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Dynamic Matching Market with Agent-Dependent Compatible Probabilities

Author
Tao, Lan

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Dynamic Matching Market with Agent-Dependent Compatible Probabilities

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by

Lan Tao

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ABSTRACT OF THE THESIS

Dynamic Matching Market with
Agent-Dependent Compatible Probabilities

by

Lan Tao
Master of Science in Statistics
University of California, Los Angeles, 2023
Professor Xiaowu Dai, Chair

Numerous dynamic matching market models pursuing different objectives have been developed for kidney exchange studies. These objectives range from minimizing waiting time and maximizing welfare to reducing the fraction of unmatched agents. Motivated by the medical observation that better matching outcomes are often achieved when donors and recipients share the same race, we extend the dynamic matching model by Akbarpour et al. (2020) to a matching market with two types of agents, in which the compatible probability depends on agent types. In this study, we examine the performance of two matching algorithms, namely Greedy algorithm and Patient algorithm, from both theoretical and empirical perspectives. Our aim is to investigate whether delaying the matching process to thicken the market can effectively decrease the fraction of unmatched agents within the market.

The thesis of Lan Tao is approved.

Jennie E. Brand

Mark S. Handcock

Xiaowu Dai, Committee Chair

University of California, Los Angeles

2023

To my parents,

who have supported and encouraged me

in my hard time

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CHAPTER 1

Introduction

1.1 Background

The dynamic matching market problem has received considerable attention over the past decade and has found applications in various domains, such as kidney exchange, sharing economy and online marketplaces, housing market, child adoption, and college admission. In a dynamic matching market, agents enter and leave the market stochastically, and only certain pairs of agents can be matched feasibly. Market planners face the challenge of determining which agent pairs to match and when to match them, taking into account their compatibility, with the ultimate goals of minimizing waiting time, maximizing welfare, or reducing the fraction of unmatched agents.

In this paper, we propose a dynamic matching market model where the probability of compatibility is influenced by the types of the agents involved. This restriction on compatibility is motivated by the observations and patterns observed in kidney transplantation markets, where the compatibility between donors and recipients plays a crucial role in the success of transplant procedures.

Kidney transplantation is considered to be the best remedy for renal disease, and the biological compatibility between patients and donors includes blood-type compatibility and tissue-type compatibility. Blood type compatibility, also known as ABO compatibility, means that the patient cannot receive a kidney from a donor who has a blood antigen (A or B) that the patient does not have (Ashlagi, Roth 2021)[AR21]. For example, an O patient is

ABO-compatible to only O donor, while an O donor is ABO-compatible to any patient. An AB patient is ABO-compatible to any donor, while an AB donor is ABO-compatible to only AB patient. Beside blood-type antigens, the donor has human leukocyte antigens (HLA), which influence the compatibility success of kidney transplantation. A patient is said to be tissue-type compatible to the donor if the patient does not have antibodies to the donor's HLA. A patient's immune system will reject the kidney if the patient have antibodies to the donor's HLA. In the operation process, A virtual match could be determined with the given data on patient and donor ABOs, the donor HLA, and the patient antibodies, and then a cross matching test would be conducted to verify whether the patient will reject the donor's kidney.

Most kidney transplants today come from deceased donors, but this source of transplants is limited. Kidney exchange arose as a way of increasing the availability of transplants from compatible living donors without violating the ban on compensation donors. Two or more incompatible patient-donor pairs might be able to exchange kidneys, so that each patient gets a kidney that is compatible with him/her, from another patient's donor (Ashlagi, Roth 2021)[AR21]. It overcame the barrier of biological incompatibility through a swap between two pairs of donors and patients who are each biologically incompatible but compatible to the other. Since 2013, over 10% of the live kidney donor transplants in the U.S. each year were accomplished through exchange. Over $1,000$ of the U.S. living donor kidney transplants in 2019 resulted from kidney exchange, and many of them were conducted through kidney exchange platforms that organize these exchanges among multiple hospitals (Ashlagi, Roth 2021)[AR21]. Consequently, it is crucial to study and design a matching mechanism between pairs of donors and patients to alleviate this dilemma and decrease the number of death caused by shortage of transplantable organs. Numerous studies have been conducted to investigate the incentive, the efficiency, the welfare of matching market under kidney exchange problem context.

Medical researchers in the field of organ sharing have recognized the significance of race as a crucial factor for achieving HLA compatibility between donors and patients. It has been observed that better cross-matching outcomes are attained when both the donors and recipients are of the same race. In particular, a study by Lazda (1992) [Laz92] revealed that kidneys from non-black donors, which are well-matched for HLA phenotypes more commonly found in black patients, were rarely allocated to black patients. Building upon this observation, we incorporate a matching market model in our study, involving two distinct agent types. In this model, the compatibility between agents of the same type is assigned a higher probability, while the compatibility between agents of different types is assigned a lower probability. We investigate the performance of two algorithms, namely Greedy algorithm and Patient algorithm, both theoretically and empirically, with the aim of demonstrating that delaying the matching process to allow for a larger participant pool can effectively reduce the proportion of unmatched agents in the market.

The paper is structured as follows: In the subsequent section, we review literature focusing on matching market models pertaining to the kidney exchange problem and sharing economy problem. Chapter 2 formulates the dynamic matching market model, including the model settings, compatibility and description of the two matching algorithms. Chapter 3 evaluates the loss of Greedy algorithm and Patient algorithm from a theoretical perspective, in which we derive the theoretical bounds of loss of both algorithms. Chapter 4 presents simulation studies that compare the performance of the two algorithms, analyze the effects of different parameters, and assess the effectiveness of the derived loss bounds. Finally, Chapter 5 concludes the study by discussing the limitations of the proposed model and suggesting potential directions for future research.

1.2 Related literature

Many studies about matching market within various context and settings with different objectives of stability, welfare or quality have been discussed during the past decade. Researchers have developed various models for problems in kidney exchange, sharing economy and online marketplaces, housing market, child adoption, and college admission.

Under the context of kidney exchange problem, it is significant to study and design a matching mechanism between pairs of donors and patients to alleviate this dilemma and decrease the number of death caused by shortage of transplantable organs. This problem was originally studied by Roth, Sönmez, and Ünver (2004, 2005) [RSU04][RSU05a][RSU05b]. Ünver(2010)[Unv10] firstly considers dynamic model with multiple types of agents and agents never perish. His model reveals that waiting to thicken the market is not helpful when only bilateral exchanges are allowed. However, when agents depart stochastically, waiting is worthwhile according to the study by Akbarpour et al. (2020)[ALG20]. Their results show that waiting to thicken the market can yield large gains if the planner can forecast departures accurately and information about departures is highly valuable if it is feasible to wait. Anderson et al. (2015)[AAG15] raise a model with the main objective to minimize the average waiting time under the setting that agents never perish. Sönmez et al. (2020)[SUY20] analyze the efficiency and equity implications of incentivized exchange in an analytically tractable dynamic large-market model of kidney transplantation. They show that compared with primary technologies, incentivized exchange increases both overall access and equity in access to transplants. Baccara et al. (2020)[BLY20] studied the trade-off between waiting time for a thicker market allowing for higher quality matches, and minimizing agents' waiting costs. Ashlagi and Roth (2021)[AR21] reviewed some of the key operational issues in the design of successful kidney exchange programs, and pointed out several research directions, including international exchanges, improvements on greedy matching, and dynamics with frictions etc. They concluded that a thick inter-hospital kidney exchange network should be built, which means assembling a sufficiently large, constantly renewed pool of patient-donor pair.

With the development of sharing economy, studies into matching market sharing economy and online marketplaces have received more attention in recent years. Ashlagi et al.

(2019)[ABD19] study the problem of matching two passengers in the context of an on-demand ride-sharing platform, in which matching agents leave after some time periods and each pair of agents can yield a different match value. They investigate the total match value over a finite time horizon, and compare several algorithms under different settings of deterministic or stochastic departures, and adversarial or random arriving order. Liu et al. (2019)[LWY19] develop a model for two-sided dynamic matching market and estimate it using data to show that Patient algorithm outperforms Greedy algorithm if drivers are not too heterogeneous. Varma et al. (2020)[MBT20] study optimal pricing and matching control in a two-sided queueing system in sharing economy and online marketplaces, where heterogeneous customers and servers arrive to the system with price-dependent arrival rates, with the objective to maximize the long-run average profits of the system while minimizing average waiting time. Aouad and Saban (2022) [AS23] focus on the online assortment optimization problem faced by a two-sided matching platform that hosts a set of suppliers waiting to match with a customer, and investigate assortment algorithms to maximize the expected number of matches.

In addition to these two most popular application problems, dynamic matching models are developed for other matching market mechanism design problems in real life. Leshno (2021)[Les22] studies waiting list allocation mechanisms in public housing problem, where waiting times serve a similar role to that of monetary prices in directing agents' choices. The study shows that waiting time fluctuations lead to misallocation and welfare loss. Inspired by children-adoption problem, Baccara et al. (2020)[BLY20] consider a market model in which one type agent is more desirable than the other, and they evaluate the effect of waiting cost on expected welfare.

Ashlagi et al. (2019)[ABJ19] study the efficiency and thickness in heterogeneous matching market where agents arrive stochastically over time. They considered two types of agents in the model, hard-to-match agents and easy-to-match agents, and investigated the scenarios and corresponding efficient matching algorithms (involving agents' priorities) to improve agents' waiting times. Akbarpour et al. (2020)[ALG20] considered model of dynamic matching in networked markets where agents arrive and depart stochastically, and the composition of the trade network depends endogeneously on the matching algorithm. They found that waiting to thicken the market substantially reduces the fraction of unmatched agents if the planner is able to identify critical agents who are about to depart. In their model, any two agents are compatible with probability $p = d/m$, where d is the density parameter of the model, and m is the Poisson rate at which new agents arrive in the market. They showed that the Greedy algorithm's loss is at least $1/(2d+1)$, while the Patient algorithm's loss is at most $e^{-d/2}/2$.

CHAPTER 2

Problem Formulation

2.1 Dynamic matching market

We construct a continuous-time model for a dynamic matching market with network constraints which runs in time interval $[0, T]$. New agents arrive at the market at Poisson rate m, in other words, m new agents are expected to arrive at the market during each time period $[t, t+1]$, and we assume $m \geq 1$. Let C_t be the set of agents in the market at time t, which is referred as *pool*, and $Z_t = |C_t|$ is called the *pool size*. Suppose all agents can be classified as two types, *type-A* agent and *type-B* agent, according to some characteristics, which are denoted by v_A and v_B respectively. C_t^A denotes the set of type-A agent in the market at time t, and $Z_t^A = |C_t^A|$ is the number of type-A agent in the pool, and called the pool size of A. Similarly, C_t^B and $Z_t^B = |C_t^B|$ can be defined in this way. Let C_t^n denote the set of agents arriving at the market at time t, and C_{t_0,t_1}^n denote the set of agents arriving at the market in time interval $[t_0, t_1]$. Note that $|C_t^n| \leq 1$ with probability 1, since we assume that no agents will arrive at the market at the exactly same time.

Each agent in the pool will become *critical*, a state that the agent will leave the pool immediately if not matched, at an independent Poisson rate λ, which can be normalized to 1 without loss of generality. An agent is said to *perish* from the market if it leaves the market unmatched. Suppose an agent v arrives at the market at time t_0, and it will become critical at time $t_0 + t_c$, where $t_c \sim Exp(1)$. If the agent v doesn't get matched during time interval $[t_0, t_0 + t_c]$, then v will perish from the market immediately at time $t_0 + t_c$. It is apparent that the agent v can be matched and leave the market at any time point t_1 during $[t_0, t_0+t_c]$, and the length of time which an agent v is in the market is referred as *sojourn*, which is defined as $s(v) = t_1 - t_0$. Let C_t^c be the set of agents who become critical at time t, and we assume that $|C_t^c| \leq 1$ with probability 1.

2.2 Compatibility

Now there are two types of agents in the market, v_A and v_B, and the probability of being compatible between any pair of agents are independent across pairs, but dependent on the type of agents. For any pair of agents of the same type, that is $v_A - v_A$ or $v_B - v_B$, they are compatible with probability p_1, while any pair of agents of different types, that is $v_A - v_B$, are compatible with probability p_2, where $0 \leq p_2 \leq p_1 \leq 1$.

