorem 3.3.0.2 and 3.3.0.3 implies that the*

Carathéodory metric on a bounded convex domain is a Hermitian metric if and only if the domain is biholomorphic to the unit ball. In the paper of Burns, Shnider and Wells [13], they constructed an example of a small generic perturbation $\mathcal{P}$ of the unit ball $\mathbb{B}_n$ such that it is convex with smooth boundary but it is not biholomorphic to the unit ball and it is not Kähler for $n \geq 2$. Then the Carathéodory metric on $\mathcal{P}$ is not a Hermitian metric for $n \geq 2$.

Example 3.4.0.5 *As described in example 2.2.2.7, the Kobayashi metric of the unit bidisk is given by*

$$K_{\mathbb{D}\times\mathbb{D}}((p,p'),(q,q')) = \max\{K_{\mathbb{D}}(p,q), K_{\mathbb{D}}(p',q')\}.$$

We see that the Kobayashi metric is not $\mathcal{C}^1$ and the domain is then not Kähler.

Example 3.4.0.6 *Dong and Treuer [15] constructed the following bounded domain*

$$\mathcal{R} := \{(z,w) \in \mathbb{C}^2 : |z|^4 + |z|^2 + |w|^2 < 1\},$$

which is strongly convex with an algebraic doundary and it is not biholomorphic to $\mathbb{B}^2$. The well-known complex analytic ellipsoid

$$E := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^{2n} < 1\}$$

is convex and not biholomorphic to the unit ball $\mathbb{B}^2$, where $1 < n \in \mathbb{Z}^+$. Then the Carathéodory metric on $\mathcal{P}$ is not a Hermitian metric.

Example 3.4.0.7 *Burns and Shnider [12] have constructed examples of non-simply connected strictly pseudoconvex domains with spherical boundaries in $\mathbb{C}^n$:*

$$\mathcal{S} := \{(z,w) \in \mathbb{C}^2 : \sin\log|z| + |w|^2 < 0, e^{-\pi} < |z| < 1\}.$$

By the uniformization theorem [41] [42], this domain is universally covered by the unit ball, so by theorem 3.4.0.1 the Kobayashi metric is Kähler, yet by our theorem 3.4.0.2 the Carathéodory *metric is not Kähler.*

Chapter 4

Rigidity of the Bergman Metric

4.1 Lu's Constant

The Lu constant comes from a significant result by Lu [37] that the Bergman metric always dominates the *Carathéodory* metric on bounded Euclidean domains. We will present a short proof by Hahn here [54].

Theorem 4.1.0.1 *Let M be a complex manifold. For each tangent vector $v \in T_z M = \mathbb{C}^n$ at $z \in M$, it holds that*

$$B_M(z,v) \geq C_M(z,v),$$

where $B_M(z,v)$ and $C_M(z,v)$ are the lengths of v respect to the Bergman metric and differential Carathéodory metric respectively if M admits them.

Proof. Let F be the set of the square integrable holomorphic $n-$forms on a complex manifold of dim n. Then F is a separable Hilbert space with inner product $(f,g) = \int f\bar{g}$. We have that the Bergman kernel form $\mathcal{K}_M(z,\bar{z}) = \max_{(f,f)=1} f \wedge \bar{f}$ at $z \in M$. Let g be the element of F with $g(z) = 0$ and $||g|| \leq 1$ which maximizes $b_g = \partial_\xi g \wedge \overline{\partial_\xi g}$ at $z \in M$, where ∂_ξ denotes a holomorphic tangent vector in the $\xi-$direction. Then we have

$$\begin{aligned}
b_g &= \max_{g(z)=0, ||g||_{L^2} \leq 1} \partial_\xi g \wedge \overline{\partial_\xi g}(z) \\
&= \max_{\int \mathcal{K}_M(z,\zeta) g(\zeta) d\zeta = 0 = g(z), ||g||_{L^2} \leq 1} |\partial_\xi g(z)|^2 \\
&= \max_{\int \mathcal{K}_M(z,\zeta) g(\zeta) d\zeta = 0, ||g||_{L^2} \leq 1} |\partial_\xi \int \mathcal{K}_M(z,\zeta) g(\zeta) d\zeta|^2 \\
&= \max_{\int \mathcal{K}_M(z,\zeta) g(\zeta) d\zeta = 0, ||g||_{L^2} \leq 1} |\int \partial_\xi \mathcal{K}_M(z,\zeta) g(\zeta) d\zeta|^2,
\end{aligned}$$

which is the maximum of the projection of $\partial_\xi \mathcal{K}_M(z,)$ onto the subspace $\mathcal{K}_M(z,)^\perp$ and by Hilbert space theory, we should get $(\partial_\xi k_M(z,), \partial_\xi k_M(z,)) - \frac{(\partial_\xi \mathcal{K}_M(z,), \mathcal{K}_M(z,))^2}{(\mathcal{K}_M(z,), \mathcal{K}_M(z,))}$. Then

$$\begin{aligned}
b_g &= \int \partial_\xi \mathcal{K}_M(z,\zeta) \overline{\partial_\xi \mathcal{K}_M(z,\zeta)} d\zeta - \frac{|\int \partial_\xi \mathcal{K}_M(z,\zeta) \overline{\mathcal{K}_M(z,\zeta)} d\zeta|^2}{\int \mathcal{K}_M(z,\zeta) \overline{\mathcal{K}_M(z,\zeta)} d\zeta} \\
&= \partial_\xi \partial_{\bar{\xi}} \int \mathcal{K}_M(z,\zeta) \overline{\mathcal{K}_M(z,\zeta)} d\zeta - \frac{|\partial_\xi \mathcal{K}_M(z,\bar{z})|^2}{\mathcal{K}_M(z,\bar{z})} \\
&= \partial_\xi \partial_{\bar{\xi}} \mathcal{K}_M(z,\bar{z}) - \frac{|\partial_\xi \mathcal{K}_M(z,z)|^2}{\mathcal{K}_M(z,\bar{z})} \\
&= \mathcal{K}_M(z,\bar{z}) (\partial_\xi \partial_{\bar{\xi}} \log \mathcal{K}_M(z,\bar{z})) \\
&= \mathcal{K}_M(z,\bar{z}) B_M^2(z,\xi).
\end{aligned}$$

The for any holomorphic function $\phi : M \to \mathbb{D}, \phi(z) = 0$, we have ϕf is an element of F such that $b_g = \mathcal{K}_M(z,\bar{z}) B_M^2(z,\xi) \geq \partial_\xi(\phi f) \wedge \overline{\partial_\xi(\phi f)} = (\partial_{xi}\phi) f \wedge \overline{(\partial_\xi) f} = |\partial_\xi \phi|^2 f \wedge \bar{f}, z \in M$. Hence, $\mathcal{K}_M(z,\bar{z}) B_M^2(z,\xi) \geq \sup |\partial_\xi \phi|^2 \mathcal{K}_M(z,\bar{z})$ and $B_M^2 \geq C_M^2$. ■

Let B_Ω and C_Ω be the Bergman metric and *Carathéodory* metric respectively. Then the Lu constant $L(\Omega)$ of a bounded domain Ω in $\mathbb{C}^n$ is defined as follow.

Definition 4.1.0.2

$$L(\Omega) = \sup_{z\in\Omega, 0\neq v\in T_z\Omega} \left(\frac{C_\Omega(z,v)}{B_\Omega(z,v)}\right).$$

The definition of $L(\Omega)$ implies that $L(\Omega)B_\Omega \geq C_\Omega$. By theorem 4.1.0.1, we see that $L(\Omega) \leq 1$. For the unit ball in $\mathbb{C}^n$, $L(\Omega) = \frac{1}{\sqrt{n+1}}$ by standard calculation of the constant Bgerman metric and *Carathéodory* metric on the unit ball.

