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One-sided prime ideals in noncommutative algebra

by

Manuel Lionel Reyes

A dissertation submitted in partial satisfaction of the
requirements for the degree of
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University of California, Berkeley

Committee in charge:

Professor Tsit Yuen Lam, Chair
Professor George Bergman
Professor Koushik Sen

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Abstract

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by

Manuel Lionel Reyes

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Tsit Yuen Lam, Chair

The goal of this dissertation is to provide noncommutative generalizations of the following theorems from commutative algebra: (Cohen's Theorem) every ideal of a commutative ring R is finitely generated if and only if every prime ideal of R is finitely generated, and (Kaplansky's Theorems) every ideal of R is principal if and only if every prime ideal of R is principal, if and only if R is noetherian and every maximal ideal of R is principal. We approach this problem by introducing certain families of right ideals in noncommutative rings, called *right Oka families*, generalizing previous work on commutative rings by T. Y. Lam and the author. As in the commutative case, we prove that the right Oka families in a ring R correspond bijectively to the classes of cyclic right R-modules that are *closed under extensions*. We define *completely prime right ideals* and prove the *Completely Prime Ideal Principle*, which states that a right ideal maximal in the complement of a right Oka family is completely prime. We exploit the connection with cyclic modules to provide many examples of right Oka families. Our methods produce some new results that generalize well-known facts from commutative algebra, and they also recover earlier theorems stating that certain noncommutative rings are domains—namely, proper right PCI rings and rings with the right restricted minimum condition that are not right artinian.

After developing the theory of right Oka families, we proceed to the generalizations of the theorems stated above. Define a right ideal P of a ring R to be *cocritical* if the module R/P has larger Krull dimension than each of its proper factors. We prove that a ring is right noetherian (resp. a principal right ideal ring) if and only if all of its (essential) cocritical right ideals are finitely generated (resp. principal). We apply our methods to prove that a (left and right) noetherian ring is a principal right ideal ring if and only if all of its maximal right ideals are principal. Examples are provided to show that the left noetherian hypothesis cannot be omitted. Finally, we compare these results with previous generalizations of these theorems, and are able to recover most of these with our methods.

To my family,

Zoë, Mom, Dad, and Eddie.

Contents

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Chapter 1

Introduction

1.1 Background: the existence of prime ideals in commutative rings

A well documented phenomenon in commutative algebra is the tendency for ideals that are maximal for certain properties to be prime. For instance, if R is a commutative ring, any ideal of one of the following types must be prime:

- A maximal ideal of R;

- An ideal maximal with respect to being disjoint from a fixed multiplicatively closed subset $S \subseteq R$;

- An ideal maximal with respect to not being finitely generated;

- An ideal maximal with respect to not being principal;

- An ideal maximal with respect to not being invertible;

- A maximal point annihilator of a fixed module M_R.

It seems that such results are ripe for inclusion as exercises in commutative algebra textbooks; for instance, see [27, pp. 1, 4–5, 8, 44, 74] or [11, pp. 84–85]. Until recently, the proofs of these results relied on various ad hoc methods. Although the similarity of the results may have been noted, there had been no attempt to find a uniform method for proving all of them.

In [37], T. Y. Lam and the author provided the first systematic analysis of these "maximal implies prime" results. The key insight there is to study the problem in terms of families of ideals. (The term *family* is used here to mean a set. We use this special word to refer to sets of ideals. Such a distinction seems especially relevant, because every ideal is itself a

set!) Given a family $\mathcal{F}$ of ideals of a ring R, the problem of unifying the above results can be rephrased as follows: find conditions on the family $\mathcal{F}$ that ensure that every ideal maximal in the complement of $\mathcal{F}$ is prime.

An *Oka family of ideals* of a commutative ring R was defined to be a family $\mathcal{F}$ of ideals of R such that $R \in \mathcal{F}$ and, for any ideal I of R and any element $a \in R$, if $I + (a) \in \mathcal{F}$ and $(I : a) \in \mathcal{F}$, then $I \in \mathcal{F}$. (Recall that $(I : a) = \{r \in R : ar \in R\}$. The central result of [37] is the *Prime Ideal Principle*, which states that for any Oka family $\mathcal{F}$ in a commutative ring R, every ideal maximal in the complement of $\mathcal{F}$ is prime. This was applied to give a unified presentation of the classical "maximal implies prime" results stated above, by reducing the proof of each result to showing that a certain ideal family is Oka. In addition, the Prime Ideal Principal led to a number of new "maximal implies prime" results. For instance, it was shown that an ideal of a commutative ring that is maximal with respect to not being idempotent must be a maximal ideal.

Results of the type listed above can sometimes be applied to show that all ideals of a commutative ring have a certain property precisely when the prime ideals of that ring have the property in question. For instance, I. S. Cohen characterized commutative noetherian rings in the following, which appeared as Theorem 2 of [8].

Theorem 1.1.1 (Cohen's Theorem). *A commutative ring R is noetherian iff every prime ideal of R is finitely generated.*

Also, we recall two characterizations of commutative principal ideal rings due to I. Kaplansky, which appeared as Theorem 12.3 of [26]. Throughout this dissertation, a ring in which all right ideals are principal will be called a *principal right ideal ring*, or PRIR. Similarly, we have principal left ideal rings (PLIRs), and a ring which is both a PRIR and a PLIR is called a *principal ideal ring*, or PIR.

Theorem 1.1.2 (Kaplansky's Theorem). *A commutative noetherian ring R is a principal ideal ring iff every maximal ideal of R is principal.*

Combining this result with Cohen's Theorem, Kaplansky deduced the following in Footnote 8 on p. 486 of [26].

Theorem 1.1.3 (Kaplansky-Cohen Theorem). *A commutative ring R is a principal ideal ring iff every prime ideal of R is principal.*

(We refer to this result as the Kaplansky-Cohen Theorem for two reasons. The primary and most obvious reason is that it follows from a combination of the above results due to Cohen and Kaplansky. But we also use this term because it is a result in the spirit of Cohen's Theorem, that was first deduced by Kaplansky.)

1.2   From commutative to noncommutative algebra

There are many natural questions in noncommutative algebra that are raised by the discussion above. What can one say about a right ideal of a noncommutative ring R that is maximal with respect to not being finitely generated? What families of right ideals are sufficient "test sets" for the property of being a right noetherian ring or a princiapl right ideal ring?

In this dissertation, we answer these questions and many more, generalizing the circle of ideas in the previous section to the setting of right ideals in noncommutative rings. As one should expect, the situation is more subtle in noncommutative algebra. However, the idea of an Oka family survives as a unifying theme among all of the various results.

While prime two-sided ideals are studied in noncommutative rings, it is safe to say that they do not control the structure of noncommutative rings in the sense of the theorems of Cohen and Kaplansky above. Part of the trouble is that many complicated rings have few two-sided ideals. A striking class of examples is given by the simple rings, which have only one prime ideal but often have complicated one-sided structure.

We begin in §2.1 by introducing and studying *completely prime right ideals*, which are a certain type of "prime one-sided ideal" illuminating the structure of a noncommutative ring. The idea of prime one-sided ideals is not new, as evidenced by numerous attempts to define such objects in the literature (for instance, see [1], [29], and [41]). However, a common theme among earlier versions of one-sided prime ideals is that they were produced by simply deforming the defining condition of a prime ideal in a commutative ring ($ab \in \mathfrak{p}$ implies a or b lies in $\mathfrak{p}$). Our approach is slightly less arbitrary, as it is inspired by the systematic analysis in [37] of results from commutative algebra in the vein of Theorems 1.1.1 and 1.1.3 above. As a result, these one-sided primes are accompanied by a ready-made theory producing a number of results that relate them to the one-sided structure of a ring.

In §2.2, after introducing Oka families of right ideals, we present the *Completely Prime Ideal Principle* 2.2.4 (CPIP). This result generalizes the Prime Ideal Principle of [37] to one-sided ideals of a noncommutative ring. It formalizes a one-sided "maximal implies prime" philosophy: *right ideals that are maximal in certain senses tend to be completely prime.* The CPIP is our main tool connecting completely prime right ideals to the (one-sided) structure of a ring. For instance, it allows us to provide a noncommutative generalization of Cohen's Theorem 1.1.1 in Theorem 4.2.5.

In order to effectively apply the CPIP, we investigate how to construct examples of right Oka families (from classes of cyclic modules that are *closed under extensions*) in §3.1. Most of the applications of the Completely Prime Ideal Principle are given in §3.2. Some highlights include a study of point annihilators of modules over noncommutative rings, conditions for a ring to be a domain, and a simple proof that a right PCI ring is a domain.

Then in §3.3 we turn our attention to a special subset of the completely prime right ideals of a ring, the set of *comonoform right ideals*. These right ideals are more well-behaved than completely prime right ideals generally are. They enjoy special versions of the "Prime

Ideal Principle" and its "Supplement," which again allow us to produce results that relate these right ideals to the one-sided structure of a noncommutative ring. Their existence is also closely tied to the well-studied right Gabriel filters from the theory of noncommutative localization.

Our work in §4.1–4.2 addresses the following question: what are some sufficient conditions for all right ideals of a ring to lie in a given right Oka family? In §4.1 we develop the idea of a *point annihilator set* in order to deal with this problem. We give a number of examples of point annihilator sets, particularly examples of *noetherian point annihilator sets.*

Then in §4.2 we prove the *Point Annihilator Set Theorem* 4.2.3, a generalized version of the Completely Prime Ideal Principle Supplement 2.2.6. As a result of these efforts, we emerge with our generalization of Cohen's Theorem in Theorem 4.2.5. This theorem is "flexible" in the sense that, in order to check whether a ring is right noetherian, one can use various test sets of right ideals. However, one important specific case of the theorem can be stated as follows. A right ideal $P \subseteq R$ is said to be *cocritical* if K. $\dim(R/P) >$ K. $\dim(R/I)$ for any right ideal $I \supsetneq P$, where K. dim denotes the (Gabriel-Rentschler) Krull dimension. Then *a ring is right noetherian iff all of its cocritical right ideals are finitely generated.* We also develop conditions for a (left) perfect ring to be right artinian.

Next we consider families of principal right ideals in §5.1. Whereas the family of principal ideals of a commutative ring is always an Oka family, it turns out that the family $\mathcal{F}_{\mathrm{pr}}$ of principal right ideals can fail to be a right Oka family in certain noncommutative rings. Nevertheless, we are able to give a characterization of the rings for which this is a right Oka family, and we provide some examples of such rings. Moreover, by defining another family $\mathcal{F}_{\mathrm{pr}}^{\circ}$ which "approximates" $\mathcal{F}_{\mathrm{pr}}$ and is *always* a right Oka family, we are able to provide a noncommutative generalization of the Kaplansky-Cohen Theorem in Theorem 5.1.11. As before, a specific version of this theorem is the following: *a ring is a principal right ideal ring iff all of its cocritical right ideals are principal.*

In §5.2 we sharpen our versions of the Cohen and Kaplansky-Cohen Theorems by considering families of right ideals that are closed under direct summands. Of course, this includes the family of finitely generated right ideals and the family of principal right ideals. This allows us to reduce the "test sets" of our the Point Annihilator Set Theorem 4.2.3 to sets of essential right ideals. For instance, to check if a ring is right noetherian or a principal right ideal ring, it suffices to test the essential cocritical right ideals. The section ends with some homological applications.

In §5.3 we work toward a noncommutative generalization of Kaplansky's Theorem 1.1.2. This requires a number of preparatory results, and incorporates the results of §5.2. All of this culminates in Theorem 5.3.9, which states that *a (left and right) noetherian ring is a principal right ideal ring iff its maximal right ideals are principal.* Notably, our analysis also implies that such a ring has right Krull dimension ≤ 1. An example shows that the left noetherian hypothesis is in fact necessary. We close with some questions raised by this theorem.

Finally, we explore the connections between our results and previous generalizations of the

Cohen and Kaplansky-Cohen theorems in §5.4. These include theorems due to V. R. Chandran, K. Koh, G. O. Michler, P. F. Smith, and B. V. Zabavs'kiĭ. Reviewing these earlier results affords us an opportunity to survey some earlier notions of "prime right ideals" studied in the literature.

1.3 Conventions

Throughout this dissertation, all rings are assumed to be associative with unit element, and all subrings, modules and ring homomorphisms are assumed to be unital. Fix a ring R. We say that R is a *semisimple ring* if R_R is a semisimple module. We say R is *Dedekind-finite* if every right invertible element is invertible; this is equivalent to the condition that R_R is not isomorphic to a proper direct summand of itself (see [36, Ex. 1.8]). We write $I_R \subseteq R$ (resp. $I \lhd R$) to mean that I is a right (resp. two-sided) ideal in R. The term *ideal* always refers to a two-sided ideal of R, with the sole exception of the phrase "Completely Prime Ideal Principle" (Theorem 2.2.4). We denote the Jacobson radical of R by $\mathrm{rad}(R)$. We say that R is *semilocal* (resp. *local*) if $R/\mathrm{rad}(R)$ is semisimple (resp. a division ring). A ring is *right duo* if all of its right ideals are two-sided ideals. The set of prime (two-sided) ideals of R is denoted by $\mathrm{Spec}(R)$. An element of R is *regular* if it is not a left or right zero-divisor. Given a family $\mathcal{F}$ of right ideals in R, we let $\mathcal{F}'$ denote the complement of $\mathcal{F}$ within the set of all right ideals of R, and we let $\mathrm{Max}(\mathcal{F}')$ denote the set of maximal elements of $\mathcal{F}'$.

Now fix an R-module M_R. We let $\mathrm{soc}(M)$ denote the socle of M (the sum of all simple submodules of M). We will write $N \subseteq_e M$ to mean that N is an essential submodule of M. A *proper factor* of M is a module of the form M/N for some nonzero submodule $N_R \subseteq M$. We use "f.g." as shorthand for "finitely generated."

The symbol ":=" is used to mean that the left-hand side of the equation is defined to be equal to the right-hand side. Occasionally, the end of a multi-paragraph example will be marked by the symbol "□" for the sake of clarity.

Chapter 2

A new type of prime right ideal

2.1   Completely prime right ideals

In this section we define the completely prime right ideals that we shall study, and we investigate some of their basic properties. Throughout this dissertation, given an element m and a submodule N of a right R-module M_R, we write

$$m^{-1}N := \{r \in R : mr \in N\},$$

which is a right ideal of R. Except in §3.3, we only deal with this construction in the form $a^{-1}I$ for an element a and a right ideal I of R. In commutative algebra, this ideal is usually denoted by $(I : a)$.

Definition 2.1.1. A right ideal $P_R \subsetneq R$ is *completely prime* if for any $a, b \in R$ such that $aP \subseteq P$, $ab \in P$ implies that either $a \in P$ or $b \in P$ (equivalently, for any $a \in R$, $aP \subseteq P$ and $a \notin P$ imply $a^{-1}P = P$).

Our use of the term "completely prime" is justified by the next result, which characterizes the two-sided ideals that are completely prime as right ideals. Recall that an ideal $P \lhd R$ is said to be *completely prime* if the factor ring R/P is a domain (equivalently, $P \neq R$ and for all $a, b \in R$, $ab \in P \implies a \in P$ or $b \in P$); for instance, see [33, p. 194].

Proposition 2.1.2. *For any ring R, an ideal $P \lhd R$ is completely prime as a right ideal iff it is a completely prime ideal. In particular, an ideal P is completely prime as a right ideal iff it is completely prime as a left ideal.*

Proof. For an ideal $P \lhd R$, we tautologically have $aP \subseteq P$ for all $a \in R$. So such $P \neq R$ is completely prime as a right ideal iff for every $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$, which happens precisely when P is a completely prime ideal. $\square$

Corollary 2.1.3. *If R is a commutative ring, then an ideal $P \lhd R$ is completely prime as a (right) ideal iff it is a prime ideal.*

Thus completely prime right ideals extend the notion of completely prime ideals in non-commutative rings, and these right ideals also directly generalize the the concept of a prime ideal of a commutative ring. Some readers may wonder whether it would be better to reserve the term "completely prime" for a right ideal $P \subsetneq R$ satisfying the following property: for all $a, b \in R$, if $ab \in P$ then either $a \in P$ or $b \in P$. (Such right ideals have been studied, for instance, in [1].) Let us informally refer to such right ideals as "extremely prime." We argue that one merit of completely prime right ideals is that they occur in situations where extremely prime right ideals are absent. We shall show in Corollary 2.1.10 that every maximal right ideal of a ring is completely prime, thus proving that every nonzero ring has a completely prime right ideal. On the other hand, there are many examples of nonzero rings that do not have any extremely prime right ideals. We thank T. Y. Lam and G. Bergman for contributing to the following.

Proposition 2.1.4. *Let R be a simple ring that has a nontrivial idempotent. Then R has no extremely prime right ideals.*

Proof. Assume for contradiction that $I \subsetneq R$ is an extremely prime right ideal. For any idempotent $e \neq 0, 1$ of R we have $RfR = R$, where $f = 1 - e \neq 0$. Since I is extremely prime and $(eRf)^2 = 0 \subseteq I$, we have $eRf \subseteq I$. Hence $eR = eRfR \subseteq I$, so that $e \in I$. Similarly, $f \in I$. Hence $1 = e + f \in I$, which is a contradiction. $\square$

Explicit examples of such rings are readily available. Let k be a division ring. Then we may take R to be the matrix ring $\mathbb{M}_n(k)$ for $n > 1$. Alternatively, if V_k is such that $\dim_k(V) = \alpha$ is any infinite cardinal, then we may take R to be the factor of $E := \operatorname{End}_k(V)$ by its unique maximal ideal $M = \{g \in E : \dim_k(g(V)) < \alpha\}$ (see [34, Ex. 3.16]). The latter example is certainly neither left nor right noetherian. More generally, large classes of rings satisfying the hypothesis of Proposition 2.1.4 include simple von Neumann regular rings that are not division rings, as well as purely infinite simple rings (a ring $R \neq 0$ that is not a division ring is *purely infinite simple* if, for every $r \in R$, there exist $x, y \in R$ such that $xry = 1$; see [2, §1]).

It follows from Proposition 2.1.2 that we can omit the modifiers "left" and "right" when referring to two-sided ideals that are completely prime. The same result shows that the completely prime right ideals can often be "sparse" among two-sided ideals in noncommutative rings. For example, there exist many rings that have no completely prime ideals, such as simple rings that are not domains. Also, it is noteworthy that a prime ideal $P \lhd R$ of a noncommutative ring is not necessarily a completely prime right ideal. Thus completely prime right ideals generalize the notion of prime ideals in commutative rings in a markedly different way than the more familiar two-sided prime ideals of noncommutative ring theory. (For further evidence of this idea, see Proposition 2.1.11.) The point is that these two types of "primes" give insight into different facets of a ring's structure, with completely prime right ideals giving a better picture of the right-sided structure of a ring as argued throughout this dissertation.

Below are some alternative characterizations of completely prime right ideals that help elucidate their nature. The *idealizer* of a right ideal $J_R \subseteq R$ is the subring of R given by

$$\mathbb{I}_R(J) := \{x \in R : xJ \subseteq J\}.$$

This is the largest subring of R in which J is a (two-sided) ideal. It is a standard fact that $\operatorname{End}_R(R/J) \cong \mathbb{I}_R(J)/J$.

Proposition 2.1.5. *For a right ideal $P_R \subsetneq R$, the following are equivalent:*

(1) *P is completely prime;*

(2) *For $a, b \in R$, $ab \in P$ and $a \in \mathbb{I}_R(P)$ imply either $a \in P$ or $b \in P$;*

(3) *Any nonzero $f \in \operatorname{End}_R(R/P)$ is injective;*

(4) *$E := \operatorname{End}_R(R/P)$ is a domain and $_E(R/P)$ is torsionfree.*

Proof. Characterization (2) is merely a restatement of the definition given above for completely prime right ideals, so we have (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (3): Let P be a completely prime right ideal, and let $0 \neq f \in \operatorname{End}_R(R/P)$. Choose $x \in R$ such that $f(1 + P) = x + P$. Because $f \neq 0$, $x \notin P$. Also, because f is an R-module homomorphism, $(1 + P)P = 0$ implies that $(x + P)P = 0$, or $xP \subseteq P$. Then because P is completely prime, $x^{-1}P = P$. But this gives $\ker f = (x^{-1}P)/P = 0$, so f is injective as desired.

(3) $\Longrightarrow$ (1): Assume that any nonzero endomorphism of R/P is injective. Suppose $x, y \in R$ are such that $xP \subseteq P$ and $xy \in P$. Then there is an endomorphism f of R/P given by $f(r+P) = xr+P$. If $x \notin P$ then $f \neq 0$, making f injective. Then $f(y+P) = xy+P = 0+P$ implies that $y + P = 0 + P$, so that $y \in P$. Hence P is a completely prime right ideal.

(3) $\Longleftrightarrow$ (4): This equivalence is still true if we replace R/P with any nonzero module M_R. If $E = \operatorname{End}_R(M)$ is a domain and $_EM$ is torsionfree, then it is clear that every nonzero endomorphism of M is injective. Assume conversely that all nonzero endomorphisms of M are injective. Given $f, g \in E \setminus \{0\}$ and $m \in M \setminus \{0\}$, injectivity of g gives $g(m) \neq 0$ and injectivity of f gives $f(g(m)) \neq 0$. In particular $fg \neq 0$, proving that E is a domain. Because g and m above were arbitrary, we conclude that M is a torsionfree left E-module. □

(As a side note, we mention that modules for which every nonzero endomorphism is injective have been studied by A. K. Tiwary and B. M. Pandeya in [56]. In [57], W. Xue investigated the dual notion of a module for which every nonzero endomorphism is surjective. These were respectively referred to as modules with the properties $(*)$ and $(**)$. In our proof of (3) $\Longleftrightarrow$ (4) above, we showed that a module M_R satisfies $(*)$ iff $E := \operatorname{End}_R(M)$ is a domain and $_EM$ is torsionfree. One can also prove the dual statement that $M \neq 0$ satisfies $(**)$ iff E is a domain and $_EM$ is divisible.)

One consequence of characterization (3) above is that the property of being completely prime depends only on the quotient module R/P. (Using language to be introduced in Definition 3.1.3, the set of all completely prime right ideals in R is closed under similarity.) It is a straightforward consequence of (2) above that *a completely prime right ideal P is a completely prime ideal in the subring $\mathbb{I}_R(P)$*. This is further evidenced from (4) because $\mathbb{I}_R(P)/P \cong \mathrm{End}_R(R/P)$ is a domain.

One might wonder whether the torsionfree requirement in condition (4) above is necessary. That is, can a cyclic module C_R have endomorphism ring E which is a domain but with ${}_E C$ not torsionfree? The next example shows that this is indeed the case.

Example 2.1.6. For an integer $n > 1$, consider the following ring and right ideal:

$$R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z}/(n) \\ 0 & \mathbb{Z} \end{pmatrix} \supseteq J_R := \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}.$$

It is easy to show that $\mathbb{I}_R(J) = \left(\begin{smallmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$, so that $E := \mathrm{End}_R(R/J) \cong \mathbb{I}_R(J)/J \cong \mathbb{Z}$ is a domain acting on $R/J \cong (\mathbb{Z},\ \mathbb{Z}/(n))_R$ by (left) multiplication. But ${}_E(R/J)$ has nonzero torsion submodule isomorphic to $(0,\ \mathbb{Z}/(n))_R$.

For a prime ideal P in a commutative ring R, the factor module R/P is indecomposable (i.e., has no nontrivial direct summand). This property persists for completely prime right ideals.

Corollary 2.1.7. *If P is a completely prime right ideal of R, then the right R-module R/P is indecomposable.*

Proof. By Proposition 2.1.5 the ring $E := \mathrm{End}_R(R/P)$ is a domain. Thus E has no nontrivial idempotents, proving that R/P is indecomposable. $\square$

For a commutative ring R and $P \in \mathrm{Spec}(R)$, the module R/P is not only indecomposable, it is uniform. (A module $U_R \neq 0$ is *uniform* if every pair of nonzero submodules of U has nonzero intersection.) However, the following example shows that for a completely prime right ideal P in a general ring R, R/P need not be uniform as a right R-module.

Example 2.1.8. Let k be a division ring and let R be the following subring of $\mathbb{M}_3(k)$:

$$R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}.$$

Notice that R has precisely three simple right modules (since the same is true modulo its Jacobson radical), namely $S_i = k$ ($i = 1, 2, 3$) with right R-action given by multiplication by the (i, i)-entry of any matrix in R.

Let $P \subseteq R$ be the right ideal consisting of matrices in R whose first row is zero. Notice that R/P is isomorphic to the module $V := (k \ k \ k)$ of row vectors over k with the usual right R-action. Then V has unique maximal submodule $M = (0 \ k \ k)$, and M in turn has precisely two proper nonzero submodules, $U = (0 \ k \ 0)$ and $W = (0 \ 0 \ k)$. Having cataloged all submodules of V, let us list the composition factors:

$$V/M \cong S_1, \quad M/W \cong U \cong S_2, \quad M/U \cong W \cong S_3.$$

Notice that S_1 occurs as a composition factor of every nonzero factor module of V, and it does not occur as a composition factor of any proper submodule of V. Thus every nonzero endomorphism of $V \cong R/P$ is injective, proving that P is a completely prime right ideal. However, V contains the direct sum $U \oplus W$, so $R/P \cong V$ is not uniform.

Proposition 2.1.2 shows how frequently completely prime right ideals occur among two-sided ideals. The next few results give us further insight into how many completely prime right ideals exist in a general ring. The first result gives a sufficient condition for a right ideal to be completely prime. Recall that a module is said to be *cohopfian* if all of its injective endomorphisms are automorphisms. For example, it is straightforward to show that any artinian module is cohopfian (see [34, Ex. 4.16]). The following is easily proved using Proposition 2.1.5(3).

Proposition 2.1.9. *If a right ideal $P \subseteq R$ is such that $E := \mathrm{End}_R(R/P)$ is a division ring, then P is completely prime. The converse holds if R/P is cohopfian.*

Corollary 2.1.10. (A) *A maximal right ideal $\mathfrak{m}_R \subseteq R$ is a completely prime right ideal.*
(B) *For a right ideal P in a right artinian ring R, the following are equivalent:*

(1) *P is a completely prime right ideal;*

(2) *$\mathrm{End}_R(R/P)$ is a division ring;*

(3) *P is a maximal right (equivalently, maximal left) ideal in its idealizer $\mathbb{I}_R(P)$.*

Proof. Part (A) follows from Schur's Lemma and Proposition 2.1.9. For part (B), (1) $\Longleftrightarrow$ (2) follows from Proposition 2.1.9 (every cyclic right R-module is artinian, hence cohopfian), and (2) $\Longleftrightarrow$ (3) follows easily from the canonical isomorphism $\mathbb{I}_R(P)/P \cong \mathrm{End}_R(R/P)$. $\square$

Because every nonzero ring has a maximal right ideal (by a familiar Zorn's lemma argument), part (A) above applies to show that a nonzero ring always has a completely prime right ideal. (This fact was already mentioned at the beginning of this section.) The same cannot be said for completely prime two-sided ideals or the aforementioned "extremely prime" right ideals! On the other hand, Example 2.1.8 shows that a completely prime right ideal in an artinian ring need not be maximal, so we cannot hope to strengthen part (B) very drastically.

To get another indication of the role of completely prime right ideals, we may ask the following natural question: when is every proper right ideal of a ring completely prime? It is straightforward to verify that a commutative ring in which every proper ideal is prime must be a field. On the other hand, there exist nonsimple noncommutative rings in which every proper ideal is prime. (For example, take $R = \text{End}(V_k)$ where V is a right vector space of dimension at least $\aleph_0$ over a division ring k; see [34, Ex. 10.6]). In contrast, the behavior of completely prime right ideals is much closer to that of prime ideals in the commutative case.

Proposition 2.1.11. *For a nonzero ring R, every proper right ideal in R is completely prime iff R is a division ring.*

Proof. ("Only if") For an arbitrary $0 \neq a \in R$, it suffices to show that a is right invertible. Assume for contradiction that $aR \neq R$. Then the right ideal $J = a^2R \subseteq aR \subsetneq R$ is proper and hence is completely prime. Certainly $a \in \mathbb{I}_R(J)$. Because $a^2 \in J$, we must have $a \in J$ (recall Proposition 2.1.5(2)). But also the ideal $0 \neq R$ is completely prime, so R is a domain by Proposition 2.1.2. Then $a \in J = a^2R$ implies $1 \in aR$, contradicting that $aR \neq R$. $\square$

An inspection of the proof above actually shows that a nonzero ring R is a division ring iff the endomorphism ring of every nonzero cyclic right R-module is a domain, iff R is a domain and the endomorphism ring of every cyclic right R-module is reduced. We mention here that certain other "prime right ideals" studied previously do not enjoy the property proved above. For instance, K. Koh showed [29, Thm. 4.2] that all proper right ideals I of a ring R satisfy

$$(aRb \in I \implies a \in I \text{ or } b \in I) \quad \text{for all } a, b \in R$$

precisely when R is simple. In this sense, a ring may have "too many" of these prime-like right ideals.

The following is an analogue of the theorem describing $\text{Spec}(R/I)$ for a commutative ring R and an ideal $I \lhd R$. We present the result in a more general context than that of completely prime right ideals because it will be applicable to other types of "prime right ideals" that we will consider later.

Remark 2.1.12. Let $\mathcal{P}$ be a module-theoretic property such that, if V_R is a module and I is an ideal of R contained in $\text{ann}(V)$, then V satisfies $\mathcal{P}$ as an R-module iff it satisfies $\mathcal{P}$ when considered as a module over R/I. For every ring R let $\mathcal{S}(R)$ denote the set of all right ideals $P_R \subseteq R$ such that R/P satisfies $\mathcal{P}$. Then it follows directly from our assumption on the property $\mathcal{P}$ that there is a one-to-one correspondence

$$\{P_R \in \mathcal{S}(R) : P \supseteq I\} \longleftrightarrow \mathcal{S}(R/I)$$

given by $P \leftrightarrow P/I$.

In particular, we may take $\mathcal{P}$ to be the property "$V \neq 0$ and every nonzero endomorphism of V is injective." Then the associated set $\mathcal{S}(R)$ is the collection of all completely prime right

ideals of R, according to characterization (3) of Proposition 2.1.5. In this case we conclude that *for any ideal $I \lhd R$ the completely prime right ideals of R/I correspond bijectively, in the natural way, to the set of completely prime right ideals of R containing I.*

In §§2.2–3.2 we will take a much closer look at the existence of completely prime right ideals in rings. To close this section, we explore how completely prime right ideals behave when "pulled back" along ring homomorphisms. One can interpret Remark 2.1.12 as demonstrating that, under a surjective ring homomorphism $f \colon R \to S$, the preimage of any completely prime right ideal of S is a completely prime right ideal of R. The next example demonstrates that this does not hold for arbitrary ring homomorphisms.

Example 2.1.13. For a division ring k, let $S := \mathbb{M}_3(k)$ and let R be the subring of S defined in Example 2.1.8. Consider the right ideals

$$Q_S := \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ d & e & f \end{pmatrix} \right\} \subseteq S \quad \text{and} \quad P_R := \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq R.$$

Because $Q_S \cong (k \ k \ k)_S^2$ has composition length 2, it is a maximal right ideal of S and thus is completely prime by Corollary 2.1.10(A). Let $f \colon R \to S$ be the inclusion homomorphism. Then $P = Q \cap R = f^{-1}(Q)$. However $(R/P)_R \cong (0 \ k \ k)_R \cong (0 \ k \ 0)_R \oplus (0 \ 0 \ k)_R$ is decomposable, so Lemma 2.1.7 shows that P_R is not completely prime.