We can regard this matching problem from the perspective of graph. At any time point $t > 0$, each agent in the market can be seen as a vertex, and all agents in the pool form a vertex set, which is just C_t. For any two distinct vertices v_1, v_2 in C_t, there exist an edge between them if and only if these two agents are compatible in the market. Note that compatible pairs persist over time. Thus, the edge set $E_t \subseteq C_t \times C_t$ represents the set of all compatible pairs of agents in the market at time t, and $G_t = (A_t, E_t)$ is the network graph at time t. For any agent $v \in C_t$, its *neighbor* is defined as the set of agents who are compatible with it, denoted by $N_t(v) \subseteq C_t$.

Let $C = \cup_{t \leq T} C_t^n$ be the set of agents who enter the market by time T, $E \subset C \times C$ be the set of possible transactions between agents in C, $G = (C, E)$. It is clear that any realization of the above stochastic process is uniquely defined, given C_t^n, C_t^c for all $t \geq 0$.

2.3 Matching algorithms

A *matching* $M_t \subseteq E_t$ is a set of edges where no two edges share the same end points. A *matching algorithm* aims to select a matching M_t in the graph G_t and the end points of edges in the matching will leave the market immediately. Note that C_t, E_t, Z_t are all functions of the underlying matching algorithm.

An ideal matching algorithm is to maximize the number of matched agents and minimize the number of perished agents. Thus, we introduce the following notation to define and measure the loss of a matching algorithm.

We use ALG(T) to denote the set of agents matched by time T under the matching algorithm ALG, that is

$$\text{ALG}(T) := \{v \in C : v \text{ is matched by ALG by time } T\}.$$

The loss of a matching algorithm is defined to be the ratio of expected total number of perished agents to expected number of agents by time T, which can be expressed as follows:

$$\begin{aligned} L(ALG) &= \frac{E[|C - ALG(T) - C_T|]}{E[|C|]} \\ &= \frac{E[|C - ALG(T) - C_T|]}{mT}. \end{aligned}$$

We introduce the following two matching algorithms, *Greedy algorithm* and *Patient algorithm.* The Greedy algorithm matches agents as soon as possible. For any new agent v enters the market at time t, match it with a random agent in its neighbors $N_t(v)$ if $N_t(v) \neq \emptyset$. The Patient algorithm matches only critical agents. If an agent v becomes critical at time t, then match it with a random agent in its neighbors $N_t(v)$ if $N_t(v) \neq \emptyset$.

Compared with Greedy algorithm, Patient algorithm extends each agent's waiting time in order to thicken the market and make better matching choices. We study the performance of these two algorithms to investigate whether the sacrifice of longer waiting time can effectively reduce the fraction of perished agents.

Algorithm 1 Greedy Algorithm for dynamic matching

for a new agent v arrives at the market at time t **do**

  if $N_t(v) \neq \emptyset$ **then**

    randomly choose $v' \in N_t(v)$ at uniform, and match v with v', $Z_t = Z_t - 1$.

  else

    agent v stays at the pool, $Z_t = Z_t + 1$.

  end if

end for

Algorithm 2 Patient Algorithm for dynamic matching

for an agent v at the pool becomes critical at time t **do**

  if $N_t(v) \neq \emptyset$ **then**

    randomly choose $v' \in N_t(v)$ at uniform, and match v with v', $Z_t = Z_t - 2$.

  else

    agent v perishes from the market, $Z_t = Z_t - 1$.

  end if

end for

CHAPTER 3

Evaluation of Loss of Two Algorithms

Let $z_t(\cdot)$ be the distribution of pool size Z_t at time t and $\mu(\cdot)$ be the stationary distribution of Markov Chain on Z_t. The mixing time of the Markov chain on Z_t is defined as $\tau_{mix}(\epsilon) = inf\{t : \sum_{k=0}^{\infty} |\mu(k) - z_t(k)| \leq \epsilon\}$. Since there are two types of agents in the market, we consider the 2-dim Markov Chain on (Z_t^A, Z_t^B) as well, whose stationary distribution is denoted by $\pi(\cdot, \cdot)$.

Let $n_t = \max\{Z_t^A, Z_t^B\}$ denote the maximum number of type-A agent or type-B agent for each (Z_t^A, Z_t^B) at time t, and $n'_t = \min\{Z_t^A, Z_t^B\}$ denote the minimum number of type-A agent or type-B agent for each (Z_t^A, Z_t^B) at time t. $\tilde{\pi}(\cdot)$ is defined as the stationary distribution of n_t. We define the proportion $s = \frac{n'_t}{n_t} = \min\{\frac{Z_t^A}{Z_t^B}, \frac{Z_t^B}{Z_t^A}\}$ $(0 < s \leq 1)$, which is not a fixed value but a random variable when the Markov chain on Z_t is under steady state. We assume the distribution of s as a Beta distribution, that is $s \sim \mathcal{B}(\alpha, \beta)$, where the hyper parameters α and β are related to model parameters m, p_1, p_2, λ and matching algorithm, but independent of Z_t^A and Z_t^B at each time period. Let $u = \frac{s \cdot p_1 + p_2}{s+1}$ denote the average compatible probability of this matching market, and $d = m \cdot u$ stands for the density in this matching model. In the following, conditional on the proportion s, we study the bound of loss of Greedy algorithm and Patient algorithm.

3.1 Bound of loss of Greedy algorithm

The Greedy algorithm matches agents as soon as possible. Let $\xi := \mathbb{E}_{Z\sim\mu}[Z]$ is the expected pool size under stationary distribution of the Markov chain Z_t. If the Markov chain on Z_t is mixed, then agents perish at the rate of ξ, since the pool is almost always an empty graph under Greedy algorithm. According to the definition of the loss of a matching algorithm,

$$\begin{aligned} L(Greedy) &= \frac{\mathbb{E}[\int_0^T Z_t dt]}{mT} \\ &= \frac{1}{mT}\int_0^T \mathbb{E}[Z_t]dt. \end{aligned}$$

Proposition 20 [ALG20] holds under this setting, and by that we have

$$E[Z_t] \leq E[\tilde{Z}_{t_0}] = m(1-e^{-t_0}) \leq m.$$

Lemma 11 [ALG20] also holds under this setting, so we have

$$L(Greedy) \leq \frac{\tau_{mix}(\epsilon)}{T} + 6\epsilon + \frac{1}{m}2^{-6m} + \frac{E_{Z\sim\mu}[Z]}{m}.$$

Here, the term $\frac{\tau_{mix}(\epsilon)}{T}$ is related to the time the Markov Chain costs to transit to stationary distribution; while the other terms are loss approximation after the Markov Chain mixes.

To upper-bound $\mathbb{E}_{Z\sim\mu}[Z]$, we find out the balance equation and combine it with transition rates to investigate the stationary distribution $\mu(\cdot)$.

Lemma 3.1.1. *Let* $u = \frac{s\cdot p_1 + p_2}{s+1}$. *For any integer* $k \geq k^*$,

$$\frac{\mu(k+1)}{\mu(k)} \leq e^{-u\cdot(k^*-k)}.$$

And, for any $k \leq k^*$, $\frac{\mu(k-1)}{\mu(k)} \leq e^{-u\cdot(k^*-k+1)}$.

Proof. The pool size $Z_t = k$ goes to either $k+1$ or $k-1$. The former situation happens only when a new agent comes to the market and the market-maker cannot match it. Since the compatible probability is related to the type of agents, we define k_0 as the number of agents in the pool which are of the same type of this new agent, obviously $k_0 \leq k$. Then we have

$$r_{k \to k+1} = m \cdot (1-p_1)^{k_0} \cdot (1-p_2)^{(k-k_0)}.$$

The latter situation happens when a new agent comes and market-maker can match it or an agent currently in the pool gets critical, thus

$$r_{k \to k-1} = k + m \cdot [1 - (1-p_1)^{k_0} \cdot (1-p_2)^{(k-k_0)}].$$

According to the balance equation of a Markov chain, $\forall S \subseteq N$,

$$\sum_{i \in S, j \notin S} \mu(i) \cdot r_{i \to j} = \sum_{i \in S, j \notin S} \mu(j) \cdot r_{j \to i}.$$

The stationary distribution should satisfy

$$\mu(k-1) \cdot r_{k-1 \to k} = \mu(k) \cdot r_{k \to k-1}.$$

If the newly coming agent is type-A, then the transition probability operator will be:

$$r_{k \to k+1} = m \cdot (1-p_1)^{k_A} \cdot (1-p_2)^{k_B},$$

$$r_{k \to k-1} = k + m \cdot [1 - (1-p_1)^{k_A} \cdot (1-p_2)^{k_B}];$$

otherwise, the newly coming agent is type-B, then we have

$$r_{k \to k+1} = m \cdot (1-p_1)^{k_B} \cdot (1-p_2)^{k_A},$$

$$r_{k \to k-1} = k + m \cdot [1 - (1-p_1)^{k_B} \cdot (1-p_2)^{k_A}].$$

Under stationary distribution, the ratio of type-A agents to type-B agents in the pool is fixed, which satisfies $\frac{k_A}{k_B} = \alpha (0 < \alpha < +\infty)$. In other words, $k_A = \frac{\alpha}{\alpha+1} k$ and $k_B = \frac{1}{\alpha+1} k$. Notice that α could be either s or $\frac{1}{s}$.

Case 1. If the newly coming agent is v_A, then it should be satisfied that

$$\mu(k-1)\cdot m\cdot(1-p_1)^{\frac{\alpha}{\alpha+1}(k-1)}\cdot(1-p_2)^{\frac{1}{\alpha+1}(k-1)} = \mu(k)\cdot\{k+m\cdot[1-(1-p_1)^{\frac{\alpha}{\alpha+1}(k-1)}\cdot(1-p_2)^{\frac{1}{\alpha+1}(k-1)}]\}.$$

Let's define a function $f(k)$ as follows:

$$\begin{aligned}
f(k) &= r_{k\to k+1} - r_{k\to k-1} \\
&= m\cdot(1-p_1)^{\frac{\alpha}{\alpha+1}k}(1-p_2)^{\frac{1}{\alpha+1}k} - [k + m\cdot(1-(1-p_1)^{\frac{\alpha}{\alpha+1}k}(1-p_2)^{\frac{1}{\alpha+1}k})] \\
&= 2m\cdot[(1-p_1)^{\alpha}(1-p_2)]^{\frac{k}{\alpha+1}} - k - m
\end{aligned}$$

Since $f'(k) = \frac{2}{\alpha+1}m\cdot[(1-p_1)^{\alpha}(1-p_2)]^{\frac{k}{\alpha+1}}\cdot(\alpha\cdot log(1-p_1)+log(1-p_2)) - 1 < 0$ and $f''(k) = \frac{2}{\alpha+1}m\cdot[(1-p_1)^{\alpha}(1-p_2)]^{\frac{k}{\alpha+1}}\cdot(\alpha\cdot log(1-p_1)+log(1-p_2))^2 > 0$, $f(k)$ is a convex decreasing function. Suppose k^* is the unique root for equation $f(k)=0$, then it can be proved that $\frac{m}{\frac{2m(\alpha\cdot p_1+p_2)}{\alpha+1}+1} \le k^* \le log2\cdot\frac{\alpha+1}{\alpha\cdot p_1+p_2}$. Let $k^*_{min} = \frac{m}{\frac{2m(\alpha\cdot p_1+p_2)}{\alpha+1}+1}$ and $k^*_{max} = log2\cdot\frac{\alpha+1}{\alpha\cdot p_1+p_2}$,

$$\begin{aligned}
f(k^*_{min}) &= 2m\cdot[(1-p_1)^{\alpha}(1-p_2)]^{\frac{k^*_{min}}{\alpha+1}} - k^*_{min} - m \\
&\ge 2m\cdot[(1-p_1\cdot\frac{\alpha}{\alpha+1}\cdot k^*_{min})(1-p_2\cdot\frac{1}{\alpha+1}\cdot k^*_{min})] - k^*_{min} - m \\
&= m - k^*_{min}(\frac{2m\cdot(\alpha\cdot p_1+p_2)}{\alpha+1}+1) + {k^*_{min}}^2\cdot\frac{2m\cdot\alpha\cdot p_1\cdot p_2}{(\alpha+1)^2} \\
&\ge 0 \\
f(k^*_{max}) &= 2m\cdot[(1-p_1)^{\alpha}(1-p_2)]^{\frac{k^*_{max}}{\alpha+1}} - k^*_{max} - m \\
&\le 2m\cdot[e^{-\frac{\alpha}{\alpha+1}k^*_{max}\cdot p_1 - \frac{1}{\alpha+1}k^*_{max}\cdot p_2}] - k^*_{max} - m \\
&\le 2m\cdot e^{-\frac{\alpha\cdot p_1+p_2}{\alpha+1}\cdot k^*_{max}} - k^*_{max} - m \\
&\le 0
\end{aligned}$$

For now, we have shown that $k^*_{min} \le k^* \le k^*_{max}$, and it remains to show that μ is highly concentrated around k^*.

