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On the Construction of Minimal Model for Some A-infinity Algebras

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ABSTRACT OF THE DISSERTATION

On the Construction of Minimal Model for Some A-infinity Algebras

By

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For a formal differential graded algebra, if extended by an odd degree element, we prove that the extended algebra has an A_∞-minimal model with only m_2 and m_3 non-trivial. As an application, the A_∞-algebras constructed by Tsai, Tseng and Yau on formal symplectic manifolds satisfy this property. Separately, we expand the result of Miller and Crowley-Nordström for k-connected manifold. In particular, we prove that if the dimension of the manifold $n \leq (l+1)k+2$, then its de Rham complex has an A_∞-minimal model with $m_p = 0$ for all $p \geq l$.

Chapter 1

Introduction

Rational homotopy theory was introduced by Sullivian [16] and Quillen [15]. Rational homotopy equivalence is an isomorphism on homology or cohomology with rational coefficients, which is called a quasi-isomorphism. Compared to homotopy theory, rational homotopy theory ignores the information of torsion. However, it makes the calculation much easier.

As an geometric application, Vigué and Sullivian proved that if the cohomology ring of a simply connected closed Riemannian manifold is not generated by at least two elements, then the manifold has infinitely many geometrically distinct closed geodesics [19].

Algebraically, the cohomology ring can be viewed as a differential algebra (dga). We say a dga is formal if it is quasi-isomorphic to its cohomology. Thus, we can study its homotopy type by its cohomology. If the de Rham complex of a manifold is a formal dga, the manifold itself is called formal. A natural question is, what conditions or characteristics ensure that a manifold is formal?

For a compact complex manifold, Deligne, Griffiths, Morgan and Sullivan proved that it is formal if the dd^c-lemma holds [6]. One may ask what would make a compact symplectic manifold formal. Babenko and Taimanov in 1998 conjectured that a simply-connected compact symplectic manifold is formal if and only if it satisfies the hard Lefschetz property [1]. Both directions of the statement are now known to be false. Gompf constructed a simply-connected 6-manifold which does not satisfy the hard Lefschetz property [8]. This example is formal because Miller proved that all simply-connected compact 6-manifolds are formal [14]. The other direction was studied in [10][13], and was further clarified by Cavalcanti who gave a simply-connected non-compact symplectic manifold with the hard Lefschetz property, but is not formal [3] (see also [4]).

Since there is no relationship between the hard Lefschetz property and the formality of a symplectic manifold, we can consider if they are related in the context of another cochain complex on symplectic manifolds. Tsai, Tseng and Yau constructed cochain complexes consisting of primitive forms, or more generally, filtered forms which are defined by the Lefschetz decomposition [18]. These complexes carry an A_∞-algebra structure, and for simplicity, we will just call them TTY-algebras.

The formality of an A_∞-algebra is defined differently, but it is equivalent to formal as a dga when the A_∞-algebra is a dga. By Kadeishvili [11][12], every A_∞-algebra (A, m^A) is quasi-isomorphic to its cohomology $H^*(A)$ equipped with an A_∞ structure m such that $m_1 = 0$ and m_2 is induced by m_2^A. $(H^*(A), m)$ is called an A_∞-minimal model of (A, m^A). Note that the A_∞-minimal model here is different than the minimal model of a dga defined by Sullivan [6], even if A itself is a dga. If A has an A_∞-minimal model $(H^*(A), m)$ such that $m_p = 0$ for all p except for $p = 2$, then we say A is formal.

As we will see, the TTY-algebra can be non-formal even when the symplectic manifold is formal. For example, it is non-formal for the torus T^{2N} (See Example 3.7). On the other hand, there are some formal manifolds such as the projective spaces and the Euclidean spaces whose TTY-algebras are also formal (See Example 3.6). So we here consider a statement for TTY-algebra which is weaker than formal, and can be induced from the formality of the manifold. In Chapter 3, we prove the following:

Theorem 1.1. *Suppose (M, ω) is a formal symplectic manifold. Its TTY-algebra is formal if the symplectic form ω is exact. When ω is non-exact, its TTY-algebra has an A_∞-minimal model with $m_p = 0$ for all p except for $p = 2$ or 3.*

By the work of Tanaka and Tseng [17], the TTY-algebra of p-filtered forms on a symplectic manifold (M, ω) is quasi-isomorphic to a dga which is the de Rham complex on M extended by an odd-degree element θ, such that $d\theta = \omega^{p+1}$. So we can consider the A_∞-minimal model of this dga instead, and we construct it explicitly in the proof. Thus, the homotopy type of the TTY-algebra is determined by its cohomology together with the given operation m_3. Theorem 1.1 follows from the more general result below for dgas.

Theorem 1.2. *Suppose A is a formal dga. $\omega_A \in A$ is an even-degree element. Extend A to $\tilde{A} = \{\alpha + \theta_A \beta | \ \alpha, \beta \in A\}$ with $d\theta_A = \omega_A$. Then $\tilde{A}$ has an A_∞-minimal model with $m_p = 0$ for all p except for $p = 2$ or 3.*

In Chapter 4, we show that given a k-connected compact manifold M, we can find an explicit way to obtain the A_∞-minimal model for its de Rham complex $\Omega^*(M)$. This generalizes Miller's result [14] that the manifold is formal if its dimension $n \leq 4k + 2$, and Crowley-Nordström's result that when $n \leq 5k+2$, the homotopy type is determined by its cohomology $H^*(M)$ together with a 3-tensor on $H^*(M)$, which they called the Bianchi-Massey tensor [5].

Theorem 1.3. *Suppose M is an n-dimensional k-connected compact manifold. If $l \geq 3$ such that $n \leq (l + 1)k + 2$, then $\Omega^*(M)$ has an A_∞-minimal model with $m_p = 0$ for $p \geq l$.*

When $l = 3$, this statement is the theorem of Miller [14]. When $l = 4$, it is the theorem of Crowley and Nordström [5]. For higher l, it implies that the homotopy type of $\Omega^*(M)$ is determined by its cohomology together with operations $m_3, \cdots, m_{l-1}$.

This paper is organized as follows. In Chapter 2, we review the definitions and some basic properties of dga, A_∞-algebra and TTY-algebra. In Chapter 3, we prove Theorem 1.1 by first proving Theorem 1.2. Chapter 4 consists of the proof of Theorem 1.3. We conclude in Chapter 5 by stating two conjectures that extend the results of this paper.

Chapter 2

Preliminaries

In this chapter we recall the definition and some basic properties of differential graded algebra, A_∞-algebra and TTY-algebra.

2.1 Differential graded algebras

Definition 2.1. A **differential graded algebra** (**dga**) over a field k is a graded k-algebra $A = \bigoplus_{i \geqslant 0} A^i$ together with a k-linear map $d : A \to A$ such that

i) $k \subset A^0$;

ii) The multiplication is graded commutative: For $x \in A^i, y \in A^j$, we have $x \cdot y = (-1)^{ij} y \cdot x$;

iii) The Leibniz product rule holds: $d(x \cdot y) = dx \cdot y + (-1)^i x \cdot dy$;

iv) $d^2 = 0$.

Example 2.2. Let M be a manifold. Its differential forms form a dga $(\Omega^*(M), d, \wedge)$, where d is the differential operator and $\wedge$ is the wedge product of differential forms.

Example 2.3. If M is a complex manifold, $(\Omega^*(M), \bar{\partial}, \wedge)$ is also a dga. Here $\bar{\partial}$ is the Dolbeault operator.

Also, the subspace $\bigoplus_{q\geq 0} \Omega^{0,q}(M)$ is another dga, since $\bar{\partial}\Omega^{0,q} \subset \Omega^{0,q+1}$ and $\Omega^{0,q} \wedge \Omega^{0,q'} \subset \Omega^{0,q+q'}$.

Example 2.4. Given a dga (A, d), its cohomology $(H^*(A), d)$ is also a dga, with $d \equiv 0$. The multiplication on $H^*(A)$ is naturally induced by the multiplication on A.

Definition 2.5. Let (A, d_A) and (B, d_B) be two dgas. A **dga-homomorphism** is a k-linear map $f : A \to B$ such that

i) $f(A^i) \subset B^i$;

ii) $f(x \cdot y) = f(x) \cdot f(y)$;

iii) $d_B \circ f = f \circ d_A$:

$$\begin{tikzcd} \cdots \arrow[r, "d_A"] & A^k \arrow[r, "d_A"] \arrow[d, "f"] & A^{k+1} \arrow[r, "d_A"] \arrow[d, "f"] & \cdots \\ \cdots \arrow[r, "d_B"] & B^k \arrow[r, "d_B"] & B^{k+1} \arrow[r, "d_B"] & \cdots \end{tikzcd}$$

Naturally, f induces a homomorphism:

$$f^* : H^*(A, d_A) \to H^*(B, d_B).$$

f is called a **dga-quasi-isomorphism** if f^* is an isomorphism.

Definition 2.6. Two dgas (A, d_A) and (B, d_B) are **equivalent** if there exists a sequence of dga-quasi-isomorphisms:

$$\begin{tikzcd}[column sep=small] & (C_1, d_{C_1}) \arrow[dl] \arrow[dr] & & \cdots\cdots \arrow[dl] \arrow[dr] & & (C_n, d_{C_n}) \arrow[dl] \arrow[dr] & \\ (A, d_A) & & (C_2, d_{C_2}) & & \cdots\cdots & & (B, d_B) \end{tikzcd}$$

Definition 2.7. A dga (A, d_A) is called **formal** if (A, d_A) is equivalent to a dga (B, d_B) with $d_B = 0$. Identically, (A, d_A) is equivalent to $(H^*(A), d = 0)$ if and only if (A, d_A) is formal.

We say a manifold M is **formal** if its de Rham complex $(\Omega^*(M), d, \wedge)$ is a formal dga.

Theorem 2.8 (Deligne-Griffiths-Morgan-Sullivan, 1975)**.** *A complex manifold where dd^c-lemma holds is formal. In particular, all Kähler manifolds are formal.*

Example 2.9. Given a set S of degree n, let $\Lambda_n(S)$ denote the polynomial algebra generated by S when n is even, or the exterior algebra generated by S when n is odd.

Suppose x, y, z are all degree 1 elements with $dx = dy = 0$ and $dz = xy$. Let $A = \Lambda_1(x, y, z)$, then A is non-formal.

Geometrically, we can construct a 3-dimensional nilmanifold as follows. Let N^3 denotes the space of upper triangular matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c are real numbers. Let $\Gamma \subset N^3$ be the subgroup of integral matrices. Then set $M^3 = N^3/\Gamma$. The de Rham complex of M^3 is quasi-isomorphic to $A = \Lambda_1(x, y, z)$ in the example above. Therefore, M^3 is non-formal.

Example 2.10. Suppose the degrees of x, y are 2 and the degrees of ϕ, ψ are 3. Let $dx = dy = 0$, $d\phi = x \wedge x$, and $d\psi = x \wedge y$. Then $\Lambda_2(x, y) \otimes \Lambda_3(\phi, \psi)$ is non-formal.

In this example, there is no element of degree 1, so $H^1 = 0$. This examples shows that simply connected does not imply formal.

2.2 A_∞-algebra

Definition 2.11. Let k be a field. An **A_∞-algebra over** k is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{n\in\mathbb{Z}} A^n$ endowed with graded k-linear maps

$$m_p : A^{\otimes p} \to A, \ p \geq 1$$

of degree $2-p$ satisfying

$$\sum_{r+s+t=l} (-1)^{r+st} m_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$

for all $l \geq 1$.

Specially, when $l = 1$, we have

$$m_1 m_1 = 0;$$

When $l = 2$, we have

$$m_1 m_2 = m_2(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1);$$

When $l = 3$, we have

$$m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) = m_1 m_3 + m_3(m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1).$$

If $m_3 = 0$, m_2 is associative.

A dga is a special A_∞-algebra, where m_1 is the differential operator d, m_2 is multiplication, and $m_p = 0$ for all $p \geq 3$.