The above example may seem surprising to the reader who recalls that completely prime (two-sided) ideals pull back along *any* ring homomorphism. The tension between this fact and Example 2.1.13 is resolved in the following result.

Proposition 2.1.14. *Let $f : R \to S$ be a ring homomorphism, let $Q_S \subsetneq S$ be a completely prime right ideal, and set $P_R := f^{-1}(Q)$. If $f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$, then P is a completely prime right ideal of R.*

Proof. Because Q is a proper right ideal of S, P must be a proper right ideal of R. Suppose that $a \in \mathbb{I}_R(P)$ and $b \in R$ are such that $ab \in P$. Then $f(a)f(b) = f(ab) \in f(P) \subseteq Q$ with $f(a) \in f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$. Because Q_S is completely prime, this means that one of $f(a)$ or $f(b)$ lies in Q. Then one of a or b lies in $f^{-1}(Q) = P$. Hence P is completely prime. $\qquad\square$

This simultaneously explains why the preimage of a completely prime right ideal is again completely prime when the original ideal is two-sided and when the ring homomorphism is surjective. When $Q_S \subseteq S$ above is a two-sided ideal, $\mathbb{I}_S(Q) = S$ and the condition $f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$ is trivially satisfied. On the other hand, if $Q \subseteq \operatorname{im}(f)$ (e.g. if f is surjective), then one can use the fact that $f(P) = Q$ to show that $f(\mathbb{I}_R(P)) \subseteq \mathbb{I}_S(Q)$ again holds.

2.2   The Completely Prime Ideal Principle

A distinct advantage that completely prime right ideals have over earlier notions of "one-sided primes" is a theorem assuring the existence of completely prime right ideals in a wide array of situations. It states that right ideals that are "maximal" in certain senses must be completely prime. This result is the *Completely Prime Ideal Principle*, or *CPIP*, and it is presented in Theorem 2.2.4 below.

A number of famous theorems from commutative algebra state that an ideal maximal with respect to a certain property must be prime. A useful perspective from which to study this phenomenon is to consider a family $\mathcal{F}$ of ideals in a commutative ring R and ask when an ideal maximal in the *complement* $\mathcal{F}'$ of $\mathcal{F}$ is prime. Some well-known examples of such $\mathcal{F}$ include the family of ideals intersecting a fixed multiplicative set $S \subseteq R$, the family of finitely generated ideals, the family of principal ideals, and the family of ideals that do not annihilate any nonzero element of a fixed module M_R. In [37] an *Oka family of ideals* in a commutative ring R was defined to be a set $\mathcal{F}$ of ideals of R with $R \in \mathcal{F}$ such that, for any ideal $I \lhd R$ and element $a \in R$, $I + (a) \in \mathcal{F}$ and $(I : a) \in \mathcal{F}$ imply $I \in \mathcal{F}$. The *Prime Ideal Principle* (or PIP) [37, Thm. 2.4] states that for any Oka family $\mathcal{F}$, an ideal maximal in the complement of $\mathcal{F}$ is prime (in short, $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$). In [37, §3] it was shown that many of the "maximal implies prime" results in commutative algebra (including those mentioned above) follow directly from the Prime Ideal Principle.

The following notion generalizes Oka families to the noncommutative setting.

Definition 2.2.1. Let R be a ring. An *Oka family of right ideals* (or *right Oka family*) in R is a family $\mathcal{F}$ of right ideals with $R \in \mathcal{F}$ such that, given any $I_R \subseteq R$ and $a \in R$,

$$I + aR, \ a^{-1}I \in \mathcal{F} \implies I \in \mathcal{F}. \tag{2.2.2}$$

If R is commutative, notice that this coincides with the definition of an Oka family of ideals in R, given in [37, Def. 2.1]. When verifying that some set $\mathcal{F}$ is a right Oka family, we will often omit the step of showing that $R \in \mathcal{F}$ if this is straightforward.

Remark 2.2.3. The fact that this definition is given in terms of the "closure property" (2.2.2) makes it clear that the collection of Oka families of right ideals in a ring R is closed under arbitrary intersections. Thus the set of right Oka families of R forms a complete lattice under the containment relation.

Without delay, let us prove the noncommutative analogue of the Prime Ideal Principle [37, Thm. 2.4], the Completely Prime Ideal Principle (CPIP).

Theorem 2.2.4 (Completely Prime Ideal Principle). *Let $\mathcal{F}$ be an Oka family of right ideals in a ring R. Then every $I \in \mathrm{Max}(\mathcal{F}')$ is a completely prime right ideal.*

Proof. Let $I \in \mathrm{Max}(\mathcal{F}')$. Notice that $I \neq R$ since $R \in \mathcal{F}$. Assume for contradiction that there exist $a, b \in R \setminus I$ such that $aI \subseteq I$ and $ab \in I$. Because $a \notin I$ we have $I \subsetneq I + aR$. Additionally $I \subseteq a^{-1}I$, and $b \in a^{-1}I$ implies $I \subsetneq a^{-1}I$. Since $I \in \mathrm{Max}(\mathcal{F}')$ we find that $I + aR$, $a^{-1}I \in \mathcal{F}$. Because $\mathcal{F}$ is a right Oka family we must have $I \in \mathcal{F}$, a contradiction. $\square$

In the original setting of Oka families in commutative rings, a result called the "Prime Ideal Principle Supplement" [37, Thm. 2.6] was used to recover results such as Cohen's Theorem [8, Thm. 2] that a commutative ring R is noetherian iff its prime ideals are all finitely generated. As with the Completely Prime Ideal Principle above, there is a direct generalization of this fact for noncommutative rings. The idea of this result is that for *certain* right Oka families $\mathcal{F}$, in order to test whether $\mathcal{F}$ contains all right ideals of R, it is sufficient to test only the completely prime right ideals. We first define the one-sided version of a concept introduced in [37].

Definition 2.2.5. A *semifilter of right ideals* in a ring R is a family $\mathcal{F}$ of right ideals such, for all right ideals I and J of R, if $I \in \mathcal{F}$ and $J \supseteq I$ then $J \in \mathcal{F}$.

Theorem 2.2.6 (Completely Prime Ideal Principle Supplement)**.** *Let $\mathcal{F}$ be a right Oka family in a ring R such that every nonempty chain of right ideals in $\mathcal{F}'$ (with respect to inclusion) has an upper bound in $\mathcal{F}'$. (This holds, for example, if every right ideal in $\mathcal{F}$ is f.g.) Let $\mathcal{S}$ denote the set of completely prime right ideals of R.*

 (1) *Let $\mathcal{F}_0$ be a semifilter of right ideals in R. If $\mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*

 (2) *For $J_R \subseteq R$, if all right ideals in $\mathcal{S}$ containing J (resp. properly containing J) belong to $\mathcal{F}$, then all right ideals containing J (resp. properly containing J) belong to $\mathcal{F}$.*

 (3) *If $\mathcal{S} \subseteq \mathcal{F}$, then all right ideals of R belong to $\mathcal{F}$.*

Proof. For (1), let $\mathcal{F}_0$ be a semifilter of right ideals and suppose that $\mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$. Assume for contradiction that there exists a right ideal $I \in \mathcal{F} \setminus \mathcal{F}_0 = \mathcal{F}' \cap \mathcal{F}_0$. The hypothesis on $\mathcal{F}'$ allows us to apply Zorn's lemma to deduce that there exists a right ideal $P \supseteq I$ with $P \in \mathrm{Max}(\mathcal{F}')$, so $P \in \mathcal{S}$ by the Completely Prime Ideal Principle 2.2.4. Because $\mathcal{F}_0$ is a semifilter containing I, we also have $P \in \mathcal{F}_0$. It follows that $P \in \mathcal{S} \cap \mathcal{F}_0 \setminus \mathcal{F}$, contradicting our hypothesis.

Parts (2) and (3) follow from (1) by taking the semifilter $\mathcal{F}_0$ to be, respectively, the set of all right ideals containing J, the set of all right ideals properly containing J, or the set of all right ideals of R. $\square$

We will largely refrain from applying the Completely Prime Ideal Principle and its Supplement until §3.2, when we will have enough tools to efficiently construct right Oka families. However, it seems appropriate to at least give one classical application to showcase these ideas at work. In [42, Appendix], M. Nagata gave a simple proof of Cohen's Theorem 1.1.1

using the following lemma: if an ideal I and an element a of a commutative ring R are such that $I + (a)$ and $(I : a)$ are finitely generated, then I itself is finitely generated. This statement amounts to saying that the family of finitely generated ideals in a commutative ring is an Oka family. Nagata cited a paper [43] of K. Oka as the inspiration for this result. (In [37, p. 3007] it was pointed out that Oka's Corollaire 2 is the relevant statement.) This was the reason for the use of the term *Oka family* in [37]. More generally, in [37, Prop. 3.16] it was shown that, for any infinite cardinal α, the family of all ideals generated by a set of cardinality $< \alpha$ is Oka. The following generalizes this collection of results to the noncommutative setting. We let $\mu(M)$ denote the smallest cardinal μ such that the module M_R can be generated by a set of cardinality μ.

Proposition 2.2.7. *Let α be an infinite cardinal, and let $\mathcal{F}_{<\alpha}$ be the set of all right ideals $I_R \subseteq R$ with $\mu(I) < \alpha$. Then $\mathcal{F}_{<\alpha}$ is a right Oka family, and any right ideal maximal with respect to $\mu(I) \geq \alpha$ is completely prime. In particular, the set of finitely generated right ideals is a right Oka family; hence a right ideal maximal with respect to not being finitely generated is completely prime.*

Proof. We first show that $\mathcal{F}_{<\alpha}$ is a right Oka family. Let $I_R \subseteq R$, $a \in R$ be such that $I + aR$, $a^{-1}I \in \mathcal{F}_{<\alpha}$. From $\mu(I + aR) < \alpha$ it is straightforward to verify that there is a right ideal $I_0 \subseteq I$ with $\mu(I_0) < \alpha$ such that $I + aR = I_0 + aR$. It follows that $I = I_0 + a(a^{-1}I)$. Because $\mu(I_0) < \alpha$ and $\mu(a(a^{-1}I)) \leq \mu(a^{-1}I) < \alpha$, we see that $\mu(I) < \alpha + \alpha = \alpha$. Thus $I \in \mathcal{F}_{<\alpha}$, proving that $\mathcal{F}_\alpha$ is right Oka.

If I is a right ideal maximal with respect to $\mu(I) \geq \alpha$ then $I \in \mathrm{Max}(\mathcal{F}')$ and the CPIP 2.2.4 implies that I is completely prime. The last sentence follows when we take $\alpha = \aleph_0$. $\qquad\square$

This leads to a noncommutative generalization of Cohen's Theorem 1.1.1 for completely prime right ideals.

Theorem 2.2.8 (A noncommutative Cohen's Theorem)**.** *A ring R is right noetherian iff all of its completely prime right ideals are finitely generated.*

Proof. This follows from Proposition 2.2.7 and the CPIP Supplement 2.2.6(3) applied to $\mathcal{F} = \mathcal{F}_{<\aleph_0}$, the family of f.g. right ideals. $\qquad\square$

Notice how quickly the last two results were proved using the CPIP and its Supplement! This highlights the utility of right Oka families as a framework from which to study such problems. Of course, other generalizations of Cohen's Theorem have been proven in the past. In Chapters 4 and 5, we will apply the methods of right Oka families developed here to improve upon our generalization of Cohen's Theorem. We will also develop noncommutative generalizations of the theorems of Kaplansky which say that a commutative ring is a principal ideal ring iff its prime ideals are principal, iff it is noetherian and its maximal

ideals are principal. See §5.4 for a comparison of our version of Cohen's Theorem to earlier generalizations in the literature.

The generalization of Cohen's Theorem 1.1.1 in Theorem 2.2.8 above does not hold if we replace the phrase "completely prime" with "extremely prime" (as defined in §2.1). Indeed, using Proposition 2.1.4 we showed that there exist rings R that are not right noetherian with no extremely prime right ideals. But for such R, it is vacuously true that every extremely prime right ideal of R is finitely generated! This strikingly illustrates the idea that completely prime right ideals control the right-sided structure of a general ring better than extremely prime right ideals.

For any cardinal β, we can also define a family $\mathcal{F}_{\leq \beta}$ of all right ideals I such that $\mu(I) \leq \beta$. Letting β^+ denote the successor cardinal of β, we see that $\mathcal{F}_{\leq \beta} = \mathcal{F}_{<\beta^+}$, so we have not sacrificed any generality in the statement of Proposition 2.2.7. In particular, taking $\beta = \aleph_0$ we see that the family of all countably generated right ideals is a right Oka family. The "maximal implies prime" result in the case where R is commutative and $\beta = \aleph_0$ was noted in Exercise 11 of [27, p. 8]. The case of larger infinite cardinals α for commutative rings was proved by Gilmer and Heinzer in [14, Prop. 3]. (This reference was unfortunately overlooked in [37, p. 3017].)

One might wonder whether the obvious analogue of Cohen's Theorem for right ideals with generating sets of higher cardinalities is also true, in light of Proposition 2.2.7. However, in the commutative case Gilmer and Heinzer [14] have already settled this in the negative. The rings which serve as their counterexamples are (commutative) valuation domains.

Here we provide a sample application of the Completely Prime Ideal Principle. It is well-known that Cohen's Theorem 1.1.1 can be used to prove that if R is a commutative noetherian ring, then the power series ring $R[[x]]$ is also noetherian. In [41], G. Michler proved a version of Cohen's Theorem and gave an analogous application of this result to power series over noncommutative rings. Our version of Cohen's Theorem can be applied in the same way.

Corollary 2.2.9. *If a ring R is right noetherian, then the power series ring $R[[x]]$ is also right noetherian.*

Proof. Let P be a completely prime right ideal of $S := R[[x]]$; by Theorem 2.2.8 it suffices to show that P is finitely generated. Let $C_R \subseteq R$ be the right ideal of R consisting of all constant terms of all power series in P. Then P is finitely generated. Choose power series $f_1, \ldots, f_n \in P$ whose constant terms generate C, and set $I := \sum f_j R \subseteq P$. If $x \in P$ then it is easy to see that $P = I + xS$ is finitely generated. So assume that $x \notin P$. In this case, we claim that $P = I$. Again we will have P finitely generated and the proof will be complete. Given $h \in P$, the constant term of $h_0 := h$ is equal to the constant term of some $g_0 = \sum a_{0j} f_j \in I$, where $a_{0j} \in R$. Then $h_0 - g_0 = xh_1$ for some $h_1 \in S$. Notice that $xh_1 = h_0 - g_0 \in P$. Because P is completely prime with $xP = Px \subseteq P$ and $x \notin P$, it

follows that $h_1 \in P$. One can proceed inductively to find $g_i = \sum_{j=1}^{n} a_{ij} f_j$ $(a_{ij} \in R)$ such that $h_i = g_i + x h_{i+1}$. Hence $h = \sum_{j=1}^{n} \left(\sum_{i=0}^{\infty} a_{ij} x^i \right) f_j \in I$. $\qquad\square$

Before moving on, we mention a sort of "converse" to the CPIP 2.2.4 characterizing exactly which families $\mathcal{F}$ of right ideals are such that $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals. It turns out that a weak form of the Oka property (2.2.2) characterizes these families.

Proposition 2.2.10. *Let $\mathcal{F}$ be a family of right ideals in a ring R. All right ideals in $\mathrm{Max}(\mathcal{F}')$ are completely prime iff, for all $I_R \subseteq R$ where every right ideal $J \supsetneq I$ lies in $\mathcal{F}$ and for all elements $a \in \mathbb{I}_R(I)$, the Oka property (2.2.2) is satisfied.*

Proof. First suppose that $\mathcal{F}$ satisfies property (2.2.2) for all I and a described above. Then the proof of the CPIP 2.2.4 applies to show that any right ideal in $\mathrm{Max}(\mathcal{F}')$ is completely prime. Conversely, suppose that $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals, and let $I_R \subseteq R$ and $a \in \mathbb{I}_R(I)$ be as described above. Assume for contradiction that $I \notin \mathcal{F}$. It follows that $I \in \mathrm{Max}(\mathcal{F}')$, so I is a completely prime right ideal. Because $I \notin \mathcal{F}$ and $I + aR \in \mathcal{F}$, we see that $a \notin I$. But then the remark at the end of Definition 2.1.1 shows that $I = a^{-1} I \in \mathcal{F}$, a contradiction. We conclude that in fact $I \in \mathcal{F}$, completing the proof. $\qquad\square$

So for a commutative ring R, this result classifies precisely which families $\mathcal{F}$ of ideals satisfy $\mathrm{Max}(\mathcal{F}') \subseteq \mathrm{Spec}(R)$. (Note: here we can replace $a \in \mathbb{I}_R(I)$ by $a \in R$.) In fact, the above result was first discovered in the commutative setting by T. Y. Lam and the present author during the development of [37], though it did not appear there.

Chapter 3

Further results on right Oka families

3.1 Right Oka families and classes of cyclic modules

In order to apply the CPIP 2.2.4, we need an effective tool for constructing right Oka families. The relevant result will be Theorem 3.1.7 below. This theorem generalizes one of the most important facts about Oka families in commutative rings: there is a correspondence between Oka families in a ring R and certain classes of cyclic R-modules (to be defined below). Throughout this paper we use $\mathfrak{M}_R$ to denote the class of all right R-modules and $\mathfrak{M}_R^c \subseteq \mathfrak{M}_R$ to denote the subclass of cyclic R-modules.

Definition 3.1.1. Let R be any ring. A subclass $\mathcal{C} \subseteq \mathfrak{M}_R^c$ with $0 \in \mathcal{C}$ is *closed under extensions* if, for every exact sequence $0 \to L \to M \to N \to 0$ of <u>cyclic</u> right R-modules, whenever L, $N \in \mathcal{C}$ it follows that $M \in \mathcal{C}$.

Specifically, it was shown in [37, Thm. 4.1] that for any commutative ring R, the Oka families in R are in bijection with the classes of cyclic R-modules that are closed under extensions. This correspondence provided many interesting examples of Oka families in commutative rings. The goal of this section is to show that the Oka families of right ideals in an arbitrary ring R correspond to the classes of cyclic right R-modules which are closed under extensions.

In a commutative ring R, the correspondence described above was given by associating to any Oka family $\mathcal{F}$ the class $\mathcal{C} := \{M_R : M \cong R/I \text{ for some } I \in \mathcal{F}\}$ of cyclic modules. Then $\mathcal{F}$ is determined by $\mathcal{C}$ because, for an ideal I of R, we may recover I from the isomorphism class of the cyclic module R/I since I is the annihilator of this cyclic module. (In fact, this works for any family $\mathcal{F}$ of ideals in R.) However, in a noncommutative ring there can certainly exist right ideals $I, J \subseteq R$ such that $I \neq J$ but $R/I \cong R/J$ (as right R-modules).

Example 3.1.2. Any simple artinian ring R has a single isomorphism class of simple right modules. Thus all maximal right ideals $\mathfrak{m}_R \subseteq R$ have isomorphic factor modules $R/\mathfrak{m}$. But if $R \cong \mathbb{M}_n(k)$ for a division ring k and if $n > 1$ (i.e. R is not a division ring), then there

exist multiple maximal right ideals: we may take $\mathfrak{m}_i$ $(i = 1, \ldots, n)$ to correspond to the right ideal of matrices whose ith row is zero. In fact, over an infinite division ring k even the ring $\mathbb{M}_2(k)$ has infinitely many maximal right ideals! This is true because, for any $\lambda \in k$, the set of all matrices of the form

$$\begin{pmatrix} a & b \\ \lambda a & \lambda b \end{pmatrix}$$

is a maximal right ideal, and these right ideals are distinct for each value of λ. (Of course, a similar construction also works over the ring $\mathbb{M}_n(k)$ for $n > 2$.)

Therefore we do not expect every family $\mathcal{F}$ of right ideals to naturally correspond to a class of cyclic modules. This prompts the following definition.

Definition 3.1.3. Two right ideals I and J of a ring R are said to be *similar* if $R/I \cong R/J$ as right R-modules. A family $\mathcal{F}$ of right ideals in a ring R is *closed under similarity* if, for any similar right ideals $I_R, J_R \subseteq R$, $I \in \mathcal{F}$ implies $J \in \mathcal{F}$. This is equivalent to $I \in \mathcal{F} \iff J \in \mathcal{F}$ whenever $R/I \cong R/J$.

The notion of similarity dates at least as far back as Jacobson's text [23, pp. 33 & 130] (although he only studied this idea in specific classes of rings). With the appropriate terminology in place, the next fact is easily verified.

Proposition 3.1.4. *For any ring R, there is a bijective correspondence*

$$\left\{ \begin{array}{c} \text{families } \mathcal{F} \text{ of right} \\ \text{ideals of } R \text{ that are} \\ \text{closed under similarity} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{classes } \mathcal{C} \text{ of cyclic right} \\ R\text{-modules that are closed} \\ \text{under isomorphism} \end{array} \right\}.$$

For a family $\mathcal{F}$ and a class $\mathcal{C}$ as above, the correspondence is given by the maps

$$\mathcal{F} \mapsto \mathcal{C}_{\mathcal{F}} := \{M_R : M \cong R/I \text{ for some } I \in \mathcal{F}\},$$
$$\mathcal{C} \mapsto \mathcal{F}_{\mathcal{C}} := \{I_R \subseteq R : R/I \in \mathcal{C}\}.$$

We will show that every right Oka family is closed under similarity with the help of the following two-part lemma. The first part describes, up to isomorphism, the cyclic submodules of a cyclic module R/I. The second part is a rather well-known criterion for two cyclic modules to be isomorphic.

Lemma 3.1.5. *Let R be a ring.*

(A) *For any right ideal $I \subseteq R$ and any element $a \in R$, there is an isomorphism*

$$R/a^{-1}I \xrightarrow{\sim} (I + aR)/I \subseteq R/I$$

given by $r + a^{-1}I \mapsto ar + I$.

(B) *Given two right ideals $I_R, J_R \subseteq R$, $R/I \cong R/J$ iff there exists $a \in R$ such that $I + aR = R$ and $a^{-1}I = J$.*

Proof. Part (A) is a straightforward application of the First Isomorphism Theorem. Proofs for part (B) can be found, for instance, in [9, Prop. 1.3.6] or [34, Ex. 1.30]. (In fact, it was already observed in Jacobson's text [23, p. 33], though in the special setting of PIDs.) In any case, the "if" direction follows from part (A) above, and the reader can readily verify the "only if" direction. $\square$

These elementary observations are very important for us. The reader should be aware that we will freely use the isomorphism $R/a^{-1}I \cong (I + aR)/I$ throughout this paper.

Proposition 3.1.6. *A family $\mathcal{F}$ of right ideals in a ring R is closed under similarity iff for any $I_R \subseteq R$ and $a \in R$, $I + aR = R$ and $a^{-1}I \in \mathcal{F}$ imply $I \in \mathcal{F}$. In particular, any right Oka family $\mathcal{F}$ is closed under similarity.*

Proof. The first statement follows directly from Lemma 3.1.5(B), and the second statement follows from Definition 2.2.1 because every right Oka family contains the unit ideal R. $\square$

Thus we see that every right Oka family will indeed correspond, as in Proposition 3.1.4, to *some* class of cyclic right modules; it remains to show that they correspond precisely to the classes that are closed under extensions. We first need to mention one fact regarding module classes closed under extensions. From the condition $0 \in \mathcal{C}$ and the exact sequence $0 \to L \to M \to 0 \to 0$ for $L_R \cong M_R$, we see that a class $\mathcal{C}$ of cyclic modules closed under extensions is also closed under isomorphisms. We are now ready to prove the main result of this section.

Theorem 3.1.7. *Given a class $\mathcal{C}$ of cyclic right R-modules that is closed under extensions, the family $\mathcal{F}_{\mathcal{C}}$ is an Oka family of right ideals. Conversely, given a right Oka family $\mathcal{F}$, the class $\mathcal{C}_{\mathcal{F}}$ of cyclic right R-modules is closed under extensions.*

Proof. First suppose that the given class $\mathcal{C}$ is closed under extensions. Then $R \in \mathcal{F}_{\mathcal{C}}$ because $0 \in \mathcal{C}$. So let $I_R \subseteq R$ and $a \in R$ be such that $I + aR$, $a^{-1}I \in \mathcal{F}_{\mathcal{C}}$. Then $R/(I + aR)$ and $R/a^{-1}I$ lie in $\mathcal{C}$. Moreover, we have an exact sequence

$$0 \to (I + aR)/I \to R/I \to R/(I + aR) \to 0,$$

where $(I + aR)/I \cong R/a^{-1}I$ lies in $\mathcal{C}$ (recall that $\mathcal{C}$ is closed under isomorphisms). Because $\mathcal{C}$ is closed under extensions, $R/I \in \mathcal{C}$. Thus $I \in \mathcal{F}_{\mathcal{C}}$, proving that $\mathcal{F}_{\mathcal{C}}$ is a right Oka family.

Now suppose that $\mathcal{F}$ is a right Oka family. That $0 \in \mathcal{C}_{\mathcal{F}}$ follows from the fact that $R \in \mathcal{F}$. Consider an exact sequence of cyclic right R-modules

$$0 \to L \to M \to N \to 0$$

where $L, N \in \mathcal{C}_{\mathcal{F}}$, so that there exist $A, B \in \mathcal{F}$ such that $L \cong R/A$ and $N \cong R/B$. We may identify M up to isomorphism with R/I for some right ideal $I \subseteq R$. Because L is cyclic and embeds in $M \cong R/I$, we have $L \cong (I + aR)/I$ for some $a \in R$. Hence $R/(I + aR) \cong N \cong R/B$, and Proposition 3.1.6 implies that $I + aR \in \mathcal{F}$. Note also that $R/a^{-1}I \cong (I + aR)/I \cong L \cong R/A$, so by Proposition 3.1.6 we conclude that $a^{-1}I \in \mathcal{F}$. Because $\mathcal{F}$ is a right Oka family we must have $I \in \mathcal{F}$. So $M \cong R/I$ implies that $M \in \mathcal{C}_{\mathcal{F}}$. $\square$

We examine one consequence of this correspondence. This will require the following lemma, which compares a class $\mathcal{C} \subseteq \mathfrak{M}_R^c$ that is closed under extensions with its closure under extensions in the larger class $\mathfrak{M}_R$.

Lemma 3.1.8. *Let $\mathcal{C}$ be a class of cyclic right R-modules that is closed under extensions (as in Definition 3.1.1), and let $\overline{\mathcal{C}}$ be its closure under extensions in the class $\mathfrak{M}_R$ of all cyclic right R-modules. Then $\mathcal{C} = \overline{\mathcal{C}} \cap \mathfrak{M}_R^c$.*

Proof. Certainly $\mathcal{C} \subseteq \overline{\mathcal{C}} \cap \mathfrak{M}_R^c$. Conversely, suppose that $M \in \overline{\mathcal{C}} \cap \mathfrak{M}_R^c$. Because $M \in \overline{\mathcal{C}}$, there is a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that each $M_j/M_{j-1} \in \mathcal{C}$. One can then prove by downward induction that the cyclic modules M/M_j lie in $\mathcal{C}$. So $M \cong M/M_0$ and $M/M_0 \in \mathcal{C}$ imply that $M \in \mathcal{C}$. $\square$

Corollary 3.1.9. *Let $\mathcal{F}$ be a right Oka family in a ring R. Suppose that $I_R \subseteq R$ is such that R/I has a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = R/I$$

where each filtration factor is cyclic and of the form $M_j/M_{j-1} \cong R/I_j$ for some $I_j \in \mathcal{F}$. Then $I \in \mathcal{F}$.

Proof. Let $\mathcal{C} := \mathcal{C}_{\mathcal{F}}$, which is closed under extensions by Theorem 3.1.7. Then the above filtration of the cyclic module $M = R/I$ has filtration factors isomorphic to the $R/I_j \in \mathcal{C}$. From Lemma 3.1.8 it follows that $R/I \in \mathcal{C}$, and thus $I \in \mathcal{F}_{\mathcal{C}} = \mathcal{F}$. $\square$

This implies, for instance, that if a right Oka family $\mathcal{F}$ in a ring R contains all maximal right ideals of R, then it contains all right ideals I such that R/I has finite length.

We close this section by applying Theorem 3.1.7 to produce a second "converse" to the Completely Prime Ideal Principle 2.2.4, distinct from the one mentioned at the end of §2.2. This result mildly strengthens the CPIP to an "iff" statement, saying that a right ideal P of a ring R is completely prime iff $P \in \mathrm{Max}(\mathcal{F}')$ for some right Oka family $\mathcal{F}$. (This was already noted in the commutative case in [38, p. 274].)

Let V_R be an R-module, and define the class

$$\mathcal{E}[V] := \{M_R : M = 0 \text{ or } M \not\hookrightarrow V\}. \tag{3.1.10}$$

We claim that $\mathcal{E}[V]$ is closed under extensions in $\mathfrak{M}_R$. Indeed, suppose that $0 \to L \to M \to N \to 0$ is a short exact sequence in $\mathfrak{M}_R$ with $L, N \in \mathcal{E}[V]$. If $L = 0$ then $M \cong N$, so that $M \in \mathcal{E}[V]$. Otherwise L cannot embed in V. Because $L \hookrightarrow M$, M cannot embed in V, proving $M \in \mathfrak{M}_R$. With the class $\mathcal{E}[V]$ in mind, we prove the second "converse" of the CPIP.

Proposition 3.1.11. *For any completely prime right ideal $P_R \subseteq R$, there exists an Oka family $\mathcal{F}$ of right ideals in R such that $P \in \mathrm{Max}(\mathcal{F}')$.*

Proof. Let $V_R = R/P$, and let $\mathcal{E}[V]$ be as above. Fixing the class $\mathcal{C} = \mathcal{E}[V] \cap \mathfrak{M}_R^c$, set $\mathcal{F} := \mathcal{F}_\mathcal{C}$. By Theorem 3.1.7, $\mathcal{F}$ is a right Oka family. Certainly $P \notin \mathcal{F}$ since $R/P = V \notin \mathcal{E}[V]$, so it only remains to show the maximality of P. Assume for contradiction that there is a right ideal $I \notin \mathcal{F}$ with $I \supsetneq P$. Then we have a natural surjection $R/P \twoheadrightarrow R/I$, and because $I \notin \mathcal{F}$ we have $0 \neq R/I \hookrightarrow V = R/P$. Composing these maps as

$$R/P \twoheadrightarrow R/I \hookrightarrow R/P$$

gives a nonzero endomorphism $f \in \mathrm{End}(R/P)$ with $\ker f = I/P \neq 0$. This contradicts characterization (3) of Proposition 2.1.5, so we must have $P \in \mathrm{Max}(\mathcal{F}')$ as desired. $\qquad\square$

3.2 Applications of the Completely Prime Ideal Principle

In this section we will give various applications of the Completely Prime Ideal Principle. Every application should be viewed as a new source of completely prime right ideals in a ring or as an application of the notion of completely prime right ideals (and right Oka families) to study the one-sided structure of a ring. The diversity of concepts that interweave with the notion of completely prime right ideals (via right Oka families) in this section showcases the ubiquity of these objects. We remind the reader that when verifying that a set $\mathcal{F}$ of right ideals in R is a right Oka family, we will often skip the step of checking that $R \in \mathcal{F}$.

