Title
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Journal
Duke Mathematical Journal, 162(11)

ISSN
0012-7094

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Publication Date
2013

DOI
10.1215/00127094-2293611

Peer reviewed

EQUIVARIANT SCHRÖDINGER MAPS IN TWO SPATIAL DIMENSIONS

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Abstract. We consider equivariant solutions for the Schrödinger map problem from $\mathbb{R}^{2+1}$ to $\mathbb{S}^2$ with energy less than 4π and show that they are global in time and scatter.

1. Introduction

In this article we consider the Schrödinger map equation in $\mathbb{R}^{2+1}$ with values into $\mathbb{S}^2 \subset \mathbb{R}^3$ (the two dimensional unit sphere in $\mathbb{R}^3$),

$$u_t = u \times \Delta u, \qquad u(0) = u_0 \tag{1.1}$$

This equation admits a conserved energy,

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx$$

and is invariant with respect to the dimensionless scaling

$$u(t,x) \to u(\lambda^2 t, \lambda x).$$

The energy is invariant with respect to the above scaling, therefore the Schrödinger map equation in $\mathbb{R}^{2+1}$ is *energy critical.*

The local theory for classical data was established in [25] and [20]. We will use the following

Theorem 1.1 (McGahagan, [20])**.** *If $u_0 \in \dot{H}^1 \cap \dot{H}^3$ then there exists a time $T > 0$, such that* (1.1) *has a unique solution in $L^\infty_t([0,T] : \dot{H}^1 \cap \dot{H}^3)$.*

The local and global in time of the Schrödinger map problem with small data has been intensely studied, see [3], [4], [5], [6], [8], [14], [15]. The state of the art result for the problem with small data was established by the authors in [6] where they proved that classical solutions (and in fact rough solutions too) with small energy are global in time. More recently Smith established, in [23] and [24], the global existence in two dimensions of smooth Schrödinger maps with energy-dispersed data (essentially $\|u_0\|_{\dot{B}^1_{2,\infty}} \ll 1$) and satisfying $E(u) < 4\pi$, see the commentaries below for the relevance of this threshold.

To understand the large data problem, one needs to describe the solitons for (1.1). The solitons for this problem are the harmonic maps, which are solutions to $u \times \Delta u = 0$. The harmonic maps cannot have arbitrary energy. The trivial solitons are of the form $u = Q$ for some $Q \in \mathbb{S}^2$ and their energy is 0. The next energy level admissible for solitons is 4π.

I.B. was supported in part by NSF grant DMS-1001676. A. I. was partially supported by a Packard Fellowship and NSF grant DMS-1065710. C.K. was supported in part by NSF grant DMS-0968742. D.T. was supported in part by the Miller Foundation and by NSF grant DMS-0801261.

In this article we confine ourselves to the class of *equivariant* Schrödinger maps. These are indexed by an integer m called the equivariance class, and consist of maps of the form

$$u(r,\theta) = e^{m\theta R}\bar{u}(r) \tag{1.2}$$

Here R is the generator of horizontal rotations, which can be interpreted as a matrix or, equivalently, as the operator below

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Ru = \overrightarrow{k} \times u.$$

Here and thereafter we denote by $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ the standard orthonormal basis in $\mathbb{R}^3$, i.e. the vectors with coordinate representation $(1,0,0), (0,1,0)$ respectively $(0,0,1)$. The case $m = 0$ corresponds to radial symmetry.

The energy for equivariant maps takes the following form:

$$E(u) = \pi \int_0^\infty \left(|\partial_r \bar{u}(r)|^2 + \frac{m^2}{r^2}(\bar{u}_1^2(r) + \bar{u}_2^2(r)) \right) rdr \tag{1.3}$$

If $m \neq 0$, then $E(u) < \infty$ implies that u_1 and u_2 have limit zero as $r \to 0$ and $r \to \infty$. For a proof see for instance [10]. Due to the restriction on the size of the energy, this implies that $u_3(t,r)$ has the same limit $+1$ or -1 both as $r \to 0$ and $r \to \infty$, see [10] or the commentaries in subsection 2.2. To fix matters we agree that this limit is -1 for all t (the fact that $u_3(0,t)$ or $u_3(\infty,t)$ cannot jump in time from -1 to $+1$ is justified in subsection 2.2).

The global regularity question in the case $m = 0$, corresponding to radial symmetry, has been considered recently by Gustafson and Koo, see [13]. In this paper we consider the case $m = 1$. More precisely, our main result is the following:

Theorem 1.2. *i) Let $u_0 \in \dot{H}^1 \cap \dot{H}^3$ be a 1-equivariant function satisfying $E(u_0) < 4\pi$. Then* (1.1) *has a unique global in time solution $u \in L^\infty(\mathbb{R} : \dot{H}^1 \cap \dot{H}^3)$. In addition ∇u, expressed in a Coulomb frame, scatters to the free solution of a suitable linear Schrödinger equation.*

ii) The above solution is Lipschitz continuous with respect to the initial data in $\dot{H}^1$. In particular if $u_0 \in \dot{H}^1$ is a 1-equivariant function satisfying $E(u_0) < 4\pi$ then there is a global solution $u(t) \in L^\infty \dot{H}^1$ defined as the unique limit of smooth solutions in $\dot{H}^1 \cap \dot{H}^3$.

The statement of the scattering cannot be made precise at this time. We need to introduce a moving frame on $\mathbb{S}^2$, write the equation of the coordinates of ∇u in that frame and identify there the linear part of the Schrödinger equation. This will be carried out in Section 2.

The result in Theorem 1.2 is sharp from the following point of view. Maps with energy less than 4π have topological degree 0, and cannot cover the full sphere. However, for energies greater than or equal to 4π there are also maps with topological degree 1 which fully cover the sphere. The solitons have least energy, i.e. 4π, among such maps, and all degree one maps with energy slightly larger than 4π must stay close to the soliton family. The global in time behavior of such maps was first studied by Bejenaru and Tataru [7], who proved that the solitons are unstable, but global solutions always exist for equivariant data which are small localized perturbations of solitons. Later Merle, Raphael and Rodnianski [22] produced examples of equivariant data which are somewhat less localized soliton perturbations, which still have energy slightly larger than 4π and for which the solutions blow up in finite time. One should note that the situation changes for larger m, namely $m \geq 3$; in that case Gustafson and collaborators, see [10], [11] and [12] have established the stability of the corresponding solitons in the m-equivariant class.

However, if one restricts attention to maps to 1-equivariant maps whose topological degree is 0, then such maps cannot cover the sphere if their energy is below 8π. Hence it is natural to conjecture that for degree zero maps the above result should be valid up to an 8π energy. This is why most of our arguments are written for the 8π threshold. While going above 4π is not very difficult, extending the proof up to 8π requires some new idea; our present argument fails due to the lack of sign for a term in the virial identities.

Our result in Theorem 1.2 extends to all other $m \neq 0$. Moreover we expect that it extends to the case when the target manifold is the hyperbolic space $\mathbb{H}^2$ in which case no restriction on the data is needed due to the absence of nontrivial solitons.

1.1. **Definitions and notations.** While at fixed time our maps into the sphere are functions defined on $\mathbb{R}^2$, the equivariance condition allows us to reduce our analysis to functions of a single variable $|x| = r \in [0, \infty)$. One such instance is exhibited in (1.2) where to each equivariant map u we naturally associate its radial component $\bar{u}$. Some other functions will turn out to be radial by definition, see, for instance, all the gauge elements in Section 2. We agree to identify such radial functions with the corresponding one dimensional functions of r. Some of these functions are complex valued, and this convention allows us to use the bar notation with the standard meaning, i.e. the complex conjugate.

Even though we work mainly with functions of a single spatial variable r, they originate in two dimensions. Therefore, it is natural to make the convention that for the one dimensional functions all the Lebesgue integrals and spaces are with respect to the rdr measure, unless otherwise specified.

Since equivariant functions are easily reduced to their one-dimensional companions via (1.2), we introduce the one dimensional equivariant version of $\dot{H}^1$,

$$\|f\|^2_{\dot{H}^1_e} = \|\partial_r f\|^2_{L^2(rdr)} + \|r^{-1} f\|^2_{L^2(rdr)}. \tag{1.4}$$

This is natural since for functions $u : \mathbb{R}^2 \to \mathbb{R}^2$ with $u(r,\theta) = e^{\theta R}\bar{u}(r)$ (here $Ru = \overrightarrow{k} \times u$ or, as a matrix, it is the upper left 2×2 block of the original matrix R) we have

$$\|u\|_{\dot{H}^1} = (2\pi)^{\frac{1}{2}} \|\bar{u}\|_{\dot{H}^1_e}.$$

It is important to note that functions in $\dot{H}^1_e$ enjoy the following properties: they are continuous and have limit 0 both at $r = 0$ and $r = \infty$, see [10] for a proof.

We introduce $\dot{H}^{-1}_e$ as the dual space to $\dot{H}^1_e$ with respect to the L^2 pairing, i.e.

$$\|f\|_{\dot{H}^{-1}_e} = \sup_{\|\phi\|_{\dot{H}^1_e}=1} \langle f, \phi \rangle$$

We will mostly be interested in elements from $\dot{H}^{-1}_e$ of the form $\partial_r g$ or $\frac{g}{r}$ with $g \in L^2$.

Three operators which are often used on radial functions are $[\partial_r]^{-1}$, $[r^{-1}\bar{\partial}_r]^{-1}$ and $[r\partial_r]^{-1}$ defined as

$$[\partial_r]^{-1} f(r) = -\int_r^\infty f(s)ds, \quad [r^{-1}\bar{\partial}_r]^{-1} f(r) = \int_0^r f(s)sds$$
$$[r\partial_r]^{-1} f(r) = -\int_r^\infty \frac{1}{s} f(s)ds$$

A direct argument shows that

$$
\begin{aligned}
&\|[r\partial_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \qquad 1 \le p < \infty, \\
&\|r^{-2}[r^{-1}\bar{\partial}_r]^{-1} f\|_{L^p} \lesssim_p \|f\|_{L^p}, \qquad 1 < p \le \infty, \\
&\|[\partial_r]^{-1} f\|_{L^2} \lesssim \|f\|_{L^1}.
\end{aligned} \tag{1.5}
$$

The equivariance properties of the functions involved in this paper require that the two-dimensional Fourier calculus is replaced by the Hankel calculus for one-dimensional functions which we recall below.

For $k \ge 0$ integer, let J_k be the Bessel function of the first kind,

$$
J_k(r) = \frac{1}{\pi}\int_0^\pi \cos(n\tau - r\sin\tau)d\tau
$$

If $H_k = \partial_r^2 + \frac{1}{r}\partial_r - \frac{k^2}{r^2}$, then J_k solves $H_k J_k = -J_k$.

We recall some formulas involving Bessel functions

$$
\partial_r J_k = \frac{1}{2}(J_{k-1} - J_{k+1}), \quad (r^{-1}\partial_r)^m \left(\frac{J_k}{r^k}\right) = (-1)^m \frac{J_{k+m}}{r^{k+m}}, \tag{1.6}
$$

where $J_{-k} = (-1)^k J_k$.

For each $k \ge 0$ integer one defines the Hankel transform $\mathcal{F}_k$ by

$$
\mathcal{F}_k f(\xi) = \int_0^\infty J_k(r\xi) f(r) r dr
$$

The inversion formula holds true

$$
f(r) = \int_0^\infty J_k(r\xi)\mathcal{F}_k f(\xi)\xi d\xi
$$

The Plancherel formula holds true, hence in particular, the Hankel transform is an isometry.

For a radial function f and for an integer k we define its two-dimensional extension

$$
R_k(r,\theta) = e^{ik\theta} f(r) \tag{1.7}
$$

If $f \in L^2$ then $R_k f \in L^2$; if $R_k f$ has additional regularity, this is easily red in terms of $\mathcal{F}_k f$. Indeed for any $s \ge 0$ integer the following holds true

$$
R_k f \in \dot{H}^s \Leftrightarrow \xi^s \mathcal{F}_k f \in L^2 \tag{1.8}
$$

For even values of s this is a consequence of $\Delta R_k f = R_k H_k f$, while for odd values of s it follows by interpolation.

By direct computation, we also have that for $k \ne 0$,

$$
R_k f \in \dot{H}^1 \Leftrightarrow f \in \dot{H}^1_e, \qquad R_0 f \in \dot{H}^1 \Leftrightarrow \partial_r f \in L^2. \tag{1.9}
$$

We will use the following result

Lemma 1.3. *i) If $f \in L^2$ is such that $H_0 f \in L^2$, then the following holds true*

$$
\|\partial_r^2 f\|_{L^2} + \|\frac{\partial_r f}{r}\|_{L^2} \lesssim \|H_0 f\|_{L^2}
$$

ii) If $f \in L^2$ is such that $H_2 f \in L^2$, then the following holds true

$$
\|\partial_r^2 f\|_{L^2} + \|\frac{\partial_r f}{r}\|_{L^2} + \|\frac{f}{r^2}\|_{L^2} \lesssim \|H_2 f\|_{L^2}
$$

Proof. i) The proof follows the same lines as the one in ii), though it is easier.

ii) Based on the representation formula

$$f = \int J_2(r\xi)\mathcal{F}_2 f(\xi)\xi d\xi$$

and by using (1.6), we compute

$$\partial_r f = \frac{1}{2}\int (J_1(r\xi)) - J_3(r\xi))\xi\mathcal{F}_2 f_0^-(\xi)\xi d\xi = g_1 - g_3$$

with $R_1 g_1, R_3 g_3 \in \dot{H}^1$, hence $\partial_r f \in \dot{H}^1_e$. Using (1.9) we obtain the conclusion for $\partial_r^2 f$ and $\frac{1}{r}\partial_r f$. Since the estimate holds true for $H_2 f$, it follows for $\frac{1}{r^2} f$ too. □

2. The Coulomb gauge representation of the equation

In this section we rewrite the Schrödinger map equation for equivariant solutions in a gauge form. This approach originates in the work of Chang, Shatah, Uhlenbeck [8]. However, our analysis is closer to the one in [5] and [7].

2.1. **The Coulomb gauge.** The computations below are at the formal level as we are not yet concerned with the regularity of the terms involved in writing various identities and equations. Implicitly we use only the information $u \in \dot{H}^1$. In subsection 2.3 we prove that if $u \in \dot{H}^3$ then all the gauge elements, their compatibility relations and the equations they obey are meaningful in the sense that they involve terms which are at least at the level of L^2.

We let the differentiation operators $\partial_0, \partial_1, \partial_2$ stand for $\partial_t, \partial_r, \partial_\theta$ respectively. Our strategy will be to replace the equation for the Schrödinger map u with equations for its derivatives $\partial_1 u$, $\partial_2 u$ expressed in an orthonormal frame $v, w \in T_u\mathbb{S}^2$. We choose $v \in T_u\mathbb{S}^2$ such that $v \cdot v = 1$ and define $w = u \times v \in T_u\mathbb{S}^2$; to summarize

$$v \cdot v = 1, \qquad v \cdot u = 0, \qquad w = v \times w \tag{2.1}$$

From this, we obtain

$$w \cdot v = 0, \qquad w \cdot w = 1, \qquad v \times w = u, \qquad w \times u = v \tag{2.2}$$

Since u is 1-equivariant it is natural to work with 1-equivariant frames, i.e.

$$v = e^{\theta R}\bar{v}(r), \qquad w = e^{\theta R}\bar{w}(r).$$

where $\bar{v}, \bar{w}$ (as well as $\bar{u}$ from (1.2)) are unit vectors in $\mathbb{R}^3$.

Given such a frame we introduce the differentiated fields ψ_k and the connection coefficients A_k by

$$\psi_k = \partial_k u \cdot v + i\partial_k u \cdot w, \qquad A_k = \partial_k v \cdot w. \tag{2.3}$$

Due to the equivariance of (u, v, w) it follows that both ψ_k and A_k are spherically symmetric (therefore subject to the conventions made in Section 1.1). Conversely, given ψ_k and A_k we can return to the frame (u, v, w) via the ODE system:

$$\begin{cases} \partial_k u = (\Re\psi_k)v + (\Im\psi_k)w \\ \partial_k v = -(\Re\psi_k)u + A_k w \\ \partial_k w = -(\Im\psi_k)u - A_k v \end{cases} \tag{2.4}$$

If we introduce the covariant differentiation

$$D_k = \partial_k + iA_k, \quad k \in \{0, 1, 2\}$$

it is a straightforward computation to check the compatibility conditions:

$$D_l \psi_k = D_k \psi_l, \quad l, k = 0, 1, 2. \tag{2.5}$$

The curvature of this connection is given by

$$D_l D_k - D_k D_l = i(\partial_l A_k - \partial_k A_l) = i\Im(\psi_l \bar{\psi}_k), \quad l, k = 0, 1, 2. \tag{2.6}$$

An important geometric feature is that ψ_2, A_2 are closely related to the original map. Precisely, for A_2 we have:

$$A_2 = (\overrightarrow{k} \times v) \cdot w = \overrightarrow{k} \cdot (v \times w) = \overrightarrow{k} \cdot u = u_3 \tag{2.7}$$

and, in a similar manner,

$$\psi_2 = w_3 - iv_3 \tag{2.8}$$

Since the (u, v, w) frame is orthonormal, it follows that $|\psi_2|^2 = u_1^2 + u_2^2$ and the following important conservation law

$$|\psi_2|^2 + A_2^2 = 1 \tag{2.9}$$

Now we turn our attention to the choice of the $(\bar{v}, \bar{w})$ frame at $\theta = 0$. Here we have the freedom of an arbitrary rotation depending on t and r. In this article we will use the Coulomb gauge, which for general maps u has the form $\operatorname{div} A = 0$. In polar coordinates this is written as $\partial_1 A_1 + r^{-2}\partial_2 A_2 = 0$. However, in the equivariant case A_2 is radial, so we are left with a simpler formulation $A_1 = 0$, or equivalently

$$\partial_r \bar{v} \cdot \bar{w} = 0 \tag{2.10}$$

which can be rearranged into a convenient ODE as follows

$$\partial_r \bar{v} = (\bar{v} \cdot \bar{u})\partial_r \bar{u} - (\bar{v} \cdot \partial_r \bar{u})\bar{u} \tag{2.11}$$

The first term on the right vanishes and could be omitted, but it is convenient to add it so that the above linear ODE is solved not only by $\bar{v}$ and $\bar{w}$, but also by $\bar{u}$. Then we can write an equation for the matrix $\mathcal{O} = (\bar{v}, \bar{w}, \bar{u})$:

$$\partial_r \mathcal{O} = M\mathcal{O}, \qquad M = \partial_r \bar{u} \wedge \bar{u} := \partial_r \bar{u} \otimes \bar{u} - \bar{u} \otimes \partial_r \bar{u} \tag{2.12}$$

with an antisymmetric matrix M.

An advantage of using the Coulomb gauge is that it makes the derivative terms in the nonlinearity disappear. Unfortunately, this only happens in the equivariant case, which is why in [6] we had to use a different gauge, namely the caloric gauge.

The ODE (2.11) needs to be initialized at some point. A change in the initialization leads to a multiplication of all of the ψ_k by a unit sized complex number. This is irrelevant at fixed time, but as the time varies we need to be careful and choose this initialization uniformly with respect to t, in order to avoid introducing a constant time dependent potential into the equations via A_0. Since in our results we start with data which converges asymptotically to $-\vec{k}$ as $r \to \infty$, and the solutions continue to have this property, it is natural to fix the choice of $\bar{v}$ and $\bar{w}$ at infinity,

$$\lim_{r \to \infty} \bar{v}(r, t) = \vec{i}, \qquad \lim_{r \to \infty} \bar{w}(r, t) = -\vec{j} \tag{2.13}$$

The existence of a unique solution $\bar{v} \in C((0,+\infty) : \mathbb{R}^3)$ of (2.11) satisfying (2.13) is standard, we skip the details. Moreover the solution is continuous with respect to u in the following sense

$$\|\bar{v} - \bar{\tilde{v}}\|_{L^\infty} \lesssim \|u - \tilde{u}\|_{\dot{H}^1} \tag{2.14}$$

2.2. **Schrödinger maps in the Coulomb gauge.** We are now prepared to write the evolution equations for the differentiated fields ψ_1 and ψ_2 in (2.3) computed with respect to the Coulomb gauge.