For $k \geq k^*$, consider

$$\begin{aligned}\frac{\mu(k)}{\mu(k+1)} &= \frac{r_{k+1\to k}}{r_{k\to k+1}} \\ &= \frac{k+1+m\cdot[1-(1-p_1)^{\frac{\alpha(k+1)}{\alpha+1}}\cdot(1-p_2)^{\frac{k+1}{\alpha+1}}]}{m\cdot(1-p_1)^{\frac{\alpha\cdot k}{\alpha+1}}\cdot(1-p_2)^{\frac{k}{\alpha+1}}},\end{aligned}$$

due to $f(k^*)=0$, we have

$$\begin{aligned}\frac{\mu(k)}{\mu(k+1)} &= \frac{k+1+m\cdot[1-(1-p_1)^{\frac{\alpha(k+1)}{\alpha+1}}\cdot(1-p_2)^{\frac{k+1}{\alpha+1}}]}{m\cdot(1-p_1)^{\frac{\alpha\cdot k}{\alpha+1}}\cdot(1-p_2)^{\frac{k}{\alpha+1}}} \\ &= \frac{k-k^*+1-m\cdot(1-p_1)^{\frac{\alpha(k+1)}{\alpha+1}}(1-p_2)^{\frac{k+1}{\alpha+1}}+2m\cdot[(1-p_1)^{\frac{\alpha\cdot k^*}{\alpha+1}}(1-p_2)^{\frac{k^*}{\alpha+1}}]}{m\cdot(1-p_1)^{\frac{\alpha\cdot k}{\alpha+1}}(1-p_2)^{\frac{k}{\alpha+1}}} \\ &\geq -(1-p_1)^{\frac{\alpha}{\alpha+1}}(1-p_2)^{\frac{1}{\alpha+1}}+\frac{2}{((1-p_1)^{\frac{\alpha}{\alpha+1}}(1-p_2)^{\frac{1}{\alpha+1}})^{k-k^*}} \\ &\geq \frac{1}{((1-p_1)^{\frac{\alpha}{\alpha+1}}(1-p_2)^{\frac{1}{\alpha+1}})^{k-k^*}} \\ &\geq \frac{1}{(e^{-p_1\cdot\frac{\alpha}{\alpha+1}}\cdot e^{-p_2\cdot\frac{1}{\alpha+1}})^{k-k^*}} \\ &= e^{-\frac{\alpha\cdot p_1+p_2}{\alpha+1}\cdot(k^*-k)}.\end{aligned}$$

It reveals that $\forall k \geq k^*$, $\frac{\mu(k+1)}{\mu(k)} \leq e^{-\frac{\alpha\cdot p_1+p_2}{\alpha+1}\cdot(k-k^*)}$. Similarly, it can be obtained that $\forall k \leq k^*$, $\frac{\mu(k-1)}{\mu(k)} \leq (1-\frac{\alpha\cdot p_1+p_2}{\alpha+1})^{k^*-k+1} \leq e^{-\frac{\alpha\cdot p_1+p_2}{\alpha+1}\cdot(k^*-k+1)}$.

Case 2. If the newly coming agent is v_B, then it should be satisfied that

$$\mu(k-1)\cdot m\cdot(1-p_1)^{\frac{1}{\alpha+1}(k-1)}\cdot(1-p_2)^{\frac{\alpha}{\alpha+1}(k-1)} = \mu(k)\cdot\{k+m\cdot[1-(1-p_1)^{\frac{1}{\alpha+1}(k-1)}\cdot(1-p_2)^{\frac{\alpha}{\alpha+1}(k-1)}]\}.$$

By comparing this equation with the equation in Case 1, we realize that we can replicate the proof in Case 1 by replacing α with $\frac{1}{\alpha}$. Similarly, the function $f(k)$ in this case can be defined as $f(k) = r_{k\to k+1} - r_{k\to k-1} = 2m\cdot[(1-p_1)(1-p_2)^{\alpha}]^{\frac{k}{\alpha+1}} - k - m$. Following the steps in Case 1, we obtain

$$\begin{cases} \dfrac{\mu(k+1)}{\mu(k)} \leq e^{-\frac{p_1+\alpha\cdot p_2}{\alpha+1}\cdot(k-k^*)}, & \forall k \geq k^* \\ \dfrac{\mu(k-1)}{\mu(k)} \leq e^{-\frac{p_1+\alpha\cdot p_2}{\alpha+1}\cdot(k^*-k+1)}, & \forall k \leq k^* \end{cases}$$

Considering these two cases together, no matter what kind of new agents arriving in the market, we always have $\frac{\mu(k+1)}{\mu(k)} \leq e^{-u\cdot(k^*-k)}$ for any integer $k \geq k^*$; and $\frac{\mu(k-1)}{\mu(k)} \leq e^{-u\cdot(k^*-k+1)}$ for any $k \leq k^*$, where $u = min\{\frac{\alpha\cdot p_1+p_2}{\alpha+1}, \frac{p_1+\alpha\cdot p_2}{\alpha+1}\} = \frac{s\cdot p_1+p_2}{s+1}$. $\square$

By repeated application of the Lemma 3.1.1, we can derive the following Greedy Concentration Proposition.

Proposition 3.1.2 (Greedy Concentration)**.** *Let* $u = \frac{s\cdot p_1+p_2}{s+1}$*, there exists* $\frac{m}{2m\cdot u+1} \leq k^* \leq \frac{1}{u}\cdot log2$*, such that for any* $\sigma > 1$*,*

$$\mathbb{P}_\pi[k^* - \sigma\cdot\sqrt{\frac{2}{u}} \leq Z \leq k^* + \sigma\cdot\sqrt{\frac{2}{u}}] \geq 1 - O(\sqrt{\frac{2}{u}})e^{-\sigma^2}.$$

Proof. For any integer $k \geq k^*$, we have

$$\mu(k) \leq \frac{\mu(k)}{\mu(\lceil k^*\rceil)} \leq exp[-u\cdot\sum_{i=\lceil k^*\rceil}^{k-1}(i-k^*)] \leq exp[-\frac{(k-k^*-1)^2\cdot u}{2}].$$

$\forall\sigma > 0$,

$$\begin{aligned}\sum_{k=k^*+1+\sigma\sqrt{\frac{2}{u}}}^{\infty}\mu(k) &\leq \sum_{k=k^*+1+\sigma\sqrt{\frac{2}{u}}}^{\infty} exp[-\frac{(k-k^*-1)^2\cdot u}{2}] \\ &= \sum_{k=0}^{\infty} exp(-\frac{(k+\sigma\sqrt{\frac{2}{u}})^2\cdot u}{2}) \\ &\leq \frac{e^{-\sigma^2}}{min\{\frac{1}{2}, \sigma\sqrt{\frac{u}{2}}\}}.\end{aligned}$$

On the other side, $\sum_{k=0}^{k^*-\sigma\sqrt{\frac{2}{u}}}$ can be upper-bounded in a similar way, this completes the proof. $\square$

Furthermore, With this Greedy Concentration Proposition, $\mathbb{E}_{Z\sim\mu}[Z]$ can be upper bounded by $k^* + O(\sqrt{\frac{1}{u}}\cdot log\frac{1}{u})$, and then the loss of Greedy algorithm can be furthermore derived as in Theorem 3.1.4.

Lemma 3.1.3. *For k^* and u as in Proposition 3.1.2,*

$$\mathbb{E}_{Z\sim\pi}[Z] \le k^* + O(\sqrt{\frac{1}{u}} \cdot log\frac{1}{u}).$$

Proof. $\mathbb{E}_{Z\sim\pi}[Z] \le k^* + \Delta + \sum_{i=k^*+\Delta+1}^{+\infty} i \cdot \pi(i)$.

$$\begin{aligned}\sum_{i=k^*+\Delta+1}^{+\infty} i \cdot \pi(i) &\le \sum_{i=\Delta+1}^{+\infty} e^{-\frac{u}{2}\cdot(i-1)^2} \cdot (i+k^*) \\ &= \sum_{i=\Delta}^{+\infty} e^{-\frac{u}{2}\cdot i^2} \cdot (i-1) + \sum_{i=\Delta}^{+\infty} e^{-\frac{u}{2}\cdot i^2} \cdot (k^*+2) \\ &\le \frac{e^{-\frac{u}{2}\cdot(\Delta-1)^2}}{u} + (k^*+2) \cdot \frac{e^{-\frac{u\Delta^2}{2}}}{min\{\frac{1}{2}, \frac{u\cdot\Delta}{2}\}}\end{aligned}$$

Let $\Delta = 1 + 2 \cdot \sqrt{\frac{1}{u}} \cdot log\sqrt{\frac{1}{u}}$, then the right hand side in the above equation is at most 1. Thus, we have $\mathbb{E}_{Z\sim\pi}[Z] \le k^* + 2 + 2 \cdot \sqrt{\frac{1}{u}} \cdot log\sqrt{\frac{1}{u}} = k^* + O(\sqrt{\frac{1}{u}} \cdot log\frac{1}{u})$. ☐

Theorem 3.1.4. *For any $\epsilon \ge 0$ and $T > 0$,*

$$\begin{aligned}L(Greedy) \le &\frac{1}{m} \cdot log2 \cdot \frac{1}{u} + \frac{\tau_{mix}(\epsilon)}{T} + 6\epsilon + \frac{1}{m}2^{-6m} \\ &+ \frac{1}{m} \cdot O(\sqrt{\frac{1}{u}} \cdot log\frac{1}{u}),\end{aligned}$$

where $u = \frac{s\cdot p_1+p_2}{s+1}$.

Here $u = \frac{s\cdot p_1+p_2}{s+1}$ stands for the average compatible probability of this matching market. When s goes to 0, which implies the pool is almost full of agents v_B (or v_A), then u goes to p_2, the upper bound of L(Greedy) is identical to the result in the paper [ALG20].

We also investigate the lower bound of loss of Greedy algorithm, which is shown as the following Theorem.

Theorem 3.1.5. *As T goes to ∞,*

$$L(Greedy) \ge \frac{1}{2m \cdot u + 1}.$$

Proof. Under Greedy algorithm, the pool is almost an empty graph without edges, since end points of each edge leave the pool immediately once there forms an edge. In empty graph, critical agents perish with probability 1, therefore, agents in the pool becomes critical at the rate Z_t, which is just the pool size. In steady state, $T \to \infty$, $L(Greedy) \approx \frac{\mathbb{E}[Z_t]}{m}$, where the numerator is the expected rate of agents perishing and the denominator is the expected rate of agents arriving.

During time interval $[t, t+1]$, m agents come to the market. For a newly coming agent v of type-A, the probability of an existing agent of type-A in pool to be compatible with it is p_1, and the probability of an existing agent of type-B in pool to be compatible with it is p_2. Then the probability for this newly coming agent v to have no compatible match equals to $(1-p_1)^{k_A} \cdot (1-p_2)^{k_B}$, and thus the probability for v to have a compatible match is $1-(1-p_1)^{k_A} \cdot (1-p_2)^{k_B}$. Similarly, for a newly coming agent v of type-B, it has no compatible match with agents in the pool with probability $(1-p_2)^{k_A} \cdot (1-p_1)^{k_B}$, and has a compatible match with an agent in the pool with probability $1-(1-p_2)^{k_A} \cdot (1-p_1)^{k_B}$.

Suppose a newly coming agent is of type-A with probability p, and is of type-B with probability $1-p$, where $0 \leq p \leq 1$. At a balance point z^*, for all $0 \leq p \leq 1$, it should satisfy that

$$\begin{aligned} &m \cdot p \cdot (1-p_1)^{k_A} \cdot (1-p_2)^{k_B} + m \cdot (1-p) \cdot (1-p_2)^{k_A} \cdot (1-p_1)^{k_B} \\ =&z^* + m \cdot p \cdot [1-(1-p_1)^{k_A} \cdot (1-p_2)^{k_B}] + m \cdot (1-p) \cdot [1-(1-p_2)^{k_A} \cdot (1-p_1)^{k_B}], \end{aligned}$$

where $k_A = \frac{\alpha}{\alpha+1} z^*$, and $k_B = \frac{1}{\alpha+1} z^*$.