Remark 3.2.1. An effective method of creating right Oka families is as follows. Consider a subclass $\mathcal{E} \subseteq \mathfrak{M}_R$ that is closed under extensions in the full class of right modules $\mathfrak{M}_R$. Then $\mathcal{C} = \mathcal{E} \cap \mathfrak{M}_R^c$ is a class of cyclic modules that is closed under extensions. Hence $\mathcal{F} := \mathcal{F}_\mathcal{C}$ is a right Oka family. (Notice that, according to Lemma 3.1.8, *every* such $\mathcal{C}$ arises this way.)

When working relative to a ring homomorphism, a similar method applies. Recall that for a ring k, a *k-ring* R is a ring with a fixed homomorphism $k \to R$. Given a k-ring R, let $\mathcal{E}_1$ be any class of right k-modules that is closed under extensions in $\mathfrak{M}_k$, and let $\mathcal{E}$ denote the subclass of $\mathfrak{M}_R$ consisting of modules that lie in $\mathcal{E}_1$ when considered as k-modules under the map $k \to R$. Then $\mathcal{E}$ is certainly closed under extensions in $\mathfrak{M}_R$, so $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_R^c$ is closed under extensions and $\mathcal{F} := \mathcal{F}_\mathcal{C}$ is a right Oka family.

3.2.A   Point annihilators and zero-divisors

Point annihilators are basic objects from commutative algebra that connect the modules over a commutative ring to the ideals of that ring. Prime ideals play an important role there in the form of associated primes of a module. Here we study these themes in the setting of noncommutative rings.

Definition 3.2.2. For a ring R and a module $M_R \neq 0$, a *point annihilator* of M is a right ideal of the form $\operatorname{ann}(m)$ for some $0 \neq m \in M$.

A standard theorem of commutative algebra states that for a module M_R over a commutative ring R, a maximal point annihilator of M is a prime ideal. The next result is the direct generalization of this fact. This application takes advantage of the construction $\mathcal{E}[V]$ presented in (3.1.10).

Proposition 3.2.3. *Let R be a ring and $M_R \neq 0$ an R-module. The family $\mathcal{F}$ of right ideals that are not point annihilators of M is a right Oka family. Thus, a maximal point annihilator of M is a completely prime right ideal.*

Proof. Following the notation of (3.1.10), let $\mathcal{C} = \mathcal{E}[M] \cap \mathfrak{M}_R^c$, which is a class of cyclic modules closed under extensions. Then $\mathcal{F}_{\mathcal{C}}$ is a right Oka family. But by definition of $\mathcal{E}[M]$, we see that

$$\begin{aligned}
\mathcal{F}'_{\mathcal{C}} &= \{I_R \subseteq R : 0 \neq R/I \hookrightarrow M\} \\
&= \{\operatorname{ann}(m) : 0 \neq m \in M\} \\
&= \mathcal{F}'.
\end{aligned}$$

So $\mathcal{F} = \mathcal{F}_{\mathcal{C}}$ is a right Oka family. The last statement follows from the CPIP 2.2.4. □

The proof that a maximal point annihilator of a module M_R is completely prime can also be achieved using the following family:

$$\mathcal{F} := \{I_R \subseteq R : \text{for } m \in M, \, mI = 0 \implies m = 0\}.$$

One can show that $\mathcal{F}$ is a right Oka family. Moreover, it is readily checked that $\operatorname{Max}(\mathcal{F}')$ consists of the maximal point annihilators of M. The CPIP again applies to show that the maximal point annihilators of M are completely prime. This was essentially the approach taken in the commutative case in [37, Prop. 3.5].

As in the theory of modules over commutative rings, one may wish to study "associated primes" of a module M over a noncommutative ring R. For a module M_R, let us say that a completely prime right ideal $P_R \subseteq R$ is *associated to* M if it is a point annihilator of M (equivalently, if $R/P \hookrightarrow M$). A famous fact from commutative algebra is that a noetherian module over a commutative ring has only finitely many associated primes; see [11, Thm. 3.1]. It is easy to show that the analogous statement for completely prime right ideals does not hold

over noncommutative rings. For instance, Example 3.1.2 provided a ring R with infinitely many maximal right ideals $\{\mathfrak{m}_i\}$ such that the modules $R/\mathfrak{m}_i$ were all isomorphic to the same simple module, say S_R. Then the $\mathfrak{m}_i$ are infinitely many completely prime right ideals that are associated to the module S (which is simple and thus noetherian).

In response to this easy example, one may ask whether a noetherian module has finitely many associated completely prime right ideals *up to similarity*. Again, the answer is negative. We recall an example used by K. R. Goodearl in [18] to answer a question by Goldie. Let k be a field of characteristic zero and let D be the derivation on the power series ring $k[[y]]$ given by $D = y\frac{d}{dy}$. Define $R := k[[y]][x; D]$, a skew polynomial extension. Consider the right module $M_R = R/xR$. Notice that $M \cong k[[y]]$ as a module over $k[[y]]$. Goodearl showed that the nonzero submodules of M are precisely the $\bar{y}^i R \cong y^i k[[y]]$ (where $\bar{y}^i = y^i + xR \in M$) and that these submodules are pairwise nonisomorphic. From the fact that each of these submodules has infinite k-dimension and finite k-codimension in M, one can easily verify that M (and its nonzero submodules) are monoform (in the sense of Definition 3.3.2). So the right ideals $\mathrm{ann}(\bar{y}^i)$ are comonoform and thus are completely prime by Proposition 3.3.3 to be proved later. But they are pairwise nonsimilar because the factor modules $R/\mathrm{ann}(\bar{y}^i) \cong \bar{y}^i R$ are pairwise nonisomorphic.

(In spite of this failure of finiteness, interested readers should note that O. Goldman developed a theory of associated primes of noncommutative rings in which every noetherian module has finitely many associated primes; see [17, Thm. 6.14]. We will not discuss Goldman's prime torsion theories here but will simply remark that they are related to monoform modules and comonoform right ideals, which are discussed in the the next section. See, for instance, [55].)

The following is an application of Proposition 3.2.3. For a nonzero module M_R, we define the *zero-divisors of M* in R to be the set of all $z \in R$ such that $mz = 0$ for some $0 \neq m \in M$. A theorem from commutative algebra states that the set of zero-divisors of a module over a commutative ring R is equal to the union of some set of prime ideals. Here we generalize this fact for noetherian right modules over noncommutative rings.

Corollary 3.2.4. *Let M_R be a module over a ring R such that R satisfies the ACC on point annihilators of M (e.g., this will hold if M_R or R_R is noetherian). Then the set of zero-divisors of M is a union of completely prime right ideals.*

Proof. Let $z \in R$ be a zero-divisor of M_R. Then there exists $0 \neq m \in M$ such that $z \in \mathrm{ann}(m)$. Because R satisfies ACC on point annihilators of M, there exists a maximal point annihilator $P_z \subseteq R$ of M containing $\mathrm{ann}(m)$, so that $z \in \mathrm{ann}(m) \subseteq P_z$. By Proposition 3.2.3, P_z is a completely prime right ideal. Choosing some such P_z for every zero-divisor z on M, we see that the set of zero-divisors of M is equal to $\bigcup_z P_z$.

If R is right noetherian, then the ACC hypothesis is certainly satisfied. Finally, let us assume that M_R is noetherian and prove that R satisfies ACC on point annihilators of M. Let $I := \mathrm{ann}(m_0) \subseteq \mathrm{ann}(m_1) \subseteq \cdots$ be an ascending chain of point annihilators of M (where $m_i \in M \setminus \{0\}$). Notice that $R/I \cong m_0 R \subseteq M$ is a noetherian module; thus R satisfies ACC

on right ideals containing I. It follows that this ascending chain of point annihilators of M must stabilize. $\qquad\square$

Next we shall investigate conditions for a ring to be a domain. The following fact from commutative algebra was recovered in [37, Cor. 3.2]: a commutative ring R is a domain iff every nonzero prime ideal of R contains a regular element. We generalize this result through a natural progression of ideas, starting with another application of Proposition 3.2.3. Given a ring R, we will use the term *right principal annihilator* to mean a right ideal of the form $I = \operatorname{ann}_r(x)$ for some $x \in R \setminus \{0\}$. This is just another name for a point annihilator of the module R_R, but we use this term below to evoke the idea of chain conditions on annihilators. Also, by a *left regular element* of R we mean an element $s \in R$ such that $\operatorname{ann}_\ell(s) = 0$.

Proposition 3.2.5. *For any nonzero ring R, the following are equivalent:*

(1) *R is a domain;*

(2) *R satisfies ACC on right principal annihilators, and for every nonzero completely prime right ideal P of R, P is not a right principal annihilator;*

(3) *R satisfies ACC on right principal annihilators, and every nonzero completely prime right ideal of R contains a left regular element.*

Proof. Certainly (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2), so it suffices to show (2) $\Longrightarrow$ (1). Let R be as in (2), and let $\mathcal{F}$ be the family of right ideals of R which are not point annihilators of the module R_R. Then $\mathcal{F}$ is a right Oka family by Proposition 3.2.3. Because every point annihilator of R_R is a right principal annihilator, the first hypothesis shows that $\mathcal{F}'$ has the ascending chain condition. Furthermore, the second assumption shows that any nonzero completely prime right ideal of R lies in $\mathcal{F}$. By the CPIP Supplement 2.2.6(2), all nonzero right ideals lie in $\mathcal{F}$. It follows that every nonzero element of R has zero right annihilator, proving that R is a domain. $\qquad\square$

A simple example demonstrates that the chain condition is in fact necessary for (1) $\Longleftrightarrow$ (2) above. Indeed, let k be a field and let R be the commutative k-algebra generated by $\{x_i : i \in \mathbb{N}\}$ with relations $x_i^2 = 0$. Clearly R is not a domain, but its unique prime ideal $(x_0, x_1, x_2, \dots)$ is not a principal annihilator.

This leaves us with the following question: If every completely prime right ideal of a ring contains a left regular element, then is R a domain? Professor G. Bergman has answered this question in the affirmative. With his kind permission, we present a modified version of his argument below.

Lemma 3.2.6. *For a ring R and a module M_R, let $\mathcal{F}$ be the family of right ideals I of R such that there exists a nonempty finite subset $X \subseteq I$ such that, for all $m \in M$, $mX = 0 \implies m = 0$. The family $\mathcal{F}$ is a right Oka family.*

Proof. To see that $R \in \mathcal{F}$, simply take $X = \{1\} \subseteq R$. Now suppose that $I_R \subseteq R$ and $a \in R$ are such that $I + aR$, $a^{-1}I \in \mathcal{F}$. Choose nonempty subsets $X_0 = \{i_1 + ar_1, \ldots, i_p + ar_p\} \subseteq I + aR$ (where each $i_k \in I$) and $X_1 = \{x_1, \ldots, x_q\} \subseteq a^{-1}I$ such that, for $m \in M$, $mX_j = 0$ implies $m = 0$ (for $j = 0, 1$). Define

$$X := \{i_1, \ldots, i_p, ax_1, \ldots, ax_q\} \subseteq I.$$

Suppose that $mX = 0$ for some $m \in M$. Then $maX_1 \subseteq mX = 0$ implies that $ma = 0$. It follows that $mX_0 = 0$, from which we conclude $m = 0$. This proves that $I \in \mathcal{F}$, hence $\mathcal{F}$ is right Oka. $\square$

Proposition 3.2.7. *For a module $M_R \neq 0$ over a ring R, the following are equivalent:*

(1) *M has no zero-divisors (i.e., $0 \neq m \in M$ and $0 \neq r \in R$ imply $mr \neq 0$);*

(2) *Every nonzero completely prime right ideal of R contains a non zero-divisor for M;*

(3) *Every nonzero completely prime right ideal P of R has a nonempty finite subset $X \subseteq P$ such that, for all $m \in M$, $mX = 0 \implies m = 0$.*

Proof. Clearly $(1) \implies (2) \implies (3)$; we prove $(3) \implies (1)$. Assume that (3) holds, and let $\mathcal{F}$ be the Oka family of right ideals defined in Lemma 3.2.6. It is easy to check that the union of any chain of right ideals in $\mathcal{F}'$ also lies in $\mathcal{F}'$. By (3), every nonzero completely prime right ideal of R lies in $\mathcal{F}$. Then the CPIP Supplement 2.2.6 implies that all nonzero right ideals of R lie in $\mathcal{F}$. It is clear that no right ideal in $\mathcal{F}$ can be a point annihilator for M. It follows immediately that M has no zero-divisors. $\square$

Corollary 3.2.8. *For a ring $R \neq 0$, the following are equivalent:*

(1) *R is a domain;*

(2) *Every nonzero completely prime right ideal of R contains a left regular element;*

(3) *Every nonzero completely prime right ideal of R has a nonempty finite subset whose left annihilator is zero.*

Here is another demonstration that completely prime right ideals control the structure of a ring better than the "extremely prime" right ideals (discussed in §2.1). Using Proposition 2.1.4 we constructed rings with no extremely prime right ideals that are not domains. But it is vacuously true that every extremely prime right ideal of such a ring contains a regular element. Thus there is no hope that the result above could be achieved using this more sparse collection of one-sided primes.

3.2.B   Homological properties

Module-theoretic properties that are preserved under extensions arise very naturally in homological algebra. This provides a rich supply of right Oka families, and consequently produces completely prime right ideals via the CPIP.

Example 3.2.9. For a ring k and a k-ring R, consider the following properties of a right ideal $I_R \subseteq R$ (which are known to be preserved by extensions of the factor module):

(1) R/I is a projective right k-module;

(2) R/I is an injective right k-module;

(3) R/I is a flat right k-module.

For each property above, the family $\mathcal{F}$ of all right ideals with that property is a right Oka family (by Remark 3.2.1); hence $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals.

We have the following immediate application, which includes a criterion for a ring to be semisimple.

Proposition 3.2.10. *The family $\mathcal{F}$ of right ideals that are direct summands of R_R is a right Oka family. A right ideal $I_R \subseteq R$ maximal with respect to not being a direct summand of R is a maximal right ideal. A ring R is semisimple iff every maximal right ideal of R is a direct summand of R_R.*

Proof. This family $\mathcal{F}$ is readily seen to be equal to the family given in Example 3.2.9(1) (with $k = R$ and the identity map $k \to R$), and thus it is right Oka. Let $P \in \mathrm{Max}(\mathcal{F}')$. Then P is completely prime, so R/P is indecomposable by Corollary 2.1.7. On the other hand, because every right ideal properly containing P is a direct summand of R_R, the module R/P is semisimple. It follows that R/P is simple, so P is maximal as claimed.

The nontrivial part of the last statement of the proposition is the "if" direction. Assume that every maximal right ideal of R is a direct summand. It suffices to show that every completely prime right ideal of R is maximal. (For if this is the case, then every completely prime right ideal will lie the right Oka family $\mathcal{F}$. Now $\mathcal{F}$ consists of principal—hence f.g.—right ideals by the classical fact that $\mathcal{F} = \{eR : e^2 = e \in R\}$. Then the CPIP Supplement 2.2.6(3) will show that every right ideal of R is a direct summand, making R semisimple.) So suppose $P_R \subsetneq R$ is completely prime. Fix a maximal right ideal $\mathfrak{m}$ of R with $\mathfrak{m} \supseteq P$. Because $\mathfrak{m}$ is a proper direct summand of R_R, $\mathfrak{m}/P$ must be a proper summand of R/P. But R/P is indecomposable by Proposition 2.1.7. Thus $\mathfrak{m}/P = 0$, so that $P = \mathfrak{m}$ is maximal. $\qquad\square$

Of course, we can also prove the "iff" statement above without any reference to right Oka families. (Suppose that every maximal right ideal of R is a direct summand. Assume for contradiction that the right socle $S_R := \mathrm{soc}(R_R)$ is a proper right ideal. Then there is

some maximal right ideal $\mathfrak{m} \subseteq R$ such that $S \subseteq \mathfrak{m}$. But by hypothesis there exists $V_R \subseteq R$ such that $R = V \oplus \mathfrak{m}$. Then $V \cong R/\mathfrak{m}$ is simple. So $V \subseteq S$, contradicting the fact that $V \cap S \subseteq V \cap \mathfrak{m} = 0$.) Although such ad hoc methods are able to recover this fact, our method involving the CPIP 2.2.4 has the desirable effect of fitting the result into a larger context. Also, the CPIP and right Oka families may point one to results that might not have otherwise been discovered without this viewpoint, even if these results could have been proven individually with other methods.

We can also use Example 3.2.9 to recover a bit of the structure theory of right PCI rings. A right module over a ring R is called a *proper cyclic module* if it is cyclic and not isomorphic to R_R. (Note that this is stronger than saying that the module is isomorphic to R/I for some $0 \neq I_R \subseteq R$, though it is easy to confuse the two notions.) A ring R is a *right PCI ring* if every proper cyclic right R-module is injective, and such a ring R is called a *proper right PCI ring* if it is not right self-injective (by a theorem of Osofsky, this is equivalent to saying that R is not semisimple). C. Faith showed in [13] that any proper right PCI ring is a simple right semihereditary right Ore domain. In Faith's own words [13, p. 98], "The reductions to the case R is a domain are long, and not entirely satisfactory inasmuch as they are quite intricate." Our next application of the Completely Prime Ideal Principle shows how to easily deduce that a proper right PCI ring is a domain with the help of a later result on right PCI rings.

Proposition 3.2.11 (Faith). *A proper right PCI ring is a domain.*

Proof. A theorem of R. F. Damiano [10] states that any right PCI ring is right noetherian. (Another proof of this result, due to B. L. Osofsky and P. F. Smith, appears in [46, Cor. 7].) In particular, any right PCI ring is Dedekind-finite.

Now let R be a proper right PCI ring. Because R is Dedekind-finite, for every nonzero right ideal I, $R/I \not\cong R_R$ is a proper cyclic module. Letting $\mathcal{F}$ denote the family of right ideals I such that R/I is injective, we have $0 \in \mathrm{Max}(\mathcal{F}')$. But $\mathcal{F}$ is a right Oka family by Example 3.2.9, so the CPIP 2.2.4 and Proposition 2.1.2(2) together show that R is a domain. $\square$

The astute reader may worry that the above proof is nothing more than circular reasoning, because the proof of Damiano's theorem in [10] seems to rely on Faith's result! This would indeed be the case if Damiano's were the only proof available for his theorem. (Specifically, Damiano cites another result of Faith—basically [13, Prop. 16A]—to conclude that over a right PCI ring, every finitely presented proper cyclic module has a von Neumann regular endomorphism ring. But Faith's result is stated only for cyclic *singular* finitely presented modules. So Damiano seems to be implicitly applying the fact that a proper right PCI ring is a right Ore domain.) Thankfully, we are saved by the fact that Osofsky and Smith's (considerably shorter) proof [46, Cor. 7] of Damiano's result does not require any of Faith's structure theory.

It is worth noting that A. K. Boyle had already provided a proof [5, Cor. 9] that a right noetherian proper right PCI ring is a domain. (This was before Damiano's theorem had been proved.) One difference between our approach and that of [5] is that we do not use any facts about direct sum decompositions of injective modules over right noetherian rings. Of course, the proof using the CPIP is also desirable because we are able to fit the result into a larger context in which it becomes "natural" that such a ring should be a domain.

As in [37], we can generalize Example 3.2.9 with items (1)–(3) below. One may think of the following examples as being defined by the existence of certain (co)resolutions of the modules. Recall that a module M_R is said to be *finitely presented* if there exists an exact sequence of the form $R^m \to R^n \to M \to 0$, and that a *finite free resolution* of M is an exact sequence of the form

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

where the F_i are finitely generated free modules.

Example 3.2.12. Let R be a k-ring, and fix any one of the following properties of a right ideal I in R (known to be closed under extensions), where n is a nonnegative integer:

(1) R/I has k-projective dimension $\leq n$ (or $< \infty$);

(2) R/I has k-injective dimension $\leq n$ (or $< \infty$);

(3) R/I has k-flat dimension $\leq n$ (or $< \infty$);

(4) R/I is a finitely presented right R-modules;

(5) R/I has a finite free resolution as a right R-module.

Then the family $\mathcal{F}$ of right ideals satisfying that property is right Oka (as in Remark 3.2.1); hence $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals.

The families in Example 3.2.9 are just (1)–(3) above with $n = 0$. Restricting to the case $n = 1$ and $k = R$, the family obtained from part (1) (resp. part (3)) of Example 3.2.12 is the family of projective (resp. flat) right ideals of R (for the latter, see [32, (4.86)(2)]). In particular, the CPIP 2.2.4 implies that *a right ideal of R maximal with respect to not being projective (resp. flat) over R is completely prime.*

The family $\mathcal{F}$ in part (4) above is actually equal to the family of *finitely generated* right ideals. Indeed, if $I_R \subseteq R_R$ is finitely generated, then the module R/I is certainly finitely presented. Conversely, if R/I is a finitely presented module, [32, (4.26)(b)] implies that I_R is f.g. This recovers the last sentence of Proposition 2.2.7 from a module-theoretic perspective.

We can also use the family in (2) above to generalize Proposition 3.2.11 about proper right PCI rings. Given a nonnegative integer n, let us say that a ring R is a *right n-PCI ring* if the supremum of the injective dimensions of all proper cyclic right R-modules is equal

to n. (Thus a right 0-PCI ring is simply a right PCI ring.) Also, call a right n-PCI ring R *proper* if R_R has injective dimension greater than n (possibly infinite). Then the following is proved as in 3.2.11, using the family from Example 3.2.12(2).

Proposition 3.2.13. *If a proper right n-PCI ring is Dedekind-finite, then it is a domain.*

Unlike the above result, Proposition 3.2.11 is not a conditional statement because Damiano's theorem guarantees that a right 0-PCI ring is right noetherian, hence Dedekind-finite. Also, it is known that right PCI rings are right hereditary, which implies that the injective dimension of R_R is 1 if R is proper right 0-PCI. Thus we pose the following questions.

Question 3.2.14. *What aspects of the Faith-Damiano structure theory for PCI rings carry over to n-PCI rings? In particular, for a proper right n-PCI ring R, we ask:*

1. *Must R_R have finite injective dimension? If so, is this dimension necessarily equal to $n + 1$?*

2. *Must R be Dedekind-finite, or even possibly right noetherian? What if we assume that R_R has finite injective dimension, say equal to $n + 1$ (if (1) above fails)?*

Further generalizing Examples 3.2.9 and 3.2.12, we have the following.

Example 3.2.15. Given a k-ring R, fix a right module M_k and a left module $_kN$, and let n be a nonnegative integer. Fix one of the following properties of a right ideal $I \subseteq R$:

(1) R/I satisfies $\operatorname{Ext}_k^n(R/I, M) = 0$;

(2) R/I satisfies $\operatorname{Ext}_k^n(M, R/I) = 0$;

(3) R/I satisfies $\operatorname{Tor}_n^k(R/I, N) = 0$.

Applying Remark 3.2.1, the family $\mathcal{F}$ of right ideals satisfying that fixed property is right Oka. Thus $\operatorname{Max}(\mathcal{F}')$ consists of completely prime right ideals.

(The fact that the corresponding classes of cyclic modules are closed under extensions follows from a simple analysis of the long exact sequences for Ext and Tor derived from a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathfrak{M}_R$.)

We can actually use these to recover the families (1)–(3) of Example 3.2.12 as follows. It is known that a module B has k-projective dimension n iff $\operatorname{Ext}_k^n(B, M) = 0$ for all right modules M_k. Then intersecting the families in Example 3.2.15(1) over all modules M_k gives the class in Example 3.2.12(1). A similar process works for (2) and (3) of Example 3.2.12.

As an application of case (1) above, we present the following interesting family of right ideals in any ring associated to an arbitrary module M_R.

Proposition 3.2.16. *For any module M_R, the family $\mathcal{F}$ of all right ideals $I_R \subseteq R$ such that any homomorphism $f : I \to M$ extends to some $\tilde{f} : R \to M$ is a right Oka family. A right ideal maximal with respect to $I \notin \mathcal{F}$ is completely prime.*

Proof. Let $\mathcal{G}$ be the family in Example 3.2.15(1) with $k = R$ and $n = 1$. We claim that $\mathcal{F} = \mathcal{G}$, from which the proposition will certainly follow. Given $I_R \subseteq R$, consider the long exact sequence in Ext associated to the short exact sequence $0 \to I \to R \to R/I \to 0$:

$$0 \to \mathrm{Hom}_R(R/I, M) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M)$$
$$\to \mathrm{Ext}^1_R(R/I, M) \to \mathrm{Ext}^1_R(R, M) = 0$$

($\mathrm{Ext}^1_R(R, M) = 0$ because R_R is projective). Thus $I \in \mathcal{F}$ iff the natural map $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M)$ is surjective, iff its cokernel $\mathrm{Ext}^1_R(R/I, M)$ is zero, iff $I \in \mathcal{G}$. $\qquad\square$

It is an interesting exercise to "check by hand" that the family $\mathcal{F}$ above satisfies the Oka property (2.2.2). When R is a right self-injective ring, one can dualize the above proof of Proposition 3.2.16, (R_R must be injective to ensure that $\mathrm{Ext}^1_R(M, R) = 0$), and a similar argument works using the functor Tor_1 in place of Ext^1. We obtain the following.

Proposition 3.2.17. (A) Let R be a right self-injective ring and let M_R be any module. The family $\mathcal{F}$ of right ideals $I \subseteq R$ such that every homomorphism $f : M \to R/I$ lifts to some $f' : M \to R$ is a right Oka family. Hence, any $I \in \mathrm{Max}(\mathcal{F}')$ is completely prime.

(B) For a ring R and a module $_R N$, let $\mathcal{F}$ be the family of right ideals $I_R \subseteq R$ such that the natural map $I \otimes_R N \to R \otimes_R N \cong N$ is injective. Then $\mathcal{F}$ is an Oka family of right ideals. Hence, any $I \in \mathrm{Max}(\mathcal{F}')$ is completely prime.

For us, what is most interesting about Propositions 3.2.16 and 3.2.17 is that they provide multiple ways to define right Oka families starting with *any given module M_R*. Thanks to the Completely Prime Ideal Principle 2.2.4, each of these families $\mathcal{F}$ gives rise to completely prime right ideals in $\mathrm{Max}(\mathcal{F}')$ whenever this set is nonempty.

3.2.C   Finiteness conditions, multiplicative sets, and invertibility

The final few applications of the Completely Prime Ideal Principle given here come from finiteness conditions on modules, multiplicatively closed subsets of a ring, and invertible right ideals.

We first turn our attention to finiteness conditions. We remind the reader that a module M_R is said to be *finitely cogenerated* provided that, for every set $\{N_i : i \in I\}$ of submodules of M such that $\bigcap_{i \in I} N_i = 0$, there exists a finite subset $J \subseteq I$ such that $\bigcap_{j \in J} N_j = 0$. This is equivalent to saying that the socle of M is finitely generated and is an essential submodule of M. See [32, §19A] for further details.

Example 3.2.18. Let R be a k-ring. Fix any one of the following properties of a right ideal I of R:

(1A) R/I is a finitely generated right k-module;

(1B) R/I is a finitely cogenerated right k-module;

 (2) R/I has cardinality $< \alpha$, where α is an infinite cardinal;

(3A) R/I is a noetherian right k-module;

(3B) R/I is an artinian right k-module;

 (4) R/I is a right k-module of finite length;

 (5) R/I is a right k-module of finite uniform dimension.

The family $\mathcal{F}$ of right ideals satisfying that fixed property is right Oka by Remark 3.2.1; hence $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals.

As a refinement of (4) above, notice that the right k-modules of finite length whose composition factors have certain prescribed isomorphism types is closed under extensions. The same is true for the right k-modules whose length is a multiple of a fixed integer d. Thus these classes give rise to two other Oka families of right ideals.

Right Oka families and completely prime right ideals also arise in connection with multiplicatively closed subsets of a ring.

Example 3.2.19. Consider a multiplicative subset S of a ring R (i.e., a submonoid of the multiplicative monoid of R). A module M_R is said to be *S-torsion* if, for every $m \in M$ there exists $s \in S$ such that $ms = 0$. It is easy to see that the class of S-torsion modules is closed under extensions. Thus the family $\mathcal{F}$ of right ideals $I_R \subseteq R$ such that R/I is S-torsion is a right Oka family. Hence $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals.

Recall that a multiplicative set S in a ring R is called a *right Ore set* if, for all $a \in R$ and $s \in S$, $aS \cap sR \neq \varnothing$. (For example, it is easy to see that any multiplicative set in a commutative ring is right Ore.) One can show that a multiplicative set S is right Ore iff for every module M_R the set

$$t_S(M) := \{m \in M : ms = 0 \text{ for some } s \in S\}$$

of *S-torsion* elements of M is a submodule of M. This makes it easy to verify that for such S, R/I is S-torsion iff $I \cap S \neq \varnothing$. So for a right Ore set $S \subseteq R$, the family of all right ideals I of R such that $I \cap S \neq \varnothing$ is equal to the family $\mathcal{F}$ above and thus is a right Oka family. In particular, a right ideal maximal with respect to being disjoint from S is completely prime. We will be able to strengthen these statements later—see Example 3.3.24.

One further right Oka family comes from the notion of invertibility of right ideals. Fix a ring Q with a subring $R \subseteq Q$. For any submodule $I_R \subseteq Q_R$ we write $I^* := \{q \in Q : qI \subseteq R\}$, which is a *left* R-submodule of Q. We will say that a right R-submodule $I \subseteq Q$ is *right invertible (in Q)* if there exist $x_1, \ldots, x_n \in I$ and $q_1, \ldots, q_n \in I^*$ such that $\sum x_i q_i = 1$. (This definition is inspired by [54, §II.4].) Notice that if I is right invertible as above, then I is necessarily finitely generated, with generating set $x_1, \ldots, x_n$. The concept of a right invertible right ideal certainly generalizes the notion of an invertible ideal in a commutative ring, and it gives rise to a new right Oka family.

Proposition 3.2.20. *Let R be a subring of a ring Q. The family $\mathcal{F}$ of right ideals of R that are right invertible in Q is a right Oka family. Hence the set $\mathrm{Max}(\mathcal{F}')$ consists of completely prime right ideals.*

Proof. Let $I_R \subseteq R$ and $a \in R$ be such that $I + aR$ and $a^{-1}I$ are right invertible. We want to show that I is also right invertible. There exist $i_1, \ldots, i_m \in I$ and $q_k,\ q \in (I + aR)^*$ such that $\sum_{i=1}^m i_k q_k + aq = 1$. Similarly, there exist $x_1, \ldots, x_n \in a^{-1}I$ and $p_j \in (a^{-1}I)^*$ such that $\sum x_j p_j = 1$. Combining these equations, we have

$$1 = \sum i_k q_k + aq$$
$$= \sum i_k q_k + a \left(\sum x_j p_j \right) q$$
$$= \sum i_k q_k + \sum (ax_j)(p_j q).$$

In this equation we have $i_k \in I$, $q_k \in (I + aR)^* \subseteq I^*$, and $ax_j \in a(a^{-1}I) \subseteq I$. Thus we will be done if we can show that every $p_j q \in I^*$.