Definition 2.12. A **morphism of A_∞-algebra** $f : A \to B$ is a family of graded maps

$f_p : A^{\otimes p} \to B$ of degree $1-p$ such that:

$$\sum (-1)^{r+st} f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s^A \otimes \mathbf{1}^{\otimes r}) = \sum (-1)^s m_r^B (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r})$$

for each $l \geq 1$, where the left hand side sum runs over all decompositions $l = r+s+t$, and the right hand side sum runs over all $1 \leq r \leq l$ and all decompositions $l = i_1 + i_2 + \cdots + i_r$. The sign on the right side is given by

$$s = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + 2(i_{r-2} - 1) + (i_{r-1} - 1).$$

Specially, when $l = 1$, we have

$$m_1 f_1 = f_1 m_1.$$

The morphism f is called a **quasi-isomorphism** if f_1^* is an isomorphism.

Theorem 2.13 (Kadeishvili [11][12]). *If (A, m^A) is an A_∞-algebra, then $H^*(A)$ has an A_∞-algebraic structure such that*

i) $m_1 = 0$ and m_2 is induced by m_2^A;

ii) there is a quasi-isomorphism f of A_∞-algebras $H^*(A) \to A$ and f_1^* is the identity of $H^*(A)$.

This structure is unique up to isomorphism of A_∞-algebras.

Definition 2.14. $H^*(A)$ with the structure above is called an A_∞**-minimal model** for A. We say A is **formal** if we can chose all m_p to be 0 for $p \geq 3$ on its A_∞-minimal model.

We will give an explicit construction of the A_∞-minimal model, and this idea will be used for the constructions in later chapters. For convenience, we use following notation:

Let $f : A \to B$ be an A_∞-morphism between algebras. For $p \geq 3$, set

$$
\begin{aligned}
F_p = & \sum_{\substack{i_1+\cdots+i_r=p \\ 2\leq r\leq p}} (-1)^{(r-1)(i_1-1)+(r-2)(i_2-1)+\cdots+1\cdot(i_{r-1}-1)} m_r^B(f_{i_1}\otimes\cdots\otimes f_{i_r}) \\
& - \sum_{\substack{r+s+t=p \\ 2\leq s\leq p-1}} (-1)^{r+st} f_{r+t+1}(1^{\otimes r}\otimes m_s^A \otimes 1^{\otimes t}).
\end{aligned}
\tag{2.1}
$$

F_p is defined by $f_1, \cdots, f_{p-1}, m_1^A, \cdots, m_{p-1}^A$, and $m_1^B, \cdots, m_p^B$. m_p^A and f_p are determined by F_p. $m_p^A = [F_p]$ and f_p needs to satisfy $f_1 m_p^A - m_1^B f_p = F_p$. Therefore, we can define f_p and m_p inductively.

Proof of Theorem 2.13: Let A^E denote the subspace of all the exact forms in A. By Zorn's Lemma, we can find a subspace A^C, such that the subspace of all closed form can be written as $A^E \oplus A^C$. By Zorn's Lemma again, we can find a subspace $A^\perp$ such that $A = A^E \oplus A^C \oplus A^\perp$. Then for each $\alpha \in A^E$, there is a unique $\beta \in A^\perp$ such that $\alpha = d\beta$. So we can define a map $Q : A^E \to A^\perp$ such that $Q\alpha = \beta$. Then $Qd = \mathbf{1}_{A^E}$ and $dQ = \mathbf{1}_{A^\perp}$, where $\mathbf{1}$ is the identity map.

For example, if M is a Riemannian manifold and $A = \Omega^*(A)$, we can set A^C be the space of harmonic forms, $A^\perp = \operatorname{im} d^*$, and $Q = d^*$.

For each $[x] \in H^*(A)$, there exists a unique $x_0 \in A^C$ such that $x_0 \in [x]$. Set $f_1([x]) = x_0$. For f_2, set $f_2 = Q(f_1 m_2 - m_2(f_1 \otimes f_1))$. To define higher m_p and f_p, suppose $m_1, \cdots, m_{p-1}$ on $H^*(A)$ and $f_1, \cdots, f_{p-1}$ are defined, then $m_p = [F_p]$ and f_n needs to satisfy $f_1 m_p - m_1 f_p = F_p$. Set $f_p = Q(f_1 m_p - F_p)$. By induction we can construct an A_∞-minimal model of A. $\square$

By the theorem below, a dga satisfying the definition of formal in the dga sense is equivalent to satisfying the definition of formal as an A_∞-algebra. So in this context we will simply say this dga is formal.

Theorem 2.15 (see [12])**.** *If A is a dga, it is formal as a dga if and only if it is formal as* *an A_∞-algebra.*

2.3 TTY-algebra

Given a $2N$-dimensional symplectic manifold (M, ω), we have three basic operators:

1. $L : \Omega^k \to \Omega^{k+2}$, sends $\alpha \mapsto \omega \wedge \alpha$.
2. $\Lambda : \Omega^k \to \Omega^{k-2}$, sends $\alpha \to \frac{1}{2} \sum (\omega^{-1})^{ij} \iota_{\partial_{x^i}} \iota_{\partial_{x^j}} \alpha$.
3. $H : \Omega^k \to : \Omega^k$, sends $\alpha \to (n-k)\alpha$.

$\{L, \Lambda, H\}$ generates an sl_2 Lie algebra acting on Ω^*.

$$[H, \Lambda] = 2\Lambda, \quad [H, L] = -2L, \quad [\Lambda, L] = H.$$

Definition 2.16. A differential form α is called **primitive** if $\Lambda\alpha = 0$. The set of all primitive k-forms is denoted by P^k.

Theorem 2.17 (Lefschetz Decomposition)**.** *On a $2N$-dimensional symplectic manifold, every $\alpha_k \in \Omega^k$ can be uniquely written as*

$$\alpha_k = \bigoplus_{\substack{j+2s=k \\ j+s \leq N}} L^j \beta_s$$

where $\beta_s \in P^s$. Thus,

$$\Omega^k = \bigoplus_{\substack{j+2s=k \\ j+s \leq N}} L^j P^s.$$

When $j + s > N$, $L^j P^s = 0$. Specially, $P^k = 0$ for $k > N$.

With Lefschetz decomposition, we can define the following operators:

4. $L^{-p} : \Omega^k \to \Omega^{k-2p}$.

$$L^{-p}(L^j \beta_s) = \begin{cases} L^{j-p}\beta_s, & \text{if } j \geq p, \\ 0, & \text{if } j < p. \end{cases}$$

i.e.

$$L^{-p}(\sum L^j \beta_{k-2j}) = \beta_{k-2j-2p} + L\beta_{k-2j-2p-2} + \cdots .$$

5. $*_r : \Omega^k \to \Omega^{2N-k}$.

$$*_r(L^j\beta_s) = L^{N-j-s}\beta_s.$$

6. $\Pi^p : \Omega^k \to \Omega^k$.

$$\Pi^p(L^j \beta_s) = \begin{cases} L^j\beta_s, & \text{if } j \leq p, \\ 0, & \text{if } j > p. \end{cases}$$

i.e.

$$\Pi^p(\sum L^j \beta_{k-2j}) = \beta_k + L\beta_{k-2} + \cdots + L^p\beta_{k-2p}.$$

Definition 2.18. The set of **p-filtered k-forms** are defined by

$$F^p\Omega^k := \Pi^p\Omega^k.$$

Note that $F^0\Omega^k = P^k$ and $F^N\Omega^k = \Omega^k$.

For each $L^j\beta_s \in L^jP^s$, $d(L^j\beta_s) \in L^jP^{s+1} \oplus L^{j+1}P^{s-1}$. Thus we can decompose d as

$$d = \partial_+ + L\partial_-,$$

where $\partial_+ : L^jP^s \to L^jP^{s+1}$ and $\partial_- : L^jP^s \to L^jP^{s-1}$. ∂_+ and ∂_- have the following properties:

$$\partial_+^2 = \partial_-^2 = 0, \quad [L, \partial_+] = [L, L\partial_-] = 0, \quad L\partial_+\partial_- = -L\partial_-\partial_+.$$

For p-filtered forms, we can define

$$
\begin{aligned}
d_+ =& \Pi^p \circ d, \\
d_- =& *_r \, d *_r,
\end{aligned}
$$

where $d_+ : F^p\Omega^k \to F^p\Omega^{k+1}$ and $d_- : F^p\Omega^k \to F^p\Omega^{k-1}$. When $p = 0$, $d_\pm$ are exactly $\partial_\pm$. We also have $d_+^2 = d_-^2 = 0$. Furthermore, $(\partial_+\partial_-)d_+ = d_-(\partial_+\partial_-) = 0$. Thus, we have the following cochain complex [18]:

$$
\begin{tikzcd}
0 \arrow[r] & F^p\Omega^0_+ \arrow[r, "d_+"] & F^p\Omega^1_+ \arrow[r, "d_+"] & \cdots \arrow[r, "d_+"] & F^p\Omega^{N+p}_+ \arrow[d, "\partial_+\partial_-"] \\
0 & F^p\Omega^0_- \arrow[l] & F^p\Omega^1_- \arrow[l, "d_-"'] & \cdots \arrow[l, "d_-"'] & F^p\Omega^{N+p}_- \arrow[l, "d_-"']
\end{tikzcd}
$$

where $F^p\Omega^k_\pm = F^p\Omega^k$.

The above cochain complex has an A_∞-algebra structure, constructed by Tsai, Tseng and Yau [18]. For simplicity we call it **TTY-algebra**. The A_∞-algebra is $(F^p\Omega^*, m_l)$, where the operations m_l are defined below:

The m_1 equation.

$$
m_1\alpha = \begin{cases} d_+\alpha, & \text{if } \alpha \in F^p\Omega^k_+ \text{ and } k < N, \\ -\partial_+\partial_-\alpha, & \text{if } \alpha \in F^p\Omega^N_+, \\ -d_-\alpha, & \text{if } \alpha \in F^p\Omega^k_-. \end{cases}
$$

The m_2 equation. For $m_2(\alpha_1, \alpha_2)$, when $\alpha_1 \in F^p\Omega^{k_1}_+$ and $\alpha_2 \in F^p\Omega^{k_2}_+$, we set

$$
m_2(\alpha_1,\alpha_2) = \Pi^p(\alpha_1\wedge\alpha_2) + \Pi^p *_r \big(-dL^{-(p+1)}(\alpha_1\wedge\alpha_2) + (L^{-(p+1)}\alpha_1)\wedge\alpha_2 + (-1)^{k_1}\alpha_1\wedge(L^{-(p+1)}\alpha_2)\big).
$$

Actually, if $k_1 + k_2 \leq N + p$, the second term is 0. If $k_1 + k_2 > N + p$, the first term is 0.

When $\alpha_1 \in F^p\Omega_+^{k_1}$ and $\alpha_2 \in F^p\Omega_-^{k_2}$, we set

$$m_2(\alpha_1, \alpha_2) = (-1)^{k_1} *_r \big(\alpha_1 \wedge (*_r\alpha_2)\big).$$

When $\alpha_1 \in F^p\Omega_-^{k_1}$ and $\alpha_2 \in F^p\Omega_+^{k_2}$, we set

$$m_2(\alpha_1, \alpha_2) = *_r\big((*_r\alpha_1) \wedge \alpha_2\big).$$

When $\alpha_1 \in F^p\Omega_-^{k_1}$ and $\alpha_2 \in F^p\Omega_-^{k_2}$, we set

$$m_2(\alpha_1, \alpha_2) = 0$$

By definition, m_2 is graded commutative:

$$m_2(\alpha_1, \alpha_2) = (-1)^{k_1k_2} m_2(\alpha_2, \alpha_1).$$

The m_3 equation. For $m_3(\alpha_1, \alpha_2, \alpha_3)$, when $\alpha_i \in F^p\Omega_+^{k_i}$ for all $1 \leq i \leq 3$ and $k_1+k_2+k_3 \geq N + p + 2$, we set

$$m_3(\alpha_1, \alpha_2, \alpha_3) = \Pi^p *_r \big(\alpha_1 \wedge L^{-(p+1)}(\alpha_2 \wedge \alpha_3) - L^{-(p+1)}(\alpha_1 \wedge \alpha_2) \wedge \alpha_3\big).$$

For the other cases, we set

$$m_3(\alpha_1, \alpha_2, \alpha_3) = 0.$$

The higher m_l equation. When $l \geq 4$, we set $m_l = 0$.