According to proof of Proposition 3.1.2, the solution to the above equation is located in this interval $\frac{m}{2m \cdot u+1} \leq z^* \leq \frac{1}{u} \cdot log2$, where $u = min\{\frac{\alpha \cdot p_1 + p_2}{\alpha+1}, \frac{p_1 + \alpha \cdot p_2}{\alpha+1}\} = \frac{s \cdot p_1 + p_2}{s+1}$. Based on Proposition 3.1.2, under stationary distribution, $Z_t \to k^*$. Therefore, $E[Z_t] \approx k^* \geq \frac{m}{2m \cdot u+1}$, and $L(Greedy) \approx \frac{E[Z_t]}{m} \geq \frac{1}{2m \cdot u+1}$. □

3.2 Bound of loss of Patient algorithm

Patient algorithm matches an agent only when it becomes critical. Once an agent a becomes critical, it has no acceptable transactions with probability

$$\begin{cases} (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B} & \text{if } a \text{ is type-A,} \\ (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A} & \text{if } a \text{ is type-B.} \end{cases}$$

By definition of loss of a matching algorithm, the loss of Patient algorithm is:

$$L(Patient) = \frac{1}{mT} \cdot E[\int_{t=0}^{T} Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B} + Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A} dt].$$

By involving the stationary distribution of the Markov Chain on (Z_t^A, Z_t^B) and n_t, the loss of Patient algorithm can be derived as follows:

Lemma 3.2.1. *For any $\epsilon > 0$ and $T > 0$,*

$$\begin{aligned} L(Patient) \leq \frac{\tau_{mix}(\epsilon)}{T} + \frac{2\epsilon}{m \cdot (1-p_2) \cdot p_1^2} + \frac{1}{m}\Bigg[& \frac{1}{(1-p_1)} E_{n_t \sim \tilde{\pi}}[n_t((1-p_1)(1-p_2)^s))^{n_t}] \\ & + \frac{s}{(1-p_1)} E_{n_t \sim \tilde{\pi}}[n_t((1-p_1)^s(1-p_2)))^{n_t}]\Bigg]. \end{aligned}$$

Proof.

$$\begin{aligned} L(Patient) &= \frac{1}{mT} \cdot E[\int_{t=0}^{T} Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B} + Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A} dt] \\ &= \frac{1}{mT} \cdot \int_{t=0}^{\tau_{mix}(\epsilon)} E[Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B}] + E[Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A}]dt \\ &+ \frac{1}{mT} \cdot \int_{t=\tau_{mix}(\epsilon)}^{T} E[Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B}] + E[Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A}]dt \\ &\leq \frac{1}{mT} \cdot \int_{t=0}^{\tau_{mix}(\epsilon)} E[Z_t^A + Z_t^B]dt \\ &+ \frac{1}{mT} \cdot \int_{t=\tau_{mix}(\epsilon)}^{T} E[Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B}] + E[Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A}]dt \\ &\leq \frac{\tau_{mix}(\epsilon)}{T} + \frac{1}{mT} \cdot \int_{t=\tau_{mix}(\epsilon)}^{T} E[Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B}] \\ &+ E[Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A}]dt \end{aligned}$$

Let $\pi(\ ,\ )$ denote the stationary distribution of Markov chain on (Z_t^A, Z_t^B). Then we can write

$$
\begin{aligned}
L(Patient) \leq & \frac{\tau_{mix}(\epsilon)}{T} + \frac{1}{mT} \cdot \int_{t=\tau_{mix}(\epsilon)}^{T} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (\pi(i,j)+\epsilon) \cdot [i \cdot (1-p_1)^{i-1}(1-p_2)^j] \\
& + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (\pi(i,j)+\epsilon) \cdot [j \cdot (1-p_1)^{j-1}(1-p_2)^i] dt \\
& \leq \frac{\tau_{mix}(\epsilon)}{T} + \frac{2\epsilon}{m} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} i(1-p_1)^{i-1}(1-p_2)^j + \frac{1}{m}\Bigg[E[Z_t^A \cdot (1-p_1)^{(Z_t^A-1)}(1-p_2)^{Z_t^B}] \\
& + E[Z_t^B \cdot (1-p_1)^{(Z_t^B-1)}(1-p_2)^{Z_t^A}]\Bigg] \\
& = \frac{\tau_{mix}(\epsilon)}{T} + \frac{2\epsilon}{m \cdot (1-p_2) \cdot p_1^2} + \frac{1}{m}\Bigg[E_{(Z_t^A, Z_t^B)\sim\pi}[Z_t^A(1-p_1)^{Z_t^A-1}(1-p_2)^{Z_t^B}] \\
& + E_{(Z_t^A, Z_t^B)\sim\pi}[Z_t^B(1-p_1)^{Z_t^B-1}(1-p_2)^{Z_t^A}]\Bigg].
\end{aligned}
$$

For the last equality, we use $\sum_{i=0}^{\infty}(1-p_2)^i = \frac{1}{1-p_2}$, and $\sum_{i=0}^{\infty} i(1-p_1)^{i-1} = \frac{1}{p_1^2}$.

Under stationary distribution and conditional on s, it could be furthermore derived as follows,

$$
\begin{aligned}
L(Patient) \leq \frac{\tau_{mix}(\epsilon)}{T} + \frac{2\epsilon}{m \cdot (1-p_2) \cdot p_1^2} + \frac{1}{m}\Bigg[& \frac{1}{(1-p_1)} E_{n_t\sim\tilde{\pi}}[n_t((1-p_1)(1-p_2)^s))^{n_t}] \\
& + \frac{s}{(1-p_1)} E_{n_t\sim\tilde{\pi}}[n_t((1-p_1)^s(1-p_2)))^{n_t}].
\end{aligned}
$$

□

This lemma demonstrates that the loss of Patient algorithm can be upper-bounded by the expectation of a function of n_t under its stationary distribution plus other approximation terms for the time before the Markov chain mixes. To investigate the stationary distribution of n_t and furthermore to bound $E_{n_t\sim\tilde{\pi}}[n_t((1-p_1)(1-p_2)^s))^{n_t}]$, we study the balance equation and transition rates of Markov chain on (Z_t^A, Z_t^B). For any (k_A, k_B), the Markov chain can transit to 7 states, including (k_A+1, k_B), (k_A, k_B+1), (k_A-1, k_B), (k_A, k_B-1), (k_A-2, k_B), (k_A, k_B-2), and (k_A-1, k_B-1). The transition paths are shown in the Figure 3.1.

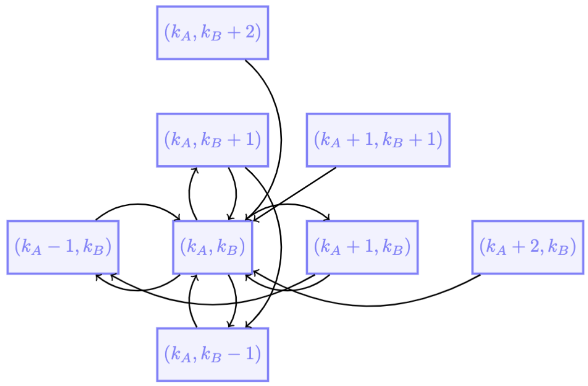

Figure 3.1: The transition paths around (k_A, k_B) of the Markov Chain on (Z_t^A, Z_t^B) under Patient Algorithm

Suppose a newly arrival agent is of type-A with probability p, and is of type-B with probability $1-p$. The Markov chain on (k_A, k_B) goes to (k_A+1, k_B) when a new type-A agent arrives at the market, then we have

$$r_{(A \to A+1, B)} = m \cdot p.$$

Similarly, (k_A, k_B) goes to (k_A, k_B+1) when a new type-B agent arrives at the market, whose probability is

$$r_{(A, B \to B+1)} = m \cdot (1-p).$$

When a type-A agent becomes critical bu gets no compatible match in the market, it will perish from the market, so we have

$$r_{(A \to A-1, B)} = k_A \cdot (1-p_1)^{k_A - 1}(1-P_2)^{k_B}.$$

Then the probability that a critical type-A agent has a compatible match in the market is $1-(1-p_1)^{k_A-1}(1-P_2)^{k_B}$. A critical type-A agent gets matched to another type-A agent with the probability that $[1-(1-p_1)^{k_A-1}(1-P_2)^{k_B}] \cdot \frac{(k_A-1)\cdot p_1}{(k_A-1)\cdot p_1 + k_B \cdot p_2}$, and at this time, two type-A agents leave the market. Thus, the transition rate $r_{(A \to A-2, B)}$ can be written as

$$r_{(A \to A-2, B)} = k_A \cdot [1-(1-p_1)^{k_A-1}(1-P_2)^{k_B}] \cdot \frac{(k_A-1)\cdot p_1}{(k_A-1)\cdot p_1 + k_B \cdot p_2}.$$

Similarly, for a critical type-B agent, it will perish from the market without compatible match with the probability $(1-p_1)^{k_B-1}(1-P_2)^{k_A}$, the transition rate $r_{(A,B\to B-1)}$ is

$$r_{(A,B\to B-1)} = k_B \cdot (1-p_1)^{k_B-1}(1-P_2)^{k_A}.$$

Two matched type-B agents will leave the market with probability $[1-(1-p_1)^{k_B-1}(1-P_2)^{k_A}] \cdot \frac{(k_B-1)\cdot p_1}{(k_B-1)\cdot p_1 + k_A \cdot p_2}$, so the transition rate $r_{(A,B\to B-2)}$ is

$$r_{(A,B\to B-2)} = k_B \cdot [1-(1-p_1)^{k_B-1}(1-P_2)^{k_A}] \cdot \frac{(k_B-1)\cdot p_1}{(k_B-1)\cdot p_1 + k_A \cdot p_2}.$$

The number of both type-A and type-B agents in the pool will decrease by 1 when either a critical type-A agent is matched with a type-B agent or a critical type-B agent is matched with a type-A agent, thus we have

$$\begin{aligned} r_{(A\to A-1,B\to B-1)} = {} & k_A \cdot [1-(1-p_1)^{k_A-1}(1-P_2)^{k_B}] \cdot \frac{k_B \cdot p_2}{(k_A-1)\cdot p_1 + k_B \cdot p_2} \\ & + k_B \cdot [1-(1-p_1)^{k_B-1}(1-P_2)^{k_A}] \cdot \frac{k_A \cdot p_2}{(k_B-1)\cdot p_1 + k_A \cdot p_2}. \end{aligned}$$

Suppose $n = max(k_A, k_B)$, the balance equation can be written as

$$\begin{aligned} & \sum_{i=1}^{n} \pi(n,i) \cdot r_{(n\to n+1,i)} + \sum_{i=1}^{n} \pi(i,n) \cdot r_{(i,n\to n+1)} \\ = & \sum_{i=1}^{n-1} \pi(n+1,i) \cdot r_{(n+1\to n-1,i)} + \sum_{i=1}^{n-1} \pi(i,n+1) \cdot r_{(i,n+1\to n-1)} \\ + & \sum_{i=1}^{n} \pi(n+1,i) \cdot r_{(n+1\to n,i)} + \sum_{i=1}^{n} \pi(n+2,i) \cdot r_{(n+2\to n,i)} + \sum_{i=1}^{n-1} \pi(n+1,i+1) \cdot r_{(n+1\to n,i+1\to i)} \\ + & \sum_{i=1}^{n} \pi(i,n+1) \cdot r_{(i,n+1\to n)} + \sum_{i=1}^{n} \pi(i,n+2) \cdot r_{(i,n+2\to n)} + \sum_{i=1}^{n-1} \pi(i+1,n+1) \cdot r_{(i+1\to i,n+1\to n)} \\ + & \pi(n,n+1) \cdot r_{(n,n+1\to n-1)} + \pi(n+1,n+1) \cdot r_{(n+1\to n,n+1\to n)} + \pi(n+1,n) \cdot r_{(n+1\to n-1,n)}. \end{aligned}$$

We found out that the Markov chain on n_t is sufficient for the balance equation. Plugging transition rates, the balance equation can be rewritten as follows:

$$
\begin{aligned}
&m \cdot n \cdot \tilde{\pi}(n) \\
=&\tilde{\pi}(n+1) \cdot (n+1) \cdot [(1-p_1)^n(1-p_2) \cdot \frac{p_2}{np_1+p_2} + \frac{np_1}{np_1+p_2}] \\
+&\sum_{i=2}^{n} \tilde{\pi}(n+1) \cdot \{(n+1) + i \cdot [1-(1-p_1)^{i-1}(1-p_2)^{n+1}] \frac{(n+1)p_2}{(i-1)p_1+(n+1)p_2}\} \\
+&\sum_{i=1}^{n} \tilde{\pi}(n+2) \cdot (n+2) \cdot [1-(1-p_1)^{n+1}(1-p_2)^{i}] \frac{(n+1)p_1}{(n+1)p_1+i \cdot p_2} \\
+&\tilde{\pi}(n+1) \cdot [1-(1-p_1)^n(1-p_2)^{n+1}] \frac{(n+1)^2 p_2}{np_1+(n+1)p_2}.
\end{aligned}
$$

Define a function $g(n)$ as follows,

$$
\begin{aligned}
g(n) =&m \cdot n - (n+1) \cdot [(1-p_1)^n(1-p_2) \frac{p_2}{np_1+p_2} + \frac{np_1}{np_1+p_2}] \\
&-\sum_{i=2}^{n} \{(n+1) + i \cdot [1-(1-p_1)^{i-1}(1-p_2)^{n+1}] \frac{(n+1)p_2}{(i-1)p_1+(n+1)p_2}\} \\
&-\sum_{i=1}^{n} \cdot (n+2) \cdot [1-(1-p_1)^{n+1}(1-p_2)^{i}] \frac{(n+1)p_1}{(n+1)p_1+i \cdot p_2} \\
&-[1-(1-p_1)^n(1-p_2)^{n+1}] \frac{(n+1)^2 p_2}{np_1+(n+1)p_2},
\end{aligned}
$$

Suppose n^* is a unique root for equation $g(n) = 0$, it can be proved that $\frac{m}{3} \leq n^* \leq m-1$, when m is sufficiently large ($m \geq 24$).