We claim that $qI \subseteq a^{-1}I$. This follows from the fact that, for any $i \in I$, $q_k i \in R$ so that

$$aqi = \left(1 - \sum i_k q_k \right) i = i - \sum i_k (q_k i) \in I.$$

Thus we find

$$(p_j q)I = p_j(qI) \subseteq (a^{-1}I)^*(a^{-1}I) \subseteq R.$$

It follows that $p_j q \in I^*$, completing the proof. $\qquad\square$

In the case that R is a right Ore ring, it is known (see [54, II.4.3]) that the right ideals of R that are right invertible in its classical right ring of quotients Q are precisely the projective right ideals that intersect the right Ore set S of regular elements of R. (Recall that a ring R is right Ore if the multiplicatively closed set of regular elements in R is right Ore. This is equivalent to the statement that R has a classical right ring of quotients Q; see [32, §10B].) We can use this to give a second proof that the family $\mathcal{F}$ of right invertible right ideals of R is a right Oka family in this case. The alternative characterization of right invertibility in this setting means that $\mathcal{F}$ is the intersection of the family $\mathcal{F}_1$ of projective right ideals

(which was shown to be a right Oka family as an application of Example 3.2.12) with the family $\mathcal{F}_2$ of right ideals that intersect the right Ore set S (which was shown to be a right Oka family in Example 3.2.19). Recalling Remark 2.2.3, we conclude that $\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2$ is a right Oka family.

Using this notion of invertibility, we can generalize the theorem of I. S. Cohen stating that a commutative ring R is a Dedekind domain iff every nonzero prime ideal of R is invertible.

Proposition 3.2.21. *For a subring R of a ring Q, every nonzero right ideal of R is right invertible in Q iff every nonzero completely prime right ideal of R is right invertible in Q. If R is a right Ore ring with classical right ring of quotients Q, then R is a right hereditary right noetherian domain iff every nonzero completely prime right ideal of R is right invertible in Q.*

Proof. First suppose that R is right Ore. According to [54, Prop. II.4.3], a right ideal of R is right invertible in Q iff it is projective and contains a regular element. Thus the right Ore ring R is a right hereditary right noetherian domain iff every nonzero right ideal of R is right invertible in the classical right quotient ring of R. So it suffices to prove the first statement.

Now for any R and Q, the family $\mathcal{F}$ of right ideals of R that are right invertible in Q is a right Oka family by Proposition 3.2.20. Once we recall that a right invertible right ideal is finitely generated, the claim follows from the CPIP Supplement 2.2.6(2). $\square$

3.3 Comonoform right ideals and divisible right Oka families

We devote the final section of this chapter to the study of a particularly well-behaved subset of the completely prime right ideals of a general ring, the *comonoform right ideals* (Definition 3.3.2). Our goal is to provide a richer understanding of the completely prime right ideals of a general ring. There is a special type of Prime Ideal Principle that accompanies this new set of right ideals, as well as new applications to the one-sided structure of rings.

These special right ideals $I_R \subseteq R$ are defined by imposing a certain condition on the factor module R/I. First we must describe the many equivalent ways to phrase this module-theoretic condition. Given an R-module M_R, a submodule $N \subseteq M$ is said to be *dense* if, for all $x, y \in M$ with $x \neq 0$, $x \cdot (y^{-1}N) \neq 0$ (recall the definition of $y^{-1}M$ from §2.1). We write $N \subseteq_d M$ to mean that N is a dense submodule of M, and we let $E(M)$ denote the injective hull of M. It is known that $N \subseteq_d M$ iff $\operatorname{Hom}_R(M/N, E(M)) = 0$, iff for every submodule U with $N \subseteq U \subseteq M$ we have $\operatorname{Hom}_R(U/N, M) = 0$. In addition, for any submodules $N \subseteq U \subseteq M$, it turns out that $N \subseteq_d M$ iff $N \subseteq_d U$ and $U \subseteq_d M$. (See [32, (8.6) & (8.7)] for details.) Finally, any dense submodule of M is *essential* in M, meaning that it has nonzero intersection with every nonzero submodule of M.

Proposition 3.3.1. *For a module $M_R \neq 0$, the following are equivalent:*

(1) *Every nonzero submodule of M is dense in M;*

(2) *Every nonzero cyclic submodule of M is dense in M;*

(3) *For any $x, y, z \in M$ with $x, z \neq 0$, $x \cdot y^{-1}(zR) \neq 0$;*

(4) *Any nonzero $f \in \operatorname{Hom}_R(M, E(M))$ is injective;*

(5) *For any submodule $C \subseteq M$, any nonzero $f \in \operatorname{Hom}_R(C, M)$ (resp. any nonzero $f \in \operatorname{Hom}_R(C, E(M))$) is injective;*

(5′) *M is uniform and for any cyclic submodule $C \subseteq M$, any nonzero $f \in \operatorname{Hom}_R(C, M)$ is injective;*

(6) *There is no nonzero R-homomorphism from any submodule of any proper factor of M to M (resp. to $E(M)$).*

Proof. Clearly (1) $\implies$ (2). For (2) $\implies$ (1), let P be any nonzero submodule of M. Then, taking some cyclic submodule $0 \neq C \subseteq P$, we have $C \subseteq_d M \implies P \subseteq_d M$.

Now (2) $\iff$ (3) is clear from the definition of density. Also, (1) $\iff$ (4) and (1) $\iff$ (5) follow from the various reformulations of density stated above. The equivalence of (5), (6), and their parenthetical formulations is straightforward.

Finally we prove (5) $\iff$ (5′). Assume (5) holds; to verify (5′), we only need to show that M is uniform. By the equivalence of (1) and (5), we see that every nonzero submodule of M is dense and is therefore essential. This proves that M is uniform. Now suppose that (5′) holds, and let $0 \neq f \in \operatorname{Hom}(C, M)$ where C is any submodule of M. Fix some cyclic submodule $0 \neq C_0 \subseteq C$ such that $C_0 \not\subseteq \ker f$, and let g denote the restriction of f to C_0. By hypothesis, $0 = \ker g = \ker f \cap C_0$. Because M is uniform this implies that $\ker f = 0$, proving that (5) is true. $\square$

An easy example shows that the requirement in (5′) that M be uniform is in fact necessary. If V_k is a vector space over a division ring k then it is certainly true that every nonzero homomorphism from a cyclic submodule of V into V is injective. However, if $\dim_k V > 1$, then V has nontrivial direct summands and cannot be uniform.

Definition 3.3.2. A nonzero module M_R is said to be *monoform* (following [20]) if it satisfies the equivalent conditions of Proposition 3.3.1. A right ideal $P_R \subsetneq R$ is *comonoform* if the factor module R/P is monoform.

As a basic example, notice that simple modules are monoform and hence maximal right ideals are comonoform. We can easily verify that the comonoform right ideals of a ring form a subset of the set of completely prime right ideals, as mentioned earlier.

Proposition 3.3.3. *If M_R is monoform, then every nonzero endomorphism of M is injective. In particular, every comonoform right ideal of R is completely prime.*

Proof. The first claim follows from Proposition 3.3.1(5) by taking $C = M$ there. Now the second statement is true by Proposition 2.1.5. $\qquad\square$

Some clarifying remarks about terminology are appropriate. Monoform modules have been given several other names in the literature. They seem to have been first investigated by O. Goldman in [17, §6]. Each monoform module is associated to a certain *prime right Gabriel filter* $\mathcal{F}$ (a term which we will not define here), and Goldman referred to such a module as a *supporting module for* $\mathcal{F}$. They have also been referred to as *cocritical modules*, $\mathcal{F}$-*cocritical modules*, and *strongly uniform modules*. The latter term is justified because, as shown in (5′) above, any monoform module is uniform. Also, comonoform right ideals have been referred to as *critical right ideals* [39] (which explains the term "cocritical module") and *super-prime right ideals* [47]. We have chosen to use the term "monoform" because we feel that it best describes the properties of these modules, and we are using the term "comonoform" rather than "critical" for right ideals in order to avoid confusion with the modules that are critical in the sense of the Gabriel-Rentschler Krull dimension.

Comonoform right ideals enjoy special properties that distinguish them from the more general completely prime right ideals. For instance, if P is a comonoform right ideal of R, then R/P is uniform by Proposition 3.3.1(5′). On the other hand, Example 2.1.8 showed that the more general completely prime right ideals do not always have this property. A second desirable property of comonoform right ideals is given in the following lemma. It is easy to verify (from several of the characterizations in Proposition 3.3.1) that a nonzero submodule of a monoform module is again monoform. Applying Lemma 3.1.5(A) yields the following result.

Lemma 3.3.4. *For any comonoform right ideal $P_R \subseteq R$ and any element $x \in R \setminus P$, the right ideal $x^{-1}P$ is also comonoform.*

It is readily verified that the lemma above does not hold if we replace the word "comonoform" with "completely prime." For instance, consider again Example 2.1.8. For the completely prime right ideal P of the ring R described there and the element $x = E_{12} + E_{13} \in R$, it is readily verified that

$$x^{-1}P = \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not a completely prime right ideal (because the module $R/x^{-1}P$ is decomposable).

Without going into details, we mention that the lemma above suggests that comonoform right ideals P can be naturally grouped into equivalence classes corresponding to the isomorphism classes of the injective hulls $E(R/P)$. (This is directly related to Goldman's notion of primes in [17], and is also investigated in [39].)

Let us consider a few ways one might find comonoform right ideals in a given ring R. First, we have already seen that *every maximal right ideal in R is comonoform.* Second,

Remark 2.1.12 shows that if $I \triangleleft R$ is an ideal contained in a right ideal J, then *J is a comonoform right ideal of R iff J/I is a comonoform right ideal of R/I.* Next, let us examine which (two-sided) ideals of R are comonoform as right ideals. The following result seems to have been first recorded (without proof) in [21, Prop. 4].

Proposition 3.3.5. *An ideal $P \triangleleft R$ is comonoform as a right ideal iff R/P is a right Ore domain.*

Proof. First suppose that P_R is comonoform. Then P is completely prime by Proposition 3.3.3, hence R/P a domain by Proposition 2.1.2. Also R/P is right uniform (see Proposition 3.3.1(5′)). Now because the domain R/P is right uniform, it is right Ore.

Conversely, suppose that $S := R/P$ is a right Ore domain, with right division ring of quotients Q. Then because $E(S_R) = Q_R$, it is easy to see that every nonzero map

$$f \in \mathrm{Hom}(S_R, E(S_R)) = \mathrm{Hom}(S_S, Q_S)$$

must be injective. Thus S_R is monoform, completing the proof. $\square$

Remark 3.3.6. This result makes it easy to construct an example of a ring with a completely prime right ideal that is not comonoform. Let R be any domain that is not right Ore (such as the free algebra generated by two elements over a field), and let $P = 0 \triangleleft R$. Then P_R is completely prime (recall Proposition 2.1.2), but it cannot be right comonoform by the above.

Incidentally, because every monoform module is uniform, the ring R constructed in Example 2.1.8 is an example of an *artinian* (hence noetherian) ring with a completely prime right ideal that is not comonoform. This is to be contrasted with the non-Ore domain example, which is necessarily non-noetherian.

Another consequence of this result is that the comonoform right ideals directly generalize the concept of a prime ideal in a commutative ring, just like the completely prime right ideals.

Corollary 3.3.7. *In a commutative ring R, an ideal $P \triangleleft R$ is comonoform iff it is a prime ideal.*

The Completely Prime Ideal Principle 2.2.4 gives us a method for exploring the existence of completely prime right ideals. We will provide a similar tool for studying the existence of the more special comonoform right ideals in Theorem 3.3.10. The idea is that comonoform right ideals occur as the right ideals that are maximal in the complement of right Oka families that satisfy an extra condition, defined below.

Definition 3.3.8. A family $\mathcal{F}$ of right ideals in a ring R is *divisible* if, for all $a \in R$,

$$I \in \mathcal{F} \implies a^{-1}I \in \mathcal{F}.$$

The next lemma is required to prove the "stronger PIP" for divisible right Oka families.

Lemma 3.3.9. *In a ring R, suppose that I and K are right ideals and that $a \in R$. Then*

$$K \supseteq a^{-1}I \iff K = a^{-1}J \text{ for some right ideal } J_R \supseteq I.$$

Proof. If $K = a^{-1}J$ for some $J \supseteq I$, then clearly $K = a^{-1}J \supseteq a^{-1}I$. Conversely, suppose that $K \supseteq a^{-1}I$. Then for $J := I + aK$, we claim that $K = a^{-1}J$. Certainly $K \subseteq a^{-1}J$. So suppose that $x \in a^{-1}J$. Then $ax \in J = I + aK$ implies that $ax = i + ak$ for some $i \in I$, $k \in K$. Because $a(x - k) = i$, we see that $x - k \in a^{-1}I$. Hence $x = (x - k) + k \in a^{-1}I + K = K$. $\square$

Theorem 3.3.10. *Let $\mathcal{F}$ be a divisible right Oka family. Then every $P \in \operatorname{Max}(\mathcal{F}')$ is a comonoform right ideal.*

Proof. Let $P \in \operatorname{Max}(\mathcal{F}')$. To show that $R/P \neq 0$ is monoform, it is sufficient by Proposition 3.3.1(1) to show that every nonzero submodule $I/P \subseteq R/P$ is dense. That is, for any $0 \neq x + P \in R/P$ and any $y + P \in R/P$, we wish to show that $(x + P) \cdot (y + P)^{-1}(I/P) \neq 0$. It is straightforward to see that $(y + P)^{-1}(I/P) = y^{-1}I$. Thus it is enough to show, for any right ideal $I \supsetneq P$ and elements $x \in R \setminus P$ and $y \in R$, that $x \cdot y^{-1}I \nsubseteq P$.

Assume for contradiction that $x \cdot y^{-1}I \subseteq P$ for such x, y, and I. Then $y^{-1}I \subseteq x^{-1}P$, and Lemma 3.3.9 shows that $x^{-1}P = y^{-1}J$ for some right ideal $J \supseteq I$. Since $P \in \operatorname{Max}(\mathcal{F}')$, the fact that $J \supseteq I \supsetneq P$ implies that $J \in \mathcal{F}$. Because $\mathcal{F}$ is divisible, $x^{-1}P = y^{-1}J \in \mathcal{F}$. Also $x \notin P$ and maximality of P give $P + xR \in \mathcal{F}$. Since $\mathcal{F}$ is right Oka we conclude that $P \in \mathcal{F}$, a contradiction. $\square$

As with the more general completely prime right ideals, there is a "Supplement" that accompanies this "stronger PIP." We omit its proof, which parallels that of Theorem 2.2.6.

Theorem 3.3.11. *Let $\mathcal{F}$ be a divisible right Oka family in a ring R such that every nonempty chain of right ideals in $\mathcal{F}'$ (with respect to inclusion) has an upper bound in $\mathcal{F}'$. Let $\mathcal{S}$ denote the set of comonoform right ideals of R.*

(1) *Let $\mathcal{F}_0$ be a semifilter of right ideals in R. If $\mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*

(2) *For $J_R \subseteq R$, if all right ideals in $\mathcal{S}$ containing J (resp. properly containing J) belong to $\mathcal{F}$, then all right ideals containing J (resp. properly containing J) belong to $\mathcal{F}$.*

(3) *If $\mathcal{S} \subseteq \mathcal{F}$, then $\mathcal{F}$ consists of all right ideals of R.*

To apply the last two theorems, we must provide some ways to construct divisible right Oka families. The first method is extremely straightforward.

Remark 3.3.12. Let $\mathcal{E} \subseteq \mathfrak{M}_R$ be a class of right R-modules that is closed under extensions and closed under passing to submodules. Then for $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_R^c$, the right Oka family $\mathcal{F}_\mathcal{C}$ is divisible. For if $I \in \mathcal{F}_\mathcal{C}$ and $x \in R$, then $R/x^{-1}I$ is isomorphic to the submodule $(I + xR)/I$ of $R/I \in \mathcal{C} \subseteq \mathcal{E}$. Then by hypothesis, $R/x^{-1}I \in \mathcal{E} \cap \mathfrak{M}_R^c = \mathcal{C}$, proving that $x^{-1}I \in \mathcal{F}_\mathcal{C}$.

The above method applies immediately to many of the families $\mathcal{F}_{\mathcal{C}}$ which we have already investigated. To begin with, for a k-ring R, all of the finiteness properties listed in Example 3.2.18 pass to submodules, with the exception of finite generation (1A). Thus a right ideal I maximal with respect to R/I not having one of those properties is comonoform.

We can apply this specifically to rings with the so-called *right restricted minimum condition*; these are the rings R such that R/I is an artinian right R-module for all right ideals $I \neq 0$. If such a ring R is not right artinian, we see that the zero ideal is in $\mathrm{Max}(\mathcal{F}')$ where $\mathcal{F}$ is the divisible Oka family of right ideals $I \subseteq R$ such that R/I is an artinian R-module. Thus the zero ideal is right comonoform by Theorem 3.3.10. Hence R is a right Ore domain by Proposition 3.3.5. The fact that such a ring is a right Ore domain was proved by A. J. Ornstein in [44, Thm. 13] as a generalization of a theorem of Cohen [8, Cor. 2].

Corollary 3.3.13 (Ornstein)**.** *If a ring R satisfies the right restricted minimum condition and is not right artinian, then R is a right Ore domain.*

In addition, for a multiplicative set $S \subseteq R$ the class of S-torsion modules (see Example 3.2.19) is closed under extensions and submodules. So a right ideal I maximal with respect to R/I not being S-torsion is comonoform. Notice that this is true *whether or not the set S is right Ore*. (However, if S is not right Ore then we do not have the characterization that R/I is S-torsion iff $I \cap S \neq \varnothing$.)

For another example, fix a multiplicative set $S \subseteq R$, which again need not be right Ore. A module M_R is said to be *S-torsionfree* if, for any $m \in M$ and $s \in S$, $ms = 0$ implies $m = 0$. The class of S-torsionfree modules is easily shown to be closed under extensions. Hence the family $\mathcal{F}$ of right ideals in R such that R/I is S-torsionfree is a right Oka family. Notice that $\mathcal{F}$ can alternatively be described as

$$\mathcal{F} = \{I_R \subseteq R : \text{for } r \in R \text{ and } s \in S, \, rs \in I \implies r \in I\}.$$

Furthermore, $\mathcal{F}$ is divisible because any submodule of a torsionfree module is torsionfree. So every right ideal $P \subseteq R$ with $P \in \mathrm{Max}(\mathcal{F}')$ is comonoform.

A second effective method of constructing a divisible right Oka family is by defining it in terms of certain families of two-sided ideals. This is achieved in Proposition 3.3.15 below. Given a right ideal I of R, recall that the largest ideal of R contained in I is called the *core of I*, denoted $\mathrm{core}(I)$. It is straightforward to check that $\mathrm{core}(I) = \mathrm{ann}(R/I)$ for any $I_R \subseteq R$.

Lemma 3.3.14. *Let $\mathcal{F}$ be a semifilter of right ideals in a ring R that is generated as a semifilter by two-sided ideals—that is to say, there exists a set $\mathcal{G}$ of two-sided ideals of R such that*

$$\mathcal{F} = \{I_R \subseteq R : I \supseteq J \text{ for some } J \in \mathcal{G}\}$$
$$= \{I_R \subseteq R : \mathrm{core}(I) \in \mathcal{G}\}.$$

Then $\mathcal{F}$ is divisible.

Proof. The equality of the two descriptions of $\mathcal{F}$ above follows from the fact that $\mathcal{G}$ is a semifilter. Suppose that $I \in \mathcal{F}$, so that there exists $J \in \mathcal{G}$ such that $I \supseteq J$. Then for any $a \in R$, $aJ \subseteq J \subseteq I$ implies that $J \subseteq a^{-1}I$. It follows that $a^{-1}I \in \mathcal{F}$, and $\mathcal{F}$ is divisible. $\square$

By analogy with Definition 2.2.5, we define a *semifilter of (two-sided) ideals* in a ring R to be a family $\mathcal{G}$ of ideals of R such that, for $I, J \lhd R$, $I \in \mathcal{G}$ and $J \supseteq I$ imply $J \in \mathcal{G}$. As in [37] we define the following property of a family $\mathcal{G}$ of two-sided ideals in R:

(P_1): $\mathcal{G}$ is a semifilter of ideals that is closed under pairwise products and that contains the ideal R (equivalently, is nonempty).

In [37, Thm. 2.7], it was shown that any (P_1) family of ideals in a commutative ring is an Oka family. The following shows how to define a right Oka family from a (P_1) family of ideals in a noncommutative ring.

Proposition 3.3.15. *Let $\mathcal{G}$ be a family of ideals in a ring R satisfying (P_1). Then the semifilter $\mathcal{F}$ of right ideals generated by $\mathcal{G}$ (as in Lemma 3.3.14) is a divisible right Oka family. Thus, every right ideal in $\mathrm{Max}(\mathcal{F}')$ is comonoform.*

Proof. Let $\mathcal{E}$ be the class of right R-modules M such that $\mathrm{ann}(M) \in \mathcal{G}$. We claim that $\mathcal{E}$ is closed under extensions in $\mathfrak{M}_R$. Indeed, let $L, N \in \mathcal{E}$ and suppose $0 \to L \to M \to N \to 0$ is an exact sequence of right R-modules. We want to conclude that $M \in \mathcal{E}$. Because $\mathrm{ann}(L)$ and $\mathrm{ann}(N)$ belong to $\mathcal{G}$, the fact that $\mathcal{G}$ is (P_1) means that $\mathrm{ann}(M) \supseteq \mathrm{ann}(N) \cdot \mathrm{ann}(L)$ must also lie in $\mathcal{G}$. Thus $M \in \mathcal{E}$ as desired.

Now any cyclic module R/I has annihilator $\mathrm{ann}(R/I) = \mathrm{core}(I)$. So for $\mathcal{C} := \mathcal{E} \cap \mathfrak{M}_R^c$ we see that our family is $\mathcal{F} = \mathcal{F}_{\mathcal{C}}$. Hence $\mathcal{F}$ is a right Oka family. Lemma 3.3.14 implies that $\mathcal{F}$ is divisible. The last sentence of the proposition now follows from Theorem 3.3.10. $\square$

We will apply the result above to a special example of such a family $\mathcal{G}$ of ideals. For a ring R, recall that a subset $S \subseteq R$ is called an *m-system* if $1 \in S$ and for any $s, t \in S$ there exists $r \in R$ such that $srt \in S$. It is well-known that an ideal $P \lhd R$ is prime iff $R \setminus P$ is an *m*-system.

Corollary 3.3.16. (1) For an *m*-system S in a ring R, the family $\mathcal{F}$ of right ideals I such that $\mathrm{core}(I) \cap S \neq \varnothing$ is a divisible right Oka family. A right ideal maximal with respect to having its core disjoint from S is comonoform.

(2) For a prime ideal P of a ring R, the family of all right ideals I_R such that $\mathrm{core}(I) \nsubseteq P$ is a divisible right Oka family. A right ideal I maximal with respect to $\mathrm{core}(I) \subseteq P$ is comonoform. In particular, if R is a prime ring, a right ideal maximal with respect to $\mathrm{core}(I) \neq 0$ is comonoform.

Proof. For (1), we can apply Proposition 3.3.15 to the family $\mathcal{G}$ of ideals having nonempty intersection with the *m*-system S, which is certainly a (P_1) family of ideals. Then (2) follows from (1) if we let $S = R \setminus P$, which is an *m*-system when P is a prime ideal. $\square$

Another application of Proposition 3.3.15 involves the notion of boundedness. Recall that a ring R is said to be *right bounded* if every essential right ideal contains a two-sided ideal that is right essential. (Another way to say this is that if $I_R \subseteq R$ is essential, then $\mathrm{core}(I)$ is right essential.) Then one can characterize whether certain types of rings are right bounded in terms of their comonoform right ideals. Given a module M_R, we write $N \subseteq_e M$ to mean that N is an essential submodule of M.

Proposition 3.3.17. *Let R be a ring in which the set of ideals $\{J \lhd R : J_R \subseteq_e R_R\}$ is closed under squaring (e.g. a semiprime ring or a right nonsingular ring), and suppose that every ideal of R that is right essential is finitely generated as a right ideal (this holds, for instance, if R is right noetherian). Then R is right bounded iff every essential comonoform right ideal of R has right essential core.*

Proof. Assume that R satisfies the two stated hypotheses. We claim that the ideal family $\{J \lhd R : J_R \subseteq_e R\}$ is in fact closed under pairwise products. Indeed, if $I, J \lhd R$ are essential as right ideals, then their product IJ contains the essential right ideal $(I \cap J)^2$ and thus is right essential. This allows us to apply Proposition 3.3.15 to say that the family $\mathcal{F}$ of right ideals with right essential core is a divisible right Oka family. Next, the assumption that every ideal that is right essential is right finitely generated implies that the union of any nonempty chain of right ideals in $\mathcal{F}'$ lies in $\mathcal{F}'$. Also, the set $\mathcal{F}_0$ of essential right ideals is a semifilter. Thus the statement of the proposition, excluding the first parenthetical remark, follows from Theorem 3.3.11(1).

It remains to verify that a semiprime or right nonsingular ring R satisfies the first hypothesis. Suppose that $J \lhd R$ is right essential, and let I_R be a right ideal such that $I \cap J^2 = 0$. Then

$$(I \cap J)^2 \subseteq I \cap J^2 = 0 \quad \text{and} \quad (I \cap J)J \subseteq I \cap J^2 = 0$$

respectively imply that $I \cap J$ squares to zero and has essential right annihilator. Thus if R is either semiprime or right nonsingular, then $I \cap J = 0$. Because J is right essential, we conclude that $I = 0$. Hence J^2 is right essential as desired. $\square$

Corollary 3.3.18. *A prime right noetherian ring R is right bounded iff every essential comonoform right ideal of R has nonzero core.*

Proof. It is a well-known (and easy to verify) fact that every nonzero ideal of a prime ring is right essential. Thus a right ideal of R has right essential core iff its core is nonzero. Because R is prime and right noetherian, we can directly apply Proposition 3.3.17. $\square$

In fact, the last result can be directly deduced from Corollary 3.3.16(2). We chose to include Proposition 3.3.17 because it seems to apply rather broadly.

Next we will show that certain well-studied families of right ideals are actually examples of divisible right Oka families, providing a third method of constructing the latter. The

concept of a *Gabriel filter* of right ideals arises naturally in the study of torsion theories and the related subject of localization in noncommutative rings. The definition of these families is recalled below.

Definition 3.3.19. A *right Gabriel filter* (or *right Gabriel topology*) in a ring R is a nonempty family $\mathcal{F}$ of right ideals of R satisfying the following four axioms (where $I_R, J_R \subseteq R$):

(1) If $I \in \mathcal{F}$ and $J \supseteq I$ then $J \in \mathcal{F}$;

(2) If $I, J \in \mathcal{F}$ then $I \cap J \in \mathcal{F}$;

(3) If $I \in \mathcal{F}$ and $x \in R$ then $x^{-1}I \in \mathcal{F}$;

(4) If $I \in \mathcal{F}$ and $J_R \subseteq R$ is such that $x^{-1}J \in \mathcal{F}$ for all $x \in I$, then $J \in \mathcal{F}$.

Notice that axiom (3) above simply states that *a right Gabriel filter is divisible.* For the reader's convenience, we outline some basic facts regarding right Gabriel filters and torsion theories that will be used here. Refer to [54, VI.1-5] for further details.

Given any right Gabriel filter $\mathcal{F}$ and any module M_R, we define a subset of M:

$$t_\mathcal{F}(M) := \{m \in M : \operatorname{ann}(m) \in \mathcal{F}\}.$$

Axioms (1), (2), and (3) of Definition 3.3.19 guarantee that this is a submodule of M, and it is called the *$\mathcal{F}$-torsion submodule* of M. A module M_R is defined to be *$\mathcal{F}$-torsion* if $t_\mathcal{F}(M) = M$ or *$\mathcal{F}$-torsionfree* if $t_\mathcal{F}(M) = 0$. One can easily verify that *for a right Gabriel filter $\mathcal{F}$, a right ideal $I \subseteq R$ lies in $\mathcal{F}$ iff R/I is $\mathcal{F}$-torsion.*

For any Gabriel filter $\mathcal{F}$, it turns out that the class

$$\mathcal{T}_\mathcal{F} := \{M_R : M \text{ is } \mathcal{F}\text{-torsion, i.e. } M = t_\mathcal{F}(M)\}$$

of all $\mathcal{F}$-torsion right R-modules satisfies the axioms of a *hereditary torsion class*. While we shall not define this term here, it is equivalent to saying that the class $\mathcal{T}_\mathcal{F}$ is *closed under factor modules, direct sums of arbitrary families, and extensions (in $\mathfrak{M}_R$).* (Thus the reader may simply take this to be the definition of a hereditary torsion class.)

With the information provided above we will prove that right Gabriel filters are examples of divisible right Oka families.

Proposition 3.3.20. *Over a ring R, any right Gabriel filter $\mathcal{F}$ is a divisible right Oka family. Any right ideal $P \in \operatorname{Max}(\mathcal{F}')$ is comonoform.*

Proof. Any right Gabriel filter is tautologically a divisible family of right ideals. The torsion class $\mathcal{T}_\mathcal{F}$ is closed under extensions in $\mathfrak{M}_R$, so the class $\mathcal{C} := \mathcal{T}_\mathcal{F} \cap \mathfrak{M}_R^c$ of cyclic $\mathcal{F}$-torsion modules is closed under extensions. A right ideal $I \subseteq R$ lies in $\mathcal{F}$ iff $R/I \in \mathcal{T}_\mathcal{F}$ (as mentioned above), iff $R/I \in \mathcal{C}$ (since $R/I \in \mathfrak{M}_R^c$), iff $I \in \mathcal{F}_\mathcal{C}$. It follows from Theorem 3.1.7 that $\mathcal{F} = \mathcal{F}_\mathcal{C}$ is a right Oka family. The last sentence is true by Theorem 3.3.10. $\square$

We pause for a moment to give a sort of "converse" to this result, in the spirit of Proposition 3.1.11. Given any injective module E_R, the class $\{M_R : \operatorname{Hom}(M, E) = 0\}$ is a hereditary torsion class. This is called the torsion class *cogenerated by E*. We will also say that the corresponding right Gabriel filter is the right Gabriel filter *cogenerated by E*. As stated in [54, VI.5.6], this is the largest right Gabriel filter with respect to which E is torsionfree. Let I be a right ideal in R. In the following, we let $\mathcal{F}_I$ denote the right Gabriel filter cogenerated by $E(R/I)$; that is, $\mathcal{F}_I$ is the set of all right ideals $J \subseteq R$ such that $\operatorname{Hom}_R(R/J, E(R/I)) = 0$. We are now ready for the promised result.

Proposition 3.3.21. *For any right ideal $P \subsetneq R$, the following are equivalent:*

(1) $P \in \operatorname{Max}(\mathcal{F}')$ *for some right Gabriel filter $\mathcal{F}$;*

(2) $P \in \operatorname{Max}((\mathcal{F}_P)')$;

(3) P *is a comonoform right ideal.*

Proof. (2) $\Longrightarrow$ (1) is clear, and (1) $\Longrightarrow$ (3) follows from Theorem 3.3.20. For (3) $\Longrightarrow$ (2), assume that R/P is monoform. Proposition 3.3.1 implies that for every right ideal $I \supsetneq P$ we have $\operatorname{Hom}_R(R/I, E(R/P)) = 0$. Then every such right ideal I tautologically lies in $\mathcal{F}_P$, proving that $P \in \operatorname{Max}((\mathcal{F}_P)')$. $\square$

We mention in passing that this result is similar to [20, Thm. 2.9], though it is not stated in quite the same way. This proposition actually provides a second, though perhaps less satisfying, proof that any comonoform right ideal is completely prime. Given a comonoform right ideal $P \subseteq R$, Proposition 3.3.21 provides a right Gabriel filter $\mathcal{F}$ with $P \in \operatorname{Max}(\mathcal{F}')$. Then because $\mathcal{F}$ is a right Oka family (by Theorem 3.3.20), the CPIP implies that P is a completely prime right ideal.

