Title
Upper Bounds on the Resolvent Degree of General Polynomials and the Families of Alternating and Symmetric Groups

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Publication Date
2022

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Upper Bounds on the Resolvent Degree of General Polynomials and the Families of Alternating and Symmetric Groups

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Alexander James Sutherland

Dissertation Committee:
Professor Jesse Wolfson, Chair
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2022

DEDICATION

This dissertation is a product of six years of difficult work: taking courses, passing qualifying exams, passing my advancement to Ph.D candidacy, conducting research, teaching, and serving my communities. In many ways, this thesis is dedicated to myself. However, this journey would not have been possible without the love and support of many people. Let me begin by thanking my family and friends outside of academia, including Debbie Sutherland, Mark Sutherland, Amanda and Andrew Arani, Michael Chang, Sean Jian, Robert Kato, Cameron McElfresh, Gregg Ratanaphanyarat, Pete and Ashley Sinn, and Robert Tseng.

Throughout my time as a grad student at UC Irvine, I have been thankful to know many wonderful peers, including:

- Kevin Barnum, Kelly Isham, and Jesse Kreger;
  For the countless hours spent together in our cohort and the friendships that will last lifetimes.
- Kat Dover and Lora Weiss:
  For being the best officemates I could ask for.
- Joshua Jordan:
  For your mathematical curiosity, unending kindness, and dedication to community.
- Anubhav Nanavaty and Sidhanth Raman:
  For your unmatched enthusiasm and for letting me play a part in your mathematical journeys.

I would like to recognize the UCI Division of Teaching Excellence and Innovation as a source of inspiration, including Danny Mann, Matthew Mahavongtrakul, and the entire *Pedagogical Fellowship* program.

I am a proud member of UAW 2865, the union ran by and for teaching assistants, graduate student instructors, tutors, and readers like myself across the entire University of California system. If you are not currently represented by a union, please consider visiting

www.aflcio.org/formaunion

to learn more about what unions do and how to start organizing in your workplace.

I would like to thank my undergraduate mentors, including Jack Calcut, Susan Colley, Courtney Gibbons, and Chris Marx; this would not have been possible without your support.

Thank you, Nathan Kaplan, for your trust as we collaborated on teaching graduate algebra during a pandemic and for creating a welcoming space in the number theory seminar.

Indeed, the final third of my time as a graduate student (March 2020 - June 2022) was remote during the COVID-19 pandemic. I would like to thank the friends that I made virtually and the remote support I received while in-person connections were impossible.

Thank you, Claudio Gómez-Gonzáles, for exemplifying the type of mathematician I strive to be: compassionate, kind, and supportive. I eagerly look forward to thriving in the communities we will continue to build around us.

Finally, I would like to thank Jesse Wolfson - this dissertation would not exist without you. I have learned so much new mathematics and have adjusted my mathematical philosophy so many times from all of our wonderful conversations. What I appreciate most, however, has been your dedication to making people feel like they belong as mathematicians. Thank you for believing in me all the times when I did not believe in myself. It has been an honor and a privilege to be your first Ph.D student; I cannot wait for everything else we will accomplish.

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ACKNOWLEDGMENTS

Much of this work has been supported in part by the National Science Foundation through the NSF CAREER Grant DMS-1944862, as well as NSF Grant DMS-1811846. This work has additionally been supported in part by a *Graduate Dean's Dissertation Fellowship* from the University of California, Irvine and an *ARCS Foundation Scholarship* from the Orange County chapter of the ARCS Foundation.

I would like to thank Jesse Wolfson for the incredible amount of helpful conversations and mathematical insights which have influenced the entirety of this dissertation.

I would like to Kenneth Ascher, Manuel Reyes, and Jesse Wolfson (Chair) for serving on my dissertation committee.

I would like to thank Vladimir Baranovsky (Mathematics), Nathan Kaplan (Mathematics), JB Manchak (Logic and Philosophy of Science), Jeffrey Streets (Mathematics), and Jesse Wolfson (Chair, Mathematics) for serving on my candidacy committee.

In regards to the work "Upper Bounds on Resolvent Degree and Its Growth Rate," I thank Benson Farb, Hannah Knight, Curt McMullen, and Zinovy Reichstein for helpful comments on a draft and I thank Kenneth Ascher, Claudio Gómez-Gonzáles, Joshua Jordan, and Roman Vershynin for helpful conversations.

Results from the work "Upper Bounds on Resolvent Degree via Sylvester's Obliteration Algorithm" are joint with Curtis Heberle. I thank David Ishii Smyth and Joshua Jordan for helpful conversations with Curtis and myself, respectively.

For the translation "On the Problem of Resolvents" by G.N. Chebotarev, I thank Ignat Soroko for helpful comments on the bibliographic information.

Additionally, I would like to thank Kenneth Ascher, Benson Farb, Daniel Litt, Eduard Looijenga, Priyam Patel, Nicholas Miller, and Sidhanth Raman for helpful conversations on a related project which did not end up in this dissertation.

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Journal of Algebra and Its Applications, DOI:10.1142/S0219498818500792
arXiv:1507.08354 [math.AC]

ARTICLES SUBMITTED FOR PUBLICATION

Upper Bounds on Resolvent Degree via Sylvester's Obliteration Algorithm (with C. Heberle) **Oct 2021**
arXiv:2110.08670 [math.AG] [math.AC] [math.HO]

Upper Bounds on Resolvent Degree and Its Growth Rate **Jul 2021**
arXiv:2107.08139 [math.AG] [math.AC]

TRANSLATIONS

G.N. Chebotarev's "On the Problem of Resolvents" **Jun 2021**
arXiv:2107.01006 [math.HO]

Anders Wiman's "On the Application of Tschirnhaus Transformations to the Reduction of Algebraic Equations" **Jun 2021**
arXiv:2106.09247 [math.HO]

Felix Klein's "About the Solution of the General Equations of Fifth and Sixth Degree (Excerpt from a letter to Mr. K. Hensel)" **Nov 2019**
arXiv:1911.02358 [math.HO] [math.AC] [math.AG]

ABSTRACT OF THE DISSERTATION

Upper Bounds on the Resolvent Degree of General Polynomials and the Families of Alternating and Symmetric Groups

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2022

Professor Jesse Wolfson, Chair

In this dissertation, we determine best-to-date upper bounds on the resolvent degree of solving general polynomials. Chapters 1 and 2 provide the necessary background information and Chapters 3 and 4 establish new results.

In Chapter 1, we provide a high-level introduction of the dissertation and provide a history of the literature on resolvent degree. We also establish standard mathematical notation and terminology.

In Chapter 2, Section 1, we recall the definitions of essential dimension and resolvent degree, as well as some standard results. In Section 2, we give an introduction to the theory of Tschirnhaus transformations and explain how we will use this theory to obtain upper bounds on resolvent degree by determining special points on Tschirnhaus complete intersections.

In Chapter 3, Section 1, we recover the classical notion of the polars of a hypersurface at a point, so that we can introduce the polar cone of a hypersurface at a point and the connection between the polar cone at a point and lines on the hypersurface through that point. We then extend these notions to intersections of hypersurfaces and introduce iterated polar cones and their connections to k-planes in Section 2. Finally, we recover the obliteration algorithm of

Sylvester and present it in a modern geometric context in Section 3.

In Chapter 4, we establish our bounds on resolvent degree and Sections 1, 2, and 3 each highlight different constructions. In Section 4.4, we indicate obstructions to further bounds on resolvent degree via iterated polar cone constructions. We then proceed to establish approximations via elementary functions and compare the bounds we obtain with previous bounds in Sections 5 and 6. Finally, we posit several questions for future research in Section 7.

Appendix A contains numerical information regarding our bounding function $G'(m)$ and the previous best bounding function $F(m)$, as well as implementations of the geometric obliteration algorithm. Appendix B contains three translations [Sut2019, Sut2021A, Sut2021B] of papers in the classical resolvent degree literature [Che1954, Kle1905, Wim1927].

Chapter 1

Overview

1.1 Introduction

Material in Section 1.3 and beyond is intended for mathematical experts.

The author explicitly intends for Sections 1.1 and 1.2 to be largely readable by the interested non-expert reader.

We begin by considering the following classical problem:

Problem 1.1.1. ***(Solving Polynomials)***
Given a general polynomial $z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$, determine a root of the polynomial in terms of the coefficients $a_1, \ldots, a_n$ in the simplest manner possible.

The quadratic formula, which provides the roots in the $n = 2$ case, was known to the Babylonians and Egyptians. For a general quadratic polynomial $z^2 + bz + c$, the quadratic formula requires one to know $\sqrt{b^2 - 4c}$, a square root of the discriminant. Further, it was known to the Greeks (and likely the the Egyptians) that there is no rational formula for the roots in the $n = 2$ case, i.e. the square root in the quadratic formula is necessary.

These two observations together completely answer Problem 1.1.1 when $n = 2$. Similar observations yield complete solutions when $n = 3, 4$; namely, each of these additional cases allow for solutions in radicals. In 1824, Abel proved that $n = 4$ is the largest case that admits a solution in radicals. For the reader who has studied finite groups, we note that this is equivalent to the fact that symmetric groups S_n (equivalently, the alternating groups A_n) are not solvable groups for $n \geq 5$. Nonetheless, we know that polynomials always have roots (more precisely, any polynomial defined over a field K splits into linear factors over an algebraic closure $\overline{K}$). The mathematical community was then left with the question: how do we proceed beyond solvability in radicals?

While Abel's theorem about the insolvability of the quintic in radicals is well-known among mathematicians, Bring's solution to the quintic is a far less standard topic. In [Bri1786] (or [CHM2017] for an English translation), Bring shows that if we first use two square roots and a cube root, we can reduce the form of the general quintic from

$$z^5 + a_1 z^4 + a_2 z^3 + a_3 z^2 + a_4 x + a_5,$$

to

$$z^5 + b_4 z + b_5.$$

To view these expressions as general polynomials, we think of the coefficients as formal variables and, from this perspective, Bring's reduced form is much simpler: we only need to deal with the two variables b_4, b_5 instead of the five variables a_1, a_2, a_3, a_4, a_5. In fact, there is a precise way in which we need only work with one variable in this form (by keeping track of only $\frac{b_4}{b_5}$, i.e. working with projective coordinates). This solution of Bring is an example of the classical theory of Tschirnhaus transformations, which will be discussed in more depth in Section 2.2.

In the late 1800's, Klein provided another solution to the quintic (see [Kle1884] for the original or [Mor1956] for the English translation). Klein's "icosahedral solution" to the quintic uses two square roots and his icosahedral function $\mathcal{I}$. We will not return to Klein's solution of the quintic in this dissertation, so we will take a moment to enjoy the beautiful scenery before continuing on our way. The key insight of Klein is that we can simultaneously identify (topologically) the regular icosahedron, the sphere, and the extended complex plane.

Figure 1.1: A Regular Icosahedron by [Web2006]

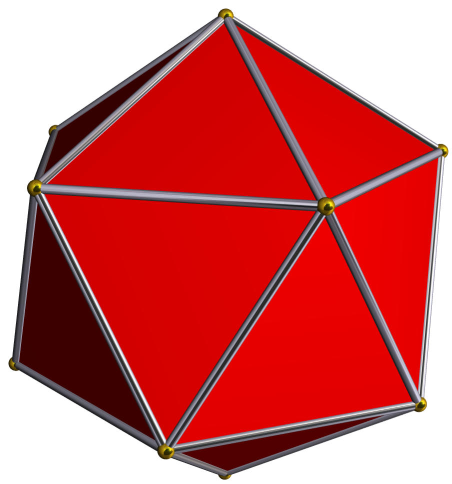

Figure 1.2: The Extended Complex Plane / A Sphere by [Amj2011]

The group of rotations in (real) 3-dimensional space which take every vertex of the icosahedron to another vertex is isomorphic to the alternating group A_5. Klein's icosahedral function is then the induced rational map $\mathcal{I} : \mathbb{P}^1_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}}/A_5$, which he works out explicitly using classical invariant theory.

Each of these solutions of the quintic use the following framework. They begin by computing a specific square root (namely, the square root of the discriminant). Next, they introduce simple auxiliary functions required to define the main function of interest. For Bring's solution, the auxiliary functions are the cube root and the other square root and the main function is derived from the form $z^5 + b_4 z + b_5$. For Klein's solution, the auxiliary function is the additional square root and the main function is the icosahedral function $\mathcal{I}$.

These solutions provide a template for how to move beyond solvability in radicals. Historically, however, the use of the auxiliary functions ("accessory irrationalities") was a point of contention. The solutions of Bring and Klein explicitly construct solutions of the quintic using only algebraic functions of a single variable. However, Kronecker disccovered and Klein proved that if accessory irrationalities are not allowed, then one must use an algebraic function of two variables. In the ending footnote of [Kle1905] (see [Sut2019] or Appendix B.1 for an English translation), Klein argues for the use of accessory irrationalities in the context of solving general polynomials:

> *There remains only a subjective difference, which I already discussed in detail on pages p.158-159 in the book on the icosahedron, but which I do not want to leave untouched here because of its importance. For the first time in his investigations into the solution of equations of the fifth degree, Kronecker begged to have a clear distinction between the natural irrationalities (which are rational functions of* $x_0, \ldots, x_4$*) and the other irrationalities (which I call accessory). Incidentally, in his first communication of 1858 [2], he himself makes an unobjectionable use of*

> *an accessory square root. Is is only in the later work of 1861 [15] that he believes that he should forbid the use of accessory irrationalities in the theory of equations altogether. In his 1885-86 lectures, he maintains this verdict:*
>
> > *...the use of accessory irrationalities is "algebraically worthless," because it "tears apart" the type.*
>
> *In order to emphasize this demand, he calls it the "Abelian postulate." In contrast to other authors of similar thinking, I have explored as far as possible in my papers printed here above, as in the book on the icosahedron, the efficacy of using naturally occurring accessory irrationalities.*

If one does not allow accessory irrationalities, the first relevant notion of complexity you arrive at is known as **essential dimension**; allowing these accessory irrationalities leads us to **resolvent degree**. While we will define essential dimension in Section 2.1 (Definitions 2.1.1, 2.1.2, 2.1.10), it is not a focus of this dissertation. The recent survey article on essential dimension and Hilbert's 13th problem by Reichstein [Rei2021] provides an introduction to essential dimension in multiple contexts and centers the connections with solving polynomials.

For each n, we denote the resolvent degree of solving the general polynomial of degree n by $\text{RD}(n)$ (see Remark 2.1.8). The quadratic, cubic, and quartic formulas in radicals yield that $\text{RD}(2) = \text{RD}(3) = \text{RD}(4) = 1$. Additionally, the formulas of Bring and Klein for the quintic show that $\text{RD}(5) = 1$. Bring's approach was extended by Hamilton [Ham1836], Hilbert [Hil1927], Segre [Seg1945], and Sylvester [Syl1887] to construct explicit formulas when $n = 6, 7, 8, 9$ to obtain the upper bounds

- $\text{RD}(6) \leq 2$,
- $\text{RD}(7) \leq 3$,

- $\mathrm{RD}(8) \leq 4$, and
- $\mathrm{RD}(9) \leq 4$.

Further, Hilbert conjectured that these inequalities are actually equalities. Hilbert's Sextic Conjecture predicts that $\mathrm{RD}(6) = 2$ and Hilbert's Octic Conjecture predicts that $\mathrm{RD}(8) = 4$ (which would imply that $\mathrm{RD}(9) = 4$). The second conjecture, that $\mathrm{RD}(7) = 3$, is one version of Hilbert's 13th problem (see [Hil1902]); Reichstein addresses the standard interpretations of Hilbert's 13th problem in [Rei2021].

In theory, one could wish to know $\mathrm{RD}(n)$ for all n. By definition, $\mathrm{RD}(n) \geq 1$ for all n. However, one of the signature open challenges regarding resolvent degree is to determine non-trivial lower bounds, i.e. to show that $\mathrm{RD}(n) > 1$ for some n. On the other hand, there is a successful history of determining upper bounds on $\mathrm{RD}(n)$, including the work of Hamilton, Hilbert, Segre, and Sylvester mentioned above. Modern upper bounds on resolvent degree have come from Brauer [Bra1975] and Wolfson [Wol2021].

In this dissertation, we provide best-to-date upper bounds on $\mathrm{RD}(n)$. We begin by recovering the theory of polars used by Chebotarev, Segre, and Wiman [Che1954, Seg1945, Wim1927]. We then extend the classical framework to iterated polar cones, which we use to establish the following bounds in Section 4.1:

Theorem 1.1.2. ***(New Bounds From Iterated Polar Cones)***

1. *For $n \geq 21$, $\mathrm{RD}(n) \leq n - 6$.*
2. *For $n \geq 109$, $\mathrm{RD}(n) \leq n - 7$.*
3. *For $n \geq 325$, $\mathrm{RD}(n) \leq n - 8$.*
4. *For $9 \leq m \leq 12$ and $n > \frac{(m-1)!}{24}$, $\mathrm{RD}(n) \leq n - m$.*

We also recover the "obliteration algorithm" of Sylvester, which was given originally given in the language of systems of homogeneous equations and which we present geometrically. We then use the geometric obliteration algorithm in the context of iterated polar cones to obtain the following collection of bounds on resolvent degree in Section 4.2:

Theorem 1.1.3. ***(New Bounds from the Geometric Obliteration Algorithm)***

1. *For* $n \geq 5,250,198$, $\mathrm{RD}(n) \leq n-13$.
2. *For each* $14 \leq m \leq 17$ *and* $n > \frac{(m-1)!}{120}$, $\mathrm{RD}(n) \leq n-m$.
3. *For* $n \geq 381,918,437,071,508,900$, $\mathrm{RD}(n) \leq n-22$.
4. *For each* $23 \leq m \leq 25$ *and* $n < \frac{(m-1)!}{720}$, $\mathrm{RD}(n) \leq n-m$.

In [Wol2021], Wolfson introduces a function $F(m)$ such that $\mathrm{RD}(n) \leq n-m$ for all $n \geq F(m)$ and thus we refer to $F(m)$ as a "bounding function." We construct a bounding function $G'(m)$ which incorporates the above bounds and has the following key properties:

Theorem 1.1.4. ***(Key Properties of*** $G'(m)$***)***

The function $G'(m)$ *of Definition 4.3.11 has the following properties:*

1. *For each* $m \geq 1$ *and* $n \geq G'(m)$, $\mathrm{RD}(n) \leq n-m$.
2. *For each* $d \geq 4$, $G'\left(2d^2+4d+4\right) < \frac{\left(2d^2+4d+3\right)!}{d!}$. *In particular, for* $d \geq 4$ *and* $n \geq \frac{\left(2d^2+4d+3\right)!}{d!}$,
$$\mathrm{RD}(n) \leq n-2d^2-4d-4.$$
3. *For each* $m \geq 6$, $G'(m) < F(m)$ *and*
$$\lim_{m\to\infty} \frac{F(m)}{G'(m)} = \infty.$$

The first statement of the main theorem is that $G'(m)$ is indeed a bounding function for $\mathrm{RD}(n)$ (Theorem 4.3.12). The second statement provides an upper bound on $\mathrm{RD}(n)$ and its growth rate using elementary functions (Theorem 4.5.1). The third statement shows that $G'(m)$ provides better bounds than $F(m)$ and does much better asymptotically (Theorem 4.6.1). Note that the equalities $G'(m) = F(m)$ for $m = 1, 2, 3, 4$ are due to classical solutions of general polynomials in low degree, which show that

$$\mathrm{RD}(2) = \mathrm{RD}(3) = \mathrm{RD}(4) = \mathrm{RD}(5) = 1.$$

Hilbert's Octic Conjecture predicts that $\mathrm{RD}(8) = 4$ and, if true, would imply that the bounds $G'(5) = F(5) = 9$ cannot be improved. We provide the values of $F(m)$ and $G'(m)$ for $m \in [1, 26]$ in Appendix A.1 (the values for $m \in [18, 26]$ are approximated for improved readability). We provide further information about the ratio $\frac{F(m)}{G'(m)}$ in Appendix A.2.

1.2 Historical Remarks

Here we largely follow "Appendix B: Historical Background" from [Wol2021], with additions from [Sut2021C] and [HS2021]. For additional information, we note the survey [Dix1993] of Dixmier on Hilbert's 13th problem and the survey [Rei2021] of Reichstein on Hilbert's 13th problem, essential dimension, and resolvent degree.

We begin with an observation of Sylvester and Hammond [SH1887] which was echoed by Wolfson in [Wol2021]:

> *The theory has been "a plant of slow growth."*

While the story starts with the Babylonians using a linear change of coordinates to reduce the general quadratic polynomial to the form $z^2 + d$, the theory of Tschirnhaus transformations comes from [Tsc1683], where Tschirnhaus introduced a transformation to show that $\mathrm{RD}(n) \leq n - 3$ for $n \geq 4$. Bring then extended this approach to show that $\mathrm{RD}(5) = 1$ in [Bri1786] (or [CHM2017] for an English translation). In [Ham1836], Hamilton constructed the first bounding function $H(m)$, which he used to show that

$$\lim_{n\to\infty} n - \mathrm{RD}(n) = \infty.$$

Sylvester then took up the mantle in [Syl1887], in which he writes

> *In the following memoir I propose to present* Hamilton's *process under what appears to me to be a clearer and more intelligible form, to extend his numerical results and to establish the principles of a more general method than that to which he has confined himself.*

The "extended numerical results" include computations to determine the following values of $H(m)$:

Table 1.1: Values of Hamilton's Bounding Function

m	4	5	6	7	8	9
$H(m)$	5	11	47	923	409,619	83,763,206,255

Sylvester sharpened these bounds slightly in [Syl1887] to obtain a bounding function with the following initial values:

Table 1.2: Values of Sylvester's Bounding Function

m	4	5	6	7	8	9
$S(m)$	5	10	44	905	409,181	83,762,797,734

Hammond, with Sylvester, gave a generating function for $S(m)$ in [SH1887, SH1888].

In [Kle1871], 6 years before [Syl1887], Klein was pursuing a new approach to solving polynomials. As we shall see prominently throughout this history, communication amongst mathematicians is not perfect. Sylvester was unaware of these prior developments by Klein, as can be seen from a longer quotation from [SH1887]:

> *The theory has been "a plant of slow growth." The LUND thesis of December, 1786 (a matter of a couple of pages), HAMILTON'S Report of 1836, with the tract of Mr. JERRARD therein referred to, and the memoir in 'CRELLE' of December, 1886, constitute, as far as we are aware, the complete bibliography of the subject up to the present date,*

who mentions only [Bri1786], [Ham1836], and [Syl1887], respectively. A central tenet of Klein's approach was a shift of focus away from presenting general polynomials specifically in forms

$$z^n + b_{m+1}z^{n-m-1} + \cdots + b_{n-1}z + b_n,$$

and instead embracing the point of view of monodromy groups acting on varieties. Klein's approach thus integrated group theory, representation theory, projective geometry, (classical) invariant theory, elliptic function theory, and automorphic function theory into solving polynomials. The crucial examples are when $n = 5, 6, 7$, for which Klein uses the action of the alternating group A_n on $\mathbb{P}^{n-4}_{\mathbb{C}}$ to reproduce the Bring and Hamilton bounds of $\mathrm{RD}(n) \leq n-4$ (see, for example, [Kle1884], [Kle1887], and [Kle1905], or the translation [Sut2019], which is also found in Appendix B.1). Notably, the solutions of Klein greatly simplified the requisite algebra and geometry. Klein was also one of the first people (potentially the first person) to explicitly consider lower bounds on $\mathrm{RD}(n)$ in [Kle1894, Kle1905] and sought to center resolvent problems in the mathematical community (including [Kle1879, Kle1888, Klein1922], as well as [Fri1926]).

In 1900, Hilbert used part of his ICM address to discuss solutions of the general septic polynomial in two variable functions (Problem 13 of [Hil1902]), building upon Enriques' 1897 ICM address [Enr1897]. Notably, he expanded the context to include formulas using analytic, or even continuous, functions and then proved that general analytic functions of three variables do not admit formulas in analytic functions of at most two variables. Further, Hilbert explicitly conjectured that $\mathrm{RD}(6) = 2$, $\mathrm{RD}(7) = 3$, and $\mathrm{RD}(8) = 4$ in [Hil1927]; he also sketched a proof that $\mathrm{RD}(9) \leq 4$ using the 27 lines on a smooth cubic surface. Implicit within his 1900 ICM address (and explicitly in [Hil1927]) was Hilbert's call for lower bounds on resolvent degree.

Despite N. Chebotarev including bounds on $\mathrm{RD}(n)$ in his 1932 ICM address [Che1932] and in several papers [Che1931a, Che1931b, Che1934, Che1943], progress came to a temporary (but noteworthy) halt after Hilbert. Indeed, much of the 19th century and some of the 20th century work appears to have been forgotten by the mid-20th century, particularly within the European mathematical community. One significant cause was the destruction of the mathematical tradition started by Klein and continued by Hilbert at Göttingen by the Nazi regime in the 1930's. While setting back new results on resolvent problems is nowhere near the most important consequences of German politics in the 1930's, the discussion at hand would be incomplete without a discussion of the events that occurred at the Göttingen Mathematical Institute between 1929 and 1933. We draw heavily on a primary source: Saunders Mac Lane's recounting of his time at Göttingen between 1931 and August 1933 in [Mac1995].

Mac Lane begins by describing Göttingen at his arrival and the beginning of his stay, which he most succinctly describes as follows:

> *... the Mathematical Institute at Göttingen in 1931-1932 was a dynamic and successful model of a top mathematical center.*

Outside of the academic post, however, the tides were turning. In fact, strong anti-Semitism had plagued Göttingen for over a decade.

> *In 1932, German politics was turbulent with street battles in Berlin and elsewhere Nazi storm troopers and communist groups. Then in January 1933, there was an election in which the Nazis made common cause with German National Party...*

. . . .

> *On March 5, 1933, the government coalition held a second election, preceded by a vast propaganda effort. It produced a much larger vote for the government ... my landlady regularly provided me with evening tea and talk; I rapidly discovered that two weeks of propaganda had converted her from mild conservative views to ardent Nazi discipleship.*

Indeed, just a month later, there was a direct impact on the Mathematical Institute:

> *In Germany, professors, Privatdozenten, and assistants are all government officials. On April 7, 1933, a new law about such officials summarily dismissed all those who were Jewish, except for those appointed before 1914 and those who served as soldiers in the First World War. In addition, dismissal awaited "all those officials who are not at every time completely committed to the National Socialist State." The effect on the Mathematical Institute was drastic. Courant, Noether, and Bernstein were immediately dismissed (on April 25). In Courant's case, his service in the First World War did not spare him; evidently his earlier political views and his wide mathematical influence (inherited from Felix Klein) made him disliked ... On April 27, Bernays, Hertz and Lewy were dismissed. Landau was advised not to lecture in the coming summer semester; he followed the advice.*

This policy and, in particular, fear of similar policies to come fueled further individual actions:

> *... my letter of May 3 to my mother read (in part):*
>
> > *So many professors and instructors have been fired or have left that the mathematics department is pretty thoroughly emasculated. It is rather hard on mathematics ...*

. . . .

> *But for the institute, there were added losses. Hermann Weyl was not Jewish, but his wife was; this meant then that their two sons were so counted. So at the end of the summer semester 1933, Weyl left for a professorship at the Institute for Advanced Study in Princeton. All told, in 1933 eighteen mathematicians left or*

were driven out from the faculty at the Mathematical Institute in Göttingen. This included Landau; he was not officially dismissed, but when he again started to lecture in the winter semester of 1933, the students organized a complete boycott of his lecture. He thereupon resigned and retired to Berlin.

. . . .

Mathematics at the University of Berlin was also seriously disrupted; there twenty-three faculty members (including Richard Brauer, Max Dehn, Hans Freudenthal, B.H. Neumann, Hanna Neumann, and Richard von Mises) left. The specific (and often less extensive) effects have been carefully tabulated by Maximilian Pinl in four articles. Detailed analysis of the situation at Göttingen has been presented by Schappacher as part of a book on Göttingen under the Nazis.

. . . .

One observer has summarized the effect on mathematics in the following words:

> *Within a few weeks this action would scatter to the winds everything that had been created over so many decades. One of the greatest tragedies experienced by human culture since the time of the Renaissance was taking place – a tragedy which a few years before would have seemed an impossibility under the twentieth century conditions.*

. . . .

As Dorothy and I left in August of 1933, I carried with me, as a treasure, something of the vision of the earlier Göttingen as the unique model of a great mathematics department. I mourned the loss, but not only for the sake of science. I did not forsee the holocaust, but I was aware of the power of state propoganda and I was actively fearful of the prospects for a world war, although prevention seemed beyond my powers.

These considerations only begin to scratch the surface of what happened within the German mathematical community in the 1930's and 40's. We have not touched on the personal actions of Bieberbach, Blaschke, Hasse, Kähler, Teichmüller, Vahlen, Witt and others (all of whom range from Nazi sympathizers to staunchly outspoken Nazis), all of whom are still acknowledged by name for various mathematical contributions, nor have we touched on many of the truly horrible human costs (such as the suicides of Felix Hausdorff, his wife Charlotte, and Charlotte's sister Edith Pappenheim prompted by the orders to be moved to the camp at Endenich). We refer the reader to Segal's book [Seg2003] for more a comprehensive account of mathematics in Nazi Germany and to Friedländer's book [Fri2018] (in particular, Chapter 2 of Volume 1) for broader context on anti-Semitism, Nazism, and the role of elites (including at universities).

In 1945, Segre provided the first rigorous proof that $\mathrm{RD}(n) \leq n-5$ for $n \geq 9$ in [Seg1945]. It is also clear that Segre was unaware of the work Hamilton and Sylvester, as he also proved that $\mathrm{RD}(n) \leq n-6$ for $n \geq 157$, despite Hamilton having established the claim for $n \geq 47$ and Sylvester having done the same for $n \geq 44$. G.N. Chebotarev (son of N. Chebotarev) gave an argument that $\mathrm{RD}(n) \leq n-6$ for $n \geq 21$ in [Che1954] by extending an argument of Wiman from [Wim1927] to establish Hilbert's bound for $\mathrm{RD}(n) \leq n-5$. However, the methods of [Wim1927] have gaps (as pointed out in [Dix1993]) and so do the methods of [Che1954]. Additionally, Segre appeared to be unaware of [Wim1927] in 1945 and G.N. Chebotarev was unaware of [Seg1945] in 1954. Note that Theorem 4.1.1 fixes the gaps in [Che1954] and provides a geometric proof of his bound; this proof can also be suitably modified to fix the argument of Wiman, complementing the algebraic proof given by Dixmier in the appendix of [Dix1993].