The TTY-algebra is quasi-isomorphic to a dga: the mapping cone of

$$\Omega^*(M)[-2p-2] \to \Omega^*(M), \quad \alpha \mapsto \omega^{p+1} \wedge \alpha,$$

Alternatively, this mapping cone can be viewed as the differential forms of a S^{2p+1} sphere bundle over M.

Algebrically, this mapping cone is an extension of $\Omega^*(M)$, by an element θ of degree $2p+1$ such that $d\theta = \omega^{p+1}$. Let $A = \Omega^*(M)$ and set the extension

$$\tilde{A} = A \otimes \Lambda_{2p+1}(\theta)$$

or equivalently write it as

$$\tilde{A} = \{\xi + \theta\eta |\ \xi, \eta \in A\}.$$

The dga structure on $\tilde{A}$ is given by

$$d(\xi + \theta\eta) = d\xi + \omega^{p+1} \wedge \eta - \theta(d\eta)$$

and

$$\begin{aligned}
\xi \wedge \theta\eta =& (-1)^{|\xi|}\theta(\xi \wedge \eta) \\
\theta\xi \wedge \eta =& \theta(\xi \wedge \eta) \\
\theta\xi \wedge \theta\eta =& 0,
\end{aligned}$$

where $\xi, \eta \in A$ and $|\xi|$ is the degree of ξ.

Theorem 2.19 (Tanaka and Tseng [17]). *$F^p\Omega^*(M)$ is quasi-isomorphic to $\tilde{A}$.*

Chapter 3

Minimal model on an extension of formal dga

Since the TTY-algebra $F^p\Omega^*$ is quasi-isomorphic to $\tilde{A} = \Omega^*(M) + \theta\Omega^*(M)$, we can consider the homotopy type of $\tilde{A}$ instead. We will show that when M is formal, $\tilde{A}$ is quasi-isomorphic to the extension of $H^*(A)$ by θ. Then we can consider a much simpler dga.

Theorem 3.1. *Suppose A, B are two dgas and $f : A \to B$ is a quasi-isomorphism. $\omega_A \in A, \omega_B \in B$ are d-closed even-degree elements such that $f^*([\omega_A]) = [\omega_B]$. Extend A, B to $\tilde{A} = \{\alpha + \theta_A\beta |\ \alpha, \beta \in A\}$ with $d\theta_A = \omega_A$ and $\tilde{B} = \{x + \theta_B y |\ x, y \in B\}$ with $d\theta_B = \omega_B$. Then there exists a quasi-isomorphism $g : \tilde{A} \to \tilde{B}$.*

Proof. Without loss of generality, we can assume $f(\omega_A) = \omega_B$. Otherwise, by assumption $\omega_B = f(\omega_A) + dr$ for some $r \in A$. Then we can consider $\omega'_B = \omega_B - dr$ and $\theta'_B = \theta_B - r$ instead of ω_B and θ_B.

Set

$$g(\alpha + \theta_A\beta) = f(\alpha) + \theta_B f(\beta).$$

It is easy to check that g is linear, preserves wedge products and $gd_A = d_B g$. It remains to show that g^* is bijective.

1) g^* is injective.

Suppose $\alpha + \theta_A \beta$ is closed in $\tilde{A}$ and $g^*[\alpha + \theta_A \beta] = 0$. There exists $x, y \in B$ such that

$$d(x + \theta_B y) = g(\alpha + \theta_A \beta).$$

Thus,

$$dx + \omega_B \wedge y - \theta_B dy = f(\alpha) + \theta_B f(\beta).$$

So we have

$$dx + \omega_B \wedge y = f(\alpha) \quad \text{and} \quad dy = -f(\beta).$$

On the other hand,

$$0 = d(\alpha + \theta_A \beta) = d\alpha + \omega_A \wedge \beta - \theta_A d\beta.$$

Hence, β is closed and $\omega_A \wedge \beta = -d\alpha$. Since $f^*[\beta] = -[dy] = 0$, β must be exact in A.

Assume $\eta \in A$ such that $d\eta = \beta$, by

$$d(y + f(\eta)) = -f(\beta) + f(d\eta) = 0$$

and f^* is surjective, there exists $\xi \in A$ and $z \in B$ such that

$$d\xi = 0 \quad \text{and} \quad f(\xi) = y + f(\eta) + dz.$$

Then

$$
\begin{aligned}
f(\alpha + \omega_A \wedge \eta) =& f(\alpha) + \omega_B \wedge f(\eta) \\
=& dx + \omega_B \wedge y + \omega_B \wedge f(\eta) \\
=& dx + \omega_B \wedge (f(\xi) - dz) \\
=& f(\omega_A \wedge \xi) + dx - dz.
\end{aligned}
$$

So $f(\alpha + \omega_A \wedge \eta - \omega_A \wedge \xi) = dx - dz$ is exact in B. Also

$$d(\alpha + \omega_A \wedge \eta - \omega_A \wedge \xi) = d\alpha + \omega_A \wedge \beta = 0.$$

Hence, $\alpha + \omega_A \wedge \eta - \omega_A \wedge \xi$ is exact since f^* is injective. Let $\gamma \in A$ such that

$$d\gamma = \alpha + \omega_A \wedge \eta - \omega_A \wedge \xi.$$

Therefore,

$$
\begin{aligned}
\alpha + \theta_A \beta =& d\gamma - \omega_A \wedge \eta + \omega_A \wedge \xi + \theta_A d\eta \\
=& d\gamma - d(\theta_A \eta) + d(\theta_A \xi),
\end{aligned}
$$

which is exact in A. That shows g^* is injective.

2) g^* is surjective.

Given arbitrary closed $x + \theta_B y \in \tilde{B}$, we have

$$dx + \omega_B \wedge y = 0 \quad \text{and} \quad dy = 0.$$

As y is closed and f^* is surjective, there exists $\beta \in A, z \in B$ such that

$$f(\beta) = y + dz \quad \text{and} \quad d\beta = 0.$$

Then

$$f(\omega_A \wedge \beta) = \omega_B \wedge (y + dz) = -dx + d(\omega_B \wedge z).$$

Since $\omega_A \wedge \beta$ is closed and f^* is injective, $\omega_A \wedge \beta$ must be exact. So there exists $\alpha \in A$ such that $d\alpha = \omega_A \wedge \beta$. Thus,

$$\begin{aligned} d(x - \omega_B \wedge z + f(\alpha)) &= -\, \omega_B \wedge y - \omega_B \wedge dz + f(d\alpha) \\ &= -\, \omega_B \wedge (y + dz) + f(\omega_A \wedge \beta) \\ &= -\, \omega_B \wedge f(\beta) + \omega_B \wedge f(\beta) \\ &= 0. \end{aligned}$$

So there exists $\xi \in A$ and $w \in B$ such that

$$f(\xi) = x - \omega_B \wedge z + f(\alpha) + dw \quad \text{and} \quad d\xi = 0.$$

Therefore,

$$\begin{aligned} & f(\xi - \alpha) + \theta_B f(\beta) \\ =& x - \omega_B \wedge z + dw + \theta_B(y + dz) \\ =& x + \theta_B \wedge y - d(\theta_B z) + dw \end{aligned}$$

i.e.

$$g^*[\xi - \alpha + \theta_A \beta] = [x + \theta_B y].$$

Thus, g^* is surjective. $\square$

When A is formal, there exists a zigzag of quasi-isomorphisms. We can extend each isomorphism by the previous theorem, and obtain the following:

Corollary 3.2. *Suppose A is a formal dga. $\omega \in A$ is a closed even-degree element. $\tilde{A} = \{\alpha + \theta_A \beta |\ \alpha, \beta \in A\}$, where $d\theta_A = \omega_A$. $\tilde{A}$ is quasi-isomorphic to the extension of $H^*(A)$: $\{x + \theta_H y |\ x, y \in H^*(A)\}$, where $d\theta_H = [\omega_A]$.*

Identically, every extension of a formal dga by an odd-degree element is quasi-isomorphic to the extension of a dga A whose differential is 0. We then construct an A_∞-minimal model for the extension of A.

Theorem 3.3. *Suppose A is a dga and $d_A = 0$. $\omega_A \in A$ is an even-degree element. Let $\tilde{A} = \{\alpha + \theta_A \beta |\ \alpha, \beta \in A\}$ with $d\theta_A = \omega_A$. Then $\tilde{A}$ has an A_∞-minimal model with $m_p = 0$ for all p except for $p = 2$ or 3.*

Proof. Since $d_A = 0$, for arbitrary $\alpha, \beta \in A$, $\alpha + \theta_A \beta$ is closed if and only if $\omega_A \wedge \beta = 0$. It is exact if and only if $\alpha \in I(\omega_A)$ and $\beta = 0$, where $I(\omega_A) = \{\omega \wedge \alpha |\ \alpha \in A\}$ is the ideal generated by ω_A in A. Thus,

$$H^*(\tilde{A}) = \big(A/I(\omega_A)\big) \oplus \ker L,$$

where $L : A \to A$ such that $L\alpha = \omega_A \wedge \alpha$.

1) Defining of f_1.

Decompose $A = I(\omega_A) \oplus A^C$ for some subspace A^C of A. For each cohomology class $[\alpha + \theta_A \beta]$ in $H^*(\tilde{A})$, by the discussion above, there exists unique $\alpha_0 \in A^C, \beta_0 \in \ker L$ such that $\alpha_0 + \theta_A \beta_0 \in [\alpha + \theta_A \beta]$. So we can set

$$\begin{aligned} f_1 : \quad & H^*(\tilde{A}) \longrightarrow \tilde{A} \\ & [\alpha + \theta_A \beta] \longmapsto \alpha_0 + \theta_A \beta_0. \end{aligned}$$

It is easy to verify f_1 is a quasi-isomorphism.

2) Defining of f_2.

Given another decomposition of A by $A = \ker L \oplus A^{\perp}$ for some subspace $A^{\perp}$ of A. For each $\alpha \in I(\omega_A)$, there exists a unique $\beta \in A^{\perp}$ such that $\alpha = \omega_A \wedge \beta$. So we can define a map $Q : I(\omega_A) \to \theta_A A$ by $Q(\alpha) = \theta_A \beta$., then define f_2:

$$f_2(x, y) = Q\big(f_1 m_2(x, y) - f_1(x) \wedge f_1(y)\big).$$

Such f_2 is well-defined. Suppose

$$f_1(x) = \alpha + \theta_A \beta, \quad f_1(y) = \xi + \theta_A \eta,$$

then

$$m_2(x, y) = [f_1(x) \wedge f_1(y)] = [\alpha \wedge \xi + \theta_A \beta \wedge \xi + (-1)^{|\alpha|} \theta_A \alpha \wedge \eta].$$

Hence,

$$f_1 m_2(x, y) = f_1([\alpha \wedge \xi]) + \theta_A \beta \wedge \xi + (-1)^{|\alpha|} \theta_A \alpha \wedge \eta,$$

and

$$f_1 m_2(x, y) - f_1(x) \wedge f_1(y) = f_1([\alpha \wedge \xi]) - \alpha \wedge \xi \in I(\omega_A).$$

As dQ is the identity map on $I(\omega_A)$, f_2 satisfies the equation

$$m_1 f_2 = dQ\big(f_1 m_2 - m_2(f_1 \otimes f_1)\big) = f_1 m_2 - m_2(f_1 \otimes f_1).$$

3) Defining of m_3 and f_3.

m_3 and f_3 need to satisfy

$$f_1 m_3 - m_1 f_3 = F_3 = m_2(f_1 \otimes f_2 - f_2 \otimes f_1) - f_2(m_2 \otimes \mathbf{1} - \mathbf{1} \otimes m_2),$$

and m_3 is the cohomology class of F_3. By the definition of f_2, its image is in $I(\theta_A)$, which is the ideal generated by θ_A in $\tilde{A}$. Hence, for any $x, y, z \in H^*(\tilde{A})$,

$$m_2(f_1 \otimes f_2 - f_2 \otimes f_1)(x, y, z) - f_2(m_2 \otimes \mathbf{1} - \mathbf{1} \otimes m_2)(x, y, z) = \theta_A \alpha$$

for some $\alpha \in A$. Thus,

$$m_3(x, y, z) = [\theta_A \alpha], \quad \text{and} \quad f_1 m_3(x, y, z) = f_1([\theta_A \alpha]) = \theta_A \alpha.$$

Therefore, $m_1 f_3(x, y, z) = 0$, and we can set $f_3 = 0$.

