ANOSOV MAPS, POLYCYCLIC GROUPS AND HOMOLOGY

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§1. INTRODUCTION

Let $M$ denote a compact Riemannian manifold. A $C^1$ map $f: M \to M$ is called Anosov if there exists a continuous splitting $TM = E^u \oplus E^s$ of the tangent bundle of $M$, and constants $C > 0, \lambda > 1$, such that $Tf(E^u) = E^u, Tf(E^s) \subset E^s$, and furthermore for all positive integers $m$ and tangent vectors $X \in TM$:

$$|Tf^m(X)| \geq C\lambda^m |X| \quad \text{if } X \in E^u,$$

$$|Tf^m(x)| \leq C^{-1}\lambda^{-m} |X| \quad \text{if } X \in E^s.$$

This condition is independent of the Riemannian metric. Anosov maps have been studied in [1, 2, 7, 11], and elsewhere.

Examples of Anosov maps can be obtained from a Lie group $G$, a discrete subgroup $\Gamma$ with $G/\Gamma$ compact, and an endomorphism $\phi: G \to G$ such that $\phi(\Gamma) \subset \Gamma$; see [8]. If the derivative of $\phi$ at the identity of $G$ has no eigenvalues of absolute value one, then the map $G/\Gamma \to G/\Gamma$ induced by $\phi$ is an Anosov map. All known Anosov maps are intimately related to maps of this type.

As an example take $G = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, the integer lattice; then $\phi$ is defined by an $n \times n$ integer matrix $A$ and $G/\Gamma$ is the $n$-torus $T^n$. In this case we can identify $A$ with the linear transformation

$$f_*: H_1(T^n; \mathbb{R}) \to H_1(T^n; \mathbb{R})$$

induced by $f$ on the first homology group with real coefficients.

John Franks [2] has proved that if $f: T^n \to T^n$ is any Anosov diffeomorphism, then $f_*$ has no root of unity among its eigenvalues.

Theorem 1 extends Franks' result to Anosov maps on a wider class of manifolds which includes all nilmanifolds. As an application, many manifolds that do not admit Anosov diffeomorphisms are constructed. For example: the Cartesian product of the Klein bottle and a torus.

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§2. STATEMENT OF RESULTS

Theorem 1. Let \( f : M \to M \) be an Anosov map of a compact manifold. Let \( u \in H^1(M; \mathbb{Z}) \) be a nonzero integral cohomology class of dimension 1 such that \( (f^*)^n u = u \), for some \( n \in \mathbb{Z}_+ \). Then the infinite cyclic covering space corresponding to \( u \) has infinite dimensional rational homology.

The proof is postponed to §3.

Corollary 2. Let \( M \) be a compact manifold such that every infinite cyclic covering space has finite dimensional rational homology. Then if \( f : M \to M \) is an Anosov map, the induced homomorphism

\[
 f_* : H_1(M; \mathbb{R}) \to H_1(M; \mathbb{R})
\]

has no root of unity for an eigenvalue.

Proof. The eigenvalues of \( f_* \) are the same as those of \( f^* : H^1(M; \mathbb{R}) \to H^1(M; \mathbb{R}) \), since the Kronecker index

\[
 H^1(M; \mathbb{R}) \times H_1(M; \mathbb{R}) \to \mathbb{R}
\]

is a dual pairing under which \( f^* \) and \( f_* \) are adjoint homomorphisms. A basis for the finitely generated free abelian group \( H^1(M; \mathbb{Z}) \) is also a vector space basis for \( H^1(M; \mathbb{R}) \) under the natural inclusion

\[
 H^1(M; \mathbb{Z}) \to H^1(M; \mathbb{R})
\]

and with this basis, \( f^* \) is defined by an integral matrix. Therefore if \( f_* : H_1(M; \mathbb{R}) \to H_1(M; \mathbb{R}) \) has an eigenvalue \( \omega \) with \( \omega^n = 1 \), there exists a nonzero vector \( x \in H^1(M; \mathbb{R}) \) having rational coordinates in the basis defined above, such that \( (f^*)^n(x) = x \). Some nonzero integer multiple of \( x \) lies in \( H^1(M; \mathbb{Z}) \), and a contradiction is reached by applying Theorem 1.