Arnold published a result in [Arn1957] which he viewed as a "complete solution of the 13th problem of Hilbert" in 1957, when he was 19 years old. Kolmogorov strengthened Arnold's theorem in [Kol1957] by showing that for any continuous function $f : [0,1]^n \to \mathbb{R}$ on the unit cube[1], there are continuous function $g_1, \ldots, g_{2n-1} : [0,1] \to \mathbb{R}$ and for each $1 \leq j \leq 2n-1$, there are continuous functions $\phi_{j,1}, \ldots, \phi_{j,n} : [0,1] \to \mathbb{R}$ such that

$$f(t_1, \ldots, t_n) = \sum_{j=1}^{2n-1} g_j \left(\sum_{k=1}^{n} \phi_{j,k}(t_k) \right).$$

However, this result only applies to an interpretation of Hilbert's 13th problem asking for solutions in single-valued continuous functions. Indeed, Hilbert's septic is still multivalued when considered as a continuous function. As indicated by his later writing, Hilbert was primarily interested in algebraic solutions of the general septic and one can view his mention of continuous functions as an indication that there might be a topological obstruction. Indeed, Arnold himself concluded that Hilbert's 13th problem was still open and continued to work on it over the next fourty years [Arn1970A, Arn1970B, Arn1970C, AS1976, Arn1999, Arn2000]. Nonetheless, further work on Hilbert's 13th problem largely stopped after the results of Arnold and Kolmogorov.

In 1970, Khovanskii proved in [Kho1970] that any formula for the quintic which did not use division (i.e. if one restricted to entire algebraic functions) required an algebraic function of two variables. Abhyankar proved an analogous result for the sextic in [Abh1995] (and seemed to be unaware of [Kho1970]). In [Lin1973, Lin1976, Lin1996], Lin also considers formulas for general polynomials which do not allow division, as well as other conditions.

Despite the classical motivations, the first rigorous definition of resolvent degree came from Brauer in 1975; Arnold and Shimura, unaware of [Bra1975], provided a definition of resolvent degree in [AS1976]. Following his definition, Brauer provided an explicit bounding function when he showed that $\mathrm{RD}(n) \leq n - m$ for $n \geq (r-1)! + 1$, which improved on the bounds of Hamilton and Sylvester for $m \geq 7$. Additionally, Buhler and Reichstein formalized the

[1] It is within this section alone that we use $[0,1]$ to denote the unit interval. As is established in Section 1.3, we use $[a,b]$ for the collection of integers $\{x \in \mathbb{Z} \mid a \leq x \leq b\}$ in all other sections.

Kronecker-Klein resolvent problem and introduced a rigorous definition of essential dimension in [BR1997, BR1999].

Substantial progress has been made on essential dimension questions since [BR1997]; we refer the readers to the surveys of Reichstein [Rei2011] and Merkurjev [Mer2013] for comprehensive summaries of the essential dimension landscape. We also note [FKW2021A, FKW2021B] as particular sources of results on essential dimension relevant for the resolvent degree questions about solving general polynomials.

For modern surveys on Hilbert's 13th problem which do not explicitly use the term "resolvent degree," we refer the reader to [Dix1993] and [Vit2004].

Recent literature on resolvent degree is still sparse; we provide a complete enumeration of the literature here. Farb and Wolfson "pick up where Brauer left off" in [FW2019], by recalling the definition of resolvent definition, proving several properties of resolvent degree, and connecting resolvent degree to several classical and enumerative problems. Farb, Kisin, and Wolfson recover many of the classical connections between modular functions and resolvent problems in [FKW2022]. Wolfson improved upon the upper bounds on $\mathrm{RD}(n)$ of Brauer in [Wol2021]. The author established the best upper bounds on $\mathrm{RD}(n)$ in [Sut2021C] (with some additional improvements in low degree cases with Heberle in [HS2021]).

Finally, recent work of Reichstein [Rei2022] establishes upper bounds on the resolvent degree of connected complex linear algebraic groups. Specifically, he establishes that $\mathrm{RD}(G) \leq 5$ for any connected complex linear algebraic group G, which serves as potential evidence against Hilbert's conjectures. In contrast, upcoming work of Farb, Kisin, and Wolfson [FKWIP] extends resolvent degree to the setting of arithmetic groups, variations of Hodge structures, and certain moduli problems. Moreover, they give examples of families $\mathcal{F}$ such that $\sup\left(\{\mathrm{RD}(f) \mid f \in \mathcal{F}\}\right) = \infty$, which serve as potential evidence for Hilbert's conjectures.

1.3 Mathematical Conventions

1. We consider only fields K which are finitely generated $\mathbb{C}$-algebras. One could also work over an arbitrary algebraically closed field F of characteristic zero and work relative to F.

2. Given $a, b \in \mathbb{Z}_{\geq 0}$, we set $[a, b] := \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

3. Similarly to [Har2010], we define a **projective variety** (respectively, an **affine variety**) to be a closed algebraic set in $\mathbb{P}^r_K$ (respectively, in $\mathbb{A}^r_K$). By a **variety**, we mean a quasi-projective variety (i.e. a locally closed subset of a projective variety). Notably, we do not require varieties to be irreducible.

4. Given a collection of homogeneous polynomials $S = \{f_1, \ldots, f_s\} \subseteq K[x_0, \ldots, x_r]$, we write $\mathbb{V}(f_1, \ldots, f_s)$ (and sometimes $\mathbb{V}(S)$) for the subvariety of $\mathbb{P}^r_K$ determined by the condition $f_1 = \cdots = f_s = 0$.

5. Given two morphisms of varieties $Y_1 \to X$ and $Y_2 \to X$, we write $Y_1 \times_X Y_2$ for the corresponding pullback.

6. For a subvariety $V \subseteq \mathbb{P}^r_K$, we write $V(K)$ for the set of K-rational points of V.

7. We write K_n to mean $\mathbb{C}(a_1, \ldots, a_n)$, a purely transcendental extension of $\mathbb{C}$ with transcendence basis $a_1, \ldots, a_n$.

8. Given points $P_0, \ldots, P_\ell \in \mathbb{P}^r(K)$, we write $\Lambda(P_0, \ldots, P_\ell)$ for the linear subvariety of $\mathbb{P}^r_K$ that they determine.

9. We refer to a linear subvariety of $\mathbb{P}^r_K$ of dimension $k \geq 3$ as a k**-plane**. We refer to linear subvarieties of dimension one and two as lines and planes, respectively.

10. We write $\mathrm{Gr}(k, r)$ to denote the Grassmannian of k-dimensional subspaces of $\mathbb{C}^r$ and $\mathcal{G}\mathrm{r}(k, r)$ for the space of k-planes in $\mathbb{P}^r_{\mathbb{C}}$. In particular, $\mathcal{G}\mathrm{r}(k, r) \cong \mathrm{Gr}(k + 1, r + 1)$.

11. Given a polynomial ring $K[x_0, \ldots, x_r]$ over a field K, we write

    - $K[x_0, \ldots, x_r]_{(d)}$ for the vector space of degree d polynomials,
    - $K[x_0, \ldots, x_r]_{(d)}^\vee$ for its dual space,
    - $S^*\left(K[x_0, \ldots, x_r]_{(d)}^\vee\right)$ for the corresponding free commutative K-algebra, and
    - $S^*\left(K[x_0, \ldots, x_r]_{(d)}^\vee\right)^{\mathrm{GL}(K,r+1)}$ for the associated graded ring of $\mathrm{GL}(K, r+1)$-invariants.

12. We write log to mean the base e logarithm.

With regard to point 4, observe that for generic $f_1, \ldots, f_s$, $\mathbb{V}(f_1, \ldots, f_s)$ is a complete intersection. However, the twisted cubic curve is an example of choices which do not yield a complete intersection. Consequently, we refer to a subvariety of the form $\mathbb{V}(f_1, \ldots, f_s)$ as an **intersection of hypersurfaces**.

Chapter 2

Background

2.1 Resolvent Degree

2.1.1 Definitions of Essential Dimension and Resolvent Degree

Resolvent degree was introduced independently by Brauer in [Bra1975], as well as by Arnold and Shimura in [AS1976]. Farb and Wolfson brought resolvent degree into the modern literature in [FW2019]. Essential dimension is a related invariant introduced by Buhler and Reichstein in [BR1997] which has the same classical motivations as resolvent degree. Note that [FW2019] is the primary reference for this chapter and we follow it closely (modulo our restriction to working over $\mathbb{C}$). For completeness, we begin by including definitions of essential dimension for finite extensions of $\mathbb{C}$-fields and generically finite, dominant, rational maps of $\mathbb{C}$-varieties.

Definition 2.1.1. (Essential Dimension of a Field Extension)

Let L/K be a finite extension of $\mathbb{C}$-fields. The **essential dimension** of L/K, denoted $\mathrm{ed}(L/K)$, is the minimal d for which there is a subfield $F \subseteq K$ with $\mathrm{tr.deg}(F) \leq d$ and a finite extension $\widetilde{F}/F$ such that

$$L \cong K \otimes_F \widetilde{F}.$$

Definition 2.1.2. (Essential Dimension of Generically Finite, Dominant Maps) Let $\pi : Y \dashrightarrow X$ be a generically finite, dominant, rational map of $\mathbb{C}$-varieties. The **essential dimension** of $Y \dashrightarrow X$, denoted $\mathrm{RD}(Y \dashrightarrow X)$, is the minimum d for which there is a dense Zariski open $X_0 \subseteq X$ and a surjective morphism $\widetilde{Z} \to Z$ with $\dim(Z) \leq d$ which admits a surjective morphism $X_0 \to Z$ such that

$$\pi^{-1}(X_0) \cong X_0 \times_Z \widetilde{Z}.$$

Remark 2.1.3. (Compatibility of Essential Dimension Definitions) We note that Definitions 2.1.1 and 2.1.2 are equivalent. For an irreducible, affine $\mathbb{C}$-variety X, the equivalence of definitions is derived from the classical equivalence which sends X to its functions field $\mathbb{C}(X)$; the general case follows from invariance of essential dimension under birational equivalence.

For concreteness, we now consider a family of examples.

Example 2.1.4. (Essential Dimension of Multiple Square Roots) Recall that $K_n = \mathbb{C}(a_1, \ldots, a_n)$ is a purely transcendental with transcendence basis $a_1, \ldots, a_n$ for each $n \geq 1$. Similarly, let $\widetilde{K}_n = \mathbb{C}\left(\sqrt{a_1}, \ldots, \sqrt{a_n}\right)$. Since $\mathrm{tr.deg}_{\mathbb{C}}(K_n) = n$, it follows immediately from Definition 2.1.1 that

$$\mathrm{ed}\left(\widetilde{K}_n/K_n\right) \leq n.$$

One can also show that

$$\mathrm{ed}\left(\widetilde{K}_n/K_n\right) \geq n;$$

we will discuss this further in Remark 2.1.11. Consequently, we conclude that

$$\mathrm{ed}\left(\widetilde{K}_n/K_n\right) = n.$$

Having taken the time to define essential dimension, we make no objection of its use in defining resolvent degree and Remark 2.1.3 again yields compatibility.

Definition 2.1.5. (Resolvent Degree of Field Extensions)
Let L/K be a finite extension of $\mathbb{C}$-fields. The **resolvent degree** of L/K, denoted $\mathrm{RD}(L/K)$, is the minimal d for which there exists a tower of finite field extensions

$$K = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_\ell$$

such L embeds into E_ℓ over K and $\mathrm{ed}(E_j/E_{j-1}) \leq d$ for all $i \in [1, \ell]$.

Definition 2.1.6. (Resolvent Degree of Generically Finite, Dominant Maps)
Let $Y \dashrightarrow X$ be a generically finite, dominant, rational map of $\mathbb{C}$-varieties. The **resolvent degree** of $Y \dashrightarrow X$, denoted $\mathrm{RD}(Y \dashrightarrow X)$, is the minimal d for which there exists a tower of generically finite, dominant, rational maps

$$E_\ell \dashrightarrow \cdots \dashrightarrow E_1 \dashrightarrow E_0 \subseteq X$$

such that $E_0 \subseteq X$ is a Zariski dense open, $E_\ell \dashrightarrow E_0$ factors through $Y \dashrightarrow X$ and $\mathrm{ed}(E_j \dashrightarrow E_{j-1}) \leq d$ for all $j \in [1, \ell]$.

Wolfson extends Definition 2.1.6 to dominant, rational maps $Y \dashrightarrow X$ which are not generically finite, under the assumption that every irreducible component of X is dominated by an irreducible component of Y, in Definition 4.6 of [Wol2021].

Next, we revisit Example 2.1.4 from the perspective of resolvent degree.

Example 2.1.7. (Resolvent Degree of Multiple Square Roots)

We continue with the notation of Example 2.1.4. Fix n and set $E_0 = K_n$. Additionally, set $L = K_n(x)$ with x transcendental over K_n and take $\widetilde{L} = K_n(\sqrt{x})$. For each $j \in [1, n]$, set $E_j = K_n\left(\sqrt{a_1}, \dots, \sqrt{a_j}, a_{j+1}, \dots, a_n\right)$ and consider the embeddings $L \hookrightarrow E_{j-1}$ where $x \mapsto a_j$. Then, we observe that

$$E_j \cong E_{j-1} \otimes_L \widetilde{L}$$

for each j. Noting that $E_n = \widetilde{K}_n$ and $\mathrm{ed}\left(\widetilde{L}/L\right) = 1$, the tower

$$K_n = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_n = \widetilde{K}_n$$

shows that $\mathrm{RD}\left(\widetilde{K}_n/K_n\right) = 1$. Combining this example with Example 2.1.4, we see that

$$\mathrm{ed}\left(\widetilde{K}_n/K_n\right) - \mathrm{RD}\left(\widetilde{K}_n/K_n\right) = n - 1$$

and hence we can construct examples for which the difference between essential dimension and resolvent degree is arbitrarily large.

We now give a precise construction of the resolvent degree of solving the general degree n polynomial.

Remark 2.1.8. ($\mathrm{RD}(n)$ **Notation)**

Set $\phi_n(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n \in K_n[z]$ and $K_n' = K_n[z]/(\phi_n(z))$. Then,

$$\mathrm{RD}(n) := \mathrm{RD}\left(K_n'/K_n\right) = \mathrm{RD}\left(\mathrm{Spec}\left(K_n'\right) \to \mathrm{Spec}\left(K_n\right)\right).$$

Shortly, we will give the extensions of essential dimension and resolvent degree to finite groups. We first recall three relevant definitions.

Definition 2.1.9. (G-Varieties)
Let G be a finite group. A **G-variety** is a variety X with an action of G by autmorphisms. A G-variety X is

- **primitive** if G acts transitively on the set of irreducible components of X, and is
- **faithful** if the representation $G \to \mathrm{Aut}(X)$ is injective.

Notably, given a primitive, faithful G-variety X with G finite, the quotient X/G is itself a variety.

Definition 2.1.10. (Essential Dimension, Resolvent Degree of a Finite Group)
Let G be a finite group. The **essential dimension** and **resolvent degree** of G, denoted respectively by $\mathrm{ed}(G)$ and $\mathrm{RD}(G)$, are given by

$$\mathrm{ed}(G) := \sup\left\{\mathrm{ed}(X \dashrightarrow X/G) \mid X \text{ is a primitive, faithful, complex } G\text{-variety}\right\},$$
$$\mathrm{RD}(G) := \sup\left\{\mathrm{RD}(X \dashrightarrow X/G) \mid X \text{ is a primitive, faithful, complex } G\text{-variety}\right\}.$$

It follows immediately that $\mathrm{RD}(X \dashrightarrow X/G) \leq \mathrm{RD}(G)$ for any primitive, faithful G-variety X; it is not necessarily true, however, that $\mathrm{RD}(G) \leq \mathrm{RD}(X \dashrightarrow X/G)$. Nonetheless, there are important classes of G-varieties X for which $\mathrm{RD}(G) = \mathrm{RD}(X \dashrightarrow X/G)$, namely those which are versal for G (see [DR2015]) and those which are solvably versal or RD-versal for

G (see [FW2019]). In particular, Farb and Wolfson establish (Theorem 3.3 and Corollary 3.17) that

$$\begin{aligned}\mathrm{RD}(n) = \mathrm{RD}(S_n) &= \mathrm{RD}\left(\mathbb{A}^n_{\mathbb{C}} \dashrightarrow \mathbb{A}^n_{\mathbb{C}}/S_n\right), \\ &= \mathrm{RD}(A_n) = \mathrm{RD}\left(\mathbb{A}^n_{\mathbb{C}} \dashrightarrow \mathbb{A}^n_{\mathbb{C}}/A_n\right),\end{aligned}$$

for $n \geq 2$. We refer the reader to Section 3 of [FW2019] for definitions of versality and extensions thereof, as well as central results and examples.

Remark 2.1.11. (Example 2.1.4, Continued)
We now wish to outline the proof that $\mathrm{ed}\left(\widetilde{K}_n/K_n\right) \geq n$. First, note that $\widetilde{K}_n/K_n$ is Galois with Galois group $G = (\mathbb{Z}/2\mathbb{Z})^n$ and this extension corresponds to the minimal faithful G-representation $\mathbb{A}^n_{\mathbb{C}}$, which is versal for G. It follows that

$$\mathrm{ed}\left(\widetilde{K}_n/K_n\right) = \mathrm{ed}\,(G) = n,$$

from Theorem 6.1 of [BR1997], which we refer the reader to for details.

2.1.2 Basic Properties

We now recall several results on resolvent degree, many of which follow directly from the relevant definitions. We will freely use these results in the following chapters, often without explicit reference.

Lemma 2.1.12. ***(Easy Upper Bounds, Lemma 2.5 of [FW2019])***
Let $Y \dashrightarrow X$ be a generically finite, dominant, rational map.

1. $\mathrm{RD}(Y \dashrightarrow X) \leq \mathrm{ed}(Y \dashrightarrow X) \leq \dim(X)$.

2. *Let* $\widetilde{Y} \dashrightarrow X$ *be any dominant rational map. Then,*

$$\mathrm{RD}\left(\widetilde{Y} \times_X Y \dashrightarrow \widetilde{Y}\right) \leq \mathrm{RD}\,(Y \dashrightarrow X)\,.$$

3. *If* $Y \dashrightarrow X$ *is birationally equivalent to* $\widetilde{Y} \dashrightarrow \widetilde{X}$*, i.e. there are birational equivalences* $Y \dashrightarrow \widetilde{Y}$ *and* $X \dashrightarrow \widetilde{X}$ *such that the induced diagram*

$$\begin{tikzcd} Y \arrow[r, dashed, "\textit{birat. equiv.}"] \arrow[d, dashed] & \widetilde{Y} \arrow[d, dashed] \\ X \arrow[r, dashed, "\textit{birat. equiv.}"] & \widetilde{X} \end{tikzcd}$$

commutes, then $\mathrm{RD}(Y \dashrightarrow X) = \mathrm{RD}\left(\widetilde{Y} \dashrightarrow \widetilde{X}\right)$.

Lemma 2.1.13. ***(Irreducible Components, Lemma 2.6 of [FW2019])***
Let $Y \dashrightarrow X$ *be a generically finite, dominant, rational map . Consider the set* $\{X_1, \ldots, X_\mu\}$ *of irreducible components of* X *and for each* j*, denote the set of irreducible components of* $Y \times_{X_j} X_j$ *by* $\left\{Y_1^j, \ldots, Y_{\nu_j}^j\right\}$*. Then,*

$$\mathrm{RD}(Y \dashrightarrow X) = \max\left\{\mathrm{RD}\left(Y_\ell^j \dashrightarrow X_j\right) \mid j \in [1,\mu], \ell \in [1,\nu_\mu]\right\}.$$

Lemma 2.1.14. ***(RD of Compositions, Lemma 2.7 of [FW2019])***
Let $Z \dashrightarrow Y \dashrightarrow X$ *be a pair of generically finite, dominant, rational maps . Then,*

$$\mathrm{RD}(Z \dashrightarrow X) = \max\left\{\mathrm{RD}(Z \dashrightarrow Y), \mathrm{RD}(Y \dashrightarrow X)\right\}.$$

Lemma 2.1.15. ***(Universality of Solving Polynomials, Lemma 2.9 of [FW2019])***
Let $\pi : Y \dashrightarrow X$ *be a generically finite, dominant, rational map with* $U \subseteq X$ *a dense Zariski open such that the restriction* $\pi^{-1}(U) \to U$ *is a surjective morphism with* n *points in each*

fiber. Then,

$$\mathrm{RD}(Y \dashrightarrow X) \leq \mathrm{RD}(n).$$

Notably, Lemma 2.1.15 implies that for any field extension L/K with $[L : K] = n$, we have that $\mathrm{RD}(L/K) \leq \mathrm{RD}(n)$. We now state another immediate consequence of Lemma 2.1.15.

Proposition 2.1.16. ***(Determining Rational Points over Extensions)***
Let $V \subseteq \mathbb{P}^r_K$ be a degree d subvariety. Then, there is an extension L/K with $\mathrm{RD}(L/K) \leq \mathrm{RD}(d)$ *over which we can determine a rational point of V.*

Proof. Set $\ell = \dim(V)$. For a generic $(r-\ell)$-plane Λ, the intersection $V \cap \Lambda$ has dimension 0 and thus has d $\overline{K}$-points (with multiplicity) in any algebraic closure $\overline{K}$ of K; we denote these points by $Q_1, \ldots, Q_d$. Observe that the polynomial

$$f(z) = (z - Q_1)(z - Q_2) \cdots (z - Q_d)$$

has coefficients defined over K. Let $m(z)$ be an irreducible factor of $f(z)$ and set $L = K[z]/(m(z))$. Note that we can determine an L-point of V by construction and Lemma 2.1.15 yields that

$$\mathrm{RD}(L/K) \leq \mathrm{RD}(\deg(m(z))) \leq \mathrm{RD}(n).$$

$\square$

Remark 2.1.17. (Extensions Given By Solving Polynomials)
Given Proposition 2.1.16 and continuing to use the same notation, we henceforth say that we can determine a point of V by solving a degree d polynomial.

In Section 2.3 of [FW2019], Farb and Wolfson discuss Galois theory for generically finite, dominant rational maps, which allows them to reduce considerations of arbitrary towers in the definitions of resolvent degree to certain towers which are "nicer" in a precise sense. Additionally, this allows for a formal characterization of the classical notion of accessory irrationalities in Section 2.4. These are important formalizations for the general theory, but they are far enough afield from the main goals of this thesis that we direct the reader to [FW2019] for details.

2.2 Tschirnhaus Transformations

We now provide an introduction to the theory of Tschirnhaus transformations. The primary reference for this section is [Wol2021]. Recall that $K_n = \mathbb{C}(a_1, \dots, a_n)$, a purely transcendental extension of $\mathbb{C}$ with transcendence basis $a_1, \dots, a_n$.

Definition 2.2.1. (General Polynomials)

The **general polynomial of degree** n is the polynomial

$$\phi_n(z) := z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n \in K_n[z].$$

Note that for any polynomial $f(w) = w^n + (x_1 + iy_1) w^{n-1} + \cdots + (x_{n-1} + iy_{n-1}) w + (x_n + iy_n) \in \mathbb{C}[w]$, there is a unique ring morphism $\mathbb{C}[a_1, \dots, a_n][z] \to \mathbb{C}[w]$ specializing $\phi_n(z)$ to $f(w)$ via $a_j \mapsto x_j + iy_j$.

Definition 2.2.2. (Tschirnhaus Transformations)

A **Tschirnhaus transformation** of the general degree n polynomial is an isomorphism of K_n fields

$$\Upsilon : K_n[z]/(\phi_n(z)) \xrightarrow{\cong} K_n[z]/(\psi(z)),$$

where

$$\psi(z) = z^n + b_1 z + \cdots + b_{n-1} z + b_n.$$

We say that Υ has **type** $(j_1, \ldots, j_k)$ if $b_{j_1} = \cdots = b_{j_k} = 0$.

Note that the upper bounds of Brauer, Bring, Hamilton, Hilbert, Segre, Sylvester, Wiman, and Wolfson are all obtained by determining Tschirnhaus transformations of a specific type. We will make this precise in Remark 2.2.6; for now, we consider the quadratic formula from the perspective of Tschirnhaus transformations.

Example 2.2.3. (The Quadratic Formula)

Consider the case where $n = 2$ and take ζ_ϕ, ζ_ψ to be primitive elements of the respective fields. We set $\psi(z) = z^2 - \frac{1}{4}a_1 + a_2$ and claim that

$$\begin{aligned} \Upsilon : K_n[z]/(\phi_n(z)) &\to K_n[z]/(\psi(z)) \\ \zeta_\phi &\mapsto -\frac{a_1}{2} + \zeta_\psi \end{aligned}$$

is an isomorphism. If Υ is a K_n-algebra morphism, it is uniquely determined by $\Upsilon(\zeta_\phi)$. Thus, it suffices to show that $\Upsilon(\zeta_\phi)$ is a root of $\phi_n(z)$. Observe that

$$\begin{aligned} \left(-\frac{a_1}{2} + \zeta_\psi\right)^2 + a_1\left(-\frac{a_1}{2} + \zeta_\psi\right) + a_2 &= \left(\frac{a_1^2}{4} - a_1\zeta_\psi + \zeta_\psi^2\right) + \left(-\frac{a_1^2}{2} + a_1\zeta_\psi\right) + a_2, \\ &= \zeta_\psi^2 - \frac{1}{4}a_1^2 + a_2, \\ &= 0, \end{aligned}$$

and Υ is in fact a Tschirnhaus transformation. Additionally, by construction, we have that

$$\zeta_\psi = \sqrt{-\frac{a_1^2}{4} + a_2},$$

from which it follows that

$$\zeta_\phi = -\frac{a_1}{2} + \zeta_\psi = -\frac{a_1}{2} + \sqrt{-\frac{a_1^2}{4} + a_2} = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}.$$

Remark 2.2.4. (Description of Tschirnhaus Transformations)

Let ζ_ϕ, ζ_ψ be primitive elements for $K_n[z]/(\phi_n(z))$ and $K_n[z]/(\psi(z))$, respectively, over K_n. Then, every K_n-algebra morphism

$$\Upsilon : K_n[z]/(\phi_n(z)) \to K_n[z]/(\psi(z)),$$

is determined by $\Upsilon(\zeta_\phi)$, which is a unique K_n-linear combination of powers of ζ_ψ:

$$\Upsilon\left(\zeta_\phi\right) = w_0 + w_1\zeta_\psi + \cdots + w_{n-1}\zeta_\psi^{n-1}, \text{ some } w_0, \ldots, w_{n-1} \in K_n.$$

Note that Υ is a Tschirnhaus transformation (i.e. an isomorphism) exactly when $w_j \neq 0$ for some $j \in [1, n-1]$. Let $\mathbb{A}^n_{K_n}$ denote the affine space with coordinates $w_0, \ldots, w_{n-1}$ and consider the w_0-axis of $\mathbb{A}^n_{K_n}$:

$$\mathbb{A}^1_{K_n,0} := \left\{(w_0, \ldots, w_{n-1}) \in \mathbb{A}^n_{K_n} \mid w_1 = \cdots = w_{n-1} = 0\right\}.$$

Then, we can geometrically describe the space of Tschirnhaus transformations as

$$\widetilde{\mathcal{T}}^n_n = \mathbb{A}^n_{K_n} \setminus \mathbb{A}^1_{K_n,0},$$

as was noted in Corollary 3.3 of [Wol2021]. However, from the perspective of resolvent degree, we need only work with Tschirnhaus transformations up to re-scaling; the corresponding space is thus the projectivization

$$\mathcal{T}_{K_n}^n = \mathbb{P}_{K_n}^{n-1} \setminus \{[1:0:\cdots:0]\}.$$

One can verify via direct computation that each b_m in Definition 2.2.2 is a homogeneous polynomial in the variables $w_0, \dots, w_{n-1}$ with coefficients in K_n (e.g. is an element of the m^{th} graded piece $K_n[w_0, \dots, w_{n-1}]_{(m)}$).

Definition 2.2.5. (Tschirnhaus Complete Intersections)

Fix $n \geq 2$. For any $m \in [1, n]$, the m^{th} **extended Tschirnhaus hypersurface** is

$$\tau_m := \mathbb{V}(b_m) \subseteq \mathbb{P}_{K_n}^{n-1}$$

and the m^{th} **extended Tschirnhaus complete intersection** is

$$\tau_{1,\dots,m} := \bigcap_{j=1}^{m} \tau_j \subseteq \mathbb{P}_{K_n}^{n-1}.$$

Similarly, the the m^{th} **Tschirnhaus hypersurface** is

$$\tau_m^\circ := \mathbb{V}(b_m) \subseteq \mathbb{P}_{K_n}^{n-1} \setminus \{[1:0:\cdots:0]\},$$

and the m^{th} **Tschirnhaus complete intersection** is

$$\tau_{1,\dots,m}^\circ := \bigcap_{j=1}^{m} \tau_j \subseteq \mathbb{P}_{K_n}^{n-1} \setminus \{[1:0:\cdots:0]\}.$$

Our word choice in Definition 2.2.5 is motivated by the language found in [Wol2021]. Nonetheless, we acknowledge the nomenclature issues. Indeed, each m^{th} extended Tschirnhaus hypersurface is a hypersurface of $\mathbb{P}^{n-1}_{K_n}$, but potentially contains the point $[1:0:\cdots:0]$, which does not correspond to a Tschirnhaus transformation. Conversely, each m^{th} Tschirnhaus hypersurface contains only points which correspond to Tschirnhaus transformations, but which is potentially not a hypersurface in $\mathbb{P}^{n-1}_{K_n}$.

In practice, these are non-issues. We will always pass to a hyperplane $\Lambda \subseteq \mathbb{P}^{n-1}_K$ which does not contain $[1:0:\cdots:0]$, in which case $\Lambda \cap \tau_m = \Lambda \cap \tau_m^\circ$ is a hypersurface in $\Lambda \cong \mathbb{P}^{n-2}_{K_n}$, all of whose points correspond to Tschirnhaus transformations.