4) Triviality of m_4 and f_4.

As $f_3 = 0$, m_4 and f_4 need to satisfy

$$f_1 m_4 - m_1 f_4 = -m_2(f_2 \otimes f_2) + f_2(m_3 \otimes \mathbf{1} + \mathbf{1} \otimes m_3).$$

We claim $m_2(f_2 \otimes f_2) = 0$ since $\operatorname{im} f_2 \in I(\theta_A)$ and $\theta_A \wedge \theta_A = 0$. On the other hand, for any $x, y, z, w \in H^*(\tilde{A})$, we can assume

$$m_3(x, y, z) = \theta\alpha \quad \text{and} \quad f_1(w) = \beta + \theta\gamma$$

for some $\alpha, \beta, \gamma \in A$. Then

$$f_1 m_2\big(m_3(x, y, z), w\big) = f_1\big([\theta_A \alpha \wedge (\beta + \theta\gamma)]\big) = f_1[\theta_A \alpha\beta] = \theta_A \alpha\beta,$$

and

$$m_2\big(f_1 m_3(x,y,z), f_1(w)\big) = f_1\big([\theta_A \alpha]\big) \wedge (\beta + \theta_A \gamma) = \theta_A \alpha\beta.$$

Hence, $m_1 f_2\big(m_3(x,y,z), w\big) = 0$. By previous discussion we have $f_2 = Q m_1 f_2$, so $f_2(m_3 \otimes \mathbf{1}) = 0$. Similarly, $f_2(\mathbf{1} \otimes m_3) = 0$.

Therefore, $m_4 = 0$ and we can set $f_4 = 0$.

5) Triviality of higher m_p and f_p.

For higher degrees, we will prove $m_p = 0$ and $f_p = 0$ by induction. Suppose $m_p = 0$ on $H^*(\tilde{A})$ for $4 \leq p \leq n-1$ and $f_p = 0$ for $3 \leq p \leq n-1$, where $n \geq 5$. m_n and f_n need to satisfy

$$\begin{aligned}
&f_1 m_n - m_1 f_n \\
=& \sum_{\substack{i_1+\cdots+i_r=n \\ r\geq 2}} (-1)^{\delta_1} m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}) - \sum_{\substack{r+s+t=n \\ 2\leq s\leq n-1}} (-1)^{\delta_2} f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) \\
=& \sum_{i=1}^{n-1} (-1)^{\delta_1} m_2(f_i \otimes f_{n-i}) - \sum_{r=0}^{1} (-1)^{\delta_2} f_2(\mathbf{1}^{\otimes r} \otimes m_{n-1} \otimes \mathbf{1}^{\otimes (1-r)})
\end{aligned}$$

where $\delta_1 = \sum_{t=1}^{r} (n-t)(i_t - 1)$ and $\delta_2 = r + st$.

Since $n \geq 5$, either $i \geq 3$ or $n - i \geq 3$, so $m_2(f_i \otimes f_{n-i}) = 0$. Also, $n - 1 \geq 4$. So $m_{n-1} = 0$. That implies $f_1 m_n - m_1 f_n = 0$. Therefore, $m_n = 0$ and we can take $f_n = 0$. $\square$

By the previous theorem, we have the following statement for formal dga.

Theorem 3.4. *Suppose A is a formal dga. $\omega_A \in A$ is an even-degree element. Extend A to $\tilde{A} = \{\alpha + \theta_A \beta |\ \alpha, \beta \in A\}$ with $d\theta_A = \omega_A$. Then $\tilde{A}$ has an A_∞-minimal model with $m_p = 0$ for all p except for $p = 2$ or 3.*

By Theorem 2.19, the TTY-algebra of a symplectic manifold M is quasi-isomorphic to $\tilde{A}$, where $A = \Omega^*(M)$. A is a formal dga when M is formal. So we have

Theorem 3.5. *Suppose (M, ω) is a formal symplectic manifold. Its TTY-algebra is formal if the symplectic form ω is exact. When ω is non-exact, its TTY-algebra has an A_∞-minimal model with $m_p = 0$ for all p except for $p = 2$ or 3.*

When ω is non-exact, the TTY-algebra may be or may not be formal. There are examples for both cases. The TTY-algebra of primitive forms ($p = 0$ in Definition 2.18) on a projective space is formal, but the TTY-algebra of primitive forms on a torus is not.

Example 3.6. Let $A = \Omega^*(\mathbb{C}P^N)$ be the space of differential forms on a complex projective space $(\mathbb{C}P^N, \omega)$. The TTY-algebra of primitive forms on $\mathbb{C}P^N$ is quasi-isomorphic to the extension $\tilde{A} = \{\alpha + \theta\beta | \alpha, \beta \in A\}$, where $d\theta = \omega$. The dga A is formal since $\mathbb{C}P^N$ is Kähler. By Corollary 3.2, $\tilde{A}$ is quasi-isomorphic to the extension of $H^*(A)$, which is $B = \{x + \theta_H y | x, y \in H^*(A)\}$ and $d\theta_H = [\omega]$. B has an A_∞-minimal model with only m_2 and m_3 non-trivial.

$$H^i(A) = \begin{cases} \langle [\omega^p] \rangle, & \text{if } i = 2p, 0 \le i \le 2N \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$B^i = \begin{cases} \langle [\omega^p] \rangle, & \text{if } i = 2p, 0 \le p \le N \\ \langle \theta_H [\omega^p] \rangle, & \text{if } i = 2p+1, 0 \le p \le N \\ 0, & i > 2N+1 \end{cases}$$

Since $[\omega^p] = d(\theta_H[\omega^{p-1}])$, $H^i(B)$ must be trivial except for $i = 0$ or $2N + 1$. For any $x, y, z \in H^*(B)$, the total degree of $m_3(x, y, z)$ can only be $k(2N + 1) - 1$ where k is the number of x, y, z in H^{2N+1}, i.e. $0 \le k \le 3$. As $k(2N+1)-1 \ne 0$ or $2N+1$, $m_3(x, y, z)$ must be 0. That implies m_3 is trivial in the A_∞-minimal model. Therefore, the TTY-algebra of primitive forms on $\mathbb{C}P^N$ is formal.

Example 3.7. We use the same notations as the previous example. Let $A = \Omega^*(T^{2N})$ be the space of differential forms on a torus (T^{2N}, ω). The TTY-algebra of primitive forms on T^{2N}, ω is quasi-isomorphic to the extension $\tilde{A} = \{\alpha + \theta\beta | \alpha, \beta \in A\}$, where $d\theta = \omega$. A is formal since T^{2N} is Kähler. By Corollary 3.2, $\tilde{A}$ is quasi-isomorphic to the extension of $H^*(A)$, which is $B = \{x + \theta_H y | x, y \in H^*(A)\}$ and $d\theta_H = [\omega]$. Let $f : H^*(B) \to B$ be an A_∞-quasi-isomorphism.

On a torus, we can find a basis $\{e_1, \cdots, e_{2N}\}$ of $H^1(A)$ such that

$$[\omega] = \sum_{j=1}^{N} e_{2j-1} \wedge e_{2j}.$$

Let

$$y = \prod_{j=2}^{N} e_{2j-1} = e_3 \wedge e_5 \wedge \cdots \wedge e_{2N-1}.$$

we will prove that $m_3([ye_1], [e_2], [e_2])$ is non-trivial.

Note that ye_1 is in $B^N = H^N(A) \oplus \theta_H H^{N-1}(A)$. Since the map $L : H^{N-1}(A) \to H^{N+1}(A)$ by wedging $[\omega]$ is injective, the only closed form in $\theta_H H^{N-1}(A)$ is 0. So the subspace of closed forms in B^N is $H^N(A)$. The subspace of exact forms in B^N is the ideal generated by $[\omega]$ in $H^N(A)$. Thus, $[ye_1]$ is a non-trivial cohomology class. It follows that $f_1([ye_1]) = ye_1 + [\omega]z$ for some $z \in H^{N-2}(A)$.

On the other hand, e_2 is in $B^1 = H^1(A) \oplus \theta_H H^0(A)$. In B^1, the subspace of closed forms is $H^1(A)$ and the only exact form is 0. So $[e_2]$ is also a non-trivial cohomology class and $f_1([e_2]) = e_2$.

By $(ye_1) \wedge e_2 = [\omega] \wedge y = d(\theta_H y)$, we have $m_2([ye_1], [e_2]) = 0$ and

$$m_1 f_2([ye_1], [e_2]) = -f_1([ye_1]) \wedge f_1([e_2]) = -((ye_1) \wedge e_2 + [\omega] z e_2) = -d(\theta_H(y + z e_2)).$$

It follows that $f_2([ye_1], [e_2]) = -\theta_H(y + ze_2) + x_1$, where x_1 is a closed form in B^N. By the discussion above, $x_1 \in H^N(A)$.

Since $e_2 \wedge e_2 = 0$, we have $m_2([e_2],[e_2]) = 0$ and $f_2([e_2],[e_2]) = x_2$ for some closed form $x_2 \in B^1$. Hence, $x_2 \in H^1(A)$. Then

$$\begin{aligned} F_3([ye_1],[e_2],[e_2]) =& \big(m_2(f_1 \otimes f_2 - f_2 \otimes f_1) - f_2(m_2 \otimes 1 - 1 \otimes m_2)\big)([ye_1],[e_2],[e_2]) \\ =&(-1)^N f_1([ye_1]) \wedge f_2([e_2],[e_2]) - f_2([ye_1],[e_2]) \wedge f_1([e_2]) \\ =&(-1)^N (ye_1 + [\omega]z)x_2 - (-\theta_H(y + ze_2) + x_1)e_2 \end{aligned}$$

All elements in $H^{N+1}(A)$ are in the ideal of $[\omega]$. So they are exact in B^{N+1}. Thus, $m_3([ye_1],[e_2],[e_2]) = F_3([ye_1],[e_2],[e_2]) = [\theta_H(y + ze_2)e_2] = [\theta_H ye_2]$. Since $\theta_H ye_2$ is closed but not exact in B^{N+1}, $m_3([ye_1],[e_2],[e_2])$ cannot be 0. Therefore, B is not formal so that the TTY-algebra of primitive forms on T^{2N} is not formal.

Actually, when $N = 1$, the TTY-algebra of primitive forms is quasi-isomorphic to Example 2.9.

Chapter 4

Minimal model on k-connected compact manifold

We here recall a result of Miller for k-connected compact manifolds. A manifold M is called k-connected if it is path-connected and its homotopy group $\pi_r(M) = 0$ for $1 \leq r \leq k$. Our goal in this chapter is to generalize Miller's result.

Theorem 4.1 (Miller [14])**.** *Let M be an n-dimensional k-connected compact manifold. If $n \leq 4k+2$, then M is formal.*

Given an A_∞-algebra (A, m_p^A), by Theorem 2.13 we can construct an A_∞-minimal model on $H^*(A)$ and a quasi-isomorphism $f : H^*(A) \to A$ such that $f_p(x_1, \cdots, x_p) = 0$ when $m_1 f_p(x_1, \cdots, x_p) = 0$. This quasi-isomorphism is well-defined because f_p needs to satisfy $f_1 m_p - m_1 f_p = F_p$, where F_p is defined in (2.1) and $m_p = [F_p]$. In this chapter we use $M(A)$ to denote the A_∞-minimal model and $f : M(A) \to A$ to denote the specific quasi-isomorphism constructed in this way, i.e. $f_p(x_1, \cdots, x_p) = 0$ when $m_1 f_p(x_1, \cdots, x_p) = 0$.