For polydisk $\mathbb{D}^n = \{\mathbb{D} \times \mathbb{D}... \times \mathbb{D} | \mathbb{D} \subset \mathbb{C}\}$, we have that the Bergman Kernel $\mathcal{K}(z,w) = \prod_{i,j=1} \frac{1}{(1-z_i\bar{w}_j)^2}$, so that the Berman metric $g(z) = \sum_{i,j=1}^n g_{ij}$

$$g_{ij}(z) = \log \mathcal{K}(z,z) = \begin{cases} 0 & \text{if } i \neq j \\ \\ \frac{2}{(1-|z_i|^2)^2} & \text{if } i = j. \end{cases}$$

Since the *Carathéodory* metric on the polydisk $C_{\mathbb{D}^n}(z,v) = \max\{C_{\mathbb{D}}(z_1,v_1), ..., C_{\mathbb{D}(z_n,v_n)}\} = \max \frac{|v_i|}{(1-|z_i|^2)^2}$, we can conclude $L(\mathbb{D}^n) = \sup_{z\in\Omega,0\neq v\in T_z\Omega}(\frac{C_\Omega(z,v)}{B_\Omega(z,v)}) = \frac{\max|v_j|}{\sqrt{2}||v||} = 1/\sqrt{2}$. If Ω is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$, then we have that $L(\Omega) \geq \frac{1}{\sqrt{n+1}}$ by the boundary behaviors of the Bergman metric and *Carathéodory* metric due to Diederich [14] and Graham [29], aS $\frac{C_\Omega}{B_\Omega}$ approaches $\frac{1}{\sqrt{n+1}}$ near the boundary. If the holomorphic sectional curvature of the Bergman metric on a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ is bounded above by $-\frac{2}{n+1}$, then $L(\Omega) = \frac{1}{\sqrt{n+1}}$ following from the Ahlfors-Schwarz lemma [2].

Definition 4.1.0.3 *The Gaussian curvature of a Hermitian metric h is defined to be*

$$G(h) = -\frac{1}{h}\frac{\partial^2 \log h}{\partial z \partial \overline{z}}.$$

Ahlfors established a beautiful connection between curvature and distortion for holomorphic maps by showing that any holomorphic map of the disk into a Riemann surface furnished with a conformal metric whose Gaussian curvature is everywhere less than or equal to -1 must be distance decreasing from the *Poincaré* metric of the unit disk:

Lemma 4.1.0.4 (Ahlfors-Schwarz) *Let $f : \mathbb{D} \to M$ be a mapping. If M is a Riemann surface equipped with a Hermitian metric ds_M^2 with curvature bounded from above by a negative number $-K$, then we have*

$$f^* ds_M^2 \leq \frac{1}{K} ds_{\mathbb{D}}^2,$$

where $ds_{\mathbb{D}}^2 = \frac{2}{(1-|z|^2)^2}(dzd\bar{z})$ is the Poincaré metric of the unit disk $\mathbb{D}$.

Proof. Since f is holomorphic, $f^* ds_M^2$ is a $(1,1)-$tensor on $\mathbb{D}$. Thus, $f^* ds_M^2 = A(z) dz d\overline{z}$ for some smooth function $A : \mathbb{D} \to \mathbb{R}$. Let $B(z) = \frac{2}{(1-|z|^2)^2}$. It suffices to show that $\frac{A(z)}{B(z)} \leq \frac{1}{K}$ for each $z \in \mathbb{D}$. We then shall divide the proof into two cases.

Special Case. Assume that the function $u(z) := \frac{A(z)}{B(z)}$ attains its maximum at $z_0 \in$. Then it is sufficient to show that $u(z_0) \leq \frac{1}{K}$. If $u(z_0)$ is 0, then we are done. Hence we may assume that it is positive. Consequently, $u(z)$ is positive in a small neighborhood of z_0.

Since the function $u(z)$ attain its maximum at z_0, then we see that

$$\nabla \log u|_{z_0} = 0 \text{ and } \Delta \log_u |_{z_0} \leq 0.$$

At z_0, we have $0 \geq \Delta \log A - \log B$. Since the curvature of the *Poincaré* metric of the unit disk is -1, we have that

$$\frac{-1}{B} \Delta \log B = -1.$$

From the upper bounded condition of the curvature of ds_M^2, we also have that

$$\frac{-1}{A} \Delta \log A \leq -K.$$

Now we obtain at z_0, $0 \geq AK - B$. Consequently, we get that $u \leq \frac{1}{K}$. However the assumption on the existence of maximum above does not hold in general, so we need to show the following:

General Case. Now $u : \mathbb{D} \to \mathbb{R}$ is just non-negative and does not have to attain its maximum anywhere on $\mathbb{D}$. Let $\zeta \in \mathbb{D}$ be fixed. We want to show that $u(\zeta) \leq \frac{1}{K}$. Then consider a constant r with $|\zeta| < r < 1$ and $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$, and endow it with the metric $ds_r^2 = B_r dz d\bar{z} = \frac{2r^2 dz d\bar{z}}{(r^2 - |z|^2)^2}$. Let $f_r = f|_{\mathbb{D}_r} : \mathbb{D}_r \to M$. Then we see that $f_r^* ds_M^2 = u_r(z) ds_r^2$ with

$$u_r(z) = 2r^{-2}(r^2 - |z|^2)^2 A(z),$$

where A is a non-negative function on the whole $\mathbb{D}$. Therefore, u_r is a non-negative function that vanishes on $\{z : |z| = r\}$. Therefore it attains its maximum on $\mathbb{D}_r$. Then we can apply the same calculation at the maximum point of u_r to obatain that

$$f^* ds_M^2|_\zeta \leq \frac{1}{K} ds_r^2|_\zeta.$$

Then, letting $r \to 1$ we get the desired result. ■

Yau and Royden generalized the Ahlfors-Schwarz lemma for more general complex manifolds, which will be applied to the main results presented in the following sections. The statements of these important results are cited below with proof can be found in [57] and [48].

Theorem 4.1.0.5 *(Yau) Let M be a complete Kähler manifold with metric ds_M^2 of Ricci curvature bounded from below by K_1. Let N be another Hermitian manifold with metric ds_N^2 of holomorphic bisectional curvature bounded from above by a negative constant K_2.*

Then if there is a non-constant holomorphic mapping f from M into N, we have $K_1 \geq 0$ and

$$f^*(ds_N^2) \leq \frac{K_1}{K_2} ds_M^2.$$

Theorem 4.1.0.6 *(Royden) Let M and N be Hermitian manifolds with M complete and holomorphic sectional curvature bounded from below by a constant $K_1 \leq 0$ and the holomorphic sectional curvature of N bounded from above by a constant $K_2 < 0$. Assume either the M has Riemann sectional curavture bounded from below or that M is Kähler with holomorphic bisectional curvature bounded from below. Then any holomorphic map $f : M \to N$ satisfies that*

$$f^*(ds_N^2) \leq \frac{K_1}{K_2} ds_M^2$$

4.2 Bergman Representative Coordinates

The Bergman kernel function gives rise not only to the Bergman metric, but also to some local holomorphic coordinate systems which play a significant role in the study of biholomorphic mappings. These local coordinate system is known as the Bergman representative coordinates. The biholomorphic mappings are linear when expressed in Bergman representative coordinates. We will follow [45] in presenting the construction of Bergman Representative coordinates.

Let $\Omega \in \mathbb{C}^n$ be a bounded domain and a point $p \in \Omega$. The diagonal Bergman kerne $\mathcal{K}_\Omega(p, p)$ is real and positive so that there is a neighborhood U of p, such that for all z, ζ in the neighborhood of p, we have $\mathcal{K}_\Omega(z, \zeta) \neq 0$.