Remark 2.2.6. (RD Bounds from Tschirnhaus Transformations)

The general degree n polynomial $f_n(z)$ defines a multi-valued algebraic function $\mathbb{C}^n \to \mathbb{C}$ with coordinates $a_1, \dots, a_n$ via the assignment

$$(a_1, \dots, a_n\} \mapsto \left\{z \in \mathbb{C} \mid z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n = 0\right\}.$$

Given a Tschirnhaus transformation of type $(1, \dots, m-1)$, we can reduce to considering the algebraic function $\mathbb{C}^{n-m+1} \to \mathbb{C}$ defined by

$$(b_m, \dots, b_n) \mapsto \left\{z \in \mathbb{C} \mid z^n + b_m z^{n-m} + \cdots + b_{n-1}z + b_n = 0\right\}.$$

However, we can always normalize the last coordinate to further reduce to an algebraic function $\mathbb{C}^m \to \mathbb{C}$ defined by

$$\{b_m, \dots, b_{n-1}) \mapsto \left\{z \in \mathbb{C} \mid z^n + b_m z^{n-m} + \cdots + b_{n-1}z + 1 = 0\right\}.$$

Consequently, if we can determine a Tschirnhaus transformation of type $(1, \dots, m-1)$ of sufficiently small resolvent degree (i.e. a point of $\tau_{1,\dots,m-1}$ over an extension L/K_n with $\mathrm{RD}(L/K_n)$ small enough), then we can conclude that $\mathrm{RD}(n) \leq n-m$.

For example, Proposition 2.1.16 yields that we can directly determine a point of $\tau_{1,\dots,m-1}$ by solving a polynomial of degree $(m-1)!$. Consequently, $\mathrm{RD}(n) \leq n-m$ for $n \geq (m-1)!+1$, which is exactly the bounding function determined by Brauer in [Bra1975].

We can improve this further by determining an $(m-d-1)$-plane $\Lambda \subseteq \tau_{1,\dots,d}^{\circ}$ over an extension of low resolvent degree, as then the degree of $\Lambda \cap \tau_{1,\dots,m-1}^{\circ}$ is $\frac{(m-1)!}{d!}$. Indeed, this is the approach we use in Chapter 4. First, however, we discuss polar cones and their connections to determining k-planes.

Chapter 3

Polars

3.1 Polars of Hypersurfaces

When constructing his upper bounds in [Seg1945], many of Segre's proofs rely on a "well-known fact" which may no longer be as well-known as it once was. We include this result as Lemma 3.1.6, which we refer to as Bertini's Lemma since the reference Segre gives for this fact is [Ber1923]. We provide our own overview of the theory of polars, with an emphasis on polar cones. For an alternative prospective which focuses more on individual polars, see [Dol2012] (p.5-6 and equations 1.8-1.11 in particular). We begin by introducing the polars of a polynomial.

Definition 3.1.1. (Polars of a Polynomial)

Consider a degree d, homogeneous polynomial $f \in K[x_0, \ldots, x_r]_{(d)}$ and $P \in \mathbb{P}^r(K)$. Note that the set

$$I_j^* := \mathrm{Hom}_{\mathrm{Set}}\left([1, j], [0, r]\right)$$

indexes the ordered j^{th} partial derivatives of f for each $j \in [0, d]$. However, here mixed partials commute and we use the shorthand

$$\partial_0^{j_0} \cdots \partial_\ell^{j_\ell} = \frac{\partial^{j_0+\cdots+j_\ell}}{\partial x_0^{j_0} \cdots \partial x_\ell^{j_\ell}},$$

for any such ordered partial derivative. For each $j \in [0, d]$, the j^{th} **polar of f at P** is the homogeneous polynomial

$$t(j, k, P) := \sum_{\iota \in I^*_{d-j}} \left(\partial_0^{\left|\iota^{-1}(0)\right|} \cdots \partial_r^{\left|\iota^{-1}(r)\right|} f \right)\Big|_P x_0^{\left|\iota^{-1}(0)\right|} \cdots x_r^{\left|\iota^{-1}(r)\right|}, \tag{3.1.1}$$

which is of degree $d - j$.

For concreteness, we work through an explicit computation of polars for a specific polynomial.

Example 3.1.2. (Computation of Polars)

We consider $f(x, y) = x^2 + 3xy + 2y^2 \in \mathbb{C}[x, y]_{(2)}$. Observe that

$$f(-1, 1) = (-1)^2 + 3(-1)(1) + 2(-1)^2 = 1 - 3 + 2 = 0.$$

We will now compute the polars of f at $P = [1 : -1]$. First is the 0^{th} polar:

$$\begin{aligned}
t(0, f, P) = &\left(\frac{\partial^2}{\partial x^2} \left(x^2 + 3xy + 2y^2\right) \right)\Big|_P x^2 + \left(\frac{\partial^2}{\partial x \partial y} \left(x^2 + 3xy + 2y^2\right) \right)\Big|_P xy \\
&+ \left(\frac{\partial^2}{\partial y \partial x} \left(x^2 + 3xy + 2y^2\right) \right)\Big|_P xy + \left(\frac{\partial^2}{\partial y^2} \left(x^2 + 3xy + 2y^2\right) \right)\Big|_P y^2, \\
&= (2)\Big|_P x^2 + (3)\Big|_P xy + (3)\Big|_P xy + (4)\Big|_P y^2, \\
&= 2x^2 + 6xy + 4y^2, \\
&= 2f(x, y, z).
\end{aligned}$$

Next is the 1^{st} polar:

$$\begin{aligned} t(1, f, P) &= \left(\frac{\partial}{\partial x} \left(x^2 + 3xy + 2y^2 \right) \right) \Big|_P x + \left(\frac{\partial}{\partial y} \left(x^2 + 3xy + 2y^2 \right) \right) \Big|_P y, \\ &= (2x + 3y) \Big|_P x + (3x + 4y) \Big|_P y, \\ &= -x - y \end{aligned}$$

Finally, the 2^{nd} polar:

$$t(2, f, P) = \left(\partial^0 f \right) \Big|_P = f(P) = 0.$$

Example 3.1.3. (Extremal Polars)

We again take $f \in K[x_0, \ldots, x_r]_{(d)}$ and $P \in \mathbb{P}^r(K)$. The extremal polars of f at P, i.e. $t(0, f, P)$ and $t(d, f, P)$, are always easily describable in terms of f. Specifically,

$$\begin{aligned} &t(0, f, P) = d! \cdot f(x_0, \ldots, x_r), \\ &t(d, f, P) = f(P) = 0. \end{aligned}$$

We now bring polars into the geometric setting.

Definition 3.1.4. (Polars of a Hypersurface)

Let $H = \mathbb{V}(f)$ be a hypersurface in $\mathbb{P}^r_K$ and $P \in \mathbb{P}^r(K)$. The j^{th} **polar of H at P** is

$$T(j, f, P) := \mathbb{V}\left(t(j, f, P)\right) \subseteq \mathbb{P}^r_K.$$

Remark 3.1.5. (Special Polars of a Hypersurface)

Using the notation of Definition 3.1.4 and recalling Example 3.1.3, we note that $T(0, f, P) =$

H. Similarly,

$$T(d,f,P) = \begin{cases} \mathbb{P}^r_K, \text{ if } P \in H(K), \\ \emptyset, \text{ otherwise.} \end{cases}$$

Finally, if H is smooth at P, one can check that $T(d-1,f,P)$ is the tangent hyperplane of H at P. Indeed, this motivates the use of $T(j,f,P)$ for polar hypersurfaces and thus the use of $t(j,f,P)$ for their defining polynomials.

We are now ready to state Segre's "well-known fact," which motivates the definition which immediately follows it.

Lemma 3.1.6. ***(Bertini's Lemma for Hypersurfaces)***

Let $H = \mathbb{V}(f) \subseteq \mathbb{P}^r_K$ be a hypersurface and $P \in H(K)$. Then,

$$\bigcap_{j=0}^{d-1} T(j,f,P) \subseteq H$$

is a cone with vertex P.

Definition 3.1.7. (Polar Cone of a Hypersurface)

Let $H = \mathbb{V}(f) \subseteq \mathbb{P}^r_K$ be a hypersurface and $P \in H(K)$. The **polar cone of H at p** is

$$\mathcal{C}(H;P) = \bigcap_{j=0}^{d-1} T(j,f,P).$$

Remark 3.1.8. (Segre's Notation) On p.292 of [Seg1945], Segre writes "the successive polars $V_r^{n-1}, V_r^{n-2}, \cdots, V_r^1$ of P at V," where V is a degree n hypersurface with P an M-rational point. We follow the indexing conventions of [Dol2012], which unfortunately yields the minor inconvenience that

$$V_r^j = T(d - j, f, P).$$

Additionally, Segre provides no general notation or terminology analogous to the polar cone $\mathcal{C}(H; P)$, instead writing "the intersection of $V_r^1, V_r^2, \cdots, V_r^{n-1}, V_r^n$" for the general case and analogously for specific cases of low degree.

Before proving Lemma 3.1.6, we first use the 27 lines on a smooth cubic surface to build intuition for polar cones.

Example 3.1.9. (Lines on Smooth Cubic Surfaces)
Let $S \subseteq \mathbb{P}_K^3$ be a smooth cubic surface. If K is algebraically closed, then S contains exactly 27 lines (recall that we work only with fields which are $\mathbb{C}$-algebras, hence $\mathrm{char}(K) = 0$). In such a case, if $P \in S(K)$ lies on exactly one line Λ, then $\mathcal{C}(S; P) = \Lambda$. If $P \in S(K)$ lies on exactly two lines Λ_1, Λ_2, then $\mathcal{C}(S; P) = \Lambda_1 \cup \Lambda_2$. Most points $P \in S(K)$ do not lie on any line and in this situation $\mathcal{C}(S; P)$ is simply the point P with multiplicity 6.

We now work through such an example explicitly. Consider the Fermat cubic

$$S = \mathbb{V}\left(f(w, x, y, z)\right) \subseteq \mathbb{P}_{\mathbb{C}}^3, \quad f(w, x, y, z) = w^3 + x^3 + y^3 + z^3.$$

Set $P = [1 : -1 : 1 : -1]$ and observe that $P \in S(\mathbb{C})$. From Remark 3.1.5, we know that

$T(0, f, P) = S$. Next, observe that all mixed partials of f are zero. Hence,

$$\begin{aligned} t(1, f, P) &= \left(\frac{\partial}{\partial w^2} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P w^2 + \left(\frac{\partial}{\partial x^2} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P x^2 \\ &+ \left(\frac{\partial}{\partial y^2} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P y^2 + \left(\frac{\partial}{\partial z^2} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P z^2, \\ &= (6w) \bigg|_P w^2 + (6x) \bigg|_P x^2 + (6y) \bigg|_P y^2 + (6z) \bigg|_P z^2, \\ &= 6w^2 - 6x^2 + 6y^2 - 6z^2, \end{aligned}$$

and thus $T(1, f, P) = \mathbb{V}\left(w^2 - x^2 + y^2 - z^2\right)$. Similarly,

$$\begin{aligned} t(2, f, P) &= \left(\frac{\partial}{\partial w} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P w + \left(\frac{\partial}{\partial x} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P x \\ &+ \left(\frac{\partial}{\partial y} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P y + \left(\frac{\partial}{\partial z} w^3 + x^3 + y^3 + z^3 \right) \bigg|_P z, \\ &= \left(3w^2\right) \bigg|_P w + \left(3x^2\right) \bigg|_P x + \left(3y^2\right) \bigg|_P y + \left(3z^2\right) \bigg|_P z, \\ &= 3w + 3x + 3y + 3z, \end{aligned}$$

and so $T(2, f, P) = \mathbb{V}(w + x + y + z)$. Substituting $w + x + y + z$ into $w^2 - x^2 + y^2 - z^2$ yields

$$(-x - y - z)^2 - x^2 + y^2 - z^2 = 2y^2 + 2xy + 2xz + 2yz = 2(x + y)(y + z).$$

Consequently, $T(2, f, P) \cap T(1, f, P)$ is the union of the two lines

$$\begin{aligned} \Lambda_1 &= \mathbb{V}(w + x + y + z, x + y) = \mathbb{V}(w + z, x + y), \\ \Lambda_2 &= \mathbb{V}(w + x + y + z, y + z) = \mathbb{V}(w + x, y + z). \end{aligned}$$

Thus, to fully characterize $\mathcal{C}(S; P)$, it suffices to determine if each of Λ_1, Λ_2 lie on S. Observe that

$$(-z)^3 + (-y)^3 + y^3 + z^3 = 0,$$
$$(-x)^3 + x^3 + (-z)^3 + z^3 = 0,$$

hence

$$\mathcal{C}(S;P) = \bigcap_{j=0}^{2} T(j,f,P) = \Lambda_1 \cup \Lambda_2.$$

Let $\zeta = e^{\frac{2\pi i}{3}}$ and consider $\mathbb{P}^1_{\mathbb{C}}$ with coordinates $[u : v]$. Then, the 27 lines on S can be parametrized as follows:

Table 3.1: The 27 Lines on the Fermat Cubic Surface

$[u : -u : v : -v]$	$[u : -u : v : \zeta v]$	$[u : -u : v : \zeta^2 v]$
$[u : \zeta u : v : -v]$	$[u : \zeta u : v : \zeta v]$	$[u : \zeta u : v : \zeta^2 v]$
$[u : \zeta^2 u : v : -v]$	$[u : \zeta^2 u : v : \zeta v]$	$[u : \zeta^2 u : v : \zeta^2 v]$
$[u : v : -u : -v]$	$[u : v : -u : \zeta v]$	$[u : v : -u : \zeta^2 v]$
$[u : v : \zeta u : -v]$	$[u : v : \zeta u : \zeta v]$	$[u : v : \zeta u : \zeta^2 v]$
$[u : v : \zeta^2 u : -v]$	$[u : v : \zeta^2 u : \zeta v]$	$[u : v : \zeta^2 u : \zeta^2 v]$
$[u : v : -v : -u]$	$[u : v : \zeta v : -u]$	$[u : v : \zeta^2 v : -u]$
$[u : v : -v : \zeta u]$	$[u : v : \zeta v : \zeta u]$	$[u : v : \zeta^2 v : \zeta u]$
$[u : v : -v : \zeta^2 u]$	$[u : v : \zeta v : \zeta^2 u]$	$[u : v : \zeta^2 v : \zeta^2 u]$

It follows that any point $Q \in S(\mathbb{C})$ which is not of any of the forms described in Table 3.1 does not lie on a line of S and has a trivial polar cone; writing $\sqrt[3]{19}$ for the unique real root of $x^3 - 19$, we see that $Q = [-3 : 2 : \sqrt[3]{19} : 0]$ is one such point.

Our proof of Lemma 3.1.6 relies on the following technical lemma, whose proof is straightforward conceptually, but which requires cumbersome notation. Consequently, we will state the technical lemma, provide a proof of Lemma 3.1.6 which is conditional on the technical lemma, and then prove the technical lemma.

Lemma 3.1.10. ***(Technical Lemma)***

Let $P, Q \in \mathbb{P}^r(K)$ and $f \in K[x_0, \ldots, x_r]_{(d)}$. Applying a projective change of coordinates as necessary, we assume that

$$P = [1 : p_1 : \cdots : p_r],$$
$$Q = [1 : q_1 : \cdots : q_r],$$

so that the line determined by P and Q is

$$\Lambda(P,Q)(K) = \left\{ [1 : \lambda p_1 + \mu q_1 : \cdots : \lambda p_r + \mu q_r] \mid [\lambda : \mu] \in \mathbb{P}^1(K) \right\}.$$

Then, for any point $R_{\lambda:\mu} = [1 : \lambda p_1 + \mu q_1 : \cdots : \lambda p_r + \mu q_r] \in \Lambda(P,Q)(K)$, we have that

$$f(R_{\lambda:\mu}) = f(\lambda P) + f(\mu Q) + \sum_{j=1}^{d-1} \frac{1}{j!} t(d-j, f, \Lambda P)(\mu Q). \tag{3.1.2}$$

Proof. **(Proof of Lemma 3.1.6)**

Recall that $H = \mathbb{V}(f)$ is a hypersurface of degree d and $P \in H(K)$. It is immediate that $\mathcal{C}(H;P) \subseteq T(0, f, P) = H$. Now, observe that for every $Q \in \mathcal{C}(H;P)$, every term on the right-hand of equation (3.1.2) vanishes. It follows that each $\Lambda(P,Q) \subseteq \mathcal{C}(H;P)$ and thus $\mathcal{C}(H;P)$ is a cone with vertex P. $\square$

Proof. **(Proof of Lemma 3.1.10)**

For simplicity, we set $p_0 = q_0 = 1$. Note that when $r = 1$, the claim follows immediately from the binomial formula. Consequently, we assume $r \geq 2$. Further, partial derivatives are linear, hence it suffices to consider the case where f is a monomial:

$$f(x_0, \ldots, x_r) = a x_0^{j_0} \cdots x_r^{j_r}.$$

As a result, for any point $R_{\lambda:\mu}$, we have

$$\begin{aligned} f\left(R_{\lambda:\mu}\right) &= a \prod_{\alpha=0}^{r} \left(\lambda p_\alpha + \mu q_\alpha\right)^{j_\alpha}, \\ &= a \prod_{\alpha=0}^{r} \sum_{\ell_\alpha=0}^{j_\alpha} \binom{j_\alpha}{\ell_\alpha} \left(\lambda p_\alpha\right)^{j_\alpha - \ell_\alpha} \left(\mu q_\alpha\right)^{\ell_\alpha}, \\ &= a \sum_{\ell_0=0}^{j_0} \cdots \sum_{\ell_r=0}^{j_r} \left(\prod_{\alpha=0}^{r} \binom{j_\alpha}{\ell_\alpha} \left(\lambda p_\alpha\right)^{j_\alpha - \ell_\alpha} \left(\mu q_\alpha\right)^{\ell_\alpha} \right). \end{aligned}$$

Extending the notation of Definition 3.1.1, we introduce the indexing set

$$I = \left\{ (k_0, \ldots, k_r) \in \mathbb{Z}^{r+1} \mid k_\alpha \in [0, j_\alpha] \right\},$$

which we then partition as follows:

$$I_k = \left\{ (k_0, \ldots, k_r) \in I \mid k_0 + \cdots + k_r = k \right\}, \quad k \in [0, d].$$

Hence, we write

$$f\left(R_{\lambda:\mu}\right) = a \sum_{I} \left(\prod_{\alpha=0}^{r} \binom{j_\alpha}{k_\alpha} \left(\lambda p_\alpha\right)^{j_\alpha - k_\alpha} \left(\mu q_\alpha\right)^{k_\alpha} \right) = a \sum_{k=0}^{d} \sum_{I_k} \left(\prod_{\alpha=0}^{r} \binom{j_\alpha}{k_\alpha} \left(\lambda p_\alpha\right)^{j_\alpha - k_\alpha} \left(\mu q_\alpha\right)^{k_\alpha} \right).$$

Observe that

$$f(\lambda P) = a\left(\prod_{\alpha=0}^{r}(\lambda p_\alpha)^{j_\alpha}\right) = a\sum_{I_0}\left(\prod_{\alpha=0}^{r}\binom{j_\alpha}{0}(\lambda p_\alpha)^{j_\alpha-0}(\mu q_j)^0\right),$$
$$f(\mu Q) = a\left(\prod_{\alpha=0}^{r}(\mu q_\alpha)^{j_\alpha}\right) = a\sum_{I_d}\left(\prod_{\alpha=0}^{r}\binom{j_\alpha}{j_\alpha}(\lambda p_\alpha)^{j_\alpha-j_\alpha}(\mu q_j)^{j_\alpha}\right),$$

and so

$$f\left(R_{\lambda:\mu}\right) - f(\lambda P) - f(\mu Q) = a\sum_{k=1}^{d-1}\sum_{I_k}\left(\prod_{\alpha=0}^{r}\binom{j_\alpha}{k_\alpha}\left(\lambda p_\alpha\right)^{j_\alpha-k_\alpha}\left(\mu q_\alpha\right)^{k_\alpha}\right).$$

As a result, it suffices to show that

$$\frac{1}{k!}t(d-k, f, \lambda P)(\mu Q) = a\sum_{I_k}\left(\prod_{\alpha=0}^{r}\binom{j_\alpha}{k_\alpha}\left(\lambda p_\alpha\right)^{j_\alpha-k_\alpha}\left(\mu q_\alpha\right)^{k_\alpha}\right),$$

for each $k \in [1, d-1]$. Recall that $I_k^* = \mathrm{Hom}_{\mathrm{Set}}([1,k],[0,r])$. Thus,

$$\frac{1}{k!}t(d-k, f, \lambda P)(\mu Q) = \frac{1}{k!}\sum_{\iota\in I_k^*}\left(\partial_0^{\left|\iota^{-1}(0)\right|}\cdots\partial_r^{\left|\iota^{-1}(r)\right|}f\right)\Big|_{\lambda P}(\mu q_0)^{\left|\iota^{-1}(0)\right|}\cdots(\mu q_r)^{\left|\iota^{-1}(0)\right|}.$$

Expanding out the right side of the previous equation and condensing yields

$$\frac{a}{k!}\sum_{\iota\in I_k^*}\left(\prod_{\alpha=0}^{r}\frac{j_\alpha}{(j_\alpha-|\iota^{-1}(\alpha)|)!}(\lambda p_\alpha)^{j_\alpha-\left|\iota^{-1}(\alpha)\right|}(\mu q_\alpha)^{\left|\iota^{-1}(\alpha)\right|}\right).$$

Note that we have a natural surjection $\pi_k : I_k^* \to I_k$ given by

$$\iota \mapsto \left(\left|\iota^{-1}(0)\right|, \ldots, \left|\iota^{-1}(r)\right|\right).$$

For each $\kappa = (k_0, \ldots, k_r) \in I_k$, we now determine the cardinality of the fiber $\pi^{-1}(\kappa)$. The S_k action on $[1, k]$ does not change the number of fibers of each cardinality for every ι. To get a unique representative of each unordered class of partial derivative, we identify permutations which fix each individual fiber, but permute the elements within any such fiber. Thus, the size of each fiber is

$$\left|\pi_k^{-1}(\kappa)\right| = \frac{|S_k|}{|S_{k_0}| \cdots |S_{k_r}|} = k! \prod_{\beta=0}^{r} \frac{1}{(k_\beta)!}.$$

Consequently,

$$\begin{aligned}
\frac{1}{k!} t(d-k, f, \lambda P)(\mu Q) &= \frac{a}{k!} \sum_{\iota \in I_k^*} \left(\prod_{j=0}^{\alpha} \frac{j_\alpha!}{(j_\alpha - |\iota^{-1}(\alpha)|)!} (\lambda p_\alpha)^{j_\alpha - \left|\iota^{-1}(\alpha)\right|} (\mu q_\alpha)^{\left|\iota^{-1}(\alpha)\right|} \right), \\
&= \frac{a}{k!} \sum_{I_k} \left(k! \prod_{\beta=0}^{r} \frac{1}{(k_\beta)!} \right) \prod_{\alpha=0}^{r} \frac{j_\alpha!}{(j_\alpha - k_\alpha)!} (\lambda p_\alpha)^{j_\alpha - k_\alpha} (\mu q_\alpha)^{k_\alpha}, \\
&= a \sum_{I_k} \prod_{\alpha=0}^{r} \binom{j_\alpha}{k_\alpha} (\lambda p_\alpha)^{j_\alpha - k_\alpha} (\mu q_\alpha)^{k_\alpha},
\end{aligned}$$

which yields the lemma. $\square$

3.2 Polars of Intersections of Hypersurfaces and Iterated Polar Cones

Let $H \subseteq \mathbb{P}_K^r$ be a hypersurface and $P \in H(K)$. For every point $Q \in \mathcal{C}(H; P)(K) \setminus \{P\}$, Lemma 3.1.6 yields that the line $\Lambda(P, Q)$ lies in H. Note that if we have an intersection of hypersurfaces $V = \mathbb{V}(f_1, \ldots, f_s) \subseteq \mathbb{P}_K^r$, a line Λ lies on V if and only if Λ lies on each $\mathbb{V}(f_j)$. The combination of these observations motivates Definition 3.2.1 and yields Lemma 3.2.2.

Definition 3.2.1. (Polar Cone of an Intersection of Hypersurfaces)

Let $V = \mathbb{V}(f_1, \ldots, f_s) \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and $P \in V(K)$. The **(first) polar cone of** V **at** P is

$$\mathcal{C}(V;P) := \bigcap_{j=1}^{s} \mathcal{C}\left(\mathbb{V}(f_j);P\right).$$

Lemma 3.2.2. ***(Bertini's Lemma for Intersections of Hypersurfaces)***

Let $V = \mathbb{V}(f_1, \ldots, f_s) \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and $P \in V(K)$. Then, $\mathcal{C}(V;P) \subseteq V$ is a cone with vertex P.

Recall from Remark 2.2.6 that we will seek to determine not just lines, but k-planes on intersections of hypersurfaces in Chapter 4. However, we can iterate the polar cone construction to do exactly this. We begin by introducing the language of iterated polar cones and k-polar points.

Definition 3.2.3. (Iterated Polar Cones and k-Polar Points)

Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and $P_0 \in V(K)$. We begin by setting

$$\mathcal{C}^1\left(V;P_0\right) := \mathcal{C}\left(V;P_0\right).$$

Given additional points $P_1, \ldots, P_{k-1} \in V(K)$ such that

$$P_\ell \in \mathcal{C}^\ell\left(V;P_0, \ldots, P_{\ell-1}\right) \setminus \Lambda\left(P_0, \ldots, P_{\ell-1}\right),$$

for $\ell \in [1, k-1]$, then the k^{th} **polar cone of** V **at** $P_0, \ldots, P_{k-1}$ is

$$\mathcal{C}^k\left(V;P_0, \ldots, P_{k-1}\right) = \mathcal{C}\left(\mathcal{C}^{k-1}\left(V;P_0, \ldots, P_{k-2}\right);P_{k-1}\right).$$

We refer to an ordered collection $(P_0, \ldots, P_k)$ of such points as a k**-polar point.**

If the points $P_0, \ldots, P_{k-1}$ have already chosen, we refer to $\mathcal{C}^k(V; P_0, \ldots, P_{k-1})$ as *the* k^{th} polar cone of V. In the event that such points exist but have not been explicitly chosen, we refer to *a* k^{th} polar cone of V. Additionally, it is occasionally useful to refer to V itself as a zeroth pole of V at any of its rational points.

Remark 3.2.4. (Iterated Polar Cones Are Nested)

Note that if $(P_0, \ldots, P_k)$ is a k-polar point of an intersection of hypersurfaces $V \subseteq \mathbb{P}^r_K$, then the identity $T(0, f, P) = \mathbb{V}(f)$ (c.f. Remark 3.1.5) yields that

$$\mathcal{C}^k(V; P_0, \ldots, P_{k-1}) \subseteq \mathcal{C}^{k-1}(V; P_0, \ldots, P_{k-2}) \subseteq \cdots \subseteq \mathcal{C}^2(V; P_0, P_1) \subseteq \mathcal{C}^1(V; P_0) = \mathcal{C}(V; P_0).$$

Lemma 3.2.5. ***(Polar Point Lemma)***

Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces and let $(P_0, \ldots, P_k)$ be a k-polar point of V. Then, $\Lambda(P_0, \ldots, P_k) \subseteq \mathcal{C}^k(V; P_0, \ldots, P_{k-1}) \subseteq V$ is a k-plane.

Proof. We prove the claim by induction on k. Note that the case of $k = 1$ follows immediately from Lemma 3.2.2. Now, consider arbitrary $k > 1$ and let $(P_0, \ldots, P_k)$ be a k-polar points of V. Then, $(P_1, \ldots, P_k)$ is a $(k-1)$-polar point of $\mathcal{C}(V; P_0)$ and hence

$$\Lambda(P_1, \ldots, P_k) \subseteq \mathcal{C}^{k-1}(\mathcal{C}(V; P_0); P_1, \ldots, P_k) = \mathcal{C}^k(V; P_0, \ldots, P_{k-1}),$$

is a $(k-1)$-plane. However, $P_0 \in \mathcal{C}^k(V; P_0, \ldots, P_{k-1})(K)$ and $\mathcal{C}^k(V; P_0, \ldots, P_{k-1}) \subseteq \mathcal{C}(V; P_0)$. Since $\Lambda(P_1, \ldots, P_k)$ does not contain P_0, Lemma 3.2.1 yields that

$$\Lambda(P_0, \ldots, P_k) \subseteq \mathcal{C}^k(V; P_0, \ldots, P_{k-1}) \subseteq V$$

is a k-plane. $\square$

Example 3.2.6. **(k-Planes on Cubic 7-Folds)**

Let $H \subseteq \mathbb{P}^8_K$ be a cubic hypersurface and assume that K is algebraically closed. Given any 1-polar point (P_0, P_1) of H, we see that $\mathcal{C}^2(H; P_0, P_1)$ is an intersection of a single cubic hypersurface (namely, H), two quadric hypersurfaces, and three hyperplanes. Thus,

$$\dim\left(\mathcal{C}^2\left(H : P_0, P_1\right)\right) \geq 8 - 6 \geq 2,$$

and it follows that $\left(\mathcal{C}^2\left(H : P_0, P_1\right) \setminus \Lambda(P_0, P_1)\right)(K)$ is non-empty. Consequently, every point of H lies on at least one plane contained in H.

We will now exhibit a smooth cubic 7-fold which exhibits multiple 3-planes through a given point. Let $H = \mathbb{V}(f) \subseteq \mathbb{P}^7_{\mathbb{C}}$, where

$$f(x_0, \ldots, x_7) = x_0^3 + \cdots + x_7^3,$$

and set $\zeta = e^{\frac{2\pi i}{3}}$. Note that $P = [1 : -1 : 1 : -1 : 1 : -1 : 1 : -1] \in H(K)$. Further, we can embed a 3-plane $\mathbb{P}^3_{\mathbb{C}}$ with coordinates $a : b : u : v$ into H via

$$[a : b : u : v] \mapsto [a : -a : b : -b : u : -u : v : -v]$$

and we see that P lies on this 3-plane (and many others, which can be obtained by permuting the coordinates and multiplying by ζ, in analogy with Example 3.1.9).

Given a degree d hypersurface, taking a polar cone introduces $d-1$ hypersurfaces, exactly one of degree j for each $j \in [1, d-1]$; iterated polar cones of intersections of hypersurfaces have even more complicated multi-degrees. We introduce notation and language in Definition 3.2.7 which cleanly presents this combinatorial data.