As a first application of Theorem 3.3.20 we explore the maximal point annihilators of an *injective* module, recovering a result of Lambek and Michler in [39, Prop. 2.7]. This should be compared with Proposition 3.2.3.

Proposition 3.3.22 (Lambek and Michler). *For any injective module E_R, a maximal point annihilator of E is comonoform.*

Proof. Let $\mathcal{F} = \{I_R \subseteq R : \operatorname{Hom}_R(R/I, E) = 0\}$ be the right Gabriel filter cogenerated by E. Then the set of maximal point annihilators of E is clearly equal to $\operatorname{Max}(\mathcal{F}')$. By Theorem 3.3.20, any $P \in \operatorname{Max}(\mathcal{F}')$ is comonoform. $\square$

Example 3.3.23. As shown in [54, VI.6], the set $\mathcal{F}$ of all dense right ideals in any ring R is a right Gabriel filter. (In fact, it is the right Gabriel filter cogenerated by the injective module $E(R_R)$.) Therefore $\mathcal{F}$ is a right Oka family, and a right ideal maximal with respect to not being dense in R is comonoform. Furthermore, in a right nonsingular ring, this family

$\mathcal{F}$ coincides with the set of all essential right ideals (see [32, (8.7)] or [54, VI.6.8]). Thus in a right nonsingular ring, the family $\mathcal{F}$ of essential right ideals is a right Gabriel filter, and a right ideal maximal with respect to not being essential is comonoform.

Example 3.3.24. Let S be a right Ore set in a ring R, and let $\mathcal{F}$ denote the family of all right ideals $I_R \subseteq R$ such that $I \cap S \neq \varnothing$. It is shown in the proof of [54, Prop. VI.6.1] that $\mathcal{F}$ is a right Gabriel filter. It follows from Theorem 3.3.20 that a right ideal maximal with respect to being disjoint from S is comonoform.

We offer an application of the example above. Let us say that a multiplicative set S in a ring R is *right saturated* if $ab \in S$ implies $a \in S$ for all $a, b \in R$.

Corollary 3.3.25. *For every right saturated right Ore set $S \subseteq R$, there exists a set $\{P_i\}$ of comonoform right ideals such that $R \setminus S = \bigcup P_i$.*

Proof. Indeed, for all $x \in R \setminus S$, we must have $xR \subseteq R \setminus S$ because S is right saturated. By a Zorn's Lemma argument, there is a right ideal P_x containing x maximal with respect to being disjoint from S. Example 3.3.24 implies that P_x is comonoform. Choosing such P_x for all $x \in R \setminus S$, we have $R \setminus S = \bigcup P_x$. $\square$

Next we apply Example 3.3.24 to show that a "nice enough" prime (two-sided) ideal must be "close to" some comonoform right ideal.

Corollary 3.3.26. *Let $P_0 \in \mathrm{Spec}(R)$ be such that R/P_0 is right Goldie. Then there exists a comonoform right ideal $P_R \supseteq P_0$ such that $\mathrm{core}(P) = P_0$. In particular, if R is right noetherian then every prime ideal occurs as the core of some comonoform right ideal.*

Proof. Remark 2.1.12 shows that, for any right ideal $L_R \subseteq R$ and any two-sided ideal $I \subseteq L$, L is comonoform in R iff L/I is comonoform in R/I. Then passing to the factor ring R/P_0, it clearly suffices to show that *in a prime right Goldie ring R there exists a comonoform right ideal P of R with zero core.* Indeed, let $S \subseteq R$ be the set of regular elements, and let $P_R \subseteq R$ be maximal with respect to $P \cap S = \varnothing$. Because R is prime right Goldie it is a right Ore ring by Goldie's Theorem, making S a right Ore set. Then P is comonoform by Example 3.3.24. We claim that $\mathrm{core}(P) = 0$. Indeed, suppose that $I \neq 0$ is a nonzero ideal of R. Then since R is prime, I is essential as a right ideal in R. It follows from the theory of semiprime right Goldie rings that $I \cap S \neq \varnothing$ (see, for instance, [32, (11.13)]). This means that we cannot have $I \subseteq P$, verifying that $\mathrm{core}(P) = 0$. $\square$

Notice that the above condition on P_0 is satisfied if R/P_0 is right noetherian. Conversely, it is not true that the core of every comonoform right ideal is prime, even in an artinian ring. For example, let R be the ring of $n \times n$ upper-triangular matrices over a division ring k for $n \geq 2$, and let $P \subsetneq R$ be the right ideal consisting of matrices in R whose first row is zero. Then one can verify that R/P is monoform (for example, using a composition series argument), so that P is comonoform. But the ideal $\mathrm{core}(P) = 0$ is not (semi)prime.

We also provide a slight variation of Corollary 3.2.8, which tested whether or not a ring R is a domain. The version below applies when R is a right Ore ring (e.g., when the multiplicative set of non-zero-divisors of R is a right Ore set).

Proposition 3.3.27. *A right Ore ring R is a domain iff every nonzero comonoform right ideal of R contains a regular element.*

Proof. ("If" direction) Let $S \subseteq R$ be the set of regular elements of R. Then S is a right Ore set, so the family $\mathcal{F} := \{I_R \subseteq R : I \cap S \neq \varnothing\}$ is a right Gabriel filter (in particular, a divisible right Oka family) by Example 3.3.24. Clearly the union of a chain of right ideals in $\mathcal{F}'$ also lies in $\mathcal{F}'$. By Theorem 2.2.6, if every nonzero comonoform right ideal of R contains a regular element, then so does every nonzero right ideal. It follows easily that R is a domain. $\qquad\square$

As a closing observation, we note that there is a second way (aside from Theorem 3.3.20) that right Gabriel filters give rise to comonoform right ideals. Given a right Gabriel filter $\mathcal{G}$ in a ring R, the class of $\mathcal{G}$-torsionfree modules is closed under extensions and submodules (just as the class of $\mathcal{G}$-torsion modules was). Thus a right ideal I of R maximal with respect to the property that R/I is not $\mathcal{G}$-torsionfree must be comonoform by Theorem 3.3.10. A similar statement was shown to be true for the S-torsionfree property, where S is a multiplicative set. However, there is a logical relation between these facts only in the case that S is right Ore, when the family $\mathcal{G}$ of right ideals intersecting S is a right Gabriel filter.

Chapter 4

Test sets for properties of right ideals

4.1 Point annihilator sets for classes of modules

In this section we develop an appropriate notion of a "test set" for certain properties of right ideals in noncommutative rings. This is required for us to state the main theorems along these lines in the next section. Recall that a *point annihilator* of a module M_R is defined to be an annihilator of a nonzero element $m \in M \setminus \{0\}$.

Definitions 4.1.1. Let $\mathcal{C}$ be a class of right modules over a ring R. A set $\mathcal{S}$ of right ideals of R is a *point annihilator set for $\mathcal{C}$* if every nonzero $M \in \mathcal{C}$ has a point annihilator that lies in $\mathcal{S}$. In addition, we make the following two definitions for special choices of $\mathcal{C}$:

- A point annihilator set for the class of all right R-modules will simply be called a *(right) point annihilator set for R*.

- A point annihilator set for the class of all noetherian right R-modules will be called a *(right) noetherian point annihilator set for R*.

Notice that a point annihilator set need not contain the unit ideal R, because point annihilators are always proper right ideals. Another immediate observation is that, for a right noetherian ring R, a right point annihilator set for R is the same as a right noetherian point annihilator set for R.

Remark 4.1.2. The idea of a point annihilator set $\mathcal{S}$ for a class of modules $\mathcal{C}$ is simply that $\mathcal{S}$ is "large enough" to contain a point annihilator of every nonzero module in $\mathcal{C}$. In particular, our definition *does not* require every right ideal in $\mathcal{S}$ to actually be a point annihilator for some module in $\mathcal{C}$. This means that any other set $\mathcal{S}'$ of right ideals with $\mathcal{S}' \supseteq \mathcal{S}$ is also a point annihilator set for $\mathcal{C}$. On the other hand, if $\mathcal{C}_0 \subseteq \mathcal{C}$ is a subclass of modules, then $\mathcal{S}$ is again a point annihilator set for $\mathcal{C}_0$.

Remark 4.1.3. Notice that $\mathcal{S}$ is a point annihilator set for a class $\mathcal{C}$ of modules iff, for every nonzero module $M_R \in \mathcal{C}$, there exists a *proper* right ideal $I \in \mathcal{S}$ such that the right module R/I embeds into M.

The next result shows that noetherian point annihilator sets for a ring R exert a considerable amount of control over the noetherian right R-modules.

Lemma 4.1.4. *A set $\mathcal{S}$ of right ideals in R is a noetherian point annihilator set iff for every noetherian module $M_R \neq 0$, there is a finite filtration of M*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that, for $1 \leq j \leq n$, there exists $I_j \in \mathcal{S}$ such that $M_j/M_{j-1} \cong R/I_j$.

Proof. The "if" direction is easy, so we will prove the "only if" part. For convenience, we will refer to a filtration like the one described above as as an $\mathcal{S}$-*filtration*. Suppose that $\mathcal{S}$ is a noetherian point annihilator set for R, and let $M_R \neq 0$ be noetherian. We prove by noetherian induction that M has an $\mathcal{S}$-filtration. Consider the set $\mathcal{X}$ of nonzero submodules of M that have an $\mathcal{S}$-filtration. Because $\mathcal{S}$ is a noetherian point annihilator set, it follows that $\mathcal{X}$ is nonempty. Since M is noetherian, $\mathcal{X}$ has a maximal element, say N. Assume for contradiction that $N \neq M$. Then $M/N \neq 0$ is noetherian, and by hypothesis there exists $I \in \mathcal{S}$ with $I \neq R$ such that $R/I \cong N'/N \subseteq M/N$ for some $N'_R \subseteq M$. But then $N \subsetneq N' \in \mathcal{X}$, contradicting the maximality of N. Hence $M = N \in \mathcal{X}$, completing the proof. $\square$

We wish to highlight a special type of point annihilator set in the definition below.

Definition 4.1.5. A set $\mathcal{S}$ of right ideals of a ring R is *closed under point annihilators* if, for all $I \in \mathcal{S}$, every point annihilator of R/I lies in $\mathcal{S}$. (This is equivalent to saying that $I \in \mathcal{S}$ and $x \in R \setminus I$ imply $x^{-1}I \in \mathcal{S}$.) If $\mathcal{C}$ is a class of right R-modules, we will say that $\mathcal{S}$ is a *closed point annihilator set for $\mathcal{C}$* if $\mathcal{S}$ is a point annihilator set for $\mathcal{C}$ and $\mathcal{S}$ is closed under point annihilators.

The idea of the above definition is that $\mathcal{S}$ is "closed under passing to further point annihilators of R/I" whenever $I \in \mathcal{S}$. The significance of these closed point annihilator sets is demonstrated by the next result.

Lemma 4.1.6. *Let $\mathcal{C}$ be a class of right modules over a ring R that is closed under taking submodules (e.g. the class of noetherian modules). Suppose that $\mathcal{S}$ is a closed point annihilator set for $\mathcal{C}$. Then for any other point annihilator set $\mathcal{S}_1$ of $\mathcal{C}$, the set $\mathcal{S}_1 \cap \mathcal{S}$ is a point annihilator set for $\mathcal{C}$.*

Proof. Let $0 \neq M_R \in \mathcal{C}$. Because $\mathcal{S}$ is a point annihilator set for $\mathcal{C}$, there exists $0 \neq m \in M$ such that $I := \mathrm{ann}(m) \in \mathcal{S}$. By the hypothesis on $\mathcal{C}$, the module mR lies in $\mathcal{C}$. Because $\mathcal{S}_1$ is also a point annihilator set for $\mathcal{C}$, there exists $0 \neq mr \in mR$ such that $\mathrm{ann}(mr) \in \mathcal{S}_1$. The fact that $\mathcal{S}$ is closed implies that $\mathrm{ann}(mr) \in \mathcal{S} \cap \mathcal{S}_1$. This proves that $\mathcal{S} \cap \mathcal{S}_1$ is also a point annihilator set for $\mathcal{C}$. $\square$

The prototypical example of a noetherian point annihilator set is the prime spectrum of a commutative ring. In fact, *every* noetherian point annihilator set in a commutative ring can be "reduced to" some set of prime ideals, as we show below.

Proposition 4.1.7. *In any commutative ring R, the set $\operatorname{Spec}(R)$ of prime ideals is a closed noetherian point annihilator set. Moreover, given any noetherian point annihilator set $\mathcal{S}$ for R, the set $\mathcal{S} \cap \operatorname{Spec}(R)$ is a noetherian point annihilator subset of S consisting of prime ideals.*

Proof. The set $\operatorname{Spec}(R)$ is a noetherian point annihilator set thanks to the standard fact that any noetherian module over a commutative ring has an associated prime; see, for example, [11, Thm. 3.1]. Furthermore, this set is closed because for $P \in \operatorname{Spec}(R)$, the annihilator of any nonzero element of R/P is equal to P. The last statement now follows from Lemma 4.1.6. □

In this sense right noetherian point annihilator sets of a ring generalize the concept of the prime spectrum of a commutative ring. However, one should not push this analogy too far: in a commutative ring R, *any* set $\mathcal{S}$ of ideals containing $\operatorname{Spec}(R)$ is also a noetherian point annihilator set! In fact, with the help of Proposition 4.1.7 it is easy to verify that any commutative ring R has smallest noetherian point annihilator set $\mathcal{S}_0 := \{P \in \operatorname{Spec}(R) : R/P \text{ is noetherian}\}$, and that a set $\mathcal{S}$ of ideals of R is a noetherian point annihilator set for R iff $\mathcal{S} \supseteq \mathcal{S}_0$.

For most of the remainder of this section, we will record a number of examples of point annihilator sets that will be useful in later applications. Perhaps the easiest example is the following: the family of all right ideals of a ring R is a point annihilator set for any class of right R-modules. A less trivial example: the family of maximal right ideals of a ring R is a point annihilator set for the class of right R-modules of finite length, or for the larger class of artinian right modules. More specifically, according to Remark 4.1.3 it suffices to take any set $\{\mathfrak{m}_i\}$ of maximal right ideals such that the $R/\mathfrak{m}_i$ exhaust all isomorphism classes of simple right modules.

Example 4.1.8. Recall that a module M_R is said to be semi-artinian if every nonzero factor module of M has nonzero socle, and that a ring R is right semi-artinian if R_R is a semi-artinian module. One can readily verify that R is right semi-artinian iff every nonzero right R-module has nonzero socle. Thus for such a ring R, the set of maximal right ideals is a point annihilator set for R, and in particular it is a noetherian point annihilator set for R.

Example 4.1.9. Let R be a left perfect ring, that is, a semilocal ring whose Jacobson radical is left T-nilpotent—see [33, §23] for details. (Notice that this class of rings includes semiprimary rings, especially right *or* left artinian rings.) By a theorem of Bass (see [33, (23.20)]), over such a ring, every *right* R-module satisfies DCC on cyclic submodules. Thus every nonzero right module has nonzero socle, and such a ring is right semi-artinian. But R

has finitely many simple modules up to isomorphism (because the same is true modulo its Jacobson radical). Choosing a set $\mathcal{S} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$ of maximal right ideals such that the modules $R/\mathfrak{m}_i$ exhaust the isomorphism classes of simple right R-modules, we conclude by Remark 4.1.3 that $\mathcal{S}$ is a point annihilator set for any class of right modules $\mathcal{C}$. Hence $\mathcal{S}$ forms a right noetherian point annihilator set for R. (The observant reader will likely have noticed that the same argument applies more generally to any right semi-artinian ring with finitely many isomorphism classes of simple right modules.)

Directly generalizing the fact that the prime spectrum of a commutative ring is a noetherian point annihilator set, we have the following fact, valid for any noncommutative ring.

Proposition 4.1.10. *The set of completely prime right ideals in any ring R is a noetherian point annihilator set.*

Proof. Let $M_R \neq 0$ be noetherian. For any point annihilator $I = \operatorname{ann}(m)$ with $0 \neq m \in M$, the module $R/I \hookrightarrow M$ is noetherian. Thus M must have a *maximal* point annihilator $P_R \supseteq I$, and P is completely prime by Proposition 3.2.3. $\square$

Recall that in any ring R, the set of comonoform right ideals of R forms a subset of the set of all completely prime right ideals of R. As we show next, the subset of comonoform right ideals is also a noetherian point annihilator set.

Proposition 4.1.11. *For any ring R, the set of comonoform right ideals in R is a closed noetherian point annihilator set.*

Proof. Because a nonzero submodule of a monoform module is again monoform, Remark 4.1.3 shows that it is enough to check that any nonzero noetherian module M_R has a monoform submodule. This has already been noted, for example, in [40, 4.6.5]. We include a separate proof for the sake of completeness.

Let $L_R \subseteq M$ be maximal with respect to the property that there exists a nonzero cyclic submodule $N \subseteq M/L$ that can be embedded in M. It is readily verified that N is monoform, and writing $N \cong R/I$ for some comonoform right ideal I, the fact that N embeds in I shows that I is a point annihilator of M. $\square$

Our most "refined" instance of a noetherian point annihilator set for a general non-commutative ring is connected to the concept of (Gabriel-Rentschler) Krull dimension. We review the relevant definitions here, and we refer the reader to the monograph [20] or the textbooks [19, Ch. 15] or [40, Ch. 6] for further details. Define by induction classes $\mathcal{K}_\alpha$ of right R-modules for each ordinal α (for convenience, we consider -1 to be an ordinal number) as follows. Set $\mathcal{K}_{-1}$ to be the class consisting of the zero module. Then for an ordinal $\alpha \geq 0$ such that $\mathcal{K}_\beta$ has been defined for all ordinals $\beta < \alpha$, define $\mathcal{K}_\alpha$ to be the class of all modules M_R such that, for every descending chain

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

of submodules of M indexed by natural numbers, one has $M_i/M_{i+1} \in \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ for almost all indices i. Now if a module M_R belongs to some $\mathcal{K}_\beta$, its Krull dimension, denoted K. $\dim(M)$, is defined to be the least ordinal α such that $M \in \mathcal{K}_\alpha$. Otherwise we say that the Krull dimension of M does not exist.

From the definitions it is easy to see that the right R-modules of Krull dimension 0 are precisely the (nonzero) artinian modules. Also, a module M_R has Krull dimension 1 iff it is not artinian and in every descending chain of submodules of M, almost all filtration factors are artinian.

One of the more useful features of the Krull dimension function is that it is an *exact* dimension function, in the sense that, given an exact sequence $0 \to L \to M \to N \to 0$ of right R-modules, one has

$$\text{K. } \dim(M) = \sup(\text{K. } \dim(L), \text{K. } \dim(N))$$

where either side of the equation exists iff the other side exists. See [19, Lem. 15.1] or [40, Lem. 6.2.4] for details.

The Krull dimension can also be used as a dimension measure for rings. We define the *right Krull dimension* of a ring R to be r. K. $\dim(R) = \text{K. } \dim(R_R)$. The left Krull dimension of R is defined similarly.

Now a module M_R is said to be α-*critical* ($\alpha \geq 0$ an ordinal) if K. $\dim(M) = \alpha$ but K. $\dim(M/N) < \alpha$ for all $0 \neq N_R \subseteq M$, and we say that M_R is *critical* if it is α-critical for some ordinal α. With this notion in place, we define a right ideal $I_R \subseteq R$ to be α-*cocritical* if the module R/I is α-critical, and we say that I is *cocritical* if it is α-cocritical for some ordinal α. Notice immediately that a 0-critical module is the same as a simple module, and the 0-cocritical right ideals are precisely the maximal right ideals.

Cocritical right ideals were already studied by A. W. Goldie in [16], though they are referred to there as "critical" right ideals. (The reader should take care not to confuse this terminology with the phrase "critical right ideal" used in a different sense elsewhere in the literature, as mentioned in §3.3.)

Remarks 4.1.12. The first two remarks below are known; for example, see [40, §6.2].

(1) *A nonzero submodule N of a critical module M is also critical, with* K. $\dim(N) = $ K. $\dim(M)$. Let M be α-critical. If K. $\dim(N) < \alpha$, then because K. $\dim(M/N) < \alpha$, the exactness of Krull dimension would imply the contradiction K. $\dim(M) < \alpha$. Hence K. $\dim(N) = \alpha$. Also, for any nonzero submodule $N_0 \subseteq N$ we have K. $\dim(N/N_0) \leq$ K. $\dim(M/N_0) < \alpha$, proving that N is α-critical.

(2) *A critical module is always monoform.* Suppose that M is α-critical and fix a nonzero homomorphism $f\colon C \to M$ where $C_R \subseteq M$. Because C and $\operatorname{im} f$ are both nonzero submodules of M, they are also α-critical by (1). Then K. $\dim(C) = $ K. $\dim(\operatorname{im} f) = $ K. $\dim(C/\ker f)$, so we must have $\ker f = 0$. Thus M is indeed monoform.

(3) *Any cocritical right ideal is comonoform and, in particular, is completely prime.* This follows immediately from the preceding remark and the fact (Proposition 3.3.3) that any comonoform right ideal is completely prime.

It is possible to classify the (two-sided) ideals that are cocritical as right ideals.

Proposition 4.1.13. *For any ring R and any ideal $P \lhd R$, the following are equivalent:*

(1) *P is cocritical as a right ideal;*

(2) *R/P is a right Ore domain with right Krull dimension;*

(3) *R/P is a domain with right Krull dimension.*

Proof. Because every cocritical right ideal is comonoform, (1) $\implies$ (2) follows from Proposition 3.3.5. Also, (2) $\implies$ (3) is trivial.

(3) $\implies$ (1): It can be shown that, given any module M_R whose Krull dimension exists and an injective endomorphism $f : M \to M$, K. $\dim(M) >$ K. $\dim(M/f(M))$ (see [19, Lem. 15.6]). Applying this to $M = S := R/P$, we see that K. $\dim(S) >$ K. $\dim(S/xS)$ for all nonzero $x \in S$ (this is also proved in [40, Lem. 6.3.9]). Thus S_S, and consequently S_R, are critical modules. $\square$

Example 4.1.14. The last proposition is useful for constructing an ideal of a ring that is (right and left) comonoform but not (right or left) cocritical. If R is a commutative domain that does not have Krull dimension, then the zero ideal of R is prime and thus is is comonoform by Corollary 3.3.7. But because R does not have Krull dimension, the zero ideal cannot be cocritical by the previous result. For an explicit example, one can take $R = k[x_1, x_2, \dots]$ for some commutative domain k. It is shown in [20, Ex. 10.1] that such a ring does not have Krull dimension, using the fact that a polynomial ring $R[x]$ has right Krull dimension iff the ground ring R is right noetherian.

The reason for our interest in the set of cocritical right ideals is that it is an important example of a noetherian point annihilator set in a general ring.

Proposition 4.1.15. *For any ring R, the set of all cocritical right ideals is a closed point annihilator set for the class of right R-modules whose Krull dimension exists. In particular, this set is a closed noetherian point annihilator set for R.*

Proof. Because any nonzero module with Krull dimension has a critical submodule (see [19, Lem. 15.8] or [40, Lem. 6.2.10]), Remark 4.1.3 shows that the set of cocritical right ideals of R is a point annihilator set for the class of right R-modules with Krull dimension. Because any noetherian module has Krull dimension (see [19, Lem. 15.3] or [40, Lem. 6.2.3]), we see by Remark 4.1.2 that this same set is a right noetherian point annihilator set for R. The fact that this set is closed under point annihilators follows from Remark 4.1.12(1). $\square$

Let us further examine the relationship between the general noetherian point annihilator sets given in Propositions 4.1.10, 4.1.11, and 4.1.15. From Proposition 3.3.3 and Remark 4.1.12(3) we see that there are always the following containment relations (where the first three sets are noetherian point annihilator sets but the last one is not, in general):

$$\left\{ \begin{array}{c} \text{completely prime} \\ \text{right ideals} \end{array} \right\} \supseteq \left\{ \begin{array}{c} \text{comonoform} \\ \text{right ideals} \end{array} \right\} \supseteq \left\{ \begin{array}{c} \text{cocritical} \\ \text{right ideals} \end{array} \right\} \supseteq \left\{ \begin{array}{c} \text{maximal} \\ \text{right ideals} \end{array} \right\}.$$

$$(4.1.16)$$

Notice that in a commutative ring R the first two sets are equal to $\operatorname{Spec}(R)$ by Corollaries 2.1.3 and 3.3.7, and when R is commutative and has Gabriel-Rentschler Krull dimension (e.g., when R is noetherian) the third set is also equal to $\operatorname{Spec}(R)$ by Proposition 4.1.13. The latter fact provides many examples where the last containment is strict: in any commutative ring R with Krull dimension > 0 there exists a nonmaximal prime ideal, which must be cocritical. We saw in Remark 3.3.6 and Example 4.1.14 that the first two inclusions can each be strict. However, the latter example was necessarily non-noetherian. Below we give an example of a noncommutative *artinian* (hence noetherian) ring over which both containments are strict. Before presenting the example, we provide a useful characterization of the semi-artinian modules that are monoform. For a module M_R, we denote the socle of M by $\operatorname{soc}(M)$.

Lemma 4.1.17. *Let M_R be a semi-artinian R-module. Then the following are equivalent:*

(1) *M is monoform;*

(2) *For any nonzero submodule $K_R \subseteq M$, $\operatorname{soc}(M)$ and $\operatorname{soc}(M/K)$ do not have isomorphic nonzero submodules;*

(3) *$\operatorname{soc}(M)$ is simple and does not embed into any proper factor module of M.*

Proof. (1) $\implies$ (3): Because M is semi-artinian, $S := \operatorname{soc}(M) \neq 0$. Suppose that M is monoform; then M is uniform (see Proposition 3.3.1(5′)). So S is a uniform semisimple module and is therefore simple. Now assume for contradiction that S embeds into M/K for some nonzero $K_R \subseteq M$. Let the image of this embedding be equal to L/K. Then because $L/K \cong S$, we have a nonzero noninjective map $L \twoheadrightarrow S \hookrightarrow M$, which contradicts the fact that M is monoform. So S cannot embed into any proper factor of M.

(3) $\implies$ (2) is clear. For (2) $\implies$ (1), fix any nonzero homomorphism $f \colon M \to E(M)$, where $E(M) \supseteq M$ is the injective hull of M. Let $K = \ker f$, so that M/K embeds in $E(M)$. Let $S = \operatorname{soc}(M)$ and $S' = \operatorname{soc}(M/K)$, which are both nonzero because M is semi-artinian. We will consider S' to be a submodule of $E(M)$ under the embedding $M/K \hookrightarrow E(M)$. Then because M is essential in $E(M)$, we have $0 \neq S' \cap M = S' \cap S$. By (2) we conclude that $K = 0$, proving that M is monoform. $\qquad\square$

Example 4.1.18. Let k be a division ring, and let R be the ring of all 3×3 matrices over k of the form

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & d \end{pmatrix}. \qquad (4.1.19)$$

One can easily verify (for example, by passing to the factor ring $R/\operatorname{rad}(R)$) that R has two simple right modules up to isomorphism. We may view these modules as $S_1 = k$ with right R-action given by right multiplication by a in (4.1.19) and $S_2 = k$ with right action given by right multiplication by d in (4.1.19). Consider the right ideals

$$P_0 := \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & d & e \\ 0 & 0 & d \end{pmatrix} \right\} \subseteq P_1 := \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & d & e \\ 0 & 0 & d \end{pmatrix} \right\} \subseteq P_2 := \left\{ \begin{pmatrix} 0 & b & c \\ 0 & d & e \\ 0 & 0 & d \end{pmatrix} \right\}.$$

Then the cyclic module $V := R/P_0$ is isomorphic to the space $(k\ k\ k)_R$ of row vectors with the natural right R-action. Notice that $V_i := P_i/P_0$ $(i = 1, 2)$ corresponds to the submodule of row vectors whose first $3 - i$ entries are zero. One can check that the only submodules of V are $0 \subseteq V_1 \subseteq V_2 \subseteq V$, which implies that this is the unique composition series of V. It is clear that

$$V_1 \cong V_2/V_1 \cong S_2 \quad \text{and} \quad V/V_2 \cong S_1.$$

We claim that P_0 is a completely prime right ideal that is not comonoform. To see that it is completely prime, it suffices to show that every nonzero endomorphism of $V = R/P_0$ is injective. Indeed, the only proper factors of V are V/V_1 and V/V_2. By an inspection of composition factors, neither of these can embed into V, proving that P_0 is completely prime. To see that P_0 is not comonoform, consider that

$$\operatorname{soc}(V) = V_1 \cong V_2/V_1 = \operatorname{soc}(V/V_1).$$

By Lemma 4.1.17 we see that $R/P_0 = V$ is not monoform and thus P_0 is not comonoform.

We also claim that P_1 is comonoform but not cocritical. Notice that over any right artinian ring, every cyclic critical module has Krull dimension 0. But a 0-critical module is necessarily simple. Thus a cocritical right ideal in a right artinian ring must be maximal. But P_1 is not maximal and thus is not cocritical. On the other hand, $R/P_1 \cong V/V_1$ has unique composition series $0 \subseteq V_2/V_1 \subseteq V/V_1$. This allows us to easily verify, using Lemma 4.1.17, that $V/V_1 \cong R/P_1$ is monoform, proving that P_1 is comonoform.

This same example also demonstrates that the set of completely prime right ideals is not always closed under point annihilators (as in Definition 4.1.5). This is because the cyclic submodule $V_2 \subseteq V = R/P_0$ certainly has a nonzero noninjective endomorphism, as both of its composition factors are isomorphic.

An example along these lines was already used in [20, p. 11] to show that a monoform module need not be critical. Notice that the completely prime right ideal P_0 above is such

that R/P_0 is uniform, even if it is not monoform. (This means that the right ideal P_0 is "meet-irreducible.") An example of a completely prime right ideal whose factor module is not uniform was already given in Example completely prime not meet-irreducible.

Given the containments of noetherian point annihilator sets in (4.1.16), one might question the need for the notion of a point annihilator set. Why not simply state all theorems below just for the family of cocritical right ideals? We already have an answer to this question in Example 4.1.9, which demonstrates that every left perfect ring has a *finite* right noetherian point annihilator set. The reason we can reduce to a finite set $\mathcal{S}$ in such rings is the fact stated in Remark 4.1.3 that a noetherian right module only needs to contain a submodule *isomorphic to R/I for some $I \in \mathcal{S}$*. In other words, $\mathcal{S}$ only needs to contain a single representative from any given similarity class. So while a left perfect ring R may have infinitely many maximal right ideals, it has only finitely many similarity classes of maximal right ideals. Thus we can reduce certain problems about all right ideals of R to a finite set of maximal ideals! This will be demonstrated in Proposition 4.2.8 and Corollary 5.1.5, below where we shall prove that a left perfect ring is right noetherian (resp. a PRIR) iff all maximal right ideals belonging to a (properly chosen) finite set are finitely generated (resp. principal).

We have also phrased the discussion in terms of general noetherian point annihilator sets to leave open the possibility of future applications to classes of rings which have nicer noetherian point annihilator sets than the whole set of cocritical right ideals, akin to the class of left perfect rings.