In the following, we will still use the balance equation and transition rates to study the stationary distribution $\tilde{\pi}(\cdot)$.

Lemma 3.2.2. *For all integers $n \leq n^*$,*

$$
\frac{\tilde{\pi}(n)}{max\{\tilde{\pi}(n+1), \tilde{\pi}(n+2)\}} \leq exp(-\frac{(n^*-n)}{2m}).
$$

Similarly, for any integer $n \geq n^*$, $\frac{min\{\tilde{\pi}(n+1), \tilde{\pi}(n+2)\}}{\tilde{\pi}(n)} \leq max\{exp(-\frac{n-n^*}{2m+(n-n^*)}), exp(-\frac{1}{m+1})\}$.

Proof. For any integer $n \le n^*$, according to the balance equation, we have

$$
\begin{aligned}
\frac{\tilde{\pi}(n)}{max\{\tilde{\pi}(n+1), \tilde{\pi}(n+2)\}} \le\ & \frac{1}{mn} \cdot \Big\{(n+1) \cdot [(1-p_1)^n (1-p_2) \frac{p_2}{np_1 + p_2} + \frac{np_1}{np_1 + p_2}] \\
& + \sum_{i=2}^{n} \{(n+1) + i \cdot [1-(1-p_1)^{i-1}(1-p_2)^{n+1}] \frac{(n+1)p_2}{(i-1)p_1 + (n+1)p_2}\} \\
& + \sum_{i=1}^{n} \cdot (n+2) \cdot [1-(1-p_1)^{n+1}(1-p_2)^{i}] \frac{(n+1)p_1}{(n+1)p_1 + i \cdot p_2} \\
& + [1-(1-p_1)^{n}(1-p_2)^{n+1}] \frac{(n+1)^2 p_2}{np_1 + (n+1)p_2} \Big\} \\
\le\ & \frac{1}{mn} \cdot \Big\{(n^*+1) \cdot [(1-p_1)^{n^*} (1-p_2) \frac{p_2}{n^*p_1 + p_2} + \frac{n^*p_1}{n^*p_1 + p_2}] \\
& + \sum_{i=2}^{n^*} \{(n+1) + i \cdot [1-(1-p_1)^{i-1}(1-p_2)^{n^*+1}] \frac{(n^*+1)p_2}{(i-1)p_1 + (n^*+1)p_2}\} \\
& + \sum_{i=1}^{n^*} \cdot (n^*+2) \cdot [1-(1-p_1)^{n^*+1}(1-p_2)^{i}] \frac{(n^*+1)p_1}{(n^*+1)p_1 + i \cdot p_2} \\
& + [1-(1-p_1)^{n^*}(1-p_2)^{n^*+1}] \frac{(n^*+1)^2 p_2}{n^*p_1 + (n^*+1)p_2} \Big\} \\
=\ & \frac{1}{mn} \cdot [mn - \sum_{i=2}^{n^*} (n^*+1) + \sum_{i=2}^{n^*} (n+1)] \\
=\ & \frac{mn + (n^*-1)(n-n^*)}{mn} \\
\le\ & 1 - \frac{n^*-n}{2m} \\
\le\ & e^{-\frac{n^*-n}{2m}}.
\end{aligned}
$$

The first equality holds due to the definition of n^*, which satisfies $g(n^*) = 0$. The third inequality holds due to $\frac{n^*-1}{n} \ge \frac{1}{2}$, just consider either $n \le n^* - 1$ or $n^* - 1 \le n \le n^*$. The last inequality holds with the use of $1 - x \le e^{-x}$.

For any integer $n \geq n^*$, according to the balance equation, we have

$$\begin{aligned}
\frac{min\{\tilde{\pi}(n+1), \tilde{\pi}(n+2)\}}{\tilde{\pi}(n)} &\leq mn \cdot 1 / \Big\{ (n+1) \cdot [(1-p_1)^n (1-p_2) \frac{p_2}{np_1 + p_2} + \frac{np_1}{np_1 + p_2}] \\
&\quad + \sum_{i=2}^{n} \{(n+1) + i \cdot [1 - (1-p_1)^{i-1} (1-p_2)^{n+1}] \frac{(n+1)p_2}{(i-1)p_1 + (n+1)p_2} \} \\
&\quad + \sum_{i=1}^{n} \cdot (n+2) \cdot [1 - (1-p_1)^{n+1} (1-p_2)^i] \frac{(n+1)p_1}{(n+1)p_1 + i \cdot p_2} \\
&\quad + [1 - (1-p_1)^n (1-p_2)^{n+1}] \frac{(n+1)^2 p_2}{np_1 + (n+1)p_2} \Big\} \\
&\leq mn \cdot 1 / \Big\{ (n^*+1) \cdot [(1-p_1)^{n^*} (1-p_2) \frac{p_2}{n^* p_1 + p_2} + \frac{n^* p_1}{n^* p_1 + p_2}] \\
&\quad + \sum_{i=2}^{n^*} \{(n+1) + i \cdot [1 - (1-p_1)^{i-1} (1-p_2)^{n^*+1}] \frac{(n^*+1)p_2}{(i-1)p_1 + (n^*+1)p_2} \} \\
&\quad + \sum_{i=1}^{n^*} \cdot (n^*+2) \cdot [1 - (1-p_1)^{n^*+1} (1-p_2)^i] \frac{(n^*+1)p_1}{(n^*+1)p_1 + i \cdot p_2} \\
&\quad + [1 - (1-p_1)^{n^*} (1-p_2)^{n^*+1}] \frac{(n^*+1)^2 p_2}{n^* p_1 + (n^*+1)p_2} \Big\} \\
&= \frac{mn}{mn - \sum_{i=2}^{n^*} (n^*+1) + \sum_{i=2}^{n^*} (n+1)} \\
&= \frac{mn}{mn + (n^*-1)(n-n^*)} \\
&\leq 1 - \frac{(n^*-1)(n-n^*)}{mn + (n^*-1)(n-n^*)} \\
&\leq exp(-\frac{n-n^*}{m \cdot \frac{n}{n^*-1} + (n-n^*)}).
\end{aligned}$$

When $n \leq 2n^* - 2$, that is $\frac{n}{n^*-1} \leq 2$, then it becomes

$$\frac{min\{\tilde{\pi}(n+1), \tilde{\pi}(n+2)\}}{\tilde{\pi}(n)} \leq exp(-\frac{n-n^*}{2m + (n-n^*)});$$

otherwise, $n > 2n^* - 2 \geq \frac{n^*}{n^*-2}$, note that $n^* > 4$ when $m \geq 12$, we have $\frac{n}{n^*-1} \leq n - n^*$, and

$$\frac{min\{\tilde{\pi}(n+1), \tilde{\pi}(n+2)\}}{\tilde{\pi}(n)} \leq exp(-\frac{1}{m+1}).$$

$\square$

With the use of the above lemma, we found that n_t is highly concentrated around $n^* \in [\frac{m}{3},\ m-1]$ under stationary distribution, which is shown as follows in a rigorous way.

Proposition 3.2.3 (Patient Concentration)**.** *There exists $\frac{m}{3} \le n^* \le m-1$, such that for any $\sigma \ge 1$,*

$$P[n^* - \sigma\sqrt{8m} \le n] \ge 1 - \sqrt{8m}e^{-\sigma^2},$$

$$P[n \le n^* + \sigma\sqrt{8m}] \ge 1 - 3 \cdot \frac{exp(-\frac{\sigma\cdot\sqrt{8m}-1}{2(m+1)})}{1-exp(-\frac{1}{2(m+1)})}.$$

Proof. Suppose fixed $n \le n^*$. Let $h_0, h_1, \cdots$ be a sequence of integers defined by: $h_0 = n$, $h_{i+1} = argmax\{\pi(h_i+1), \pi(h_i+2)\}$, $\forall i \ge 1$. Then we obtain

$$\begin{aligned}\pi(n) \le \prod_{i:h_i\le n^*} \frac{\pi(h_i)}{\pi(h_i+1)} &\le exp(-\sum_{i:h_i\le n^*}\frac{(n^*-h_i)}{2m})\\ &\le exp(-\sum_{i=0}^{(n^*-n)/2}\frac{2i}{2m})\\ &\le exp(-\frac{(n^*-n)^2}{8m})\end{aligned}$$

Here we use $\sum_{i=0}^{(n^*-n)/2}\frac{2i}{2m} \le \sum_{i:h_i\le n^*}\frac{(n^*-h_i)}{2m}$, which is guaranteed by $|h_i - h_{i-1}| \le 2$.

For $\sigma > 1$, let $\Delta = \sigma\sqrt{8m}$, then

$$\sum_{i=0}^{n^*-\Delta}\pi(i) \le \sum_{i=\Delta}^{\infty} e^{-i^2/8m} \le \frac{e^{-\Delta^2/8m}}{min\{\frac{\Delta}{8m}, \frac{1}{2}\}} \le \sqrt{8m}\cdot e^{-\sigma^2}.$$

Now we fix $n \ge n^* + 2$, we construct the following sequence of integers, $h_0 = \lfloor n^* + 2\rfloor$, $h_{i+1} := argmin\{\pi(h_i+1), \pi(h_i+2)\}$, $\forall i \ge 1$. Let h_j be the largest number in the sequence which is at most n. With observation, $h_j = n-1$ or $h_j = n$. Then we obtain

$$\begin{aligned}\pi(n) \le \frac{m\cdot(n-1)}{(n-3)\cdot n}\pi(h_j) \le \frac{m}{(n-3)}\pi(h_j) &\le 3\prod_{i=0}^{j-1}\frac{\pi(h_{i+1})}{\pi(h_i)}\\ &\le 3\cdot exp(-\sum_{i=0}^{j-1}\frac{1}{m+1})\\ &\le 3\cdot exp(-\frac{1}{m+1}\cdot j)\\ &\le 3\cdot exp(-\frac{n-n^*-1}{2(m+1)}).\end{aligned}$$

Here the first equality is derived by the balance equation; the fourth inequality is due to the fact that $exp(-\frac{n-n^*}{2m+(n-n^*)}) < exp(-\frac{1}{m+1})$ when $n - n^* > 2$. The last inequality is guaranteed by $|h_i - h_{i-1}| \leq 2$.