Definition 3.2.7. (Type of an Intersection of Hypersurfaces)

Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces. We say that V is **of type** $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ if V has multi-degree

$$(\underbrace{d, \ldots, d}_{\ell_d \text{ many}}, \underbrace{d-1, \ldots, d-1}_{\ell_{d-1} \text{ many}}, \ldots, \underbrace{2, \ldots, 2}_{\ell_2 \text{ many}}, \underbrace{1, \ldots, 1}_{\ell_1 \text{ many}}).$$

We additionally incorporate the following notational abbreviations:

- If $\ell_j = 0$ for any $j \in [1, d-1]$, we may omit it in the the presentation; e.g. an intersection of four quadrics is of type $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$.
- When $d \geq 2$ and each $\ell_j = 1$, we say V is of type $(1, \ldots, d)$.
- When $d \geq 3$, $\ell_1 = 0$, and $\ell_j = 1$ for $j \in [2, d]$, we say that V is of type $(2, \ldots, d)$.

Remark 3.2.8. (Type of a System of Equations)

In Subsection 3.3.2, we work with systems of homogeneous equations and adopt the same notation. More specifically, given a system of homogeneous equations S, we say that S is of type $\begin{bmatrix} d & \cdots & 1 & \\ \ell_d & \ell_{d-1} & \cdots & \ell_1 \end{bmatrix}$ if and only if $\mathbb{V}(S)$ is of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$.

Remark 3.2.9. (Types of Tschirnhaus Intersections)

Note that the extended Tschirnhaus complete intersections $\tau_{1,\ldots,m-1}$ are intersections of hypersurfaces of type $(1, \ldots, m-1)$. In Section 4.3, we will observe that τ_1 is a hyperplane and consider $\tau_{1,\ldots,m-1}$ as an intersection of hypersurfaces of type $(2, \ldots, d)$ inside τ_1.

Next, we characterize the type of an iterated polar cone of an intersection of hypersurfaces V of type $(1, \ldots, d)$ (in particular, this characterization applies when V is an extended Tschirnhaus complete intersection).

Proposition 3.2.10. ***(Type of a k^{th} Polar Cone of an Intersection of Type $(1,\ldots,d)$)*** *Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $(1,\ldots,d)$ and take $k \geq 1$. Then, a k^{th} polar cone $\mathcal{C}^k(V;P_0,\ldots,P_{k-1})$ of V is of type*

$$\begin{bmatrix} d & d-1 & d-2 & \cdots & 3 & 2 & 1 \\ 1 & k+1 & \binom{k+2}{2} & \cdots & \binom{k+d-3}{d-3} & \binom{k+d-2}{d-2} & \binom{k+d-1}{d-1} \end{bmatrix}$$

for $r \geq \binom{k+d}{d+1}$.

Proof. We proceed by induction on k. The case of $k=1$ follows exactly from Definition 3.2.1 and the observation that $\binom{1+j}{1} = 1+j$ for each $j \in [2, d-1]$.

Now, suppose the claim is true for an arbitrary k. The number of hypersurfaces of degree j in a $(k+1)^{st}$ polar cone, ℓ_j, is exactly the number of hypersurfaces of degree at least j in a k^{th} polar cone. By induction, we have that

$$\ell_j = \sum_{\mu=0}^{j} \binom{k+\mu}{\mu} = \binom{k+j+1}{j} = \binom{(k+1)+j}{j},$$

where the second equality follows from induction on j and the observation that

$$\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$$

for positive integers a, b. This same arguement yields that $\sum\limits_{\mu=0}^{d-1} \binom{k+\mu}{\mu} = \binom{k+d}{d-1}$. □

We end this section with an explicit application of Lemma 3.2.5 which will be of use in Section 4.1.

Proposition 3.2.11. ***(k-Polar Points of Intersections of Quadrics)*** *Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} 2 \\ \ell \end{bmatrix}$ and take $k \geq 1$. For $r \geq (k+1)\ell + k$, we can determine a k-polar point $(P_0, \ldots, P_k)$ of V over an extension L/K with $\mathrm{RD}(L/K) \leq \mathrm{RD}\left(2^\ell\right)$. Moreover, for any point $P \in V(K)$, we can determine a k-plane containing P over such an extension L.*

Proof. When $r > (k+1)\ell + k$, we can restrict to an arbitrary $(k+1)\ell + k$ plane in $\mathbb{P}^r_K$ and so it suffices to consider the case where $r = (k+1)\ell + k$. Note that for a quadric hypersurface H, direct computation yields that a k^{th} polar cone is of type $\begin{bmatrix} 2 & 1 \\ 1 & k \end{bmatrix}$. Consequently, Definition 3.2.1 yields that a k^{th} polar cone of V has type $\begin{bmatrix} 2 & 1 \\ \ell & k\ell \end{bmatrix}$. We proceed by induction on k.

When $k = 1$, $r = 2\ell + 1$ and so $\dim(V) \geq \ell + 1 > 0$. Consequently, we can determine a point P_0 of $V(L_1)$ by solving a polynomial of degree 2^ℓ over K. The polar cone $\mathcal{C}(V; P_0)$ is of type $\begin{bmatrix} 2 & 1 \\ \ell & \ell \end{bmatrix}$, hence

$$\dim\left(\mathcal{C}\left(V; P_0\right)\right) \geq (2\ell + 1) - 2\ell = 1 > 0.$$

As a result, we can determine a point of $\mathcal{C}(V; P_0)(L) \setminus \{P\}$ by solving a polynomial of degree at most 2^ℓ over L_1. Note that (P_0, P_1) is a polar point by construction.

Now, consider the case of an arbitrary $k > 1$. By induction, we assume that we have a $(k-1)$-polar point $(P_0, \ldots, P_{k-1})$ of V over an extension L_{k-1}/K. Note that

$$\dim\left(\mathcal{C}^k\left(V; P_0, \ldots, P_{k-1}\right)\right) \geq ((k+1)\ell + k) - (k+1) \geq k.$$

Consequently, we can determine a point of $\mathcal{C}^k(V; P_0, \dots, P_{k-1}) \setminus \Lambda(P_0, \dots, P_k)$ by solving a polynomial of degree at most 2^ℓ over L_{k-1}. Again, $(P_0, \dots, P_k)$ is a k-polar point of V by construction.

Finally, note that we can replace P_0 with P in the original construction with no change and Lemma 3.2.5 yields that $\Lambda(P, P_1, \dots, P_k)$ is the requisite k-plane. $\square$

3.3 The Obliteration Algorithms

Proposition 3.2.11 provides a key example of how to naively obtain k-planes on intersections of hypersurfaces. However, for any intersection of hypersurfaces V which contains a hypersurface of degree at least 3, $\deg\left(\mathcal{C}^k(V; P_0, \dots, P_k)\right)$ grows exponentially in k. For example, Proposition 3.2.10 yields that if V is of type $(1,2,3)$, then

$$\deg\left(\mathcal{C}^k(V; P_0, \dots, P_{k-1})\right) = 3 \cdot 2^{k+1}.$$

Similarly, if V is of type $(1,2,3,4)$, then

$$\deg\left(\mathcal{C}^k(V; P_0, \dots, P_{k-1})\right) = 4 \cdot 3^{k+1} \cdot 2^{\frac{k^2+3k+2}{2}}.$$

Indeed, these methods obtain bounds $\mathrm{RD}(n) \leq n - m$ most effectively for small m; see Section 4.1.

In [Syl1887], Sylvester gives an algorithm to determine an upper bound on the number of variables required to determine a non-trivial solution for a system of homogeneous polynomials of degrees $d_1, \dots, d_s$ by only solving polynomials of degree at most $\max\{d_1, \dots, d_s\}$. This approach is relatively orthogonal to the naive approach discussed earlier, where the number of variables is minimized and the degrees of the polynomials at hand grows.

Sylvester's algorithm centers on his "formula of obliteration" (see Proposition 3.3.15), and so we refer to the method as the "obliteration algorithm." In Subsection 3.3.1, we give a modern description of the obliteration algorithm via geometry (in terms of varieties, rational points, and polar cones). In Subsection 3.3.2, we describe the obliteration algorithm in terms of systems of homogeneous polynomials and explain Sylvester's classical language.

3.3.1 The Geometric Obliteration Algorithm

Given an intersection of hypersurfaces $V \subseteq \mathbb{P}_K^r$ of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$, we give a bound on the ambient dimension required to be able to determine a point of V over an extension L/K given by solving polynomials of degree at most d. Note that this bound will depend only on the type of V.

Definition 3.3.1. (Minimal Dimension Bound)

The **minimial dimension bound of type** $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$, denoted $r\left(d;\ell_d,\ldots,\ell_1\right)$ is the minimal $r' \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ such that whenever $r \geq r'$, we can determine a point of any intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ in $\mathbb{P}_K^r$ over an extension L/K determined by solving polynomials of degree at most d.

Given a complete intersection V of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$, we set $r(V) := r\left(d;\ell_d,\ldots,\ell_1\right)$.

Remark 3.3.2. (Finiteness of the Minimal Dimension Bound)

The main goal of this section is to establish an upper bound on $r\left(d;\ell_d,\ldots,\ell_1\right)$. More specifically, we introduce a recursive, combinatorial bound $g\left(d;\ell_d,\ldots,\ell_1\right)$ in Definition 3.3.3

which we will show satisfies

$$r\,(d:\ell_d,\ldots,\ell_1) \leq g\,(d;\ell_d,\ldots,\ell_1)\,. \tag{3.3.1}$$

The proof of inequality (2) is exactly the geometric version of the obliteration algorithm.

We now give Definition 3.3 and note that the underlying geometric intuition is explained in Lemma 3.3.5 and Remark 3.3.6.

Definition 3.3.3. (Geometric Dimension Bound)
The **geometric dimension bound of type** $\begin{bmatrix} 1 \\ \ell_1 \end{bmatrix}$ is $g\,(1;\ell_1) := \ell_1$. Similarly, the geometric dimension bound of type $\begin{bmatrix} 2 & 1 \\ 1 & \ell_1 \end{bmatrix}$ is $g\,(2;1,\ell_1) := 1+\ell_1$. The geometric dimension bound of type $\begin{bmatrix} 2 & 1 \\ \ell_2 & \ell_1 \end{bmatrix}$ with $\ell_2 \geq 2$ is

$$g\,(2;\ell_2,\ell_1) := g\,(2;\ell_2-1,\ell_2+\ell_1+1)\,.$$

For $d \geq 3$, the geometric dimension bound of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ 1 & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ is

$$g\,(d;1,\ell_{d-1},\ldots,\ell_2,\ell_1) := g\left(d-1;\ell_{d-1},(\ell_{d-1}+\ell_{d-2})\,,\ldots,\sum_{j=2}^{d-1}\ell_j,\left(\sum_{j=1}^{d-1}\ell_j\right)+1\right).$$

For $d \geq 3$ and $\ell_d \geq 2$, the geometric dimension bound of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ is

$$g\left(d; \ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1\right) := g\left(d; \ell_d - 1, (\ell_d + \ell_{d-1}) - 1, \ldots, \left(\sum_{j=2}^{d} \ell_j\right) - 1, \sum_{j=1}^{d} \ell_j\right).$$

Finally, given an intersection of hypersurfaces V of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$, we set

$$g(V) := g\left(d; \ell_d, \ldots, \ell_1\right).$$

Remark 3.3.4. (Hyperplane Identities)

The definitions of both the minimal and geometric dimension bounds admit a "hyperplane identity", which we use without explicit reference:

$$1 + r\left(d; \ell_d, \ldots, \ell_2, \ell_1\right) = r\left(d; \ell_d, \ldots, \ell_2, \ell_1 + 1\right),$$
$$1 + g\left(d; \ell_d, \ldots, \ell_2, \ell_1\right) = g\left(d; \ell_d, \ldots, \ell_2, \ell_1 + 1\right).$$

We next state Lemma 3.3.5, which is the technical underpinning of the geometric obliteration algorithm and which specializes to give the geometric version of Sylvester's "formula of reduction."

Lemma 3.3.5. ***(The Reduction Lemma)***

Let V be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ which is not a hyperplane. Take V_d to be a degree d hypersurface and V^{red} to be an intersection of hypersurfaces of type

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d - 1 & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$$

if $\ell_d \geq 2$ and of type

$$\begin{bmatrix} d-1 & d-2 & \cdots & 2 & 1 \\ \ell_{d-1} & \ell_{d-2} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$$

if $\ell_d = 1$ such that $V = V^{red} \cap V_d$. Let $P \in V^{red}(K)$ and take H to be a hyperplane which does not contain P. Then,

$$g(V) = g\left(H \cap \mathcal{C}\left(V^{red}; P\right)\right) = g\left(\mathcal{C}\left(V^{red}; P\right)\right) + 1.$$

Remark 3.3.6. (Geometric Insight for the Reduction Lemma) The proof of Lemma 3.3.5 will follow immediately from Definition 3.3.3, but we first address the geometric reasoning underlying the lemma (and hence Definition 3.3.3).

We continue using the notation of Lemma 3.3.5. Suppose our goal is to determine a point Q of V over an extension of bounded resolvent degree. If we can determine a line $\Lambda \subseteq V^{\mathrm{red}}$, then we need only solve a degree d polynomial to determine a point of V. As V^{red} is V with V_d removed, it is already "less difficult" to determine the point $P \in V^{\mathrm{red}}(K)$ given by assumption $\left(\text{e.g. } g(V) \geq g\left(V^{\mathrm{red}}\right)\right)$. Further, we can determine a line $\Lambda \subseteq V^{\mathrm{red}}$ by determining a point $P' \neq P$ of $\mathcal{C}\left(V^{\mathrm{red}}; P\right)$. As H is taken to be a hyperplane which does not contain P, it suffices to determine any point of $\mathcal{C}\left(V^{\mathrm{red}}; P\right) \cap H$. This will be advantageous for us, as we have reduced the number of hypersurfaces of maximal degree.

Proof. **(Proof of Lemma 3.3.5)**

We first consider the case when $\ell_d \geq 2$. From Definition 3.3.3, it follows that

$$\begin{aligned} g(V) &= g\left(d; \ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1\right), \\ &= g\left(d; \ell_d - 1, (\ell_d + \ell_{d-1}) - 1, \ldots, \left(\sum_{j=2}^{d} \ell_j\right) - 1, \sum_{j=1}^{d} \ell_j\right), \\ &= g\left(\mathcal{C}\left(V^{\mathrm{red}}; P\right)\right) + 1, \\ &= g\left(H \cap \mathcal{C}\left(V^{\mathrm{red}}; P\right)\right). \end{aligned}$$

Similarly, when $\ell_d = 1$, we have

$$\begin{aligned} g(V) &= g\left(d; 1, \ell_{d-1}, \ldots, \ell_2, \ell_1\right), \\ &= g\left(d-1; \ell_{d-1}, (\ell_d + \ell_{d-1}), \ldots, \sum_{j=2}^{d-1} \ell_j, \left(\sum_{j=1}^{d-1} \ell_j\right) + 1\right), \\ &= g\left(\mathcal{C}\left(V^{\mathrm{red}}; P\right)\right) + 1, \\ &= g\left(H \cap \mathcal{C}\left(V^{\mathrm{red}}; P\right)\right). \end{aligned}$$

$\square$

As in Lemma 3.3.5, we frequently want to split an intersection of hypersurfaces V into parts analogous to V^{red} and V_d.

Definition 3.3.7. (Reduction and Complement)

Given an intersection of hypersurfaces V of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $\ell_d \geq 2$, a **reduction** of V is an intersection of hypersurfaces V^{red} of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d - 1 & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ such that $V = V^{\mathrm{red}} \cap V_d$ for some degree d hypersurface V_d; we refer to V_d as a **complement** of V^{red}

for V.

Similarly, in the case where $\ell_d = 1$, a reduction of V is an intersection of hypersurfaces of type $\begin{bmatrix} d-1 & \cdots & 1 \\ \ell_{d-1} & \cdots & \ell_1 \end{bmatrix}$ such that $V = V^{\mathrm{red}} \cap V_d$ for some degree d hypersurface V_d, which we again refer to as a complement of V^{red} for V.

We are now ready to state the geometric version of Sylvester's "formula of reduction," as a specialization of the Reduction Lemma.

Corollary 3.3.8. ***(Geometric Formula of Reduction)***

Let W be an intersection of hypersurfaces. Then, for any $P_0 \in W(K)$, any reduction $\mathcal{C}(W; P_0)^{red}$, and any $P_1 \in \mathcal{C}(W; P_0)^{red}(K)$, we have

$$g\left(\mathcal{C}\left(W; P_0\right)\right) = g\left(\mathcal{C}\left(\mathcal{C}(W; P_0)^{red}; P_1\right)\right) + 1.$$

Proof. This follows immediately as a special case of Lemma 3.3.5 applied to $V = \mathcal{C}(W; P_0)$. $\square$

In Proposition 3.3.10, we will successively iterate Lemma 3.3.5 to eliminate all of the hypersurfaces of largest degree from a given intersection of hypersurfaces. We first introduce notation to facilitate this iteration.

Definition 3.3.9. (Sylvester Reductions)

Let V be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$ which is not a hypersurface. A **first partial reduction of** V is

$$V^{\mathrm{Syl}}(d; 1) := \mathcal{C}\left(V^{\mathrm{red}}; P_0\right),$$

where V^{red} is any reduction of V and $P_0 \in V^{\mathrm{red}}(K)$. Proceeding inductively, for any $j \in$

$[2, \ell_d]$, a j^{th} **partial Sylvester reduction of** V is

$$V^{\mathrm{Syl}}(d;j) := \mathcal{C}\left(H_{k-1} \cap V^{\mathrm{Syl}}(d;k-1); P_k\right) = H_{k-1} \cap \mathcal{C}\left(V^{\mathrm{Syl}}(d;k-1); P_k\right)$$

where H_{k-1} is a hyperplane which does contain P_{k-1} and $P_k \in \left(H_{k-1} \cap V^{\mathrm{Syl}}(d;k-1\right)(K)$.

Further, when $d \geq 3$, a **first Sylvester reduction of** V is

$$V_1^{\mathrm{Syl}} := V^{\mathrm{Syl}}\left(d;\ell_d\right).$$

For each $j \in [2, d-1]$, let λ_{d-j+1} be the number of degree $d-j+1$ hypersurfaces defining a $(j-1)^{st}$ Sylvester reduction V_{j-1}^{Syl}. Then, a j^{th} **Sylvester reduction of** V **is**

$$V_j^{\mathrm{Syl}} := \left(V_{j-1}^{\mathrm{Syl}}\right)^{\mathrm{Syl}}\left(d-j+1; \lambda_{d-j+1}\right).$$

Proposition 3.3.10. ***(The Obliteration Proposition)***
Let V be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 3$ which is not a hypersurface. Then,

$$g(V) = g\left(V_1^{Syl}\right),$$

for any first Sylvester reduction V_1^{Syl} of V.

Proof. From Lemma 3.3.5 and Definition 3.3.9, it follows immediately that

$$g\left(V^{\mathrm{Syl}}(d;j)\right) = g\left(V^{\mathrm{Syl}}(d;j+1)\right),$$

for each $j \in [1, \ell_d - 1]$. Consequently, applying Lemma 3.3.5 to V and its partial Sylvester reductions yields

$$g(V) = g\left(V^{\mathrm{Syl}}(d;1)\right) = \cdots = g\left(V^{\mathrm{Syl}}(d;\ell_d - 1)\right) = g\left(V^{\mathrm{Syl}}(d;\ell_d)\right) = g\left(V_1^{\mathrm{Syl}}\right).$$

$\square$

Note that the core ideas of Definition 3.3.9 and Proposition 3.3.10 hold when $d = 2$, however there is no need to actually do another reduction once $\ell_2 = 1$; instead we can just solve a quadratic polynomial directly. This will be done explicitly in the geometric obliteration algorithm.

Remark 3.3.11. (Geometric Dimension Bound via Obliteration)
From the definition of the j^{th} Sylvester reductions, we can iteratively apply Proposition 3.3.10 to observe that

$$g(V) = g\left(V_1^{\mathrm{Syl}}\right) = \cdots = g\left(V_{d-3}^{\mathrm{Syl}}\right) = g\left(V_{d-2}^{\mathrm{Syl}}\right),$$

which provides the most succinct description of the central argument of the geometric obliteration algorithm.

Next, we arrive at the geometric version of Sylvester's "formula of obliteration" as a specialization of Proposition 3.3.10.

Corollary 3.3.12. ***(Geometric Formula of Obliteration)***
Let W be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 3$. For any $P_0 \in W(K)$ and any Sylvester reduction $\mathcal{C}\left(W; P_0\right)_1^{Syl}$, we have

$$g\left(\mathcal{C}\left(W; P_0\right)\right) = g\left(\mathcal{C}\left(W; P_0\right)_1^{Syl}\right). \tag{3.3.2}$$

Proof. This follows immediately as a special case of Proposition 3.3.10 with $V = \mathcal{C}(W; P_0)$.

$\square$

Remark 3.3.13. (Explicit Numerics of the Formula of Obliteration)
Sylvester's "formula of obliteration" (Proposition 3.3.15) is given numerically and, for notational reasons, he chooses to write the statement in terms of "linear solutions" of $\mathcal{C}(W; P_0)^{\mathrm{Syl}}(d; \ell_d - 1)$ instead of $g\left(\mathcal{C}(W; P_0)_1^{\mathrm{Syl}}\right)$. For this reason, we delay the discussion of numerics of the formuila of obliteration to Subsection 3.3.2.

Having established the reduction lemma and the obliteration proposition which we used to recover Sylvester's formula of reduction and formula of obliteration, we proceed to prove inequality (3.3.1).

Proposition 3.3.14. ***(Minimal vs. Geometric Dimension Bound)***
For every type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ *of an intersection of hypersurfaces,*

$$r(d; \ell_d, \ldots, \ell_1) \leq g(d; \ell_d, \ldots, \ell_1) < \infty.$$

Proof. **(The Geometric Obliteration Algorithm)**
We proceed by induction on d. First, observe that when $d = 1$, it is immediate that

$$r(1; \ell_1) = \ell_1 = g(1; \ell_1).$$

We additionally consider the case $d = 2$ before considering the general case. Restricting to $d = 2$, we proceed via induction on ℓ_2. When $\ell_2 = 1$, $\deg(V) = 2$ and thus we can determine a point of V by solving a quadratic polynomial when

$$\dim(V) \geq r - (\ell_1 + 1) = 0.$$

It follows that

$$r\left(2;1,\ell_1\right)=\ell_1+1=g\left(2;1,\ell_1\right).$$

Now, consider the case where $\ell_2\geq 2$ is arbitrary. Our inductive hypothesis yields that

$$r\left(2;\ell_2-1,\lambda_1\right)\leq g\left(2;\ell_2-1,\lambda_1\right)$$

for any $\lambda_1\geq 0$. Let V^{red} be a reduction of V with complement V_2. As V^{red} is of type $\begin{bmatrix} 2 & 1 \\ \ell_2-1 & \ell_1 \end{bmatrix}$, we can determine a point P_0 of V^{red} over an iterated quadratic extension whenever $r\geq g\left(V^{\mathrm{red}}\right)$. Let H be a hyperplane which does not contain P_0. Then, $H\cap\mathcal{C}\left(V^{\mathrm{red}};P_0\right)$ is of type $\begin{bmatrix} 2 & 1 \\ \ell_2-1 & \ell_2+\ell_1 \end{bmatrix}$ and thus we can determine a point P_1 of $H\cap\mathcal{C}\left(V^{\mathrm{red}};P_0\right)$ over an iterated quadratic extension whenever $r\geq g\left(\mathcal{C}\left(V^{\mathrm{red}};P_0\right)\right)+1$. From Lemma 3.2.5, we have that

$$\Lambda\left(P_0,P_1\right)\subseteq\mathcal{C}\left(V^{\mathrm{red}};P_0\right)\subseteq V^{\mathrm{red}}.$$

Hence, we can determine a point of $\Lambda\left(P_0,P_1\right)\cap V_2\subseteq V$ over an additional quadratic extension. From Lemma 3.3.5, we have that

$$r\left(2;\ell_2,\ell_1\right)\leq\max\left\{g\left(2;\ell_2-1,\ell_1\right),g\left(2;\ell_2-1,\ell_2+\ell_1\right)\right\}=g\left(2;\ell_2-1,\ell_2+\ell_1\right)=g\left(2;\ell_2,\ell_1\right).$$

We now return to the case of arbitrary $d\geq 3$. Our inductive hypothesis for d yields that

$$r\left(d-1;\lambda_{d-1},\ldots,\lambda_1\right)\leq g\left(d-1;\lambda_{d-1},\ldots,\lambda_1\right),$$

for every $\lambda_{d-1} \geq 1$ and every $\lambda_j \geq 0$, $j \in [1, d-2]$. We take V^{red} to be a reduction of V with complement V_d and proceed by induction on ℓ_d.

When $\ell_d = 1$, our inductive hypothesis on d yields that we can determine a point P_0 of V^{red} by solving polynomials of degree at most $d-1$ when $r \geq g\left(V^{\text{red}}\right)$. Taking H to be a hyperplane which does not contain P_0, we can similarly determine a point P_1 of $H \cap \mathcal{C}\left(V^{\text{red}}; P_0\right)$ by solving polynoimals of degree at most $d-1$ when $r \geq g\left(\mathcal{C}\left(V^{\text{red}}; P_0\right)\right) + 1$. It follows that

$$\Lambda\left(P_0, P_1\right) \subseteq \mathcal{C}\left(V^{\text{red}}; P_0\right) \subseteq V^{\text{red}},$$

is a line and so we can determine a point of $\Lambda\left(P_0, P_1\right) \cap V_d \subseteq V$ by solving a degree d polynomial. As a result,

$$\begin{aligned}
r\left(d; 1; \ell_{d-1}, \ldots, \ell_1\right) &\leq \max\left\{g\left(V^{\text{red}}\right), g\left(\mathcal{C}\left(V^{\text{red}}; P_0\right)\right) + 1\right\}, \\
&= g\left(\mathcal{C}\left(V^{\text{red}}; P_0\right)\right) + 1, \\
&= g(V), \\
&= g\left(d; 1, \ell_{d-1}, \ldots, \ell_1\right).
\end{aligned}$$

Next, we consider the case of arbitrary $\ell_d \geq 2$. Our inductive hypothesis for ℓ_d yields that

$$r\left(d; \ell_d - 1, \lambda_{d-1}, \ldots, \lambda_1\right) \leq g\left(d; \ell_d - 1, \lambda_{d-1}, \ldots, \lambda_1\right),$$

for all $\lambda_j \geq 0$, $j \in [1, d-1]$. Consequently, we can determine a point P_0 of V^{red} by solving polynomials of degree at most d when $r \geq g\left(V^{\text{red}}\right)$. Taking H to be a hyperplane which does not contain P_0, we can determine a point P_1 of $H \cap \mathcal{C}\left(V^{\text{red}}; P_0\right)$ by solving polynomials of degree at most d when $r \geq g\left(\mathcal{C}\left(V^{\text{red}}; P_0\right)\right) + 1$. As a result, we have the line

$$\Lambda\left(P_0, P_1\right) \subseteq \mathcal{C}\left(V^{\text{red}}; P_0\right) \subseteq V^{\text{red}},$$

and so we can determine a point P_1 of $\Lambda(P_0, P_1) \cap V_d \subseteq V$ by solving an additional degree d polynomial. Therefore,

$$\begin{aligned} r\left(d; \ell_d, \ldots, \ell_1\right) &\leq \max\left\{g\left(V^{\text{red}}\right), g\left(\mathcal{C}\left(V^{\text{red}}; P_0\right)\right) + 1\right\}, \\ &= g\left(\mathcal{C}\left(V^{\text{red}}; P_0\right)\right) + 1, \\ &= g(V), \\ &= g\left(d; \ell_d, \ldots, \ell_1\right). \end{aligned}$$

Finally, we note that the polar cone construction introduces only finitely many hypersurfaces, all of which are of strictly smaller degree. Conssequently, iterating Lemma 3.3.5 yields that $g\left(d; \ell_d, \ldots, \ell_1\right)$ is finite for every type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$.

$\square$

3.3.2 Sylvester's Obliteration Algorithm

We recall the following excerpt from [Syl1887] which was previously given in Section 1.2:

> *In the following memoir I propose to present* Hamilton's *process under what appears to me to be a clearer and more intelligible form, to extend his numerical results and to establish the principles of a more general method than that to which he has confined himself.*

In what follows, we propose to serve the analogous role for Sylvester that Sylvester served for Hamilton. We note that [Syl1887] begins with "a somewhat more extended statement of the Law of Intertia (Trägheitzgesetz) for quadratic forms," which we omit here, but is well-known (see e.g. [Ost1959]) and unnecessary for our purposes. He also presents a brief history of Tschirnhaus transformations; we refer the reader to Section 1.2 for a more comprehensive survey.

Throughout this subsection, we consider a system of homogeneous polynomials $S = \{f_1, \ldots, f_s\}$. Given a solution P_0 of S, the "first emanant" of S at P_0 is

$$S(1; P_0) := \{t(j, f_\ell, P_0) \mid \ell \in [1, s], j \in [0, \deg(f_\ell) - 1]\},$$

where $t(j, f_\ell, P_0)$ is as in equation (3.1.1). Sylvester's sub-lemma states that any linear combination $\lambda_0 P_0 + \lambda_1 P_1$ (what he calls an "alliance" of P_0 and P_1) is a solution of $S(1; P_0)$, where $[\lambda_0 : \lambda_1] \in \mathbb{P}^1(K)$. Consequently, Sylvester says that P_0 and P_1 define a "linear solution" of $S(1; P_0)$ (and thus also of S, since $S \subseteq S(1; P_0)$.

Note that the geometric version of Sylvester's sub-lemma is Lemma 3.2.2. Further, the core algebraic computation reduces to the case of hypersurfaces, which is handled by Lemma 3.1.10. Sylvester analogously introduces "k^{th} emanants" and his lemma is the analogue of Lemma 3.2.5 and his proof follows from iterating the sublemma.

Having acknowledged the general case, Sylvester focuses now on linear solutions of systems of equations. First, he introduces "completed emanants" to ensure that P_1 is distinct from P_0, and so P_0 and P_1 determine a genuine linear solution. More specifically, a completed emanant is a system of equations $T = S(1; P_0) \cup \{g\}$, where g is a homogeneous linear polynomial such that $g(P_0) \neq 0$.