Writing the Laplacian in polar coordinates, a direct computation using the formulas (2.3) shows that we can rewrite the Schrödinger Map equation (1.1) in the form

$$\psi_0 = i\left(D_1\psi_1 + \frac{1}{r}\psi_1 + \frac{1}{r^2}D_2\psi_2\right) \tag{2.15}$$

Applying the operators D_1 and D_2 to both sides of this equation and using the relation (2.6) for $l,k = 1,2$ we obtain

$$\begin{aligned} D_1\psi_0 &= i\left(D_1(D_1 + \frac{1}{r})\psi_1 + \frac{1}{r^2}D_2D_1\psi_2\right) - \frac{1}{r^2}\Im(\psi_1\bar{\psi}_2)\psi_2 \\ D_2\psi_0 &= i\left((D_1 + \frac{1}{r})D_2\psi_1 + \frac{1}{r^2}D_2D_2\psi_2\right) - \Im(\psi_2\bar{\psi}_1)\psi_1 \end{aligned} \tag{2.16}$$

Using now (2.5) for $(k,l) = (0,1)$ respectively $(k,l) = (0,2)$ on the left and for $(k,l) = (1,2)$ on the right we can derive the evolution equations for ψ_m, $m = 1,2$:

$$\begin{aligned} D_0\psi_1 &= i\left(D_1(D_1 + \frac{1}{r}) + \frac{1}{r^2}D_2D_2\right)\psi_1 - \frac{1}{r^2}\Im(\psi_1\bar{\psi}_2)\psi_2 \\ D_0\psi_2 &= i\left((D_1 + \frac{1}{r})D_1 + \frac{1}{r^2}D_2D_2\right)\psi_2 - \Im(\psi_2\bar{\psi}_1)\psi_1 \end{aligned} \tag{2.17}$$

In our set-up all functions are radial and we are using the the Coulomb gauge $A_1 = 0$. Then these equations take the simpler form

$$\begin{aligned} \partial_t\psi_1 + iA_0\psi_1 =& i\Delta\psi_1 - i\frac{1}{r^2}A_2^2\psi_1 - i\frac{1}{r^2}\psi_1 + \frac{2}{r^3}A_2\psi_2 - \frac{1}{r^2}\Im(\psi_1\bar{\psi}_2)\psi_2 \\ \partial_t\psi_2 + iA_0\psi_2 =& i\Delta\psi_2 - i\frac{1}{r^2}A_2^2\psi_2 - \Im(\psi_2\bar{\psi}_1)\psi_1 \end{aligned}$$

The two variables ψ_1 and ψ_2 are not independent. Indeed, the relations (2.5) and (2.6) for $(k,l) = (1,2)$ give

$$\partial_r A_2 = \Im(\psi_1\bar{\psi}_2), \qquad \partial_r\psi_2 = iA_2\psi_1 \tag{2.18}$$

which at the same time describe the relation between ψ_1 and ψ_2 and determine A_2.

From the compatibility relations involving A_0, we obtain

$$\partial_r A_0 = -\frac{1}{2r^2}\partial_r(r^2|\psi_1|^2 - |\psi_2|^2) \tag{2.19}$$

from which we derive

$$A_0 = -\frac{1}{2}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) - [r\partial_r]^{-1}\left(|\psi_1|^2 - \frac{1}{r^2}|\psi_2|^2\right) \tag{2.20}$$

This is where the initialization of the Coulomb gauge at infinity is important. It guarantees that $A_0 \in L^p$, provided that $|\psi_1|^2 - r^{-2}|\psi_2|^2 \in L^p$ for $1 \leq p < \infty$. In particular, without any additional regularity assumptions, we know that $A_0 \in L^1$. A direct computation using integration by parts gives that

$$\int A_0(r) r dr = 0. \tag{2.21}$$

The system satisfied by ψ_1 and $\frac{\psi_2}{r}$ (this being in fact the correct variable instead of ψ_2) is given by:

$$\begin{aligned}
(i\partial_t + \Delta - \frac{2}{r^2})\psi_1 + \frac{4i}{r^2}\frac{\psi_2}{r} &= A_0\psi_1 + \frac{A_2^2 - 1}{r^2}\psi_1 + 2i\frac{A_2+1}{r^3}\psi_2 - i\Im(\psi_1\frac{\bar{\psi}_2}{r})\frac{\psi_2}{r} \\
(i\partial_t + \Delta - \frac{2}{r^2})\frac{\psi_2}{r} - \frac{4i}{r^2}\psi_1 &= A_0\frac{\psi_2}{r} + \frac{A_2^2 - 1}{r^2}\frac{\psi_2}{r} - 2i\frac{A_2+1}{r^2}\psi_1 - i\Im(\frac{\psi_2}{r}\bar{\psi}_1)\psi_1
\end{aligned}$$

The problem with this system is that its linear part is not decoupled. This can be remedied by a change of variables. Indeed consider

$$\psi^- = \psi_1 - i\frac{\psi_2}{r}, \qquad \psi^+ = \psi_1 + i\frac{\psi_2}{r}$$

It turns out that $\psi^\pm$ satisfy a similar system (described below) whose linear part is decoupled. The relevance of the variables $\psi^\pm$ comes also from the following reinterpretation. If $\mathcal{W}^\pm$ is defined as the vector

$$\mathcal{W}^\pm = \partial_r u \pm \frac{1}{r} u \times Ru \in T_u(\mathbb{S}^2)$$

then $\psi^\pm$ is the representation of $\mathcal{W}^\pm$ with respect to the frame (v, w). The vector field $\mathcal{W}^+$ vanishes if and only if u is an equivariant soliton which starts at the south pole at $r = 0$ and goes to the north pole at $r = \infty$. For $\mathcal{W}^-$ the poles get interchanged. On the other hand, a direct computation leads to

$$\begin{aligned}
E(u) &= \pi \int_0^\infty \left(|\partial_r \bar{u}|^2 + \frac{1}{r^2}|\bar{u} \times R\bar{u}|^2 \right) r dr \\
&= \pi \|\bar{\mathcal{W}}^\pm\|_{L^2}^2 \mp 2\pi(\bar{u}_3(\infty) - \bar{u}_3(0))
\end{aligned}$$

where we recall that $u(r,\theta) = e^{m\theta R}\bar{u}(r)$ and $\bar{u}_3(\infty) = \lim_{r\to\infty} \bar{u}_3(r), \bar{u}_3(0) = \lim_{r\to 0} \bar{u}_3(r)$ are well-defined since $\bar{u}_1, \bar{u}_2 \in \dot{H}^1_e$ and if $f \in \dot{H}^1_e$ then $\lim_{r\to 0} f(r) = \lim_{r\to 0} f(r) = 0$, see [10] or [7]. From the above computation and that $E(u) < 4\pi$ it follows that $\bar{u}_3(0) = \bar{u}_3(\infty) \in \{-1, +1\}$ and we have made the choice $\bar{u}_3(0) = \bar{u}_3(\infty) = -1$. This limit is invariant under the evolution. Indeed since $\bar{u}, \Delta\bar{u} \in \dot{H}^1$ it follows that $\bar{u} \times \Delta\bar{u} \in \dot{H}^1_e$, where we use the fact that $\dot{H}^1_e \subset L^\infty$ is an algebra. Therefore $\bar{u}_t \in \dot{H}^1_e$ hence $\bar{u}(0,t), \bar{u}(\infty,t)$ are continuous in time.

In our setup we have the following identity

$$\|\psi^\pm\|_{L^2}^2 = \|\bar{\mathcal{W}}^\pm\|_{L^2}^2 = \frac{E(u)}{\pi}. \tag{2.22}$$

From (2.14) it follows that the following continuity property holds true

$$\|\psi^\pm - \tilde{\psi}^\pm\|_{L^2} \lesssim \|u - \tilde{u}\|_{\dot{H}^1} \tag{2.23}$$

A direct computation yields the following system for $\psi^{\pm}$:

$$(i\partial_t + H^-)\psi^- = \left(A_0 - 2\frac{A_2+1}{r^2} + \frac{A_2^2-1}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}_1)\right)\psi^-$$
$$(i\partial_t + H^+)\psi^+ = \left(A_0 + 2\frac{A_2+1}{r^2} + \frac{A_2^2-1}{r^2} + \frac{1}{r}\Im(\psi_2\bar{\psi}_1)\right)\psi^+$$

where

$$H^- = \Delta - \frac{4}{r^2}, \qquad H^+ = \Delta.$$

Here and whenever Δ acts on radial functions, it is known that $\Delta = \partial_r^2 + \frac{1}{r}\partial_r$. By replacing $\psi_1 = \psi^- + ir^{-1}\psi_2$ and using $A_2^2 + |\psi_2|^2 = 1$, we obtain the key evolution system we work with in this paper,

$$\begin{cases} (i\partial_t + H^-)\psi^- = (A_0 - 2\frac{A_2+1}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}^-))\psi^- \\ (i\partial_t + H^+)\psi^+ = (A_0 + 2\frac{A_2+1}{r^2} + \frac{1}{r}\Im(\psi_2\bar{\psi}^+))\psi^+ \end{cases} \tag{2.24}$$

We will use this system in order to obtain estimates for $\psi^{\pm}$. The old variables ψ_1 and $\frac{\psi_2}{r}$ are recovered from

$$\psi_1 = \frac{\psi^+ + \psi^-}{2}, \qquad \frac{\psi_2}{r} = \frac{\psi^+ - \psi^-}{2i} \tag{2.25}$$

From the compatibility conditions (2.18) we derive the formula for A_2

$$A_2(r) + 1 = -\int_0^r \frac{|\psi^+|^2 - |\psi^-|^2}{4} sds \tag{2.26}$$

From (2.20) A_0 is given by

$$A_0 = -\frac{1}{2}\Re(\overline{\psi}^+\psi^-) + [r\partial_r]^{-1}\Re(\overline{\psi}^+\psi^-) \tag{2.27}$$

The compatibility condition (2.18) reduces then to

$$\partial_r r(\psi^+ - \psi^-) = -A_2(\psi^+ + \psi^-) \tag{2.28}$$

Next assume that $\psi^{\pm} \in L^2$ are given such that $\|\psi^-\|_{L^2}, \|\psi^+\|_{L^2} < 2\sqrt{2}$ and satisfy the compatibility conditions (2.28). We reconstruct A_2, ψ_2, ψ_1 using the (2.25) and (2.26). From (2.26) and (2.28) it follows that (2.18) hold true. From (2.26) it follows that $A_2 \in L^\infty$ and it is continuous and has limits both at 0 and ∞. From the definition of ψ_2 we have $\frac{\psi_2}{r} \in L^2$ and from (2.28) we derive $\partial_r\psi_2 \in L^2$, hence $\psi_2 \in \dot{H}^1_e$. From this and (2.26) it follows that $\partial_r A_2 \in L^2$, while by invoking (1.5) we obtain $\frac{A_2+1}{r} \in L^2$, therefore $A_2+1 \in \dot{H}^1_e$. In particular $A_2(\infty) = \lim_{r\to\infty} A_2(r) = -1$ which implies that $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2}$.

If $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2} < 2$ it follows from (2.26) that

$$\sup_{r\in(0,\infty)} A_2(r) \leq -1 + \frac{\|\psi^-\|_{L^2}^2}{4} < 0 \tag{2.29}$$

If $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2} < 2\sqrt{2}$ it follows from (2.26) that

$$\sup_{r\in(0,\infty)} A_2(r) \leq -1 + \frac{\|\psi^-\|_{L^2}^2}{4} < 1 \tag{2.30}$$

Most of the arguments in this paper will work for $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2} < 2\sqrt{2}$, but when involving the virial identities we will need the stronger conclusion in (2.29), hence the limitation of our final result to the case $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2} < 2$.

In fact one can keep track of a single variable, ψ^- or ψ^+ since it contains all the information about the map, provided that the choice of gauge (2.13) was made. To be more precise, (2.18) gives the following

$$\partial_r A_2 = \Im(\psi^- \bar{\psi}_2) + \frac{1}{r}|\psi_2|^2, \qquad \partial_r \psi_2 = iA_2\psi^- - \frac{1}{r}A_2\psi_2 \tag{2.31}$$

We will show that if $\|\psi^-\|_{L^2} < 2\sqrt{2}$, this system has a unique solution $A_2+1, \psi_2 \in \dot{H}^1_e$. From this we can reconstruct ψ_1, ψ^+, A_0. Then we can return to the map u via the system (2.4) with the boundary condition at infinity given by (2.13). A similar procedure can completely reconstruct u from ψ^+.

Keeping track of both variables $\psi^\pm$ (instead of just one) via the system (2.24) is advantageous for the dynamical (in time) properties of the problem. Understanding how to recover all the information from only one variable, say ψ^-, is advantageous for the elliptic part of the profile decomposition in Proposition 4.3.

2.3. **Regularity of the gauge elements.** In this section we clarify the regularity of the gauge elements. We extend $\psi^\pm$ to two-dimensional functions by

$$R_+\psi^+ := R_0\psi^+, \qquad R_-\psi^- = R_2\psi^- \tag{2.32}$$

Essentially we claim that if $u \in \dot{H}^1 \cap \dot{H}^3$ then the two-dimensional functions $R_\pm\psi^\pm \in H^2$. Our main claim is the following

Proposition 2.1. *If $u \in \dot{H}^3$ then $R_\pm\psi^\pm \in H^2$ and*

$$\|u\|_{\dot{H}^1\cap\dot{H}^3} \approx \|R_+\psi^+\|_{H^2} + \|R_-\psi^-\|_{H^2} \tag{2.33}$$

The proof of this result will be provided in the Appendix.

Therefore, in the context of $u \in \dot{H}^1 \cap \dot{H}^3$, we have that $R_\pm\psi^\pm \in H^2 \subset L^\infty$. The H^2 regularity cannot be extended to (two-dimensional extensions of) ψ_1 and $\frac{\psi_2}{r}$ since the ψ^+ and ψ^- require different phases for regularity. However, all the Sobolev embeddings are inherited by ψ_1 and $\frac{\psi_2}{r}$, in particular $\psi_1, \frac{\psi_2}{r} \subset L^\infty$. Since $A_2 = u_3$ it follows that $A_2 \in \dot{H}^1 \cap \dot{H}^3$ and $\partial_t A_2 \in H^1$. Finally by differentiating with respect to t the system (2.11), one can show that $\partial_t \bar{v} \in H^1$, hence $A_0 \in H^1$ which in turn gives $\partial_r A_0 \in L^2$. Therefore all the compatibility conditions in the previous two subsections are at least at the level of L^2.

2.4. **Recovering the map from ψ^-.** In this section we address the issue of re-constructing the Schrödinger map u together with its gauge elements from only one of its reduced variables, say ψ^-. Reconstructing ψ_2, A_2 such that $\psi_2, A_2 + 1 \in \dot{H}^1_e$ is a unique process; however, the reconstruction of the actual map with its frame, i.e. of (u, v, w) is unique provided one prescribes conditions at ∞. The map u satisfies $u(\infty) = -\vec{k}$, while the gauge is subjected to the choice (2.13).

The main result of this section is the following

Proposition 2.2. *Given $\psi^- \in L^2$, such that $\|\psi^-\|_{L^2} < 2\sqrt{2}$, there is a unique map u : $\mathbb{R}^2 \to \mathbb{S}^2$ with the property that ψ^- is the representation of $\mathcal{W}^-$ relative to a Coulomb gauge satisfying* (2.13)*. This also satisfies $E(u) = \pi\|\psi^-\|^2_{L^2}$.*

The map $\psi^- \to u$ is Lipschitz continuous in the following sense

$$\|u - \tilde{u}\|_{\dot{H}^1} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2} \tag{2.34}$$

Here ψ^+ can be reconstructed from ψ^-. Moreover the equations (2.31) which we use for reconstruction force the compatibility condition (2.28) between $\psi^\pm$. The result remains true if we start from ψ^+ just that we would start the reconstruction (described below) from the corresponding (2.31) written for ψ^+. The two problems are in effect equivalent via an inversion. The uniqueness of the reconstruction guarantees that starting from either ψ^+ or ψ^- (which are assumed to be compatible) gives the same u.

The proof consists of several steps. The first one deals with recovering the two gauge elements ψ_2, A_2 from ψ^- by using the system (2.31).

Lemma 2.3. *Given $\psi^- \in L^2$, such that $\|\psi^-\|_{L^2} < 2\sqrt{2}$, the system (2.31) has a unique solution (A_2, ψ_2) satisfying $\psi_2, A_2 + 1 \in \dot{H}^1_e$. This solution satisfies*

$$\|\psi_2\|_{\dot{H}^1_e} + \|A_2 + 1\|_{\dot{H}^1_e} + \|\frac{A_2 + 1}{r}\|_{L^1(dr)} \lesssim \|\psi^-\|_{L^2} \tag{2.35}$$

In addition we have the following properties:

i) given $\epsilon > 0$, and R such that $\|\psi^-\|_{L^2(\mathbb{R}\setminus[R^{-1},R])} \leq \epsilon$, then the following holds true

$$\|\psi_2\|_{\dot{H}^1_e(\mathbb{R}\setminus[\epsilon R^{-1},R])} + \|A_2 + 1\|_{\dot{H}^1_e(\mathbb{R}\setminus[\epsilon R^{-1},R])} \lesssim \epsilon \tag{2.36}$$

ii) if $(\tilde{A}_2, \tilde{\psi}_2)$ is another solution (as above) to (2.31) corresponding to $\tilde{\psi}^-$, then

$$\|\psi_2 - \tilde{\psi}_2\|_{\dot{H}^1_e} + \|A_2 - \tilde{A}_2\|_{\dot{H}^1_e} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2} \tag{2.37}$$

iii) if $(\tilde{A}_2, \tilde{\psi}_2)$ satisfy $\tilde{\psi}_2, \tilde{A}_2 + 1 \in \dot{H}^1_e$ and solve

$$\begin{aligned} \partial_r \tilde{\psi}_2 &= i\tilde{A}_2 \tilde{\psi}^- - \frac{1}{r}\tilde{A}_2 \tilde{\psi}_2 + E_1 \\ \partial_r \tilde{A}_2 &= \Im(\tilde{\psi}^- \bar{\tilde{\psi}}_2) + \frac{1}{r}(1 - \tilde{A}_2^2) + E_2 \end{aligned} \tag{2.38}$$

where $\||E_1| + |E_2|\|_{L^1(dr)+L^2} \lesssim \epsilon$ then

$$\|\psi_2 - \tilde{\psi}_2\|_{\dot{H}^1_e} + \|A_2 - \tilde{A}_2\|_{\dot{H}^1_e} \lesssim \|\psi^- - \tilde{\psi}^-\|_{L^2} + \epsilon \tag{2.39}$$

iv) if $\psi^- \in L^p$ with $1 \leq p < \infty$ then $\psi^+, \frac{\psi_2}{r}, \frac{1+A_2}{r} \in L^p$ and

$$\|\psi^+\|_{L^p} + \|\frac{\psi_2}{r}\|_{L^p} + \|\frac{1 + A_2}{r}\|_{L^p} \lesssim \|\psi^-\|_{L^p} \tag{2.40}$$

v) if $R_- \psi^- \in H^s$ then $R_+ \psi^+ \in H^s$ for any $s \in \{1, 2, 3\}$.

The reason for having the second type of statement in (2.39) is of technical nature and will be apparent in Section 4. The equation for $\tilde{A}_2$ in (2.38) is more convenient in that form when taking differences. For the original system (2.31) it does not matter how one writes the equation for A_2 thanks to the conservation law $|\psi_2|^2 + A_2^2 = 1$; however in the case of (2.38) this conservation law does not hold true, hence we write the system in the more convenient form (2.38).

Proof. Our strategy is to solve the ode system (2.31) from infinity. Since $\psi_2, 1 + A_2 \in \dot{H}^1_e$, it follows that $\lim_{r\to\infty} \psi_2 = 0, \lim_{r\to\infty} A_2 = -1$ These two conditions play the role of boundary conditions at infinity. Since $\partial_r(|\psi_2|^2 + A_2^2) = 0$, it follows from the conditions at ∞ that $|\psi_2|^2 + A_2^2 = 1$ holds on all of $\mathbb{R}_+$. We define

$$\psi_1 = \psi^- + i\frac{\psi_2}{r}, \qquad \psi^+ = \psi^- + 2i\frac{\psi_2}{r}$$

and note that

$$\partial_r A_2 = \frac{1}{4} r(|\psi^+|^2 - |\psi^-|^2)$$

The function A_2 we seek must satisfy $1 + A_2 \in \dot{H}^1_e$, therefore it needs to have limit zero at both $r = 0$ and $r = \infty$. Hence integrating from infinity it gives

$$1 + A_2 \leq \frac{1}{4}\|\psi^-\|^2_{L^2} < 2$$

Thus we must have $\sup_{r\in(0,\infty)} A_2(r) < 1$ everywhere (which is to say that the corresponding map u cannot reach the north pole).

To establish existence and uniqueness for (2.31) it suffices to do so in a neighbourhood $[R, +\infty)$ of infinity. The extension of this down to $r = 0$ follows from standard arguments since $L^2(rdr) \subset L^1_{loc}(dr)$.