For $\sigma \geq 1$, $\Delta = \sigma \cdot \sqrt{8m}$,

$$\begin{aligned}\sum_{i=n^*+\Delta}^{\infty} \pi(i) \leq 3 \cdot \sum_{i=n^*+\Delta}^{\infty} exp(-\frac{i-n^*-1}{2(m+1)}) &\leq 3 \cdot \sum_{i=\Delta-1}^{\infty} exp(-\frac{i}{2(m+1)}) \\ &= 3 \cdot \frac{exp(-\frac{\Delta-1}{2(m+1)})}{1-exp(-\frac{1}{2(m+1)})} \\ &= 3 \cdot \frac{exp(-\frac{\sigma\cdot\sqrt{8m}-1}{2(m+1)})}{1-exp(-\frac{1}{2(m+1)})}\end{aligned}$$

□

Lemma 3.2.4. *For any $0 < p_2 \leq p_1 < 1$ and sufficiently large m,*

$$E_{n_t\sim\tilde{\pi}}[n_t((1-p_1)(1-p_2)^s))^{n_t}] \leq max_{[m/3,m]}((n_t + \tilde{O}(\sqrt{m}))((1-p_1)(1-p_2)^s))^{n_t}) + 1.$$

Proof. Let $\Delta = 5m \cdot logm$, $v = (1-p_1)(1-p_2)^s$, and $\beta = max_{[\frac{m}{3}-\Delta,m+\Delta]} n_t \cdot v^{n_t}$, then

$$E_{n_t\sim\tilde{\pi}}[n_t \cdot v^{n_t}] \leq \beta + \sum_{i=0}^{\frac{m}{3}-\Delta} \frac{m}{3} \cdot \tilde{\pi}(i) \cdot v^i + \sum_{i=m+\Delta}^{\infty} i \cdot \tilde{\pi}(i) \cdot v^m.$$

Firstly, we claim that $\beta = max_{[\frac{m}{3}-\Delta,m+\Delta]} n_t \cdot v^{n_t} \leq max_{[\frac{m}{3},m]} n_t \cdot v^{n_t} + \frac{m}{3} \cdot v^{\frac{m}{3}} \cdot (v^{-\Delta} - 1) + v^m \cdot \Delta$.

Let $\hat{n} = argmax_{n_t\in[\frac{m}{3}-\Delta,m+\Delta]} n_t \cdot v^{n_t}$. If $\hat{n} \in [\frac{m}{3}, m]$, then the inequality holds naturally. If $\hat{n} \in [m, m+\Delta]$, then $max_{n_t\in[\frac{m}{3},m]} n_t \cdot v^{n_t} = m \cdot v^m$, and

$$\begin{aligned}\beta &= \hat{n} \cdot v^{\hat{n}} \\ &\leq (m+\Delta) \cdot v^m \\ &= max_{n_t\in[\frac{m}{3},m]} n_t \cdot v^{n_t} + \Delta \cdot v^m \\ &\leq max_{[\frac{m}{3},m]} n_t \cdot v^{n_t} + \frac{m}{3} \cdot v^{\frac{m}{3}} \cdot (v^{-\Delta} - 1) + v^m \cdot \Delta.\end{aligned}$$

Otherwise, $\hat{n} \in [\frac{m}{3} - \Delta, \frac{m}{3}]$, then $max_{n_t \in [\frac{m}{3}-\Delta, \frac{m}{3}]} n_t \cdot v^{n_t} = \frac{m}{3} \cdot v^{\frac{m}{3}}$,

$$\beta = v^{-\Delta} \cdot \hat{n} \cdot v^{\hat{n}+\Delta} \leq v^{-\Delta} \cdot (\hat{n} + \Delta) \cdot v^{\hat{n}+\Delta} \leq \frac{m}{3} \cdot v^{\frac{m}{3}-\Delta} = max_{[\frac{m}{3},m]} n_t \cdot v^{n_t} + \frac{m}{3} \cdot v^{\frac{m}{3}} \cdot (v^{-\Delta} - 1).$$

Therefore, the claim holds.

Let $\Delta' = 6(log 2m + 1) \cdot \Delta$, if $v^{-\Delta} \leq 1 + \frac{\Delta'}{m}$, then

$$\begin{aligned}
\beta &\leq max_{[\frac{m}{3},m]} n_t \cdot v^{n_t} + \frac{m}{3} \cdot v^{\frac{m}{3}} \cdot (v^{-\Delta} - 1) + v^m \cdot \Delta \\
&\leq max_{[\frac{m}{3},m]} n_t \cdot v^{n_t} + \frac{\Delta'}{3} \cdot v^{\frac{m}{3}} + v^m \cdot \Delta \\
&\leq max_{[\frac{m}{3},m]} (n_t + \Delta' + \Delta) \cdot v^{n_t} + 1
\end{aligned}$$

otherwise, $v^{-\Delta} > 1 + \frac{\Delta'}{m}$, assuming that m is sufficiently large such that $\Delta' \leq m$, this implies $v^{\Delta} \leq \frac{1}{1+\frac{\Delta'}{m}} \leq 1 - \frac{\Delta'}{2m}$, then we have

$$\beta \leq (m + \Delta) \cdot v^{\frac{m}{3}-\Delta} \leq 2m \cdot (1 - \frac{\Delta'}{2m})^{\frac{m}{3\Delta}-1} \leq 2m \cdot e^{1-\frac{\Delta'}{6\Delta}} \leq 1.$$

Then we bound the second term $\sum_{i=0}^{\frac{m}{3}-\Delta} \frac{m}{3} \cdot \tilde{\pi}(i) \cdot v^i$,

$$\sum_{i=0}^{\frac{m}{3}-\Delta} \frac{m}{3} \cdot \tilde{\pi}(i) \cdot v^i \leq \frac{m}{3} \cdot \sum_{i=0}^{\frac{m}{3}-\Delta} \tilde{\pi}(i) = \frac{m}{3} \cdot \sqrt{8m} \cdot e^{-\Delta^2/8m} \leq \frac{\sqrt{8m}}{3}.$$

Lastly, we bound the last term as follows,

$$\begin{aligned}
\sum_{i=m+\Delta}^{\infty} i \cdot \tilde{\pi}(i) \cdot v^m \leq \sum_{i=m+\Delta}^{\infty} i \cdot \tilde{\pi}(i) &= \sum_{i=\Delta}^{\infty} (m+i) \cdot \tilde{\pi}(m+i) \\
&\leq m \cdot \sum_{i=\Delta}^{\infty} \tilde{\pi}(m+i) + \sum_{i=\Delta}^{\infty} i \cdot \tilde{\pi}(m+i) \\
&\leq \frac{3m \cdot exp(-\frac{\Delta-1}{2(m+1)})}{1 - exp(-\frac{1}{2(m+1)})} + \frac{3 \cdot exp(-\frac{\Delta-1}{2(m+1)}) \cdot (\frac{\Delta}{m+1} + 4)}{1 - exp(-\frac{1}{2(m+1)}) \cdot (\frac{1}{2(m+1)^2})} \\
&\leq 3 \cdot exp(-\frac{\Delta - 1}{2(m+1)}) \cdot [\frac{m}{1 - exp(-\frac{1}{2(m+1)})} \\
&\quad + \frac{\frac{\Delta}{m+1} + 4}{1 - exp(-\frac{1}{2(m+1)}) \cdot (\frac{1}{2(m+1)^2})}] \\
&\leq \frac{5}{3} log m \cdot \sqrt{m}.
\end{aligned}$$

$\square$

Corollary 3.2.5. *Similarly, for any $0 < p_2 \leq p_1 < 1$ and sufficiently large m,*

$$E_{n_t \sim \tilde{\pi}}[n_t((1-p_1)^s(1-p_2)))^{n_t}] \leq max_{[m/3,m]}((n_t + \tilde{O}(\sqrt{m}))((1-p_1)^s(1-p_2)^))^{n_t}) + 1.$$

With this Patient concentration proposition, we find that $\mathbb{E}_{n_t \sim \tilde{\pi}}[n_t((1-p_1)(1-p_2)^s))^{n_t}] \leq max_{[m/3,m]}((n_t + \tilde{O}(\sqrt{m}))((1-p_1)(1-p_2)^s))^{n_t}) + 1$. Furthermore, the loss of Patient algorithm can be upper-bounded as follows:

Theorem 3.2.6. *For any $\epsilon > 0$ and $T > 0$,*

$$\begin{aligned} L(Patient) \leq (s+1) max_{n_t \in [\frac{1}{3},1]} \left((n_t + \tilde{O}(\frac{1}{\sqrt{m}})) e^{-[(s+1)m \cdot u \cdot n_t - p_1]} \right) \\ + \frac{\tau_{mix}(\epsilon)}{T} + \frac{2\epsilon}{m \cdot (1-p_2) \cdot p_1^2} + \frac{1+s}{m \cdot (1-p_1)}, \end{aligned}$$

where $u = \frac{s \cdot p_1 + p_2}{s+1}$.

When m is sufficiently large and ϵ small enough, the loss of Patient algorithm can be bounded as follows:

$$L(Patient) \leq (s+1) max_{n_t \in [\frac{1}{3},1]} \left(n_t \cdot e^{-[(s+1)d \cdot n_t - p_1]} \right).$$

Here u stands for the average compatibility probability, $d = m \cdot u$ is the density parameter of our model. When $d \geq 3$, it can be bounded by $\frac{s+1}{3} \cdot e^{-\frac{(s+1) \cdot d}{3} + p_1}$.

CHAPTER 4

Simulation Studies

4.1 Loss of two algorithms with different parameter settings

Firstly, we simulate the market with different parameter settings under Greedy algorithm and Patient algorithm. We consider situations when new agents arrive in the market at a rate $m = 30, 50, 70, 100$, the compatible probability between the same type of agents $p_1 \in [0.03, 0.8]$, the compatible probability between distinct types of agents $p_2 = 0.01, 0.1$, and the simulated results are shown in Figure 4.1-4.6.

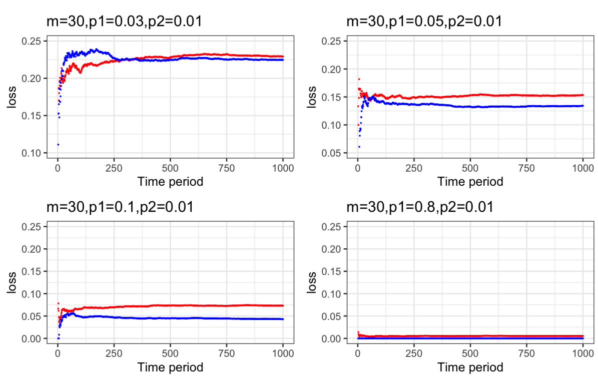

Figure 4.1: The change of loss of two algorithms (Red: Greedy algorithm; Blue: Patient algorithm) with parameter settings: $m = 30$, $p_2 = 0.01$, $p_1 = 0.03/0.05/0.1/0.8$.

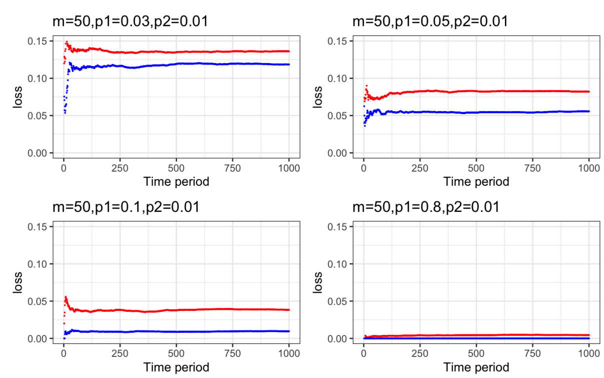

Figure 4.2: The change of loss of two algorithms (Red: Greedy algorithm; Blue: Patient algorithm) with parameter settings: $m = 50$, $p_2 = 0.01$, $p_1 = 0.03/0.05/0.1/0.8$.

According to plots in Figure 4.1-4.6, the loss of both Greedy algorithm and Patient algorithm converges as T increases, and the loss of Greedy algorithm is always higher than that

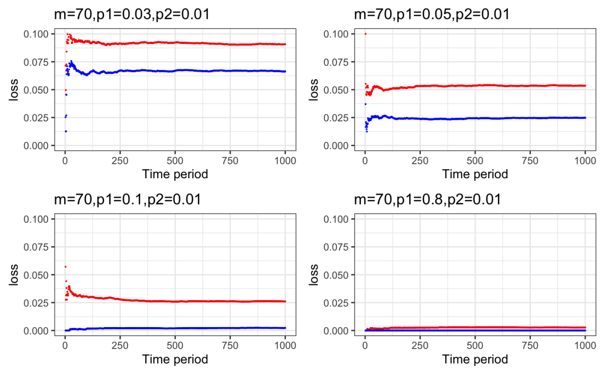

Figure 4.3: The change of loss of two algorithms (Red: Greedy algorithm; Blue: Patient algorithm) with parameter settings: $m = 70$, $p_2 = 0.01$,$p_1 = 0.03/0.05/0.1/0.8$.