Now, let S be of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ (c.f. Remark 3.2.8). Sylvester introduces the notation $[d; \ell_d, \dots, \ell_1]$ to denote the number of variables necessary to determine a linear solution of S, i.e.

$$[d; \ell_d, \dots, \ell_1] = r\left(\mathcal{C}\left(\mathbb{V}(S); P_0\right)\right) + 1,$$

for any $P_0 \in \mathbb{V}(S)(K)$. It follows that Sylvester's formula of reduction is

$$[d; \ell_d, \dots, \ell_1] \leq \left[d; \ell_d - 1, \ell_d + \ell_{d-1}, \dots, \sum_{j=2}^{d} \ell_j, \sum_{j=1}^{d} \ell_j\right] + 1,$$

when $\ell_d \geq 2$. When $\ell_d = 1$, let d' be the largest $j \leq d-1$ such that ℓ_j is non-zero. Then, Sylvester's formula of reduction is

$$[d; \ell_d, \dots, \ell_1] \leq \left[d'; \ell_{d'}, \ell_{d'} + \ell_{d'-1}, \dots, \sum_{j=2}^{d'} \ell_j, \sum_{j=1}^{d'} \ell_j\right] + 1.$$

Sylvester then claims his formula of obliteration without proof. We state his formula of obliteration and provide a proof, for the sake of completeness.

Proposition 3.3.15. ***(Sylvester's Formula of Obliteration)***
Let S be a system of homogeneous polynomials of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ with $d \geq 2$ and $\ell_d \geq 2$. Then,

$$\begin{aligned} [d; \ell_d, \dots, \ell_1] &\leq [d-1; \lambda_{d-1}, \lambda_{d-2}, \dots, \lambda_2, \lambda_1] + \ell_d, \\ &= [d-1; \lambda_{d-1}, \lambda_{d-2}, \dots, \lambda_2, \lambda_1 + \ell_d], \end{aligned}$$

where

$$\lambda_{d-j} = \binom{\ell_d + j - 1}{j} \frac{j\ell_d + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{\ell_d + \nu - 1}{\nu} \ell_{d-j+\nu}.$$

Proof. It is straightforward to see that iteratively applying Sylvester's formula of reduction allows us to reduce to a system of equations of degree at most $d-1$. For the explicit numerics, we give a proof via induction on ℓ_d. Note that to determine a linear solution of S, it suffices to determine a point solution of a completed emanant T_0 of S at some point solution P_0. Additionally, we note that the type of T_0 is

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_d + \ell_{d-1} & \cdots & \sum_{j=2}^{d} \ell_j & \left(\sum_{j=1}^{d} \ell_j\right) + 1 \end{bmatrix}.$$

Now, suppose that $\ell_d = 1$. We can determine a point solution P_1 of T_0 by determining a linear solution of the subsystem T_0', which is of type

$$\begin{bmatrix} d-1 & \cdots & 2 & 1 \\ 1 + \ell_{d-1} & \cdots & 1 + \sum_{j=2}^{d-1} \ell_j & \left(1 + \sum_{j=1}^{d-1} \ell_j\right) + 1 \end{bmatrix}.$$

Futhermore, we see that

$$\lambda_{d-j} = \binom{1 + j - 1}{j} \frac{j(1) + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{1 + \nu - 1}{\nu} \ell_{d-j+\nu} = 1 + \sum_{\nu=0}^{j-1} \ell_{d-j+\nu} = 1 + \sum_{\mu=d-j}^{d-1} \ell_\mu,$$

so the claim holds when $\ell_d = 1$. Now, consider the case where $\ell_d \geq 2$ is arbitrary. To determine a point solution of T_0, it suffices to determine a linear solution of a subsystem T_0', which is of type

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d - 1 & \ell_d + \ell_{d-1} & \cdots & \sum_{j=2}^{d} \ell_j & \left(\sum_{j=1}^{d} \ell_j\right) + 1 \end{bmatrix}.$$

Thus,

$$[d; \ell_d, \ldots, \ell_1] \leq \left[d; \ell_d - 1, (\ell_d + \ell_{d-1}), \ldots, \left(\sum_{j=2}^{d} \ell_d\right), \left(\sum_{j=1}^{d} \ell_j\right) + 1\right].$$

By induction, however, we have that

$$\left[d; \ell_d - 1, (\ell_d + \ell_{d-1}), \ldots, \left(\sum_{j=2}^{d} \ell_d\right), \left(\sum_{j=1}^{d} \ell_j\right) + 1\right] \leq [d-1; \theta_{d-1}, \ldots, \theta_1 + \ell_d],$$

where

$$\begin{aligned} \theta_{d-j} &= \binom{(\ell_d - 1) + j - 1}{j} \frac{j(\ell_d - 1) + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{(\ell_d - 1) + \nu - 1}{\nu} \left(\sum_{\mu=0}^{j} \ell_{d-j+\mu}\right), \\ &= \binom{\ell_d + j - 2}{j} \frac{j\ell_d - j + 1}{j+1} + \sum_{\nu=0}^{j-1} \binom{\ell_d + \nu - 2}{\nu} \left(\sum_{\mu=0}^{j} \ell_{d-j+\mu}\right). \end{aligned}$$

Note that for each $\mu' \in [0, j-1]$, there are exactly $\mu' + 1$ summands containing $\ell_{d-j+\mu'}$, namely

$$\binom{\ell_d - 2}{0} \ell_{\mu'}, \binom{\ell_d - 1}{1} \ell_{\mu'}, \ldots, \binom{\ell_d + \mu' - 2}{\mu'} \ell_{\mu'}.$$

Additionally, there are exactly j summands containing ℓ_d, namely

$$\binom{\ell_d - 2}{0}\ell_d, \binom{\ell_d - 1}{1}\ell_d, \dots, \binom{\ell_d + j - 3}{j-1}\ell_d.$$

As a result,

$$\begin{aligned}\theta_{d-j} = &\binom{\ell_d + j - 2}{j}\frac{j\ell_d - j + 1}{j+1} + \sum_{\nu'=0}^{j-1}\binom{\ell_d + \nu' - 2}{\nu'}\ell_d \\ &+ \sum_{\mu_1=0}^{j-1}\left(\sum_{\mu_2=0}^{\mu_1}\binom{\ell_d + \mu_2 - 2}{\mu_2}\right)\ell_{d-j+\mu_1},\end{aligned}$$

$$= \binom{\ell_d + j - 2}{j}\frac{j\ell_d - j + 1}{j+1} + \binom{\ell_d + j - 2}{j-1}\ell_d + \sum_{\mu_1=0}^{j-1}\binom{\ell_d + \mu_1 - 1}{\mu_1}\ell_{d-j+\mu_1}.$$

Next, we see that

$$\binom{\ell_d + j - 2}{j}\frac{j\ell_d - j + 1}{j+1} = \binom{\ell_d + j - 2}{j}\frac{j\ell_d + 1}{j+1} - \binom{\ell_d + j - 2}{j}\frac{j}{j+1},$$

and

$$\binom{\ell_d + j - 2}{j-1}\ell_d = \binom{\ell_d + j - 2}{j-1}\frac{j\ell_d + 1}{j+1} + \binom{\ell_d + j - 2}{j-1}\frac{\ell_d - 1}{j+1}.$$

Noting that $\binom{\ell_d+j-2}{j} + \binom{\ell_d+j-2}{j-1} = \binom{\ell_d+j-1}{j}$, it follows that

$$\begin{aligned}\theta_{d-j} = &\binom{\ell_d + j - 1}{j}\frac{j\ell_d + 1}{j+1} + \binom{\ell_d + j - 2}{j-1}\frac{\ell_d - 1}{j+1} - \binom{\ell_d + j - 2}{j}\frac{j}{j+1} \\ &+ \sum_{\mu_1=0}^{j-1}\binom{\ell_d + \mu_1 - 1}{\mu_1}\ell_{d-j+\mu_1}.\end{aligned}$$

However,

$$
\begin{aligned}
&\binom{\ell_d+j-2}{j-1}\frac{\ell_d-1}{j+1}-\binom{\ell_d+j-2}{j}\frac{j}{j+1}\\
&=\frac{(\ell_d+j-2)!(\ell_d-1)}{(j-1)!(\ell_d-1)!(j+1)}-\frac{(\ell_d+j-2)!j}{j!(\ell_d-2)!(j+1)},\\
&=\frac{(\ell_d+j-2)!}{(j-1)!(\ell_d-2)!(j+1)}-\frac{(\ell_d+j-2)!}{(j-1)!(\ell_d-2)!(j+1)},\\
&=0,
\end{aligned}
$$

and thus

$$\theta_{d-j}=\binom{\ell_d+j-1}{j}\frac{j\ell_d+1}{j+1}+\sum_{\mu_1=0}^{j-1}\binom{\ell_d+\mu_1-1}{\mu_1}\ell_{d-j+\mu_1}=\lambda_{d-j},$$

which proves the claim. $\square$

Sylvester then applies his formula of obliteration to the question of determining non-zero solutions of equations which define the Tschirnhaus complete intersections $\tau_{1,\dots,m-1}$, including his Triangle of Obliteration. We omit his discussion here as the bounds he obtains are succeeded by the bounds of [Bra1975], [Wol2021], [Sut2021C], and Chapter 4.

Chapter 4

Upper Bounds on Resolvent Degree

We are now ready to establish our bounds on $\mathrm{RD}(n)$ and will use the general approach outlined in Remark 2.2.6 throughout the chapter. In Section 4.1, we will apply the iterated polar cone constructions of Section 3.2 directly to obtain bounds for $m \in [6, 12]$. At a macro level, the bounds in this small m region minimize the ambient dimension required and at the cost of solving larger degree polynomials. In Section 4.2, we use the geometric obliteration algorithm of Section 3.3 to obtain bounds for $m \in [13, 17] \cup [22, 25]$. In this intermediate range, we minimize the degree of the polynomials we solve at the cost of letting ambient dimension grow. In Section 4.3, we introduce a general approach using a combinatorial existence condition for k-planes on intersections of hypersurfaces; this approach gives our bounds for $m \in [18, 21]$ and $m \geq 26$. For the general case, the general existence condition and the dimension of the corresponding moduli space outperform any of our explicit constructions using iterated polar cones.

Note that the geometric obliteration algorithm can also be used to obtain the bounds for $m \in [18, 21]$. However, the bounds for $m \in [18, 21]$ were established in [Sut2021C] before the recovery of the obliteration algorithms in [HS2021].

In Section 4.4, we discuss obstructions to obtaining further bounds from iterated polar cone methods. In Section 4.5, give an upper bound on $G'(m)$ in terms of elementary functions. In Section 4.6, we compare our bounding function $G'(m)$ with the previous best bounding function $F(m)$ constructed by Wolfson in [Wol2021]. Finally, in Section 4.7, we address several remaining questions.

4.1 Using Iterated Polar Cones Directly

The results obtained in this section were first established in Section 3.2 of [Sut2021C].

Theorem 4.1.1. ***(The*** $n-6$ ***Bound)***
For $n \geq 21$, $\mathrm{RD}(n) \leq n-6$.

Proof. First, suppose that we can determine a plane $\Lambda \subseteq \tau_{1,2,3}^{\circ}$ over an extension of K_n of low resolvent degree. Then, $\Lambda \cap \tau_{1,2,3,4,5}$ has degree 20 and we can determine a point of $\Lambda \cap \tau_{1,2,3,4,5} \subseteq \tau_{1,2,3,4,5}^{\circ}$ by solving a polynomial of degree 20.

Given a 2-polar point (P_0, P_1, P_2) of $\tau_{1,2,3}$, Lemma 3.2.5 yields that $\Lambda(P_0, P_1, P_2) \subseteq \tau_{1,2,3}$ is a plane. We will show that we can determine such a 2-polar point by solving polynomials of degree at most 12 whenever $n \geq 19$; indeed, it suffices to consider the $n = 19$ case.

Recall that when $n = 19$, we work in $\mathbb{P}^{18}_{K_n}$ (as is established in Remark 2.2.4 and Definition 2.2.5). We begin by passing to a hyperplane H that does not contain $[1 : 0 : \cdots : 0]$, hence $H \cap \tau_{1,2,3} = H \cap \tau_{1,2,3}^{\circ}$. We can determine a point P_0 of $H \cap \tau_{1,2,3}$ by solving at most a sextic polynomial. Further, the polar cone $\mathcal{C}(H \cap \tau_{1,2,3}; P_0)$ has type $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and thus

$$\dim(\mathcal{C}(H \cap \tau_{1,2,3}; P_0)) \geq 18 - 7 = 11.$$

Consequently, we can determine a point P_1 of $\mathcal{C}\left(H \cap \tau_{1,2,3}; P_0\right) \setminus \{P\}$ by solving a degree 12 polynomial.

The second polar cone $\mathcal{C}^2\left(H \cap \tau_{1,2,3}; P_0, P_1\right)$ has type $\begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 7 \end{bmatrix}$. We work inside the $\mathbb{P}^{11}$ given by the vanishing of the seven hyperplanes. Note that τ_2 is a quadric hypersurface in this $\mathbb{P}^{11}$ and thus Proposition 3.2.11 yields that we can determine a 5-plane $\Lambda' \subseteq \tau_2 \cap \mathbb{P}^{11}$ over a quadratic extension. Further, $\Lambda' \cap \mathcal{C}^2\left(H \cap \tau_{1,2,3}; P_0, P_1\right)$ has type $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ inside Λ'. Since

$$\dim\left(\Lambda' \cap \mathcal{C}^2\left(H \cap \tau_{1,2,3}; P_0, P_1\right)\right) \geq 5 - 3 = 2,$$

we can determine a point P_2 of $\Lambda' \cap \mathcal{C}^2\left(H \cap \tau_{1,2,3}; P_0, P_1\right) \setminus \Lambda\left(P_0, P_1\right)$ by solving a degree 12 polynomial. We observe that $\left(P_0, P_1, P_2\right)$ is a 2-polar point of $H \cap \tau_{1,2,3}$ by construction, which finishes the proof. $\square$

Remark 4.1.2. (Fixing Gaps in [Che1954])

As was noted in Section 1.2, Chebotarev gave an argument that $\mathrm{RD}(n) \leq n-6$ for $n \geq 21$ in [Che1954] (see Appendix B.3 for an English translation). However, his argument had gaps; indeed, similar gaps are present in [Wim1927] (see Appendix B.2 for an English translation), which was the primary literature cited in [Che1954]. More specifically, there are multiple situations where Chebotarev and Wiman assume without proof that certain intersections of hypersurfaces in affine space are generic. The proof of Theorem 4.1.1 obtains Chebotarev's bound using similar geometric intuition; this argument can be suitably modified to provide a geometric proof of Wiman's claims as well. Additionally, Dixmier gave an algebraic proof of Wiman's bound in the appendix of [Dix1993].

Remark 4.1.3. (Notation for Theorem 4.1.4)

The crux of the previous theorem was determining a plane on $\tau^{\circ}_{1,2,3}$; we now proceed to determining k-planes on $\tau^{\circ}_{1,2,3,4}$. We now introduce notation which will improve fluency in the proof of Theorem 4.1.4. Specifically, we introduce two functions $\rho, \eta : [1,7] \to \mathbb{Z}$ such that for each k, $\rho(k)$ is the ambient dimension needed in our method to determine a k-plane on $\tau^{\circ}_{1,2,3,4}$ over an extension determined by solving polynomials of degree at most $\eta(k)$. The values of ρ and η are as follows:

Table 4.1: Auxiliary Functions for Theorem 4.1.4

k	1	2	3	4	5	6	7
$\rho(k)$	25	60	264	806	1773	8905	34,546
$\eta(k)$	36	108	324	972	2916	8748	26,244

Additionally, we will often need to refer to a subcollection of hypersurfaces defining an intersection of hypersurfaces V. When

$$V = \bigcap_{j=1}^{d} \bigcap_{\ell=1}^{k_\ell} \mathbb{V}\left(f_{j,\ell}\right),$$

with $\deg\left(f_{j,\ell}\right) = j$ for each ℓ, we set

$$V_j := \bigcap_{\ell=1}^{k_j} \mathbb{V}\left(f_{j,\ell}\right).$$

for each $j \in [1,d]$. In the proof of Theorem 4.1.4, we will use this notation in the context of iterated polar cones. For example, $\mathcal{C}^k\left(V; P_0, \ldots, P_{k-1}\right)_2$ would be the intersection of all of the quadric hypersurfaces defining $\mathcal{C}^k\left(V; P_0, \ldots, P_{k-1}\right)$.

Theorem 4.1.4. ***(The Bounds for** $m \in [7, 12]$**)***

1. *For* $n \geq 109$*,* $\mathrm{RD}(n) \leq n - 7$*.*

2. *For* $n \geq 325$*,* $\mathrm{RD}(n) \leq n - 8$*.*

3. *For each* $m \in [9, 12]$ *and* $n > \frac{(m-1)!}{24}$*,* $\mathrm{RD}(n) \leq n - m$*.*

We note that Theorem 4.1.4 first appears as Theorem 3.10 in [Sut2021C], which includes the cases of $m = 13, 14$. The bounds obtained for $m = 13, 14$ are valid, but are improved upon using techniques in Section 4.2. Additionally, while the core proof ideas hold in these cases, they are also the most notationally cumbersome cases and with both these observations in mind, we omit these cases here and refer the interested reader to the original source.

Proof. We claim for each $k \in [1, 7]$, we can determine a k-plane $\Lambda \subseteq \tau^{\circ}_{1,2,3,4}$ over an extension of K_n of resolvent degree at most $\mathrm{RD}(\eta(k))$. Given this claim, $\Lambda \cap \tau_{1,\dots,k+4}$ has degree $\frac{(k+4)!}{24}$ and so we can determine a point of $\Lambda \cap \tau_{1,\dots,k+4} \subseteq \tau^{\circ}_{1,\dots,k+4}$ by solving a polynomial of degree at most $\frac{(d+4)!}{24}$, which will yield the necessary bounds. We note explicitly that the $n - 7$ and $n - 8$ bounds correspond to the cases of $k = 2, 3$, respectively, and that in these cases $\eta(k) > \frac{(k+4)!}{24}$.

It remains to show that we can determine the requisite k-planes; from Lemma 3.2.5, it suffices to construct a k-polar point of $\tau_{1,2,3,4}$ which avoids $[1 : 0 : \cdots : 0]$. For each k, we need only consider the case where $n = \rho(k)$, as we can always restrict to a $\mathbb{P}^{\rho(k)-1}_{K_n} \subseteq \mathbb{P}^{n-1}_{K_n}$ whenever $n > \rho(k)$. Further, we always immediately pass to a hyperplane which does not contain $[1 : 0 : \cdots : 0]$ and, to simplify notation, all computations that follow begin within this hyperplane. More explicitly, for each k, we begin in a $\mathbb{P}^{\rho(k)-2}_{K_n}$. Additionally, the extensions will be enumerated as L_j and we begin with L_1/K_n for each k. Similarly, the k-planes will be enumerated as Λ_ℓ and we always begin with Λ_1. Proposition 3.2.10 yields the type of each iterated polar cone $\mathcal{C}^k(\tau_{1,2,3,4}; P_0, \dots, P_{k-1}) \subseteq \mathbb{P}^{\rho(k)-2}_{K_n}$. We now handle each case individually.

Case: $(k = 1)$**.** Recall that $\rho(1) = 25$ and so thus we work in $\mathbb{P}^{23}_{K_n}$. We first determine a point P_0 of $\tau_{1,2,3,4}$ by solving a polynomial of degree at most 24. Now, the polar cone $\mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)$ is of type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. Note that $\mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_1$ is an intersection of 4 hyperplanes and so $\mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_1 \cong \mathbb{P}^{19}$. Further, $\mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_2 \cap \mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_1$ is of type $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ inside $\mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_1$ and

$$19 = (4+1)(3) + 4,$$

hence Proposition 3.2.11 allows us to determine a 4-plane $\Lambda_1 \subseteq \mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_2 \cap \mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)_1$ by solving a polynomial of degree at most 8. Observe that $\Lambda_1 \cap \mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)$ has type $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ inside Λ_1. Consequently,

$$\dim\left(\Lambda_1 \cap \mathcal{C}\left(\tau_{1,2,3,4}; P_0\right)\right) \geq 4 - 3 \geq 1$$

and we can determine a point P_1 of $\mathcal{C}\left(\tau_{1,2,3,4}; P_0\right) \setminus \{P_0\}$ by solving a polynomial of degree at most $36 = 4 \cdot 3^2$. In particular, we have determined a 1-polar point (P_0, P_1) of $\tau_{1,2,3,4}$ by solving polynomials of degree at most 36.

Case: $(k = 2)$**.** Note that $\rho(2) = 60$, so we work in $\mathbb{P}^{58}_{K_n}$. From the $k = 1$ case, we pass to an extension L_1/K_n with $\mathrm{RD}(L_1/K_n) \leq \mathrm{RD}(36)$ and assume that we have a 1-polar point (P_0, P_1). The second polar cone $\mathcal{C}^2\left(\tau_{1,2,3,4}; P_0, P_1\right)$ has type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 3 & 6 & 10 \end{bmatrix}$. Observe that

$\mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1)_2 \cap \mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1)_1$ has type $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ inside $\mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1)_1 \cong \mathbb{P}^{48}$. Noting that

$$48 = (6+1)(6) + 6,$$

we can apply Proposition 3.2.11 to determine a 6-plane $\Lambda_1 \subseteq \mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1)_2 \cap \mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1)_1$ by solving a polynomial of degree at most $64 = 2^6$. Further, $\Lambda_1 \cap \mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1)$ has type $\begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}$ inside Λ_1, hence

$$\dim\left(\mathcal{C}^2(\Lambda_1 \cap \tau_{1,2,3,4}; P_0, P_1)\right) \geq 6 - 4 \geq 2.$$

As a result, we can determine a point P_2 of $\mathcal{C}^2(\tau_{1,2,3,4}; P_0, P_1) \setminus \Lambda(P_0, P_1)$ by solving a polynomial of degree at most $108 = 4 \cdot 3^3$ and (P_0, P_1, P_2) is a 2-polar point of $\tau_{1,2,3,4}$ by construction.

Case: $(k = 3)$. Recall that $\rho(3) = 264$ and hence we work in $\mathbb{P}^{262}_{K_n}$. Using the $k = 2$ case, we pass to an extension L_1/K_n with $\mathrm{RD}(L_1/K_n) \leq \mathrm{RD}(108)$ and assume that we have a 2-polar point (P_0, P_1, P_2). The third polar cone $\mathcal{C}^3(\tau_{1,2,3,4}; P_0, P_1, P_2)$ is of type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & 10 & 20 \end{bmatrix}$. Note that $\mathcal{C}^3(\tau_{1,2,3,4}; P_0, P_1, P_2)_2 \cap \mathcal{C}^3(\tau_{1,2,3,4}; P_0, P_1, P_2)_1$ has type $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$ inside $\mathcal{C}^3(\tau_{1,2,3,4}; P_0, P_1, P_2)_1 \cong \mathbb{P}^{242}$; however, $256 = 2^8$ is the largest power of 2 less than $324 = 4 \cdot 3^4$. As a result, we split $\mathcal{C}^3(\tau_{1,2,3,4}; P_0, P_1, P_2)_2$ into two intersections of quadrics:

$$\mathcal{C}^3(\tau_{1,2,3,4}; P_0, P_1, P_2)_2 = W_1 \cap W_2,$$

where W_1 has type $\begin{bmatrix} 2 \\ \\ 2 \end{bmatrix}$ and W_2 has type $\begin{bmatrix} 2 \\ \\ 8 \end{bmatrix}$. Observe that

$$242 = (80+1)(2) + 80$$

and thus Proposition 3.2.11 allows us to determine an 80-plane $\Lambda_1 \subseteq W_1 \cap \mathcal{C}^3\left(\tau_{1,2,3,4}; P_0, P_1, P_2\right)_1$ by solving at most a quartic polynomial. Further, $\Lambda_1 \cap W_2$ has type $\begin{bmatrix} 2 \\ \\ 8 \end{bmatrix}$ inside Λ_1 and

$$80 = (8+1)(8) + 8,$$

so Proposition 3.2.11 allows us to determine an 8-plane $\Lambda_2 \subseteq \Lambda_1 \cap W_2$ by solving a polynomial of degree at most 256. Now, $\Lambda_2 \cap \mathcal{C}^3\left(\tau_{1,2,3,4}; P_0, P_1, P_2\right)$ has type $\begin{bmatrix} 4 & 3 \\ & \\ 1 & 4 \end{bmatrix}$ inside Λ_2. It follows that

$$\dim\left(\Lambda_2 \cap \mathcal{C}^3\left(\tau_{1,2,3,4}; P_0, P_1, P_2\right)\right) \geq 8 - 5 \geq 3$$

and we can determine a rational point P_3 of $\mathcal{C}^3\left(\tau_{1,2,3,4}; P_0, P_1, P_2\right) \setminus \Lambda\left(P_0, P_1, P_2\right)$ by solving a polynoimal of degree at most 324. By construction, (P_0, P_1, P_2, P_3) is a suitable 3-polar point.

Case: $(k = 4)$. Observe that $\rho(4) = 806$ and so we work in $\mathbb{P}^{804}_{K_n}$. By the $k = 3$ case, we pass to an extension L_1/K_n with $\mathrm{RD}\left(L_1/K_n\right) \leq \mathrm{RD}(324)$ and assume that we have a 3-polar point (P_0, P_1, P_2, P_3). The fourth polar cone $\mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)$ has type

$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 5 & 15 & 35 \end{bmatrix}$. As a result, $\mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)_2 \cap \mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)_1$ is of type $\begin{bmatrix} 2 \\ 15 \end{bmatrix}$ inside $\mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)_1 \cong \mathbb{P}^{769}$. We see that $512 = 2^9$ is the largest power of 2 smaller than $972 = 4 \cdot 3^5$ and so we split $\mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)_2$ into two intersections of quadrics:

$$\mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)_2 = W_1 \cap W_2,$$

where W_1 has type $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and W_2 has type $\begin{bmatrix} 2 \\ 9 \end{bmatrix}$. We have that

$$769 = (109 + 1)(6) + 109,$$

and so Proposition 3.2.11 yields a 109-plane $\Lambda_1 \subseteq W_1 \cap \mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)_1$ by solving a polynomial of degree at most 64. Similarly, $\Lambda_1 \cap W_2$ has type $\begin{bmatrix} 2 \\ 9 \end{bmatrix}$ inside Λ_1 and

$$109 = (10 + 1)(9) + 10,$$

hence Proposition 3.2.11 allows us to determine a 10-plane $\Lambda_2 \subseteq \Lambda_1 \cap W_2$ by solving a polynomial of degree at most 512. Consequently, $\Lambda_2 \cap \mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)$ has type $\begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix}$ inside Λ_2 and

$$\dim\left(\Lambda_2 \cap \mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right)\right) \geq 10 - 6 \geq 4.$$

Accordingly, we can determine a point P_4 of $\mathcal{C}^4\left(\tau_{1,2,3,4}; P_0, \ldots, P_3\right) \backslash \Lambda\left(P_0, \ldots, P_3\right)$ by solving a polynomial of degree at most 972 and $(P_0, \ldots, P_4)$ is a 4-polar point of $\tau_{1,2,3,4}$ by construction.

Case: **($k = 5$).** Observe that $\rho(5) = 1773$, hence we work in $\mathbb{P}^{1771}_{K_n}$. From the $k = 4$ case, we pass to an extension L_1/K_n with $\mathrm{RD}\left(L_1/K_n\right) \leq \mathrm{RD}(972)$ and assume that we have a 4-polar point $(P_0, \ldots, P_4)$. The fifth polar cone $\mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)$ has type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 6 & 21 & 56 \end{bmatrix}$. Note that $\mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)_2 \cap \mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)_1$ has type $\begin{bmatrix} 2 \\ 21 \end{bmatrix}$ inside $\mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)_1 \cong \mathbb{P}^{1715}$. We have that $2048 = 2^{11}$ is the largest power of 2 less than $2916 = 4 \cdot 3^6$ and so we split $\mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)$ into two intersections of quadrics:

$$\mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right) = W_1 \cap W_2,$$

where W_1 has type $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$ and W_2 has type $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$. Observe that

$$1715 = (155 + 1)(10) + 155,$$

and thus Proposition 3.2.11 yields a 155-plane $\Lambda_1 \subseteq W_1 \cap \mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)_1$ by solvinga polynomial of degree at most 1024. Further, $\Lambda_1 \cap W_2$ has type $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$ inside Λ_1 and

$$155 = (12 + 1)(11) + 12,$$

hence we can apply Proposition 3.2.11 to obtain a 12-plane $\Lambda_2 \subseteq \Lambda_1 \cap W_2$ by solving a polynomial of degree at most 2048. It follows that $\Lambda_2 \cap \mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)$ has type $\begin{bmatrix} 4 & 3 \\ 1 & 6 \end{bmatrix}$ inside Λ_2 and

$$\dim\left(\Lambda_2 \cap \mathcal{C}^5\left(\tau_{1,2,3,4}; P_0, \ldots, P_4\right)\right) \geq 12 - 7 \geq 5.$$

Consequently, we can determine a point P_5 of $\mathcal{C}^5(\tau_{1,2,3,4}; P_0, \ldots, P_4) \setminus \Lambda(P_0, \ldots, P_4)$ by solving a polynomial of degree at most 2916 and $(P_0, \ldots, P_5)$ is a 5-polar point of $\tau_{1,2,3,4}$ by construction.

Case: $(k = 6)$. Recall that $\rho(6) = 8905$ and so we work in $\mathbb{P}^{8903}_{K_n}$. Using the $k = 5$ case, we pass to an extension L_1/K_n with $\mathrm{RD}(L_1/K_n) \leq \mathrm{RD}(2916)$ and assume that we have a 5-polar point $(P_0, \ldots, P_5)$. The sixth polar cone $\mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)$ has type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 7 & 28 & 84 \end{bmatrix}$. It follows that $\mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)_2 \cap \mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)_1$ has type $\begin{bmatrix} 2 \\ 28 \end{bmatrix}$ inside $\mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)_1 \cong \mathbb{P}^{8819}$. Note that $8192 = 2^{13}$ is the largest power of 2 less than $8748 = 4 \cdot 3^7$ and thus we split $\mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)_2$ into three intersections of quadrics:

$$\mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)_2 = W_1 \cap W_2 \cap W_3,$$

where W_1 has type $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and both W_2, W_3 have type $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$. We have that

$$8819 = (2939 + 1)(2) + 2939,$$

and so Proposition 3.2.11 yields that we can determine a 2939-plane $\Lambda_1 \subseteq W_1 \cap \mathcal{C}^6(\tau_{1,2,3,4}; P_0, \ldots, P_5)_1$ by solving at most a quartic polynomial. Additionally, $\Lambda_1 \cap W_2$ has type $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$ inside Λ_1 and

$$2939 = (209 + 1)(13) + 209.$$

We then apply Proposition 3.2.11 to determine a 209-plane $\Lambda_2 \subseteq \Lambda_1 \cap W_2$ by solving a polynomial of degree at most 8192. Similarly, $\Lambda_2 \cap W_3$ has type $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$ inside Λ_2 and

$$209 = (14+1)(13) + 14,$$

hence we can determine a 14-plane $\Lambda_3 \subseteq \Lambda_2 \cap W_3$ by solving a polynomial of degree at most 8192, from Proposition 3.2.11. Note that $\Lambda_3 \cap \mathcal{C}^6(\tau_{1,2,3,4}; P_0, \dots, P_5)$ has type $\begin{bmatrix} 4 & 3 \\ 1 & 7 \end{bmatrix}$ inside Λ_3 and

$$\dim\left(\Lambda_3 \cap \mathcal{C}^6(\tau_{1,2,3,4}; P_0, \dots, P_5)\right) \geq 14 - 8 \geq 6.$$

As a result, we can determine a point P_6 of $\mathcal{C}^6(\tau_{1,2,3,4}; P_0, \dots, P_5) \setminus \Lambda(P_0, \dots, P_5)$ by solving a polynomial of degree at most 8748 and $(P_0, \dots, P_6)$ is a 6-polar point of $\tau_{1,2,3,4}$ by construction.