To prove existence, by choosing R large enough we can assume without any restriction in generality that

$$\|\psi^-\|_{L^2(R,\infty)} \leq \epsilon \tag{2.41}$$

and seek (ψ_2, A_2) with the property that

$$\|\psi_2\|_{\dot{H}^1_e(R,\infty)} \lesssim \epsilon \tag{2.42}$$

This implies that $|\psi_2| \lesssim \epsilon$ and, by the relation $|\psi_2|^2 + A_2^2 = 1$, it also gives $A_2 + 1 \lesssim \epsilon^2$. Hence we are allowed to substitute $A_2(r) = -\sqrt{1 - |\psi_2(r)|^2}$ in the ψ_2 equation and discard the dependent A_2 equation. We rewrite the ψ_2 equation as

$$(\partial_r - \frac{1}{r})\psi_2 = -i\psi^- + i\frac{A_2 + 1}{r}\psi^- - \frac{(A_2 + 1)\psi_2}{r^2}$$

or equivalently

$$r\partial_r \frac{\psi_2}{r} = -i\psi^- + i(A_2 + 1)\psi^- - \frac{(A_2 + 1)\psi_2}{r}$$

and further

$$\psi_2 = -ir[r\partial_r]^{-1}\psi^- + r[r\partial_r]^{-1}(i(A_2 + 1)\psi^- - \frac{(A_2 + 1)\psi_2}{r})$$

We know from (1.5) that $[r\partial_r]^{-1}$ maps L^2 to L^2, which easily implies that

$$r[r\partial_r]^{-1} : L^2 \to \dot{H}^1_e$$

Hence in order to obtain ψ_2 via the contraction principle it suffices to show that for ψ as in (2.41) and ψ_2 as in (2.42) the map

$$\psi_2 \to i(A_2 + 1)\psi^- - \frac{(A_2 + 1)\psi_2}{r}$$

is Lipschitz from $\dot{H}^1_e \to L^2$ with a small ($O(\epsilon)$ in this case) Lipschitz constant. But this is straightforward due to the embedding $\dot{H}^1_e \subset L^\infty$. Thus the existence of ψ_2 in $[R, \infty)$ follows, and the corresponding A_2 is recovered via $A_2(r) = -\sqrt{1 - |\psi_2(r)|^2}$. The same argument also gives Lipschitz dependence of ψ_2 on ψ^- in $[R, \infty)$.

Once we have the global solution (ψ_2, A_2) we obtain the bound (2.35) via an energy type estimate. Precisely, denoting

$$F = \frac{\psi_2}{1 - A_2}$$

its derivative satisfies

$$\left| \frac{d}{dr}|F|^2 - \frac{2}{r}|F|^2 \right| \lesssim \frac{|\psi^-|}{1 - A_2}|F|$$

Since $1 - A_2$ is bounded from below, this further leads to

$$\left| \frac{d}{dr}\frac{|F|}{r} \right| \lesssim \frac{|\psi^-|}{r}$$

Integrating from infinity we obtain

$$|F| \lesssim r[r\partial_r]^{-1}|\psi^-|$$

Returning to ψ_2 we get the pointwise bound

$$|\psi_2| \lesssim r[r\partial_r]^{-1}|\psi^-| \tag{2.43}$$

By the L^2 boundedness of $[r\partial_r]^{-1}$ this gives the L^2 bound on $\frac{\psi_2}{r}$. Then the L^2 bounds for $\partial_r \psi_2$ and $\partial_r A_2$ follow directly from (2.31), while the L^2 and the $L^1(dr)$ bound for $\frac{1+A_2}{r}$ are consequences of the compatibility relation $|\psi_2|^2 + A_2^2 = 1$.

The above argument already gives the $[R, \infty)$ part of (2.36) since (2.43) holds on any such interval. Getting the $(0, \epsilon R^{-1}]$ part of (2.36) is slightly more delicate. It suffices to get the L^2 bound for $\frac{\psi_2}{r}$. From (2.43) we have

$$|\psi_2| \lesssim r[r\partial_r]^{-1}(1_{(0,R^{-1}]}|\psi^-|) + r[r\partial_r]^{-1}(1_{[R^{-1},\infty)}|\psi^-|)$$

For the first term we use the smallness of ψ_2 in the hypothesis. For the second we instead produce a pointwise bound using Cauchy-Schwarz:

$$r[r\partial_r]^{-1}(1_{[R^{-1},\infty)}|\psi^-|) \lesssim r\int_{R^{-1}}^\infty s^{-1}|\psi^-(s)|ds \lesssim rR\|\psi_2\|_{L^2}, \qquad r < R^{-1}$$

This implies the desired L^2 bound.

The reason for this difference between the argument near infinity and the one near zero has to do with the following observation. If $\psi^- = 0$, then the system (2.31) has many solutions which are completely described by $(\psi_2(r), A_2(r)) = (e^{i\theta}h_1(\lambda r), h_3(\lambda r))$, for some $\lambda \in [0, \infty]$ where

$$h_1(r) = \frac{2r}{1 + r^2}, \qquad h_3 = \frac{r^2 - 1}{r^2 + 1}.$$

These solutions are generated by the equivariant harmonic maps (which solve $u \times \Delta u = 0$) written in their Coulomb basis as described above. The main equivariant harmonic map is $Q(r, \theta) = e^{\theta R}(h_1(r), 0, h_3(r))^T$ and all the other ones are of the form $Q(\lambda r, \theta + \alpha)$. Constructing the associated Coulomb gauge, as described above, we obtain $(\psi_2(r), A_2(r)) = (e^{i\theta}h_1(\lambda r), h_3(\lambda r))$ See [7] for more details. These are not good global candidates for our system, unless $\lambda = 0$ since $\lim_{r\to\infty} h_3(r) = 1$. But they can be good local candidates near $r = 0$.

Next we turn our attention to (2.37) and (2.39). In fact, in the case of (2.37), in light of the conservation law $|\tilde{\psi}_2|^2 + \tilde{A}_2^2 = 1$, (2.37) follows from (2.39) with $E_1 = E_2 = 0$. Hence we focus our attention on (2.39). We denote

$$\delta\psi = \tilde{\psi} - \psi, \qquad \delta A_2 = \tilde{A}_2 - A_2, \qquad \delta\psi_2 = \tilde{\psi}_2 - \psi_2$$

Without any restriction in generality we can make the assumption $\|\delta\psi\|_{L^2} \ll 1$ and the bootstrap assumption

$$\|\delta\psi_2\|_{L^\infty} + \|\delta A_2\|_{L^\infty} + \|\frac{\delta\psi_2}{r}\|_{L^\infty} + \|\frac{\delta A_2}{r}\|_{L^\infty} \lesssim \epsilon^{\frac{1}{2}} + \|\delta\psi\|_{L^2}^{\frac{1}{2}} \tag{2.44}$$

Then we derive the equations for them modulo error terms. We have

$$\partial_r \delta\psi_2 = i\delta A_2 \tilde{\psi}^- + iA_2\delta\psi - \frac{1}{r}A_2\delta\psi_2 - \frac{1}{r}\delta A_2\tilde{\psi}_2 + E_1$$
$$\partial_r \delta A_2 = \Im(\psi^-\overline{\delta\psi_2}) + \Im(\delta\psi\overline{\tilde{\psi}_2}) - \frac{2}{r}A_2\delta A_2 - \frac{1}{r}(\delta A_2)^2 + E_2$$

The following terms $iA_2\delta\psi, \Im(\delta\psi\overline{\tilde{\psi}_2})$ can be directly included into the error terms E_1, E_2, while the quadratic term $\frac{1}{r}(\delta A_2)^2$ can be included in the error term E_2 based on (2.44). We obtain the following linear system for $(\delta\psi_2, \delta A_2)$:

$$\partial_r \delta\psi_2 = \frac{1}{r}\delta\psi_2 + i\tilde{\psi}^-\delta A_2 - \frac{1}{r}(A_2+1)\delta\psi_2 - \frac{1}{r}\delta A_2\tilde{\psi}_2 + E_1$$
$$\partial_r \delta A_2 = \frac{2}{r}\delta A_2 + \Im(\psi^-\overline{\delta\psi_2}) - \frac{2}{r}(1+A_2)\delta A_2 + E_2$$

By considering the $\Re\delta\psi_2, \Im\delta\psi_2$ separately, this is a system of the form

$$\partial_r X = \frac{1}{r}LX + BX + F, \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

where the matrix B, F satisfy $B \in L^2$ and $F \in L^2 + L^1(dr)$. This system needs to be solved with zero Cauchy data at infinity. For this system we need to establish the bound

$$\|X\|_{L^\infty} + \|\frac{X}{r}\|_{L^2} \lesssim \|F\|_{L^2+L^1(dr)} \tag{2.45}$$

If $B = 0$ then

$$X = \begin{pmatrix} r[r\partial_r]^{-1} & 0 & 0 \\ 0 & r[r\partial_r]^{-1} & 0 \\ 0 & 0 & r^2[r^2\partial_r]^{-1} \end{pmatrix} F$$

and the conclusion easily follows from argument of type (1.5). If B is small in either $L^2(rdr)$ or in $r^{-1}L^\infty$ then we can treat the BX term perturbatively. If B is large then some more work is needed. We decompose $B = B_1 + B_2$ where $B_1 \in L^1(dr)$ and $|B_2| \ll \frac{1}{r}$. We can construct the bounded matrix $e^{\int B_1}$ as a solution of $\partial_r e^{\int B_1} = e^{\int B_1} B_1$ which also has a bounded inverse. Then we can eliminate B_1 by conjugating with respect to $e^{\int B_1}$, and then treat the part with B_2 perturbatively.

iv) From (2.43) and (1.5) we obtain

$$\|\frac{\psi_2}{r}\|_{L^p} \lesssim \|\psi^-\|_{L^p}$$

from which (2.40) follows since $\psi^+ = 2i\frac{\psi_2}{r} + \psi^-$ and $1 + A_2 = \frac{|\psi_2|^2}{1-A_2}$.

v) Throughout this argument, the use of Sobolev embedding refers to the two-dimensional standard Sobolev embeddings which apply to $R_\pm\psi^\pm$, which then can be read in terms of $\psi^\pm$.

If $s = 1$ then we have

$$\partial_r\psi^+ = 2i\partial_r\frac{\psi_2}{r} + \partial_r\psi^- = \frac{2i}{r}(iA_2\psi^- - \frac{1}{r}(A_2 + 1)\psi_2) + \partial_r\psi^-$$

Using (2.40) we estimate

$$\begin{aligned}\|\partial_r\psi^+\|_{L^2} &\lesssim \|A_2\|_{L^\infty}\|\frac{\psi^-}{r}\|_{L^2} + \|\frac{A_2+1}{r}\|_{L^4}\|\frac{\psi_2}{r}\|_{L^4} + \|\partial_r\psi^-\|_{L^2} \\ &\lesssim \|\psi^-\|_{\dot{H}^1_e} \approx \|R_-\psi^-\|_{H^1}\end{aligned}$$

which implies that $\|R_+\psi^+\|_{H^1} \lesssim \|R_-\psi^-\|_{H^1}$.

If $s = 2$ then,

$$\begin{aligned}\partial_r^2\psi^+ &= \partial_r[\frac{2i}{r}(iA_2\psi^- - \frac{1}{r}(A_2+1)\psi_2) + \partial_r\psi^-] \\ &= (\partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2})\psi^- + 2i\partial_r[\frac{1+A_2}{r}(i\psi^- - \frac{\psi_2}{r})] \\ &= (\partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2})\psi^- - 2i\frac{1+A_2}{r^2}(i\psi^- - \frac{\psi_2}{r}) \\ &+ 2i(\Im(\psi^-\frac{\bar{\psi}_2}{r}) + \frac{|\psi_2|^2}{r^2})(i\psi^- - \frac{\psi_2}{r}) \\ &+ 2i\frac{1+A_2}{r}[i\partial_r\psi^- - \frac{1}{r}(iA_2\psi^- - \frac{1}{r}(A_2+1)\psi_2)]\end{aligned}$$

From part ii) of Lemma 1.3 we have that $(\partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2})\psi^- \in L^2$. All the other terms are estimated in L^2 by using that $\frac{\psi^-}{r^2} \in L^2$, the Sobolev embedding $\psi^- \in L^4 \cap L^6$ which implies by (2.40) that $\frac{\psi_2}{r} \in L^4 \cap L^6$ and that $1 + A_2 = \frac{|\psi_2|^2}{1-A_2}$.

The term $\frac{1}{r}\partial_r\psi^+$ is treated via a similar but easier argument. This shows that $(\partial_r^2 + \frac{1}{r}\partial_r)\psi^+ \in L^2$ which amounts to $R_+\psi_+ \in L^2$.

If $s = 3$ then it is enough to show that $\partial_r\Delta\psi^+ \in L^2$ in order to conclude that $R_+\psi^+ \in \dot{H}^3$. A direct computation gives

$$\partial_r(\partial_r + \frac{1}{r})\partial_r\psi^+ = (\partial_r + \frac{2}{r})(\partial_r - \frac{1}{r})\partial_r\psi^+ = (\partial_r + \frac{2}{r})(\Delta - \frac{4}{r^2})\psi^- + (\partial_r + \frac{2}{r})S$$

where

$$\begin{aligned}\frac{1}{2i}S = &- 2\frac{1+A_2}{r^2}(i\psi^- - \frac{\psi_2}{r}) + (\Im(\psi^-\frac{\bar{\psi}_2}{r}) + \frac{|\psi_2|^2}{r^2})(i\psi^- - \frac{\psi_2}{r}) \\ &+ \frac{1+A_2}{r}[i\partial_r\psi^- - \frac{1}{r}(iA_2\psi^- - \frac{1}{r}(A_2+1)\psi_2)]\end{aligned}$$

From the hypothesis that $\Delta R_-\psi^- \in H^1$ we obtain that $(\Delta - \frac{4}{r^2})\psi^- \in \dot{H}^1_e$, hence $(\partial_r + \frac{2}{r})(\Delta - \frac{4}{r^2})\psi^- \in L^2$. The term $(\partial_r + \frac{2}{r})S$ is further expanded by using the above computation for S, the system (2.31) and the fact that $(\partial_r + \frac{2}{r})\frac{1}{r^2} = 0$. By using Sobolev embeddings one shows that $(\partial_r + \frac{2}{r})S \in L^2$, the details are left to the reader.

□

Proof of Proposition 2.2. With ψ_2, A_2 constructed above, we can reconstruct $\psi_1 = \psi^- + i\frac{\psi_2}{r}$. Then we solve the system (2.4) at the level of $(\bar{u}, \bar{v}, \bar{w})$. We would like to solve this system with condition at ∞, $\bar{u} = -\overrightarrow{k}, \bar{v} = \overrightarrow{i}, \bar{w} = \overrightarrow{j}$. But this cannot be done apriori. Indeed, consider the coefficient matrix in (2.4)

$$M = \begin{pmatrix} 0 & \Re\psi_1 & \Im\psi_1 \\ -\Re\psi_1 & 0 & 0 \\ -\Im\psi_1 & 0 & 0 \end{pmatrix}$$

Since $M \notin L^1(dr)$, it is not meaningful to initialize the problem (2.4) at ∞. However M has another structure which is a consequence of (2.18) rewritten as $\psi_1 = (A_2 + 1)\psi_1 + i\partial_r\psi_2$. Therefore $M = N + \partial_r K$ and, by (2.35), N, K satisfy

$$\|N\|_{L^1(dr)} + \|K\|_{\dot{H}^1_e} \lesssim \|\psi^-\|_{L^2}$$

This inequality localizes on intervals $[r, \infty)$ due to (2.36). This allows us to construct solutions with data at $r = \infty$ by using the iteration scheme

$$X = \sum_i X_i, \qquad X_0 = X(\infty), \qquad X_i(r) = \int_r^\infty M(s)X_{i-1}ds$$

We run the iteration scheme in the space $C([r, \infty])$ of continuous functions on (r, ∞) which have limits at ∞. Under the assumption that $X_{i-1} \in C([r, \infty])$ we obtain

$$\begin{aligned} X_i(r) &= \int_r^\infty (N(s) + \partial_s K(s))X_{i-1}ds \\ &= \int_r^\infty N(s)X_{i-1}ds - K(r)X_{i-1}(r) - \int_r^\infty K(s)\partial_s X_{i-1}(s)ds \end{aligned}$$

and further that

$$\|\partial_r X_i\|_{L^2([r,\infty))} + \|X_i\|_{C([r,\infty])} \lesssim \|\psi^-\|_{L^2([r,\infty))}(\|X_{i-1}\|_{L^\infty([r,\infty])} + \|\partial_r X_{i-1}\|_{L^2([r,\infty))})$$

Therefore, inductively, we obtain

$$\|\partial_r X_i\|_{L^2([r,\infty))} + \|X_i\|_{C([r,\infty])} \lesssim \|\psi^-\|^i_{L^2([r,\infty))}$$

By choosing R large such that $\|\psi\|_{L^2([R,\infty))}$ is small, we can rely on an iteration scheme to construct the solution X on $[R, \infty)$.

The uniqueness of this solution is guaranteed by the conservation law $\|X\| = \textit{constant}$ which follows from the antisymmetry of M.

This also guarantees that the orthonormality conditions imposed at ∞ are preserved (recall that ∞, $\bar{u} = -\overrightarrow{k}, \bar{v} = \overrightarrow{i}, \bar{w} = \overrightarrow{j}$). The solution constructed above can be extended to $(0, \infty)$ by running a similar argument on intervals where $\|\psi^-\|_{L^2(I)}$ is small, where the last interval is of the form $(0, r]$.

The above argument leads to an estimate of the form

$$\|X\|_{C([0,\infty])} + \|\partial_r X\|_{L^2} \lesssim \|\psi^-\|_{L^2}$$

where by $C([0, \infty])$ we mean continuous functions on $(0, \infty)$ which have limits at 0 and ∞.

Additional information on $\bar u, \bar v, \bar w$ will be obtained in a different manner. Notice that $\bar u_3$ and $\zeta = \bar w_3 - i\bar v_3$ solve the system

$$\partial_r \bar u_3 = \Im(\psi_1 \bar\zeta), \qquad \partial_r \zeta = i\bar u_3 \psi_1$$

which is the same as the one satisfied by A_2, ψ_2. Since they obey the same conditions at ∞ we conclude that $\bar u_3 = A_2, \zeta = \psi_2$. From this and the fact that $|\psi_2|^2 + A_2^2 = 1$ it follows also that $|\bar u_1|^2 + |\bar u_2|^2 = |\psi_2|^2$.

Next, we extend the system of vectors to u, v, w using the equivariant setup, i.e. by multiplying them with $e^{\theta R}$. Using the identification just described above and the orthonormality conditions, it follows that (2.4) is satisfied for $k = 2$. Therefore we have just established the existence of an equivariant map u whose vector field $\mathcal{W}^-$ in the gauge (v, w) is ψ^- and whose gauge elements are ψ_1, ψ_2, A_2. Moreover, we have that

$$E(u) = \pi \|\psi^-\|_{L^2}$$

Given two fields $\psi^-, \tilde\psi^-$ we reconstruct X and $\tilde X$ as above. Since the construction is iterative it also follows that

$$\|X - \tilde X\|_{C[0,\infty]} + \|\partial_r (X - \tilde X)\|_{L^2} \lesssim \|\psi - \tilde\psi\|_{L^2}$$

from estimate for of the derivative part in $E(u - \tilde u)$ follows. Since $u_1 = v_2 w_3 - v_3 w_2$, $\tilde u_1 = \tilde v_2 \tilde w_3 - \tilde v_3 \tilde w_2$, $\psi_2 = \bar w_3 - i\bar v_3$ and $\tilde\psi_2 = \bar{\tilde w}_3 - i\bar{\tilde v}_3$ it follows that

$$\|\frac{u_1 - \tilde u_1}{r}\|_{L^2} \lesssim \|\frac{\psi_2 - \tilde\psi_2}{r}\|_{L^2} \|X\|_{L^\infty} + \|X - \tilde X\|_{L^\infty} \|\frac{\tilde\psi_2}{r}\|_{L^2} \lesssim \|\psi^- - \tilde\psi^-\|_{L^2}$$

A similar argument shows that $\|\frac{u_2 - \tilde u_2}{r}\|_{L^2} \lesssim \|\psi^- - \tilde\psi^-\|_{L^2}$ which completes the proof of (2.34).

□

3. The Cauchy problem

In this section we are concerned with the nonlinear system of equations (2.24) which we recall here

$$\begin{cases} (i\partial_t + H^-)\psi^- = (A_0 - 2\frac{A_2+1}{r^2} - \frac{1}{r}\Im(\psi_2 \bar\psi^-))\psi^- \\ (i\partial_t + H^+)\psi^+ = (A_0 + 2\frac{A_2+1}{r^2} + \frac{1}{r}\Im(\psi_2 \bar\psi^+))\psi^+ \end{cases}$$

where ψ_2, A_2, A_0 are given by (2.25), (2.26), respectively (2.27). The problem comes with an initial data $\psi^\pm(t_0) = \psi_0^\pm$ and we would like to understand its well-posedness on intervals $I \subset \mathbb{R}$ with $t_0 \in I$.

We will be mainly interested in solutions of this system which come from Schrödinger maps with energy below the critical threshold, i.e. they satisfy the compatibility conditions (2.28) and with $\|\psi^+\|_{L^2} = \|\psi^-\|_{L^2} < 2\sqrt{2}$.