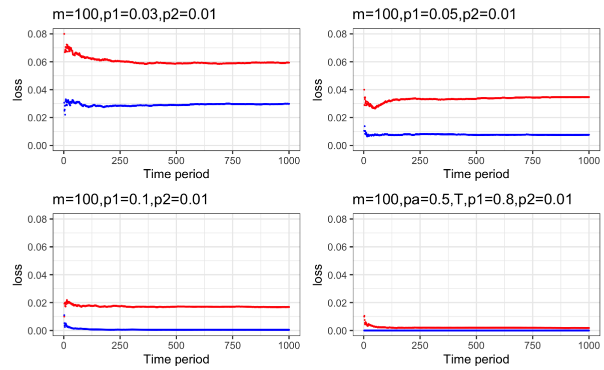

Figure 4.4: The change of loss of two algorithms (Red: Greedy algorithm; Blue: Patient algorithm) with parameter settings: $m = 100$, $p_2 = 0.01$, $p_1 = 0.03/0.05/0.1/0.8$.

of Patient algorithm, which reveals that Patient algorithm outperforms Greedy algorithm. With time period T increasing, the loss of Greedy algorithm decreases to the stationary state, while the loss of Patient algorithm grows to the stationary state, or remain as a horizontal line when compatible probability is large enough.

Additionally, these plots show that the rate of coming agents arriving at the market m and compatible probabilities p_1 and p_2 influence the difference between the loss of two algorithms. The loss of both algorithms decrease with larger compatible probabilities. Higher rate of coming agents arriving at the market shortens the mixing time, and decreases the loss as well.

Observing the difference between loss of Greedy and Patient algorithms, we found that when both compatible probabilities p_1 and p_2 are quite small or when one of them are large enough, the advantages of Patient algorithm is not significant. Except for these scenarios, Patient algorithm has significant advantages over Greedy algorithm.

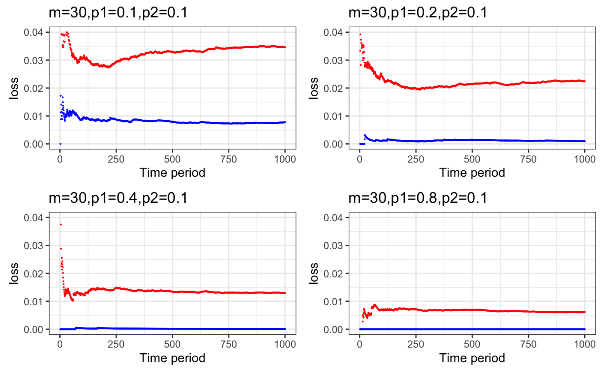

Figure 4.5: The change of loss of two algorithms (Red: Greedy algorithm; Blue: Patient algorithm) with parameter settings: $m = 30$, $p_2 = 0.1$, $p_1 = 0.1/0.2/0.4/0.8$.

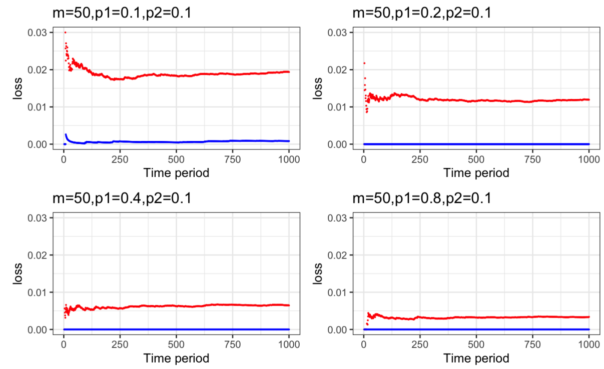

Figure 4.6: The change of loss of two algorithms (Red: Greedy algorithm; Blue: Patient algorithm) with parameter settings: $m = 50$, $p_2 = 0.1$, $p_1 = 0.1/0.2/0.4/0.8$.

4.2 Effects of new agents coming rate and compatible probabilities on algorithm performance

In the following we conduct more simulations to specify the scenarios under which Patient algorithm will have significant advantages over Greedy algorithm. We chose 0.005 as the significance value. If the difference between Greedy algorithm and Patient algorithm is larger than 0.005, then we believe Patient algorithm has significant advantages over Greedy algorithm.

We consider the situation when the rate of coming agents arriving at the market m is fixed from $\{10, 30, 50, 70\}$, the compatible probability between distinct types of agents p_2 is set to be $\{0.01, 0.05, 0.1, 0.2\}$, and the compatible probability between the same type of agents p_1 is adjusted according to $p_1/p_2 = 1, 4, 10$ when $p_2 = 0.01$ or 0.05, and $p_1/p_2 = 1, 2, 3$ when $p_2 = 0.1$ or 0.2.

Figure 4.7 reveals how the rate of new agents arriving at the market m influences the difference between loss of two algorithms. We found that with larger m, the difference

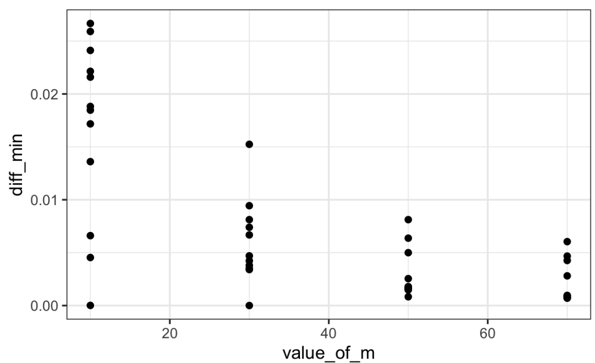

Figure 4.7: The relationship between the rate of new agents arriving at the market m and minimum difference between loss of Greedy algorithm and Patient algorithm

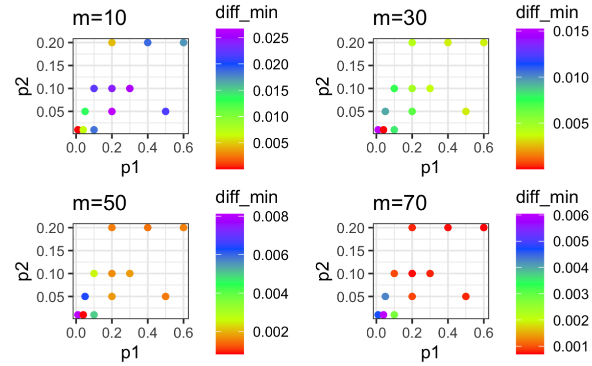

Figure 4.8: How compatible probabilities (p_1, p_2) influence the minimum difference between loss of Greedy algorithm and Patient algorithm under different values of m

between loss of two algorithms significantly decreases. When the rate $m = 10$, almost all values of differences are significant, while when the rate $m \geq 50$, most of values of differences are not significant (less than 0.005).

Figure 4.8 describes how compatible probabilities p_1 and p_2 influence the differences between loss of two algorithms under different values of m. When new agents arrive in the market at the rate $m = 10$, almost all values of differences are significant, except for the case when both compatible probabilities are really small ($p_1 = p_2 = 0.01$). When new agents arrive in the market at the rate $m = 30$, the differences are significant when $0.05 \leq p_1 \leq 0.3$ and $0.01 \leq p_2 \leq 0.1$. Otherwise, when either compatible probability is large ($p_1 \geq 0.4$, $p_2 \geq 0.2$) or when both of them are extremely small (around 0), the loss of both algorithms are very close. When new agents arrive in the market at the rate $m = 50$, the difference is significant only when $p_1 = p_2 = 0.05$. In other situations, differences are all less than 0.005. When large amounts of new agents come to the market, e.g. $m = 70$, there are no significant differences between loss of two algorithms.

4.3 Distribution of the proportion of different types of agents in steady state and the bounds of loss conditional on proportion

Since the proportion of different types of agents in steady state s plays an important role in our theoretical results, and we assumed that s is following Beta distribution with support $[0, 1]$, we conduct simulations to observe the real distribution of the proportion s. Figure 4.9 to 4.12 show the distribution of the proportion of different types of agents s in steady state with market size $m = 70, 100$ under Greedy algorithm and Patient algorithm. Under Greedy algorithm, the distribution at most cases is shown to be left-skewed, and the proportion s is concentrated around $[0.7, 1]$. However, in cases where one of the compatible probabilities becomes significantly larger, the steady-state value of the proportion s tends to fall into specific values of 0, 0.5, or 1. The proportion of different types of agents tends to be either 0 or 1 when the compatible probability is sufficiently large, which reveals that in steady state the market will be full of only one type of agent or both types of agents will be evenly distributed in the market under Greedy algorithm. As for Patient algorithm, all the proportion of of different types of agents are left-skewed distributed within $[0, 1]$ under various parameter settings.

We used Beta distribution to fit the distribution of s in the market with market size $m = 70$, compatible probability $p_1 = 0.05$ and $p_2 = 0.01$, and also the market with market size $m = 100$, compatible probability $p_1 = 0.1$ and $p_2 = 0.01$ under both algorithms. The fitted result of both markets under Greedy algorithm is shown in Figure 4.13 and 4.15. Observing these plots, we found that the predicted peak value or highly concentrated interval is biased compared with the real distribution of s. Figure 4.14 and 4.16 depict the highly concentrated interval of pool size Z^* in steady state conditional on proportion s. Compared with true value, the estimated interval would be more narrower than the real interval, since the estimated concentrated interval of proportion is larger than the real one.

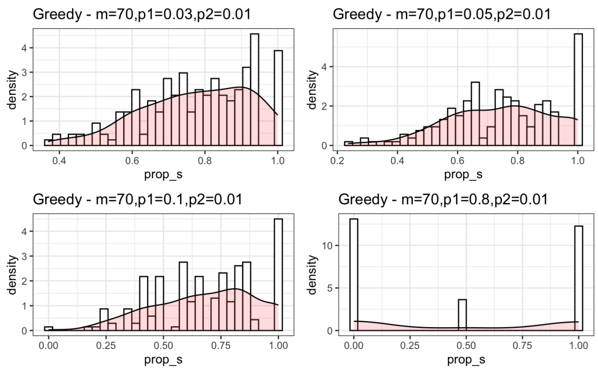

Figure 4.9: The distribution of the proportion of different types of agents s under steady state with market size $m = 70$ under Greedy algorithm.

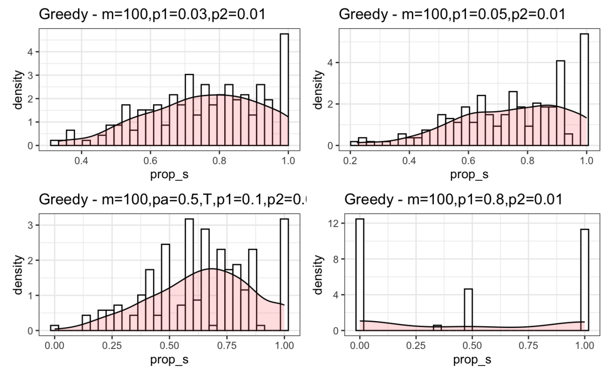

Figure 4.10: The distribution of the proportion of different types of agents s under steady state with market size $m = 100$ under Greedy algorithm.

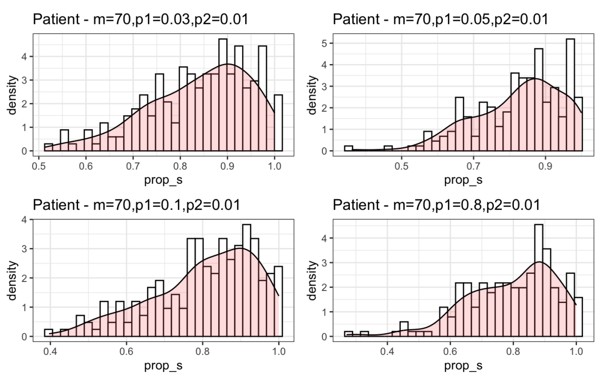

Figure 4.11: The distribution of the proportion of different types of agents s under steady state with market size $m = 70$ under Patient algorithm.

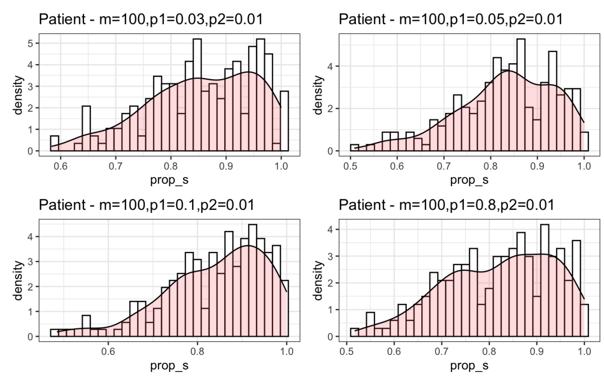

Figure 4.12: The distribution of the proportion of different types of agents s under steady state with market size $m = 100$ under Patient algorithm.