Case: $(k = 7)$. Note that $\rho(7) = 34546$, so we work in $\mathbb{P}^{34544}_{K_n}$. By the $k = 6$ case, we pass to an extension L_1/K_n with $\mathrm{RD}(L_1/K_n) \leq \mathrm{RD}(8748)$ and assume that we have a 6-polar point $(P_0, \dots, P_6)$. The seventh polar cone $\mathcal{C}^7(\tau_{1,2,3,4}; P_0, \dots, P_6)$ has type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 8 & 36 & 120 \end{bmatrix}$. Observe that $\mathcal{C}^7(\tau_{1,2,3,4}; P_0, \dots, P_6)_2 \cap \mathcal{C}^7(\tau_{1,2,3,4}; P_0, \dots, P_6)_1$ is of type $\begin{bmatrix} 2 \\ 26 \end{bmatrix}$ inside $\mathcal{C}^7(\tau_{1,2,3,4}; P_0, \dots, P_6)_1 \cong \mathbb{P}^{34424}$. Further, $16384 = 2^{14}$ is the largest power of 2 less than $26244 = 4 \cdot 3^8$. Correspondingly, we split $\mathcal{C}^7(\tau_{1,2,3,4}; P_0, \dots, P_6)$ into three intersection of quadrics:

$$\mathcal{C}^7(\tau_{1,2,3,4}; P_0, \dots, P_6)_2 = W_1 \cap W_2 \cap W_3,$$

where W_1 has type $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ and both W_2, W_3 have type $\begin{bmatrix} 2 \\ 14 \end{bmatrix}$. Observe that

$$34424 = (3824 + 1)(8) + 3824,$$

hence Proposition 3.2.11 yields a 3824-plane $\Lambda_1 \subseteq W_1 \cap \mathcal{C}^7\left(\tau_{1,2,3,4}; P_0, \ldots, P_6\right)_1$ by solving a polynomial of degree at most 256. Similarly, $\Lambda_1 \cap W_2$ has type $\begin{bmatrix} 2 \\ 14 \end{bmatrix}$ inside Λ_1 and

$$3824 = (254 + 1)(14) + 254.$$

As a result, we can apply Proposition 3.2.11 to determine a 254-plane $\Lambda_2 \subseteq \Lambda_1 \cap W_2$ by solving a polynomial of degree at most 16384. Further, $\Lambda_2 \cap W_3$ has type $\begin{bmatrix} 2 \\ 14 \end{bmatrix}$ inside Λ_2 and

$$254 = (16 + 1)(14) + 16,$$

and so Proposition 3.2.11 yields that we can determine a 16-plane $\Lambda_3 \subseteq \Lambda_2 \cap W_3$ by solving a polynomial of degree at most 16384. Finally, $\Lambda_3 \cap \mathcal{C}^7\left(\tau_{1,2,3,4}; P_0, \ldots, P_6\right)$ has type $\begin{bmatrix} 4 & 3 \\ 1 & 8 \end{bmatrix}$ inside Λ_3 and

$$\dim\left(\Lambda_3 \cap \mathcal{C}^7\left(\tau_{1,2,3,4}; P_0, \ldots, P_6\right)\right) \geq 16 - 9 \geq 7,$$

from which it follows that we can determine a point P_7 of $\mathcal{C}^7\left(\tau_{1,2,3,4}; P_0, \ldots, P_6\right) \setminus \Lambda\left(P_0, \ldots, P_6\right)$ by solving a polynomial of degree at most 26244. By construction, $(P_0, \ldots, P_7)$ is a 7-polar point of $\tau_{1,2,3,4}$. $\square$

4.2 Using the Obliteration Algorithm

We will now obtain bounds of the form $\mathrm{RD}(n) \leq n - m$ for $m \in [13, 17] \cup [22, 25]$. For $m \in [13, 17]$, our bounds will come from determining an $(m-6)$-plane on $\tau^{\circ}_{1,2,3,4,5}$. Similarly, for $m \in [22, 25]$, our bounds will come from determining an $(m-7)$-plane on $\tau^{\circ}_{1,2,3,4,5,6}$. In each case, the core idea is to apply the geometric obliteration algorithm to the iterated polar cone in question. However, we will use a slight modification which better suits our purpose.

Remark 4.2.1. (A Modification of the Geometric Obliteration Algorithm) Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. Recall that successive uses of Proposition 3.3.10 yield that

$$g(V) = g\left(V_1^{\mathrm{Syl}}\right) = \cdots = g\left(V_{d-3}^{\mathrm{Syl}}\right) = g\left(V_{d-2}^{\mathrm{Syl}}\right),$$

and that V_{d-2}^{Syl} is an intersection of type $\begin{bmatrix} 2 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$, where each V_j^{Syl} is the j^{th} Sylvester reduction of V, as in Definition 3.3.9. We could continue to iteratively compute partial Sylvester reductions of V_{d-2}^{Syl} until there is a single quadric left, at which point we would only need to solve one final quadratic polynomial.

However, we also note that $\deg\left(V_{d-2}^{\mathrm{Syl}}\right)$ is 2^{λ_2} and so we could determine a point of V_{d-2}^{Syl} by solving a polynomial of degree 2^{λ_2} whenever $r \geq \lambda_2 + \lambda_1$. Consequently, we obtain a slight improvement in the forthcoming bounds by reducing only to a j^{th} partial Sylvester reduction of V_{d-2}^{Syl} for some $j < \lambda_2$, instead of running the geometric obliteration algorithm exactly. With this in mind, we introduce the following notation.

Definition 4.2.2. (Optimal Reduction of Tschirnhaus Complete Intersections) For each $d \geq 3$ and $m \geq d+2$, consider

$$W = \left(\mathcal{C}^{m-d-1}\left(\tau_{1,\dots,d}; P_0, \dots, P_{m-d-2}\right)\right)_{d-2}^{\mathrm{Syl}},$$

a $(d-2)^{nd}$ Sylvester reduction of an $(m-d-1)^{st}$ polar cone of $\tau_{1,\dots,d}$. Note that W is of type $\begin{bmatrix} 2 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$. For each $j \in [1, \lambda_2 - 1]$, a j^{th} partial Sylvester reduction $W(2;j)$ of W has type $\begin{bmatrix} 2 & 1 \\ \lambda_2 - j & \lambda_1 + \sum\limits_{\nu=\lambda_2-j}^{\lambda_2-1} \nu \end{bmatrix}$. In particular, $\deg\left(W(2;j)\right) = 2^{\lambda_2 - j}$. For each $j \in [0, \lambda_2 - 1]$, set

$$\xi(m,d;j) := \max\left\{(m-d-+1) + (\lambda_2 - j) + \left(\lambda_1 + \sum_{\nu=\lambda_2-j}^{\lambda_2-1} \nu\right), 2^{\lambda_2-j} + 1\right\}.$$

We can now define **optimal reduction bound of** $\tau_{1,\dots,d}$ **for** m as

$$\Xi(m,d) := \min\left\{\xi(m,d;j) \mid j \in [0, \lambda_2 - 1]\right\}.$$

In particular, $\Xi(m,d)$ is defined exactly so that for $n \geq \Xi(m,d)$, we can determine an $(m-d-1)^{st}$ polar point of $\tau_{1,\dots,d}^{\circ}$ in $\mathbb{P}_{K_n}^{n-1}$ over an extension K'/K_n with $\mathrm{RD}\left(K'/K_n\right) \leq \mathrm{RD}\left(\Xi(m,d)\right)$.

Additionally, we see that for fixed d, $\Xi(m,d)$ is non-decreasing in m. This can be seen geometrically from the fact that if $(P_0, \dots, P_{m-d-1})$ is an $(m-d-1)$-polar point of $\tau_{1,\dots,d}$, then $(P_0, \dots, P_{m-d-2})$ must be an $(m-d-2)$-polar point of $\tau_{1,\dots,d}$.

We are now ready to state and prove our bounds.

Theorem 4.2.3. ***(Bounds from the Geometric Obliteration Algorithm)***

1. *For* $n \geq 5,250,198$*,* $\mathrm{RD}(n) \leq n-13$*.*

2. *For each* $m \in [14,17]$ *and* $n > \frac{(m-1)!}{120}$*,* $\mathrm{RD}(n) \leq n-m$*.*

3. *For* $n \geq 381,918,437,071,508,900$*,* $\mathrm{RD}(n) \leq n-22$*.*

4. *For each* $m \in [23,25]$ *and* $n > \frac{(m-1)!}{720}$*,* $\mathrm{RD}(n) \leq n-m$*.*

Proof. For each $m \in [13,17]$, we set

$$n_m = \max\left\{\Xi(m,5), \frac{(m-1)!}{120}+1\right\}.$$

Similarly, for $m \in [22,25]$, we set

$$n_m = \max\left\{\Xi(m,6), \frac{(m-1)!}{720}+1\right\}.$$

In each case, it suffices to show the claim when $n = n_m$, as we can always restrict to a projective space of dimension $n_m - 1$ in $\mathbb{P}^{n-1}$ for $n > n_m$. Further, note that $n_m = \Xi(m,5)$ exactly when $m = 13$ and $n_m = \Xi(m,6)$ exactly when $m = 22$.

Let us first consider the case where $m \in [13,17]$ and take $H \subseteq \mathbb{P}^{n_m-1}_{K_n}$ to be a hyperplane which does not contain $[1:0:\cdots:0]$. Note that $H \cap \tau_{1,\ldots,5} = H \cap \tau^\circ_{1,\ldots,5}$ inside $H \cong \mathbb{P}^{n_m-2}_{K_n}$. Since $\Xi(m,5) \geq \Xi(m-1,5)$, we can assume that we have an $(m-7)$-polar point $(P_0,\ldots,P_{m-7})$ of $H \cap \tau^\circ_{1,\ldots,5}$. Consider the minimal j such that $\Xi(m,5) = \xi(m,5;j)$. By definition of $\xi(m,5;j)$, we have that

$$\dim\left(\left(\left(\mathcal{C}^{m-6}\left(H \cap \tau^\circ_{1,\ldots,5}; P_0,\ldots,P_{m-7}\right)\right)_3^{\mathrm{Syl}}\right)^{\mathrm{Syl}}(2;j)\right) \geq m-6.$$

Since $\dim\left(\Lambda\left(P_0,\ldots,P_{m-7}\right)\right)=m-7$, we can determine a point P_{m-6} of

$$\mathcal{C}^{m-6}\left(H\cap\tau^{\circ}_{1,\ldots,5};P_0,\ldots,P_{m-7}\right)\setminus\Lambda\left(P_0,\ldots,P_{m-7}\right),$$

by solving polynomials of degree at most $\Xi(m;5)$. By construction, $(P_0,\ldots,P_{m-6})$ is an $(m-6)$-polar point of $\tau^{\circ}_{1,\ldots,5}$ and $\Lambda=\Lambda\left(P_0,\ldots,P_{m-6}\right)\subseteq\tau^{\circ}_{1,\ldots,5}$ is an $(m-6)$-plane. Consequently, we can determine a point of $\Lambda\cap\tau_{1,\ldots,m-1}\subseteq\tau^{\circ}_{1,\ldots,m-1}$ by solving a polynomial of degree $\frac{(m-1)!}{120}$.

We now consider the similar case of $m\in[22,25]$. Take $H\subseteq\mathbb{P}^{n_m-1}_{K_n}$ to be a hyperplane which does not contain $[1:0:\cdots:0]$. Observe that $H\cap\tau_{1,\ldots,6}=H\cap\tau^{\circ}_{1,\ldots,6}$ inside $H\cong\mathbb{P}^{n_m-2}_{K_n}$. As $\Xi(m,6)\geq\Xi(m-1,6)$, we assume we have an $(m-8)$-polar point $(P_0,\ldots,P_{m-8})$ of $H\cap\tau^{\circ}_{1,\ldots,6}$. Consider the minimal j such that $\Xi(m,6)=\xi(m,6;j)$; then, we have that

$$\dim\left(\left(\left(\mathcal{C}^{m-7}\left(H\cap\tau^{\circ}_{1,\ldots,6};P_0,\ldots,P_{m-8}\right)\right)^{\mathrm{Syl}}_4\right)^{\mathrm{Syl}}(2;j)\right)\geq m-7.$$

Since $\dim\left(\Lambda\left(P_0,\ldots,P_{m-8}\right)\right)=m-8$, we can determine a point P_{m-7} of

$$\mathcal{C}^{m-7}\left(H\cap\tau^{\circ}_{1,\ldots,6};P_0,\ldots,P_{m-8}\right)\setminus\Lambda\left(P_0,\ldots,P_{m-8}\right),$$

by solving polynomials of degree at most $\Xi(m;6)$. Note that $(P_0,\ldots,P_{m-7})$ is an $(m-7)$-polar point of $\tau^{\circ}_{1,\ldots,6}$ by construction and thus $\Lambda=\Lambda\left(P_0,\ldots,P_{m-7}\right)\subseteq\tau^{\circ}_{1,\ldots,6}$ is an $(m-7)$-plane. As a result, we can determine a point of $\Lambda\cap\tau_{1,\ldots,m-1}\subseteq\tau^{\circ}_{1,\ldots,m-1}$ by solving a polynomial of degree $\frac{(m-1)!}{720}$.

In the following tables, we note the values of $\Xi(m,5)$ and $\frac{(m-1)!}{120}+1$ for $m\in[13,17]$ and the approximate values of $\Xi(m,6)$ and $\frac{(m-1)!}{720}+1$ for $m\in[22,25]$. The exact values of $\Xi(m,5)$ for $m\in[13,17]$ and of $\Xi(m,6)$ for $m\in[22,25]$ were computed using Algorithm A.3.6, which can be found in Appendix A.3.

Table 4.2: Values of $\Xi(m,5)$ and $\frac{(m-1)!}{120}+1$ for $m \in [13,17]$

m	$\Xi(m,5)$	$\frac{(m-1)!}{120}+1$
13	5,250,198	3,991,681
14	12,253,482	51,891,841
15	26,357,165	726,485,761
16	53,008,668	10,897,286,401
17	100,769,994	174,356,582,401

Table 4.3: Approximate Values of $\Xi(m,6)$ and $\frac{(m-1)!}{720}+1$ for $m \in [22,25]$

m	$\Xi(m,5)$	$\frac{(m-1)!}{120}+1$
22	$\sim 3.819 \times 10^{17}$	$\sim 7.096 \times 10^{16}$
23	$\sim 9.526 \times 10^{17}$	$\sim 1.561 \times 10^{18}$
24	$\sim 2.262 \times 10^{18}$	$\sim 3.591 \times 10^{19}$
25	$\sim 5.137 \times 10^{18}$	$\sim 8.617 \times 10^{20}$

$\square$

4.3 Using Moduli Spaces

The method from this section is inspired by the approach found in [Wol2021]; we will discuss this in more detail in Sections 4.4 and 4.6. This approach centers certain moduli spaces, which we now introduce.

Definition 4.3.1. (Parameter and Moduli Spaces of Hypersurfaces)

Fix $d \geq 2$. The **parameter space of degree d hypersurfaces in $\mathbb{P}^r_{\mathbb{C}}$** is

$$\mathcal{H}(d;r) \cong \mathbb{P}_{\mathbb{C}}^{\binom{r+d}{d}-1}.$$

There is a natural action of $\mathrm{PGL}(\mathbb{C}, r+1)$ on $\mathbb{P}^r_{\mathbb{C}}$ which identifies hypersurfaces which are projectively equivalent. Letting $Z \subseteq \mathcal{H}(d; r)$ denote the locus where all $\mathrm{PGL}(\mathbb{C}, r+1)$-invariant polynomials simultaneously vanish, the **parameter space of semi-stable, degree d hypersurfaces** in $\mathbb{P}^r_{\mathbb{C}}$ is the complement $\mathcal{S}(d; r) := \mathcal{H}(d; r) \setminus Z$. Hence, the **coarse moduli space of semi-stable, degree d hypersurfaces in $\mathbb{P}^r_{\mathbb{C}}$** is

$$\mathcal{M}(d; r) := \mathcal{S}(d; r) / \mathrm{PGL}(\mathbb{C}, r+1).$$

The semi-stable locus $\mathcal{S}(d; r)$ is a dense Zariski open of $\mathcal{H}(d; r)$ which is $\mathrm{PGL}(\mathbb{C}, r+1)$-invariant. It contains another dense, $\mathrm{PGL}(\mathbb{C}, r+1)$-invariant Zariski open $\mathcal{S}^\circ(d; r) \subseteq \mathcal{S}(d; r)$ which parametrizes the smooth hypersurfaces. The **coarse moduli space of smooth, degree d hypersurfaces in $\mathbb{P}^r_{\mathbb{C}}$** is

$$\mathcal{M}^\circ(d; r) := \mathcal{S}^\circ(d; r) / \mathrm{PGL}(\mathbb{C}, r+1).$$

Each of the above parameter and moduli spaces classifies certain objects. We will additionally need to refer to the spaces classifying these objects with an associated choice of k-plane.

Definition 4.3.2. (Parameter and Moduli Spaces of Hypersurfaces with k-Planes) Continuing with the notation of Definition 4.3.1 and recalling that $\mathcal{G}\mathrm{r}(k, r)$ is the variety of k-planes in $\mathbb{P}^r_{\mathbb{C}}$, we denote the **parameter space of degree d hypersurfaces with choice of k-plane in $\mathbb{P}^r_{\mathbb{C}}$** by $\mathcal{H}(d; r, k)$; it is the incidence variety

$$\mathcal{H}(d; r, k) = \{(H, \Lambda) \mid \Lambda \subseteq H\} \subseteq \mathcal{H}(d; r) \times \mathcal{G}\mathrm{r}(k, r).$$

Similarly, we write $\mathcal{S}(d; r, k), \mathcal{M}(d; r, k)$, and $\mathcal{M}^\circ(d; r, k)$ for the analogous spaces which additionally classify a choice of k-plane. They similarly arise as incidence varieties or as quotients of indcidence varieties by $\mathrm{PGL}(\mathbb{C}, r+1)$.

We now expand these definitions from classifying hypersurfaces to classifying intersections of hypersurfaces of type $(2,\ldots,d)$.

Definition 4.3.3. (Parameter and Moduli Spaces of Intersections of Hypersurfaces)
The **parameter space of intersections of hypersurfaces of type** $(2,\ldots,d)$ **in** $\mathbb{P}^r_{\mathbb{C}}$ is

$$\mathcal{H}(2,\ldots,d;r) = \mathcal{H}(2;r) \times \cdots \times \mathcal{H}(d;r).$$

The natural action of $\mathrm{PGL}(\mathbb{C}, r+1)$ on $\mathbb{P}^r_{\mathbb{C}}$ induces a diagonal action on $\mathcal{H}(2,\ldots,d;r)$ and the **parameter space of semi-stable intersections of hypersurfaces of type** $(2,\ldots,d)$ **in** $\mathbb{P}^r_{\mathbb{C}}$ is the analogous Zariski open $U \subsetneq \mathcal{H}(2,\ldots,d;r)$ and is $\mathrm{PGL}(\mathbb{C}, r+1)$-invariant; it is denoted by $\mathcal{S}(2,\ldots,d;r)$.

The **moduli space of semi-stable intersections of hypersurfaces of type** $(2,\ldots,d)$ **in** $\mathbb{P}^r_{\mathbb{C}}$ is

$$\mathcal{M}(2,\ldots,d;r) = \mathcal{S}(2,\ldots,d;r)/\,\mathrm{PGL}(\mathbb{C}, r+1).$$

In analogy with Definition 4.3.2, we write $\mathcal{H}(2,\ldots,d;r,k)$, $\mathcal{S}(2,\ldots,d;r,k)$, and $\mathcal{M}(2,\ldots,d;r,k)$ for the respective spaces which additionally classify a choice of k-plane; they analogously arise as incidence varieties or as quotients of incidence varieties by $\mathrm{PGL}(\mathbb{C}, r+1)$, as well.

Remark 4.3.4. (Parameter and Moduli Spaces as Schemes)
For the interested reader, we note that these spaces can be constructed as schemes via classical invariant theory, beginning with

$$\begin{aligned}\mathcal{H}(d;r) &= \mathrm{Proj}\left(S^*\left(\mathbb{C}[x_0,\ldots,x_r]^\vee_{(d)}\right)\right),\\ \mathcal{M}(d;r) &= \mathrm{Proj}\left(S^*\left(\mathbb{C}[x_0,\ldots,x_r]^\vee_{(d)}\right)^{\mathrm{GL}(\mathbb{C},r+1)}\right).\end{aligned}$$

The remaining spaces arise analogously from the same constructions as in the variety case.

We will use the dimension of these parameter and moduli spaces, so we recall the dimensions in the case of hypersurfaces and give the dimensions in the case of intersections of hypersurfaces of type $(2, \ldots, d)$. In the proof of Proposition 3.2.10, we observed the combinatorial identity

$$\sum_{j=0}^{d} \binom{r+j}{j} = \binom{r+d+1}{d}.$$

In Remark 4.3.5 and Definition 4.3.7, we use the slight variation:

$$\sum_{j=2}^{d} \binom{r+j}{j} = \binom{r+d+1}{d} - (r+2). \tag{4.3.1}$$

Remark 4.3.5. (Dimension of Parameter and Moduli Spaces)
For fixed $d, r \geq 1$, we have

$$\dim(\mathcal{S}(d;r)) = \dim(\mathcal{H}(d;r)) = \binom{r+d}{d} - 1.$$

When $\binom{r+d}{d} - (r+1)^2 < 0$, $\mathcal{M}(d;r)$ is empty. When $\binom{r+d}{d} - (r+1)^2 \geq 0$, we have

$$\dim(\mathcal{M}^\circ(d;r)) = \dim(\mathcal{M}(d;r)) = \binom{r+d}{d} - (r+1)^2.$$

Similarly, for d, r for which the following spaces are non-empty, we have

$$\dim(\mathcal{S}(2, \ldots, d;r)) = \dim(\mathcal{H}(2, \ldots, d;r)) = \left(\sum_{j=2}^{d} \binom{d+j}{j}\right) - (d-1),$$
$$\dim(\mathcal{M}(2, \ldots, d;r)) = \left(\sum_{j=2}^{d} \binom{d+j}{j}\right) - (r+1)^2 - (d-2).$$

Using equation (4.3.1), we re-write these quantities as

$$\dim\left(\mathcal{S}(2,\ldots,d;r)\right) = \dim\left(\mathcal{H}(2,\ldots,d;r)\right) = \binom{r+d+1}{d} - (r+d+1),$$
$$\dim\left(\mathcal{M}(2,\ldots,d;r)\right) = \binom{r+d+1}{d} - (r+1)^2 - (r+d).$$

In [DeMa1998], Debarre and Manivel gave an explicit combinatorial condition for an intersection of hypersurfaces in $\mathbb{P}^r_{\mathbb{C}}$ to contain a k-plane:

Theorem 4.3.6. ***(Theorem 2.1 of [DeMa1998])***
Let $V \subseteq \mathbb{P}^r_{\mathbb{C}}$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$, which is not a quadric hypersurface. When r and k are such that

$$(k+1)(r-k) - \sum_{j=1}^{d} \ell_j \binom{k+j}{j} \geq 0,$$

then the natural maps

$$\mathcal{H}(2,\ldots,d;r,k) \to \mathcal{H}(2,\ldots,d;r),$$
$$\mathcal{H}(2,\ldots,d;r,k) \to \mathcal{H}(2,\ldots,d;r),$$

are surjective.[1]

We will use this condition to guarantee the existence of k-planes on $\tau^{\circ}_{1,\ldots,d}$ in Lemma 4.3.8. First, however, we introduce the following notation.

[1]This is not the entirety of Debarre and Manivel's result; see Theorem 2.1 of [DeMa1998] for more details.

Definition 4.3.7. (Notation for Lemma 4.3.8)

We define a new set function

$$\vartheta : \mathbb{Z}_{\geq 3} \times \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1},$$

where $\vartheta(d,k)$ is the smallest positive integer r such that

$$(k+1)(r-k) - \sum_{j=2}^{d} \binom{k+j}{j} \geq 0.$$

Using the combinatorial identity (4.3.1), we equivalently write

$$\vartheta(d,k) = k + \left\lceil \frac{1}{k+1} \left(\binom{k+d+1}{d} - (k+2) \right) \right\rceil.$$

Lemma 4.3.8. ***(k-Planes on an Intersection of Hypersurfaces of Type $(2,\dots,d)$)*** *Let $d \geq 3$ and $V \in \mathcal{S}(2,\dots,d;r)(K)$ for some $\mathbb{C}$-field K. For any $k \geq 1$ and $r \geq \vartheta(d,k)$, then we can determine a k-plane on V over an extension L/K with*

$$\mathrm{RD}(L/K) \leq \dim\left(\mathcal{M}(2,\dots,d;\vartheta(d,k))\right).$$

Proof. First, observe that it suffices to prove the claim when $r = \vartheta(d,k)$, by restriction. Theorem 4.3.6 then yields that V contains a k-plane. We identify V with $\mathbb{A}^0_K \to \mathcal{M}(2,\dots,d;\vartheta(d,k))$ and note that the resolvent degree of determining a k-plane on V is exactly the resolvent degree of the map

$$\pi_K : \mathbb{A}^0_K \times_{\mathcal{M}(2,\dots,d;\vartheta(d,k))} \mathcal{M}(2,\dots,d;\vartheta(d,k),k) \to \mathbb{A}^0_K$$

determined by the pullback square

$$\begin{tikzcd}
\mathbb{A}^0_K \times_{\mathcal{M}(2,\dots,d;\vartheta(d,k))} \mathcal{M}(2,\dots,d;\vartheta(d,k),k) \arrow[r, "\pi_{\mathcal{M}}"] \arrow[d, "\pi_K"'] & \mathcal{M}(2,\dots,d;\vartheta(d,k),k) \arrow[d] \\
\mathbb{A}^0_K \arrow[r] & \mathcal{M}(2,\dots,d;\vartheta(d,k))
\end{tikzcd}$$

Consequently, Lemma 2.1.12 yields

$$\begin{aligned}
\mathrm{RD}\left(\pi_K\right) &\leq \mathrm{RD}\left(\mathcal{M}(2,\dots,d;\vartheta(d,k),k) \to \mathcal{M}(2,\dots,d;\vartheta(d,k))\right), \\
&\leq \dim\left(\mathcal{M}(2,\dots,d;\vartheta(d,k))\right).
\end{aligned}$$

□

To apply Lemma 4.3.8 to the case at hand, we must first show that the Tschirnhaus complete intersections are semi-stable.

Proposition 4.3.9. ***(Semi-Stability of Tschirnhaus Complete Intersections)*** *For each $d \geq 3$ and $n \geq d+2$, $\tau_{1,\dots,d} \in \mathcal{S}(2,\dots,d;n-2)(K_n)$.*

Proof. First, note that $\tau_1 \subseteq \mathbb{P}^{n-1}_{K_n}$ is a hyperplane, so we can consider $\tau_{1,\dots,d} \subseteq \tau_1 \cong \mathbb{P}^{n-2}_{K_n}$, e.g. as a K_n-point of $\mathcal{H}(2,\dots,d;n-2)$. Now, from the definition of $\mathcal{S}(2,\dots,d;n-2)$, it suffices to show that there is some $\mathrm{PGL}(K_n,n-1)$-invariant polynomial in $K_n[x_0,\dots,x_{n-1}]$ which does not vanish at $\tau_{1,\dots,d}$. Theorem 2.12 of [Wol2021] yields that $\tau_{1,2,3}$ is generically smooth, hence semi-stable. In particular there is a $\mathrm{PGL}(K_n,n-1)$-invariant polynomial $f \in K_n[x_0,\dots,x_{n-1}]$ which does not vanish as $\tau_{1,2,3} \in \mathcal{H}(2,3;n-2)(K_n)$. When pulled back to $\mathcal{H}(2,\dots,d;n-2)$ via the standard projection map, f does not vanish at $\tau_{1,\dots,d}$ as well, which yields the claim. □

We are now ready to state our general construction.

Theorem 4.3.10. ***(Determining a Point on*** $\tau^\circ_{1,\dots,d+k}$***)***

Fix $d, k \geq 1$. *For* $n \geq \vartheta(d,k) + 3$, *we can determine a point of* $\tau^\circ_{1,\dots,d+k}$ *over an extension* L/K_n *with*

$$\mathrm{RD}\left(L/K_n\right) \leq \max\left\{\dim\left(\mathcal{M}(2,\dots,d;\vartheta(d,k))\right), \frac{(d+k)!}{d!}\right\}.$$

Proof. By restriction, it suffices to prove the case where $n = \vartheta(d,k) + 3$. As such, we work in $\mathbb{P}^{\vartheta(d,k)+2}_{K_n}$. We then pass to a hypersurface H which does not contain $[1 : 0 : \cdots : 0]$ and $\tau^\circ_1 = \tau_1 \cap H \cong \mathbb{P}^{\vartheta(d,k)}_{K_n}$. Consequently, Lemma 4.3.8 and Propsition 4.3.8 allow us to determine a k-plane $\Lambda \subseteq \tau^\circ_{1,\dots,d}$ over an extension L_1/K_n with

$$\mathrm{RD}\left(L/K_n\right) \leq \dim\left(\mathcal{M}(2,\dots,d;\vartheta(d,k))\right).$$

Thus, $\deg\left(\Lambda \cap \tau_{1,\dots,d+k}\right) = \frac{(d+k)!}{d!}$ and we can determine a point of $\Lambda \cap \tau_{1,\dots,d+k} \subseteq \tau^\circ_{1,\dots,d+k}$ by solving a polynomial of degree at most $\frac{(d+k)!}{d!}$. $\square$

Having established our general construction, we are now ready to define our bounding function.