Given such M_A and f, suppose that $x_1, \cdots, x_p \in M(A)$ such that $|x_1| + \cdots + |x_p| \geq n+p-1$ for some $p \geq 2$. Then $|F_p(x_1, \cdots, x_p)| \geq n+1$. Hence, $F_p(x_1, \cdots, x_p) = 0$, and $f_p(x_1, \cdots, x_p) = 0$. Therefore, $f_p = 0$ when the total degree is greater than or equal to $n+p-1$. That is, $f_p(x_1, \cdots, x_p) = 0$ for $|x_1| + \cdots + |x_p| \geq n+p-1$. Based on this, we have the following lemma:

Lemma 4.2. *Suppose A is a dga. Let $x_1, \cdots, x_p \in M(A)$ with degree $r_1, \cdots, r_p$ respectively. $r_1 + \cdots + r_p = n+p-2$. Then the following cyclic sum is 0.*

$$\sum_{j=1}^{p} (-1)^{\alpha_j} F_p(x_j, \cdots, x_n, x_1, \cdots, x_{j-1}) = 0,$$

where

$$\alpha_j = (j-1)(p+1) + (r_1 + \cdots + r_{j-1})(n-p+1).$$

So when $p \geq 3$,

$$\sum_{j=1}^{p} (-1)^{\alpha_j} m_p(x_j, \cdots, x_n, x_1, \cdots, x_{j-1}) = 0.$$

Proof: For convenience, set $x_{j+p} = x_j$ for each j. Then we need to show

$$\sum_{j=1}^{p} (-1)^{\alpha_j} F_p(x_j, \cdots, x_{j+p-1}) = 0.$$

For each j,

$$\begin{aligned} & F_p(x_j, \cdots, x_{j+p-1}) \\ =& \Big(\sum_{a=1}^{p-1} (-1)^{a-1} m_2^A (f_a \otimes f_{p-a}) - \sum_{\substack{a+b+c=p \\ 2 \leq b \leq p-1}} (-1)^{a+bc} f_{a+c+1} (\mathbf{1}^{\otimes a} \otimes m_b \otimes \mathbf{1}^{\otimes c}) \Big)(x_j, \cdots, x_{j+p-1}). \end{aligned}$$

The degree of $(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c})(x_j, \cdots, x_{j+p-1})$ is

$$(2-b) + (n+p-2) = n+p-b = n + (a+c+1) - 1.$$

Thus, $f_{a+c+1}(\mathbf{1}^{\otimes a} \otimes m_b \otimes \mathbf{1}^{\otimes c})(x_j, \cdots, x_{j+p-1}) = 0$. Then

$$\begin{aligned}
&\sum_{j=1}^{p} (-1)^{\alpha_j} F_p(x_j, \cdots, x_{j+p-1}) \\
= &\sum_{1 \leq a \leq p-1} \sum_{j=1}^{p} (-1)^{a-1+\alpha_j} m_2^A(f_a \otimes f_{p-a})(x_j, \cdots, x_{j+p-1})
\end{aligned}$$

Set

$$\Phi_{a,j} = (-1)^{a-1+\alpha_j} m_2^A(f_a \otimes f_{p-a})(x_j, \cdots, x_{j+p-1}).$$

Since $x_{j+p} = x_j$, we have $\Phi_{a,j+p} = \Phi_{a,j}$. We will show that $\Phi_{a,j} + \Phi_{p-a,j+a} = 0$.

For each $1 \leq a \leq p-1$ and $1 \leq j \leq p$, $\Phi_{a,j}$ can be written as

$$(-1)^{\xi} f_a(x_j, \cdots, x_{j+a-1}) \wedge f_{p-a}(x_{j+a}, \cdots, x_{j+p-1}),$$

where

$$\xi = a - 1 + \alpha_j + (1 - (p-a))(r_j + \cdots + r_{j+a-1}).$$

On the other hand, we can write $\Phi_{p-a,j+a}$ as

$$(-1)^{\eta} f_a(x_{j+p}, \cdots, x_{j+p+a-1}) \wedge f_{p-a}(x_{j+a}, \cdots, x_{j+p-1}).$$

We determine η now. By definition,

$$\Phi_{p-a,j+a} = (-1)^{p-a-1+\alpha_{j+a}} m_2^A(f_{p-a} \otimes f_a)(x_{j+a}, \cdots, x_{j+a+p-1}).$$

Since

$$
\begin{aligned}
&m_2^A(f_{p-a}\otimes f_a)(x_{j+a},\cdots,x_{j+a+p-1})\\
=&(-1)^{(1-a)(r_{j+a}+\cdots r_{j+p-1})}f_{p-a}(x_{j+a},\cdots,x_{j+p-1})\wedge f_a(x_{j+p},\cdots,x_{j+p+a-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
&f_{p-a}(x_{j+a},\cdots,x_{j+p-1})\wedge f_a(x_{j+p},\cdots,x_{j+p+a-1})\\
=&(-1)^{(1-(p-a)+r_{j+a}+\cdots+r_{j+p-1})(1-a+r_{j+p}+\cdots+r_{j+p+a-1})}\\
&\cdot f_a(x_{j+p},\cdots,x_{j+p+a-1})\wedge f_{p-a}(x_{j+a},\cdots,x_{j+p-1}),
\end{aligned}
$$

we have

$$
\begin{aligned}
\eta =&p-a-1+\alpha_{j+a}+(1-a)(r_{j+a}+\cdots r_{j+p-1})\\
&+(1-(p-a)+r_{j+a}+\cdots+r_{j+p-1})(1-a+r_{j+p}+\cdots+r_{j+p+a-1})
\end{aligned}
$$

As $r_j+\cdots+r_{j+p-1}=n+p-2$, the last term of η is

$$
\begin{aligned}
&(1-(p-a)+r_{j+a}+\cdots+r_{j+p-1})(1-a+r_{j+p}+\cdots+r_{j+p+a-1})\\
=&(1-p+a+r_{j+a}+\cdots+r_{j+p-1})(1-a)\\
&+(1-p+a+r_{j+a}+\cdots+r_{j+p-1})(r_j+\cdots+r_{j+a-1})\\
=&(1-p+a)(1-a)+(r_{j+a}+\cdots+r_{j+p-1})(1-a)\\
&+(1-p+a+n+p-2-r_j-\cdots-r_{j+a-1})(r_j+\cdots+r_{j+a-1})\\
=&(1-p+a)(1-a)+(r_{j+a}+\cdots+r_{j+p-1})(1-a)\\
&+(n+a-1)(r_j+\cdots+r_{j+a-1})-(r_j+\cdots+r_{j+a-1})^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(-1)^{\eta}\\
=&(-1)^{p-a-1+\alpha_{j+a}+(1-a)(r_{j+a}+\cdots r_{j+p-1})+(1-p+a)(1-a)+(r_{j+a}+\cdots+r_{j+p-1})(1-a)}\\
&\cdot(-1)^{(n+a-1)(r_j+\cdots+r_{j+a-1})-(r_j+\cdots+r_{j+a-1})^2}\\
=&(-1)^{-(1-p+a)+\alpha_{j+a}+(1-p+a)(1-a)+(n+a-1)(r_j+\cdots+r_{j+a-1})-(r_j+\cdots+r_{j+a-1})}\\
=&(-1)^{-a(1-p+a)+\alpha_{j+a}+(n+a)(r_j+\cdots+r_{j+a-1})}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(-1)^{\xi+\eta}=&(-1)^{a-1+\alpha_j+(1-p+n+2a)(r_j+\cdots+r_{j+a-1})-a(1-p)+a^2+\alpha_{j+a}}\\
=&(-1)^{-1+a(p-1)+(n-p+1)(r_j+\cdots+r_{j+a-1})+\alpha_j+\alpha_{j+a}}.
\end{aligned}
$$

As

$$
\begin{aligned}
&\alpha_j+\alpha_{j+a}\\
=&(j-1)(p+1)+(r_1+\cdots+r_{j-1})(n-p+1)+(j+a-1)(p+1)\\
&+(r_1+\cdots+r_{j+a-1})(n-p+1)\\
=&(2j+a-2)(p+1)+2(r_1+\cdots+r_{j-1})(n-p+1)+(r_j+\cdots+r_{j+a-1})(n-p+1),
\end{aligned}
$$

we have

$$
\begin{aligned}
(-1)^{\alpha_j+\alpha_{j+a}}=&(-1)^{a(p+1)+(r_j+\cdots+r_{j+a-1})(n-p+1)}\\
=&(-1)^{a(p-1)+(n-p+1)(r_j+\cdots+r_{j+a-1})}
\end{aligned}
$$

Therefore, $(-1)^{\xi+\eta}=-1$. Then $(-1)^{\xi}\Phi_{a,j}=(-1)^{\eta}\Phi_{p-a,j+a}$ and $\Phi_{a,j}+\Phi_{p-a,j+a}=0$.

When $a \neq \frac{p}{2}$,

$$\sum_{j=1}^{p} \Phi_{a,j} + \sum_{j=1}^{p} \Phi_{p-a,j} = \sum_{j=1}^{p} \Phi_{a,j} + \sum_{j=1-a}^{p-a} \Phi_{p-a,j+a} = \sum_{j=1}^{p} \Phi_{a,j} + \sum_{j=1}^{p} \Phi_{p-a,j+a} = 0.$$

When $a = \frac{p}{2}$,

$$\sum_{j=1}^{p} \Phi_{a,j} = \sum_{j=1}^{a} \Phi_{a,j} + \sum_{j=a+1}^{p} \Phi_{p-a,j} = \sum_{j=1}^{a} \Phi_{a,j} + \sum_{j=1}^{a} \Phi_{p-a,j+a} = 0.$$

By the discussion above,

$$\sum_{j=1}^{p} (-1)^{\alpha_j} F_p(x_j, \cdots, x_{j+p-1}) = \sum_{1 \leq a \leq p-1} \sum_{j=1}^{p} \Phi_{a,j} = 0.$$

$\square$

Lemma 4.3. *Let A be a dga. $1 \in A^0$ is the identity such that $m_1^A 1 = 0$ and $m_2(1, \alpha) = m_2(\alpha, 1) = \alpha$ for all $\alpha \in A$. For simplicity, in $H^0(A)$ we use 1 to denote $[1]$, the cohomology class of 1. For $x_1, \cdots, x_p \in M(A)$, $m_p(x_1, \cdots, x_p) = 0$ if some $x_j = 1$ and $p \neq 2$. Also, when $p \neq 1$, $f_p(x_1, \cdots, x_p) = 0$ if some $x_j = 1$.*

Proof: Since the only exact form in A^0 is 0, $f_1(1)$ must be the identity 1 in A^0. $m_2(1, x) = m_2(x, 1) = x$ for all $x \in M(A)$ because m_2 on $M(A)$ is induced by m_2^A.

For each $x \in M(A)$, we have

$$m_1 f_2(1, x) = (f_1 m_2 - m_2(f_1 \otimes f_1))(1, x) = f_1 m_2(1, x) - m_2(1, f_1(x)) = f_1(x) - f_1(x) = 0$$

Thus $f_2(1, x) = 0$, similarly $f_2(x, 1) = 0$.

When $p \geq 3$, we will show that $F_p(x_1, \cdots, x_p) = 0$ if some $x_j = 0$ by induction. Assume the statement is true for all k when $2 \leq k < p$, i.e. $f_k(x_1, \cdots, x_k) = 0$ and $m_k(x_1, \cdots, x_k) = 0$ if some $x_j = 0$.

For $F_p(x_1, \cdots, x_p)$, since $m_k^A = 0$ for $k \geq 3$, F_p can be simplified as:

$$F_p = \sum_{r=1}^{p-1} (-1)^{r-1} m_2^A(f_r \otimes f_{p-r}) - \sum_{\substack{r+s+t=p \\ 2 \leq s \leq p-1}} (-1)^{r+st} f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t})$$

When $x_j = 1$, there are 3 cases.

Case 1. $j = 1$.