For simplicity we denote the nonlinearities by

$$\begin{aligned} N^-(\psi^-) &= (A_0 - 2\frac{A_2+1}{r^2} - \frac{1}{r}\Im(\psi_2 \bar\psi^-))\psi^- \\ N^+(\psi^+) &= (A_0 + 2\frac{A_2+1}{r^2} + \frac{1}{r}\Im(\psi_2 \bar\psi^+))\psi^+ \end{aligned} \tag{3.1}$$

We define the mass of a function f by $M(f) := \|f\|_{L^2}^2$. The system (2.24) formally conserves the mass, i.e. $M(\psi^-(t)) = M(\psi^-(0))$ and $M(\psi^+(t)) = M(\psi^+(0))$ for all t in the interval of existence. Moreover, as discussed in subsection 2.2, a compatible pair also satisfies $\|\psi^+(0)\|_{L^2} = \|\psi^-(0)\|_{L^2}$.

3.1. **Strichartz estimates.** We begin our analysis with the linear equation

$$(i\partial_t + H_k)u = f, \qquad u(0) = u_0 \tag{3.2}$$

where we recall $H_k = \partial_r^2 + \frac{1}{r}\partial_r - \frac{k^2}{r^2}$. Note that $H^+ = H_0$ and $H^- = H_2$. For consistency in writing we may also choose to use sometimes $R_+ = R_0$ and $R_- = R_2$ (recall (1.7)).

Our first claim is that, for each k, u satisfies the standard Strichartz estimates

$$\||\nabla|^s R_k u\|_{L_t^p L_r^q} \lesssim \||\nabla|^s R_k u_0\| + \||\nabla|^s R_k f\|_{L_t^{\tilde{p}'} L_r^{\tilde{q}'}} \tag{3.3}$$

where $|\nabla|^s = (-\Delta)^{\frac{s}{2}}$ (defined in the usual manner), $(p,q), (\tilde{p}, \tilde{q})$ are admissible pairs in two dimensions ($\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, 2 < p \leq \infty$) and $(\tilde{p}', \tilde{q}')$ is the dual pair of $(\tilde{p}, \tilde{q})$. Indeed, $R_k u$ satisfies the following equation

$$(i\partial_t + \Delta)R_k u = R_k f, \qquad R_k u(0) = R_k u_0$$

Then (3.3) follows from the Strichartz estimates for the Schrödinger equation in two dimensions. We need to be able to read the Strichartz estimates at the level of the radial functions. For even powers of s we use the identity $\Delta R_k v = R_k H_k v$, hence

$$\|H_k v\|_{L_t^p L_r^q} = \|\Delta R_k v\|_{L_t^p L_x^q} \tag{3.4}$$

and this can be extended to higher regularity but we will not need it.

For odd values of s we use that $|\nabla|^s = |\nabla|(-\Delta)^{\frac{s-1}{2}}$ and that for $k \neq 0$

$$\|\partial_r v\|_{L_t^p L_r^q} + \|\frac{v}{r}\|_{L_t^p L_r^q} \lesssim \||\nabla| R_k v\|_{L_t^p L_x^q} \tag{3.5}$$

while for $k = 0$

$$\|\partial_r v\|_{L_t^p L_r^q} \lesssim \||\nabla| R_k v\|_{L_t^p L_x^q} \tag{3.6}$$

In the context of additional regularity, we need improved versions of the Strichartz estimates as described below.

Lemma 3.1. *Assume that $u^\pm$ satisfy* (3.2) *with initial data $u_0^\pm$ and forcing $f^\pm$.*

i) If $u_0^+ \in L^2$ is such that $H^+ u_0^+ \in L^2$, then the following holds true

$$\||\partial_r^2 u^+| + |\frac{\partial_r u^+}{r}|\|_{L^\infty L^2 \cap L^4 L^4 \cap L^3 L^6} \lesssim \|H^+ u_0^+\|_{L^2} + \|H^+ f^+\|_{L^1 L^2}$$

ii) If $u_0^- \in L^2$ is such that $H^- u_0^- \in L^2$, then the following holds true

$$\||\partial_r^2 u^-| + |\frac{\partial_r u^-}{r}| + |\frac{u^-}{r^2}|\|_{L^\infty L^2 \cap L^4 L^4 \cap L^3 L^6} \lesssim \|H^- u_0^-\|_{L^2} + \|H^- f^-\|_{L^1 L^2}$$

iii) If $u_0^- \in L^2$ is such that $H^- u_0^- \in \dot{H}_e^1$, then the following holds true

$$\|\frac{1}{r}(\partial_r - \frac{1}{r})\partial_r u^-\|_{L^\infty L^2 \cap L^4 L^4 \cap L^3 L^6} \lesssim \|H^- u_0^-\|_{\dot{H}_e^1} + \|H^- f^-\|_{L^1 \dot{H}_e^1}$$

These are improved versions of Strichartz estimates from the following point of view. In ii) the inequality for $(\partial_r^2 + \frac{1}{r}\partial_r - \frac{4}{r^2})u = H_2 u$ is the Strichartz estimate for $H_2 u$ which follows from (3.3) and (3.4); our statement is stronger in saying that each term satisfies the Strichartz estimate. A similar remark is in place for part i).

Proof. i) The proof follows the same lines as the one in ii), though it is easier.

ii) Without restricting the generality of the argument we can assume that $f^- = 0$. Based on the representation formula

$$u^- = \int e^{it\xi^2} J_2(r\xi)\mathcal{F}_2 u_0^-(\xi)\xi d\xi$$

and by using (1.6), we compute

$$\partial_r u^- = \frac{1}{2}\int e^{it\xi^2}(J_1(r\xi)) - J_3(r\xi))\xi\mathcal{F}_2 u_0^-(\xi)\xi d\xi = e^{itH_1}g_1 - e^{itH_3}g_3$$

with $R_1 g_1, R_3 g_3 \in \dot{H}^1$. Using (3.3) and (3.5) we obtain the conclusion for $|\partial_r^2 u|$ and $|\frac{1}{r}\partial_r u|$. Since the estimate holds true for $H^- f^-$, it follows for $\frac{|f^-|}{r^2}$.

iii) We note that, by using (1.6), $(\partial_r - \frac{1}{r})J_1 = r\frac{1}{r}\partial_r J_1 = -r\frac{J_2}{r^2} = -\frac{J_2}{r}$. Using this and again (1.6) we start in part i) and compute

$$\begin{aligned}(\partial_r - \frac{1}{r})\partial_r u^- = &-\frac{1}{2r}\int e^{it\xi^2} J_2(r\xi)\xi\mathcal{F}_2 u_0^-(\xi)\xi d\xi \\ &-\frac{1}{4}\int e^{it\xi^2}(J_2(r\xi) - J_4(r\xi))\xi^2\mathcal{F}_2 u_0^-(\xi)\xi d\xi\end{aligned}$$

The conclusion follows then from (3.3) and (3.5). □

3.2. **Setup and the Cauchy theory.** In order to make estimates shorter, we make the following notation convention $\|f^\pm\| = \|f^+\| + \|f^-\|$ for various f's and $\|\cdot\|$ involved in the rest of the paper.

Since our non-linear analysis relies mostly on the $L^4_{t,r}$ norm, we define the Strichartz norm of $f : I \times \mathbb{R}^2 \to \mathbb{C}$ by $S_I(f) := \|f\|^4_{L^4(I\times\mathbb{R})}$. If $t_0 \in I$ then we define $S_{I,\leq t_0} f = \|1_{I\cap(-\infty,t_0]} f\|^4_{L^4}$ and $S_{I,\geq t_0} f = \|1_{I\cap[t_0,\infty)} f\|^4_{L^4}$.

We say that a solution $\psi^\pm : I \times \mathbb{R} \to \mathbb{C}$ blows up forward in time if $S_{I,\geq t}\psi^\pm = +\infty, \forall t \in I$. Similarly $\psi^\pm$ blows up backward in time if $S_{I,\leq t}\psi^\pm = +\infty, \forall t \in I$.

A possibility that may occur is that for some interval I, $S_{I,\geq t_0}\psi^+ = +\infty$ while $S_{I,\geq t_0}\psi^- < \infty$, or any other combination. However from (3.9) it follows that solutions satisfying the compatibility condition (2.28) we have that $S_J(\psi^+) \approx S_J(\psi^-)$ on any time interval J. Therefore for such solutions (which we will be mainly interested in) the above scenario is ruled out.

Let $\psi^\pm_+ \in L^2$. We say that the solution $\psi^\pm : I \times \mathbb{R} \to \mathbb{C}$ scatters forward in time to $\psi^\pm_+$ iff $supI = +\infty$ and $\lim_{t\to\infty} M(\psi^\pm(t) - e^{itH^\pm}\psi^\pm_+) = 0$. We say that the solution $\psi^\pm : I \times \mathbb{R} \to \mathbb{C}$ scatters backward in time to $\psi^\pm_-$ iff $infI = -\infty$ and $\lim_{t\to-\infty} M(\psi^\pm(t) - e^{itH^\pm}\psi^\pm_-) = 0$.

Our first theorem provides the general Cauchy theory for (2.24).

Theorem 3.2. *Consider the problem* (2.24) *(with ψ_2, A_2, A_0 given by* (2.25), (2.26), (2.27)*) with $\psi^\pm_0 \in L^2$. Then there exists a unique maximal-lifespan solution pair $(\psi^+, \psi^-) : I \times \mathbb{R}^2$ with $t_0 \in I$ and $\psi^\pm(t_0) = \psi^\pm_0$ with the additional properties:*

i) I is open.

ii) (Forward scattering) If $\psi^\pm$ do not blow up forward in time, then $I_+ = [0, \infty)$ and $\psi^\pm$ scatters forward in time to $e^{itH^\pm}\psi^\pm_+$ for some $\psi^\pm_+ \in L^2$.

Conversely, if $\psi^\pm_+ \in L^2$, then there exists a unique maximal-lifespan solution $\psi^\pm$ which scatters forward in time to $e^{itH^\pm}\psi^\pm_+$.

iii) (Backward scattering) A similar statement to ii) holds true for the backward in time problem.

iv) (Small data scattering) There exist $\epsilon > 0$ such that if $M(\psi_0^\pm) \leq \epsilon$ then $S_{\mathbb{R}}(\psi^\pm) \lesssim M(\psi_0^\pm)$. In particular, the solution does not blow up and we have global existence and scattering in both directions.

v) (Uniformly continuous dependence) For every $A > 0$ and $\epsilon > 0$ there is $\delta > 0$ such that if $\psi^\pm$ is a solution satisfying $S_J(\psi^\pm) \leq A$ and $t_0 \in J$, and such that $M(\psi_0^\pm - \tilde{\psi}_0^\pm) \leq \delta$, then there exists a solution such that $S(\psi^\pm - \tilde{\psi}^\pm) \leq \epsilon$ and $M(\psi^\pm(t) - \tilde{\psi}^\pm(t)) \leq \epsilon, \forall t \in J$.

vi) (Stability result) For every $A > 0$ and $\epsilon > 0$ there exists $\delta > 0$ such that if $S_J(\psi^\pm) \leq A$, $\psi^\pm$ approximate (2.24) in the following sense

$$\|(i\partial_t + H^\pm)\psi^\pm - N^\pm(\psi^\pm)\|_{L^{\frac{4}{3}}(J\times\mathbb{R})} \leq \delta,$$

$t_0 \in J, \tilde{\psi}_0^\pm \in L^2$ and $S_J(e^{i(t-t_0)H^\pm}(\psi^\pm(t_0) - \tilde{\psi}_0^\pm)) \leq \delta$, then there exists a solution $\tilde{\psi}^\pm$ on I to (2.24) with $\tilde{\psi}^\pm(t_0) = \tilde{\psi}_0^\pm$ and $S_J(\psi^\pm - \tilde{\psi}^\pm) \leq \epsilon$.

vii) (Additional regularity) Assume that, in addition, $R_\pm\psi_0^\pm \in H^s$ (recall (2.32)) for $s \in \{1, 2, 3\}$. If J is an interval such that $S_J(\psi^\pm) \leq A < +\infty$, then the solution $\psi^\pm$ satisfies

$$\|R_\pm\psi^\pm(t)\|_{H^s} \lesssim_A \|R_\pm\psi_0^\pm\|_{H^s}, \qquad \forall t \in J \tag{3.7}$$

and it has Lipschitz dependence with respect to the initial data.

The above results are concerned with general solutions of (2.24). However, our interest lies in solutions which correspond to geometric maps. The next result completes the Cauchy theory for solutions of (2.24) which satisfy the compatibility condition (2.28). The system (2.24) does not directly involve the variable ψ_0 which is defined in this context by (2.15).

Theorem 3.3. *i) If $\psi_0^\pm \in L^2$ satisfying the compatibility condition (2.28), then $\psi^\pm(t)$ satisfies the compatibility condition (2.28) for each $t \in I$. If, in addition, $R_\pm\psi_0^\pm \in H^3$ then (2.5) and (2.6) are satisfied.*

ii) If the solution satisfies the compatibility condition (2.28) and it does not blow up in time then the two scattering states (described in ii)) are related by

$$\partial_r r(\psi_+^+ - \psi_+^-) = \psi_+^+ + \psi_+^- \tag{3.8}$$

Conversely, if $\psi_+^\pm \in L^2$ satisfy (3.8), then the unique maximal-lifespan solution $\psi^\pm$ which scatters to $e^{itH^\pm}\psi_+^\pm$ (constructed in part ii)) satisfy the compatibility condition (2.28). A similar statement holds true for the backward in time scattering.

iii) If $\psi^\pm$ satisfy the compatibility conditions, then for every interval $J \subset I$ (I being the maximal-lifespan interval) the following holds true

$$\|\psi^+\|_{L^4(J)} \approx \|\psi^-\|_{L^4(J)} \tag{3.9}$$

where the constants involved in the use $\approx$ are independent of the interval J.

As a consequence of these theorems we are able to prove the following result

Proposition 3.4. *If $\psi_0^\pm \in L^2$ satisfies the compatibility conditions (2.28), $R_\pm\psi_0^\pm \in H^2$ and $\psi^\pm(t)$ is the solution of (2.24) on I then the map $u(t)$ constructed in Proposition 2.2 (for each t) is a Schrödinger map.*

Proof of Theorem 3.2. Given the form of the nonlinearities $N^{\pm}(\psi^{\pm})$ and the formulas for ψ_2, A_2, A_0, we obtain using (1.5)

$$\|\frac{\psi_2}{r}\|_{L^4} \lesssim \|\psi^{\pm}\|_{L^4}, \quad \|\frac{A_2+1}{r^2}\|_{L^2} + \|A_0\|_{L^2} \lesssim \|\psi^{\pm}\|_{L^4}^2$$

Altogether, these estimates imply that

$$\|N^{\pm}(\psi^{\pm})\|_{L^{\frac{4}{3}}} \lesssim \|\psi^{\pm}\|_{L^4}^3 \tag{3.10}$$

In a similar fashion, we also obtain that

$$\|N^{\pm}(\psi^{\pm}) - N^{\pm}(\tilde{\psi}^{\pm})\|_{L^{\frac{4}{3}}} \lesssim \|\psi^{\pm} - \tilde{\psi}^{\pm}\|_{L^4}(\|\psi^{\pm}\|_{L^4}^2 + \|\tilde{\psi}^{\pm}\|_{L^4}^2) \tag{3.11}$$

Based on a standard theory, by using the Strichartz estimates and (3.11), the system (2.24) has a unique solution with $\psi^{\pm}(t_0) = \psi_0^{\pm}$ on some maximal life-time open interval I with $t_0 \in I$.

If $\|\psi^{\pm}\|_{L^4(\mathbb{R}_+\times\mathbb{R})}$ is finite, then $\psi^{\pm}$ scatters at infinity to $e^{itH^{\pm}}\psi_+^{\pm}$. Constructing solutions to scattering states satisfying (3.8) is done in the usual manner. Precisely, we start with linearizing near the assypmtotic states

$$\psi^- = e^{itH^-}\psi_+^- + e^-, \qquad \psi^+ = e^{itH^+}\psi_+^+ + e^+$$

and generate the following system for e^-, e^+:

$$\begin{cases} (i\partial_t + H^-)e^- = (A_0 - 2\frac{A_2+1}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}^-))\psi^- \\ (i\partial_t + H^+)e^+ = (A_0 + 2\frac{A_2+1}{r^2} + \frac{1}{r}\Im(\psi_2\bar{\psi}^+))\psi^+ \end{cases}$$

with zero data at ∞. By choosing T large enough so that $\|e^{itH^-}\psi_+^-\|_{L^4([T,+\infty)}+\|e^{itH^+}\psi_+^+\|_{L^4([T,+\infty)} \ll 1$, the solution to this system is obtained by a fixed point argument, which ensures existence and uniqueness. This finishes the proof of part i)-iii). Part iv)-vi) are standard in light of (3.11).

Parts vii) is usually standard, but our nonlinearity cannot be completely reduced to a form where we can invoke the usual arguments. We rewrite the nonlinear terms as follows

$$A_0 - 2\frac{A_2+1}{r^2} - \frac{1}{r}\Im(\psi_2\bar{\psi}^-) = -\frac{|\psi^-|^2}{2} + [r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-) + \frac{1}{2r^2}\int_0^r (|\psi^+|^2 - |\psi^-|^2)sds$$
$$A_0 + 2\frac{A_2+1}{r^2} + \frac{1}{r}\Im(\psi_2\bar{\psi}^+) = -\frac{|\psi^+|^2}{2} + [r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-) - \frac{1}{2r^2}\int_0^r (|\psi^+|^2 - |\psi^-|^2)sds$$

Without the term $[r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)$, the analysis would be standard. Indeed, the terms $|\psi^{\pm}|^2\psi^{\pm}$ can be extended to their two-dimensional regular terms $|R_{\pm}\psi^{\pm}|^2 R_{\pm}\psi^{\pm}$, and for the integral term $\frac{1}{2r^2}\int_0^r(|\psi^+|^2 - |\psi^-|^2)sds$ one notices that the operator $\frac{1}{r^2}\int_0^r \cdot sds$ keeps the two dimensional frequency localization; then one could use frequency envelopes techniques to deal with the regularity of these terms. However, the term $[r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)$ cannot be extended to a two-dimensional variant with regular terms since $\psi^{\pm}$ require different phases for completion to regular two-dimensional terms.

We provide a full analysis of the term

$$N_1^{\pm} = [r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\psi^{\pm}$$

This analysis can be extended to the other two terms in $N^{\pm}(\psi^{\pm})$.

Since $S_I(\psi^\pm) \le A$, from (3.10) and (3.3) we obtain

$$\|\psi^\pm\|_{L^3L^6(I\times\mathbb{R})} \lesssim_A 1$$

Therefore it makes sense to define

$$\begin{aligned} B &= \|\partial_r\psi^\pm\|_{L^3L^6} + \|\frac{\psi^-}{r}\|_{L^3L^6} \\ C &= \|\partial_r^2\psi^\pm\|_{L^3L^6} + \|\frac{1}{r}\partial_r\psi^\pm\|_{L^3L^6} + \|\frac{\psi^-}{r^2}\|_{L^3L^6} \\ D &= \|\partial_r H^\pm\psi^\pm\|_{L^3L^6} + \|\frac{1}{r}H^\pm\psi^-\|_{L^3L^6} + \|\frac{1}{r}(\partial_r - \frac{1}{r})\partial_r\psi^-\|_{L^3L^6} \end{aligned}$$

We prove the following estimates

$$\begin{aligned} &\|\partial_r N_1^\pm\|_{L^1L^2} + \|\frac{1}{r}N_1^-\|_{L^1L^2} \lesssim_A B \\ &\|H^\pm N_1^\pm\|_{L^1L^2} \lesssim_A C + B^2 \\ &\|\partial_r H^\pm N_1^\pm\|_{L^1L^2} + \|\frac{1}{r}H^- N_1^-\|_{L^1L^2} \lesssim_A D + BC \end{aligned} \tag{3.12}$$

Similar estimates can be established for the other two terms in $N^\pm(\psi^\pm)$, and when combined with the results in Lemma 3.1, a standard argument establishes the conclusion in (3.7).

We now turn to the proof of (3.12). We compute

$$\partial_r N_1^\pm = \partial_r\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\psi^\pm + [r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\partial_r\psi^\pm$$

and estimate

$$\|\partial_r N_1^\pm\|_{L^1L^2} \lesssim \|\psi^+\|_{L^3L^6}\|\frac{\psi^-}{r}\|_{L^3L^6}\|\psi^\pm\|_{L^3L^6} + \|\psi^\pm\|^2_{L^3L^6}\|\partial_r\psi^\pm\|_{L^3L^6}$$

from which half of the first estimate in (3.12) follows; the second half follows in a similar manner.