Definition 4.3.11. (Our Bounding Function)

We define $\varphi : \mathbb{Z}_{\geq 15} \times \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ by

$$\varphi(d,k) = \max\left\{\dim\left(\mathcal{M}(2,\dots,d;\vartheta(d,k))\right), \frac{(d+k)!}{d!}\right\}.$$

We now define $G' : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ for $m \in [1,17] \cup [22,25]$ by giving explicit values:

For $m \in [18,21]$ and $m \geq 26$, we define $G'(m)$ by

$$G'(m) = 1 + \min\left\{\varphi(d, m-d-1) \mid d \in [4, m-1]\right\}.$$

Table 4.4: Classical Bounds

m	1	2	3	4	5
$G(m)$	2	3	4	5	9

Table 4.5: Bounds from Theorems 4.1.1 and 4.1.4

m	6	7	8	9	10	11	12
$G(m)$	21	109	325	1681	15,121	151,201	1,663,201

Table 4.6: Bounds from Theorem 4.2.3, I

m	13	14	15	16	17
$G(m)$	5,250,198	51,891,841	726,485,761	10,897,286,401	174,356,582,401

Table 4.7: Bounds from Theorem 4.2.3, II

m	$G(m)$
22	381,918,437,071,508,900
23	1,561,112,121,913,344,001
24	35,905,578,804,006,912,001
25	861,733,891,296,165,888,001

Theorem 4.3.12. ***(Upper Bounds on*** $\mathrm{RD}(n)$***)***

For each $m \geq 1$ *and* $n \geq G(m)$*,* $\mathrm{RD}(n) \leq n - m$*.*

Proof. The claim for $m \in [1, 5]$ is classical, see Section 1.2 or [Wol2021]. For $m \in [6, 17] \cup [22, 25]$, the claim is handled by Theorems 4.1.1, 4.1.4, and 4.2.3.

Now, consider $m \in [18, 21] \cup [26, \infty)$. Theorem 4.3.10 yields that we can determine a point of $\tau^{\circ}_{1,\dots,m-1}$ over an extension L/K with

$$\mathrm{RD}\,(L/K) \leq \varphi(d, m - d - 1)$$

when $n \geq \vartheta(d,k)+3$. Note that

$$
\begin{aligned}
&\vartheta(d, m-d-1) \\
&< \binom{\vartheta(d, m-d-1)+d+1}{d} - (\vartheta(d, m-d-1)+1)^2 - (\vartheta(d, m-d-1)+d)\,, \\
&= \dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1))\right), \\
&\leq \varphi(d, m-d-1),
\end{aligned}
$$

and so the condition $n \geq G'(m)$ implies $n \geq \vartheta(d, m-d-1)+3$. Finally, it suffices to minimize over all such d and thus the definition of $G'(m)$ yields the claim. □

Remark 4.3.13. (Notes on $G'(m)$)

First, we explain the notation of the bounding function $G'(m)$. Prior to the work in this dissertation, the best bounding function was established by Wolfson in [Wol2021] and was denoted by F. This was first improved upon in [Sut2021C], where I introduced a new bounding function G (Definition 3.26). In [HS2021], we improved on G for $m \in [13,17] \cup [22,25]$. Given the limited range of updated values and Wolfson uses of H for Hamilton's bounding function in Appendix B of [Wol2021], we thus opted for G'.

Note that Theorem 4.2.3 can be extended to handle the cases of $m \in [18,21]$, but one obtains the same bound of $\frac{(m-1)!}{720}+1$ in each case. Given that these bounds were first established by the method of Theorem 4.3.10 in [Sut2021C], we continue to use this construction for the $[18,21]$ range when defining $G'(m)$.

4.4 Obstructions to Further Bounds Using Iterated Polar Cones

We will now give a brief description of Wolfson's construction for determining k-planes on intersections of hypersurfaces in [Wol2021]. Instead of Theorem 4.3.6, which provides a combinatorial condition for the existence of k-planes on intersections of hypersurfaces, Wolfson uses the following theorem of Waldron, which is only for hypersurfaces:

Theorem 4.4.1. ***(Theorem 1.6 of [Wal2008])***
Fix $d \geq 3$. When r and k are such that

$$(k+1)(r-k) - \binom{k+d}{d} \geq 0,$$

then the natural maps

$$\mathcal{H}(d;r,k) \to \mathcal{H}(d;r),$$
$$\mathcal{M}^\circ(d;r,k) \to \mathcal{M}^\circ(d;r),$$

are surjective.[2]

Wolfson repeatedly uses Theorem 4.4.1 to find k-planes on individual hypersurfaces. We now provide an example which is indicative of the general construction.

Example 4.4.2. (Wolfson's Method)
For $n \geq 1560$, we can determine an 8-plane on $\tau^\circ_{1,2,3,4}$ over an extension L/K_n with

$$\mathrm{RD}\,(L/K_n) \leq 78,485,029.$$

[2]This is not the entirety of Waldron's result; see Theorem 1.6 of [Wal2008] for more details.

Proof. We work in $\mathbb{P}^{1559}_{K_n}$ and pass to a hyperplane $H \cong \mathbb{P}^{1558}$ which does not contain $[1 : 0 : \cdots : 0]$. Similarly, τ_1 is a hyperplane and $H \cap \tau_1^\circ = H \cap \tau_1 \cong \mathbb{P}^{1557}$. Now, τ_2 is a quadric hypersurface in $H \cap \tau_1$ and thus it is known classically that there is a 778-plane $\Lambda_2 \subseteq H \cap \tau_{1,2}$ over an iterated quadratic extension L_1/K_n (see Proposition 2.13 and Corollary 2.14 of [Wol2021] for details). Now, $\Lambda_2 \cap \tau_{1,2,3}$ is a cubic hypersurface in Λ_2 and

$$(63+1)(778-63) - \binom{63+3}{3} = 0,$$

hence Theorem 4.4.1 yields we can determine a 63-plane $\Lambda_3 \subseteq \Lambda_2 \cap \tau_{1,2,3}$ over an extension L_2/L_1 with

$$\mathrm{RD}\,(L_2/L_1) \leq \mathrm{RD}\,(\mathcal{M}^\circ(3; 778, 63) \to \mathcal{M}^\circ(3; 778)) \leq \dim\,(\mathcal{M}^\circ(3; 778)) = 78,485,029.$$

Finally, $\Lambda_3 \cap \tau_{1,2,3,4}$ is a quartic hypersurface in Λ_3 and

$$(8+1)(63-8) - \binom{8+4}{4} = 0,$$

hence we can determine an 8-plane $\Lambda_4 \subseteq \Lambda_3 \cap \tau_{1,2,3,4}$ over an extension L_3/L_2 with

$$\mathrm{RD}\,(L_3/L_2) \leq \mathrm{RD}\,(\mathcal{H}(4; 63, 8) \to \mathcal{H}(4; 63)) \leq \dim\,(\mathcal{H}(4; 63)) = 766,479.$$

$\square$

We next provide an example which highlights the limitations of using the direct methods (as used in proving Theorems 4.1.1 and 4.1.4) to determine further upper bounds.

Remark 4.4.3. (Limitations of Direct Methods)

We can determine a 9-plane on $\tau_{1,2,3,4,5}$ over an extension L/K_n with

$$\begin{aligned} \mathrm{RD}\left(L/K_n\right) \leq \dim\left(\mathcal{M}^\circ\left(3;\psi(5,9)_3\right)\right) &= 3,298,353,885,918,738,132,194,252,727,911, \\ &\approx 3\cdot 10^{30}, \end{aligned}$$

as long as the ambient dimension is at least $\psi(5,9)_4 + 1 = 54,097,786,526 \approx 5\cdot 10^{10}$, where the ψ notation is from Notation 5.2 of [Wol2021]. See the proof of Theorem 5.6 of [Wol2021] for details.

Now, observe that a 9^{th} polar cone $\mathcal{C}^9\left(\tau_{1,2,3,4,5};P_0,\dots,P_8\right)$ of $\tau_{1,2,3,4,5}$ has type

$$\begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 1 & 10 & 55 & 220 & 715 \end{bmatrix}.$$

Thus, even for n large enough that we can determine a suitable 66-plane Λ inside

$$\mathcal{C}^9\left(\tau_{1,2,3,4,5};P_0,\dots,P_8\right)_1 \cap \mathcal{C}^9\left(\tau_{1,2,3,4,5};P_0,\dots,P_8\right)_2,$$

which we recall is the intersection of the 220 quadrics and 715 hyperplanes defining $\mathcal{C}^9\left(\tau_{1,2,3,4,5};P_0,\dots,P_8\right)$, it still follows that

$$\begin{aligned} \deg\left(\Lambda\cap\mathcal{C}^9\left(\tau_{1,2,3,4,5};P_0,\dots,P_8\right)\right) &= 5\cdot 4^{10}\cdot 3^{55}, \\ &\approx 9\cdot 10^{32}, \\ &> 3\cdot 10^{30}, \\ &\approx \dim\left(\mathcal{M}^\circ\left(3;\psi(5,9)_3\right)\right). \end{aligned}$$

Indeed, the degree of the intersections analogous to $\mathcal{C}^9(\tau_{1,2,3,4,5}; P_0, \dots, P_8)$ grow exponentially in k, whereas the dimension of the moduli spaces only grow polynomially in k.

Remark 4.4.4. (Comparing the Three Constructions used in Defining $G'(m)$) Building off Remark 4.4.3, we will now compare the advantages of the three methods at a high level. The strategies for proving Theorems 4.1.1 and 4.1.4 minimize the ambient dimension required, at the cost of the degrees of the polynomials that need to be solved. For Theorem 4.2.3, we minimize the degrees of the polynomial that need to be solved, at the cost of the ambient dimension required. Determining the threshold ($m = 13$) at which point it was optimal to switch strategies followed from direct computation.

We now compare the proof strategies for Theorems 4.2.3 and 4.3.10. Recall that in light of Theorem 4.3.10, we introduced the notation

$$\varphi(d,k) = \max\left\{\dim\left(\mathcal{M}(2,\dots,d;\vartheta(d,k))\right), \frac{(d+k)!}{d!}\right\}.$$

We are concerned with $d \geq 4$ and for each such d, the set

$$\{m \in \mathbb{Z}_{\geq 1} \mid G'(m) = 1 + \phi(d, m-d-1)\},$$

is a set of consecutive integers $[m_d, m'_d]$ (see Lemma 4.5.7 for more details). Analogously for the bounds from the geometric obliteration algorithm, we write

$$\varrho(d,k) = \max\left\{\Xi(d+k+1,d), \frac{(d+k)!}{d!}\right\},$$

for $d \geq 4$ and $k \geq 1$, as well as

$$G''(m) = \min\left\{\varrho(d, m-d-1) \mid d \in [4, m-1]\right\},$$

for $m \geq 13$. For a fixed d, $\Xi(m,d)$ is a polynomial in m, whereas $\frac{(d+k)!}{d!} = \frac{(m-1)!}{d!}$ grows factorially. It follows that for each d, there are positive integers M_d and M_d' such that $G''(m) = \varrho(d, m-d-1)$ if and only if $m \in [M_d, M_d']$. In the following table, we compare the values m_d and M_d for $d = 5, 6, 7, 8$.

Table 4.8: Values of m_d and M_d for $d \in [5, 8]$

d	m_d	M_d
5	17	13
6	25	22
7	34	41
8	44	78

The numerics of Table 4.8 provide a good heuristic for an obstruction to further bounds from the geometric obliteration algorithm. We build upon this further by establishing a lower bound on $\Xi(m,d)$.

Lemma 4.4.5. ***(Lower Approximation)***

Let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} d \\ \ell_d \end{bmatrix}$ with $d \geq 3$ and $\ell_d \geq 2$. Denote the type of a $(d-2)^{nd}$ Sylvester reduction V_{d-2}^{Syl} by $\begin{bmatrix} 2 & 1 \\ \lambda_2 & \lambda_1 \end{bmatrix}$. Then,

$$\lambda_1 \geq \lambda_2 \geq \left\lceil 2^{5-2d} (\ell_d - 1)^{2d-4} \right\rceil .$$

Proof. Note that the number of degree $d-1$ hypersurfaces of V_1^{Syl} is

$$\theta_{d-1} = \sum_{j=1}^{\ell_d - 1} \ell_d - j = \frac{1}{2} (\ell_d - 1) \ell_d \geq \left\lceil \frac{1}{2} (\ell_d - 1)^2 \right\rceil .$$

The same argument yields that the number of degree $d-2$ hypersurfaces of V_2^{Syl} is

$$\theta_{d-2} \geq \left\lceil \frac{1}{2} \left\lceil \frac{1}{2} \left(\ell_d - 1\right)^2 \right\rceil^2 \right\rceil \geq \left\lceil 2^{-3} \left(\ell_d - 1\right)^4 \right\rceil .$$

Proceeding similarly, we see that

$$\lambda_2 \geq \theta_2 \geq \left\lceil 2^{5-2d} \left(\ell_d - 1\right)^{2d-4} \right\rceil .$$

Finally, note that $\lambda_1 \geq \lambda_2$ follows immediately from the polar cone construction. $\square$

Corollary 4.4.6. ***(Lower Bound for*** $\Xi(m,d)$***)***

Let $d \geq 4$ *and* $m \geq d+2$*. Then,*

$$\Xi(m,d) \geq \left\lceil 4 \left(\frac{m-d-1}{2} \right)^{2d-4} \right\rceil .$$

Proof. Recall that Proposition 3.2.10 yields that an $(m-d-1)^{st}$ polar cone of $\tau_{1,\ldots,d}$ is of type

$$\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ 1 & \binom{m-1}{1} & \cdots & \binom{m-3}{d-2} & \binom{m-2}{d-1} \end{bmatrix}.$$

Thus, the number of degree $d-1$ hypersurfaces of $(\tau_{1,\ldots,d})_1^{\mathrm{Syl}}$ is $m-d$; we write V for the intersection of these $m-d$ degree $d-1$ hypersurfaces. Applying Lemma 4.4.5 to V, we see that

$$\lambda_1 \geq \lambda_2 \geq \left\lceil 2^{5-2d}(m-d-1)^{2d-4} \right\rceil .$$

Moreover, for each j,

$$\xi(m, d; j) \geq \lambda_1 + \lambda_2 \geq \left\lceil 4\left(\frac{m-d-1}{2}\right)^{2d-4} \right\rceil,$$

and thus it follows that

$$\Xi(m, d) \geq \left\lceil 4\left(\frac{m-d-1}{2}\right)^{2d-4} \right\rceil.$$

$\square$

While we do not provide a full comparison here, we note that the key obstruction to obtaining further bounds using the geometric obliteration algorithm is that $\Xi(m, d)$ has a lower bound which grows exponentially in d and that $m-d-1$ is always much larger than 2 for the cases in question.

We compare exact values of $F(m)$, $G(m)$, and $G'(m)$ for $m \in [2, 17]$ and approximate values of $F(m)$, $G(m)$, and $G'(m)$ for $m \in [18, 26]$ in Appendix A.1. We provide information on the ratio $\frac{F(m)}{G'(m)}$ in Appendix A.2. We will provide a general comparison of $F(m)$ and $G(m)$ in Section 4.6, but we first provide an approximation of $G'(m)$ via elementary functions.

4.5 Approximating $G'(m)$ via Elementary Functions

While the construction determining $G'(m)$ for $m \geq 26$ is qualitatively simpler than the process underlying $F(m)$, Definition 4.3.11 is not given explicitly in terms of elementary functions. Nonetheless, we now provide such an approximation.

Theorem 4.5.1. ***(Upper Bound on the Growth Rate of*** $\mathrm{RD}(n)$***)***
For every positive integer $d \geq 4$*,* $G'\left(2d^2+4d+4\right) \leq \frac{\left(2d^2+4d+3\right)!}{d!}$*. Hence, for* $n \geq \frac{\left(2d^2+4d+3\right)!}{d!}$*, it follows that*

$$\mathrm{RD}(n) \leq n - 2d^2 - 4d - 4.$$

To prove Theorem 4.5.1, we establish a simple critertion for m, in terms of d, so that we can conclude that $G'(m) < \frac{(m-1)!}{d!}$ when that criterion is met.

Recall that $\vartheta(d,k)$ is defined such that an intersection of hypersurfaces of type $(2,\ldots,d)$ in $\mathbb{P}^r$ contains a k-plane when $r \geq \vartheta(d,k)$. We begin by approximating $\vartheta(d,m-d-1)$ above.

Lemma 4.5.2. ***(Upper Bound on*** ϑ***)***
Fix $m > d \geq 4$*. Then,*

$$\vartheta(d,m-d-1) \leq m-d-2+\binom{m}{d}.$$

Proof. We first recall Definition 4.3.7:

$$\vartheta(d,k) = k + \left\lceil \frac{1}{k+1}\left(\binom{k+d+1}{d} - (k+2)\right)\right\rceil.$$

Using the identification $k = m-d-1$, we observe that

$$\begin{aligned}\vartheta(d,m-d-1) &= (m-d-1) + \left\lceil \frac{1}{m-d}\left(\binom{m}{d} - (m-d+1)\right)\right\rceil,\\ &\leq m-d-2+\binom{m}{d}.\end{aligned}$$

□

Corollary 4.5.3. ***(Upper Bound on Parameter Space Dimension)***

Fix $m > d \geq 4$. Then,

$$\dim\left(\mathcal{H}(2,\ldots,d;\vartheta(d,m-d-1))\right) \leq \binom{m-1+\binom{m}{d}}{d} - \left(m-1+\binom{m}{d}\right).$$

Proof. Remark 4.3.5 established that

$$\dim\left(\mathcal{H}(2,\ldots,d;\vartheta(d,m-d-1))\right) = \binom{\vartheta(d,m-d-1)+d+1}{d} - \left(\vartheta(d,m-d-1)+d+1\right),$$

which is non-decreasing in $\vartheta(d,m-d-1)$. Thus, Lemma 4.5.2 yields

$$\dim\left(\mathcal{H}(2,\ldots,d;\vartheta(d,m-d-1))\right) \leq \binom{m-1\binom{m}{d}}{d} - \left(m-1+\binom{m}{d}\right).$$

$\square$

We will now introduce a constant C_d and give a bound on $\log\left(\frac{C_d}{d+1}\right)$, both of which will be useful in the proof of Lemma 4.5.7. We also remind the reader that every use of log in this paper refers to the base e logarithm.

Definition 4.5.4. (Constant for the Proof of Lemma 4.5.7)

For each $d \geq 4$, we set

$$C_d := \max\left\{\binom{d+1}{j} \mid j \in [0,d+1]\right\}.$$

Lemma 4.5.5. ***(Bound on*** $\log(C_d)$***)***

For each $d \geq 4$, it follows that

$$\log\left(\frac{C_d}{d+1}\right) \leq d + \frac{3}{2}.$$

We will frequently use Stirling's approximations for factorials, including in the proof of Lemma 4.5.5 and thus state the version we use explicitly (a stronger version of which can be found in [Rob1955]):

Lemma 4.5.6. ***(Stirling's Approximations)***

Let $a \in \mathbb{Z}_{\geq 1}$. Then,

$$\sqrt{2\pi} a^{a+\frac{1}{2}} e^{-a} \leq a! \leq a^{a+\frac{1}{2}} e^{1-a}. \tag{4.5.1}$$

Proof. **(Proof of Lemma 4.5.5)**

Our proof depends on the parity of $d+1$; we begin with the case where $d+1 = 2\ell$ is even. Then,

$$C_d = \binom{2\ell}{\ell} = \frac{(2\ell)!}{(\ell!)^2},$$

and Stirling's approximation yields

$$C_d \leq \frac{(2\ell)^{2\ell+\frac{1}{2}} e^{1-2\ell}}{\left(\sqrt{2\pi}\ell^{\ell+\frac{1}{2}} e^{-\ell}\right)^2} = \frac{2^{d+\frac{1}{2}} e}{\pi\sqrt{\ell}}.$$

Consequently,

$$\log\left(\frac{C_d}{d+1}\right) \leq \log\left(\frac{2^{d+\frac{1}{2}} e}{\pi\sqrt{\ell}(d+1)}\right) \leq \log\left(\frac{e}{\pi(d+1)\sqrt{\ell}}\right) + \log\left(e^{d+\frac{1}{2}}\right) \leq d + \frac{1}{2}.$$

When $d+1 = 2\ell+1$ is odd, we observe

$$C_d = \binom{2\ell+1}{\ell} \leq \binom{2\ell+2}{\ell+1} \leq \frac{2^{d+\frac{3}{2}} e}{\pi\sqrt{\ell+1}},$$

and thus

$$\log\left(\frac{C_d}{d+1}\right) \leq \log\left(\frac{2^{d+\frac{3}{2}}e}{\pi\sqrt{\ell+1}}\right) \leq d+\frac{3}{2}.$$

$\square$

Recall that for each $d \in [1, m-1]$ and when n is large enough, we can determine an $(m-d-1)$-plane $\Lambda \subseteq \tau^{\circ}_{1,\ldots,d}$ over an extension L_1/K_n with

$$\mathrm{RD}\left(L_1/K_n\right) \leq \dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1))\right).$$

In such a case, we can determine a point of $\Lambda \cap \tau_{1,\ldots,m-1} \subseteq \tau^{\circ}_{1,\ldots,m-1}$ by solving a polynomial of degree at most $\frac{(m-1)!}{d!}$. Hence, we set

$$\varphi(d,m-d-1) = \max\left\{\frac{(m-1)!}{d!}, \dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1))\right)\right\},$$

in Definition 4.3.11. We next give a condition relating $\varphi(d,m-d-1)$ and $\varphi(d+1,m-d-2)$.

Lemma 4.5.7. ***(The φ Condition)***
Fix $d \geq 4$. For all $m \geq 2d^2+4d+4$, it follows that

$$\varphi(d+1,m-d-2) < \varphi(d,m-d-1). \tag{4.5.2}$$

Proof. For any such d and m, it is always true that

$$\frac{(m-1)!}{(d+1)!} < \frac{(m-1)!}{d!}.$$

As a result, to conclude (4.5.2), we need only show that

$$\binom{m-1+\binom{m}{d+1}}{d+1} < \frac{(m-1)!}{d!}, \tag{4.5.3}$$

since

$$\begin{aligned}\dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1))\right) &< \dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1))\right), \\ &< \binom{m-1+\binom{m}{d+1}}{d+1}.\end{aligned}$$

Observe that

$$\begin{aligned}\binom{m-1+\binom{m}{d+1}}{d+1} &= \frac{\left(m-1+\binom{m}{d+1}\right)!}{(d+1)!\left(m-d-2+\binom{m}{d+1}\right)!}, &(4.5.4)\\ &= \frac{1}{(d+1)!}\prod_{j=1}^{d+1}\left(m-j+\binom{m}{d+1}\right). &(4.5.5)\end{aligned}$$

Next, we approximate $\binom{m}{d+1}$:

$$\binom{m}{d+1} = \frac{m!}{(d+1)!(m-d-1)!} = \frac{1}{(d+1)!}\prod_{j=0}^{d}(m-j) \leq \frac{1}{(d+1)!}m^{d+1} \leq m^{d+1}. \tag{4.5.6}$$

By substituting inequality (4.5.6) into equation (4.5.4) and using that $m-j \leq m$, we obtain the approximation

$$\binom{m-1+\binom{m}{d+1}}{d+1} \leq \frac{1}{(d+1)!}\prod_{j=1}^{d+1}\left(m+m^{d+1}\right).$$

We now substitute this approximation into (4.5.3) and multiply both sides by $d!$ to obtain the sufficient condition

$$\frac{1}{d+1}\prod_{j=1}^{d+1}\left(m+m^{d+1}\right) < (m-1)!\ . \tag{4.5.7}$$

Note that the main term of the left side of inequality (4.5.7) is of the form $\prod_{j=1}^{d+1}(a+b) = (a+b)^{d+1}$, hence applying the binomial theorem yields

$$\begin{aligned}
\prod_{j=1}^{d+1}\left(m+m^{d+1}\right) &= \sum_{j=0}^{d+1}\binom{d+1}{j}\left(m^{d+2}\right)^{j} m^{d+1-j},\\
&= \sum_{j=0}^{d+1}\binom{d+1}{j} m^{dj+j} m^{d+1-j},\\
&= \sum_{j=0}^{d+1}\binom{d+1}{j} m^{dj+d+1},\\
&= m^{d+1}\sum_{j=0}^{d+1}\binom{d+1}{j}\left(m^{d}\right)^{j},
\end{aligned}$$

However, for any $a \in \mathbb{Z}_{\geq 1}$ and $x \geq a$, it follows from induction that $\sum_{j=0}^{a+1} x^j \leq 2x^{a+1}$. Recalling that $C_d = \max\left\{\binom{d+1}{j} \mid j \in [0, d+1]\right\}$, we thus conclude that

$$m^{d+1}\sum_{j=0}^{d+1}\binom{d+1}{j}\left(m^{d}\right)^{j} \leq m^{d+1} C_d\left(2\left(m^{d}\right)^{d+1}\right). \tag{4.5.8}$$

Substituting inequality (4.5.8) into inequality (4.5.7) and simplifying thus yields the condition

$$\frac{2C_d}{d+1} m^{d^2+2d+1} < (m-1)!\ .$$

Next, we apply Stirling's approximation and re-arrange terms to arrice at the condition

$$\frac{2C_d}{\sqrt{2\pi}(d+1)} m^{d^2+2d+1} < \frac{(m-1)^{m-\frac{1}{2}}}{e^{m-1}}.$$

Observe that $a^{a-1} < (a-1)^{a-\frac{1}{2}}$ for positive integers $a \geq 8$. By requiring $m > 8$, we need only consider when

$$\frac{2C_d}{\sqrt{2\pi}(d+1)} m^{d^2+2d+1} < \frac{m^{m-1}}{e^{m-1}}. \tag{4.5.9}$$

Multiplying both sides of inequality (4.5.9) by e^{d^2+2d+1}, dividing both sides by m^{d^2+2d+1}, and simplifying, we arrive at the condition

$$\frac{2C_d}{\sqrt{2\pi}(d+1)} e^{d^2+2d+1} < \left(\frac{m}{e}\right)^{m-d^2-2d-2}. \tag{4.5.10}$$

We now take the log of both sides of inequality (4.5.10), which yields

$$\log\left(\frac{2C_d}{\sqrt{2\pi}(d+1)} e^{d^2+2d+1}\right) < \left(m-d^2-2d-2\right)\log\left(\frac{m}{e}\right).$$

Requiring $m > e^2$, it suffices to have

$$m-d^2-2d-2 > \log\left(\frac{2}{\sqrt{2\pi}}\right) + \log\left(\frac{C_d}{d+1}\right) + \left(d^2+2d+\right)).$$

We note $\log\left(\frac{2}{\sqrt{2\pi}}\right) < 0$ and apply the bound $\log\left(\frac{C_d}{d+1}\right) \leq d + \frac{3}{2}$ from Lemma 4.5.5 to obtain the condition

$$m-d^2-2d-2 > \left(d+\frac{3}{2}\right) + \left(d^2+2d+2\right),$$

which we-arranges to yield our initial supposition

$$m \geq 2d^2 + 4d + 4.$$

Finally, note that for all $d \geq 4$, we have that $2d^2 + 4d + 4 > 8 > e^2$, hence the requirements $m \geq 8$ and $m > e^2$ used in the proof are rendered superfluous. $\square$

Corollary 4.5.8. ***(Upper Bound on the Growth Rate of** $G'(m)$**)***
For any $d \geq 4$ and $m \geq 2d^2 + 4d + 4$, it follows that

$$G'(m) < \frac{(m-1)!}{d!}.$$

Proof. Recall that for any $m \geq 26$,

$$G'(m) = \min\{\varphi(d, m-d-1) \mid d \in [4, m-1]\} + 1.$$

Consequently, for any $d \geq 4$ and $m \geq 2d^2 + 4d + 4$, we have

$$G(m) \leq \varphi(d+1, m-d-2) + 1 < \frac{(m-1)!}{d!},$$

from Lemma 4.5.7. $\square$

It is now straightforward to prove Theorem 4.5.1.

Proof. **(Proof of Theorem 4.5.1)**
Fix $d \geq 4$. From Corollary 4.5.8, we have that

$$G'(m) < \frac{(2d^2 + 4d + 3)!}{d!},$$

for $m \geq 2d^2 + 4d + 4$. From Theorem 4.3.12, it follows that

$$\mathrm{RD}(n) \leq n - 2d^2 - 4d - 4,$$

for $n \geq \frac{(2d^2+4d+3)!}{d!}$. $\square$

4.6 Quantitative Comparison With Prior Bounds

We now give a precise sense of how the bounds from $G'(m)$ improve upon the bounds of $F(m)$.

Theorem 4.6.1. ***(Comparing** $G'(m)$ **and** $F(m)$**)***
Let F be the function of Definition 4.6.3 (which is originally Definition 5.4 of [Wol2021]).

1. *For every $m \geq 1$, $G'(m) \leq F(m)$ with equality if and only if $m \in [1, 5]$.*
2. *$G(m)$ provides asymptotic improvements on $F(m)$, in the sense that*

$$\lim_{m\to\infty} \frac{F(m)}{G'(m)} = \infty.$$

Remark 4.6.2. Despite Theorem 4.6.1, $G('m)$ does not yield a strictly better bound on $\mathrm{RD}(n)$ for all n. As an example,

$$G'(16) = 10,897,286,401 \qquad F(16) = 54,486,432,001,$$
$$G'(17) = 174,356,582,401 \qquad F(17) = 871,782,912,001,$$

so $\frac{F(16)}{G'(16)} \approx 5$ and $\frac{(F(17)}{G'(17)} \approx 5$. However, for any $n \in [F(16), G'(17)]$, F and G' yield the same upper bound; namely, $\mathrm{RD}(n) \leq n - 16$.

The remainder of this section is spent proving Theorem 4.6.1. With this in mind, we now define Wolfson's function $F(m)$. Our presentation will vary slightly from Wolfson's and we refer to Section 5 of [Wol2021] for details of the construction.