As $x_1 = 1$, $f_r(x_1, \cdots, x_r) = 0$ when $r > 1$. Thus, the first term can be written as

$$\sum_{r=1}^{p-1} (-1)^{r-1} m_2^A(f_r \otimes f_{p-r})(x_1, \cdots, x_p) = m_2^A\big(f_1(1), f_{p-1}(x_2, \cdots, x_p)\big) = f_{p-1}(x_2, \cdots, x_p).$$

For the second term, if $r > 0$, then $1 < r + t + 1 \leq p - 1$ and

$$\begin{aligned} & f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t})(x_1, \cdots, x_p) \\ =& \pm f_{r+t+1}(1, x_2, \cdots, x_r, m_s(x_{r+1}, \cdots, x_{r+s}), x_{r+s+1}, \cdots, x_p) \\ =& 0. \end{aligned}$$

When $r = 0$, $m_s(x_1, \cdots, x_s) = 0$ if $s \neq 2$. Thus, the second term can be simplified as

$$\begin{aligned} & \sum_{\substack{r+s+t=p \\ 2 \leq s \leq p-1}} (-1)^{r+st} f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) \\ =& \sum_{\substack{s+t=p \\ 2 \leq s \leq p-1}} (-1)^{st} f_{t+1}(m_s(1, x_2, \cdots, x_s), x_{s+1}, \cdots, x_p) \\ =& (-1)^{2(p-2)} f_{(p-2)+1}(m_2(1, x_2), x_3, \cdots, x_p) \\ =& f_{p-1}(x_2, \cdots, x_p). \end{aligned}$$

Therefore,

$$F_p(x_1, \cdots, x_p) = f_{p-1}(x_2, \cdots, x_p) - f_{p-1}(x_2, \cdots, x_p) = 0.$$

Case 2. $2 \leq j \leq p-1$.

Consider the first term. When $r < j$, $1 \leq p-j < p-r \leq p-1$. Hence, $f_{p-r}(x_{r+1}, \cdots, \mathbf{x_j}, \cdots, x_p) = 0$. When $r \geq j \geq 2$, $f_r(x_1, \cdots, \mathbf{x_j}, \cdots, x_r) = 0$ since $r \leq p-1$. So

$$\sum_{r=1}^{p-1} (-1)^{r-1} m_2^A (f_r \otimes f_{p-r})(x_1, \cdots, x_p) = 0.$$

For the second term, when $r+s < j$, $r+t+1 > t > p-j \geq 1$. As $s > 1$, $r+t+1 \leq p-1$. Hence,

$$f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \pm f_{r+t+1}\big(x_1, \cdots, x_r, m_s(x_{r+1}, \cdots, x_{r+s}), x_{r+s+1}, \cdots, \mathbf{x_j}, \cdots, x_p\big) = 0.$$

Similarly, when $r \geq j$, $r+t+1 > j \geq 1$. So

$$f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \pm f_{r+t+1}\big(x_1, \cdots, \mathbf{x_j}, \cdots, x_r, m_s(x_{r+1}, \cdots, x_{r+s}), x_{r+s+1}, \cdots, x_p\big) = 0.$$

Therefore, the second term is non-trivial only when $r < j \leq r+s$. Furthermore, in this case, $m_s(x_{r+1}, \cdots, \mathbf{x_j}, \cdots, x_{r+s}) = 0$ for all $3 \leq s \leq p-1$. So the only non-trivial cases are $r = j-2$ or $j-1$, and $s = 2$. Then the second term is

$$\begin{aligned}
&\sum_{\substack{r+s+t=p \\ 2 \leq s \leq p-1}} (-1)^{r+st} f_{r+t+1}(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) \\
=&(-1)^{(j-2)+2(p-j)}(-1)^{(|x_1|+\cdots+|x_{j-2}|)\cdot 2} f_{p-1}\big(x_1, \cdots, x_{j-2}, m_2(x_{j-1}, 1), x_{j+1}, \cdots, x_p\big) \\
&+ (-1)^{(j-1)+2(p-j-1)}(-1)^{(|x_1|+\cdots+|x_{j-1}|)\cdot 2} f_{p-1}\big(x_1, \cdots, x_{j-1}, m_2(1, x_{j+1}), x_{j+2}, \cdots, x_p\big) \\
=&(-1)^{j-2} f_{p-1}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_p) + (-1)^{j-1} f_{p-1}(x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_p) \\
=&0.
\end{aligned}$$

Case 3. $j = p$. It is similar to Case 1.

Therefore, $F_p(x_1, \cdots, x_p) = 0$ if some $x_j = 0$. Then $m_p(x_1, \cdots, x_p) = 0$ and $f_p(x_1, \cdots, x_p) = 0$ in $M(A)$. $\square$

Lemma 4.4. *Suppose M is an n-dimensional connected compact orientable manifold. If $H^r(M) = 0$ for $1 \leq r \leq k$, and $l \geq 3$ is an integer such that $n \leq (l+1)k + 2$, then $\Omega^*(M)$ has an A_∞-minimal model with $m_p = 0$ for $p \geq l$.*

Proof: Let $A = \Omega^*(M)$. A is an A_∞-algebra with $m_1^A = d$, $m_2^A = \wedge$, and $m_p^A = 0$ when $p \geq 3$.

We start from the A_∞-minimal model $M(A)$ and the quasi-isomorphism $f : M(A) \to A$ constructed at the beginning of this chapter. Our goal is to obtain another A_∞-algebra structure $(M''(A), m'')$ on $H^*(A)$ and a quasi-isomorphism $h : M''(A) \to A$ such that $m''_p = 0$ for $p \geq l$.

The idea is modifying f_{l-1} to some g_{l-1}, then we can get an A_∞-minimal model $(M'(A), m')$ and a quasi-isomorphism $g : M'(A) \to A$ such that $m'_l = 0$. Next we modify g_l to some h_l, and get another A_∞-minimal model $(M''(A), m'')$ together with a quasi-isomorphism $h : M''(A) \to A$, such that $m''_{l+1} = 0$.

We use the following notations: Let $\{x_{r,1}, \cdots, x_{r,b_r}\}$ be a basis of $H^r(A)$, where b_r is the dimension of $H^r(A)$. Then $H^n(A)$ is generated by $\mu = x_{n,1}$. By Poincaré duality, there exists $y_{n-r,1}, \cdots, y_{n-r,b_r} \in H^{n-r}(M)$ such that $x_{r,i} \wedge y_{n-r,j} = \delta_{ij}\mu$.

For arbitrary $p \in \mathbb{Z}$ and $z_1, \cdots, z_p \in H^*(A)$ such that $|z_1| + \cdots + |z_p| = n + p - 2$, $m_p(z_1, \cdots, z_p) \in H^n(A)$. So there exist a constant number $C(z_1, \cdots, z_p)$ such that $m_p(z_1, \cdots, z_p) = C(z_1, \cdots, z_p)\mu$.

i) Defining of m' and g.

Now we define m' and g. Set $m'_p = m_p$ for $p \leq l-1$ and $g_p = f_p$ for $p \leq l-2$. Then $g_1 m'_p - m_1^A g_p = f_1 m_p - m_1^A f_p = F_p = G_p$ when $p \leq l-2$. Also, $m'_{l-1} = [G_{l-1}] = [F_{l-1}] = m_{l-1}$. So they are well defined.

For $g_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}})$, when $r_1+\cdots+r_{l-1} < n+l-2$, let $s = (n+l-2)-(r_1+\cdots+r_{l-1})$. Then $f_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) \in H^{n-s}(A)$. Also for $x_{s,t} \in H^s(A)$, $m_k(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}, x_{s,t}) \in H^n(A)$. So we can set

$$\begin{aligned}
&g_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) \\
=&f_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) + \sum_{t=1}^{b_s} \sum_{j=1}^{l-1} (-1)^{\phi(j)} \frac{l-j}{l} C(j,t) f_1(y_{n-s,t}),
\end{aligned}$$

where

$$\phi(j) = j(l+1) + s(n-1) + (r_1 + \cdots + r_{j-1})(n-l+1)$$

and

$$C(j,t) = C(x_{r_j,i_j}, \cdots, x_{r_{l-1},i_{l-1}}, x_{s,t}, x_{r_1,i_1}, \cdots, x_{r_{j-1},i_{j-1}}).$$

When $r_1 + \cdots + r_{l-1} \geq n+l-2$, we simply set $g_{l-1} = f_{l-1}$.

We will prove G_l is exact by the definition above. So $m'_k = 0$. Then we can define g_l satisfying $m'_1 g_l = -G_l$. Such g_l may not make G_{l+1} be exact. So we define m'' and h.

ii) Defining of m'' and h.

Set $m''_p = m'_p$ for $p \leq l$ and $h_p = g_p$ for $p \leq l-1$. Similar to the discussion for m'_p and g_p, we can show that they are well defined.

For $h_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l})$, when $n = (l+1)k+2$ and $r_j = k+1$ for all $1 \leq j \leq l$, its total degree is $n-(k+1)$. So we set

$$
\begin{aligned}
&h_l(x_{k+1,i_1},\cdots,x_{k+1,i_k})\\
=&g_l(x_{k+1,i_1},\cdots,x_{k+1,i_k})+\sum_{t=1}^{b_{k+1}}\sum_{j=1}^{l}(-1)^{\beta_j}\frac{l+1-j}{l+1}C''(j,t)g_1(y_{n-k-1,t}).
\end{aligned}
$$

where $\beta_j = jkl+(k+1)(l+1)$ and

$$C''(j,t)=C''(j,i_{l+1})=C(x_{k+1,i_j},\cdots,x_{k+1,i_k},x_{k+1,t},x_{k+1,i_1},\cdots,x_{k+1,i_{j-1}}).$$

The definition of β_j comes from the definition of $\phi(j)$ above. If we replace l by $l+1$ for $\phi(j)$, we get

$$\phi'(j)=j((l+1)+1)+s(n-1)+(r_1+\cdots+r_{j-1})(n-(l+1)+1).$$

This $\phi'(j)$ satisfies $(-1)^{\phi'(j)}=(-1)^{\beta_j}$ when $r_1=\cdots=r_{j-1}=k+1$ and $n=(l+1)k+2$.

For other cases, we simply set $h_l=g_l$.

To prove the theorem, we need to verify the following statements.

1. g_{l-1} is well defined.
2. $m'_l=0$.
3. h_l is well defined.
4. $m''_{l+1}=0$.
5. $m''_p=0$ for $p\geq l+2$.

1. g_{l-1} is well defined

Since $g_p=f_p$ when $p\leq l-2$, we have $G_{l-1}=F_{l-1}$ and $m'_{l-1}=m_{l-1}$. The image of $g_{l-1}-f_{l-1}$

is a linear combination of $f_1(y_{s,t})$, which are all m_1^A-closed. Hence,

$$g_1 m'_{l-1} - m_1^A g_{l-1} = f_1 m_{l-1} - m_1^A f_{l-1} = F_{l-1} = G_{l-1}.$$

2. $m'_l = 0$

For $m'_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l})$, if some $r_j = 0$, x_{r_j,i_j} must be a constant number because $H^0(A)$ is generated by 1. By Lemma 4.3, $m'_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) = 0$.

When all $r_j > 0$ and x_{r_j,i_j} are non-zero, r_j must be greater than k since $H^r(A) = 0$ for $1 \leq r \leq k$. Thus, the degree of $m'_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l})$ is at least $l(k+1) + (2-l) = lk + 2$. By Poincaré duality, $H^r(A) = 0$ when $r \geq n-k$ except for $r = n$. As $n \leq (l+1)k + 2$, $lk + 2 \geq n - k$. Therefore, $m'_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) = 0$ if its degree is not n. So we only need to consider the case that the degree is n, i.e. $r_1 + \cdots + r_l = n + l - 2$.