Definition 4.2.0.1 *For all $z, \zeta \in U$, we define*

$$b_j(z) = b_{j,p}(z) = \frac{\partial}{\partial \bar{\zeta}_j} \log \frac{\mathcal{K}(z,\zeta)}{\mathcal{K}(\zeta,\zeta)}\bigg|_{\zeta=p}.$$

Note that these coordinates are well-defined, independent of the choice of logarithmic branches and each $b_j(z)$ is a holomorphic function of z. Notice that it is necessary to have some restriction on z in a neighborhood U of p, since $\mathcal{K}_\Omega(z,\zeta)$ be vanish for some pair $\Omega \times \Omega$. The mapping

$$z \longmapsto (b_1(z), ..., b_n(z)) \in \mathbb{C}^n$$

is defined and holomorphic in a neighborhood of p. Also, note that $(b_1(p), ..., b_n(p)) = 0$.

By the holomorphic inverse function theorem, these functions give local coordinates if the Jacobian $\det(\frac{\partial b_j}{\partial z_k})_{j,k=1,\dots,n}$ is non-zero at p. The nonvanishing of this determinant of p is an immediate consequence of the fact that the Bergman metric is positive definite, i.e.,

$$\frac{\partial b_j}{\partial z_k}\bigg|_{z=p} = \frac{\partial}{\partial z_k}\left(\frac{\partial}{\partial \bar{\zeta}_j} \log \mathcal{K}(z,\zeta)\right)\bigg|_{z=\zeta=p} = \frac{\partial^2}{\partial z_k \partial \bar{z}_j} \log \mathcal{K}(z,z)\bigg|_{z=p}.$$

The last term is the Hermitian inner product $\langle \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_j} \rangle|_p$ with respect to the Bergman metric. Thus $\det(\frac{\partial b_j}{\partial z_k})|_p$ is the determinant of the inner product matrix of a positive definite Hermitian inner product. We can use the new coordinates to study biholomorphic mappings.

Lemma 4.2.0.2 *Let Ω_1 and Ω_2 be two bounded domains in $\mathbb{C}^n$ with $p_1 \in \Omega_1$ and $p_2 \in \Omega_2$. Denote by $(b_1^1, ..., b_n^1)$ the Bergman coordinates as defined near $p_1 \in \Omega_1$ with Bergman kernel for Ω_1 and $(b_1^2, ..., b_n^2)$ the Bergman coordinates defined in the same way near $p_2 \in \Omega_2$ with Bergman kernel for Ω_2. Suppose that there is a biholomorphic mapping $F : \Omega_1 \to \Omega_2$ with $F(p_1) = p_2$. Then the function defined near $0 \in \mathbb{C}^n$ by*

b^1 coordinate $\alpha = (\alpha_1, ..., \alpha_n) \longmapsto b^2$ coordinate of the image of $\alpha-$point under F

is a $\mathbb{C}$-linear transformation.

In short, we way that the biholomorphic mapping F is linear when expressed in the Bergman representative coordinates b^j.

Proof. Let $(z_1, ..., z_n)$ and $(\zeta_1, ..., \zeta_n)$ be the $\mathbb{C}^n$ coordinates in Ω_1 and $(z'_1, ..., z'_n)$ and $(\zeta'_1, ..., \zeta'_n)$ for the $\mathbb{C}^n$ coordinates in Ω_2. Since we have

$\frac{\mathcal{K}_{\Omega_2}(F(z),F(\zeta))}{\mathcal{K}_{\Omega_2}(F(z),F(\zeta))} = \frac{\mathcal{K}_{\Omega_1}(z,\zeta)}{\mathcal{K}_{\Omega_1}(\zeta,\zeta)} \times$ (a holomorphic functions of z) $\times$ (a holomorphic function of ζ)

then we observe that

$$\frac{\partial}{\partial \bar{\zeta}_j} \log \frac{\mathcal{K}_{\Omega_2}(F(z), F(\zeta))}{\mathcal{K}_{\Omega_2}(F(z), F(\zeta))} = \frac{\partial}{\partial \bar{\zeta}_j} \log \frac{\mathcal{K}_{\Omega_1}(z, \zeta)}{\mathcal{K}_{\Omega_1}(\zeta, \zeta)}.$$

By the complex chain rule, we see that

$$\begin{aligned} b^1_j(z) &= \frac{\partial}{\partial \bar{\zeta}_j} \log \frac{\mathcal{K}_{\Omega_1}(z, \zeta)}{\mathcal{K}_{\Omega_1}(\zeta, \zeta)}\Bigg|_{\zeta = p_1} \\ &= \frac{\partial}{\partial \bar{\zeta}_j} (\log \mathcal{K}_{\Omega_2}(F(z), F(\zeta)) - \log \mathcal{K}_{\Omega_2}(F(\zeta), F(\zeta)))\Bigg|_{\zeta = p_1} \\ &\sum_k \frac{\partial \overline{F}^k}{\partial \bar{\zeta}_j} \cdot \frac{\partial}{\partial \overline{\zeta'}_k} \cdot \log \frac{\mathcal{K}_{\Omega_2}(F(z), \zeta')}{\mathcal{K}_{\Omega_2}(\zeta', \zeta')}\Bigg|_{\zeta' = F(p_1)}, \end{aligned}$$

where F^k is the $k-$th coordinate of the function $F(\zeta_1, ..., \zeta_n)$, and the last expression is then equal to

$$\sum_k \frac{\partial \overline{F}^k}{\partial \bar{\zeta}_j}\Bigg|_{\zeta = p_1} \cdot b^2_k(F(z)).$$

Since the Jacobian matrix $(\partial F_k)/\partial \zeta_j$ of F is invertible at p, it follows that $b^2_k(F(z))$ are linear functions of the $b^1_j(z)$ coordinates. ∎

Then we can define the Bergman representative coordinate to be certain constant-coefficient linear combinations of $b_j(z)$.

Definition 4.2.0.3 *The Bergman represnetative coordinate $T(z) = (w_1, ..., w_n)$ relative to $p \in \Omega$ is defined as*

$$w_j(z) = \sum_{i=1}^{n} g^{ij}(p) \frac{\partial}{\partial \overline{\zeta}_i} \log \frac{\mathcal{K}(z,\zeta)}{\mathcal{K}(\zeta,\zeta)} \bigg|_{\zeta = p},$$

where g^{ij} is the inverse matrix of the Bergman metric $g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \mathcal{K}(z,z)$.

Example 4.2.0.4 *Let $\Omega_1 = \Omega_2$ be the unit disk $\mathbb{D} \subset \mathbb{C}$. Let $a \in \Omega_1$ and $0 \in \Omega_2$. Consider the biholomorphic mapping $F(z) : \Omega_1 \to \Omega_2$ defined by $F(z) = \lambda \frac{z-a}{1-\bar{a}z}$ for some $\lambda \in \mathbb{C}$ and $|\lambda| = 1$. Then we see that the b^1-coordinate at a is*

$$\begin{aligned}
b^1(z) &= \frac{\partial}{\partial \overline{\zeta}} \log \frac{1/(1-z\overline{\zeta})^2}{1/(1-\zeta\overline{\zeta})^2} \\
&= -2\frac{\partial}{\partial \overline{\zeta}} (\log(1-z\overline{\zeta}) - \log(1-\zeta\overline{\zeta})) \\
&= -2(\frac{-z}{1-z\overline{\zeta}} + \frac{\zeta}{1-\zeta\overline{\zeta}}) \\
&= 2(\frac{z}{1-z\overline{a}} - \frac{a}{1-a\overline{a}}) \\
&= (\frac{z-a}{1-\overline{a}z}) \cdot \frac{2}{1-a\overline{a}}.
\end{aligned}$$

We can do the same calculation with $a = 0$ to get the b^2-coordinate $b^2(z) = 2z$. We then get that under the biholomorphic mapping F, z of b^1-coordinate $(\frac{z-a}{1-\overline{a}z}) \cdot \frac{2}{1-a\overline{a}}$ is mapped to $F(z)$ of b^2-coordinate $\frac{2\lambda(z-a)}{1-\bar{a}z}$, which is equal to linear functions of b^1,i.e., $\lambda(1-\overline{a}a)b^1(z)$.