We continue with

$$\begin{aligned} H^\pm N_1^\pm =&\Delta\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\psi^\pm + 2\partial_r\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\partial_r\psi^\pm \\ &+ \left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)H^\pm\psi^\pm \end{aligned}$$

The last term is estimated by $\lesssim_A C$, the second one is estimated by $\lesssim_A B^2$, while the first one equals

$$(\partial_r + \frac{1}{r})\frac{\Re(\bar{\psi}^+\psi^-)}{r}\cdot\psi^\pm = \frac{\Re(\partial_r\bar{\psi}^+\cdot\psi^-) + \Re(\bar{\psi}^+\cdot\partial_r\psi^-)}{r}\cdot\psi^\pm$$

and its L^1L^2 norm is estimated by

$$\lesssim (\|\partial_r\psi^+\|_{L^3L^6}\|\frac{\psi^-}{r}\|_{L^3L^6} + \|\psi^+\|_{L^3L^6}\|\frac{\partial_r\psi^-}{r}\|_{L^3L^6})\|\psi^\pm\|_{L^3L^6}$$

from which the second estimate in (3.12) follows.

For the third estimate we start with

$$\begin{aligned} \partial_r H^\pm N_1^\pm =& \partial_r\Delta\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\psi^\pm + \Delta\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\partial_r\psi^\pm \\ &+ 2\partial_r^2\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\partial_r\psi^\pm + 2\partial_r\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\partial_r^2\psi^\pm \\ &+ \partial_r\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)H^\pm\psi^\pm + \left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right)\partial_r H^\pm\psi^\pm \end{aligned}$$

The L^1L^2 norm of the sixth terms above is bounded by $\lesssim_A D$. Using the previous arguments, the L^1L^2 norm of the second, fourth and fifth term is bounded by $\lesssim_A BC$. Since

$$\partial_r^2 \left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right) = \frac{\Re(\partial_r\bar{\psi}^+ \cdot \psi^-) + \Re(\bar{\psi}^+ \cdot \partial_r\psi^-)}{r} - \frac{\Re(\bar{\psi}^+\psi^-)}{r^2}$$

it follows that the L^1L^2 norm of the third term above is bounded by $\lesssim_A BC$.

Finally we compute

$$\begin{aligned}\partial_r\Delta\left([r\partial_r]^{-1}\Re(\bar{\psi}^+\psi^-)\right) &= \partial_r\left(\frac{\Re(\partial_r\bar{\psi}^+ \cdot \psi^-) + \Re(\bar{\psi}^+ \cdot \partial_r\psi^-)}{r}\right)\\ &= \frac{\Re(\partial_r^2\bar{\psi}^+ \cdot \psi^-) + 2\Re(\partial_r\bar{\psi}^+ \cdot \partial_r\psi^-)}{r} + \Re(\bar{\psi}^+\frac{1}{r}(\partial_r - \frac{1}{r})\partial_r\psi^-)\end{aligned}$$

and this allows us to estimate the first term by $BC + D$. This finishes the argument for (3.12). □

Proof of Theorem 3.3. i) It is useful to rephrase this in terms of ψ_1, ψ_2, which are recovered linearly from $\psi^\pm$. Reverting the algebraic computation from Sections 2.1 and 2.2, ψ_1, ψ_2 solve the system (2.17). Then we seek to show that the relation $D_1\psi_2 = D_2\psi_1$ is preserved along the flow. For this we will derive an equation for the quantity

$$F = D_2\psi_1 - D_1\psi_2$$

It is convenient to use covariant notations. Given ψ_1, ψ_2 and $A_1 = 0$, A_2, we define ψ_0 via (2.15). Then we consider the connection (A_0, A_1, A_2) and identify its curvature. The equation (2.26) reads

$$\partial_1 A_2 - \partial_2 A_1 = \Im(\psi_1\bar{\psi}_2)$$

The equation (2.27) yields (2.19), which in turn gives

$$\partial_1 A_0 - \partial_0 A_1 = \Im(\psi_1\bar{\psi}_0) - \frac{1}{r^2}\Re(F\bar{\psi}_2)$$

For the remaining curvature component we use (2.24) to compute

$$\begin{aligned}\partial_0 A_2(r) &= \frac{1}{2}\int_r^\infty \Re(\psi^+\bar{\psi}^+_t) - \Re(\psi^-\bar{\psi}^-_t) s ds\\ &= \frac{1}{2}\int_r^\infty \Im(\psi^+\Delta\bar{\psi}^+) - \Im(\psi^-\Delta\bar{\psi}^-) s ds\\ &= \frac{1}{2}\int_r^\infty \partial_s\Im(s\psi^+\partial_s\bar{\psi}^+) - \partial_s\Im(s\psi^-\partial_s\bar{\psi}^-) ds\\ &= -\frac{r}{2}\left(\Im(\psi^+\partial_r\bar{\psi}^+) - \Im(\psi^-\partial_r\bar{\psi}^-)\right)\\ &= r\Re(\partial_r\psi_1\frac{\bar{\psi}_2}{r} - \psi_1\partial_r\frac{\bar{\psi}_2}{r})\end{aligned}$$

On the other hand we expect to have

$$\partial_0 A_2(r) = \Im(\psi_0\bar{\psi}_2) = \Re((\partial_r + \frac{1}{r})\psi_1 + i\frac{A_2}{r^2}\psi_2)\bar{\psi}_2) = \Re((\partial_r + \frac{1}{r})\psi_1\bar{\psi}_2)$$

Taking the difference, what we actually get is

$$\partial_0 A_2 - \partial_2 A_0 = \Im(\psi_0\bar{\psi}_2) - \Re[F\bar{\psi}_1]$$

The next step is to identify the remaining two quantities in the compatibility conditions. We claim that

$$D_1\psi_0 - D_0\psi_1 = -\frac{i}{r^2}D_2F$$

$$D_2\psi_0 - D_0\psi_2 = i(D_1 + \frac{1}{r})F$$

For this it suffices to compare the two equations in (2.16), derived as before from (2.15), with the equations (2.17) obtained algebraically from (2.24). Now we can compute

$$\begin{aligned}
D_0F &= D_0D_2\psi_1 - D_0D_1\psi_2 \\
&= D_2D_0\psi_1 - D_1D_0\psi_2 + i\Im(\psi_0\bar{\psi}_2)\psi_1 - i\Re(F\bar{\psi}_1)\psi_1 \\
&\quad + i\Im(\psi_1\bar{\psi}_0)\psi_2 - i\frac{1}{r^2}\Re(F\bar{\psi}_2)\psi_2 \\
&= D_2D_1\psi_0 - D_1D_2\psi_0 + i(\frac{1}{r^2}D_2D_2 + D_1(D_1 + \frac{1}{r}))F \\
&\quad + i\Im(\psi_0\bar{\psi}_2)\psi_1 - i\Re(F\bar{\psi}_1)\psi_1 + i\Im(\psi_1\bar{\psi}_0)\psi_2 - i\Re(F\bar{\psi}_2)\psi_2 \\
&= i(\frac{1}{r^2}D_2D_2 + D_1(D_1 + \frac{1}{r}))F - i\Re(F\bar{\psi}_1)\psi_1 - i\frac{1}{r^2}\Re(F\bar{\psi}_2)\psi_2 \\
&\quad + i\Im(\psi_1\bar{\psi}_1)\psi_0 + i\Im(\psi_0\bar{\psi}_2)\psi_1 + i\Im(\psi_1\bar{\psi}_0)\psi_2
\end{aligned}$$

The last line algebraically vanishes, so we have our equation for F:

$$iD_0F = (\frac{A_2^2}{r^2} - \partial_1(\partial_1 + \frac{1}{r}))F + \Re(F\bar{\psi}_1)\psi_1 + \frac{1}{r^2}\Re(F\bar{\psi}_2)\psi_2$$

It is more convenient to recast this as an equation for

$$\frac{F}{r} = (\partial_r + \frac{1}{r})\frac{\psi_2}{r} - \frac{iA_2}{r}\psi_1$$

which is exactly the quantity in (2.28). We obtain

$$(i\partial_t + \Delta - \frac{1}{r^2})\frac{F}{r} = (A_0 + \frac{A_2^2 - 1}{r^2})F + \Re(\frac{F}{r}\bar{\psi}_1)\psi_1 + \frac{1}{r^2}\Re(\frac{F}{r}\bar{\psi}_2)\psi_2 \tag{3.13}$$

In view of the L^4 Strichartz bounds for ψ_1 and ψ_2 and the derived L^2 bounds for A_0 and $\frac{A_2^2-1}{r^2}$, standard arguments show that this linear equation is well-posed in L^2. Hence the conclusion follows provided that $\frac{F}{r}$ has sufficient regularity. Indeed, we have

$$\frac{F}{r} = -\frac{i}{2}\left(\partial_r\psi^+ + \frac{1 + A_2}{r}\psi^+ - \partial_r\psi^- - \frac{1 - A_2}{r}\psi^-\right)$$

It is obvious that if $R_\pm\psi^\pm \in H^1$ then $\frac{F}{r} \in L^2$.

If $R_\pm\psi^\pm \in H^2$ then by using the results in Lemma 1.3 and Sobolev embeddings one easily shows that $\frac{F}{r} \in \dot{H}^1_e$.

We will show in detail that if $R_\pm\psi^\pm \in H^3$, then $(\Delta - \frac{1}{r^2})\frac{F}{r} \in L^2$. Indeed,

$$\begin{aligned}
2i(\Delta - \frac{1}{r^2})\frac{F}{r} &= \partial_r\Delta\psi^+ + \frac{1 + A_2}{r}\Delta\psi^+ + \frac{\Delta A_2}{r}\psi^+ + 2(\frac{\partial_r A_2}{r} - \frac{1 + A_2}{r^2})\partial_r\psi^+ - 2\frac{\partial_r A_2}{r^2}\psi^+ \\
&\quad - (\partial_r + \frac{2}{r})(\Delta - \frac{4}{r^2})\psi^- + \frac{\Delta A_2}{r}\psi^- + 2(\frac{\partial_r A_2}{r} - \frac{1 + A_2}{r^2})\partial_r\psi^- - 2\frac{\partial_r A_2}{r^2}\psi^-
\end{aligned}$$

We have that $\|\partial_r \Delta \psi^+\|_{L^2} \lesssim \|R_+\psi^+\|_{H^3}$. For some terms, direct computations combined with Sobolev embeddings give

$$\|\frac{1+A_2}{r}\Delta\psi^+\|_{L^2} \lesssim \|\psi^\pm\|_{L^4}^2 \|\Delta\psi^+\|_{H^2} \lesssim \|R_\pm\psi^\pm\|_{H^2}^3$$

$$\|(\frac{\partial_r A_2}{r} - \frac{1+A_2}{r^2})\partial_r\psi^+\|_{L^2} \lesssim \|\psi^\pm\|_{L^\infty}^2 \|\partial_r\psi^+\|_{L^2} \lesssim \|R_\pm\psi^\pm\|_{H^2}^3$$

while for others, we rearrange them

$$\frac{1}{r}(\Delta A_2 - 2\frac{\partial_r A_2}{r})\psi^+ = \frac{1}{4r}(\partial_r - \frac{1}{r})[r(|\psi^-|^2 - |\psi^+|^2)]\psi^+ = \frac{1}{4}\partial_r(|\psi^-|^2 - |\psi^+|^2)\psi^+$$

and then estimate by

$$\|\frac{1}{r}(\Delta A_2 - 2\frac{\partial_r A_2}{r})\psi^+\|_{L^2} \lesssim \|\psi^\pm\|_{L^\infty}^2 \|\partial_r\psi^\pm\|_{L^2} \lesssim \|R_\pm\psi^\pm\|_{H^2}^3$$

The terms corresponding to ψ^- are treated in a similar manner.

Hence we can conclude that $(\Delta - \frac{1}{r^2})\frac{F}{r} \in L^2$. This allows us to run a standard energy argument by pairing the equation, with $\bar{F}$, to conclude that

$$\partial_t \|\frac{F}{r}\|_{L^2}^2 \lesssim (\|\psi_1\|_{L^\infty}^2 + \|\frac{\psi_2}{r}\|_{L^\infty}^2)\|\frac{F}{r}\|_{L^2}^2$$

which by using the Gronwall inequality and the fact that $F(0) = 0$ leads to $F(t) = 0$ for all $t \in I$.

In order to run the energy argument it suffices to have $\frac{F}{r} \in \dot{H}_e^1$ and use the pairing of $\dot{H}_e^{-1}$ and $\dot{H}_e^1$. This is useful in the proof of Proposition 3.4 where we assume only $R_\pm\psi^\pm \in H^2$.

In the general case when $\psi_0^\pm \in L^2$ only, we regularize them as follows. We produce $R_\pm\psi_{n,0}^- \in H^3$ so that $\|\psi_0^- - \psi_{n,0}^-\|_{L^2} \leq \frac{1}{n}$. By using Lemma (2.3), and particularly part v), we obtain that the compatible pair $R_+\psi_{n,0}^+ \in H^3$ and $\|\psi_0^+ - \psi_{n,0}^+\|_{L^2} \leq \frac{1}{n}$. We also recast the compatibility condition to

$$\psi^+ - \psi^- = -[r\partial_r]^{-1}\left(\psi^+ - \psi^- + A_2(\psi^+ + \psi^-)\right)$$

so that all terms involved belong to L^2. Using the conservation of the compatibility condition for $\psi_n^\pm(t)$ under the flow (2.24) and part v) of the Theorem, we obtain the desired result.

ii) The key observation is that the equation for ψ_2 in (2.31) becomes linear in the following sense:

$$\lim_{t\to\infty} \|\partial_r\psi_2 + i\psi^- - \frac{\psi_2}{r}\|_{L^2} = 0 \tag{3.14}$$

under the hypothesis that $\lim_{t\to\infty} \|\psi^-(t) - e^{itH^-}\psi_+^-\|_{L^2} = 0$. This is easily shown to follow from the following estimate

$$\lim_{t\to\infty} \sup_{r\in(0,\infty)} |r\int_r^\infty \frac{e^{itH^-}f}{s^2} ds| = 0 \tag{3.15}$$

which holds true for $f \in L^2$ and will be proved in the Appendix. Based on this, it follows that $\lim_{t\to\infty} \|\psi_2(t)\|_{L^\infty} = 0$, and that

$$\lim_{t\to\infty} \|i(A_2+1)\psi^- - \frac{1}{r}(A_2+1)\psi_2\|_{L^2} = 0$$

which justifies (3.14).

With the notation

$$f(t) = \partial_r(e^{itH^+}\psi_+^+ - e^{itH^-}\psi_+^-) - 2\frac{e^{itH^-}\psi_+^-}{r},$$

the scattering relation (3.8) can be rewritten as $\lim_{t\to\infty}\|f(t)\|_{\dot H_e^{-1}} = 0$. A direct computation gives that f obeys the equation

$$\begin{aligned}(i\partial_t + \Delta)f &= [\Delta, \partial_r](e^{itH^+}\psi_+^+ - e^{itH^-}\psi_+^-) - 2[\Delta, \frac{1}{r}]e^{itH^-}\psi_+^- - \frac{8}{r^2}\frac{e^{itH^-}\psi_+^-}{r} - \partial_r\frac{4}{r^2}e^{itH^-}\psi_+^- \\ &= \frac{1}{r^2}\partial_r e^{it\Delta}\psi_+^+ - 2(-2\frac{\partial_r}{r^2} + \frac{1}{r^3})e^{itH^-}\psi_+^- - \frac{1}{r^2}\partial_r e^{itH^-}\psi_+^- - \frac{4}{r^2}\partial_r e^{itH^-}\psi_+^- \\ &= \frac{1}{r^2}f(t)\end{aligned}$$

Since $\lim_{t\to\infty}\|f(t)\|_{\dot H_e^{-1}} = 0$ it follows from the conservation of the $\dot H_e^{-1}$ norm that $f(0) = 0$ which is (3.8). Alternatively, one could carry out this argument as we did in viii).

Assume now that given $\psi_+^\pm$ satisfying (3.8) we construct (as in ii)) solutions $\psi^\pm(t)$ to (2.24) on some $[T, +\infty)$ which scatter forward to $e^{itH^\pm}\psi_+^\pm$. Following the argument in part i) we construct F which satisfies (3.13). Assuming additional regularity on the states $\psi_+^\pm$, $R_\pm\psi_+^\pm \in H^3$, we have by part vii) of Theorem 3.2 that $R_\pm\psi^\pm(t) \in H^3$, hence by the argument in i) $(\Delta - \frac{1}{r^2})\frac{F}{r} \in L^2$ and the right-hand side of (3.13) belongs to L^2. Then the Duhamel formula applies to (3.13) and in turn the Strichartz estimate

$$\|\frac{F}{r}\|_{L^4([T,\infty)\times\mathbb{R})} \lesssim \|\psi^\pm\|^2_{L^4([T,\infty)\times\mathbb{R})}\|\frac{F}{r}\|_{L^4([T,\infty)\times\mathbb{R})}$$

where we have used that $\lim_{t\to\infty}\|\frac{F(t)}{r}\|_{L^2} = 0$ (this follows as above because of (3.15)). Next, by taking T large enough, we obtain that $F(t) \equiv 0$ for $t \geq T$ and the conclusion follows by invoking part i).

For general states $\psi_+^\pm \in L^2$ satisfying (3.8) we proceed as above. We approximate them by sequences $\psi_{n,+}^\pm$ with $R_\pm\psi_{n,+}^\pm \in H^3$; this can be done by regularizing $R_-\psi_+^-$ first and then showing that the corresponding $R_+\psi_+^+$ has the same regularity as we did in Lemma 2.3 part v) - in fact this argument involves only the linear part of the argument there. Then we write (3.8) at the level of L^2

$$\psi^+ - \psi^- = 2[r\partial_r]^{-1}\psi^-,$$

use the above argument and a limiting argument.

iii) One side of (3.9) follows from the fixed time bound (2.40). The other side is similar, and it consists in replicating the result of Lemma 2.3 starting from ψ^+ instead.

□

Proof of Proposition 3.4. With the given $\psi_0^\pm$ we reconstruct $u_0 \in \dot H^1 \cap \dot H^3$ as in Proposition 2.2. The additional regularity $R_\pm\psi_0^\pm \in H^2$ implies, by (2.33), that $u_0 \in \dot H^1 \cap \dot H^3$. For the classical Schrödinger Map $u(t)$ with data u_0 (whose existence follows from Theorem 1.1) we construct its Coulomb gauge, its field components and write the system (2.24) whose initial data is $\psi_0^\pm$. Invoking the uniqueness part of Theorem 3.2, it follows that $\psi^\pm(t)$ are the gauge representation of $\mathcal{W}^\pm(t)$, hence the reconstruction in Proposition 2.2 gives the Schrödinger Map $u(t)$ for each t.

□

We can now identify the critical threshold for global well-posedness and scattering. For any $m \geq 0$, we define $A(m)$ by

$$A(m) := sup\{S_{I_{max}}(\psi^-) : M(\psi^\pm) \leq m \text{ where } \psi^\pm \text{ is a solution to (2.24) satisfying(2.28)}\}$$

where $\psi^\pm$ is assumed to be a solution of (2.24), satisfying the compatibility condition (2.28) and I_{max} is its maximal interval of existence. Note that the compatibility condition forces $M(\psi^+) = M(\psi^-)$ so the use of $M(\psi^\pm) \leq m$ is unambiguous.

Obviously A is a monotone increasing functions, it is bounded for small m by part iv) and it is left-continuous by part v) of Theorem 3.2. Therefore there exists a critical mass $0 < m_0 \leq +\infty$ such that $A(m)$ is finite for all $m < m_0$ and it is infinite $m \geq m_0$. Also any solution $\psi^\pm$ with $M(\psi^\pm) < m_0$ is globally defined and scatters.

Note that from (3.9) it follows that we could have used $S_{I_{max}}(\psi^+)$ in the definition of $A(m)$ and arrive to the same conclusion as above with the same critical mass m_0.

4. Concentration compactness

We start by exhibiting the symmetries of the system (2.24). First, the system is invariant under the time reversal transformation $\psi^\pm(r,t) \to \bar{\psi}^\pm(r,-t)$. This allows us to focus our attention on positive times, i.e. $t \geq 0$.