Polycyclic groups are introduced to ensure the validity of the topological hypothesis on \( M \) in Corollary 2.

Definition. A group \( \pi \) is polycyclic if it is solvable and every subgroup is finitely generated.

The following lemma collects some facts about polycyclic groups; see [3, 9, 12].

Lemma 3. The following conditions are pairwise equivalent:

(a) \( \pi \) is polycyclic.
(b) \( \pi \) is solvable and every Abelian subgroup is finitely generated.
(c) There exists an exact sequence

\[
 0 \to N \to \pi \to A \to 0
\]

where \( N \) is a finitely generated nilpotent group and \( A \) is a finitely generated Abelian group.

(d) \( \pi \) is built from cyclic groups by forming a finite number of extensions.
(e) \( \pi \) is isomorphic to a solvable multiplicative group of integer matrices.

The following is our main result:

Theorem 4. Let \( M \) be a compact manifold such that (a) the fundamental group has a polycyclic subgroup of finite index; and (b) the universal covering \( M \) has finite dimensional rational homology.
Then if \( f: M \to M \) is an Anosov map, there is no root of unity among the eigenvalues of \( f_+: H_1(M; \mathbb{R}) \to H_1(M; \mathbb{R}) \).

**Proof.** Follows from Corollary 2 and Lemma 6, below.

A useful property of polycyclic groups is the following.

**Lemma 5.** Let \( Q^n \) denote a rational vector space of finite dimension \( n \). Given a representation \( \rho \) of a polycyclic group \( \pi \) in the group \( GL(Q^n) \) of automorphisms of \( Q^n \), the corresponding homology group

\[
H_*(\pi; Q^n, \rho)
\]

is a finite dimensional vector space over \( Q \).

**Proof.** Let \( B_\pi \) be a classifying space for \( \pi \). That is, \( B_\pi = E_\pi/\pi \) where \( E \) is a contractible space on which \( \pi \) operates freely and \( B_\pi \) is its orbit space; the natural map \( E_\pi \to B_\pi \) defines a covering space. Then \( B_\pi \) is connected, \( \pi_i(B_\pi) \cong \pi \) and \( \pi_i(B_\pi) = 0 \) for \( i > 1 \); these properties characterize \( B_\pi \) up to weak homotopy type.

By definition,

\[
H_*(\pi; Q^n, \rho) = H_*(B_\pi; Q^n, \rho),
\]

the total homology group of the space \( B_\pi \) with coefficients twisted by the homomorphism \( \rho: \pi_i(B_\pi) \cong \pi \to Aut(Q^n) \).

If it is known that \( B_\pi \) has the homotopy type of a finite simplicial complex, the lemma follows immediately. This is not the case in general but \( \pi \) has a subgroup \( \pi_0 \) of finite index whose classifying space \( B_{\pi_0} \) has the homotopy type of a finite complex. We now show this.

It is proved in Malcev [3] that \( \pi \) has a subgroup \( \pi_0 \) of finite index such that in the exact sequence

\[
0 \to N_0 \to \pi_0 \to A_0 \to 0
\]

of Lemma 3(c), \( N_0 \) and \( A_0 \) are torsion free. For any exact sequence of groups \( 0 \to \alpha \to \beta \to \gamma \to 0 \), the homomorphisms \( i \) and \( j \) determine maps \( B_\pi \to B_\gamma \to B_\beta \) which, with proper choice of classifying spaces, may be taken to be a fibration. Therefore \( B_{\pi_0} \) is a fibre space over \( B_{A_0} \) with fibre \( B_{N_0} \). Now \( B_{A_0} \) can be chosen to be an \( n \)-torus, and \( B_{N_0} \) has the homotopy type of a nilmanifold [4]. It is easily proved by induction on the number of cells in \( B \) that if \( F \to E \to B \) is a fibration over a finite complex \( B \) whose fibre \( F \) has the homotopy type of a finite complex, then so has the total space \( E \). Therefore \( B_{\pi_0} \) has the homotopy type of a finite complex.