One of the most important application of the Bergman representative coordinates is to the proof of a famous theorem by Lu Qi-Keng [37] on domains with a Bergman metric of constant holomorphic sectional curvature.

Theorem 4.2.0.5 *Let Ω be a bounded domain in $\mathbb{C}^n$ with a complete Bergman metric B^2_Ω. If the holomorphic sectional curvature is equal to a negative constant $-c^2$, then Ω is* *biholomorphic to the unit ball* $\mathbb{B}_n$ *and* $c^2 = \frac{2}{n+1}$.

Proof. We first observe that the holomorphic sectional curvature must be negative. Since, if it is positive, then Ω would be a complete Riemannian manifold with all sectional curvatures greater than or equal $c^2/2 > 0$. Hence Ω would be compact by the Myers' theorem that: a complete Riemannain manifold with sectional curvature everywhere $\geq \epsilon > 0$ has diameter $\leq \pi/\sqrt{\epsilon}$ and is hence compact. If c^2 were zero, then the universal cover of Ω would be biholomorphic to $\mathbb{C}^n$. However, since Ω is bounded, the covering map into Ω would be constant by Liouville's theorem. This would contradict the surjectivity of the covering map.

Let g_Ω be the Bergman metirc of Ω, then the metric $g := \frac{c^2(n+1)}{2} g_\Omega$ has constant negative holomorphic sectional curvature $-2/(n+1)$. Thus, the simply connected covering space $\tilde{\Omega}$ of Ω with the pullback $\tilde{g}$ of the metric g is biholomorphically isometric to the unit ball $\mathbb{B}_n$ with its Bergman metric. Then we obtain the holomorphic covering map $\pi : \mathbb{B}_n \to \Omega$, which is locally isometric for the Bergman metric on $\mathbb{B}_n$ and g on Ω, respectively. To prove the theorem, we want to show that the covering map π is injective.

Let $p = \pi(0)$. Since F is a covering map, there exists an open neighborhood of U of p and a neighborhood V of 0 such that $\pi|_v : V \to U$ is a biholomorphism. Let i be the inverse of $\pi|_V$. For $z, \zeta \in U$, we have

$$\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log \mathcal{K}_{\mathbb{B}_n}(i(z), i(z)) = g_{jk} = \lambda g_{\Omega, jk},$$

where $\lambda = \frac{c^2(n+1)}{2}$. This implies that

$$\frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log \mathcal{K}_{\mathbb{B}_n}(i(z), i(z)) - \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \lambda \log \mathcal{K}_\Omega(z, z) = 0,$$

for all $z \in U$ and furthermore that

$$\log \mathcal{K}_{\mathbb{B}_n}(i(z), i(\zeta)) - \lambda \log \mathcal{K}_\Omega(z, \zeta) = \psi(z) + \overline{\psi(\zeta)},$$

for some holomorphic function $\psi : U \to \mathbb{C}$. We may need to replace U by a smaller, simply connected neighborhood but it can be done without loss of generality. Consequently one obtains the following

$$\frac{\partial}{\partial \overline{w}_j} \log \frac{\mathcal{K}_{\mathbb{B}_n}(z, w)}{\mathcal{K}_{\mathbb{B}_n}} - \frac{\partial}{\partial \overline{w}_j} \lambda \log \frac{\mathcal{K}_\Omega(z, w)}{\mathcal{K}_{\mathbb{B}_n}(w, w)} = 0$$

for all $z, w \in U$. Then we can compute the Bergman representative coordinate system for V and U. By the lemma 4.2.0.2, we obtain that

$$i(\zeta) = (\pi|_V)^{-1} = (b^1)^{-1} \circ A \circ b^2(\zeta)$$

for $\zeta \in U$, where A is the linear map represented by the matrix $\lambda \overline{\frac{\partial \pi_k}{\partial z_j}|_0}$. The map b^1 is a constant multiple of the Euclidean coordinate system and thus it extends to all of $\mathbb{C}^n$ holomorphically. So does the linear map A. The map $\zeta \to b^2(\zeta)$ extends to a holomorphic mapping of $\Omega \backslash Z_p$, where $Z_p = \{\zeta \in \Omega | \mathcal{K}_\zeta(\zeta, p) = 0\}$. Since $\mathcal{K}_\Omega(\cdot, p)$ is a holomorphic function on Ω with $\mathcal{K}_\Omega(p, p) \neq 0$, the set Z_p is an analytic variety whose complex codimension in Ω is 1. Hence $\Omega \backslash Z_p$ is a connected, dense, and open subset of Ω. Therefore, the map i extends to a holomorphic mapping of $\Omega \backslash Z_p$ into $\mathbb{C}^n$, denoted $\tilde{i}$.

Consider $\pi^{-1}(Z_p)$. We see that $X := \pi^{-1}(Z_p) = \{z \in \mathbb{B}_n | \mathcal{K}_\Omega(i(z), p) = 0\}$. Since $\mathcal{K}_\Omega(i(0), p) = \mathcal{K}_\Omega(p, p) \neq 0$, we have that X is also a complex analytic subvariety of $\mathbb{B}_n$ with complex codimension 1. Thus $\mathbb{B}_n \backslash X$ is a connected, dense, and open subset of $\mathbb{B}_n$. Furthermore, $\tilde{i} \circ \pi : \mathbb{B}_n \backslash X \to \mathbb{C}^n$ is holomorphic with $\tilde{i} \circ \pi(z) = z$ for all $z \in V$ as $i = \tilde{i}$ on

V. It follows that $\tilde{i} \circ \pi(z) = z$ for all $z \in \mathbb{B}_n \backslash X$. Then for every $\zeta \in \Omega \backslash Z_p$, we can choose $w \in \mathbb{B}_n$ such that $\pi(w) = \zeta$. Then we get

$$\tilde{i}(\zeta) = \tilde{i}(\pi(w)) = w.$$

It follows that $\tilde{i}(\Omega \backslash Z_p) \subset \mathbb{B}_n$. We see that $\tilde{i}$ is holomorphic on $\Omega \backslash Z_p$. By the Riemann extension theorem, $\tilde{i}$ extends to a holomorphic mapping of Ω into $\mathbb{C}^n$. Since $\tilde{i}$ is the inverse of π, we see that π is injective. ■

From the classical uniformization theorem 2.5.0.7, we only know that Ω is universally covered by the unit ball. Once we can show that Ω is biholomorphic to the unit ball, then $c^2 = \frac{2}{n+1}$ follows from the fact that the Bergman metric on the unit ball equals to $-\frac{2}{n+1}$.

4.3 Local Rigidity of the Bergman Metric

In this section, we would like to present a main result that generalizes the Lu's uniformization theorem of Bergman metric to bounded domains with a complete Bergman metric of constant holomorhpic sectional curvature in a small neighborhood near the boundary. Our result shows that the same conclusion holds under weaker hypotheses of localized condition. Before stating the main result, we will introduce a couple of technical results used in the proof.

Lu [37] proved the following important lemma in order to prove Theorem 4.2.0.5.

Lemma 4.3.0.1 *If the holomorphic sectional curvature of a Kähler manifold M is constant $-c^2$, then for every point $p \in M$, there exists a neighborhood U_p such that in U_p, with* *respect to the Bergman representative coordinate $T(z)$, we have*

$$h(T(z), \overline{T(z)}) = \frac{-2}{c} \log(1 - \frac{c^2}{2}|T(z)^2|) + f(T(z)) + \overline{f(T(z))},$$

where $f(T(z))$ is analytic in a neighborhood of $T(z) = 0$ and $h(z, \overline{z})$ is analytic function in $z, \overline{z}$ for some Kähler metric $ds^2 = \frac{\partial h(z,\overline{z})}{\partial z \partial \overline{z}} dz_i d\overline{z}_j$.