As $m_p^A = 0$ for $p \geq 3$, G_l can be divided by the following four parts. We will talk about them separately.

$$\begin{aligned}
G_l =& \sum_{a=1}^{l-1} (-1)^{a-1} m_2^A(g_a \otimes g_{l-a}) - \sum_{\substack{a+b+c=l \\ 2 \leq b \leq l-1}} (-1)^{a+bc} g_{a+c+1}(1^{\otimes a} \otimes m'_b \otimes 1^{\otimes c}) \\
=& \big(m_2^A(g_1 \otimes g_{l-1}) + (-1)^{l-2} m_2^A(g_{l-1} \otimes g_1)\big) && (1) \\
&+ \sum_{a=2}^{l-2} (-1)^{a-1} m_2^A(g_a \otimes g_{l-a}) && (2) \\
&- \sum_{a+c=l-2} (-1)^a g_{l-1}(1^{\otimes a} \otimes m'_2 \otimes 1^{\otimes c}) && (3) \\
&- \sum_{\substack{a+b+c=l \\ 3 \leq b \leq l-1}} (-1)^{a+bc} g_{a+c+1}(1^{\otimes a} \otimes m'_b \otimes 1^{\otimes c}) && (4)
\end{aligned} \tag{4.1}$$

For part (1) of (4.1),

$$\begin{aligned}
&\big(m_2^A(g_1 \otimes g_{l-1}) + (-1)^{l-2} m_2^A(g_{l-1} \otimes g_1)\big)(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) \\
=&(-1)^{r_1(2-l)} g_1(x_{r_1,i_1}) \wedge g_{l-1}(x_{r_2,i_2}, \cdots, x_{r_l,i_l}) + (-1)^l g_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) \wedge g_1(x_{r_l,i_l}).
\end{aligned}$$

Consider the second term. Since $r_1 + \cdots + r_{l-1} + 2 - l = n - r_l$, we have

$$\begin{aligned}
&g_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) \\
=&f_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) + \sum_{t=1}^{b_{r_l}} \sum_{j=1}^{l-1} (-1)^{\phi(j)} \frac{l-j}{l} C(j,t) f_1(y_{n-r_l,t}).
\end{aligned}$$

As $y_{n-r_l,t} \wedge x_{r_l,i_l} = \delta_{ij}\mu$, $f_1(y_{n-r_l,t}) \wedge f_1(x_{r_l,i_l})$ is exact when $t \neq i_l$, and is $(-1)^{r_l(n-r_l)} f_1(\mu)$ plus some exact form when $t = i_l$. So we have

$$\begin{aligned}
&g_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) \wedge g_1(x_{r_l,i_l}) \\
=&f_{l-1}(x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}) \wedge f_1(x_{r_l,i_l}) + \sum_{j=1}^{l-1} (-1)^{\phi(j) + r_l(n-r_l)} \frac{l-j}{l} C(j,i_l) f_1(\mu) + R_1,
\end{aligned}$$

where R_1 is some exact form,

$$\phi(j) = j(l+1) + r_l(n-1) + (r_1 + \cdots + r_{j-1})(n-l+1),$$

and $C(j,t)$ is defined by

$$C(x_{r_j,i_j}, \cdots, x_{r_{l-1},i_{l-1}}, x_{r_l,t}, x_{r_1,i_1}, \cdots, x_{r_{j-1},i_{j-1}}).$$

When $t = i_l$,

$$C(j, i_l) = C(x_{r_j,i_j}, \cdots, x_{r_l,i_l}, x_{r_1,i_1}, \cdots, x_{r_{j-1},i_{j-1}}).$$

Similarly, for the first term

$$
\begin{aligned}
&g_1(x_{r_1,i_1}) \wedge g_{l-1}(x_{r_2,i_2}, \cdots, x_{r_l,i_l}) \\
=&f_1(x_{r_1,i_1}) \wedge f_{l-1}(x_{r_2,i_2}, \cdots, x_{r_l,i_l}) + \sum_{j=1}^{l-1}(-1)^{\psi(j)}\frac{l-j}{l}C'(j,i_1)f_1(\mu) + R_2,
\end{aligned}
$$

where R_2 is some exact form,

$$
\psi(j) = j(l+1) + r_1(n-1) + (r_2 + \cdots + r_j)(n-l+1),
$$

and $C'(j,t)$ is defined by

$$
C(x_{r_{j+1},i_{j+1}}, \cdots, x_{r_l,i_l}, x_{r_1,t}, x_{r_2,i_2}, \cdots, x_{r_j,i_j}).
$$

When $t = i_1$,

$$
C'(j,i_1) = C(x_{r_{j+1},i_{j+1}}, \cdots, x_{r_l,i_l}, x_{r_1,i_1}, \cdots, x_{r_j,i_j}).
$$

Therefore,

$$
\begin{aligned}
&\big(m_2^A(g_1 \otimes g_{l-1}) + (-1)^{l-2}m_2^A(g_{l-1} \otimes g_1)\big)(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) \\
=&\big(m_2^A(f_1 \otimes f_{l-1}) + (-1)^{l-2}m_2^A(f_{l-1} \otimes f_1)\big)(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) \\
&+ \sum_{j=1}^{l-1}(-1)^{r_1 l + \psi(j)}\frac{l-j}{l}C'(j,i_1)f_1(\mu) + \sum_{j=1}^{l-1}(-1)^{l+\phi(j)+r_l(n-r_l)}\frac{l-j}{l}C(j,i_l)f_1(\mu) + R
\end{aligned}
$$

where $R = R_1 + R_2$ is exact.

Observe when $1 \le j \le l-2$,

$$
C'(j,i_1) = C(x_{r_{j+1},i_{j+1}}, \cdots, x_{r_l,i_l}, x_{r_1,i_1}, \cdots, x_{r_j,i_j}) = C(j+1,i_l).
$$

We can also let $C(l, i_l)$ denote

$$C'(l-1, i_1) = C(x_{r_l,i_l}, x_{r_1,i_1}, \cdots, x_{r_{l-1},i_{l-1}}).$$

Then $C'(j, i_1) = C(j+1, i_l)$ for $1 \le j \le l-1$. On the other hand,

$$\begin{aligned}
&\psi(j) - \phi(j+1) \\
=&j(l+1) + r_1(n-1) + (r_2 + \cdots + r_j)(n-l+1) \\
&-(j+1)(l+1) - r_l(n-1) - (r_1 + \cdots + r_j)(n-l+1) \\
=&-(l+1) + r_1(l-2) - r_l(n-1).
\end{aligned}$$

So we have

$$(-1)^{r_1 l + \psi(j)} = (-1)^{r_1(2l-2)-(l+1)-r_l(n-1)+\phi(j+1)} = -(-1)^{l - r_l(n-r_l)+\phi(j+1)},$$

and they are equal to

$$(-1)^{r_1 l + j(l+1) + r_1(n-1) + (r_2+\cdots+r_j)(n-l+1)} = (-1)^{\alpha_{j+1}}$$

where

$$\alpha_{j+1} = j(l+1) + (r_1 + \cdots + r_j)(n-l+1)$$

is the notation in Lemma 4.2. Therefore, we can write

$$
\begin{aligned}
&\sum_{j=1}^{l-1}(-1)^{r_1 l+\psi(j)}\frac{l-j}{l}C'(j,i_1)f_1(\mu)+\sum_{j=1}^{l-1}(-1)^{l+\phi(j)+r_l(n-r_l)}\frac{l-j}{l}C(j,i_l)f_1(\mu)\\
=&\sum_{j=1}^{l-1}(-1)^{r_1 l+\phi(j+1)}\frac{l-j}{l}C(j+1,i_1)f_1(\mu)+\sum_{j=1}^{l-1}(-1)^{l+\phi(j)+r_l(n-r_l)}\frac{l-j}{l}C(j,i_l)f_1(\mu)\\
=&\sum_{j=2}^{l}(-1)^{\alpha_j}\frac{l-j+1}{l}C(j,i_l)f_1(\mu)-\sum_{j=1}^{l-1}(-1)^{\alpha_j}\frac{l-j}{l}C(j,i_l)f_1(\mu)\\
=&\sum_{j=2}^{l-1}(-1)^{\alpha_j}(\frac{l-j+1}{l}-\frac{l-j}{l})C(j,i_l)f_1(\mu)+(-1)^{\alpha_l}\frac{1}{l}C(l,i_l)f_1(\mu)\\
&-(-1)^{\alpha_1}\frac{l-1}{l}C(1,i_l)f_1(\mu)\\
=&\sum_{j=1}^{l}(-1)^{\alpha_j}\frac{1}{l}C(j,i_l)f_1(\mu)-(-1)^{\alpha_1}C(1,i_l)f_1(\mu)
\end{aligned}
$$

For part (2) of (4.1), since $2 \le a \le l-2$, we have $l-a \le l-2$. Then $g_a = f_a$, $g_{l-a} = f_{l-a}$, and

$$\sum_{a=2}^{l-2}(-1)^{a-1}m_2^A(g_a \otimes g_{l-a}) = \sum_{a=2}^{l-2}(-1)^{a-1}m_2^A(f_a \otimes f_{l-a})$$

For part (3) of (4.1), the total degree of $(1^{\otimes a}\otimes m_2' \otimes 1^{\otimes c})(x_{r_1,i_1},\cdots,x_{r_l,i_l})$ is $r_1+\cdots+r_l = n+l-2$. In this case $g_{l-1} = f_{l-1}$. Thus,

$$
\begin{aligned}
&\sum_{a+c=l-2}(-1)^a g_{l-1}(1^{\otimes a}\otimes m_2'\otimes 1^{\otimes c})(x_{r_1,i_1},\cdots,x_{r_l,i_l})\\
=&\sum_{a+c=l-2}(-1)^a f_{l-1}(1^{\otimes a}\otimes m_2\otimes 1^{\otimes c})(x_{r_1,i_1},\cdots,x_{r_l,i_l}).
\end{aligned}
$$

For part (4) of (4.1), as $3 \le b \le l-1$, we have $a+c+1 \le l-2$. So $g_{a+c+1} = f_{a+c+1}$,

$m'_b = m_b$, and

$$\sum_{\substack{a+b+c=l \\ 3\le b\le l-1}} (-1)^{a+bc} g_{a+c+1}(1^{\otimes a} \otimes m'_b \otimes 1^{\otimes c}) = \sum_{\substack{a+b+c=l \\ 3\le b\le l-1}} (-1)^{a+bc} f_{a+c+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c})$$

By the discussion above, we get

$$\begin{aligned} &G_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) \\ =&F_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) + \sum_{j=1}^{l} (-1)^{\alpha_j} \frac{1}{l} C(j, i_l) f_1(\mu) - (-1)^{\alpha_1} C(1, i_l) f_1(\mu) + R. \end{aligned}$$

Then

$$\begin{aligned} &m'_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) \\ =&m_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) + \sum_{j=1}^{l} (-1)^{\alpha_j} \frac{1}{l} C(j, i_l)\mu - (-1)^{\alpha_1} C(1, i_l)\mu \end{aligned}$$

As $C(j, i_l)\mu = m_l(x_{r_j,i_j}, \cdots, x_{r_l,i_l}, x_{r_1,i_1}, \cdots, x_{r_{j-1},i_{j-1}})$, we have

$$\sum_{j=1}^{l} (-1)^{\alpha_j} \frac{1}{l} C(j, i_l)\mu = \frac{1}{l} \sum_{j=1}^{l} (-1)^{\alpha_j} m_l(x_{r_j,i_j}, \cdots, x_{r_l,i_l}, x_{r_1,i_1}, \cdots, x_{r_{j-1},i_{j-1}}) = 0$$

by Lemma 4.2. On the other hand,

$$(-1)^{\alpha_1} C(1, i_l)\mu = m_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}).$$

Therefore, we have proved

$$m'_l(x_{r_1,i_1}, \cdots, x_{r_l,i_l}) = 0.$$

3. h_l is well defined

Similar to the discussion for g_{l-1}, we have $H_l = G_l$, $m''_l = m'l$ and $h_l - g_l$ is closed. Thus,

$$h_1 m''_l - m_1^A h_l = g_1 m'_l - m_1^A g_l = G_l = H_l.$$

4. $m''_{l+1} = 0$

For $m''_{l+1}(x_{r_1,i_1}, \cdots, x_{r_{l+1},i_{l+1}})$, it is 0 when some $r_j = 0$. If all $r_j > 0$, then $r_j \geq l+1$. The total degree is at least $(l+1)k+2$. So it is 0 except for $n = (l+1)k+2$ and the total degree is n, i.e. every $r_j = k+1$. Thus, we only need to consider this special case.