4.2   The Point Annihilator Set Theorem

Having introduced the notion of a point annihilator set, we can now state our fundamental result, the Point Annihilator Set Theorem 4.2.1. This theorem gives conditions under which one may deduce that one family $\mathcal{F}_0$ of right ideals is contained in a second family $\mathcal{F}$ of right ideals. We will most often use it as a sufficient condition for concluding that all right ideals of a ring lie in a particular right Oka family $\mathcal{F}$.

Certain results in commutative algebra state that when every prime ideal in a commutative ring has a certain property, then all ideals in the ring have that property. As mentioned in the introduction, the two motivating examples are Cohen's Theorem 1.1.1 and Kaplansky's Theorem 1.1.3. In [37, p. 3017], these theorems were both recovered in the context of Oka families and the Prime Ideal Principle. The useful tool in that context was the "Prime Ideal Principle Supplement" [37, Thm. 2.6]. We have already provided one noncommutative generalization of this tool in the CPIP Supplement 2.2.6, which we used to produce a noncommutative extension of Cohen's Theorem in Theorem 2.2.8.

The CPIP Supplement states that for certain right Oka families $\mathcal{F}$, if the set $\mathcal{S}$ of completely prime right ideals lies in $\mathcal{F}$, then all right ideals lie in $\mathcal{F}$. The main goal of this

section is to improve upon this result by allowing the set $\mathcal{S}$ to be *any* point annihilator set. This is achieved in Theorem 4.2.3 as an application of the Point Annihilator Set Theorem.

The Point Annihilator Set Theorem basically formalizes a general "strategy of proof." For the sake of clarity, we present an informal sketch of this proof strategy before stating the theorem. Suppose that we want to prove that *every module with the property $\mathcal{P}$ also has the property $\mathcal{Q}$*. Assume for contradiction that there is a counterexample. Use Zorn's Lemma to pass to a counterexample M satisfying $\mathcal{P}$ that is "critical" with respect to not satisfying $\mathcal{Q}$, in the sense that every proper factor module of M satisfies $\mathcal{Q}$ but M itself does not satisfy $\mathcal{Q}$. Argue that M has a nonzero submodule N that satisfies $\mathcal{Q}$. Finally, use the fact that N and M/N have $\mathcal{Q}$ to deduce the contradiction that M has $\mathcal{Q}$.

Our theorem applies in the specific case where one's attention is restricted to cyclic modules. In the outline above, we may think of the properties $\mathcal{P}$ and $\mathcal{Q}$ to be, respectively, "$M = R/I$ where $I \in \mathcal{F}_0$" and "$M = R/I$ where $I \in \mathcal{F}$."

Theorem 4.2.1 (The Point Annihilator Set Theorem). *Let $\mathcal{F}$ be a right Oka family such that every nonempty chain of right ideals in $\mathcal{F}'$ (with respect to inclusion) has an upper bound in $\mathcal{F}'$.*

(1) *Let $\mathcal{F}_0$ be a semifilter of right ideals in R. If $\mathcal{F}$ is a point annihilator set for the class of modules $\{R/I : I_R \in \mathrm{Max}(\mathcal{F}') \cap \mathcal{F}_0\}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*

(2) *For any right ideal $J_R \subseteq R$, if $\mathcal{F}$ is a point annihilator set for the class of modules R/I such that $I \in \mathrm{Max}(\mathcal{F}')$ and $I \supseteq J$ (resp. $I \supsetneq J$), then all right ideals containing (resp. properly containing) J belong to $\mathcal{F}$.*

(3) *If $\mathcal{F}$ is a point annihilator set for the class of modules $\{R/I : I_R \in \mathrm{Max}(\mathcal{F}')\}$, then $\mathcal{F}$ consists of all right ideals of R.*

Proof. Suppose that the hypotheses of (1) hold, and assume for contradiction that there exists $I_0 \in \mathcal{F}_0 \setminus \mathcal{F}$. The assumptions on $\mathcal{F}'$ allow us to apply Zorn's Lemma to find $I \in \mathrm{Max}(\mathcal{F}')$ with $I \supseteq I_0$. Then $I \in \mathcal{F}_0$ because $\mathcal{F}_0$ is a semifilter. The point annihilator hypothesis implies that there is a nonzero element $a + I \in R/I$ such that $a^{-1}I = \mathrm{ann}(a+I) \in \mathcal{F}$. On the other hand, $a + I \neq 0 + I$ implies that $I + aR \supsetneq I$. By maximality of I, this means that $I + aR \in \mathcal{F}$. Because $\mathcal{F}$ is a right Oka family, we arrive at the contradiction $I \in \mathcal{F}$.

Parts (2) and (3) follows from (1) by taking $\mathcal{F}_0$ to be, respectively, the set of all right ideals of R (properly) containing J or the set of all right ideals of R. $\square$

Notice that part (1) above remains true if we weaken the condition on chains in $\mathcal{F}'$ to the following: every nonempty chain in $\mathcal{F}' \cap \mathcal{F}_0$ has an upper bound in $\mathcal{F}'$. The latter condition holds if every $I \in \mathcal{F}_0$ is such that R/I is a noetherian module, or more generally if $\mathcal{F}_0$ satisfies the ascending chain condition (as a partially ordered set with respect to inclusion). However, we shall not make use this observation in the present work.

The following is an illustration of how Theorem 4.2.1 can be applied in practice. It is well-known that every finitely generated artinian module over a commutative ring has finite length. However, there exist finitely generated (even cyclic) artinian right modules over noncommutative rings that do not have finite length; for instance, see [34, Ex. 4.28]. Here we provide a sufficient condition for all finitely generated artinian right modules over a ring to have finite length.

Proposition 4.2.2. *If all maximal right ideals of a ring R are finitely generated, then every finitely generated artinian right R-module has finite length.*

Proof. It suffices to show that every cyclic artinian right R-module has finite length. Let $\mathcal{F}_0$ be the semifilter of right ideals I_R such that R/I is right artinian, and let $\mathcal{F}$ be the right Oka family of right ideals I such that R/I has finite length. Our goal is then to show that $\mathcal{F}_0 \subseteq \mathcal{F}$. Because every nonzero cyclic artinian module has a simple submodule, we see that $\mathcal{F}$ is a point annihilator set for the class $\{R/I : I \in \mathcal{F}_0\} \supseteq \{R/I : I \in \mathrm{Max}(\mathcal{F}') \cap \mathcal{F}_0\}$. To apply Theorem 4.2.1(1) we will show that every nonempty chain in $\mathcal{F}'$ has an upper bound in $\mathcal{F}'$. For this, it is enough to check that $\mathcal{F}$ consists of finitely generated right ideals. The hypothesis implies that all simple right R-modules are finitely presented. If $I \in \mathcal{F}$ then R/I, being a repeated extension of finitely many simple modules, is finitely presented. It follows that I is finitely generated. (The fact that $\mathcal{F}$ consists of f.g. right ideals can also be deduced from Corollary 3.1.9.) $\qquad\square$

In light of the result above, it would be interesting to find a characterization of the rings R over which every finitely generated artinian right R-module has finite length. How would such a characterization unite both commutative rings and the rings in which every maximal right ideal is finitely generated?

For our purposes, it will often best to use a variant of the theorem above. This variant keeps with the theme of Cohen's and Kaplansky's results (Theorems 1.1.1–1.1.3) of "testing" a property on special sets of right ideals.

Theorem 4.2.3. *Let $\mathcal{F}$ be a right Oka family such that every nonempty chain of right ideals in $\mathcal{F}'$ (with respect to inclusion) has an upper bound in $\mathcal{F}'$. Let $\mathcal{S}$ be a set of right ideals that is a point annihilator set for the class of modules $\{R/I : I_R \in \mathrm{Max}(\mathcal{F}')\}$.*

(1) *Let $\mathcal{F}_0$ be a divisible semifilter of right ideals in R. If $\mathcal{F}_0 \cap \mathcal{S} \subseteq \mathcal{F}$, then $\mathcal{F}_0 \subseteq \mathcal{F}$.*

(2) *For any ideal $J \lhd R$, if all right ideals in $\mathcal{S}$ that contain J belong to $\mathcal{F}$, then every right ideal containing J belongs to $\mathcal{F}$.*

(3) *If $\mathcal{S} \subseteq \mathcal{F}$, then all right ideals of R belong to $\mathcal{F}$.*

Proof. As in the previous result, parts (2) and (3) are special cases of part (1). To prove (1), Theorem 4.2.1 implies that it is enough to show that $\mathcal{F}$ is a point annihilator set for the class of modules $\{R/I : I_R \in \mathrm{Max}(\mathcal{F}') \cap \mathcal{F}_0\}$. Fixing such R/I, the hypothesis of part (1) ensures that R/I has a point annihilator in $\mathcal{S}$, say $A = \mathrm{ann}(x+I) \in \mathcal{S}$ for some $x+I \in R/I \setminus \{0+I\}$. Because $I \in \mathcal{F}_0$ and $\mathcal{F}_0$ is divisible, the fact that $A = x^{-1}I$ implies that $A \in \mathcal{F}_0$. Thus $A \in \mathcal{S} \cap \mathcal{F}_0 \subseteq \mathcal{F}$, providing a point annihilator of R/I that lies in $\mathcal{F}$. $\square$

We also record a version of Theorem 4.2.3 adapted especially for families of finitely generated right ideals. Because of its easier formulation, it will allow for simpler proofs as we provide applications of Theorem 4.2.3.

Corollary 4.2.4. *Let $\mathcal{F}$ be a right Oka family in a ring R that consists of finitely generated right ideals. Let $\mathcal{S}$ be a noetherian point annihilator set for R. Then the following are equivalent:*

(1) *$\mathcal{F}$ consists of all right ideals of R;*

(2) *$\mathcal{F}$ is a noetherian point annihilator set;*

(3) *$\mathcal{S} \subseteq \mathcal{F}$.*

Proof. Given any $I \in \mathrm{Max}(\mathcal{F}')$, any nonzero submodule of R/I is the image of a right ideal properly containing I, which must be finitely generated; thus R/I is a noetherian right R-module. Stated another way, the class $\{R/I : I \in \mathrm{Max}(\mathcal{F}')\}$ consists of noetherian modules. Thus (1) $\Longleftrightarrow$ (2) follows from Theorem 4.2.1(3) and (1) $\Longleftrightarrow$ (3) follows from Theorem 4.2.3(3). $\square$

As our first application of the simplified corollary above, we will finally present our noncommutative generalization of Cohen's Theorem 1.1.1, improving upon Theorem 2.2.8.

Theorem 4.2.5 (A noncommutative Cohen's Theorem). *Let R be a ring with a right noetherian point annihilator set $\mathcal{S}$. The following are equivalent:*

(1) *R is right noetherian;*

(2) *Every right ideal in $\mathcal{S}$ is finitely generated;*

(3) *Every nonzero noetherian right R-module has a finitely generated point annihilator;*

(4) *Every nonzero noetherian right R-module has a nonzero cyclic finitely presented submodule.*

In particular, R is right noetherian iff every cocritical right ideal is finitely generated.

Proof. The family of finitely generated right ideals is a right Oka family by Proposition 2.2.7. The equivalence of (1), (2), and (3) thus follows directly from Corollary 4.2.4. Also, (3) $\iff$ (4) comes from the observation that a right ideal I is a point annihilator of a module M_R iff there is an injective module homomorphism $R/I \hookrightarrow M$, as well as the fact that R/I is a finitely presented module iff I is a finitely generated right ideal [32, (4.26)(b)]. The last statement follows from Proposition 4.1.15. □

In particular, if we take the set $\mathcal{S}$ above to be the completely prime right ideals of R, we recover Theorem 2.2.8. Our version of Cohen's Theorem will be compared and contrasted with earlier such generalizations in §5.4.

The result above suggests that one might wish to drop the word "cyclic" in characterization (4). This is indeed possible. We present this as a separate result since it does not take advantage of the "formalized proof method" given in Theorem 4.2.1. However, this result does follow the informal "strategy of proof" outlined at the beginning of this section.

Proposition 4.2.6. *For a ring R, the following are equivalent:*

(1) *R is right noetherian;*

(5) *Every nonzero noetherian right R-module has a nonzero finitely presented submodule.*

Proof. Using the numbering from Theorem 4.2.5, we have (1) $\implies$ (4) $\implies$ (5). Suppose that (5) holds, and assume for contradiction that there exists a right ideal of R that is not finitely generated. Using Zorn's Lemma, pass to $I_R \subseteq R$ that is maximal with respect to not being finitely generated. Then because every right ideal properly containing I is f.g., the module R/I is noetherian. By hypothesis, there is a finitely presented submodule $0 \neq J/I \subseteq R/I$. Then $J \supsetneq I$ implies that J is finitely generated, so that R/J is finitely presented. Because R/I is an extension of the two finitely presented modules J/I and R/J, R/I is finitely presented [36, Ex. 4.8(2)]. But if R/I is finitely presented then I_R is finitely generated [32, (4.26)(b)]. This is a contradiction. □

A well-known theorem in commutative algebra states that a commutative ring R is artinian iff it is noetherian and every prime ideal is maximal. Recall that a module M_R is *finitely cogenerated* if any family of submodules of M whose intersection is zero has a finite subfamily whose intersection is zero. In [37, (5.17)] consideration of the class of finitely cogenerated right modules led to the following "artinian version" of Cohen's theorem: a commutative ring R is artinian iff for all $P \in \mathrm{Spec}(R)$, P is finitely generated and R/P is finitely cogenerated. Here we generalize both of these results to the noncommutative setting.

Proposition 4.2.7. *For a ring R with right noetherian point annihilator set $\mathcal{S}$, the following are equivalent:*

(1) *R is right artinian;*

(2) *R is right noetherian and for all $P \in \mathcal{S}$, $(R/P)_R$ has finite length;*

(3) *For all $P \in \mathcal{S}$, P_R is finitely generated and $(R/P)_R$ has finite length;*

(4) *For all $P \in \mathcal{S}$, P_R is finitely generated and $(R/P)_R$ is finitely cogenerated;*

(5) *R is right noetherian and every cocritical right ideal of R is maximal;*

(6) *Every cocritical right ideal of R is finitely generated and maximal.*

Proof. (1) $\iff$ (2) $\iff$ (3): It is well-known that R is right artinian iff R_R has finite length. This equivalence then follows from Corollary 4.2.4, Theorem 4.2.5, and the fact (see Example 3.2.18(4)) that $\mathcal{F} := \{I_R \subseteq R : R/I_R \text{ has finite length}\}$ is a right Oka family.

(1) $\iff$ (4): It is known that a module M_R is artinian iff every quotient of M is finitely cogenerated (see [36, Ex. 19.0]). Then (1) $\iff$ (4) follows from Corollary 4.2.4 and Example 3.2.18(1B) with $k = R$.

We get (1) $\iff$ (5) $\iff$ (6) by applying the equivalence of (1), (2), and (3) to the case where $\mathcal{S}$ is the set of cocritical right ideals of R, noting that every artinian critical module is necessarily simple. $\qquad\square$

Of course, the fact that a right noetherian ring is right artinian iff all of its cocritical right ideals are maximal follows from a direct argument involving Krull dimensions of modules. Indeed, given a right noetherian ring R with right Krull dimension α, choose a right ideal $I \subseteq R$ maximal with respect to K. $\dim(R/I) = \alpha$. Then for any right ideal $J \supseteq I$, K. $\dim(R/J) < \alpha = $ K. $\dim(R/I)$; hence I is cocritical. So

$$\text{r. K. }\dim(R) = \sup\{\text{K. }\dim(R/I) : I_R \subseteq R \text{ is cocritical}\}.$$

The result now follows once we recall that the 0-critical modules are precisely the simple modules.

Another application of Theorem 4.2.5 tells us when a right semi-artinian ring, especially a *left* artinian ring, is right artinian. (The definition of a right semi-artinian ring was recalled in Example 4.1.8.)

Proposition 4.2.8. (1) A right semi-artinian ring R is right artinian iff every maximal right ideal of R is finitely generated.

(2) Let R be a left perfect ring (e.g. a semiprimary ring, such as a left artinian ring) and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be maximal right ideals such that $R/\mathfrak{m}_i$ exhaust all isomorphism classes of simple right modules. Then R is right artinian iff all of the $\mathfrak{m}_i$ are finitely generated.

Proof. It is easy to check that a right semi-artinian ring R is right artinian iff it is right noetherian. The proposition then follows from Theorem 4.2.5 and Examples 4.1.8–4.1.9. $\quad\square$

A result of B. Osofsky [45, Lem. 11] states that a left or right perfect ring R with Jacobson radical J is right artinian iff J/J^2 is finitely generated as a right R-module. This applies, in particular, to left artinian rings. D. V. Huynh characterized which (possibly nonunital) left artinian rings are right artinian in [22, Thm. 1]. In the unital case, his characterization recovers Osofsky's result above for the special class of left artinian rings. We can use our previous result to recover a weaker version of Osofsky's theorem that implies Huynh's result for unital left artinian rings.

Corollary 4.2.9. *Let R be a ring with $J := \mathrm{rad}(R)$. The the following are equivalent:*

(1) *R is right artinian;*

(2) *R is left perfect and J is a finitely generated right ideal;*

(3) *R is perfect and J/J^2 is a finitely generated right R-module.*

In particular, if R is semiprimary (for instance, if it is left artinian), then R is right artinian iff J/J^2 is finitely generated on the right.

Proof. Because any right artinian ring is both perfect and right noetherian, we have (1) $\implies$ (3). For (3) $\implies$ (2), suppose that R is perfect and that J/J^2 is right finitely generated. Then for some finitely generated submodule $M_R \subseteq J_R$, $J = M + J^2$. Since R is right perfect, J is right T-nilpotent. Then by "Nakayama's Lemma" for right T-nilpotent ideals (see [33, (23.16)]) implies that $J_R = M_R$ is finitely generated.

Finally we show (2) $\implies$ (1). Suppose that R is left perfect and that J_R is finitely generated. For any maximal right ideal $\mathfrak{m}$ of R, we have $J \subseteq \mathfrak{m}$. Now $\mathfrak{m}/J$ is a right ideal of the semisimple ring R/J and is therefore finitely generated. Because J_R is also finitely generated, we see that $\mathfrak{m}_R$ itself is finitely generated. Since this is true for all maximal right ideals of R, Proposition 4.2.8(2) implies that R is right artinian. $\qquad\square$

Next we give a condition for every finitely generated right module over a ring R to have a finite free resolution (FFR). Notice that such a ring is necessarily right noetherian. Indeed, any module with an FFR is necessarily finitely presented. Thus if every f.g. right R-module has an FFR, then for every right ideal $I \subseteq R$ the module R/I must have an FFR and therefore must be finitely presented. It follows (from Schanuel's Lemma [32, (5.1)]) that I_R is finitely generated, and R is right noetherian.

Proposition 4.2.10. *Let $\mathcal{S}$ be a right noetherian point annihilator set for a ring R (e.g. the set of cocritical right ideals). Then the following are equivalent.*

(1) *Every finitely generated right R-module has a finite free resolution;*

(2) *For all $P \in \mathcal{S}$, R/P has a finite free resolution;*

(3) *Every right ideal in $\mathcal{S}$ has a finite free resolution.*

Proof. (1) $\Longrightarrow$ (3): As mentioned before the proposition, if every f.g. right R-module has a finite free resolution then R is right noetherian. So every right ideal $I_R \subseteq R$ is finitely generated and therefore has a finite free resolution.

Next, (3) $\Longrightarrow$ (2) follows from the easy fact that, given $I_R \subseteq R$, if I has a finite free resolution then so does R/I. For (2) $\Longrightarrow$ (1), let $\mathcal{F}$ be the family of right ideals I such that R/I has a finite free resolution and assume that $\mathcal{S} \subseteq \mathcal{F}$. This is a right Oka family according to Example 3.2.12(5). Moreover, if $I \in \mathcal{F}$ then R/I is finitely presented. As noted earlier, this implies that I_R must be finitely generated [32, (4.26)(b)]. It follows from Corollary 4.2.4 that every right ideal of R lies in $\mathcal{F}$. Because any finitely generated right R-module is an extension of cyclic modules and because the property of having an FFR is preserved by extensions, we conclude that (1) holds. $\qquad\square$

Chapter 5

When are all right ideals principal?

5.1 Families of principal right ideals

We will use $\mathcal{F}_{\mathrm{pr}}(R)$ to denote the family of principal right ideals of a ring R. If the ring R is understood from the context, we may simply use $\mathcal{F}_{\mathrm{pr}}$ to denote this family.

A theorem of Kaplansky [26, Thm. 12.3 & Footnote 8] states that a commutative ring is a principal ideal ring iff its prime ideals are all principal. In [37, (3.17)] this theorem was recovered via the "PIP supplement." It is therefore reasonable to hope that the methods presented here will lead to a generalization of this result. Specifically, we would like to know whether a ring R is a principal right ideal ring (PRIR) if, say, every cocritical right ideal is principal. It turns out that this is in fact true, but the path to proving the result is not as straightforward as one might imagine. The obvious starting point is to ask whether the family $\mathcal{F}_{\mathrm{pr}}$ of principal right ideals in an arbitrary ring R is a right Oka family. Suppose that R is a ring such that $\mathcal{F}_{\mathrm{pr}}$ is a right Oka family. Then Corollary 4.2.4 readily applies to $\mathcal{F}_{\mathrm{pr}}$. However, it is not immediately clear whether or not $\mathcal{F}_{\mathrm{pr}}(R)$ is necessarily right Oka for every ring R. The following proposition provides some guidance in this matter.

Proposition 5.1.1. *Let $S \subseteq R$ be a multiplicative set. Then $\mathcal{F} := \{sR : s \in S\}$ is a right Oka family iff it is closed under similarity. In particular, for any ring R, the family $\mathcal{F}_{\mathrm{pr}}$ of principal right ideals is a right Oka family iff it is closed under similarity.*

Proof. By Proposition 3.1.6, any right Oka family is closed under similarity. On the other hand, assume that the family $\mathcal{F}$ in question is closed under similarity. Suppose that $I + aR$, $a^{-1}I \in \mathcal{F}$, and write $I + aR = sR$ for some $s \in S$. In the short exact sequence of right R-modules

$$0 \to \frac{I + aR}{I} \to \frac{R}{I} \to \frac{R}{I + aR} \to 0,$$

observe that $R/(a^{-1}I) \cong (I + aR)/I = sR/I \cong R/(s^{-1}I)$. Because $\mathcal{F}$ is closed under similarity and $a^{-1}I \in \mathcal{F}$, we must also have $s^{-1}I \in \mathcal{F}$. Fix $t \in S$ such that $s^{-1}I = tR$. Then because $I \subseteq I + aR = sR$ we have $I = s(s^{-1}I) = stR$, and $st \in S$ implies that $I \in \mathcal{F}$. ☐

In particular, we have the following "first approximation" to our desired theorem.

Corollary 5.1.2. *Let $\mathcal{S}$ be a right noetherian point annihilator set for R. The following are equivalent:*

(1) *R is a principal right ideal ring;*

(2) *$\mathcal{F}_{\mathrm{pr}}$ is closed under similarity and every right ideal in $\mathcal{S}$ is principal;*

(3) *$\mathcal{F}_{\mathrm{pr}}$ is closed under similarity and is a right noetherian point annihilator set.*

Proof. If R is a PRIR, then $\mathcal{F}_{\mathrm{pr}}$ is equal to the family of *all* right ideals in R and therefore is closed under similarity. Also, by Proposition 5.1.1, if $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity then it is a right Oka family. These observations along with Corollary 4.2.4 establish the equivalence of (1)–(3). □

This provides some motivation to explore for which rings the family $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity (and consequently is a right Oka family). It is easy to see that in any right duo ring, and particularly in any commutative ring, every family of right ideals is closed under similarity. This is because in such a ring R, any right ideal I is necessarily a two-sided ideal, so that $I = \mathrm{ann}(R/I)$ can be recovered from the isomorphism class of R/I. Thus Proposition 5.1.1 applies to show that $\mathcal{F}_{\mathrm{pr}}$ is a right Oka family whenever R is a right duo ring, such as a commutative ring. For commutative rings R, the fact that $\mathcal{F}_{\mathrm{pr}}$ is an Oka family was already noted in [37, (3.17)].

Another collection of rings in which $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity is the class of local rings. To show that this is the case, we apply Proposition 3.1.6. Suppose that R is local, and that $I_R \subseteq R$ and $a \in R$ are such that $I + aR = R$ and $a^{-1}I = xR$ is principal. We want to conclude that I is principal. Write $1 = i_0 + ar$ for some $i_0 \in I$ and $r \in R$. Let $U(R)$ denote the group of units of R. If $i_0 \in U(R)$, then $I = R$ is principal. Else $i_0 \notin U(R)$ implies that $1 - i_0 = ar \in U(R)$ and hence $a \in U(R)$ (R local implies that right invertible elements are invertible). But then $a^{-1}I = a^{-1} \cdot I$, so that $I = a(a^{-1}I) = axR$ is principal as desired.

Remark 5.1.3. In any ring R, let $I_R, J_R \subseteq R$ be right ideals such that $J = xR$ is principal and $R/I \cong R/J$. Then I is generated by at most two elements. To see this, apply Schanuel's Lemma (for instance, see [32, (5.1)]) to the exact sequences

$$0 \to I \to R \to R/I \to 0 \quad \text{and}$$
$$0 \to J \to R \to R/J \to 0$$

to get $R \oplus I \cong R \oplus J$. The latter module is generated by at most two elements. Therefore I, being isomorphic to a direct summand of this module, is generated by at most two elements. Thus we see that such I is "not too far" from being principal. (Of course, the same argument shows that if $J_R \subseteq R$ is generated by at most n elements and if $I_R \subseteq R$ is similar to J, then I is generated by at most $n+1$ elements.)

The analysis above also provides the following useful fact: if the module R_R is cancellable in the category of (finitely generated) right R-modules (or even in the category of finite direct sums of f.g. right ideals), then the family $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity (and hence is a right Oka family). Indeed, if this is the case, suppose that $R/I \cong R/J$ for right ideals I and J with J principal. By the remark above, we have I finitely generated and $R \oplus I \cong R \oplus J$. With the assumption on R_R we would have $I_R \cong J_R$ principal, proving $\mathcal{F}_{\mathrm{pr}}$ to be closed under similarity. (In fact one can similarly show that, over such rings, the minimal number of generators $\mu(I)$ of a f.g. right ideal $I \subseteq R$ is an invariant of the similarity class of I.)

This provides another class of rings for which $\mathcal{F}_{\mathrm{pr}}$ is a right Oka family, as follows. Recall that a ring R is said to have *(right) stable range 1* if, for $a, b \in R$, $aR + bR = R$ implies that $(a + br)R = R$ for some $r \in R$ (see [35, §1] for details). In [12, Thm. 2] E. G. Evans showed that for any ring with stable range 1, R_R is cancellable in the full module category $\mathfrak{M}_R$. Thus for any ring R with stable range 1, $\mathcal{F}_{\mathrm{pr}}(R)$ is a right Oka family. The class of rings with stable range 1 includes all semilocal rings (see [33, (20.9)] or [35, (2.10)]), so that this generalizes the case of local rings discussed above.

A similar argument applies in the class of 2-firs. A ring R is said to be a 2-*fir* (where "fir" stands for "free ideal ring") if the free right R-module of rank 2 has invariant basis number and every right ideal of R generated by at most two elements is free. We claim that $\mathcal{F}_{\mathrm{pr}}(R)$ is closed under similarity if R is a 2-fir. Suppose that $I_R \subseteq R$ is similar to a principal right ideal J. As before, we have $R \oplus I \cong R \oplus J$, and I is generated by at most two elements. So $I \cong R^m$, and $J \cong R^n$ where $n \leq 1$ because J is princpal. Thus $R^{m+1} \cong R^{n+1}$ with $n+1 \leq 2$, and the invariant basis number of the latter free module implies that $m = n \leq 1$. Hence $I_R \cong R^m$ is a principal right ideal.

There is yet another way in which $\mathcal{F}_{\mathrm{pr}}(R)$ can be closed under similarity. Suppose that every finitely generated right ideal of R is principal; rings satisfying this property are often called *right Bèzout rings*. Then $\mathcal{F}_{\mathrm{pr}}$ is equal to the set of all f.g. right ideals of R and is therefore a right Oka family by Proposition 2.2.7. A familiar class of examples of such rings is the class of von Neumann regular rings; in such rings, every finitely generated right ideal is a direct summand of R_R, and therefore is principal.

We present a summary of the examples above.

Examples 5.1.4. In each of the following types of rings, the family $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity and thus is a right Oka family:

(1) Right duo rings (including commutative rings);

(2) Rings with stable range 1 (including semilocal rings);

(3) 2-firs;

(4) Right Bèzout rings (including von Neumann regular rings).

One collection of semilocal rings that we have already mentioned is the class of left perfect rings. An application of Corollary 5.1.2 in this case gives the following.

Corollary 5.1.5. *Let R be a left perfect ring (e.g. a semiprimary ring, such as a one-sided artinian ring), and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subseteq R$ be maximal right ideals such that the $R/\mathfrak{m}_i$ represent all isomorphism classes of simple right R-modules. Then R is a PRIR iff all of the $\mathfrak{m}_i$ are principal right ideals.*

Proof. By Example 5.1.4(2), $\mathcal{F}_{\mathrm{pr}}$ is an Oka family of right ideals in R. By Example 4.1.9, the set $\{\mathfrak{m}_i\}$ is a right noetherian point annihilator set. The claim then follows from Corollary 5.1.2. $\qquad\square$

As it turns out, the family $\mathcal{F}_{\mathrm{pr}}$ can indeed fail to be right Oka, even in a noetherian domain! This will be shown in Example 5.1.7 below, with the help of the following lemma.

Lemma 5.1.6. *Let R be a ring with an element $x \in R$ that is not a left zero-divisor.*

(A) *If J and K are right ideals of R with $J \subseteq xR$, then*
$$x^{-1}(J + K) = x^{-1}J + x^{-1}K.$$

(B) *For any $f \in R$,*
$$x^{-1}(xfR + (1 + xy)R) = fR + (1 + yx)R.$$

Proof. (A) The containment "$\supseteq$" holds without any assumptions on x, J, or K because $x(x^{-1}J + x^{-1}K) = x \cdot (x^{-1}J) + x \cdot (x^{-1}K) \subseteq J + K$. To show "$\subseteq$" let $f \in x^{-1}(J + K)$, so that there exist $j \in J$ and $k \in K$ such that $xf = j + k$. Because $J \subseteq xR$, there exists j_0 such that $j = xj_0$; notice that $j_0 \in x^{-1}J$. Then we have $k = xk_0$ for $k_0 = f - j_0 \in x^{-1}K$. Now $xf = xj_0 + xk_0$, and because x is not a left zero-divisor we have $f = j_0 + k_0 \in x^{-1}J + x^{-1}K$.

(B) Setting $J = xfR$ and $K = (1 + xy)R$, one may compute that $x^{-1}J = fR$ and $x^{-1}K = (1 + yx)R$ (using the fact that x is not a left zero divisor). The claim follows directly from part (A). $\qquad\square$

Example 5.1.7. *A ring in which $\mathcal{F}_{\mathrm{pr}}$ is not a right Oka family.* Let k be a field and let $R := A_1(k) = k\langle x, y : xy = yx + 1\rangle$ be the first Weyl algebra over k. Then R is known to be a noetherian domain (which is simple if k has characteristic 0). Define the right ideal
$$I_R := x^2 R + (1 + xy)R \subseteq R,$$
which is shown to be nonprincipal in [40, 7.11.8]. Because $I + xR$ contains both $1 + xy \in I$ and $xy \in xR$, we must have $1 \in I + xR = R$.