Lu proved the well-known uniformization theorem for a bounded domain with a complete Bergman metric of constant holomorphic sectional curvature. For a bounded domain $\Omega \subset \mathbb{C}^n$, fix a point $z_0 \in \Omega$ and let $A_{z_0} := \{z \in \Omega | \mathcal{K}(z, z_0) = 0\}$ be the zero set of the Bergman kernel $\mathcal{K}(\cdot, z_0)$. Since A_{z_0} is an analytic variety, as domains $\Omega \backslash A_{z_0}$ and Ω have the same Bergman kernel $\mathcal{K}$ and Bergman metric g.

Definition 4.3.0.2 *The Kahler potential on $\Omega \backslash A_{Z_0}$ is defined to be*

$$\Phi_{z_0}(z) := \log \frac{\mathcal{K}(z,z)\mathcal{K}(z_0, z_))}{|\mathcal{K}(z_0, z_))|^2}$$

for the Bergman metric $g = \partial\overline{\partial}\Phi_{z_0}$

Theorem 4.3.0.3 *Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that the holomorphic sectional curvature of the Bergman metric B^2_Ω is identically equal to a constant $-c^2$ on an open set $U \subset \Omega$. Then for any $p \in \Omega$, the Bergman representative coordinate $T(z) = (w_1, ..., w_n)^\tau$ is a bounded holomorphic map from Ω to the ball $\mathcal{B} := \{(w_1, ..., w_n)^\tau : \sum_{i,j=1}^n w_i g_{ij}(p)\overline{w}_j < 2c^{-2}\}$. Moreover, $T : (\Omega, B^2_\Omega) \to (\mathcal{B}, B^2_\mathcal{B})$ is a holomorphic isometry, where $B^2_\mathcal{B}$ is the Bergman metric for $\mathcal{B}$. Consequently, the holomorphic sectional curvature of the Bergman metric B^2_Ω is identically equal to a negative constant $-c^2$ on Ω.*

Proof.

First, we assume that the Bergman metric at $p \in U$ satisfies

$$g_{\alpha\bar{\beta}}(p) = \delta_{\alpha\beta}, \tag{4.1}$$

and in this case we will show that $T(z)$ defined by 4.2.0.3 is a bounded holomorphic map from Ω to the ball $\mathbb{B}^n := \{w \in \mathbb{C}^n : |w|^2 < 2c^{-2}\}$.

At any $p \in U$, by the uniqueness of the Taylor expansion shown in Bochner [6] and Lu [37], there exists a smaller neighborhood U_p such that the Bergman kernel can be locally decomposed as

$$K(z,z) = \left(1 - \frac{c^2}{2}|T(z)|^2\right)^{\frac{-2}{c^2}} e^{f(T(z))+\overline{f(T(z))}}, \quad z \in U_p, \tag{4.2}$$

where f is holomorphic on U_p. The map T defined by 4.2.0.3 is holomorphic on $\Omega \setminus A_p$, where $A_p := \{z \in \Omega \,|\, K(z,p) = 0\}$ is the zero set of the Bergman kernel $K(\cdot,p)$. Let $\Omega' := \{z \in \Omega \setminus A_p : T(z) \in \mathbb{B}^n\}$ be the set of points in $\Omega \setminus A_p$ that are mapped into the ball. In particular, $U_p \subset \Omega'$. By (4.2) and the theory of power series, one may duplicate the variable with its conjugate so that the full Bergman kernel can be complex analytically continued as

$$K(z,z_0) = \left(1 - \frac{c^2}{2}\sum_{\alpha=1}^{n} w_\alpha(z)\overline{w_\alpha(z_0)}\right)^{\frac{-2}{c^2}} e^{f(T(z))+\overline{f(T(z_0))}}, \quad z, z_0 \in U_p.$$

Consider two Kähler potentials

$$\Phi_p(z) := \log \frac{K(z,z)K(p,p)}{|K(z,p)|^2}, \quad z \in \Omega \setminus A_p$$

and

$$\Phi_{z_0}(z) := \log \frac{K(z,z)K(z_0,z_0)}{|K(z,z_0)|^2}, \quad z \in \Omega \setminus A_{z_0}$$

for the Bergman metric $B_\Omega^2 = \partial\overline{\partial}\Phi_{z_0} = \partial\overline{\partial}\Phi_p$, for $z \in (\Omega \setminus A_p) \cap (\Omega \setminus A_{z_0})$. When U_p is small enough, for any $z_0 \in U_p$, it holds that

$$\begin{aligned}\Phi_{z_0}(z) &= \log \frac{\left(1-\frac{c^2}{2}|T(z)|^2\right)^{\frac{-2}{c^2}} e^{f(T(z))+\overline{f(T(z))}} \left(1-\frac{c^2}{2}|T(z_0)|^2\right)^{\frac{-2}{c^2}} e^{f(T(z_0))+\overline{f(T(z_0))}}}{\left|1-\frac{c^2}{2}\sum_{\alpha=1}^n w_\alpha(z)\overline{w_\alpha(z_0)}\right|^{\frac{-4}{c^2}} |e^{f(T(z))+\overline{f(T(z_0))}}|^2} \\ &= \frac{-2}{c^2}\log\left[\left(1-\frac{c^2}{2}|T(z)|^2\right)\left(1-\frac{c^2}{2}|T(z_0)|^2\right)\left|1-\frac{c^2}{2}\sum_{\alpha=1}^n w_\alpha(z)\overline{w_\alpha(z_0)}\right|^{-2}\right], \quad z \in U_p,\end{aligned}$$

which yields that

$$\Phi_p(z_0) = \Phi_{z_0}(p) = \frac{-2}{c^2}\log\left(1-\frac{c^2}{2}|T(z_0)|^2\right).$$

Note that $\Phi_p(z)$ is defined on $\Omega \setminus A_p$ and thus on Ω', where $\frac{-2}{c^2}\log\left(1-\frac{c^2}{2}|T(z)|^2\right)$ can be defined. Since these two real-analytic functions coincide on U_p, by the identity theorem they are identical to each other on Ω'. That is,

$$\Phi_p(z) = \frac{-2}{c^2}\log\left(1-\frac{c^2}{2}|T(z)|^2\right), \quad z \in \Omega'. \tag{4.3}$$

1) We claim that no point in $\Omega \setminus A_p$ is mapped outside the ball $\mathbb{B}^n$ by T.

If not, suppose there exists some point $q \in \Omega \setminus A_p$ that is mapped to $\{w \in \mathbb{C}^n : |w|^2 \geq 2c^{-2}\}$. Choose some point $q_0 \in \Omega'$. Since $\Omega \setminus A_p$ is path-connected, one can choose a path γ that connects q_0 and q. Suppose under T the image of γ intersects $\partial\mathbb{B}^n$ firstly at some point $T(q_1)$.

Along the path γ take a sequence of points $(q_l)_{l\in\mathbb{N}} \subset \Omega'$ such that $q_l \to q_1$. Then by (4.3),

$$\Phi_p(q_l) = \frac{-2}{c^2}\log\left(1-\frac{c^2}{2}|T(q_l)|^2\right).$$

Here, as $q_l \to q_1$, the left hand side is finite but the right hand side blows up to infinity. This is a contradiction, so we have thus proved our claim, which says that $\Omega' = \Omega \setminus A_p$.

Therefore, (4.3) in fact holds on $\Omega \setminus A_p$.