Now the map \( p: B_{\pi_0} \to B_\pi \) corresponding to the inclusion homomorphism \( \pi_0 \to \pi \) can be chosen to be a finite covering space. It is well known that the homomorphism induced by \( p \) maps \( H_*(B_{\pi_0}; Q^n, \rho_0) \) onto \( H_*(B_\pi; Q^n, \rho) \), where \( \rho_0 \) denotes the composite homomorphism

\[
\pi_0 \to \pi^n \to GL(Q^n).
\]
(A right inverse to \( p_* \) can be constructed by considering the inverse image by \( p \) of a chain in \( B_{\pi} \).) Therefore \( H_*^\pi(X; \mathbb{Q}^\beta, \rho) \) is finite dimensional. This proves Lemma 5.

**Remark.** It follows immediately from Wang [10; Corollary 2, p. 17] that a polycyclic group \( \pi \) has a normal subgroup \( \pi_0 \) of finite index such that \( B_{\pi_0} \) is a compact smooth manifold.

**Lemma 6.** Let \( X \) be a finite simplicial complex. Suppose (a) \( \pi_1(M) \) has a polycyclic subgroup of finite index, and (b) \( H_*^\pi(X; \mathbb{Q}) \) is finite dimensional where \( X \) is the universal covering of \( X \). Then \( H_*^\pi(X; \mathbb{Q}) \) is finite dimensional for any covering \( \tilde{X} \) of \( X \).

**Proof.** By passing to a finite covering of \( X \), we may assume \( \pi_1(M) \) is polycyclic. Then so is \( \pi_1(X) = \pi \). Now \( \pi \) acts freely on \( \tilde{X} \) with orbit space \( \tilde{X} \). In this situation there is a spectral sequence with \( E^2 = H_*(\pi; H_*(\tilde{X}; \mathbb{Q}), \rho) \), converging to \( H_*(\tilde{X}; \mathbb{Q}) \); see MacLane [5]. Here \( \rho: \pi \rightarrow \text{Aut} H_*(\tilde{X}; \mathbb{Q}) \) is the homomorphism determined by the action of \( \pi \) on \( \tilde{X} \).

By assumption \( H_*(\tilde{X}; \mathbb{Q}) \) is finite dimensional, and then by Lemma 4, \( E^2 \) is finite dimensional. Lemma 6 follows by the usual spectral sequence argument: \( H_*(\tilde{X}; \mathbb{Q}) \) is exhibited as being obtained from \( E^2 \) by passing alternately to subspaces and quotient spaces a finite number of times. This completes the proof of Lemma 6.

### §3. PROOF OF THEOREM 1

It is well known that if \( f: M \rightarrow M \) is an Anosov map then some iterate \( f^m \) of \( f \) has a fixed point, \( m \geq 1 \); compare Proposition 1.7 of Franks [1]. Since all the fixed points of an Anosov map have the same index \( \pm 1 \), this means that every iterate \( f^{mk} \) of \( f^m \), \( k \geq 1 \), has nonzero Lefschetz number \( L(f^{mk}) \). It suffices to prove Theorem 1 for an iterate of \( f \); therefore, we may suppose \( L(f) \neq 0 \). Theorem 1 is thus a corollary of

**Lemma 7.** Let \( X \) be a finite complex, \( f: X \rightarrow X \) a continuous map, and \( u \in H_1(M; \mathbb{Z}) \) a nonzero class such that \( f^*u = u \). Let \( p: Y \rightarrow X \) be the infinite cyclic covering corresponding to \( u \). If \( H_*(Y; \mathbb{Q}) \) is finite dimensional then \( L(f) = 0 \).