Because $1 + yx = xy \in xR$, Lemma 5.1.6(B) above (with $f = x$) implies that $x^{-1}I = xR + (1 + yx)R = xR$. Therefore we have $I + xR = R$ and $x^{-1}I = xR$ both members of

$\mathcal{F}_{\mathrm{pr}}$ with $I \notin \mathcal{F}_{\mathrm{pr}}$ proving that $\mathcal{F}_{\mathrm{pr}}$ is not a right Oka family. In fact we have $R/I \cong R/xR$ where I is not principal (the isomorphism follows from Lemma 3.1.5(A), showing explicitly that $\mathcal{F}_{\mathrm{pr}}$ is not closed under similarity as predicted by Proposition 5.1.1. In agreement with Remark 5.1.3, I is generated by two elements.

Notice that $R/xR \cong k[y]$, where $k[y] \subseteq R$ acts by right multiplication and $x \in R$ acts as $-\partial/\partial y$. If k has characteristic 0 then this module is evidently simple, and because $R/I \cong R/xR$ we see that I is a maximal right ideal. If instead $\mathrm{char}(k) = p > 0$, then $R/xR \cong k[y]$ is evidently not simple, and not even artinian (the submodules $y^{np}k[y]$ form a strictly descending chain for $n \geq 0$). But every proper factor of this module has finite dimension over k and is therefore artinian. So we see that $R/I \cong R/xR$ is 1-critical, making I a 1-cocritical right ideal. Thus regardless of the characteristic of k, the nonprincipal right ideal I is cocritical.

On the other hand, when $\mathrm{char}\, k = 0$ the ring $\mathbb{M}_2(R)$ is known to be a principal (right and left) ideal ring—see [40, 7.11.7]. Then $\mathcal{F}_{\mathrm{pr}}(\mathbb{M}_2(R))$ is equal to the set of *all* right ideals in $\mathbb{M}_2(R)$ and thus is a right Oka family. So we see that the property "$\mathcal{F}_{\mathrm{pr}}(R)$ is a right Oka family" is not Morita invariant. $\qquad\square$

It would be very desirable to eliminate the condition in Corollary 5.1.2 that $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity. It turns out that a suitable strengthening of the hypothesis on the point annihilator set $\mathcal{S}$ will in fact allow us to discard that assumption. The following constructions will help us achieve this goal in Theorem 5.1.11 below. Recall that for right ideals I and J of a ring R, we write $I \sim J$ to mean that I and J are similar.

Definition 5.1.8. For any ring R, we define

$$\mathcal{F}_{\mathrm{pr}}^{\circ}(R) := \{I_R \subseteq R : \forall J_R \subseteq R, \ I \sim J \implies J \in \mathcal{F}_{\mathrm{pr}}\}$$
$$= \{I_R \subseteq R : I \text{ is only similar to principal right ideals}\}.$$

Alternatively, $\mathcal{F}_{\mathrm{pr}}^{\circ}$ is the largest subset of $\mathcal{F}_{\mathrm{pr}}$ that is closed under similarity.

As with $\mathcal{F}_{\mathrm{pr}}$, we will often write $\mathcal{F}_{\mathrm{pr}}^{\circ}$ in place of $\mathcal{F}_{\mathrm{pr}}^{\circ}(R)$ when the ring R is understood from the context. We saw in Proposition 5.1.1 that certain families of principal right ideals are right Oka precisely when they are closed under similarity. But $\mathcal{F}_{\mathrm{pr}}^{\circ}$ is the *largest* family of principal right ideas that is closed under similarity. Thus one might wonder whether $\mathcal{F}_{\mathrm{pr}}^{\circ}$ might be a right Oka family. As it turns out, we are very fortunate and this is in fact true in every ring!

Proposition 5.1.9. *For any ring R, $\mathcal{F}_{\mathrm{pr}}^{\circ}(R)$ is an Oka family of right ideals.*

Proof. We will denote $\mathcal{F} := \mathcal{F}_{\mathrm{pr}}^{\circ}(R)$. Because $I_R \sim R_R$ implies $I = R \in \mathcal{F}_{\mathrm{pr}}$, we see that $R \in \mathcal{F}$. Suppose that $I_R \subseteq R$ and $a \in R$ are such that $I + aR$, $a^{-1}I \in \mathcal{F}$. Set $C_1 := R/a^{-1}I$ and $C_2 := R/(I + aR)$, so that we have an exact sequence

$$0 \to C_1 \to R/I \to C_2 \to 0.$$

To prove that $I \in \mathcal{F}$, let $J_R \subseteq R$ be such that $R/J \cong R/I$. We need to show that J is principal. There is also an exact sequence

$$0 \to C_1 \to R/J \to C_2 \to 0.$$

Thus there exists $x \in R$ with $C_1 \cong (J+xR)/J$ and $C_2 \cong R/(J+xR)$. But then $R/(I+aR) = C_2 \cong R/(J+xR)$ and $I + aR \in \mathcal{F}$ imply that $J + xR = cR$ for some $c \in R$. Now

$$\frac{R}{a^{-1}I} = C_1 \cong \frac{J+xR}{J} = \frac{cR}{J} \cong \frac{R}{c^{-1}J}$$

and $a^{-1}I \in \mathcal{F}$, so we find that $c^{-1}J$ is principal. Then $J \subseteq J + xR = cR$ gives $J = c(c^{-1}J)$, proving that J is principal. $\qquad\square$

The following elementary observation will be useful in a number of places. It is simply a convenient restatement of the fact that $\mathcal{F}_{\mathrm{pr}}^{\circ}$ is the largest set of principal right ideals that is closed under similarity.

Lemma 5.1.10. *Let $\mathcal{S}$ be a set of right ideals of a ring R that is closed under similarity. If $\mathcal{S} \subseteq \mathcal{F}_{\mathrm{pr}}$, then $\mathcal{S} \subseteq \mathcal{F}_{\mathrm{pr}}^{\circ}$ (and, of course, conversely).*

We are finally ready to state and prove our noncommutative generalization of the Kaplansky-Cohen Theorem 1.1.3.

Theorem 5.1.11 (A noncommutative Kaplansky-Cohen Theorem). *For any ring R, let $\mathcal{S}$ be a right noetherian point annihilator set that is closed under similarity. The following are equivalent:*

(1) *R is a principal right ideal ring;*

(2) *Every right ideal in $\mathcal{S}$ is principal;*

(3) *$\mathcal{F}_{\mathrm{pr}}^{\circ}$ is a right noetherian point annihilator set.*

In particular, R is a principal right ideal ring iff every cocritical right ideal of R is principal.

Proof. The set of cocritical right ideals of R is a noetherian point annihilator set that is closed under similarity, so it suffices to prove the equivalence of (1)–(3). It is easy to see that (1) is equivalent to the claim that all right ideals lie in $\mathcal{F}_{\mathrm{pr}}^{\circ}$. Also, it follows from Lemma 5.1.10 that (2) holds precisely when $\mathcal{S} \subseteq \mathcal{F}_{\mathrm{pr}}^{\circ}$. The equivalence of (1)–(3) now follows from Corollary 4.2.4 and Proposition 5.1.9. $\qquad\square$

As with Cohen's Theorem, there exist previous noncommutative generalizations of the Kaplansky-Cohen theorem in the literature. In §5.4 we relate our theorem with these earlier results.

Comparing our two versions of the Kaplansky-Cohen Theorem, we see that Corollary 5.1.2 follows from Theorem 5.1.11, at least if we consider condition (3) in each equivalence. (Recall Remark 4.1.2, and the fact that $\mathcal{F}_{\mathrm{pr}}^{\circ} \subseteq \mathcal{F}_{\mathrm{pr}}$.) However, this does not mean that Corollary 5.1.2 is obsolete. It is clear that Theorem 5.1.11 is preferable to Corollary 5.1.2 if we have enough knowledge about the point annihilator set $\mathcal{S}$ but we do not know whether the family $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity. On the other hand, if we are working in a class of rings for which we know that $\mathcal{F}_{\mathrm{pr}}$ is closed under similarity, then Corollary 5.1.2 may be of more use. This proved to be the case in Corollary 5.1.5, where we were able to reduce the point annihilator set $\mathcal{S}$ to a finite set.

Notice that our earlier examination of the Weyl algebra $A_1(k)$ in Example 5.1.7 fits nicely with Theorem 5.1.11, because the nonprincipal right ideal discussed in that example was shown to be cocritical.

As a simple application of Theorem 5.1.11, we can show that *a domain R with right Krull dimension ≤ 1 is a principal right ideal domain iff its maximal right ideals are principal.* Indeed, by Proposition 4.1.13 the zero ideal of R is 1-cocritical as a right ideal (and it is, of course, principal). Thus any nonzero cocritical right ideal of R is 0-critical and therefore is a maximal right ideal. The claim then follows from Theorem 5.1.11. However, we will prove a substantially more general version of this fact in Proposition 5.3.1.

5.2 Families closed under direct summands

In this section we will develop further generalizations of Cohen's Theorem and the Kaplansky-Cohen theorems by further reducing the set of right ideals in a ring which we are required to "test." In particular, where our previous theorems stated that it was sufficient to check that every right ideal in some noetherian point annihilator set $\mathcal{S}$ is finitely generated (or principal), we will further reduce the task to checking that every *essential* right ideal in $\mathcal{S}$ is finitely generated (or principal). We begin with a definition, temporarily digressing to families of submodules of a given modules other than R_R.

Definition 5.2.1. Let M_R be a module over a ring R. We will say that a family $\mathcal{F}$ of submodules of M is *closed under direct summands* if for any $N \in \mathcal{F}$, any direct summand of N also lies in $\mathcal{F}$.

Notice that a family $\mathcal{F}$ of submodules of M that is closed under direct summands necessarily has $0 \in \mathcal{F}$ as long as $\mathcal{F} \neq \varnothing$. The following result is the reason for our interest in families that are closed under direct summands. It shows the link between such families and the essential submodules of M.

Lemma 5.2.2. *In a module M_R, let $\mathcal{F}$ be a family of submodules that is closed under direct summands. Then all submodules of M lie in $\mathcal{F}$ iff all essential submodules of M lie in $\mathcal{F}$. In*

particular, if $\mathcal{F}$ is a family of right ideals in a ring R that is closed under direct summands, then all right ideals of R lie in $\mathcal{F}$ iff all essential right ideals of R lie in $\mathcal{F}$.

Proof. ("If" direction) Suppose that every essential submodule of M lies in $\mathcal{F}$, and let $L_R \subseteq M$. By Zorn's lemma there exists a submodule N_R maximal with respect to $L \cap N = 0$ (in the literature, such N is referred to as a *complement* to L). We claim that $N \oplus L$ is an essential submodule of M. Indeed, assume for contradiction that $0 \neq K \subseteq M$ is a submodule such that $(L \oplus N) \cap K = 0$. Then we have the direct sum $L \oplus N \oplus K$ in M. It follows that $L \cap (N \oplus K) = 0$, contradicting the maximality of N.

By assumption, $N \oplus L \subseteq_e M$ implies that $N \oplus L \in \mathcal{F}$. Then because $\mathcal{F}$ is closed under direct summands, we conclude that $N \in \mathcal{F}$. $\qquad\square$

With this result as our motivation, let us consider a few examples of families of right ideals that are closed under direct summands.

Example 5.2.3. In any module M_R, the easiest nontrivial example of a family that is closed under direct summands is the family $\mathcal{F}$ of all direct summands of M! The application of Lemma 5.2.2 in this case says that a module M is semisimple iff every essential submodule of M is a direct summand. However, it is easy to check that a direct summand of M is essential in M iff it is equal to M. So this says that a module is semisimple iff it has no proper essential submodules. This is a known result; for instance, see [36, Ex. 3.9].

Example 5.2.4. The family of finitely generated submodules of a module M_R is certainly closed under direct summands. It follows that a module M is right noetherian iff all of its essential submodules are finitely generated. Again, this fact can be found, for instance, in [36, Ex. 6.11].

We can generalize the result above as follows. Let α be any cardinal (finite or infinite), and let $\mathcal{F}$ be the family of all submodules of M that have a generating set of size $< \alpha$. Then $\mathcal{F}$ is again closed under direct summands. So every submodule of M is generated by $< \alpha$ elements iff the essential submodules of M are all generated by $< \alpha$ elements.

Taking $M_R = R_R$ and $\alpha = 2$, we see in particular that $\mathcal{F}_{\mathrm{pr}}$ is closed under direct summands, and Lemma 5.2.2 implies that R is a PRIR iff its essential right ideals are principal.

Here we end our digression into families of submodules of arbitrary modules and focus our attention on families of right ideals in a ring R that are closed under direct summands. The next two examples are of a homological nature.

Example 5.2.5. For a module M_R, let $\mathcal{F}$ be the family of right ideals $I \subseteq R$ such that every module homomorphism $f \colon I \to M$ extends to a homomorphism $R \to M$. This was shown to be a right Oka family in Proposition 3.2.16. We claim that $\mathcal{F}$ is closed under direct summands. For if $I \oplus J \in \mathcal{F}$ and $f \colon I \to M$ is any homomorphism, then we may extend f trivially to $I \oplus J \to M$. This morphism in turn extends to $R \to M$ because $I \oplus J \in \mathcal{F}$. Hence $I \in \mathcal{F}$.

By Baer's Criterion, every right ideal lies in $\mathcal{F}$ precisely when M is injective. So applying Lemma 5.2.2, we find that M is injective iff every essential right ideal of R lies in $\mathcal{F}$. This "essential version" of Baer's Criterion has been noticed before; for instance, see [36, Ex. 3.26].

More generally, for any module M_R and integer $n \geq 0$, let $\mathcal{F}_M^n$ denote the family of right ideals $I \subseteq R$ such that $\operatorname{Ext}_R^{n+1}(R/I, M) = 0$. The family $\mathcal{F}$ above was shown to be equal to $\mathcal{F}_M^0$ in the proof of Proposition 3.2.16. We claim that the families $\mathcal{F}^n$ are closed under direct summands. The case $n = 0$ is covered above, so suppose that $n \geq 1$. Note that $\operatorname{Ext}_R^n(R, M) = \operatorname{Ext}_R^{n+1}(R, M) = 0$ because R_R is projective. So for any right ideal $K \subseteq R$, the long exact sequence in Ext provides isomorphisms $\operatorname{Ext}_R^n(K, M) \cong \operatorname{Ext}_R^{n+1}(R/K, M)$. Thus for any direct sum of right ideals $I \oplus J \subseteq R$, combining this observation with a standard fact about Ext and direct sums gives

$$
\begin{aligned}
\operatorname{Ext}_R^{n+1}(R/(I \oplus J), M) &\cong \operatorname{Ext}_R^n(I \oplus J, M) \\
&\cong \operatorname{Ext}_R^n(I, M) \oplus \operatorname{Ext}_R^n(J, M) \\
&\cong \operatorname{Ext}_R^{n+1}(R/I, M) \oplus \operatorname{Ext}_R^{n+1}(R/J, M).
\end{aligned}
$$

This makes it clear that if $I \oplus J \in \mathcal{F}_M^n$, then $I \in \mathcal{F}_M^n$.

Extending Baer's Criterion, one can show that a module M_R has injective dimension $\leq n$ iff $\operatorname{Ext}_R^{n+1}(R/I, M) = 0$ for all right ideals I of R (this is demonstrated in the proof of [50, Thm. 8.16]). If we apply Lemma 5.2.2 to the family $\mathcal{F}_M^n$, we see that for any module M_R we have $\operatorname{id}(M) \leq n$ iff $\operatorname{Ext}_R^{n+1}(R/I, M) = 0$ for all essential right ideals I of R. $\qquad\square$

Example 5.2.6. As an application of Example 5.2.5 above, we produce another example of a family that is closed under direct summands. Let $\mathcal{F}^n$ be the family of all right ideals of R such that $\operatorname{pd}(R/I) \leq n$. Because R/I has projective dimension $\leq n$ iff $\operatorname{Ext}_R^{n+1}(R/I, M) = 0$ for all modules M, we see that $\mathcal{F}^n$ is equal to the intersection of all of the families $\mathcal{F}_M^n$ as M ranges over all right R-modules. Since all of these families are closed under direct summands, $\mathcal{F}^n$ is also closed under summands. In this case we can apply Lemma 5.2.2 to say that a ring R has r. gl. $\dim(R) \leq n$ iff $\operatorname{pd}(R/I) \leq n$ for all essential right ideals $I_R \subseteq R$. Notice that when $n = 0$, $\mathcal{F}^0$ is the family of right ideal direct summands mentioned in Example 5.2.3.

Before continuing to the heart of this section, we require a small observation as well as a new definition.

Remark 5.2.7. Notice that the set of essential right ideals is a divisible semifilter, and is closed under similarity. It is easy to see that the set is a semifilter. To see that it is divisible, we will use the following fact about essential submodules: for any homomorphism of modules $f : M_R \to N_R$ and any essential submodule $N_0 \subseteq N$, the preimage $f^{-1}(N_0)$ is an essential submodule of M (see [36, Ex. 3.7] for a proof of this fact). Now given a right ideal $I \subseteq R$, $x^{-1}I$ is the preimage of the right ideal I under the homomorphism $R_R \to R_R$ given by left multiplication by x. Thus if I is an essential right ideal, so is $x^{-1}I$. Finally, to see that this set is closed under similarity, one only needs to realize that $I_R \subseteq R$ is essential iff R/I is a singular module; see [36, Ex. 2(b)].

Definition 5.2.8. Let $\mathcal{F}$ be a family of right ideals in a ring R. We define

$$\widetilde{\mathcal{F}} := \{I_R \subseteq R : I \oplus J \in \mathcal{F} \text{ for some } J_R \subseteq R\}.$$

This is the smallest family of right ideals containing $\mathcal{F}$ that is closed under direct summands

The next result, which is fundamental to this section, is a variation of Theorem 4.2.3 and Corollary 4.2.4.

Theorem 5.2.9. *Let $\mathcal{F}$ be an Oka family of right ideals in a ring R.*

(1) *Assume that every chain of right ideals in $\mathcal{F}'$ has an upper bound in $\mathcal{F}'$, and let $\mathcal{S}$ be a point annihilator set for the class of modules $\{R/I : I \in \mathrm{Max}(\mathcal{F}')\}$. If every essential right ideal in $\mathcal{S}$ lies in $\mathcal{F}$, then all right ideals of R lie in $\widetilde{\mathcal{F}}$.*

(2) *Let $\mathcal{S}$ be a noetherian point annihilator set for R, and assume that $\mathcal{F}$ consists of finitely generated right ideals. If every essential right ideal in $\mathcal{S}$ lies in $\mathcal{F}$, then all right ideals of R lie in $\widetilde{\mathcal{F}}$.*

Proof. To prove (1), let $\mathcal{S}$ and $\mathcal{F}$ satisfy the given hypotheses. Let $\mathcal{F}_0$ denote the divisible semifilter of essential right ideals of R. By assumption we have $\mathcal{F}_0 \cap \mathcal{S} \subseteq \mathcal{F}$, so it follows from Theorem 4.2.3 that $\mathcal{F}_0 \subseteq \mathcal{F}$. Then all essential right ideals of R lie in $\widetilde{\mathcal{F}} \supseteq \mathcal{F}$, and it follows from Lemma 5.2.2 that all right ideals lie in $\widetilde{\mathcal{F}}$.

Now (2) follows from (1) because the fact that $\mathcal{F}$ consists of finitely generated right ideals implies both that every chain of right ideals in $\mathcal{F}'$ has an upper bound in $\mathcal{F}'$ and that the class $\{R/I : I \in \mathrm{Max}(\mathcal{F}')\}$ consists of noetherian modules (as in the proof of Corollary 4.2.4). $\square$

In particular, if the right Oka family $\mathcal{F}$ in the theorem above is in fact closed under direct summands, then $\widetilde{\mathcal{F}} = \mathcal{F}$. Thus in this case Theorem 5.2.9 is a generalization of Theorem 4.2.3. Our first application of this result will be a strengthening of the noncommutative Cohen's Theorem 4.2.5.

Theorem 5.2.10. *For a ring R, let $\mathcal{S}$ be a right noetherian point annihilator set (such as the set of cocritical right ideals). Then R is right noetherian iff every essential right ideal in $\mathcal{S}$ is finitely generated.*

Proof. ("If" direction) This follows directly from Example 5.2.4 and Theorem 5.2.9(2) by taking $\mathcal{F} = \widetilde{\mathcal{F}}$ to be the family of finitely generated right ideals of R. $\square$

Our next application of Theorem 5.2.9 will strengthen our noncommutative version of the Kaplansky-Cohen Theorem 5.1.11. The careful statement of Theorem 5.2.9 will pay off here.

Theorem 5.2.11. *Let R be a ring with noetherian point annihilator set $\mathcal{S}$ that is closed under similarity (such as the set of cocritical right ideals). Then R is a principal right ideal ring iff every essential right ideal in $\mathcal{S}$ is principal.*

Proof. ("If" direction) Suppose that every essential right ideal in $\mathcal{S}$ is principal, and set $\mathcal{F} := \mathcal{F}_{\mathrm{pr}}^{\circ}$. If $\mathcal{S}_0 \subseteq \mathcal{S}$ is the set of essential right ideals in $\mathcal{S}$, then $\mathcal{S}_0$ is closed under similarity because both $\mathcal{S}$ and the set of essential right ideals are closed under similarity (recall Remark 5.2.7). By hypothesis $\mathcal{S}_0 \subseteq \mathcal{F}_{\mathrm{pr}}$, so Lemma 5.1.10 gives $\mathcal{S}_0 \subseteq \mathcal{F}_{\mathrm{pr}}^{\circ} =: \mathcal{F}$. That is, every essential right ideal in $\mathcal{S}$ lies in $\mathcal{F}$. Now Theorem 5.2.9(2) implies that all right ideals of R lie in $\widetilde{\mathcal{F}}$. But $\mathcal{F}_{\mathrm{pr}}$ is closed under direct summands by Example 5.2.4, so $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{pr}}$ implies that $\widetilde{\mathcal{F}} \subseteq \mathcal{F}_{\mathrm{pr}}$. Hence every right ideal of R is principal. $\qquad\square$

Our final applications of Theorem 5.2.9 show how to reduce the test sets for various homological properties in a right noetherian ring.

Theorem 5.2.12. *Let R be a right noetherian ring, and let $\mathcal{S}$ be a right (noetherian) point annihilator set for R (such as the set of cocritical right ideals).*

(1) *A module M_R has injective dimension $\leq n$ iff $\mathrm{Ext}_R^{n+1}(R/P, M) = 0$ for all essential right ideals $P \in \mathcal{S}$.*

(2) *All finitely generated right R-modules have finite projective dimension iff $\mathrm{pd}(R/P) < \infty$ for all essential right ideals $P \in \mathcal{S}$.*

(3) $\mathrm{r.\,gl.\,dim}(R) = \sup\{\mathrm{pd}(R/P) : P \in \mathcal{S}$ is an essential right ideal$\}$.

Proof. For a module M_R and a nonnegative integer n, let $\mathcal{F}_M^n$ and $\mathcal{F}^n$ be the families introduced in Examples 5.2.5 and 5.2.6, where they were shown to be closed under direct summands. These familes were shown to be right Oka families in §3.2.B. Defining $\mathcal{F}^{\infty} := \bigcup_{n=1}^{\infty} \mathcal{F}^n$, it follows that $\mathcal{F}^{\infty}$ is also a right Oka family closed under direct summands.

For part (1), we note that a module M_R has injective dimension $\leq n$ iff $\mathcal{F}_M^n$ consists of all right ideals of R, which happens iff all essential right ideals in $\mathcal{S}$ lie in $\mathcal{F}_M^n$ according to Theorem 5.2.9(2). Next we prove part (2). Because every finitely generated right R-module is has a finite filtration with cyclic filtration factors, and because the finiteness of projective dimension is preserved by extensions, we see that every finitely generated right R-module has finite projective dimension iff every cyclic right R-module does, iff $\mathcal{F}^{\infty}$ consists of all right ideals. By Theorem 5.2.9, this occurs iff all essential right ideals in $\mathcal{S}$ lie in $\mathcal{F}^{\infty}$.

Part (3) similarly follows from Theorem 5.2.9 applied to the family $\mathcal{F}^n$, noting that R has right global dimension $\leq n$ iff $\mathcal{F}^n$ consists of all right ideals. $\qquad\square$

The above joins a whole host of results stating that certain homological properties can be tested on special sets of ideals. We mention only a few relevant references here. When R is commutative, $\mathcal{S} = \mathrm{Spec}(R)$, and $n = 0$ in part (1), the theorem above recovers a result of J. A. Beachy and W. D. Weakley in [3]. Part (2) generalizes a result characterizing commutative regular rings, the "globalizations" of regular local rings (see [32, (5.94)]). Many results along the lines of part (3) are known. For instance, a result of J. J. Koker in [30,

Lem. 2.1] implies that if a ring R has right Krull dimension, then its right global dimension is equal to the supremum of the projective dimensions of the right modules R/P, where P ranges over the cocritical right ideals of R. On the other hand, for a commutative noetherian ring R the global dimension of R is equal to the supremum of $\mathrm{pd}(R/\mathfrak{m})$, where $\mathfrak{m}$ ranges over the maximal ideals of R (see [32, (5.92)]). It has also been shown by K. R. Goodearl [18, Thm. 16] and S. M. Bhatwadekar [4, Prop. 1.1] that for a (left and right) noetherian ring R whose global dimension is finite, the global dimension of R is the supremum of $\mathrm{pd}(R/\mathfrak{m})$ where $\mathfrak{m}$ ranges over the maximal right ideals of R. It is an open question whether the finiteness of the global dimension can be dropped [19, Appendix].

5.3 A noncommutative generalization of Kaplansky's Theorem

The goal of this section is to prove a noncommutative generalization of Kaplansky's Theorem 1.1.2. Specifically, we shall show in Theorem 5.3.9 that *a noetherian ring whose maximal right ideals are all principal is a principal right ideal ring.* To motivate our approach, we shall recall a result [15, Theorem C] of A. W. Goldie: a left noetherian principal right ideal ring is a direct sum of a semiprime ring and an artinian ring. Inspired by this fact, our proof of Theorem 5.3.9 will proceed by taking noetherian ring whose maximal right ideals are principal and decomposing it as a direct sum of a semiprime ring and an artinian ring. This should seem reasonable because we have already shown in Corollary 5.1.5 that, in order to test whether an artinian ring is a PRIR, it suffices to test only its maximal right ideals.

With Goldie's result in mind, we begin this section by investigating under what conditions one can check the PRIR condition on a semiprime ring by testing only its maximal right ideals. The first result applies to semiprime rings with small right Krull dimension.

Proposition 5.3.1. *Let R be a semiprime ring with* r. K. $\dim(R) \leq 1$. *Then R is a principal right ideal ring iff its maximal right ideals are principal.*

Proof. ("If" direction) By Theorem 5.2.11, it suffices to show that the essential cocritical right ideals of R are principal. Thus it is enough to show that every essential cocritical right ideal of R is maximal. According to [40, 6.3.10] the fact that R is semiprime with right Krull dimension means that, for every $E_R \subseteq R$, K. $\dim(R/E) <$ K. $\dim(R_R) = 1$. So K. $\dim(R/E) \leq 0$, and if E is also cocritical then it is 0-cocritical and thus is maximal. This completes the proof. ☐

In Example 5.3.11 below we will show that the hypothesis on the right Krull dimension cannot be relaxed. Of course, it is not the case that *every* semiprime PRIR has right Krull dimension ≤ 1. In fact, in [20, Ex. 10.3] it is shown (using a construction of A. V. Jategaonkar from [25]) that there exist principal right ideal domains whose right Krull dimension is equal to any prescribed ordinal! So while Proposition 5.3.1 gives a sufficient condition for semiprime

rings to be PRIRs, it is certainly not a necessary condition. However, with some additional effort we will use this result to formulate a precise characterization of semiprime left *and* right principal ideal rings in Corollary 5.3.5 below.

We will show in Proposition 5.3.4 below that if a semiprime ring with a certain finiteness condition on the left has all maximal right ideals principal, it must have small right Krull dimension. We take this opportunity to recall that a multiplicatively closed subset $S \subseteq R$ is *saturated* if, for any $a, b \in R$, $ab \in S$ implies $a, b \in S$.

Lemma 5.3.2. *Let R be a ring in which the multiplicative set of (resp. left) regular elements is saturated and which satisfies the ascending chain condition on left ideals of the form Rs where $s \in R$ is a (resp. left) regular element. Furthermore, suppose that every maximal right ideal of R is principal. If $b \in R$ is a (resp. left) regular element, then R/bR has finite length.*

Proof. This argument adapts some of the basic ideas of factorization in noncommutative domains, as in Prop. 0.9.3 and Thm. 1.3.5 of [9]. However, we do not assume any of those results here.

If our fixed $b \in R$ is not right invertible, then $bR \neq R$. If bR is not maximal, choose a maximal right ideal $a_1 R \subsetneq R$ such that $bR \subsetneq a_1 R$. Then $b = a_1 b_1$ for some $b_1 \in R$. We claim that $Rb \subseteq Rb_1$ is strict. Indeed, assume for contradiction that $Rb = Rb_1$. Then we may write $b_1 = ub$ for some $u \in R$. Thus $b = a_1 b_1 = a_1 ub$, and because b is (left) regular we have $a_1 u = 1$. This contradicts the fact that $a_1 R$ is maximal. Hence $Rb \subsetneq Rb_1$.

Because the set of (left) regular elements is saturated, we may now replace b above by b_1 and proceed inductively to write $b_{i-1} = a_i b_i$ (if $b_{i-1}R$ is not maximal) where b_i is (left) regular and $a_i R$ is a maximal right ideal. By the ACC condition on R, the chain

$$Rb \subsetneq Rb_1 \subsetneq Rb_2 \subsetneq \cdots$$

cannot continue indefinitely. So the process must terminate, say at $b_{n-1} = a_n b_n$. This means that $b_n R$ is a maximal right ideal. Writing $a_{n+1} := b_n$, we have a factorization $b = a_1 \cdots a_{n+1}$ where the right ideals $a_i R$ are maximal. Then in the filtration

$$bR = (a_1 \cdots a_{n+1})R \subseteq (a_1 \cdots a_n)R \subseteq \cdots \subseteq a_1 R \subseteq R,$$

each factor module $(a_1 \cdots a_{j-1})R/(a_1 \cdots a_j)R$ is a homomorphic image of the simple module $R/a_j R$ (via left multiplication by $a_1 \cdots a_{j-1}$) and thus is simple. This proves that R/bR has finite length, as desired. $\square$

In light of the hypotheses assumed above, the following definition will be useful.

Definition 5.3.3. We will say that a ring R satisfies *left ACC-reg* if it satisfies the ascending chain condition on left ideals of the form Rs where $s \in R$ is a regular element.