2) Since T maps $\Omega \setminus A_p$ to the ball $\mathbb{B}^n$ and satisfies

$$|T(z)|^2 < 2c^{-2}, \tag{4.4}$$

by the Riemann removable singularity theorem, T extends across the analytic variety A_p to the whole domain Ω with $|T(z)|^2 \leq 2c^{-2}$. The maximum modulus principle yields that (4.4) in fact holds on Ω.

Lastly, for general $p \in U$, one performs a possible linear transformation F from Ω to Ω_1 such that the Bergman metric on Ω_1 satisfies (4.1) at $F(p)$. Since F is a biholomorphism, the Bergman metric on Ω_1 locally has the same holomorphic sectional curvature.

Denote by $\omega_{\mathcal{B}}$ the Bergman metric on $\mathcal{B}$. From the above proof, we see that the Kähler potential for the Bergman metric has the following formula.

$$\Phi_p(z) = \frac{-2}{c^2} \log \left(1 - \frac{c^2}{2} \sum_{\alpha,\beta=1}^{n} w_\alpha(z) g_{\alpha\bar{\beta}}(p) \overline{w_\beta(z)} \right), \quad z \in \Omega.$$

Direct computations then yield that $T : \Omega \to \mathcal{B}$ is an isometry. In fact, if the Bergman metric at $p \in U$ satisfies (4.1), then the potential Φ_p simplifies to

$$\Phi_p(z) = \frac{-2}{c^2} \log \left(1 - \frac{c^2}{2} |T(z)|^2 \right), \quad z \in \Omega.$$

So,

$$\begin{aligned} g_{\alpha\bar{\beta}}(z) &= \frac{\partial^2 \Phi_p(z)}{\partial z_\alpha \partial \overline{z_\beta}} \\ &= \sum_{i,j=1}^{n} \left(1 - \frac{c^2}{2} |T(z)|^2 \right)^{-2} \left[\delta_{ij} \left(1 - \frac{c^2}{2} |T(z)|^2 \right) + \frac{c^2}{2} \overline{w_i(z)} w_j(z) \right] \frac{\partial w_i(z)}{\partial z_\alpha} \frac{\partial \overline{w_j(z)}}{\partial \overline{z_\beta}} \\ &= \sum_{i,j=1}^{n} G_{i\bar{j}}(w) \frac{\partial w_i(z)}{\partial z_\alpha} \frac{\partial \overline{w_j(z)}}{\partial \overline{z_\beta}}, \quad z \in \Omega, \end{aligned}$$

where

$$G_{i\bar{j}}(w) := \frac{\partial^2}{\partial w_i \partial \overline{w_j}} \left(\frac{-2}{c^2} \log(1 - \frac{c^2}{2}|w|^2) \right)$$

gives the Bergman metric on $\mathbb{B}^n$. The general case follows in a similar manner. And the curvature part also follows. ■

The above Theorem directly implies a stronger version of Lu's uniformaization of Bergman metric with constant holomorphic sectional curvature, requiring constant curvature only on an open set rather than globally:

Theorem 4.3.0.4 *Let Ω be a bounded domain in $\mathbb{C}^n$. Suppose that the Bergman metric B^2_Ω is complete and its holomorphic sectional curvature is identically equal to a negative constant $-c^2$ on an open set $U \subset \Omega$. Then Ω is biholomorphic to the unit ball $\mathbb{B}_n$.*

Remark: Hang Xu pointed out that since the Bergman metric and the holomorphic sectional curvature are real analytic functions, then if they agree on a open subset locally, by the identity theorem then they agree globally. Note that it is only true for the Bergman metric as not all Kähler metric is real analytic.

4.4 Comparison Between the *Carathéodory* Metric and Bergman Metric

There are a couple of observations by Cheung and Wong [11] from this uniformization theorem of the Bergman metric with constant holomorphic sectional curvatures, which concludes rigidity results on the comparison between the Bergman metric, *Carathéodory* metric and Kobayashi metric.

Remark 1. By Theorem 3.3.0.1, it concludes that the sectional curvature of the Kobayashi metric is bounded below by -1 and that of the differential *Carathéodory* metric is bounded above by -1. Combining these observations and Lu's uniformization theorem, we can conclude the following fact for a bounded domain $\Omega \in \mathbb{C}^n$.: if $C_\Omega = K_\Omega = kB_\Omega$, where k is a positive constant, and the metrics are complete, then Ω is biholomorphic to the unit ball $\mathbb{B}_n$ and $k = \sqrt{\frac{1}{n+1}}$.

Remark 2. Let Ω be a convex bounded domain in $\mathbb{C}^n$ with the Lu constant $L(\Omega)$. Suppose the holomorphic sectional curvature of the Bergman metric B_Ω^2 on Ω is bounded above by $-(L(\Omega))^2$, then Ω is biholomorphic to the unit ball.

Proof. From the upper bound of the holomorphic sectional curvature and the Ahlfors-Schwartz lemma, we can obtain the inequality

$$K_\Omega \geq L(\Omega)B_\Omega \geq C_\Omega,$$

where the second inequality comes from the definition of Lu constant. By the Theorem of Lempert stating that $K_\Omega = C_\Omega$ on bounded convex domains, we get the equality

$$K_\Omega = L(\Omega)B_\Omega = C_\Omega.$$

Then by the Theorems 3.3.0.3 and 3.3.0.2, it is clear that Ω is biholomorphic to the unit ball. ■

Motivated by the past results comparing intrinsic metrics, we are able to prove a couple of similar observation connecting Bergman metric and *Carathéodory* metric on different domains. Before presenting the statement of the theorems, we will first introduce some technical definitions used in the proof.

The notion of analytic capacity was first introduced by Ahlfors [1] in the 1940's to study the removability of singularities of bounded analytic functions. A compact set $K \subset \mathbb{C}$ is said to be removable for bounded analytic functions if for any open set Ω containing K, every bounded function analytic on $\Omega \backslash K$ has an analytic extension to Ω. It is shown that the compact set K is removable if and only if the analytic capacity of K is 0. We can now define the analytic capacity of a Riemann surface.

Definition 4.4.0.1 *Let X be an open hyperbolic Riemann surface. The analytic capacity of a Riemann surface X is defined as*

$$c_B(z_0) = \sup\{|f'(z_0)| : f \in Hol(X, \mathbb{D}), f(z_0) = 0\}.$$

We see that the analytic capacity is the same as the *Carathéodory* metric and the *Carathéodory* metric corresponds to the analytic capacity for an open Riemann surface is described in [53] as $c_B^2|dz|^2$.

Definition 4.4.0.2 *Let M be a complex manifold and $u : M \to \mathbb{R}$ be an upper continuous function. Assume that for any open set $U \subset M$ and any harmonic function f on U such that if $f \geq u$ on the boundary of U, then $f \geq u$ on all of U. Then u is called subharmonic.*

Let $SH^-(X)$ denote the set of negative subharmonic functions on an open Riemann surface X. For $z_0 \in X$, let w be a local coordinate in a neighborhood of z_0 such that $w(z_0) = 0$. We then can define the Green's function.

Definition 4.4.0.3 *The (negative) Green's function is defined to be*

$$G(z, z_0) = \sup\{u(z) : u \in SH^-(X), \limsup_{z \to z_0} u(z) - \log|w(z)| < \infty\}.$$

An open Riemann surface admits a Green's function if and only if there is non-constant, negative subharmonic function defined on it. The Green's function is strictly negative on the surface and harmonic except on the diagonal. The Green's function on a Riemann surface induced the logarithmic capacity, which is used in potential theory to measure the size of an non-empty set in complex space.