**Proof.** The covering space corresponding to \( u \) is the one whose fundamental group is annihilated by the composite homomorphism

\[
\pi_1(X) \xrightarrow{h} H_1(X; \mathbb{Z}) \xrightarrow{\phi} \mathbb{Z}
\]

where \( h \) is the Hurewicz map and \( \phi \) denotes the Kronecker index with \( u \).

Since \( f^*u = u \), there is a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Y \\
\downarrow p & & \downarrow p \\
X & \xrightarrow{f} & X
\end{array}
\]

Let \( t: Y \rightarrow Y \) be a generator for the group of deck transformations of \( Y \) over \( X \); then \( t \circ g = g \circ t \). Imitating Milnor [6] we consider a commutative diagram of chain complexes and chain maps:
The rows are exact. There corresponds a commutative ladder of homomorphisms

\[ \begin{array}{cccccc}
0 & \longrightarrow & C_*(Y; \mathbb{Q}) & \longrightarrow^i & C_*(Y; \mathbb{Q}) & \longrightarrow^{f_*} & C_*(X; \mathbb{Q}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_*(Y; \mathbb{Q}) & \longrightarrow & C_*(Y; \mathbb{Q}) & \longrightarrow & C_*(X; \mathbb{Q}) & \longrightarrow & 0 \\
\end{array} \]

with exact rows. A well-known computation with traces shows that for such a ladder, the Lefschetz number of the middle endomorphism equals the sum of the Lefschetz numbers of the outer endomorphisms; in this case:

\[ L(g) + L(f) = L(g). \]

Hence \( L(f) = 0 \). This proves Lemma 7 and thereby completes the proofs of Theorems 1, 2 and 4.

§4. APPLICATIONS

The following is an immediate consequence of Theorem 4:

**Theorem 8.** Suppose \( M \) is a compact manifold such that

(a) \( \pi_1(M) \) has a polycyclic subgroup of finite index,
(b) \( H_*(\tilde{M}; \mathbb{Q}) \) is finite dimensional, and
(c) \( H^1(M; \mathbb{Z}) \cong \mathbb{Z} \).

Then \( M \) does not admit an Anosov diffeomorphism.

**Proof.** If \( f: M \rightarrow M \) is any diffeomorphism, condition (c) forces \( f^* \) to have \( \pm 1 \) for an eigenvalue.

Examples of such manifolds are easy to construct; for example, the Cartesian product of a simply connected manifold with a circle or Klein bottle. Another type comprises the solvmanifolds constructed as follows. Let \( A \) be an invertible \( n \times n \) integer matrix acting as a diffeomorphism of the \( n \)-torus \( T^n \) in the usual way. Define \( M_A = (T^n \times \mathbb{R})/\mathbb{Z} \) where the generator of \( \mathbb{Z} \) acts on \( T^n \times \mathbb{R} \) by \((x, y) \mapsto (A(x), y + 1)\). The fundamental group of \( M_A \) is the semidirect product \( \mathbb{Z} \cdot \mathbb{Z}^* \), the generator of \( \mathbb{Z} \) acting on \( \mathbb{Z}^* \) by \( A \); its commutator subgroup is the range of \( A - I \) in \( \mathbb{Z} \). If 1 is not an eigenvalue of \( A \), then

\[ H_1(M_A; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \cong H^1(M_A; \mathbb{Z}). \]

Clearly \( \tilde{M}_A \approx \mathbb{R}^{n+1} \), and the exact sequence

\[ 0 \rightarrow \mathbb{Z}^n \rightarrow \pi_1(M_A) \rightarrow \mathbb{Z} \rightarrow 0 \]

proves \( \pi_1(M_A) \) polycyclic. Therefore by Theorem 4: \( M_A \) does not admit an Anosov diffeomorphism if 1 is not an eigenvalue of \( A \).
The following theorem gives a different method of exploiting the fundamental group of $\mathcal{M}_A$.