Proposition 5.3.4. *Let R be a semiprime ring with right Krull dimension that satisfies left ACC-reg. If all of the maximal right ideals of R are principal, then $\mathrm{r. K. dim}(R) \leq 1$ and R is a principal right ideal ring.*

Proof. Because R is semiprime and has right Krull dimension, it is right Goldie (see [40, 6.3.5]). This has two important consequences. First, the set of regular elements of R is saturated (because it is the intersection of R with the group of units in its semisimple right ring of quotients). Second, the essential right ideals of R are precisely the right ideals containing a regular element (see [32, (11.13)]). Thus, for every $E_R \subseteq_e R$, R/E has finite length by Lemma 5.3.2 and thus has Krull dimension at most 0. Now [40, 6.3.10] provides us with the following equation for r. K. dim(R) (which is valid because R is semiprime with right Krull dimension):

$$\text{r. K. dim}(R) = \sup\{\text{K. dim}(R/E) + 1 : E_R \subseteq_e R\} \leq 1.$$

Applying Proposition 5.3.1, we see that R is a principal right ideal ring. $\qquad\square$

An immediate consequence is the aforementioned characterization of semiprime PIRs.

Corollary 5.3.5. *Let R be a semiprime ring.*

(1) *R is a principal ideal ring iff its left and right Krull dimensions are both at most 1 and the maximal left ideals and maximal right ideals of R are all principal.*

(2) *Suppose that R satisfies left ACC-reg. Then R is a principal right ideal ring iff r. K. dim$(R) \leq 1$ and the maximal right ideals of R are principal.*

It is possible to strengthen Proposition 5.3.4 to show that more general types of rings must have small right Krull dimension.

Corollary 5.3.6. *Let R be a ring with right Krull dimension, and let N be its prime radical. Suppose that one of the following two conditions holds:*

(A) *R/N satisfies left ACC-reg;*

(B) *R/P satisfies left ACC-reg for every minimal prime ideal $P \lhd R$.*

If the maximal right ideals of R are principal, then r. K. dim$(R) \leq 1$. In particular, a noetherian ring whose maximal right ideals are principal has right Krull dimension at most 1.

Proof. According to [40, 6.3.8], the ring R with right Krull dimension has finitely many minimal prime ideals $P_1, \ldots, P_n$ and

$$\text{r. K. dim}(R) = \text{r. K. dim}(R/N) = \max\{\text{r. K. dim}(R/P_i)\}.$$

Because every factor ring of R again has principal maximal right ideals, we may now apply Proposition 5.3.4. $\qquad\square$

It is an open question whether the left and right Krull dimensions of a general noetherian ring must be equal [19, Appendix]. However, another application of Proposition 5.3.4 shows that the Krull dimension of a noetherian PRIR must is symmetric.

Corollary 5.3.7. *A left noetherian principal right ideal ring R has*

$$\mathrm{l.\,K.\,dim}(R) = \mathrm{r.\,K.\,dim}(R) \leq 1.$$

Proof. As mentioned before, the Krull dimension of R is not changed upon factoring out its nilradical [40, 6.3.8]; thus we may assume that R is semiprime. In this case, a result of J. C. Robson [48, Cor. 3.7] states that because R is a noetherian PRIR, it must also be a PLIR. According to Proposition 5.3.4, both $\mathrm{l.\,K.\,dim}(R)$ and $\mathrm{r.\,K.\,dim}(R)$ are at most 1. Now R has Krull dimension 0 on either side precisely when R is artinian on that side. But a noetherian ring is artinian on one side iff it is artinian on the other side. (This follows, for instance, from the Hopkins-Levitzki Theorem [33, (4.15)].) Thus we see that the left and right Krull dimensions of R must coincide, both equal to 0 when R is artinian and both equal to 1 when R is not artinian. □

The next preparatory result provides a method of testing whether a module over a semilocal ring is zero. One may think of this as a variation of Nakayama's Lemma (even though the latter is used in the proof below).

Lemma 5.3.8. *Let R be a semilocal ring, and let $_RB$ be a finitely generated left module. If $B = \mathfrak{m}B$ for all maximal right ideals $\mathfrak{m}$ of R, then $B = 0$.*

Proof. Let R and $_RB$ be as above, and let $J = \mathrm{rad}(R)$. We claim that B/JB satisfies the same hypotheses over the semisimple ring R/J. Indeed, the maximal right ideals of R/J are the right ideals of the form $\mathfrak{m}/J$ for a maximal right ideal $\mathfrak{m}$ of R. For such $\mathfrak{m}/J$ we have

$$(\mathfrak{m}/J) \cdot (B/JB) = \mathfrak{m}B/JB = B/JB.$$

Also, B/JB is finitely generated over R/J. So B/JB indeed satisfies the same hypotheses over R/J. If we knew the lemma to hold over all semisimple rings, it would follow that $B/JB = 0$. Nakayama's Lemma would then imply that $B = 0$.

So we may assume that R is semisimple. Choose orthogonal idempotents $e_1, \ldots, e_n$ in R whose sum is 1 such that $R_R = \bigoplus e_i R$ is a decomposition of R into minimal right ideals. Then for any k, $(1-e_k)R = \bigoplus_{i \neq k} e_i R$ is a maximal right ideal of R. By hypothesis, we have $B = (1-e_k)RB = (1-e_k)B$. Because the e_i are orthogonal,

$$(1-e_1)\cdots(1-e_k) = 1 - (e_1 + \cdots + e_k)$$

In particular, $(1-e_1)\cdots(1-e_n) = 1 - (e_1 + \cdots + e_n) = 0$. It follows that

$$B = (1-e_1)B = (1-e_1)(1-e_2)B = \cdots = (1-e_1)\cdots(1-e_n)B = 0. \qquad \square$$

Let us review some relevant results on noetherian rings. For an ideal I of a ring R, we let $\mathcal{C}(I)$ denote the set of elements $c \in R$ such that $c + I$ is a regular element of R/I. A theorem of J. C. Robson [49] states that a noetherian ring R with prime radical N is a direct sum of a

semiprime ring and an artinian ring iff, for every $c \in \mathcal{C}(N)$, $N = cN = Nc$. However, Robson commented in [49, p. 346] that if one only assumes that $N = cN$ for all $c \in \mathcal{C}(N)$, one can still conclude that there exists an idempotent $e \in R$ such that eRe is semiprime, $(1-e)R(1-e)$ is artinian, and $eR(1-e) = 0$. This gives a useful "triangular decomposition" of such a ring. In particular it can be used to derive the result of Goldie, mentioned at the beginning of this section, that a left noetherian principal right ideal ring is a direct sum of a semiprime ring and an artinian ring. The first paragraph of our argument below borrows from the proof of this last statement given in [49, Thm. 4]. With Robson's decomposition result in hand, we are finally ready to prove our noncommutative generalization of Kaplansky's Theorem 1.1.2.

Theorem 5.3.9 (A noncommutative Kaplansky's Theorem). *A noetherian ring is a principal right ideal ring iff its maximal right ideals are principal.*

Proof. ("If" direction) Suppose R is a noetherian ring whose maximal right ideals are principal. Notice that every factor ring of R satisfies the same hypotheses. Let $N \lhd R$ be the prime radical of R. We claim that $N = cN$ for every $c \in \mathcal{C}(N)$. Let $x \mapsto \bar{x}$ denote the canonical map $R \to R/N =: \overline{R}$. By Proposition 5.3.4, r. K. $\dim(\overline{R}) \leq 1$. For $c \in \mathcal{C}(N)$, the element $\bar{c} \in \overline{R}$ is regular. So by [40, 6.3.9] we must have K. $\dim(\overline{R}/\bar{c}\overline{R}) <$ K. $\dim(\overline{R}) \leq 1$. So the right R-module $R/(N + cR) \cong \overline{R}/\bar{c}\overline{R}$ has Krull dimension at most 0 and thus has finite length. Hence $R/(N + cR)$ has a finite filtration with factors isomorphic to $R/\mathfrak{m}_i$ for some maximal right ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_p$ of R. The set of maximal right ideals of R is certainly closed under similarity (it is the set of right ideals whose factor module is simple), so by Lemma 5.1.10 all maximal right ideals lie in the right Oka family $\mathcal{F}_{\mathrm{pr}}^\circ$. It follows from Corollary 3.1.9 that we have $N + cR \in \mathcal{F}_{\mathrm{pr}}^\circ$. Choose $d \in R$ such that $N + cR = dR$. Now in $\overline{R}$, $\bar{c}\overline{R} = \bar{d}\overline{R}$ means that $\bar{c} = \bar{d}\bar{r}$ for some $r \in R$. Because the set of regular elements in the semiprime noetherian ring $\overline{R}$ is saturated, the fact that $c \in \mathcal{C}(N)$ implies that $d \in \mathcal{C}(N)$. Now $N \subseteq dR$ implies that $N = d(d^{-1}N)$, and $d \in \mathcal{C}(N)$ gives $d^{-1}N = N$. Thus $N = d(d^{-1}N) = dN = (cR + N)N = cN + N^2$, and we conclude from Nakayama's Lemma [33, (4.22)] (or by induction and the fact that N is nilpotent) that $N = cN$.

Now according to Robson's decomposition result [49, p. 346] the ring R is (up to isomorphism) of the form

$$R = \begin{pmatrix} A & B \\ 0 & S \end{pmatrix},$$

where A is an artinian ring, S is a semiprime ring, and $_A B_S$ is a (left and right noetherian) bimodule. Given any maximal right ideal $\mathfrak{m}$ of A, we will show that $B = \mathfrak{m}B$. The following is a maximal right ideal of R, and is therefore principal:

$$\begin{pmatrix} \mathfrak{m} & B \\ 0 & S \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \cdot R$$

for some $x \in \mathfrak{m}$, $y \in B$, and $z \in S$. It is easy to see that $zS = S$. Because S is noetherian,

z must be a unit. Now for any $\beta \in B$, there exists $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in R$ such that

$$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \in \begin{pmatrix} \mathfrak{m} & B \\ 0 & S \end{pmatrix}.$$

Since $zc = 0$ and z is a unit, we must have $c = 0$. Thus $\beta = xb \in \mathfrak{m}B$. Because $\beta \in B$ was arbitrary, this proves that $B = \mathfrak{m}B$. Since this holds for every maximal right ideal $\mathfrak{m}$ of A, we conclude from Lemma 5.3.8 that $B = 0$.

Hence $R = A \oplus S$ where A is an artinian ring and S is a semiprime ring. The maximal right ideals of both S and A must also be principal. The artinian ring A is a PRIR according to Corollary 5.1.5, and it follows from Proposition 5.3.4 that the semiprime ring S is a PRIR. It follows that $R = A \oplus S$ is a PRIR. $\qquad\square$

It is interesting to notice that, in the commutative setting, Kaplansky's Theorem 1.1.2 is "stronger" than the Kaplansky-Cohen Theorem 1.1.3, in the sense that Kaplansky originally derived Theorem 1.1.3 as a consequence of Theorem 1.1.2. This is opposite from our present situation, where the noncommutative version of the Kaplansky Theorem 5.3.9 in fact follows from (the "essential version" of) the noncommutative Kaplansky-Cohen Theorem 5.2.11 (through a series of other intermediate results).

The following example shows that Kaplansky's Theorem does not generalize if we remove the left noetherian hypothesis.

Example 5.3.10. *A local right noetherian ring R with right Krull dimension 1 whose unique maximal right ideal is principal, but which is not a principal right ideal ring.* This construction is based on an exercise given in [34, Ex. 19.12]. Let k be a field such that there exists a field isomorphism $\theta \colon k(x) \to k$ (which certainly does *not* fix k), such as $k = \mathbb{Q}(x_1, x_2, \dots)$. Consider the discrete valuation ring $A = k[x]_{(x)}$. Given a finitely generated module M_A, we define a ring $R := A \oplus M$ with multiplication given by

$$(a, m) \cdot (a', m') := (aa', m'\theta(a) + ma').$$

Let $\mathfrak{m} = xA \oplus M$ and $N = 0 \oplus M$, both of which are ideals of R. Notice that $N^2 = 0$ while $\bar{R} := R/N \cong A$ is a domain. This means that N is the prime radical of R. Thus N is contained in the Jacobson radical $\mathrm{rad}(R)$. Because $R/\mathrm{rad}(R) \cong \bar{R}/\mathrm{rad}(\bar{R})$ is a field, the ring R is local with Jacobson radical equal to $\mathfrak{m}$. Using the fact that $\theta(x) \in k$ is a unit in A, it is easy to conclude that $\mathfrak{m} = (x, 0) \cdot R$ is a principal right ideal.

Next we show that R is right noetherian. Because the ring $R/N \cong A$ is noetherian, it is noetherian as a right R-module. Also, because $N^2 = 0$, the right R-action on $N_R = (0 \oplus M)_R$ factors through $R/N \cong A$. Because A is noetherian and M_A is finitely generated, this means that N_R is noetherian. So R_R is an extension of the noetherian right modules R/N and N, proving that R is right noetherian. Because the prime radical of R is N, r. K. $\dim(R) = $ r. K. $\dim(R/N) = $ K. $\dim(A) = 1$ (see [40, 6.3.8]). Finally, because the R-action on $N_R = (0 \oplus M)_R$ factors through $R/N \cong A$, if M_A is any noncyclic A-module then

N is not principal as a right ideal in R. In fact, because the minimal number of generators of $\mu(N_R)$ is equal to $\mu(M_A) < \infty$, this number can be made as large as one desires.

Notice that the example above is not semiprime, in accordance with Proposition 5.3.1. With some extra work, we can produce a similar example R that is a domain. By Proposition 5.3.1 again, we expect such R to have right Krull dimension > 1. (We thank G. M. Bergman for helping to correct an error in an earlier version of this example.)

Example 5.3.11. *A local right noetherian domain R with right Krull dimension 2 whose unique maximal right ideal is principal, but which is not a principal right ideal ring.* Let k, $\theta\colon k(x) \xrightarrow{\sim} k$, and $A = k[x]_{(x)}$ be as in Example 5.3.10. Let $B = A[[y;\theta]] \supseteq A$, the ring of skew power series over A subject to the relation $ay = y\theta(a)$. Consider the ideal $I = y^2 B$, and define the subring $R := A \oplus I \subseteq B$. (Notice that R is the subring of B consisting of power series in which y does not appear with exponent 1. We can suggestively write $R = A[[y^2, y^3; \theta]]$, with the understanding that the equation $ay = y\theta(a)$ only has meaning via its consequences $ay^n = y\theta^n(a)$ for $n \geq 2$.) Being a subring of the domain B, R itself is a domain.

We claim that $I \subseteq \mathrm{rad}(R)$. It suffices to show that $1+I \subseteq U(R)$ (see [33, (4.5)]). Let $i \in I$; then $1 + i$ is a unit of B because $I \subseteq yB = \mathrm{rad}(B)$. For $i' := -(1+i)^{-1}i = -i(1+i)^{-1} \in I$ (note: $(1+i)^{-1}$ commutes with i because $1 + i$ does), we have

$$(1+i) \cdot (1+i') = 1 + i + (1+i)i' = 1,$$

and similarly $(1 + i')(1 + i) = 1$. So $1 + i \in U(R)$ as desired. One can now proceed as in Example 5.3.10 to show that R is a local ring whose unique maximal right ideal $\mathfrak{m} := xA \oplus I = xR$ is principal.

It is easy to see that R is a free right module over the subring $A[[y^2;\theta]] \cong A[[t;\theta]] =: S$ with basis $\{1, y^3\}$. Because S is right noetherian (in fact, a principal right ideal domain, according to [24]), R is also right noetherian. We claim that r. K. $\dim(S) = 2$. First we show that for every $f \in S \setminus \{0\}$, K. $\dim(S/fS) \leq 1$. Indeed, we can write $f = t^m x^n u$ for some unit $u \in S$. It follows from the filtration

$$S \supseteq tS \supseteq t^2 S \supseteq \cdots \supseteq t^m S \supseteq t^m x S \supseteq t^m x^2 S \supseteq \cdots \supseteq t^m x^n S = fS$$

that S/fS has a filtration whose factors are isomorhpic to either $S/tS \cong A$ or $S/xS \cong A/xA$. These filtration factors have submodule lattices isomorphic to that of A_A or $(A/xA)_A$, and thus respectively have Krull dimension 1 or 0. Hence K. $\dim(S/fS) \leq 1$ as claimed. Because S has right Krull dimension and is a domain, we see from Proposition 4.1.13 that S_S is a critical module. We conclude that r. K. $\dim(S) = 2$. Thus K. $\dim(R_S) = $ K. $\dim(S_S^2) = $ K. $\dim(S_S) = 2$ (the second equality follows from the exact sequence $0 \to S \to S^2 \to S \to 0$), which implies that K. $\dim(R_R) \leq$ K. $\dim(R_S) = 2$. On the other hand, the descending chain $I \supseteq I^2 \supseteq I^3 \supseteq \cdots$ of right ideals in R has filtration factors $I^m/I^{m+1} = y^{2m}B/y^{2m+2}B \cong A \oplus A$. These have Krull dimension 1, so we find K. $\dim(R_R) > 1$ and thus r. K. $\dim(R) = 2$.

Finally, we show that I is not a principal right ideal of R. It suffices to show that $I/I\mathfrak{m}$ is not a cyclic right module over $R/\mathfrak{m} \cong k$. Notice that $I\mathfrak{m} = I(Ax + I) = Ix + I^2$. Now $By \subseteq yB$ implies that $I^2 = (y^2B)^2 = y^4B$. Also, $Ix = y^2xA \oplus y^3xA \oplus y^4xA \cdots$. Thus $I\mathfrak{m} = Ix + I^2 = y^2xA \oplus y^3xA \oplus y^4B$. It follows that

$$\frac{I}{I\mathfrak{m}} \cong \frac{y^2A[[y;\theta]]}{y^2xA \oplus y^3xA \oplus y^4A[[y;\theta]]} \cong y^2k \oplus y^3k$$

is not a cyclic k-vector space, as desired. □

We conclude this section with some questions that arise in light of the results above. Examples 5.3.10 and 5.3.11 show that the left noetherian hypothesis in Theorem 5.3.9 cannot simply be dropped. While it seems somehow unnatural to try to omit the right noetherian hypothesis, we have not found an example showing this to be impossible. Thus we ask the following.

Question 5.3.12. *Does there exist a left (but not right) noetherian ring R whose maximal right ideals are all principal, but which is not a principal right ideal ring? What if we assume, in addition, that R has right Krull dimension?*

While reading an earlier draft of this work, G. M. Bergman kindly pointed out to us that no such example exists if we assume further that R is a *domain*. We were able to generalize this to include semiprime right Goldie rings as follows.

Proposition 5.3.13. *Let R be a semiprime left noetherian ring in which every essential right ideal contains a regular element (the latter hypothesis is satisfied if R is a domain or if R is right Goldie—in particular, if R has right Krull dimension). If every maximal right ideal of R is principal, then R is a principal right ideal ring.*

Proof. By Example 5.2.4, it is enough to show that every essential right ideal of R is principal. To this end, fix $E_R \subseteq_e R$. Because R has a semisimple left ring of quotients, the multiplicative set of regular elements of R is saturated. Thus the hypotheses of Lemma 5.3.2 are satisfied. Since E contains a regular element, that lemma implies that R/E has finite length. So R/E has a finite filtration whose factors are isomorphic to $R/\mathfrak{m}_i$ for some maximal right ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R. Since the set of maximal right ideals is closed under similarity, Lemma 5.1.10 implies that all maximal right ideals of R lie in $\mathcal{F}_{\mathrm{pr}}^\circ$. In particular, Corollary 3.1.9 implies that $E \in \mathcal{F}_{\mathrm{pr}}^\circ \subseteq \mathcal{F}_{\mathrm{pr}}$, so that E is principal. □

We also wonder to what extent the PRIR condition can be tested up to similarity.

Question 5.3.14. *Suppose that R is a noetherian ring each of whose maximal right ideals is similar to a principal right ideal. Is R a principal right ideal ring? If not, then is every right ideal of R similar to a principal right ideal?*

It would be interesting to test the status of the first Weyl algebra $R := A_1(k)$ with respect to this question. Is every maximal right ideal of R similar to a principal right ideal? Does R have any right ideals that are not similar to principal right ideals?

5.4 Previous generalizations of the Cohen and Kaplansky theorems

In this final section we will discuss how Theorem 4.2.5 and Theorem 5.1.11 relate to earlier noncommutative generalizations of the Cohen and Kaplansky-Cohen theorems in the literature. (We are not aware of any previous generalizations of Kaplansky's Theorem 1.1.2.) In [29], K. Koh generalized both of these theorems. He defined a right ideal $I_R \subsetneq R$ to be a "prime right ideal" if, for any right ideals $A, B \subseteq R$ such that $AI \subseteq I$, $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. Notice that this is equivalent to the condition that for $a, b \in R$, $aRb \subseteq I$ with $aRI \subseteq I$ imply that either $a \in I$ or $b \in I$. We will refer to such a right ideal as a *Koh-prime right ideal*. Koh showed that a ring R is right noetherian (resp. a PRIR) iff all of its Koh-prime right ideals are finitely generated (resp. principal). Independently, in [6, 7] V. R. Chandran also gave generalizations of the Cohen and Kaplansky theorems, showing that a right duo ring is right noetherian (resp. a PRIR) iff all prime ideals of R are finitely generated (whether this is f.g. as an ideal or f.g. as a right ideal is irrelevant, since R is right duo). But Koh's result implies Chandran's result, since a two-sided ideal is Koh-prime as a right ideal iff it is a prime ideal in the usual sense.

Notice that our completely prime right ideals are necessarily Koh-prime right ideals. For suppose that $P_R \subseteq R$ is completely prime and that $A, B \subseteq R$ are such that $AP \subseteq P$ and $AB \subseteq P$. If $A \nsubseteq P$, then there exists $a \in A \setminus P$. Now $aP \subseteq P$, and for any $b \in B$ we have $ab \in P$. It follows that $b \in P$ because P is completely prime. So $B \subseteq P$, proving that P is Koh-prime. *It follows that Theorem 4.2.5 and Theorem 5.1.11, with the set $\mathcal{S}$ taken to be the set of completely prime right ideals, imply Koh's theorems, which in turn imply Chandran's theorems.*

On the other hand, G. O. Michler offered another noncommutative generalization of Cohen's Theorem in [41]. He defined a right ideal $I \subsetneq R$ to be "prime" if $aRb \subseteq I$ implies that either $a \in I$ or $b \in I$. This is equivalent to saying that, for right ideals $A, B \subseteq R$, $AB \subseteq I$ implies that one of A or B lies in I. We will refer to such right ideals as *Michler-prime right ideals*. Michler proved in [41] that a ring is right noetherian iff its Michler-prime right ideals are all finitely generated. Notice immediately that the Michler-prime right ideals of a given ring form a subset of the set of all Koh-prime right ideals of that ring; thus Michler's version of Cohen's Theorem generalizes Koh's version.

If we were to try to recover Michler's theorem directly from Theorem 4.2.5, we would need to check that the Michler-prime right ideals form a noetherian point annihilator set over an arbitrary ring R. In order to settle whether or not this is true, we offer an alternate description of the Michler-prime right ideals below. Recall that a module $M_R \neq 0$ is said to be a *prime module* if, for every nonzero submodule $N \subseteq M$, $\mathrm{ann}(N) = \mathrm{ann}(M)$. One can show that the annihilator of a prime module is a prime ideal (for example, as in [32, (3.54)]).

Proposition 5.4.1. *A right ideal $P \subsetneq R$ is Michler-prime iff R/P is a prime module.*

Proof. First suppose that P is Michler-prime. To see that R/P is a prime module, consider a nonzero submodule $A/P \subseteq R/P$ (so that the right ideal A properly contains P). Denote $B := \operatorname{ann}(A/P) \lhd R$. Then $(A/P) \cdot B = 0$ implies that $AB \subseteq P$. Because P is Michler-prime, this means that $B \subseteq P$, so that $(R/P) \cdot B = (P + B)/P = 0$. So $B = \operatorname{ann}(R/P)$, proving that the module R/P is prime.

Conversely, suppose that R/P is a prime module. Let $a, b \in R$ be such that $aRb \subseteq P$ and $a \notin P$. It follows that b annihilates $(P + aR)/P \neq 0$, so that $b \in \operatorname{ann}((P + aR)/P) = \operatorname{ann}(R/P)$. In particular, $(R/P) \cdot b = 0$ implies that $b \in P$. This proves that P is Michler-prime. $\qquad\square$

Corollary 5.4.2. *For a ring R, the set $\mathcal{S}$ of Michler-prime right ideals is a noetherian point annihilator set iff every nonzero noetherian right R-module has a prime submodule. This is satisfied, in particular, if R has the ACC on ideals.*

Proof. The "only if" direction is clear from Proposition 5.4.1. For the "if" direction, let M_R be any module with a prime submodule N. Notice that a nonzero submodule of a prime module is prime. Thus for any nonzero element $m \in N$, $R/\operatorname{ann}(m) \cong mR \subseteq N$ is a prime module. By Proposition 5.4.1, $\operatorname{ann}(m)$ is a Michler-prime right ideal. So if every nonzero noetherian module has a prime submodule, the set $\mathcal{S}$ is a noetherian point annihilator set.

If R satisfies ACC on ideals, then *every* nonzero right R-module has a prime submodule—see [32, (3.58)]. So in this case $\mathcal{S}$ is a point annihilator set, hence a noetherian point annihilator set. $\qquad\square$

We conclude from Corollary 5.4.2 and Theorem 4.2.5 that *a ring is right noetherian iff it satisfies ACC on ideals and all of its Michler-prime right ideals are finitely generated.* This is actually a slight generalization of [41, Lem. 3], which Michler used as a "stepping stone" to prove his main result.

Nevertheless, there do exist nonzero noetherian modules over some (large) rings which do not have any prime submodules. Thus Michler's primes do not form a noetherian point annihilator set in every ring. We include an example below.

Example 5.4.3. Let k be a division ring, and let R be the ring of $\mathbb{N} \times \mathbb{N}$ row-finite upper triangular matrices over k. Let $M_R = \bigoplus_{\mathbb{N}} k$, viewed as row vectors over k, with the obvious right R-action. Let M_i denote the submodule of M consisting of row vectors whose first i entries are zero. Then one can show that

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$$

are the only nonzero submodules of M. This visibly shows that M is noetherian. (Indeed, one can say more: every submodule of M is actually principal, generated by one of the "standard basis vectors." We omit the details because we will not use this fact.) However,

one can see that ann(M_i) is equal to the set of all matrices in R whose first i rows are arbitrary and whose other rows are zero. So the fact that

$$\text{ann}(M_0) \subsetneq \text{ann}(M_1) \subsetneq \text{ann}(M_2) \subsetneq \cdots$$

makes it clear that M has no prime submodules.

Incidentally, M_R is also an example of a cyclic 1-critical module that is not a prime module. Thus, choosing a right ideal $I_R \subseteq R$ such that $R/I \cong M$ (such as the right ideal of matrices in R whose first row is zero), we see that I is cocritical but not Michler-prime.

In spite of this complication, *it is in fact possible to derive Michler's Theorem from Theorem 4.2.5.* The key observation that makes this possible is a lemma [51, Lem. 2] due to P. F. Smith. This result states that if every ideal of a ring R contains a finite product of prime ideals each containing that ideal, and if R satisfies the ACC on prime ideals, then every nonzero right R-module has a prime submodule.

Theorem 5.4.4 (Michler). *A ring R is right noetherian iff all of the Michler-prime right ideals of R are finitely generated.*

Proof. ("If" direction.) Suppose that the Michler-prime right ideals of R are all finitely generated. Every prime (two-sided) ideal of R is Michler-prime, and thus is finitely generated as a right ideal. By [41, Lemmas 4 & 5] the following two conditions hold:

1. Every ideal $I \lhd R$ contains a product of finitely many prime ideals of R, where each of these ideals contains I;

2. R satisfies the ascending chain condition on prime ideals.

It follows from [51, Lem. 2] that every nonzero right R-module has a prime submodule. So by Corollary 5.4.2, the set of Michler-prime right ideals is a noetherian point annihilator set for R. Now it follows from Theorem 4.2.5 that R is a right noetherian ring. $\square$

In addition, our methods allow us to produce a generalization of the Kaplansky-Cohen Theorem that is in the spirit of Michler's Theorem! Note that this was not proved in [41], and in fact seems to be a new result.

Theorem 5.4.5. *A ring R is a principal right ideal ring iff the all of the Michler-prime right ideals of R are principal.*

Proof. ("If" direction.) Suppose that all of the Michler-prime right ideals of R are principal. As in the proof of Theorem 5.4.4 above, it follows that the set $\mathcal{S}$ of Michler-prime right ideals of R is a noetherian point annihilator set for R. This set is closed under similarity thanks to Proposition 5.4.1, so Theorem 5.1.11 implies that R is a principal right ideal ring. $\square$

For a given ring R, the effectiveness of Michler's Theorem versus Theorem 4.2.5 with $\mathcal{S}$ taken to be the set of completely prime right ideals of R depends on the scarcity or abundance of right ideals in R from the "test set" in either theorem. For example, over a simple ring R, every nonzero right R-module is certainly prime. So every proper right ideal of R will be Michler-prime. (In fact, Koh [28, Thm. 4.2] has shown even more: a ring R is simple iff all of its proper right ideals are Michler-prime.) Thus for a simple ring R, Michler's theorem provides no advantage, as we would still need to test *every* right ideal to see whether R is right noetherian. On the other hand, Proposition 2.1.11 states that all right ideals of a ring R are completely prime only if R is a division ring. So outside of this trivial class of rings, we are guaranteed that Theorem 4.2.5 with $\mathcal{S} = \{$completely prime right ideals$\}$ reduces the set of right ideals which we need to test in order to determine whether a ring is right noetherian. We can expect Theorem 4.2.5 to be increasingly effective when we take $\mathcal{S}$ to be either of the two smaller test sets in (4.1.16).

There is another variant of Cohen's Theorem for right fully bounded rings. (Recall that R is right fully bounded if, for every prime ideal $P \lhd R$, every essential right ideal of R/P contains a nonzero ideal of R/P). This result says that *a right fully bounded ring is right noetherian iff all of its prime ideals are finitely generated as right ideals*. A statement of this theorem is given in [31, p. 95], and it is attributed to G. O. Michler and L. W. Small independently. P. F. Smith provided a proof using homological methods in [52, Cor. 5] and an elementary proof in [53, Thm. 1]. We wonder whether it is possible to recover this result using the methods developed in the present work, such as the Point Annihilator Set Theorem 4.2.1 or its adaptation in Theorem 5.2.9. On a related note, [53] also features a version of Cohen's Theorem for modules over commutative rings.

In a more recent paper [58], B. V. Zabavs'kiĭ also studied noncommutative versions of the Cohen and Kaplansky-Cohen theorems. Theorem 1 of that paper states that, for a right chain ring R (i.e., a ring whose right ideals are totally ordered under inclusion), if every Michler-prime right ideal is principal, then R is a principal right ideal ring. This is clearly generalized by Theorem 5.4.5 above. To prove a version of the Kaplansky-Cohen Theorem for general rings, he defined a right ideal P to be *weakly primary* if, for any $a, b \in R$, $(a + P)R(b + P) \subseteq P$ implies that one of a or b is in P. In [58, Thm. 2] it is shown that a ring is a principal right ideal ring iff every weakly primary right ideal is principal. It is straightforward to verify that completely prime right ideals are weakly primary, so this theorem is generalized by Theorem 5.1.11 with the set $\mathcal{S}$ taken to be the completely prime right ideals. There is a version of Cohen's Theorem in [58, Thm. 5] that is subsumed by Michler's Theorem; we do not include its statement here. Finally, there are also some results in [58] investigating when every two-sided ideal of a ring is either finitely generated or principal when considered as a right ideal.

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