Definition 4.4.0.4 *Let X be an open hyperbolic Riemann surface. The logarithmic capacity c_β is defined to be*

$$c_\beta(z_0) = \lim_{z \to z_0} \exp\left(G(z, z_0) - \log|w(z)|\right)$$

If $f : X_1 \to X_2$ is a biholomorphism, then for analytic capacity and logarithmic capacity is then

$$c_{B,x_1}(Z) = |f'(z)|c_{B,X_2}(h(z))$$

$$c_{\beta,x_1}(Z) = |f'(z)|c_{\beta,X_2}(h(z)).$$

Same as the intrinsic metrics, the logarithm capacity and the analytic capacity are also invariant under the changes of local coordinates. Minda [38] gave a proof of the logarithmic capacity metric dominates the the *Poincaré* metric $\lambda|dz|^2 = \frac{2}{(1-|z|^2)^2}(dzd\bar{z})$.

Theorem 4.4.0.5 *Let X be a hyperbolic Riemann surface. Then $c_\beta(z)|dz| \leq \lambda|dz|$. If equality holds at a single point, then X is simply connected.*

Proof. Let $\mathcal{A}_z(X)$ denote the family of all multiple=valued analytic functions f defined on X such that $|f|$ is single-valued, $f(z) = 0$, and $f'(z) = 1$ for one of the branches. If X does not have a Green's function, then there is nothing to prove. Fix $\zeta \in X$ and a local coordinates w of ζ with $w(\zeta) = 0$. Let $f_x \in \mathcal{A}_z(X)$ be the unique extremal function relative to the local coordinate w. Let $\pi : \mathbb{D} \to X$ be the universal covering such that $\pi(0) = \zeta$. Then $c_\beta(\zeta) f_X \circ \pi$ is a single-valued analytic mapping fo $\mathbb{D}$ into itself that fixed the origin. Also, this function vanishes at each point of the set $\pi^{-1}(\zeta)$. The by the Schwarz' lemma, we have

$$c_\beta(\zeta)|f'_X(\zeta)||\pi'(0)| \leq 1,$$

where $\pi'(0)$ denotes the derivative of $w \circ \pi$ at the origin. Thus. we get that

$$c_\beta(\zeta) \geq \frac{1}{|\pi'(0)|} = \lambda.$$

Equality implies that $c_\beta(\zeta) f_X \circ \pi$ is a rotation of $\mathbb{D}$ about the origin. In particular, it vanishes just once, so $\pi^{-1}(\zeta) = 0$ and X is then simply connected. ■

Theorem 4.4.0.6 *1. Let X be a hyperbolic Riemann surface. Assume that the Bergman metric $B(z)|dz|^2$ is complete with holomorphic sectional curvature no less than -1 and the Carathéodory metric is written as $c_B^2|dz|^2$, under some local coordinate z. Then it holds on X that*

$$B \geq c_B^2;$$

moreover, X is biholomorphic to a disk is and only if there exists some $z_0 \in X$ such that

$$B(z_0) = c_B^2(z_0).$$

2. Let $\Omega \subset \mathbb{C}^n$ be a domain whose universal cover is biholomorphic to a unit ball. Assume that the Bergman metric B_Ω^2 is complete on Ω with holomorphic sectional curvature *no less than* $-(L(\Omega))^2$*. Then* Ω *is biholomorphic to a ball if and only if there exists some* $z_0 \in \Omega$ *such that*

$$L(\Omega)B_\Omega(z_0, v) = C_\Omega(z_0, v),$$

for all $v \in T_{z_0}\Omega$

Proof. Part 1. Since Ω is universally covered by the disk, it has the natural *Poincaré* metric $\lambda^2(z)|dz|^2$ under some local coordinate z with constant holomorphic sectional curvature -1. By the Ahlfors-Schwarz lemma and the fact that Kobayashi metric dominates the *Carathéodory* metric, we get the inequality

$$\frac{B}{z} \geq \lambda \geq c_B^2.$$

If there exists some $z_0 \in X$ such that $B(z_0) = c_B^2(z_0)$, then we see that $\lambda(z_0) = c_B(z_0)$. Since the Kobayashi metric and *Carathéodory* metric coincide at one point, then by theorem 3.3.0.2, the space X is biholomorphic to a disk. The converse direction is straight forward since the Kobayashi metric and *Carathéodory* metric equal to one another on a disk.

Part 2. Since Ω is holomrophically covered by a ball, by theorem 3.4.0.1 the Kobayashi metric on Ω is a *Kähler* metric with constant holomorphic sectional curvature. Since the holomorphic sectional curvature of the Bergman metric is bounded below by $-(L(\Omega))^2$, by the generalized Ahlfors-Schwarz lemma we get the inequality

$$L(\Omega)B_\Omega \geq K_\Omega \geq C_\Omega.$$

If there exists some $z_0 \in \Omega$ such that $L(\Omega) = B_\Omega(z_0, v) = C_\Omega(z_0, v)$ for all $v \in T_{z_0}\Omega$, then it follows that $K_\Omega(z_0, v) = C_\Omega(z_0, v)$. Similarly, since the Kobayashi metric and

Carathéodory metric coincide at one point, then by theorem 3.3.0.2, the domain Ω is biholomorphic to a disk. The converse direction is straight forward since the Kobayashi metric and *Carathéodory* metric equal to one another on a disk. ■

Theorem 4.4.0.7 *Let X be a hyperbolic Riemann surface whose Bergman metric $B(z)|dz|^2$ is complete with holomorphic sectional curavture no less that -1. In a local coordinate at $a \in X$, let*

$$\lambda^2(z)|dz|^2,\ c_\beta^2|dz|^2, \text{ and } c_B^2|dz|^2$$

denote the Poincaré metric $\lambda^2|dz|^2 = \frac{2}{(1-|z|^2)^2}(dzd\bar{z})$ with constant holomorphic sectional curvature -1, the logarithmic capacity, and the analytic capacity, respectively. Then it holds on X that

$$B \geq \lambda^2 \geq c_B^2 \geq B^2.$$

Moreover, X is biholomorphic to a disk if and only if there exists some $z_0 \in X$ such that

$$B(z_0) = c_\beta^2(z_0) \text{ or } c_B^2(z_0)$$

Proof. From the lower bound of the holomorphic sectional curvature of the Bergman metric and the Ahlfors-Schwatz lemma, we get the first inequality. The second inequality is a result of Theorem 4.4.0.5, which also shows that the *Poincaré* metric with constant holomorphic sectional curvature coincide with the logarithmic capacity at one point if and only if the surface is simply connected. The last inequality is a well-known result.

If there exists some $z_0 \in X$ such that

$$B(z_0) = c_\beta^2(z_0) \text{ or } c_B^2(z_0),$$

then $\lambda(z_0) = c_\beta(z_0)$ and we can conclude by Theorem 4.4.0.5 that X is simply connected. The converse is straightforward as we just have the equalities $B = \lambda^2 = c_B^2 = B^2$ ■

Lastly, we can extend a previous result of **Remark 2** by Cheung and Wong from bounded convex domain to strictly pseudoconvex domains.