Next, the system is invariant under two other transformations: scaling, $\psi^\lambda = \lambda^{-1}\psi(\lambda^{-1}r, \lambda^{-2}t)$ with $\lambda \in \mathbb{R}$, and phase multiplication, $\psi^\alpha(r,t) = e^{i\alpha}\psi(r,t)$ with $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$. The phase multiplication can be ignored as the group generated is compact. This way we generate the first (non-compact) group G of transformations g_λ defined by

$$g_\lambda f(r) = \lambda^{-1} f(\lambda^{-1} r)$$

From (2.25), (2.26) and (2.27), the effect of the action g_λ on $\psi^\pm$ is translated in the action of g_λ^1 on ψ_2, A_2 and g_λ^2 on A_0 where

$$g_\lambda^1 f(r) = f(\lambda^{-1} r), \qquad g_\lambda^2 f(r) = \lambda^{-2} f(\lambda^{-1} r)$$

The action of g_λ is extended to space-time functions by

$$T_{g_\lambda} f(r,t) = \lambda^{-1} f(\lambda^{-1} r, \lambda^{-2} t)$$

The equations in (2.24) are also time translation invariant and this suggests enlarging the group G to G' as follows. Given $\lambda > 0$ and $t_0 \in \mathbb{R}$, we define

$$g_{\lambda,t_0} f = \lambda^{-1}[e^{it_0 H^-} f](\lambda^{-1} r)$$

We denote by G' the groups generated by these transformations. We also define the extend the action of g_{λ,t_0} to space-time functions by

$$T_{g_{\lambda,t_0}} f = T_{g_\lambda} f(\cdot, \cdot + t_0)$$

Given two sequences $g^n, \tilde{g}^n \in G'$, we say that they are asymptotically orthogonal iff

$$\frac{\lambda_n}{\tilde{\lambda}_n} + \frac{\tilde{\lambda}_n}{\lambda_n} + |t_n \lambda_n^2 - \tilde{t}_n \tilde{\lambda}_n^2| = \infty \tag{4.1}$$

The asymptotic orthogonality will be mostly exploited as follows. Given two asymptotically orthogonal sequences $g^n, \tilde{g}^n \in G'$

$$\lim_{n\to\infty} \langle g^n f, \tilde{g}^n \tilde{f} \rangle = 0, \quad \forall f, \tilde{f} \in L^2 \tag{4.2}$$

(4.3) $$\lim_{n\to\infty} \||T_{g^n}h|^{\frac{1}{2}}|T_{\tilde{g}^n}\tilde{h}|^{\frac{1}{2}}\|_{L^4} = 0, \quad \forall h, \tilde{h} \in L^4(\mathbb{R}\times\mathbb{R}).$$

(4.4) $$\lim_{n\to\infty} \|T_{g^n}f \cdot [r\partial_r]^{-1}\left(T_{\tilde{g}^n}hT_{\tilde{g}^n}\tilde{h}\right)\|_{L^{\frac{4}{3}}} = 0, \quad \forall f, h, \tilde{h} \in L^4(\mathbb{R}\times\mathbb{R}).$$

(4.5) $$\lim_{n\to\infty} \|T_{g^n}f \cdot r^{-2}[r^{-1}\bar{\partial}_r]^{-1}\left(\int_0^r T_{\tilde{g}^n}hT_{\tilde{g}^n}\tilde{h}\right)\|_{L^{\frac{4}{3}}} = 0, \quad \forall f, h, \tilde{h} \in L^4(\mathbb{R}\times\mathbb{R}).$$

These estimate are inspired by the similar ones in [1], [17], [21], [26], [2] to which we also refer the reader for further details on the subject. They can also be proved directly using the Riemann-Lebesgue characterization of L^p spaces.

We are now ready to state the two main results of this section.

Theorem 4.1. *Assume that the critical mass $m_0 < 8$. Then there exists a critical element, i.e. a maximal-lifespan solution $\psi^\pm$ to (2.24) and satisfying (2.28), with mass m_0 which blows up forward in time. In addition this solution has the following compactness property: there exists a continuous function $\lambda(t) : I_+ = [0, T_+) \to \mathbb{R}_+$ such that the sets*

$$K^\pm := \left\{\frac{1}{\lambda(t)}\psi^\pm(\frac{r}{\lambda(t)}, t), t \in I_+\right\}$$

are precompact in L^2.

Remark. *As a consequence of the compactness property it follows that there exists a function $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that the above critical element satisfies*

(4.6) $$\int_{r\geq C(\eta)\lambda(t)^{-1}} |\psi^\pm(t,r)|^2 r dr \leq \eta, \quad \forall t \in I_+.$$

One can construct critical elements whose function $\lambda(t)$ has more explicit behavior.

Theorem 4.2. *Assume that the critical mass $m_0 < 8$. Then we can construct a critical element as in Theorem 4.1 such that one of the two scenarios holds true:*

i) $T_+ = \infty$ and $\lambda(t) \geq c > 0, \forall t \geq 0$.

ii) $T_+ < \infty$ and $\lim_{t\to T_+} \lambda(t) = \infty$.

The rest of this section is spent on sketching the proof of the two theorems above. The statements of Theorema 4.1 and 4.2 are similar to the corresponding ones in the work of Kenig and Merle, see [16].

It is standard, see for instance [16] and [26] that the result in Theorem 4.1 follows from the following

Proposition 4.3. *Assume $m_0 < 8$. Let $\psi_n^\pm : I_{n+} = [0, T_{n+}) \times \mathbb{R} \to \mathbb{C}, n \in \mathbb{N}$ be a sequence of solutions to (2.24), satisfying (2.28) and such that $\lim_{n\to\infty} M(\psi_n^\pm) = m_0$ and $\lim_{n\to\infty} S_{I_{n+}}(\psi_n^\pm) = \infty$. Then there are group elements $g_n \in G$ such that the sequence $g_n\psi_n^\pm(t_n)$ has a subsequence which converges in L^2.*

One of the main ingredients in the proof of Proposition 4.3 is the classical linear profile decomposition result. These type of results originate in the work of Bahouri and Gerard [1], for the case of nonlinear wave equation and independently, and in the work of Merle and Vega [21], for the case of the nonlinear Schrödinger equation. For the case of nonlinear Schrödinger equations see also [2], [17], [26].

Proposition 4.4. *Let $\psi_0^n, n \in \mathbb{N}$ a bounded sequence in L^2. Then (after passing to a subsequence if necessary) there exists a sequence $\phi^j, j \in \mathbb{N}$ of functions in L^2 and $g^{n,j} \in G', n, j \in \mathbb{N}$ such that we have the decomposition*

$$\psi_0^n = \sum_{j=1}^{l} g^{n,j}\phi^j + w^{n,l}, \qquad \forall l \in \mathbb{N} \tag{4.7}$$

where $w^{n,l}$ satisfies

$$\lim_{l\to\infty}\lim_{n\to\infty} S(e^{itH^-}w^{n,l}) = 0 \tag{4.8}$$

Moreover $g^{n,j}$ and $g^{n,j'}$ are asymptotically orthogonal for any $j \neq j'$ and we have the following orthogonality condition

$$weak \lim_{n\to\infty}(g^{n,j})^{-1}w^{n,l} = 0, \quad \forall 1 \leq j \leq l \tag{4.9}$$

As a consequence the mass decoupling property holds

$$\lim_{n\to\infty}(M(u^n) - \sum_{j=1}^{l} M(\phi^j) - M(w^{n,l})) = 0 \tag{4.10}$$

The same statement holds true also for the operator H^+. The general version of this result was established originally in [21] and it has the same statement with the following adjustments: the sequence of functions $\psi_0^n \in L^2(\mathbb{R}^2)$ and the group G is enlarged to capture the additional symmetries available in the general case, i.e. spatial and frequency translation, and the Galilean transformation. Our statement is closer in spirit to the one in [26], see Theorem 7.3, where the statement is restricted to the radial case. Our statement is the corresponding one for the case of "equivariant" functions, i.e. functions with the property that $u(r,\theta) = e^{2i\theta}v(r)$ (in the case of H^+ the result is needed for radial function and then the Theorem 7.3 in [26] applies verbatim). Indeed, as we already noticed before, the transformation $R_- e^{itH^-}\psi^- = e^{it\Delta}(R_-\psi^-)$ takes one-dimensional homogeneous solutions of (3.2) into two-dimensional homogeneous "equivariant "solutions of the free Schrödinger equation. In [26], see the proof of Theorem 7.3, a robust argument derives the radial statement from the general case. The argument there can be easily replicated for the "equivariant" solutions.

Proof of Proposition 4.3. A key ingredient in this proof is to produce a geometric profile decomposition in the spirit of Proposition 4.4. The challenge is that the profiles cannot be arbitrary, but geometrically related via the compatibility condition (2.28).

One choice is to produce the profile decomposition for $\psi_n^{\pm}(0)$ and then perform additional analysis on the profiles so that they satisfy the compatibility relations (2.28). This route had been implemented in the case of Wave Maps by Krieger and Schlag in [19].

We choose a different, more explicit route. We use the profile decomposition for $\psi_n^-(0)$ and then use (2.31) to produce a geometric profile decomposition for $\psi_n^+(0)$. We apply Proposition 4.4 to $\psi_n^-(0)$ to obtain the linear profile decomposition

$$\psi_n^-(0) = \sum_{j=1}^{l} g^{n,j}\phi_j^- + w_{n,l}^-$$

as described in the Proposition 4.4. This can be further factorized to $g^{n,j} = h^{n,j} e^{it_{n,j}H^-}$ with $t_{n,j} \in \mathbb{R}$ and $h^{n,j} \in G$. Next, a standard diagonalisation argument, allows us to reduce the problem to the case when for each j, the sequence $\{t_{n,j}\}_{n\in\mathbb{N}}$ converges to some limit in $[-\infty, +\infty]$. If for some j, $\lim_{n\to\infty} t_{n,j} = t_j$ is finite then one can shift ϕ_j by the propagator $e^{it_jH^-}$ and reduce the problem to the case $t_j = 0$. Moreover, by absorbing the $e^{it_{n,j}H^-}\phi_j^- - \phi_j^-$ into the error $w_{n,l}^-$ one can further assume that $t_{n,j} = 0, \forall n \in \mathbb{N}$.

Next, it follows from (4.10) that

$$\sum_{j\geq 1} M(\phi_j^-) \leq \lim_{n\to\infty} M(\psi_n^-(0)) = m_0 \tag{4.11}$$

Therefore $\sup_j M(\phi_j^-) \leq m_0$. Assume first that, for some $\epsilon > 0$,

$$\sup_j \|M(\phi_j^-)\| \leq m_0 - \epsilon \tag{4.12}$$

We will show that this leads to a contradiction. Since $A(m)$ is monotone increasing and finite on $[0, m_0 - \epsilon]$, by iv) in Proposition 3.2 it follows that

$$A(m) \leq Bm, \quad 0 \leq m \leq m_0 - \epsilon \tag{4.13}$$

where B depends on m_0 and ϵ only.

Then we introduce the nonlinear profiles $v_j^- : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ associated to ϕ_j^- and $t_j = \lim_{n\to\infty} t_{n,j}$ as follows

- if $t_j = 0$ which implies that $t_{n,j} = 0$, then we define ϕ_j^+ the compatible function in the sense of (2.28); the construction is done in part a) in Proposition 2.2 and Lemma 2.3. Then we define $v_j^\pm$ to be the maximal-lifespan solution of (2.24) with initial data $v_j^\pm(0) = \phi_j^\pm$.

- if $t_j = +\infty$, we construct ϕ^+ to be the compatible function in the sense of (3.8). Then we define $v_j^\pm$ to be the the maximal-lifespan solution of (2.24) which scatters forward in time to $e^{itH^\pm}\phi_j^\pm$; the existence of this solution is guaranteed by part ii) of Theorem 3.2.

- if $t_j = -\infty$, the construction is similar to the one above.

Using part a) in Proposition 2.2 and Lemma 2.3, we construct $w_{n,l}^+$, the compatible pair to $w_{n,l}^-$ in the sense of (2.28). Then we consider $w_{n,l}^\pm(t)$ the solution to (2.24) with $w_{n,l}^\pm$ initial data. By taking l and n large enough, it follows from (4.8) and part vi) of Theorem 3.2 (the approximating solution is simply the free flow) that $w_{n,l}^\pm(t)$ is global in time.

From (4.13) and Proposition 3.2 it follows that, for each j, $v_j^\pm$ are globally defined in time and satisfy

$$M(v_j^\pm) = M(\phi_j^\pm) \leq m_0 - \epsilon, \quad S(v_j^\pm) \leq A(M(\phi_j^\pm)) \leq BM(\phi_j^\pm) \tag{4.14}$$

Based on these nonlinear profiles we define the approximate global solutions

$$\psi_{n,l}^\pm(t) = \sum_{j=1}^{l} T_{h^{n,j}} v_j^\pm(t + t_{n,j}) + w_{n,l}^\pm(t).$$

Our first claim is the following

$$\lim_{l\to\infty} \lim_{n\to\infty} S(\psi_{n,l}^\pm) = \lim_{l\to\infty} \sum_{j=1}^{l} S(v_j^\pm) \leq Bm_0 \tag{4.15}$$

Indeed, by using (4.3) and the asymptotic orthogonality of $g^{n,j}$ we obtain that

$$\lim_{n\to\infty}\|\sum_{j=1}^{l}T_{h^{n,j}}v_j^{\pm}(\cdot+t_{n,j})\|_{L^4}^4=\sum_{j=1}^{l}\|v_j^{\pm}\|_{L^4}^4$$

Then using (4.8), (4.11), (4.12) and (4.13) we obtain (4.15).

Our next claim is that

$$\lim_{n\to\infty}M(\psi_{n,l}^{\pm}(0)-\psi_n^{\pm}(0))=0,\qquad \forall l\in\mathbb{N}. \tag{4.16}$$

We have from the construction that

$$\psi_{n,l}^{-}(0)=\psi_n^{-}(0)$$

Therefore we need to show only the $+$ part in (4.16). We prove this by using the variable ψ_2, hence we involve the compatibility conditions (2.31). For this we define for each $j\in\{1,..,l\}$

$$\begin{aligned}\frac{\psi_{2,n,j}(t)}{r}:=&\frac{T_{h^{n,j}}v_j^{+}(t+t_{n,j})-T_{h^{n,j}}v_j^{-}(t+t_{n,j})}{2i}\\ A_{2,n,j}(t,r):=&-1+\int_0^r\frac{|T_{h^{n,j}}v_j^{+}(t+t_{n,j})|^2-|T_{h^{n,j}}v_j^{-}(t+t_{n,j})|^2}{4}sds\end{aligned}$$

and

$$\begin{aligned}\frac{\psi_{2,er}(t)}{r}:=&\frac{w_{n,l}^{+}(t)-w_{n,l}^{-}(t)}{2i}\\ A_{2,er}(t,r):=&-1+\int_0^r\frac{|w_{n,l}^{+}(t)|^2-|w_{n,l}^{-}(t)|^2}{4}sds\end{aligned}$$

At time $t=0$ it is more convenient to define

$$\psi_{n,l,2}=\sum_{j=1}^{l}\psi_{2,n,j}+\psi_{2,er},\quad 1+\tilde{A}_{n,l,2}=\sum_{j=1}^{l}(1+A_{2,n,j})+(1+A_{2,er})$$

Indeed, from (2.26), the definition of $A_{n,l,2}$ should have been different, but the above one works better for us for technical purposes. We make the following claims (at $t=0$):

$$\lim_{n\to\infty}\|\psi_{2,n,j_1}\psi_{2,n,j_2}\|_{L^\infty}+\|\psi_{2,n,j_1}\psi_{2,er}\|_{L^\infty}=0,\quad \forall j_1\neq j_2 \tag{4.17}$$

$$\lim_{n\to\infty}\|\frac{\psi_{2,n,j_1}\psi_{2,n,j_2}}{r}\|_{L^2}+\|\frac{\psi_{2,n,j_1}\psi_{2,er}}{r}\|_{L^2}=0,\quad \forall j_1\neq j_2 \tag{4.18}$$

$$\lim_{n\to\infty}\|\frac{\psi_{2,n,j_1}\psi_{2,n,j_2}}{r^2}\|_{L^1}+\|\frac{\psi_{2,n,j_1}\psi_{2,er}}{r^2}\|_{L^1}=0,\quad \forall j_1\neq j_2 \tag{4.19}$$

and the similar ones involving $1+A_{2,n,j},1+A_{2,er}$ instead of $\psi_{2,n,j},\psi_{2,er}$.

We will establish (4.17), the other two being similar. We will need the following estimate, whose proof is provided in the Appendix. If $g^n=g_{l_n,t_n},n\in\mathbb{N}$ is asymptotically orthogonal to the constant sequence $g_{1,0}$ then for every $f\in L^2$ and any compact set $K\subset(0,+\infty)$ the following holds true for

$$\lim_{n\to\infty}\|g^nf\|_{L^2(K)}=0. \tag{4.20}$$

Let $j_1 \neq j_2$. (4.17) is scale invariant, hence by rescaling in space and by shifting time we can reduce the problem to the case $l_{n,j_1} = 1$ and $t_{n,j_1} = 0$ for all $n \in \mathbb{N}$. We obviously have from (2.35) that

$$\|\psi_{2,n,j_1}\|_{L^\infty} + \|\psi_{2,n,j_2}\|_{L^\infty} \lesssim \|\phi^{j_1}\|_{L^2} + \|\phi^{j_2}\|_{L^2} \lesssim 1$$

We need to localize this estimate. Given $\epsilon > 0$ choose R such that

$$\|\phi^{j_1}\|_{L^2(\mathbb{R}_+ \setminus [R^{-1},R])} \leq \epsilon$$

Using (2.36) we obtain that

$$\|\psi_{2,n,j_1}\|_{L^\infty(\mathbb{R}_+ \setminus [\epsilon R^{-1},R])} \lesssim \epsilon.$$

From (4.20) it follows that there exists $N(\epsilon)$ such that if $n \geq N(\epsilon)$ the following holds true

$$\|g^{n,j_2}\phi^{j_2}\|_{L^2[\epsilon R^{-1},\epsilon^{-1}R]} \leq \epsilon$$

and hence by invoking (2.36) it follows that

$$\|\psi_{2,n,j_2}\|_{L^\infty([\epsilon R^{-1},R])} \lesssim \epsilon$$

From the above estimates we obtain

$$\begin{aligned}
&\|\psi_{2,n,j_1}\psi_{2,n,j_2}\|_{L^\infty} + \|\psi_{2,n,j_1}\psi_{2,er}\|_{L^\infty} \\
\lesssim &\|\psi_{2,n,j_1}\|_{L^\infty(\mathbb{R}_+ \setminus [\epsilon R^{-1},R])}\|\psi_{2,n,j_2}\|_{L^\infty} + \|\psi_{2,n,j_1}\|_{L^\infty}\psi_{2,er}\|_{L^\infty([\epsilon R^{-1},R])} \\
\lesssim &\epsilon
\end{aligned}$$

which is true provided that $n \geq N(\epsilon)$ (which was defined earlier). This implies the first half of (4.17).

The second part of (4.17), which contains the terms involving $\psi_{2,er}$, follows by involving similar arguments together with (4.9).

Now we return to the proof of (4.16). $\psi_{2,n,l}(0)$ obeys the following differential equation:

$$\begin{aligned}
\partial_r \psi_{n,l,2} &= i\tilde{A}_{n,l,2}\psi^-_{n,l} - \frac{1}{r}\tilde{A}_{n,l,2}\psi_{n,l,2} + E^1_{n,l} \\
\partial \tilde{A}_{n,l,2} &= \Im(\psi^-_{n,l}\overline{\psi_{n,l,2}}) + \frac{1}{r}(1 - \tilde{A}^2_{n,l,2}) + E^2_{n,l}
\end{aligned}$$

where, based on (4.17)-(4.19),

$$\lim_{n\to\infty} \||E^1_{n,l}| + |E^2_{n,l}|\|_{L^2(rdr)\cap L^1(dr)} = 0$$

Based on these, we can now use (2.39) for $\psi_{n,2}$ and $\psi_{n,l,2}$ (recall that $\psi^-_{n,l}(0) = \psi^-_n(0)$), and obtain that

$$\lim_{n\to\infty} \|\frac{\psi_{n,2}(0) - \psi_{n,l,2}(0)}{r}\|_{L^2} = 0$$

In turn this implies (4.16).

Next we turn our attention to the dynamical properties of our approximate solutions. In this case, using the convention made on the definition of $N^\pm(\psi^\pm_{n,l})$, we remark that

$$A_{n,l,2} = -1 + \int_0^r \frac{|\psi^+_{n,l}|^2 - |\psi^-_{n,l}|^2}{4} sds$$

with is different than the $\tilde{A}_{n,l,2}$ (at time $t = 0$) we have used above.

We claim that $\psi_{n,l}^{\pm}$ approximately solve (2.24) in the following sense

$$\lim_{l\to\infty} \limsup_{n\to\infty} \|(i\partial_t + H^{\pm})\psi_{n,l}^{\pm} - N^{\pm}(\psi_{n,l}^{\pm})\|_{L^{\frac{4}{3}}} = 0 \tag{4.21}$$

Using the equations that the components of $\psi_{n,l}^{\pm}$ obey, this can be broken into two statements:

$$\lim_{l\to\infty} \limsup_{n\to\infty} \|\sum_{j=1}^{l} N^{\pm}(T_{h^{n,j}} v_j^{\pm}(t+t_{n,j})) - N^{\pm}(\sum_{j=1}^{l} T_{h^{n,j}} v_j^{\pm}(t+t_{n,j}))\|_{L^{\frac{4}{3}}} = 0$$

and

$$\lim_{l\to\infty} \limsup_{n\to\infty} \|N^{\pm}(\psi_{n,l}^{\pm} - w_{n,l}^{\pm}) - N^{\pm}(\psi_{n,l}^{\pm})\|_{L^{\frac{4}{3}}} = 0$$

The second statement follows from (3.11), (4.8) and (4.15).