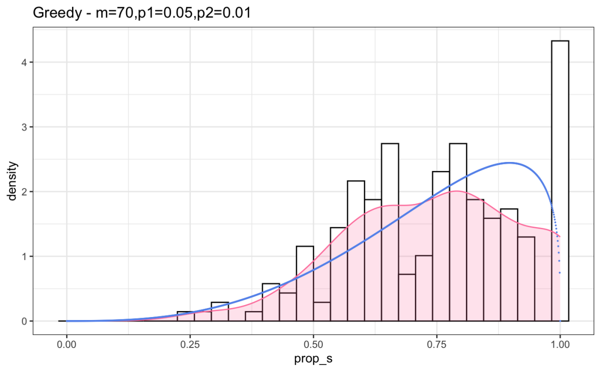

Figure 4.13: Fitting the distribution of the proportion with the use of Beta distribution $\mathcal{B}(3.8, 1.3)$.

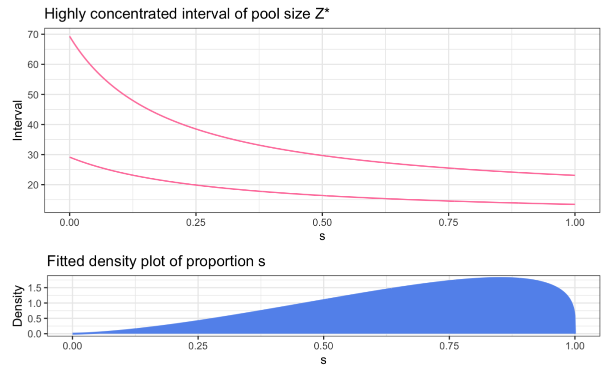

Figure 4.14: Highly concentrated interval of pool size Z^* under steady state conditional on proportion s.

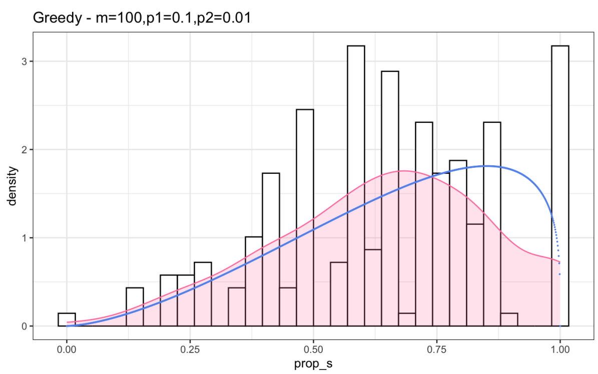

Figure 4.15: Fitting the distribution of the proportion with the use of Beta distribution $\mathcal{B}(2.6, 1.3)$.

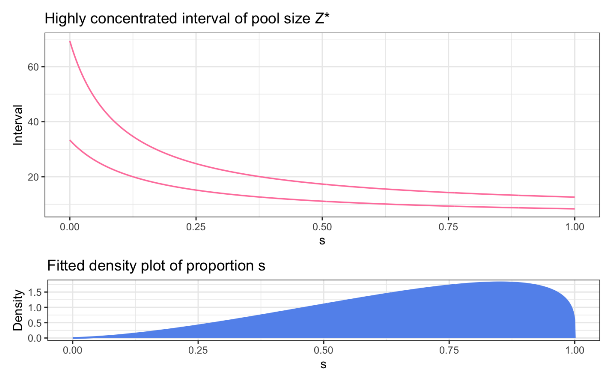

Figure 4.16: Highly concentrated interval of pool size Z^* under steady state conditional on proportion s.

Figure 4.17 and 4.19 visualize the fitted result of distribution of proportion in both markets under Patient algorithm. We found that the Beta distribution fits the distribution of proportion s under Patient algorithm much better than that under Greedy algorithm. The predicted highly concentrated interval of proportion is very close to the actual interval, and the predicted peak value of s is also near the actual result. Based on the Beta fitted result, we can draw the theoretical upper bound of the loss of Patient algorithm in steady state conditional on proportion s. The predicted results are shown in Figure 4.18 and 4.20. Compared with the actual loss of these two markets in previous simulation, we found this bound conditional on the peak value of proportion s is consistent with the actual observation. When the market size m is larger, the upper bound is more precise; while the upper bound would become loose when the market size is not sufficiently large.

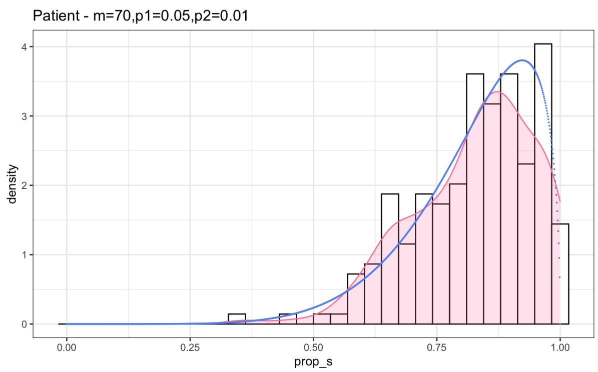

Figure 4.17: Fitting the distribution of the proportion with the use of Beta distribution $\mathcal{B}(7.1, 1.5)$.

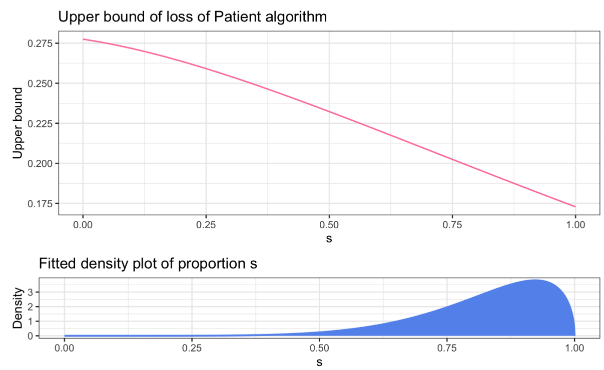

Figure 4.18: The upper bound of loss of Patient algorithm under steady state conditional on proportion s.

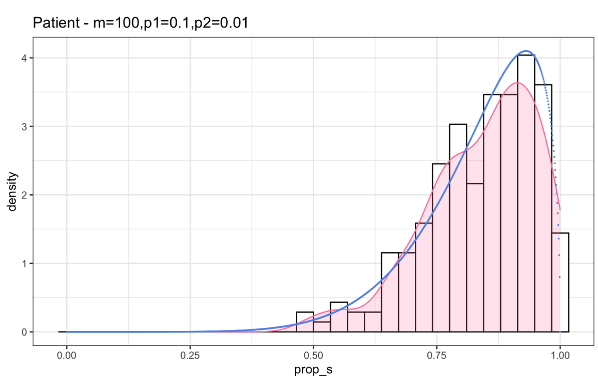

Figure 4.19: Fitting the distribution of the proportion with the use of Beta distribution $\mathcal{B}(7.7, 1.5)$.

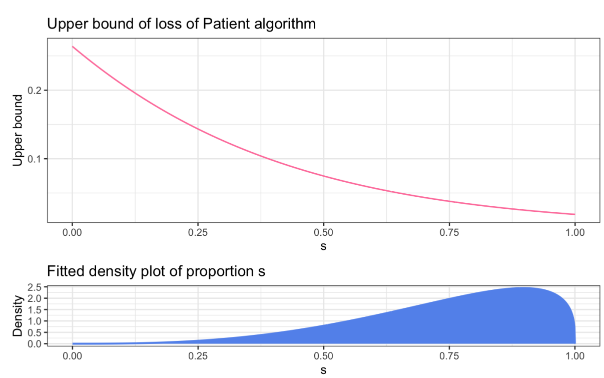

Figure 4.20: The upper bound of loss of Patient algorithm under steady state conditional on proportion s.

4.4 Simulation findings

Based on the simulation results, we observed that Patient algorithm more significantly outperforms Greedy algorithm when the rate of coming agents arriving at the market is smaller (around 30). The value of compatible probabilities (both p_1, p_2 with assumption that $p_1 \geq p_2$) will influence the performance of algorithms as well. When both of them are extremely small or when one of them are large enough, the advantages of Patient algorithm will be less significant. The advantages of Patient algorithm outperforms than that of Greedy algorithm the most when $0.05 \leq p_1 \leq 0.3$ and $0.01 \leq p_2 \leq 0.1$.

These observation are reasonable and explainable. When more agents enter the matching market each time, the market capacity becomes larger, and thus there are more compatible choices for each agent under both algorithms, whenever we match only newly arriving agents or critical agents. When the arriving rate of new agents is small, the number of compatible options to match is limited. At this time, the strategy of matching algorithm matters, and thus the advantage of Patient algorithm will be more significant under this situation when the number of agents in the market is limited. Likewise, when the compatible probabilities for agent matching are significantly high, there is little distinction between the outcomes of Greedy algorithm and Patient algorithm. When both compatible probabilities p_1 and p_2 are extremely small (close to 0), agents face substantial challenges in finding suitable matches under both algorithms.

The distribution of the proportion of different types of agents s in steady state performs slightly differently under both algorithms. The distribution of proportion s under Greedy algorithm changes with varying market parameter settings. In most cases, it shows left-skewed distribution, while the value of proportion tends to fall into either 0 or 1 when compatible probabilities become larger. In other words, our assumption that the proportion s in steady state is following Beta distribution would not be held in an easy-to-match matching market under Greedy algorithm. On the contrary, the Beta distribution fits the distribution proportion s very well in the market under Patient algorithm. This implies our assumption about the distribution of s works for Patient algorithm. Based on this, our theoretical result of upper bound of the loss of Patient algorithm is consistent with the actual value in simulation, and will be more precise when the market size m is larger.

CHAPTER 5

Discussion

This paper introduces a dynamic matching market model that incorporates two agent types, where the compatibility probability between agent pairs depends on their types. The model builds upon the dynamic matching market model by Akbarpour et al. (2020) [ALG20], extending it to include agent-dependent compatible probabilities. The proportion of different types of agents in the market, denoted as s, is incorporated into the model, allowing for the description of the average compatible probability (u) of the matching market based on s.

Theoretical investigations are conducted to determine the upper and lower bounds of loss for both the Greedy algorithm and the Patient algorithm, given the proportion s. It is demonstrated that the loss of Greedy algorithm can be lower bounded by $\frac{1}{2m \cdot u+1}$, while the loss of Patient algorithm can be upper bounded by $\frac{s+1}{3} \cdot e^{-\frac{(s+1)\cdot m \cdot u}{3}+p_1}$ when $d = m \cdot u \geq 3$.

Empirical analysis involves studying the distribution of the proportion s under both algorithms. It is found that s can generally be modeled by a Beta distribution. By conditioning on s, highly concentrated intervals of the pool size in steady state and the bounds of loss for both algorithms can be determined. The fitting of the Beta distribution to the distribution of s under the Patient algorithm is successful, and the theoretical upper bound of loss for the Patient algorithm, based on this fitting, aligns with actual simulation results. However, the fitting results for the market under the Greedy algorithm are not as satisfactory due to the frequent occurrence of extreme values (e.g., 0 or 1) in the actual data. This leads to a narrower estimated highly concentrated interval of pool size in steady state.

Furthermore, through multiple experiments with various parameter settings, it is observed that the impact of the Poisson rate m (representing the rate at which new agents enter the market) is more significant than that of the compatible probabilities p_1 and p_2), as it directly determines the market size. Based on both theoretical and empirical analyses, it is concluded that the Patient algorithm outperforms the Greedy algorithm in terms of maximizing the proportion of matched agents in the market. Hence, the study confirms that waiting can effectively reduce the fraction of unmatched agents.

A limitation of our model is that the theoretical bounds of loss for both algorithms are conditional on the proportion of different types of agents, and the assumption that the proportion follows a Beta distribution does not fit well with the Greedy algorithm in an easy-to-match market. In future research, it would be valuable to further investigate the relationship between the proportion of different types of agents and the model parameters. By revealing this relationship, more precise theoretical bounds for both the Greedy and Patient algorithms can be established and compared. Additionally, other parameters that influence the market in steady state, such as the proportions of different types of agents entering the market at each time and the prioritization of matching agents, can be incorporated into the model to enhance its complexity and authenticity.

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