Theorem 4.4.0.8 *Let $\Omega \subset \mathbb{C}^n, n \geq 2$, be a bounded strongly pseudoconvex domain with a $\mathcal{C}^2$ boundary and the Lu constant $L(\Omega)$. Suppose the holomorphic sectional curvature of the Bergman metric B_Ω^2 on Ω is bounded above by $-(L(\Omega))^2$. Then Ω is biholomorphic to a ball and $(L(\Omega))^2 = \frac{2}{n+1}$.*

Proof. From the upper bound of the holomorphic sectional curvature of the Bergman metric and the Ahlfors-Schwatz lemma, we observe that

$$K_\Omega \geq L(\Omega)B_\Omega \geq C_\Omega,$$

where the second inequality is straight froward from the definition of the Lu constant. By Theorem 3.2.0.5, there exists some $\epsilon > 0$ such that if $z \in \Omega$ and $\delta_\Omega(z) < \epsilon$, then there exists some open set such that the Kobayashi metric and the *Carathéodory* metric agree in some direction $v \neq 0$ in $E_\Omega(z) = T_z\Omega$. Hence, we see that the following equality holds on the open set described

$$K_\Omega = L(\Omega)B_\Omega = C_\Omega.$$

By Theorem 3.3.0.1, we can conclude that the holomorphic sectional curvature $H(z;v)$ of the Bergman metric is equal to $-(L(\Omega))^2$ for any non-zero vector $v \in E_\Omega(z)$. For a fixed z, since the map $v \in T_z\Omega \to H(z;v)$ is algebraic, we obtain that $H(z;v) = -L(\Omega)^2$ for all non-zero $v \in T_z\Omega$. Let $K := \{z \in \Omega | \delta_\Omega(z) \geq \epsilon\}$ to be a compact set. Then the holomorphic sectional curvature of B_Ω^2 equal $-L(\Omega)^2$ on $\Omega\backslash K$. Since the Bergman metric B_Ω^2 is complete Ω, Theorem 4.3.0.4 yields that Ω msut be biholomorphic to the ball $\mathbb{B}_n$, and the Lu constant $(L(\Omega))^2$ is naturally equal to $\frac{2}{n+1}$ because the holomorphic sectional curvature pf the unit ball is $-\frac{2}{n+1}$. ■

Chapter 5

Conclusions and Future Direction

In previous chapters, we have explored two significant directions in the study of intrinsic metrics on complex domains and primarily focused on characterizing bounded domains in $\mathbb{C}^n$ with intrinsic metrics.

The first main result, Theorem 3.4.0.2, builds upon the foundational work of Lempert and Zimmer by investigating the Kähler condition for the Carathéodory metric on bounded strongly pseudoconvex domains. Our result extends Zimmer's characterization of domains with Kähler Kobayashi metrics by establishing a similar property for the Carathéodory metric. This contribution offers further insight into the geometric rigidity of such domains and deepens the connection between intrinsic metrics and biholomorphic classification. Key tools in the analysis include the existence and behavior of complex geodesics near the boundary, originally studied by Lempert [35] and later extended by Bracci, Fornæss, Wold [16], and Zimmer [21].

The second main result, Theorem 4.3.0.4, extends Lu's classical uniformization theorem by relaxing the assumption of globally constant holomorphic sectional curvature of the Bergman metric to a local condition. We largely use the Bergman representative coordinates developed by Lu [37]. Then we were able to apply this generalization to **Remark 2** by Cheung and Wong [11] to relax the condition from convex domain to pseudoconvex domain. This generalization broadens the class of domains to which Lu's theorem applies and highlights the strength of local geometric conditions in determining the global biholomorphic type of a domain.

We see that intrinsic metrics can serve as powerful tools in characterizing the unit ball and understanding the complex structure of strongly pseudoconvex domains. However, our attempt to relax the strongly pseudoconvex domain condition to pseudoconvex is futile and not able to answer Yau's question completely.

5.1 Future Work

The characterization of complex domains through intrinsic metrics remains a vibrant area of research, with many open questions rooted in foundational problems. The results in this thesis suggest several promising directions for further investigation.

One major avenue for future work involves a deeper exploration of the longstanding conjecture posed by Yau [58], which concerns the characterization of domains of holomorphy whose Bergman metrics are complete Kähler-Einstein. Cheng and Yau [9] demonstrated the existence of complete Kähler-Einstein metrics on smoothly bounded strongly pseudoconvex domains in $\mathbb{C}^n$, and Mok and Yau extended this to bounded pseudoconvex domains. Subsequently, Cheng conjectured that any bounded strongly pseudoconvex domain whose

Bergman metric is Kähler-Einstein must be biholomorphic to the unit ball. This conjecture was resolved for $\mathcal{C}^\infty$-smooth domains by Huang and Xiao [28]:

Theorem 5.1.0.1 (Huang–Xiao) *Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with $\mathcal{C}^\infty$ boundary. Then the Bergman metric on Ω is Kähler-Einstein if and only if Ω is biholomorphic to the unit ball $\mathbb{B}_n$.*

In dimension $n = 2$, this result was previously obtained by Fu and Wong [20], and further investigations by Savale and Xiao have extended the understanding to finite-type domains in dimension 2. Future research may aim to broaden these results by relaxing regularity assumptions, or by identifying alternative characterizations based on other intrinsic metrics. In this context, our results on the coincidence and curvature properties of the Carathéodory and Kobayashi metrics may provide valuable tools, offering a complementary perspective to metric-based rigidity questions.

Another closely related question is the Ramadanov conjecture , which is deeply connected to the structure of the Bergman kernel and the geometry of the domain boundary. Before stating the conjecture, we briefly recall Fefferman's asymptotic expansion of the Bergman kernel [17, 18, 4]. He showed that if two strongly pseudoconvex domain Ω_1, Ω_2 have a piece of overlapping boundary near some point $p \in \partial\Omega_1 \cap \partial\Omega_2$, then the Bergman kernels on Ω_1, Ω_2 differ by a smooth function near p. With this fact and the 4th order contact of strongly pseudoconvex domains with the sphere, Fefferman shows that On a bounded strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, the Bergman kernel $K(z, z)$ near the boundary of a strongly pseudoconvex domain has the form:

$$K(z, z) = \frac{\phi(z)}{\rho^{n+1}(z)} + \psi(z) \log \rho(z),$$

where ρ is a smooth defining function for the domain, and ϕ, ψ are smooth functions on $\overline{\Omega}$. These coefficients are biholomorphically invariant up to terms of infinite order. The logarithmic term ψ encodes subtle geometric information about the boundary.

Ramadanov's conjecture [46] states:

Conjecture 5.1.0.2 (Ramadanov) *Let $\Omega \subset \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain with $\mathcal{C}^\infty$ boundary. If the logarithmic coefficient ψ in Fefferman's expansion vanishes to infinite order at the boundary, i.e., $\psi = \mathcal{O}(\rho^\infty)$, then Ω has spherical boundary.*

In dimension $n = 2$, this conjecture has been proven. Graham [22] showed that if $\psi = \mathcal{O}(\rho^2)$ near a boundary point $p \in \partial\Omega$, then $\partial\Omega$ is locally biholomorphic to the unit sphere near p:

Theorem 5.1.0.3 *Let $\Omega \subset \mathbb{C}^2$ be a bounded strongly pseudoconvex domain with $\mathcal{C}^\infty$ boundary. If $\psi = \mathcal{O}(\rho^2)$ near a point $p \in \partial\Omega$, then the boundary is spherical near p.*

Despite these advancements, the conjecture remains open in dimensions $n \geq 3$. Fu and Wong [20] further showed that if the Bergman metric is Kähler-Einstein, then $\psi = \mathcal{O}(\rho^\infty)$, providing a link between Ramadanov's conjecture and the Kähler-Einstein condition. Building on this, Nemirovski and Shafikov [40] showed that the Cheng–Yau conjecture follows from the Ramadanov conjecture, thus interrelating these major questions in complex geometry. Yet Ramadanov conjecture is still open while the other one has been proved.

Given these deep connections, it is natural to ask whether other intrinsic metrics—particularly the Kobayashi and Carathéodory metrics—can also be used to formulate analogous expansions or curvature-based characterizations. Our results suggest that comparing these metrics, especially when they coincide or share similar curvature behavior, might yield new approaches to problems like the Ramadanov conjecture. For instance, one could investigate whether domains on which these metrics coincide up to a scalar factor necessarily exhibit rigidity properties akin to those in the Bergman setting.

Moreover, studying the interaction between complex geodesics, boundary regularity, and the behavior of intrinsic metrics could help illuminate new criteria for biholomorphic equivalence to standard models such as the unit ball. These efforts could also be extended beyond smoothly bounded domains to consider finite-type or weakly pseudoconvex domains, where the analytic and geometric structures are more intricate.

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