The first statement follows from (4.1), (4.3), (4.4) and (4.5) by identifying all terms involved:

- cubic cross terms of type $T_{h^{n,j_1}} v_{j_1}^{\pm}(t+t_{n,j_1}) T_{h^{n,j_2}} v_{j_2}^{\pm}(t+t_{n,j_2}) T_{h^{n,j_3}} v_{j_3}^{\pm}(t+t_{n,j_3})$ with at least two of the j_1, j_2, j_3 distinct coming either from the local part of A_0 $(-\frac{1}{2}\Re(\overline{\psi}^{+}\psi^{-})\psi^{\pm})$ or from or from the ψ_2 part $(\frac{1}{r}\Im(\psi_2\bar{\psi}^{\pm})\psi^{\pm})$; these are treated with (4.3).

- nonlocal cubic terms coming from the nonlocal part of A_0 of type $T_{h^{n,j_1}} v_{j_1}^{\pm}(t+t_{n,j_1})[r\partial_r]^{-1}\left(T_{h^{n,j_2}} v_{j_2}^{\pm}(t+t_{n,j_2}) T_{h^{n,j_3}} v_{j_3}^{\pm}(t+t_{n,j_3})\right)$; they are treated using (4.3) and (4.4).

- nonlocal cubic terms coming from $1 + A_2$; they are of type

$$T_{h^{n,j_1}} v_{j_1}^{\pm}(t+t_{n,j_1}) r^{-2}[r^{-1}\bar{\partial}_r]^{-1}\left(T_{h^{n,j_2}} v_{j_2}^{\pm}(t+t_{n,j_2}) T_{h^{n,j_3}} v_{j_3}^{\pm}(t+t_{n,j_3})\right)$$

and are treated using (4.3) and (4.5).

Because of (4.16) and (4.21) we can invoke the stability result in part vi) of Proposition 3.2, and obtain a contradiction with the fact that $\lim_{n\to\infty} S(\psi_n^{\pm}) = \infty$. Therefore the assumption (4.12) is false. In light of (4.11), it follows that the only possibilities are

i) $\phi^j = 0, \forall j$; this is impossible due to global in time well-posedness of the solution $w_n^{\pm}$ (earlier denoted by $w_{n,l}^{\pm}$).

ii) after a relabeling, $\phi^1 = \phi, M(\phi) = m_0, \phi^j = 0, \forall j \geq 2$. In this case the linear profile decomposition simplifies to

$$\psi_n^{-}(0) = h^n e^{it_n H^-}\phi + w_n$$

with $\lim_{n\to\infty} M(w_n) = 0$ (which obviously implies (4.8)). If $\lim_{n\to\infty} t_n = 0$ then we are done. We are left with the case when $\lim_{n\to\infty} t_n = \infty$, as the case with $-\infty$ is entirely similar. An easy argument shows that

$$\lim_{n\to\infty} S_{\geq 0}(e^{itH^-} h^n e^{it_n H^-}\phi) = 0$$

and since $\lim_{n\to\infty} S(e^{itH^-} w_n) = 0$, we can invoke the stability argument with 0 as the approximate solution and $\psi_n^{\pm}(0)$ the initial data to derive that for n large enough $S(e^{itH^{\pm}}\psi_n^{\pm}(0))$ is finite which contradicts the hypothesis. □

We end the section with the proof of Theorem 4.2. Here we adapt the corresponding argument in [16].

Proof of Theorem 4.2. i) Consider a global forward in time critical element $\psi_C^\pm$ which satisfies Theorem 4.1, i.e.

$$\psi_{comp}^\pm(t) = \frac{1}{\lambda(t)}\psi_C^\pm(\frac{r}{\lambda(t)}, t) \in K$$

where K is compact.

Assume that there is a sequence of times $t_n \to \infty$ such that $\lim_{n\to\infty} \lambda(t_n) = 0$. Given the statement on Theorem 4.1, we can select a subsequence (which we label the same way) such that $\lim_{n\to\infty} t_n = +\infty$ and

$$\lambda(t_n) \leq 2 \inf_{t\in[0,t_n]} \lambda(t)$$

Using again Theorem 4.1, we obtain (after passing again to a subsequence labeled the same way) that

$$\psi_{0,n}^\pm := \frac{1}{\lambda(t_n)}\psi_C^\pm(\frac{r}{\lambda(t_n)}, t_n) \to \psi_0^\pm$$

in L^2. Using the mass conservation it follows that $M(\psi_0^\pm) = m_0 > 0$. The fact that $\psi_0^\pm$ satisfy the compatibility relations can be done in an indirect way. We redefine ψ_0^+ to be the compatible pair of ψ_0^-, and by using the last statement in Lemma 2.3, see (2.39), it follows that $\psi_{0,n}^+ \to \psi_0^+$ in L^2.

Let $\psi_n^\pm, \psi^\pm$ be the solution of (2.24) with initial data $\psi_{0,n}^\pm, \psi_0^\pm$ at $t = 0$. From part v) of Proposition 3.2 it follows that $T_\pm(\psi_0^\pm) \geq \underline{\lim}_{n\to\infty} T_\pm(\psi_{0,n}^\pm)$. Since the former are ∞, it follows that $T_\pm(\psi_0^\pm) = \infty$, hence $\psi^\pm$ is global. By the Cauchy theory $\|\psi^\pm\|_{S(\mathbb{R}\times\mathbb{R})} = +\infty$, since otherwise the stability part of the Cauchy theory would imply that $\|\psi_C^\pm\|_{S(\mathbb{R}\times\mathbb{R})} = \|\psi_n\|_{S(\mathbb{R}\times\mathbb{R})} < +\infty$, a contradiction.

From the Cauchy theory we also have that for every $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \psi_n^\pm(t) = \psi^\pm(t) \tag{4.22}$$

By uniqueness of the Cauchy problem (2.24) it follows that

$$\psi_n^\pm(r,t) = \frac{1}{\lambda(t_n)}\psi_C^\pm(\frac{r}{\lambda(t_n)}, t_n + \frac{t}{\lambda^2(t_n)})$$

We claim that $\lim_{n\to\infty} t_n\lambda^2(t_n) = \infty$. If this were not true, then, after passing to a subsequence, we would have $\lim_{n\to\infty} t_n\lambda^2(t_n) = t_0$. Then, by (4.22),

$$\psi_n^\pm(\cdot, -t_n\lambda^2(t_n)) = \frac{1}{\lambda(t_n)}\psi_C^\pm(\frac{\cdot}{\lambda(t_n)}, 0) \to \psi^\pm(\cdot, -t_0)$$

in L^2. Since $\lim_{n\to\infty} \lambda(t_n) = 0$, this implies $\psi^\pm(t_0) \equiv 0$ which is in contradiction with $M(\psi^\pm(-t_0)) = M(\psi_0^\pm) = m_0 > 0$.

Next, fix $t \in (-\infty, 0]$ and note that for n large enough $t_n + \frac{t}{\lambda^2(t_n)} > 0$, hence

$$\tilde{\lambda}_n(t) = \frac{\lambda(t_n + \frac{t}{\lambda^2(t_n)})}{\lambda(t_n)} \geq \frac{1}{2}$$

We claim that $\limsup_{n\to\infty} \tilde{\lambda}_n(t) < +\infty$. Assume this is not the case; then there is an increasing subsequence, still denoted by $\tilde{\lambda}_n(t)$, such that $\lim_{n\to\infty} \tilde{\lambda}_n(t) = +\infty$. On the other hand we have

$$\frac{1}{\tilde{\lambda}_n(t)}\psi_n^{\pm}(\frac{r}{\tilde{\lambda}_n(t)},t) = \frac{1}{\lambda(t_n+\frac{t}{\lambda^2(t_n)})}\psi_C^{\pm}(\frac{r}{\lambda(t_n+\frac{t}{\lambda^2(t_n)})},t_n+\frac{t}{\lambda^2(t_n)})$$
$$= \psi_{comp}^{\pm}(r,t_n+\frac{t}{\lambda^2(t_n)}) \in K$$

Since $\|\frac{1}{\tilde{\lambda}(t_n)}\psi_n^{\pm}(\frac{r}{\tilde{\lambda}(t_n)},t) - \frac{1}{\tilde{\lambda}(t_n)}\psi_0^{\pm}(\frac{r}{\tilde{\lambda}(t_n)},t)\|_{L^2} = \|\psi_n^{\pm}(t)-\psi^{\pm}(t)\|_{L^2}$ and $\frac{1}{\tilde{\lambda}(t_n)}\psi_0^{\pm}(\frac{r}{\tilde{\lambda}(t_n)},t) \to 0$ in L^2 (here we use that $\tilde{\lambda}_n(t) \to \infty$), it follows that $\psi_{comp}^{\pm}(r,t_n+\frac{t}{\lambda^2(t_n)}) \to 0$ in L^2 which violates the fact fact that $M(\psi_{comp}^{\pm}(t')) = m_0$ for all $t' \in \mathbb{R}$.

Therefore we have established that $\frac{1}{2} \leq \tilde{\lambda}_n(t) < +\infty$ for each $t<0$. On a subsequence we have that $\tilde{\lambda}_n(t) \to \tilde{\lambda}(t)$ and by (4.22)

$$\frac{1}{\tilde{\lambda}(t)}\psi^{\pm}(\frac{r}{\tilde{\lambda}(t)},t) \in K$$

with $\tilde{\lambda}(t) \geq \frac{1}{2}$ which creates the solution claimed by part i) after applying the time reversal transformation $t \to -t$.

ii) It suffices to consider the case $T_+ < \infty$ in which case the forward maximal life-span of the solution is $I_+ = [0,T_+)$. If $\lim_{t\to T_+}\lambda(t) \neq \infty$, then there is an increasing sequence $t_n \to T_+$ such that $\lambda(t_n) \to \lambda_0 \in [0,+\infty)$.

By the compactness of $\bar{K}$, it follows that $\psi_n^{\pm} = \frac{1}{\lambda(t_n)}\psi_C^{\pm}(\frac{r}{\lambda(t_n)},t_n) \to \psi_0^{\pm}$. Let $\psi^{\pm},\psi_n^{\pm}$ be the solutions of (2.24) with initial data $\psi_0^{\pm},\psi_{0,n}^{\pm}$ at time $t=T_+$. $\psi^{\pm}$ is defined on some $[T_+-\delta,T_++\delta]$ with $\delta>0$ and, by the Cauchy theory, $\psi_n^{\pm}$, for large enough n, is defined on $[T_+-\frac{\delta}{2},T_++\frac{\delta}{2}]$ with $\|\psi_n\|_{S([T_+-\frac{\delta}{2},T_++\frac{\delta}{2}])} < \infty$. On the other hand,

$$\psi_C^{\pm}(r,t) = \lambda(t_n)\psi_n^{\pm}(\lambda(t_n)r,T_++t-t_n)$$

and by choosing n large enough so that $T_+ \leq t_n + \frac{\delta}{2}$ it follows that $\|\psi_C^{\pm}\|_{S[t_n,T_+)} < \infty$ which is a contradiction with the assumption that $\psi_C^{\pm}$ blows-up at time T_+.

□

5. Momentum and localized momentum.

In this section we rule out the possible scenarios exhibited in Theorem 4.2. With the language used in Section 4, we claim the following

Theorem 5.1. *If $m_0 < 4$ critical elements do not exist.*

This will be based on virial type identities. Virial identities for the Schrödinger Map problem originate in the work of Grillakis and Stefanopoulos via a Lagrangian approach, see [9]. In their work the formulation of these identities is at the level of the conformal coordinate, obtained by using the stereographic projection. Our approach is different in the sense that we derive the virial identities at the level of the gauge components. However our results can be derived from [9].

5.1. **Virial type identities.** This section is concerned with identities involving solutions of (2.24) which satisfy the compatibility condition (2.28).

Given $a : \mathbb{R}_+ \to R$ a smooth function, i.e. $|(r\partial_r)^\alpha| \lesssim_\alpha 1$, and which decays at infinity we claim that

$$\frac{d}{dt}\int a(r)(1+A_2)rdr = -\int r\partial_r a(r)\Re(\psi_1\frac{\bar{\psi}_2}{r})rdr \tag{5.1}$$

By using part i) of Theorem 3.3, the proof of (5.1) goes as follows

$$\begin{aligned}
\frac{d}{dt}\int a(r)(1+A_2)rdr &= \int a(r)\partial_t A_2 rdr = \int a(r)\Im(\psi_0\bar{\psi}_2)rdr \\
&= \int a(r)\Im(i(\partial_r\psi_1 + \frac{1}{r}\psi_1 + \frac{iA_2}{r^2}\psi_2)\bar{\psi}_2)rdr \\
&= \int a(r)\Im(i(r\partial_r\psi_1 + \psi_1)\bar{\psi}_2)dr \\
&= \int a(r)\left(\Im(i\partial_r(r\psi_1\bar{\psi}_2)) - \Im(ir\psi_1\partial_r\bar{\psi}_2)\right)dr \\
&= -\int \partial_r a(r)\Im(i\psi_1\bar{\psi}_2)rdr = -\int r\partial_r a(r)\Re(\psi_1\frac{\bar{\psi}_2}{r})rdr
\end{aligned}$$

This computation is valid in a classical sense provided that $R_\pm\psi^\pm \in H^2$. For general functions $\psi^\pm$ this is done by using a regularization argument as we did in the proof of part i) of Theorem 3.2. Note that the quantities involved on both sides of (5.1) are meaningful in light of the fact that $\psi_0 \in \dot{H}_e^{-1}$ and $a\psi_2 \in \dot{H}_e^1$.

We now introduce the two momenta, the radial and the temporal one, as follows

$$M_1 = \frac{\Re(\psi_1\bar{\psi}_2)}{1-A_2}, \qquad M_0 = -\frac{\Re(\psi_0\bar{\psi}_2)}{1-A_2}$$

Using the covariant calculus, the time momentum can be further written as follows

$$\begin{aligned}
-(1-A_2)M_0 &= \Re(\psi_0\bar{\psi}_2) \\
&= \Re\left(i(D_1\psi_1 + \frac{1}{r}\psi_1 + \frac{1}{r^2}D_2\psi_2)\bar{\psi}_2\right) \\
&= -\Im(\partial_r\psi_1\bar{\psi}_2) - \frac{1}{r}\Im(\psi_1\bar{\psi}_2) - \frac{A_2}{r^2}|\psi_2|^2 \\
&= -\partial_r\Im(\psi_1\bar{\psi}_2) - \Im(\psi_1\partial_r\bar{\psi}_2) - \frac{1}{r}\partial_r A_2 - \frac{A_2}{r^2}|\psi_2|^2 \\
&= -\partial_r^2 A_2 - \frac{1}{r}\partial_r A_2 - A_2(|\psi_1|^2 + \frac{|\psi_2|^2}{r^2})
\end{aligned}$$

which leads to

$$M_0 = -\Delta\ln(1-A_2) - \left(\frac{\partial_r A_2}{1-A_2}\right)^2 + \frac{A_2}{1-A_2}(|\psi_1|^2 + \frac{|\psi_2|^2}{r^2}) \tag{5.2}$$

The following identity plays a fundamental role in our analysis

$$\partial_t M_1 - \partial_r M_0 = -\partial_r A_0 \tag{5.3}$$

This is established by using the covariant rules of calculus,

$$\begin{aligned}
\partial_t M_1 &= \frac{\Re(D_0\psi_1\bar{\psi}_2)}{1-A_2} + \frac{\Re(\psi_1\overline{D_0\psi_2})}{1-A_2} + \frac{\Re(\psi_1\bar{\psi}_2)}{(1-A_2)^2}\partial_t A_2 \\
&= \frac{\Re(D_1\psi_0\bar{\psi}_2)}{1-A_2} + \frac{\Re(\psi_1\overline{D_2\psi_0})}{1-A_2} + \frac{\Re(\psi_1\bar{\psi}_2)}{(1-A_2)^2}\Im(\psi_0\bar{\psi}_2) \\
&= \partial_r M_0 - \frac{\Re(\psi_0\partial_r\bar{\psi}_2)}{1-A_2} - \frac{\Re(\psi_0\bar{\psi}_2)}{(1-A_2)^2}\partial_r A_2 + \frac{\Re(\psi_1\overline{D_2\psi_0})}{1-A_2} + \frac{\Re(\psi_1\bar{\psi}_2)}{(1-A_2)^2}\Im(\psi_0\bar{\psi}_2) \\
&= \partial_r M_0 - \frac{A_2\Im(\psi_0\bar{\psi}_1)}{1-A_2} - \frac{\Re(\psi_0\bar{\psi}_2)}{(1-A_2)^2}\Im(\psi_1\bar{\psi}_2) + \frac{A_2\Im(\psi_1\overline{\psi_0})}{1-A_2} + \frac{\Re(\psi_1\bar{\psi}_2)}{(1-A_2)^2}\Im(\psi_0\bar{\psi}_2) \\
&= \partial_r M_0 - 2\frac{A_2\Im(\psi_0\bar{\psi}_1)}{1-A_2} + \frac{|\psi_2|^2\Im(\psi_0\bar{\psi}_1)}{(1-A_2)^2} \\
&= \partial_r M_0 + \Im(\psi_0\bar{\psi}_1) \\
&= \partial_r M_0 - \partial_r A_0
\end{aligned}$$

The above computation is meaningful provided that $R_\pm\psi^\pm \in H^3$.

Next we derive a localized version of (5.3) which has also the advantage that it makes sense for $\psi^\pm \in L^2$ only. We take $a : \mathbb{R}_+ \to \mathbb{R}$ to be a smooth function which decays at infinity and satisfies also $|\frac{1}{r}\partial_r a| \lesssim 1$ and $|\partial_r^2| \lesssim 1$. As a consequence we have that if $f \in \dot{H}^1_e$ then $\frac{1}{r} f \partial_r a \in \dot{H}^1_e$.

We multiply (5.3) by a and integrate by parts as follows

$$\int a(r)M_1(r)dr\,\big|_0^T + \int_0^T \int \partial_r a(r) M_0 dr = \int_0^T \int \partial_r a(r) A_0 dr \tag{5.4}$$

This identity is now meaningful for $\psi^\pm \in L^2$. Indeed each term is well-defined for the following reasons:

- the first since a is bounded and $\frac{1}{r}M_1 \in L^2$,
- the second since $\psi_0 \in \dot{H}^{-1}_e$ and $\frac{1}{r}\partial_r a \cdot \psi_2 \in \dot{H}^1_e$,
- the third since $\frac{1}{r}\partial_r a$ is bounded and $A_0 \in L^1$.

The justification of (5.4) for general $\psi^\pm \in L^2$ is done by regularizing $\psi^\pm$ as above.

It will be useful to rewrite the second term on the left-hand side as follows

$$\begin{aligned}
\int \partial_r a(r) M_0 dr &= \int \frac{1}{r}\partial_r a(r)\left(-\Delta \ln(1-A_2) - \left(\frac{\partial_r A_2}{1-A_2}\right)^2 + \frac{A_2}{1-A_2}(|\psi_1|^2 + \frac{|\psi_2|^2}{r^2})\right) r dr \\
&= \int \partial_r(\frac{1}{r}\partial_r a(r))\partial_r \ln(1-A_2) r dr \\
&\quad - \int \frac{1}{r}\partial_r a(r)\left(\left(\frac{\partial_r A_2}{1-A_2}\right)^2 - \frac{A_2}{1-A_2}(|\psi_1|^2 + \frac{|\psi_2|^2}{r^2})\right) r dr
\end{aligned}$$

5.2. **Proof of Theorem 5.1.** The argument is in the spirit of the corresponding one in [16].

Based on a localized version of (5.1) and (5.4) we rule out the possibilities exhibited in parts i) and ii) of Theorem 4.2.

In order to do so, we make a few important remarks. Since we are in the setup of $E(u) < 4\pi$, which translates into $\|\psi^{\pm}\|_{L^2} < 2$, we obtain from (2.29) that

$$\frac{-A_2}{1-A_2} \gtrsim 1$$

Next, by using (2.25) and (2.36), the concentration property (4.6) implies

$$\int_{r \gtrsim C(\eta)c^{-1}} (|\psi_1(r)|^2 + \frac{|\psi_2(r)|}{r^2} + \frac{(A_2(r)+1)^2}{r^2}) r dr \lesssim \eta, \qquad \forall t \in I_+.$$

We start by ruling out the existence of a critical element from part i) of Theorem 4.2, i.e. the global element with $\lambda(t) \geq c > 0, \forall t > 0$. In (5.4), we take $a(r) = r^2 \phi(\frac{r}{R})$ where ϕ is smooth and equals 1 for $r \leq 1$ and 0 for $r \leq 2$, and obtain

$$\begin{aligned} \int a(r) M_1(r) dr \Big|_0^T &= \int_0^T \int \partial_r (\frac{1}{r} \partial_r a(r)) \partial_r \ln(1-A_2) r dr dt \\ &+ \int_0^T \int \frac{1}{r} \partial_r a(r) \left(\left(\frac{\partial_r A_2}{1-A_2} \right)^2 - \frac{A_2}{1-A_2} (|\psi_1|^2 + \frac{|\psi_2|^2}{r^2}) \right) r dr dt \\ &+ \int_0^T \int \partial_r a(r) A_0 dr dt \end{aligned} \tag{5.5}$$

In this identity there are two main terms which we compare against each other: the first on the left-hand side and the second on the right-hand side. All the other terms are controlled by one of the two main terms just mentioned.