**Theorem 9.** Suppose $A$ is an invertible integer matrix such that $A^k \neq 1$ for all $k \neq 0$. Then:

(a) $\mathcal{M}_A \times S^1$ does not admit an Anosov diffeomorphism;

(b) If $1$ is not an eigenvalue of $A$ then $\mathcal{M}_A \times T^m$ does not admit an Anosov diffeomorphism for any $m \geq 0$.

Proof. The hypothesis on $A$ implies that the center of $\pi_1(\mathcal{M}_A) \cong \mathbb{Z} \cdot \mathbb{Z}^a$ is trivial. Therefore the center $C$ of $\pi_1(\mathcal{M}_A \times T^m) \cong \pi_1(\mathcal{M}_A) \times \mathbb{Z}^m$ is $\mathbb{Z}^m$, and $C$ meets the commutator subgroup of $\pi_1(\mathcal{M}_A \times T^m)$ only in the identity element. Therefore under the Hurewicz map $C$ maps isomorphically onto a subgroup $C \subset H_1(\mathcal{M}_A \times T^m; \mathbb{Z})$ which is invariant under every diffeomorphism of $\mathcal{M}_A \times T^m$. Under assumption (a), $C \cong \mathbb{Z}$ and therefore the automorphism $f^*$ of $H_1(M \times S^1; \mathbb{Z})/\text{torsion}$ has $\pm 1$ as an eigenvalue. Under assumption (b), $C \cong \mathbb{Z}^m$ and $H_1(M \times S^1; \mathbb{Z}) \cong \mathbb{Z}^{m+1}$, and again $f^*$ has an eigenvalue $\pm 1$. Therefore Theorem 4 proves Theorem 9.

Next we consider the Klein bottle $K^2$. The fundamental group embeds in an exact sequence $0 \to \mathbb{Z} \to \pi_1(K^2) \to \mathbb{Z}_2 \to 0$ and is therefore polycyclic.

**Theorem 10.** Let $M$ be a compact orientable manifold such that

(a) $\pi_1(M)$ has a polycyclic subgroup of finite index

(b) $H_*(M; \mathbb{Q})$ is finite dimensional.

Then $K^2 \times M$ does not admit an Anosov diffeomorphism.

Proof. Let $p : T^2 \to K^2$ be the double covering of the Klein bottle by the torus. The subgroup $p_*(\pi_1(T^2)) \times \pi_1(M) \subset \pi_1(K^2 \times M)$ is represented by loops in $K^2 \times M$ that preserve local orientation; it is therefore preserved by every diffeomorphism $f$ of $K^2 \times M$. Consequently $f$ is covered by a diffeomorphism $g$ of $T^2 \times M$. Now $g^*$ preserves the subgroup $G \subset H^1(T^2 \times M; \mathbb{Z})$ that is the image of $(p \times I)^*: H^1(K^2 \times M; \mathbb{Z}) \to H^1(T^2 \times M; \mathbb{Z})$. By the Künneth theorem,

$$H^1(K^2 \times M; \mathbb{Z}) \cong \mathbb{Z} \oplus H^1(M; \mathbb{Z})$$

and

$$H^1(T^2 \times M; \mathbb{Z}) \cong \mathbb{Z}^2 \oplus H^1(M; \mathbb{Z}).$$

Thus $G$ has corank 1, which forces $g^*: H^1(T^2 \times M; \mathbb{Z}) \to H^1(T^2 \times M; \mathbb{Z})$ to have $\pm 1$ for an eigenvalue. By Theorem 4, $g$ cannot be an Anosov diffeomorphism. Therefore $f$ cannot be one either.

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