Similar to the proof of $m'_l = 0$, we divide H_{l+1} by three parts.

$$\begin{aligned} H_{l+1} =& m_2^A(h_1 \otimes h_l) + (-1)^{l-1} m_2^A(h_l \otimes h_1) && (1) \\ &+ \sum_{a=2}^{l-1} (-1)^{a-1} m_2^A(h_a \otimes h_{l+1-a}) && (2) \\ &- \sum_{\substack{a+b+c=l+1 \\ 2 \leq b \leq l}} (-1)^{a+bc} h_{a+c+1}(1^{\otimes a} \otimes m''_b \otimes 1^{\otimes c}) && (3) \end{aligned} \tag{4.2}$$

For part (1) of (4.2),

$$\begin{aligned} & m_2^A(h_l \otimes h_1)(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) \\ =& \Big(g_l(x_{k+1,i_1}, \cdots, x_{k+1,i_l}) + \sum_{t=1}^{b_{k+1}} \sum_{j=1}^{l} (-1)^{\beta_j} \frac{l+1-j}{l+1} C''(j,t) g_1(y_{n-k-1,t})\Big)\Big) \wedge g_1(x_{k+1,i_{l+1}}) \\ =& g_l(x_{k+1,i_1}, \cdots, x_{k+1,i_l}) \wedge g_1(x_{k+1,i_{l+1}}) + \sum_{j=1}^{l} (-1)^{\beta_j + (k+1)(n-k-1)} \frac{l+1-j}{l+1} C_j g_1(\mu) + R_3 \end{aligned}$$

where R_3 is exact and

$$C_j = C''(j, i_{l+1}) = C(x_{k+1,i_j}, \cdots, x_{k+1,i_{k+1}}, x_{k+1,i_1}, \cdots, x_{k+1,i_{j-1}}).$$

On the other hand,

$$
\begin{aligned}
& m_2^A(h_1 \otimes h_l)(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) \\
=& (-1)^{(1-l)(k+1)} g_1(x_{k+1,i_1}) \wedge \big(g_k(x_{k+1,i_2}, \cdots, x_{k+1,i_{l+1}}) \\
& + \sum_{t=1}^{b_{k+1}} \sum_{j=1}^{l} (-1)^{\beta_j} \frac{l+1-j}{l+1} C'''(j,t) g_1(y_{n-k-1,t}) \big) \\
=& (-1)^{(1-l)(k+1)} g_1(x_{k+1,i_1}) \wedge g_l(x_{k+1,i_2}, \cdots, x_{k+1,i_{l+1}}) \\
& + \sum_{j=1}^{l} (-1)^{(1-l)(k+1)+\beta_j} \frac{l+1-j}{l+1} C_{j+1} g_1(\mu) + R_4
\end{aligned}
$$

where R_4 is exact and

$$
C'''(j,t) = C(x_{k+1,i_{j+1}}, \cdots, x_{k+1,i_{l+1}}, x_{k+1,t}, x_{k+1,i_2}, \cdots, x_{k+1,i_{j-1}}).
$$

Then $C'''(j, i_1) = C_{j+1}$.

By the discussion above

$$
\begin{aligned}
& \big(m_2^A(h_1 \otimes h_l) + (-1)^{l-1} m_2^A(h_l \otimes h_1) \big)(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) \\
=& \big(m_2^A(g_1 \otimes g_l) + (-1)^{l-1} m_2^A(g_l \otimes g_1) \big)(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) \\
& + \sum_{j=1}^{l} (-1)^{(1-l)(k+1)+\beta_j} \frac{l+1-j}{l+1} C_{j+1} g_1(\mu) \\
& + \sum_{j=1}^{l} (-1)^{l-1+\beta_j+(k+1)(n-k-1)} \frac{l+1-j}{l+1} C_j g_1(\mu) + R'
\end{aligned}
$$

where $R' = R_3 + R_4$ is exact.

Using the notation of Lemma 4.2,

$$
\alpha_j = (j-1)((l+1)+1) + r_1 + \cdots + r_{j-1}(n-(l+1)+1) = (j-1)(l+2) + (j-1)(k+1)(n-l).
$$

By definition $n = (l+1)k+2$, we have $(-1)^{n(k+1)} = 1$. So

$$(-1)^{\alpha_j} = (-1)^{(j-1)l-(j-1)(k+1)l} = (-1)^{(j-1)kl}.$$

Also,

$$(-1)^{(1-l)(k+1)+\beta_j} = (-1)^{(1-l)(k+1)+jkl+(k+1)(l+1)} = (-1)^{jkl} = (-1)^{\alpha_{j+1}}.$$

It follows that

$$(-1)^{l-1+\beta_j+(k+1)(n-k-1)} = (-1)^{l-1+jkl+(k+1)(l+1)-(k+1)} = (-1)^{jkl+kl+2l-1} = (-1)^{\alpha_j-1}.$$

Hence,

$$\begin{aligned}
&\sum_{j=1}^{l}(-1)^{(1-l)(k+1)+jk}\frac{l+1-j}{l+1}C_{j+1}g_1(\mu) + \sum_{j=1}^{l}(-1)^{l-1+jk+(k+1)(n-k-1)}\frac{l+1-j}{l+1}C_jg_1(\mu)\\
=&\sum_{j=1}^{l}(-1)^{\alpha_{j+1}}\frac{l+1-j}{l+1}C_{j+1}g_1(\mu) + \sum_{j=1}^{l}(-1)^{\alpha_j-1}\frac{l+1-j}{l+1}C_jg_1(\mu)\\
=&\sum_{j=2}^{l+1}(-1)^{\alpha_j}\frac{l+1-(j-1)}{l+1}C_jg_1(\mu) - \sum_{j=1}^{l}(-1)^{\alpha_j}\frac{l+1-j}{l+1}C_jg_1(\mu)\\
=&(-1)^{\alpha_{l+1}}\frac{1}{l+1}C_{l+1}g_1(\mu) + \sum_{j=2}^{l}(-1)^{\alpha_j}(\frac{l+1-(j-1)}{l+1} - \frac{l+1-j}{l+1})C_jg_1(\mu)\\
&-(-1)^{\alpha_1}\frac{1}{l+1}C_1g_1(\mu)\\
=&\sum_{j=1}^{l+1}(-1)^{\alpha_j}\frac{1}{l+1}C_jg_1(\mu) - (-1)^{\alpha_1}C_1g_1(\mu).
\end{aligned}$$

For part (2) of (4.2), when $2 \le a \le l$, $l+1-a \le l-1$. Hence, $h_a = g_a$ and $h_{l+1-a} = g_{l+1-a}$. Then

$$\sum_{a=2}^{l-1}(-1)^{a-1}m_2^A(h_a \otimes h_{l+1-a}) = \sum_{a=2}^{l-1}(-1)^{a-1}m_2^A(g_a \otimes g_{l+1-a}).$$

For part (3) of (4.2), when $b \ge 3$, $a+c+1 \le l-1$ so that $h_{l+1-a} = g_{l+1-a}$. When $b = 2$,

$a + c + 1 = l$. In this case we also have

$$h_l(1^{\otimes a} \otimes m_2 \otimes 1^{\otimes c})(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) = g_l(1^{\otimes a} \otimes m_2 \otimes 1^{\otimes c})(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}})$$

because h_l is not acting on $(H^{k+1}(A))^{\otimes l}$.

Therefore,

$$\begin{aligned}
&H_{l+1}(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) \\
=&G_{l+1}(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) + \sum_{j=1}^{l+1}(-1)^{\alpha_j}\frac{1}{l+1}C_j g_1(\mu) - (-1)^{\alpha_1}C_1 g_1(\mu) + R'.
\end{aligned}$$

That implies

$$\begin{aligned}
&m''_{l+1}(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) \\
=&m'_{l+1}(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) + \sum_{j=1}^{l+1}(-1)^{\alpha_j}\frac{1}{l+1}m'_{l+1}(x_{k+1,i_j}, \cdots, x_{k+1,i_{l+1}}, x_{k+1,i_1}, \cdots, x_{k+1,i_{j-1}}) \\
&- (-1)^{\alpha_1}m'_{l+1}(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}).
\end{aligned}$$

By Lemma 4.2 again the second term is

$$m'_{l+1}(x_{k+1,i_j}, \cdots, x_{k+1,i_{l+1}}, x_{k+1,i_1}, \cdots, x_{k+1,i_{j-1}}) = 0.$$

Since $\alpha_1 = 0$, we have the conclusion

$$m''_{l+1}(x_{k+1,i_1}, \cdots, x_{k+1,i_{l+1}}) = 0.$$

5. $m''_p = 0$ *for* $p \geq l + 2$

For $m''_p(x_1, \cdots, x_p)$ with $p \geq l+2$, either the degree of some x_j is 0, or the total degree is at least $(2-p)+p(k+1) = pk+2 \geq (l+2)k+2 > (l+1)k+2 \geq n$. Hence, $m''_p = 0$ in both cases.

In conclusion, $(M(A), m'')$ is the minimal model we want. □

By Hurewicz Theorem, k-connected compact orientable manifolds satisfy the condition of Lemma 4.4, so the statement in the Lemma is true for them. We will show that the statement in Lemma 4.4 is also true when the manifold is not orientable.

Theorem 4.5. *Suppose M is an n-dimensional k-connected compact manifold. If $l \geq 3$ such that $n \leq (l+1)k+2$, then $\Omega^*(M)$ has an A_∞-minimal model with $m_p = 0$ for $p \geq l$.*

Proof. By the defition of k-connected, $\pi_r(M) = 0$ for all $1 \leq r \leq k$. It follows that $H^r(M) = 0$ by Hurewicz Theorem.

When M is orientable, by Lemma 4.4, $\Omega^*(M)$ has an A_∞-minimal model with $m_p = 0$ for $p \geq l$.

When M is not orientable, $H^n(M) = 0$. Let $\tilde{M}$ be the orientation bundle of M. We also have $\tilde{M}$ is connected and $\pi_r(\tilde{M}) = 0$ for $1 \leq r \leq k$. So when $1 \leq r \leq k$, $H_r(\tilde{M}) = 0$. By twisted Poincaré duality, $H^{n-r}(M) \simeq H_r(\tilde{M}) = 0$. Therefore, $H^r(M) = 0$ for all $r \geq n-k$.

We use the minimal model $M(A)$ of $A = \Omega^*(M)$ and the quasi-isomorphism $f : M(A) \to A$ at the beginning of this chapter. For any $p \geq l$, $m_p(x_1, \cdots, x_p) = 0$ when some $|x_j| = 0$. If all $|x_j| > 0$, $|x_j|$ is at least $k+1$. The total degree of $m_p(x_1, \cdots, x_p)$ is at least $p(k+1)+2-p = pk+2 \geq lk+2 = (l+1)k+2-k \geq n-k$, so m_p must be 0. □

Chapter 5

Two Conjectures

In the proof of Lemma 4.4, g_l and h_{l+1} are constructed in a similar way. It would be interesting to construct A_∞-minimal models for other types of dgas or A_∞-algebras following this way. For example, we may be able to extend Cavalcanti's result that a compact orientable k-connected manifold of dimension $4k+3$ or $4k+4$ with $b_{k+1}=1$ is formal [2].

Conjecture 1. *Suppose M is an oreintable n-dimensional k-connected compact manifold, with $b_{k+1}=1$. If $j \geq 3$ such that $n \leq (j+1)k+4$, then $\Omega^*(M)$ has an A_∞-minimal model with $m_t = 0$ for $t \geq j$.*

Another conjecture is based on Theorem 3.4, which would fit as the $k=3$ case of the following broader statement.

Conjecture 2. *Suppose A is a dga. $\omega_A \in A$ is an even-degree element. Extend A to $\tilde{A} = \{\alpha + \theta_A \beta |\ \alpha, \beta \in A\}$ with $d\theta_A = \omega_A$. If A has an A_∞-minimal model with $m_p = 0$ for $p \geq k$, then $\tilde{A}$ has an A_∞-minimal model with $m_p = 0$ for $p \geq k+1$.*

Furthermore, it is interesting to consider what properties can be implied for A_∞-minimal models with $m_p = 0$ for $p \geq k$. If there are some geometric meanings associated to each k, this may lead to a different type of classification of manifolds.

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