We choose $\eta \ll 1$ small enough (the exact choice is derived from the inequalities on the error terms below) and $R = C(\eta)c^{-1} \gg c^{-1}$; we estimate the main terms in the above expression by

$$|\int a(r) M_1 dr| \lesssim \int r^2 |\psi_1| |\frac{\psi_2}{r}| r dr \lesssim R^2 \|\psi_1\|_{L^2} \|\frac{\psi_2}{r}\|_{L^2} \lesssim R^2 E$$

which is valid both at $t = 0$ and $t = T$, and

$$\int_0^T \int \frac{1}{r} \partial_r a(r) \left(\left(\frac{\partial_r A_2}{1-A_2} \right)^2 - \frac{A_2}{1-A_2} (|\psi_1|^2 + \frac{|\psi_2|^2}{r^2}) \right) r dr dt \gtrsim TE$$

By choosing $T \gg R^2$ we obtain a contradiction, provided that we establish that all the other terms involved in (5.5) are of error type.

The first term on the left-hand side of (5.5) is bounded as follows

$$|\int_0^T \int \partial_r (\frac{1}{r} \partial_r a(r)) \partial_r \ln(1-A_2) r dr dt| \lesssim \int_0^T \int_{r \approx R} |\partial_r A_2| dr dt \lesssim T \eta^{\frac{1}{2}} \ll TE$$

For the third term on the right-hand side of (5.5) we use (2.21) and write

$$|\int_0^T \int \partial_r a(r) A_0 dr dt| = |\int_0^T \int (-2 + \frac{1}{r} \partial_r a(r)) A_0 r dr dt|$$

The coefficient $-2 + \frac{1}{r}\partial_r a(r)$ is supported in $r \geq R$, so the above expression is further bounded by

$$\lesssim \int_0^T \|\psi_1\|_{L^2[R,\infty)} \|\frac{\psi_2}{r}\|_{L^2[R,\infty)} dt \lesssim T\eta \ll TE$$

We have just shown that the other two terms in (5.5) are of error type and this finishes the contradiction argument. Therefore we have shown that the scenario exhibited in part i) of Theorem 4.2 cannot happen.

Next we rule out the critical element of type exhibited in part ii). In this case the assumption is that we have a critical element with $T_+ < \infty, \lim_{t \to T_+} \lambda(t) = +\infty$.

For fixed R we claim that

$$\lim_{t \to T_+} \int \phi(\frac{r}{R})(1 + A_2) r dr = 0 \tag{5.6}$$

Indeed, for given $\epsilon > 0$, pick η such that $\eta R^2 < \epsilon$. Recalling that $|u_1|^2 + |u_2|^2 = |\psi_2|^2$ and using (2.36) we obtain

$$\begin{aligned}
&\|\phi(\frac{r}{R})(1 + A_2)\|_{L^1} \\
\lesssim &(C(\eta)\lambda(t)^{-1})^2 \|\frac{1 + A_2}{r}\|_{L^2(0, C(\eta)\lambda^{-1}(t)]} + R^2 \|\frac{1 + A_2}{r}\|_{L^2[C(\eta)\lambda^{-1}(t), R]} \\
\lesssim &(C(\eta)\lambda(t)^{-1})^2 E + \eta R^2
\end{aligned}$$

By choosing t close enough to T_+, we obtain $(C(\eta)\lambda(t)^{-1})^2 E < \epsilon$, and this establishes (5.6).

Next we choose $a(r) = \phi(\frac{r}{R})$, fix $\eta > 0$, integrate (5.1) on $[t, T_+)$ and use (5.6) to obtain

$$\int \phi(\frac{x}{R})(1 + A_2(r,t)) r dr \lesssim (T_+ - t)\|\psi_1(t)\|_{L^2(|x| \approx R)} \|\frac{\psi_2(t)}{r}\|_{L^2(|x| \approx R)} \lesssim (T_+ - t)\eta$$

provided that $R \gtrsim C(\eta)\lambda(t)^{-1}$. By fixing t and taking $\eta \to 0$ (which also forces $R \to \infty$), it follows that

$$\int (1 + A_2(r,t)) r dr = 0$$

which implies $A_2(t) \equiv -1$ hence, by (2.9) and then by (2.18) it follows that $\psi_2(t) \equiv 0$ and $\psi_1(t) \equiv 0$. Finally this implies by (2.25) that $\psi^{\pm}(t) = 0$ which contradicts the blow-up hypothesis at time T_+ (since the solution is globally in time $\equiv 0$).

6. Proof of the main result

This section is dedicated to the proof of Theorem 1.2. Given an initial data $u_0 \in \dot{H}^1 \cap \dot{H}^3$, by using Theorem 1.1 it follows that it has a unique local solution on $[0, T]$ for some $T > 0$. On this interval we use sections 2.1 and 2.2 to construct the associated fields $\psi^{\pm}$ obeying the system (2.24) and whose mass satisfies $\|\psi_0^+\|_{L^2} = \|\psi_0^-\|_{L^2} < 2$. By using Theorem 5.1 (and the previous reduction from Section 4) it follows that the solution $\psi^{\pm}$ is globally defined on $[0, +\infty)$ and with $\|\psi^{\pm}\|_{L^4(\mathbb{R}_+ \times \mathbb{R}_+)} < +\infty$. By part vii) of Theorem 3.2 the H^2 regularity of $R_{\pm}\psi_0^{\pm}$ is propagated at all times $t \geq 0$. Invoking Proposition 2.1 this implies that $u(t) \in \dot{H}^1 \cap \dot{H}^3$ with bounds depending on $\|\psi^{\pm}\|_{L^4(\mathbb{R}_+ \times \mathbb{R}_+)}, \|R_{\pm}\psi_0^{\pm}\|_{H^2}$ and t. Using again Theorem 1.1, this means that the solution $u(t)$ can be continued past time T and in fact for all times $t \geq 0$ with $u(t) \in L_t^{\infty}(\mathbb{R}_+ : \dot{H}^1 \cap \dot{H}^3)$. The scattering part mentioned refers to the scattering for $\psi^{\pm}(t)$, which follows from the Cauchy theory for the syste (2.24), see Theorem 3.2.

Next we continue with part ii) of the Therorem 1.2. From (2.23) we obtain the Lipschitz continuity of $\psi^{\pm}$ with respect u as a map from $\dot{H}^1$ to L^2. From the Cauchy theory for the system (2.24) which $\psi^{\pm}(t)$ obey, see Theorem 3.2, we obtain the Lipschitz continuity of the $\psi^{\pm}(t)$ with respect to its initial data. Finally by invoking (2.34), for each t, we obtain the Lipschitz continuity $u(t)$ with respect to u_0 in $\dot{H}^1$.

7. Appendix

Proof of Proposition 2.1. We write the arguments below in a qualitative fashion in order to have a concise argument. However one easily sees that the argument below provides quantitive bounds which lead to (2.33).

We first read the information $u \in \dot{H}^2$. Since $\Delta u \in L^2$ it follows that $\partial_r^2 u, \frac{1}{r}(\partial_r + \frac{1}{r}\partial_\theta^2)u \in L^2$, from which, using the equivariance property, we obtain

$$\partial_r^2 u, \frac{1}{r}(\partial_r - \frac{1}{r})(u_1, u_2), \frac{1}{r}\partial_r u_3 \in L^2. \tag{7.1}$$

Since $|u| = 1$ it follows that

$$\frac{u_1 \partial_r u_1 + u_2 \partial_r u_2}{r} = -\frac{u_3 \partial_r u_3}{r} \in L^2$$

and by invoking $\frac{1}{r}(\partial_r - \frac{1}{r})(u_1, u_2) \in L^2$, we obtain $\frac{u_1^2 + u_2^2}{r^2} \in L^2$.

Next we read the information that $u \in \dot{H}^3$ then $\partial_x \Delta u, \partial_y \Delta u \in L^2$ from which, using that $r\partial_r = x\partial_x + y\partial_y$ it follows

$$\partial_r(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u_1, \partial_r(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u_2, \partial_r(\partial_r^2 + \frac{1}{r}\partial_r)u_3 \in L^2 \tag{7.2}$$

and, using that $\partial_\theta = x\partial_y - y\partial_x$ and the equivariance of Δu,

$$\frac{1}{r}(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u_1, \frac{1}{r}(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})u_2 \in L^2. \tag{7.3}$$

Since $D_r(v + iw) = 0$ it follows that

$$\partial_r \psi^{\pm} = \partial_r \left(W^{\pm} \cdot (v + iw) \right) = (\partial_r W^{\pm}) \cdot (v + iw)$$

A direct computation gives

$$\begin{aligned}
\partial_r W^{\pm} &= \partial_r^2 u \pm \partial_r(\frac{u \times Ru}{r}) \\
&= \partial_r^2 u \mp \frac{\partial_r u_3 \cdot u + u_3 \cdot \partial_r u}{r} \mp \frac{\overrightarrow{k} - u_3 \cdot u}{r^2} \\
&= (\partial_r^2 \pm \frac{1}{r}\partial_r \mp \frac{1}{r^2})u \mp \frac{1 + u_3}{r}\partial_r u \mp \frac{\overrightarrow{k}}{r^2} + f^{\pm} u
\end{aligned}$$

where

$$f^{\pm} = \mp \frac{\partial_r u_3}{r} \pm \frac{1 + u_3}{r^2}$$

We then continue with

$$\partial_r\psi^\pm = \left((\partial_r^2 \pm \frac{1}{r}\partial_r \mp \frac{1}{r^2})u_1, (\partial_r^2 \pm \frac{1}{r}\partial_r \mp \frac{1}{r^2})u_2, (\partial_r^2 \pm \frac{1}{r}\partial_r)u_3\right)\cdot(v+iw)$$
$$\mp \frac{1+u_3}{r}\psi_1 \mp i\frac{(1+u_3)\psi_2}{r^2}$$
$$= F^\pm - \frac{1+u_3}{r}\psi^+$$

where $F^\pm \in L^2$ from (7.1). Since $\frac{1+u_3}{r} = \frac{1}{1-u_3}\frac{u_1^2+u_2^2}{r} \in L^1(dr)$, by using the integrating factor it follows that $\partial_r(e^{\int \frac{1+u_3}{r}dr}\psi^+) \in L^2$. By Sobolev embedding, in two dimensions, we obtain $\psi^+ \in L^4$ and since, by above, $\frac{1+u_3}{r} \in L^4$ we obtain $\partial_r\psi^+ \in L^2$. It also follows that $\partial_r\psi^- \in L^2$.

To show that $\frac{\psi^-}{r} \in L^2$ we write

$$\frac{1}{r}W^- = \frac{1}{r}\left((\partial_r - \frac{1}{r})u_1, (\partial_r - \frac{1}{r})u_2, \partial_r u_3\right) + (\frac{(1+u_3)u_1}{r^2}, \frac{(1+u_3)u_2}{r^2}, \frac{u_1^2+u_2^2}{r^2})$$

and use (7.1) and the fact that $\frac{u_1^2+u_2^2}{r^2} \in L^2$.

Hence we have just established that $R_\pm\psi^\pm \in H^1$. The procedure can be easily reversed, i.e. if $R_\pm\psi^\pm \in H^1$ then $u \in \dot{H}^2$. Indeed, the additional regularity gives, by Sobolev embedding, that $\psi^\pm \in L^4$, hence $\frac{\psi_2}{r} \in L^4$ which implies $\frac{1+u_3}{r} \in L^4$. Therefore $\frac{1+u_3}{r}\psi^+ \in L^2$ and this implies that $F^\pm \in L^2$. This takes care of the covariant part of Δu. The part of Δ in the normal space is $f^\pm = \mp\frac{\Im(\psi_1\bar{\psi}_2)}{r} \pm \frac{1}{1-u_3}\frac{u_1^2+u_2^2}{r^2}$ which belongs to L^2 since $\psi^\pm \in L^4$ by using Sobolev embeddings.

Next we transfer third derivatives of u to second derivatives for $\phi^\pm$ and vice-versa. Using the above computation for $\partial_r\psi^+$, we have

$$\partial_r^2\psi^+ = \partial_r F^+ - \frac{1+u_3}{r}\partial_r\psi^+ - \frac{\partial_r A_2}{r}\psi^+ + \frac{1+A_2}{r^2}\psi^+$$

From (7.2) and the fact that $D_r v = D_r w = 0$ we obtain that $\partial_r(F_1^+, F_2^+, F_3^+) \in L^2$. Using the Sobolev embedding, we have that $\psi^\pm \in L^6$, hence $\frac{\psi_2}{r} \in L^6$ and $\frac{1+A_2}{r^2} = \frac{1}{1-A_2}\frac{|\psi_2|^2}{r^2} \in L^3$. Therefore, by gauging the middle terms we obtain

$$\partial_r(e^{\int \frac{1+u_3}{s}}\partial_r\psi^+) \in L^2$$

which further gives $e^{\int \frac{1+u_3}{s}}\partial_r\psi^+ \in L^4$ (here it is important that we already have the information that $\partial_r\psi^+ \in L^2$). Since $\frac{1+A_2}{r} = \frac{1}{1-A_2}\frac{|\psi_2|^2}{r} \in L^4$ we obtain that in fact $\frac{1+u_3}{r}\partial_r\psi^+ \in L^2$, therefore $\partial_r^2\psi^+ \in L^2$.

Next we have

$$\frac{1}{r}\partial_r\psi^+ = \frac{1}{r}F^+ - \frac{1+A_2}{r^2}\psi^+$$

From the above arguments we have that $\frac{1+A_2}{r^2} \in L^3, \psi^+ \in L^6$, hence the last term belongs to L^2. From (7.3) we obtain the first two components of $\frac{1}{r}F^+$ in L^2. As for $\frac{1}{r}F_3^+$ we write

$$\frac{1}{r}F_3^+ = i\frac{\psi_2}{r}(\partial_r^2 + \frac{1}{r}\partial_r)u_3$$

and, bu Sobolev embeddings, $\Delta u_3 \in L^4$ and $\frac{\psi_2}{r} \in L^4$, hence $\frac{1}{r}F_3^+ \in L^2$. This finishes the argument for $\partial_r^2\psi^+, \frac{1}{r}\partial_r\psi^+ \in L^2$ which implies that $R_+\psi_+ \in H^2$.

Next, we have

$$\begin{aligned}
H^-\psi^- &= (\partial_r + \frac{1}{r})\partial_r\psi^- - \frac{4}{r^2}\psi^- \\
&= (\partial_r + \frac{1}{r})F^- + \frac{1+u_3}{r}\partial_r\psi^+ + \frac{\partial_r A_2}{r}\psi^+ - \frac{4}{r^2}\psi^- \\
&= \left((\partial_r^3 - \frac{3}{r^2}\partial_r + \frac{3}{r^3})u_1, (\partial_r^3 - \frac{3}{r^2}\partial_r + \frac{3}{r^3})u_2, (\partial_r^3 - \frac{1}{r^2}\partial_r)u_3\right)\cdot(v+iw) \\
&+ \frac{1+u_3}{r}\partial_r\psi^+ + \frac{\partial_r A_2}{r}\psi^+ - 4\frac{1+u_3}{r^3}(u_1,u_2,1-u_3)\cdot(v+iw)
\end{aligned}$$

Since

$$\partial_r^3 - \frac{3}{r^2}\partial_r + \frac{3}{r^3} = \partial_r(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}) - \frac{1}{r}(\partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2})$$

we obtain from (7.2) and (7.3) that $(\partial_r^3 - \frac{3}{r^2}\partial_r + \frac{3}{r^3})u_1, (\partial_r^3 - \frac{3}{r^2}\partial_r + \frac{3}{r^3})u_2 \in L^2$. Concerning the term $(\partial_r^3 - \frac{1}{r^2}\partial_r)u_3(v_3+iw_3)$ we use (7.2) so that it is enough to show that $\frac{1}{r}\psi_2\partial_r^2 u_3 \in L^2$. By Sobolev embedding $\Delta u_3 \in L^4$ and $\frac{\psi_2}{r} \in L^4$, hence it is enough to show that $\frac{1}{r^2}\psi_2\partial_r u_3 \in L^2$ which follows since, by (7.1), $\frac{\partial_r u_3}{r} \in L^2$ and $\frac{\psi_2}{r} \in L^\infty$; the last follows since $\psi^+ \in H^2 \subset L^\infty$ (here we refer to the two dimensional extension) and $\psi^- \in \dot{H}^1_e \subset L^\infty$.

Next, from the argument for $\Delta\psi^+$ we have that $\frac{1+u_3}{r}\partial_r\psi^+ + \frac{\partial_r A_2}{r}\psi^+ \in L^2$. For the last term we use the inequality

$$\left|\frac{1+u_3}{r^3}(u_1,u_2,1-u_3)\cdot(v+iw)\right| \lesssim \frac{|\psi_2|^3}{r^3}$$

which can then be shown to be in L^2 by the above arguments since $\frac{\psi_2}{r} \in L^4 \cap L^\infty$. This finishes the proof of the fact that $H^-\psi^- \in L^2$, and hence of the H^2 regularity of $R_-\psi^-$.

Finally, one needs to show that if $R_\pm\psi^\pm \in H^2$ then $u \in \dot{H}^3$. This is done by backtracking the previous argument with the final goal of establishing (7.2) and (7.3). The direct backtracking argument provides the covariant version of (7.2) and (7.3), i.e. their part in the tangent bundle. For the part in the normal bundle, i.e. in the direction of u, one need to involve $f^\pm$ as we did before. This part involves one derivative less, hence it is easier. The details are left to the reader. □

Lemma 7.1. *Assume that $f \in L^2$. Then the following holds true*

$$\lim_{t\to\infty}\sup_{r\in(0,\infty)}\left|r\int_r^\infty \frac{e^{itH^-}f}{s}ds\right| = 0 \tag{7.4}$$

If $g^n = g_{l_n,t_n}, n \in \mathbb{N}$ is asymptotically orthogonal to the constant sequence $g_{1,0}$ and $K \subset (0,+\infty)$ is compact, then the following holds true

$$\lim_{n\to\infty}\|g^n f\|_{L^2(K)} = 0. \tag{7.5}$$

Proof. By the Cauchy-Schwarz inequality we have the uniform bound

$$\left|r\int_r^\infty \frac{e^{itH^-}f}{s}ds\right| \lesssim \|e^{itH^-}f\|_{L^2} = \|f\|_{L^2}$$

Hence for both (7.4) and (7.5) it suffices to prove that they hold for f in a dense subset of L^2. To choose a suitable dense subset we first restrict ourselves to f with compact support.

Secondly, we recall that the functions $e^{2i\theta}e^{itH^-}f$ are solutions to the linear Schrödinger equations with initial data $e^{2i\theta}f$. We localize $e^{2i\theta}f(r)$ in frequency away from frequency 0 and infinity. If we use spherically symmetric multipliers this keeps the frequency localized functions of the form $e^{2i\theta}f(r)$ as desired. The outcome of this is to reduce the problem to the case when $e^{2i\theta}f(r)$ is a Schwartz function with all moments equal to zero. This also implies that f vanishes of infinite order at zero. Furthermore, due to the frequency localization away from zero we also know that $e^{2i\theta}f(r)$ can be represented as

$$e^{2i\theta}f(r) = \Delta(e^{2i\theta}g(r))$$

with g in the same class.

For initial data as described above, the solution to the linear Schrödinger equation is easily seen to satisfy bounds of the form

$$|e^{it\Delta}(e^{2i\theta}f(r))| \lesssim \frac{\langle \frac{|t|-r}{1+\min(r,|t|)}\rangle^{-N}}{\langle t\rangle}$$

along with all its derivatives. Since $e^{it\Delta}(e^{2i\theta}f(r)) = e^{2i\theta}e^{itH^-}f$ this implies the bound

$$|e^{itH^-}f| \lesssim \frac{\langle \frac{|t|-r}{1+\min(r,|t|)}\rangle^{-N}}{\langle t\rangle} \tag{7.6}$$

which holds for $e^{itH^-}f$ and all its r derivatives. We also have the representation

$$e^{itH^-}f = H^-e^{itH^-}g$$

where $e^{itH^-}g$ satisfies similar bounds.

For such f we now proceed to establish (7.4) and (7.5). Indeed, (7.5) follows directly from (7.6). For (7.4) we use the above representation in terms of g. Integrating by parts we have

$$\begin{aligned} r\int_r^\infty \frac{e^{itH^-}f}{s}ds &= r\int_r^\infty H^-(e^{itH^-}g)\cdot\frac{1}{s^2}sds \\ &= -2r^{-1}e^{itH^-}g(r) - \partial_r(e^{itH^-}g(r)) + r\int_r^\infty e^{itH^-}g\ H^-\left(\frac{1}{s^2}\right)sds \\ &= -2r^{-1}e^{itH^-}g(r) - \partial_r(e^{itH^-}g(r)) \end{aligned}$$

The last expression is easily seen to decay uniformly like t^{-1} in view of (7.6) applied to g.

□

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