Definition 4.6.3. (Wolfson's Functions)

Given $d \geq 3$ and $k \geq 1$, set $\psi(d,k)_0 = k$. For $0 \leq j \leq d-2$, set

$$\psi(d,k)_{j+1} := \psi(d,k)_j + \left\lceil \frac{1}{\psi(d,k)_j} \cdot \binom{\psi(d,k)_j + d - j}{d-j} \right\rceil,$$

along with $\psi(d,k)_{d-1} := 2\psi(d,k)_{d-1} + 1$. Additionally, define

$$\Phi(d,k) := \max\left\{ \frac{(d+k)!}{d!}, \dim\left(\mathcal{M}\left(3; \psi(d,k)_{d-2}\right)\right) + d + k + 1 \right\}.$$

Finally, for $m \leq 3$, set $F(m) := m+1$ and for $m \geq 4$, set

$$F(m) := 2\left\lfloor \frac{1}{2}\left(\min_{d \in [1, m-2]} \Phi(d, m-d-1) \right) \right\rfloor + 1.$$

Remark 4.6.4. (Outline for Section 4.6)

Our goal is to establish a criterion for m, in terms of d, so that we can conclude $F(m) > \frac{(m-1)!}{d!}$ when this criterion is met. We do this by examining when

$$\Phi(d, m-d-1) < \Phi(d+1, m-d-2).$$

We first show that for any $m \geq d+3$,

$$\dim\left(\mathcal{M}\left(3; \psi(d,k)_{d-2}\right)\right) \leq \dim\left(\mathcal{M}\left(3; \psi(d+1, m-d-2)_{d-1}\right)\right), \tag{4.6.1}$$

which will then leave us to consider when

$$\frac{(m-1)!}{d!} < \dim\left(\mathcal{M}\left(3; \psi(d+1, m-d-2)_{d-1}\right)\right). \tag{4.6.2}$$

We begin by introducing auxiliary functions which will be useful for proving inequality (4.6.1) holds for $m \geq d+3$.

Definition 4.6.5. (Auxiliary Functions $\Psi(d;j)$)

For any $d \geq 2$, any $j \in [1, d-2]$, and $x \in \mathbb{Z}_{\geq 1}$, we set

$$\Psi(d,j)(x) := x + \left\lfloor \frac{1}{x+1} \cdot \binom{x+d-j}{d-j} \right\rfloor.$$

Remark 4.6.6. (Key Properties of $\Psi(d;j)$)

The functions $\Psi(d;j)$ satisfy

1. $\Psi(d;j+1)(\psi(d;m-d-1)_j) = \psi(d,m-d-1)_{j+1}$ for $j \leq d-3$, and
2. $\Psi(d+1;j+1) = \Psi(d,j)$.

Lemma 4.6.7. ***(** $\Psi(d;j)$ **are Non-Decreasing)***

For each $m \geq d+1$ with $d \geq 2$ and each $j \in [1, d-2]$, the function $\Psi(d,j)(x)$ is non-decreasing.

Proof. First, observe that

$$\Psi(d,j)(x+1) - \Psi(d,j)(x) = 1 + \left\lfloor \frac{1}{x+2} \cdot \binom{x+1+d-j}{d-j} \right\rfloor - \left\lfloor \frac{1}{x+1} \cdot \binom{x+d-j}{d-j} \right\rfloor,$$

and thus

$$\Psi(d,j)(x+1) - \Psi(d,j)(x) \geq \frac{1}{x+2} \cdot \binom{x+1+d-j}{d-j} - \frac{1}{x+1} \cdot \binom{x+d-j}{d-j}.$$

Now, observe that

$$\frac{1}{x+2}\cdot\binom{x+1+d-j}{d-j}=\frac{(x+1+d-j)!}{(x+2)(d-j)!(x+1)!}=\frac{(x+d-j)!}{(d-j)!(x+1)!}\left(\frac{x+1+d-j}{(x+2)}\right),$$

and

$$\frac{1}{x+1}\cdot\binom{x+d-j}{d-j}=\frac{(x+d-j)!}{(x+1)(d-j)!x!}=\frac{(x+d-j)!}{(x+2)(d-j)!(x+1)!}.$$

Hence,

$$\Psi(d,j)(x+1)-\Psi(d,j)(x)\geq\frac{(x+d-j)!}{(d-j)!(x+1)!}\left(\frac{x+1+d-j}{(x+2)}-1\right)$$

and the right side is positive when $j \leq d-2$. □

Lemma 4.6.8. ***($\psi(d, m-d-1)_{d-2}$ is Non-Decreasing in d)***
Fix $m \geq 4$. Then,

$$\psi(2,m-3)_0\leq\psi(3,m-4)_1\leq\cdots\psi(m-3,2)_{m-5}\leq\psi(m-2,1)_{m-4}.$$

Furthermore, for each $d \in [2, m-3]$,

$$\dim\left(\mathcal{M}\left(3;\psi(d,m-d-1)_{d-2}\right)\right)\leq\dim\left(\mathcal{M}\left(3;\psi(d+1,m-d-2)_{d-1}\right)\right).$$

Proof. In light of Remark 4.6.6 and Lemma 4.6.7, it suffices to show that

$$\psi(d,m-d-1)_0\leq\psi(d+1,m-d-2)_1,$$

to prove the claim. However,

$$\psi(d,m-d-1)_0 = m-d-1 \leq m-d-2+\left\lceil \frac{1}{m-d-1} \cdot \binom{m-2}{d-1} \right\rceil = \psi(d+1,m-d-1)_1.$$

$\square$

Having proved Lemma 4.6.8, we now begin to work towards the second task outlined in Remark 4.6.4; namely, establishing a criterion for m in terms of d, which, when met, implies

$$\frac{(m-1)!}{d!} < \dim\left(\mathcal{M}\left(3; \psi(d+1, m-d-2)_{d-1}\right)\right).$$

Next, we establish elementary functions which we will use to approximate $\psi(d,m-d-1)_{d-2}$ from below.

Definition 4.6.9. (Auxiliary Functions Ω and $\omega(d,j)$)

For each $d \geq 4$ and $j \in [1, d-3]$, define $\omega(d,j) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\omega(d,j) = \frac{1}{(d-j)!} x^{d-j-1}.$$

Similarly, for each pair $m > d$ with $d \geq 4$, define the function Ω by

$$\Omega(d,m) = \left(\omega(d,d-3) \circ \cdots \circ \omega(d,1)\right)(m-d-1).$$

Remark 4.6.10. (Bounding Properties of Ω and $\omega(d,j)$)

Observe that for each j,

$$\omega(d,j+1)\left(\psi(d,m-d-1)_j\right) \leq \psi(d,m-d-1)_{j+1}.$$

In particular,

$$\Omega(d,m) \le \psi(d, m-d-1)_{d-2}.$$

Example 4.6.11. (Sample Ω Bounds)

Consider the cases where $d = 5$ and $m = 10$, $m = 100$. Note that

$$\Omega(5,10) \approx 1.185, \qquad \Omega(5,100) \approx 1.996 \times 10^8,$$
$$\psi(5,10)_3 = 133, \qquad \psi(5,100)_4 \approx 3.633 \times 10^8.$$

Lemma 4.6.12. ***(Explicit Form of Ω)***

For any $d \ge 4$, set

$$\mathfrak{C}_d = \prod_{j=3}^{d-1} \frac{1}{(j!)^{(j-2)!}}.$$

For all $m \ge d+2$,

$$\Omega(d,m) = \mathfrak{C}_d (m-d-1)^{(d-2)!}.$$

Proof. We proceed by induction on d. When $d = 4$, we have

$$\Omega(4,m) = \omega(4,1)(m-5) = \frac{1}{3!}(m-5)^2 = \mathfrak{C}_4 (m-5)^{2!}.$$

For arbitrary d, recall

$$\Omega(d,m) = (\omega(d,d-3) \circ \cdots \circ \omega(d,1))\,(m-d-1),$$

and

$$\Omega(d-1, m-1) = \left(\omega(d-1, d-4) \circ \cdots \circ \omega(d-1, 1)\right)(m-d-1).$$

Note that $\omega(d+1, j+1) = \omega(d, j)$ by definition. Consequently,

$$\begin{aligned}
\Omega(d, m) &= \left(\omega(d, d-3) \circ \cdots \circ \omega(d, 2) \circ \omega(d, 1)\right)(m-d-1), \\
&= \left(\omega(d, d-3) \circ \cdots \circ \omega(d, 2)\right)\left(\omega(d, 1)(m-d-1)\right), \\
&= \left(\omega(d-1, d-4) \circ \cdots \circ \omega(d-1, 1)\right)\left(\omega(d, 1)(m-d-1)\right), \\
&= \Omega(d-1, \omega(d, 1)(m-d-1) + d + 1).
\end{aligned}$$

By induction, we know that

$$\begin{aligned}
\Omega(d-1, \omega(d, 1)(m-d-1) + d + 1) &= \mathfrak{C}_{d-1}\left(\omega(d, 1)(m-d-1)\right)^{(d-3)!}, \\
&= \mathfrak{C}_{d-1}\left(\frac{1}{(d-1)!}(m-d-1)^{d-2}\right)^{(d-3)!}, \\
&= \mathfrak{C}_{d-1}\left(\frac{1}{(d-1)!}\right)^{(d-3)!}\left((m-d-1)^{d-2}\right)^{(d-3)!}, \\
&= \mathfrak{C}_{d}(m-d-1)^{(d-2)!}.
\end{aligned}$$

Consequently, we have that

$$\Omega(d, m) = \Omega(d-1, \omega(d, 1)(m-d-1) + d + 1) = \mathfrak{C}_{d}(m-d-1)^{(d-2)!}.$$

$\square$

In the following lemma, we prove an inequality that will be useful in the proof of Proposition 4.6.14, the proposition which establishes the criterion we seek.

Lemma 4.6.13. ***(Bounding*** $\log(\mathfrak{C}_d)$***)***

For each $d \geq 4$*,*

$$\log(\mathfrak{C}_d) \geq 2(d-2)! - 2(d-2)!\log(d-1) - 2(d-3)!\log(d-1).$$

Proof. Observe that

$$\log(\mathfrak{C}_d) = -\sum_{j=3}^{d-1}(j-2)!\log(j!) \geq -\sum_{j=3}^{d-1}(j-2)!\log\left(e^{1-j}j^j\right),$$

where the approximation is due to Stirling's approximation (4.5.1). Hence,

$$\begin{aligned}
\log(\mathfrak{C}_d) &\geq -\sum_{j=3}^{d-1}(j-2)!\left(1-j+j\log(j)\right), \\
&= \sum_{j=3}^{d-1}(j-1)! - \sum_{j=3}^{d-1}(j-2)!(j)\log(j), \\
&= \sum_{j=1}^{d-3}(j+1)! - \sum_{j=1}^{d-3}(j)!(j+2)\log(j+2), \\
&= \sum_{j=1}^{d-3}(j+1)! - \sum_{j=1}^{d-3}(j)!(j+1)\log(j+2) - \sum_{j=1}^{d-3}(j)!\log(j+2), \\
&\geq \sum_{j=1}^{d-3}(j+1)! - \log(d-1)\sum_{j=1}^{d-3}(j+1)! - \log(d-1)\sum_{j=1}^{d-3}j!, \\
&= (1-\log(d-1))\sum_{j=1}^{d-3}(j+1)! - \log(d-1)\sum_{j=1}^{d-3}j!\ .
\end{aligned}$$

Recall that for any positive integer a, $\sum_{j=1}^{a} j! \leq 2(a!)$. Since $d \geq 4$, $1-\log(d-1)$ and

$-\log(d-1)$ are both negative, hence

$$\log\left(\mathfrak{C}_d\right) \geq \left(1-\log(d-1)\right)\left(2(d-2)!\right) - \log(d-2)\left(2(d-3)!\right),$$

$$= 2(d-2)! - 2(d-2)!\log(d-1) - 2(d-3)!\log(d-1).$$

□

Proposition 4.6.14. ***(The Ω Condition)***

Fix $d \geq 6$. For any $m \geq d^2-d+4$ such that

$$m^2 - \frac{5}{2}m + \frac{1}{2} < (d+1) + \log(d)\left(d+\frac{1}{2}\right) + 6(d-3)!(d-2-\log(d-1)),$$

it follows that

$$\Phi(d, m-d-1) < \Phi(d+1, m-d-2).$$

Proof. In light of Lemma 4.6.8, it suffices to establish

$$\frac{(m-1)!}{d!} < \dim\left(\mathcal{M}\left(3, \psi(d+1, m-d-2)_{d-1}\right)\right) + m. \tag{4.6.3}$$

First, we approximate $\psi(d+1, m-d-2)_{d-1}$ below by $\lceil \Omega(d+1, m-1) \rceil$ to get

$$\dim\left(\mathcal{M}\left(3, \psi(d+1, m-d-2)_{d-1}\right)\right) \geq \dim\left(\mathcal{M}\left(3, \Omega(d+1, m)\right)\right).$$

Further,

$$\dim\left(\mathcal{M}\left(3,\Omega(d+1,m)\right)\right)+m$$

$$=\frac{1}{6}\left(\lceil\Omega(d+1,m-1)\rceil^3+6\lceil\Omega(d+1,m-1)\rceil^2+11\lceil\Omega(d+1,m-1)\rceil+6\right)$$
$$-\left(\lceil\Omega(d+1,m-1)\rceil+1\right)^2+m,$$

$$\geq\frac{1}{6}\left(\Omega(d+1,m-1)^3+6\Omega(d+1,m-1)^2+11\Omega(d+1,m-1)+6\right)$$
$$-\left(\Omega(d+1,m-1)+1\right)^2+m,$$

$$=\frac{1}{6}\Omega(d+1,m-1)^2-\frac{1}{6}\Omega(d+1,m-1)+m.$$

Since $d\geq 6$ and $m\geq d^2-d+4\geq 34$, it follows that

$$\Omega(d+1,m)\geq\Omega(6,m)\geq\Omega(6,34)>5.6\cdot 10^{12},$$

and consequently

$$\frac{1}{6}\Omega(d+1,m-1)^2-\frac{1}{6}\Omega(d+1,m-1)+m>\frac{1}{7}\Omega(d+1,m-1)^2. \tag{4.6.4}$$

Substituting inequality (4.6.4) into inequality (4.6.3) and re-arraning, we arrive at the sufficient criterion

$$\frac{7(m-1)!}{d!}<\Omega\left(d+1,m-d-2\right)^3=\mathfrak{C}_d^3(m-d-2)^{3(d-2)!}.$$

Next, we apply Stirling's approximation (4.5.1) and obtain

$$\frac{7(m-1)^{m-\frac{1}{2}}e^{2-m}}{\sqrt{2\pi}d^{d+\frac{1}{2}}e^{-d}} < \mathfrak{C}_d^3(m-d-2)^{3(d-2)!},$$

which we re-arrange as

$$\frac{(m-1)^{m-\frac{1}{2}}}{e^m(m-d-2)^{3(d-2)!}} < \frac{\sqrt{2\pi}}{7} \cdot e^{d+2} \cdot d^{d+\frac{1}{2}} \cdot \mathfrak{C}_d^3. \tag{4.6.5}$$

We take the log of both sides of inequality (4.6.5) and examine them individually. First, the right side of inequality (4.6.5) becomes

$$\log\left(\frac{\sqrt{2\pi}}{7}\right) + (d+2) + \left(d+\frac{1}{2}\right)\log(d) + 3\log\left(\mathfrak{C}_d\right). \tag{4.6.6}$$

By applying Lemma 4.6.13, observing $\log\left(\frac{\sqrt{2\pi}}{7}\right) \geq -1$, and simplifying, it suffices to replace expression (4.6.6 with

$$(d+1) + \left(d+\frac{1}{2}\right)\log(d) + 6(d-2)! - 6(d-2)!\log(d-1) - 6(d-3)!\log(d-1). \tag{4.6.7}$$

The left side of inequality (4.6.5 becomes

$$\left(m-\frac{1}{2}\right)\log(m-1) - m - 3(d-2)!\log(m-d-2). \tag{4.6.8}$$

For $m \geq d^2 - d + 4$,

$$\log(m-d-2) > \log\left(d^2-d+1\right) = 2\log(d-1),$$

and multiplying by $3(d-2)!$ yields

$$3(d-2)!\log(m-d-2) > 6(d-2)!\log(d-1). \tag{4.6.9}$$

By combining equations (4.6.7) and (4.6.8) with inequality (4.6.9), we obtain

$$\left(m-\frac{1}{2}\right)\log(m-1)-m < (d+1)+\left(d+\frac{1}{2}\right)\log(d)+6(d-2)!-6(d-3)!\log(d-1).$$

Using the approximation $\log(m-1) < m-1$, we finally arrive at the condition

$$m^2-\frac{5}{2}m+\frac{1}{2} < (d+1)+\left(d+\frac{1}{2}\right)\log(d)+6(d-3)!(d-2-\log(d-1)),$$

as claimed above. $\square$

Using the simple approximations

$$\log(d) \geq \log(6) > 1,$$
$$d-2-\log(d-1) > 1,$$

we arrive at a simplified condition.

Corollary 4.6.15. ***(The Simplified Omega Condition)***
Fix $d \geq 6$. For any $m \geq d^2-d+4$ such that

$$m^2-\frac{5}{2}m \leq 6(d-3)!+2d+1,$$

it follows that

$$\Phi(d, m-d-1) < \Phi(d+1, m-d-2),$$

and so $F(m) \geq \frac{(m-1)!}{d!}$.

Proof. Together, Lemma 4.6.8 and Proposition 4.6.14 yield that

$$\Phi(d, m-d-1) < \Phi(d', m-d-' 1),$$

for each $d' \in [d+1, m-2]$. For any $d'' < d$, we have

$$\frac{(m-1)!}{d!} < \frac{(m-1)!}{(d')!} \leq \Phi(d'', m-d''-1).$$

Hence,

$$F(m) \geq 2\lfloor \Phi(d, m-d-1) \rfloor + 1 \geq \frac{(m-1)!}{d!}.$$

□

We now state and prove a corollary, and then prove Theorem 4.6.1.

Corollary 4.6.16. ***(Bounding the Ratio*** $\frac{F(m)}{G'(m)}$***)***
For $d \geq 11$ *and* $m \geq 2d^2 + 8d + 10$, $\frac{F(m)}{G'(m)} > d + 1$.

Remark 4.6.17. We expect that better estimates of $\frac{F(m)}{G'(m)}$ could reasonably be obtained. However, Corollary 4.6.16 suffices to prove Theorem 4.6.1, which establishes that $G'(m)$ is the better bounding function and thus we do not need additional data on the growth rate of $F(m)$.

Proof. **(Proof of Corollary 4.6.16)**

Let $d \geq 4$. Recall that Corollary 4.5.8 applies for $m \geq 2d^2 + 4d + 4$ and so we set $\theta_d = 2d^2 + 4d + 4$. Similarly, Corollary 4.6.15 applies for $m \geq d^2 - d + 4$ such that

$$m^2 - \frac{5}{2}m \leq 6(d-3)! + 2d + 1.$$

Correspondingly, we set

$$\Theta_d = \max\left\{m \in \mathbb{Z} \mid m^2 - \frac{5}{2}m \leq 6(d-3)! + 2d + 1\right\}.$$

Observe that $2d^2 + 4d + 4 \geq d^2 - d + 4$ and for $d \geq 11$, we have

$$\theta_{d+1} = 2d^2 + 8d + 10 < \Theta_d.$$

Consequently, Corollary 4.5.8 yields $G'(\theta_{d+1}) < \frac{(\theta_{d+1}-1)!}{(d+1)!}$ and Corollary 4.6.15 yields that $F(\theta_{d+1}) \geq \frac{(\theta_{d+1}-1)!}{d!}$. As a result, we have

$$\frac{F(\theta_{d+1})}{G'(\theta_{d+1})} > \frac{\left(\frac{(\theta_{d+1}-1)!}{(d+1)!}\right)}{\left(\frac{(\theta_{d+1}-1)!}{d!}\right)} = \frac{(d+1)!}{d!} = d + 1.$$

In fact,

$$(6(d-3)! + 2d + 1) - \left(\theta_{d+2}^2 - \frac{5}{2}\theta_{d+2}\right)$$

is positive and strictly increasing for $d \geq 11$, so $\theta_{d+2} < \Theta_d$. Hence, Corollaries 4.5.8 and

4.6.15 yield that

$$\frac{F(m)}{G'(m)} > \frac{\left(\frac{(m-1)!}{(d+1)!}\right)}{\left(\frac{(m-1)!}{d!}\right)} = d+1,$$

for all $m \geq \theta_{d+1} = 2d^2 + 8d + 10$. $\square$

Proof. **(Proof of Theorem 4.6.1)**

First, observe that Corollary 4.6.16 implies that

$$\lim_{m\to\infty} \frac{F(m)}{G'(m)} = \infty,$$

and that $G'(m) < F(m)$ for

$$m \geq 2\left(11^2\right) + 8(11) + 10 = 340.$$

The claim for $m \in [1, 59]$ comes from explicit computation and the relevant data is provided in Appendices A.1 and A.2. Finally, we address the cases of $m \in [60, 339]$. Recall that Lemma 4.5.2 yields

$$\vartheta(d, m-d-1) \leq m-d-2+\binom{m}{d}.$$

For a fixed d, $\vartheta(m, d-m-1)$ is bounded above by a polynomial of degree d in m with positive coefficients. Remark 4.3.5 yields that $\dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1)\right)$ is also a polynomial in m, hence there is a polynomial $p_d(m)$ with positive coefficients that bounds $\dim\left(\mathcal{M}(2,\ldots,d;\vartheta(d,m-d-1)\right)$ above. Consequently, there is a minimal positive integer

a_d such that $\frac{(a_d-1)!}{d!} > p_d(a_d)$. Moreover, $\frac{(m-1)!}{d!} > p_d(m)$ and so

$$G'(m) \leq \frac{(m-1)!}{d!}$$

for all $m \geq a_d$.

We compute explicitly that for all $m \geq 57$,

$$G'(m) \leq \max\left\{\dim\left(\mathcal{M}(2,\ldots,9;\vartheta(9,m-10)\right) \leq, \frac{(m-1)!}{9!}\right\} = \frac{(m-1)!}{9!}.$$

Additionally, we explicitly compute that for $m \leq 339$,

$$\frac{(m-1)!}{6!} < \dim\left(\mathcal{M}\left(3;\psi(7,m-d-1)_5\right)\right),$$

and hence the argument proving Corollary 4.6.15 also yields that

$$F(m) \geq \frac{(m-1)!}{6!}.$$

As a consequence,

$$\frac{F(m)}{G'(m)} \geq \frac{\left(\frac{(m-1)!}{6!}\right)}{\left(\frac{(m-1)!}{9!}\right)} = \frac{9!}{6!} = 504,$$

for all $m \in [60, 340]$, which yields the theorem. $\square$

4.7 Remaining Questions

To the best of the author's knowledge, the construction of the bounding function $G'(m)$ of Definition 4.3.11 exhausts the methods and techniques for determining upper bounds on resolvent degree from the classical literature, including [Bri1786, Che1954, Ham1836, Hil1927, Seg1945, Syl1887, SH1887, SH1888, Tsc1683, Wim1927], as well as the modern insights from [Bra1975, HS2021, Sut2021C, Wol2021].

The bounding functions of Sutherland and Wolfson are explicitly constructed by determining points on the Tschirnhaus complete intersections $\tau^{\circ}_{1,\dots,m-1}$ and we can similarly rephrase the constructions underlying the bounding functions of Brauer, Hamilton, and Sylvester. However, there are solutions of the quintic, the sextic, and the septic which use alternative constructions of Tschirnhaus transformations (see [Kle1884] and [Kle1905] for the respective original works or [Mor1956] and [Sut2019] for the respective English translations). We believe it would be insightful to understand whether one can reduce the general question of determining $\mathrm{RD}(n)$ to the more specific question of determining points on the Tschirnhaus complete intersections $\tau^{\circ}_{1,\dots,m-1}$.

Question 4.7.1. ***(Optimal Formulas via Tschirnhaus Complete Intersections)*** *For every n, let m_n be such that $\mathrm{RD}(n) \leq n - m_n$. Is there a formula for the general degree n polynomial obtained by determining a point of $\tau^{\circ}_{1,\dots,m_n-1}$ over an extension L/K_n of bounded resolvent degree?*

For general m, the definition of $G'(m)$ uses the combinatorial condition of Theorem 2.1 of [DeMa1998] to guarantee the existence of k-planes on the $\tau^{\circ}_{1,\dots,d}$ and then uses the dimension of the relevant moduli space (see Section 4.3 for details). Notably, this combinatorial condition is non-constructive and relies only on the type of $\tau^{\circ}_{1,\dots,d}$. One might hope that such formulas could be determined using more explicit constructions and one approach may be to leverage the specific geometry of the $\tau^{\circ}_{1,\dots,d}$.

Question 4.7.2. ***(RD Bounds via Explicit Constructions of k-Planes)***

Is there a bounding function $\mathfrak{G}(m)$ with $\mathfrak{G}(m) \leq G'(m)$ which arises from an explicit construction of k-planes on the $\tau^{\circ}_{1,\dots,d}$? If so, is it possible to determine the bounding function $\mathfrak{G}(m)$ such that

$$\lim_{m\to\infty} \frac{G'(m)}{\mathfrak{G}(m)} = \infty?$$

In Subsection 3.3.1, we consider an intersection of hypersurfaces $V \subseteq \mathbb{P}^r_K$ of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$ and ask how large does r need to be so that we can determine a point of V over an extension L/K with $\mathrm{RD}(L/K) \leq \mathrm{RD}(d)$, i.e. over an extension given by solving polynomials of degree at most d. We denoted the minimal such r by $r(V)$ and showed that $r(V) \leq g(V)$, where $g(V)$ is the bound obtained by the geometric obliteration algorithm.

Question 4.7.3. ***(Minimal Dimension Bounds vs. Geometric Dimension Bound)***

For which intersections of hypersurfaces V is the inequality $r(V) \leq g(V)$ strict? Are there classical examples of types of intersections of hypersurfaces where the inequality is not strict?

There is a tension which underlies the constructions determining $G'(m)$, as we must balance the ambient dimension necessary and the resolvent degree of the required extension. We now give an example which is indicative of the general phenomenon.

Example 4.7.4. (Cubic Hypersurfaces)

Let us now briefly consider a smooth cubic hypersurface $H = \mathbb{V}(f) \subseteq \mathbb{P}^r_K$. When $r = 3$, H is a smooth cubic surface and the Cayle-Salmon theorem yields that H contains exactly 27 lines. Theorem 8.2 of [FW2019] yields that the resolvent degree of finding a line on a smooth cubic surface is at most 3. Additionally, that H has 27 lines is consistent with Theorem 2.1 of [DeMa1998], which states the Fano variety of lines on H is non-empty and has dimension 0. Notably, when $r = 3$, most points $P \in H(K)$ do not lie on a line of H over an algebraic closure $\overline{K}$.

When $r = 4$, however, any polar cone $\mathcal{C}(H; P)$ has dimension at least one and thus every point $P \in H(K)$ lies on at least one line $\Lambda = \Lambda(P, Q) \subseteq H$ over an algebraic closure $\overline{K}$. To determine such a point Q directly, we must solve a polynomial of degree

$$\deg\left(\mathcal{C}(H; P)\right) = 3! = 6.$$

Hence, we can determine a line through any point P over an extension L/K with $\mathrm{RD}\left(L/K\right) \leq \mathrm{RD}(6) \leq 2$.

Finally, observe that

$$g\left(\mathcal{C}(V; P)\right) = g(3; 1, 1, 1) = g(2; 1, 3) = 5.$$

Thus, when $r \geq 5$, we can determine a point $Q \in \mathcal{C}(V; P) \setminus \{P\}$ over a solvable extension.

Now, let $V \subseteq \mathbb{P}^r_K$ be an intersection of hypersurfaces of type $\begin{bmatrix} d & \cdots & 1 \\ \ell_d & \cdots & \ell_1 \end{bmatrix}$. For each $k \geq 1$, take $s_k(V)$ to be the minimal s such that

$$(k + 1(s - k) - \sum_{j=1}^{d} \ell_j \binom{k+j}{j} \geq 0.$$

One implication of Theorem 2.1 of [DeMa1998] is that V contains a k-plane for all $r \geq s_k(V)$. We expect $s_k(V)$ to be the minimal ambient dimension required for V to contain a k-plane; however, we expect the resolvent degree of determining such a k-plane to be large. Conversely, we expect $r\left(\mathcal{C}^k\left(V; P_0, \ldots, P_{k-1}\right)\right) + k$, the ambient dimension required to determine a k-polar point over an extension L/K with $\mathrm{RD}(L/K) \leq \mathrm{RD}(d)$, to be large.

Question 4.7.5. ***(Minimizing Ambient Dimensions vs. Minimixing RD of Extensions)***
Let V be an intersection of hypersurfaces. How do

- $g\left(\mathcal{C}^k\left(V; P_0, \ldots, P_{k-1}\right)\right) + k$,
- $r\left(\mathcal{C}^k\left(V; P_0, \ldots, P_{k-1}\right)\right) + k$, *and*
- $s_k(V)$,

compare ?

As was discussed in Section 2.1, the resolvent degree of a finite group is well-defined and

$$\mathrm{RD}(n) = \mathrm{RD}\left(S_n\right) = \mathrm{RD}\left(A_n\right).$$

In [FW2019], Theorem 3.3 emphasizes the role of simple groups in the theory of resolvent degree of finite groups and Theorems 8.1, 8.2, 8.3, 8.4, and 8.5 connect the resolvent degree of S_6, S_7, S_8 and $W(E_6), W(E_7)$ to the resolvent degree of certain enumerative problems. The essential dimension of finite simple groups has been studied further, including the works [BF2003, BR1997, Dun2010, Mor2021, Rei2011]; similarly, the essential p-dimension of finite simple groups has been addressed in [BMKS2016, BF2020, FKW2021B, Kni2021, RS2020]. While the following question is not new (see Problem 3.5 of [FW2019]), we emphasize it here as well.

Question 4.7.6. ***(Resolvent Degree of Finite Simple Groups)***
Let G be a finite simple group. What is $\mathrm{RD}(G)$*?*

As far as the author is aware, Question 4.7.6 has only been discussed in literature when $G = W(E_6)^+, W(E_7)^+$ (in [FW2019]), $G = \mathrm{PSL}(2,7)$ (in [FKW2022]), or when G is an alternating group.

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Appendix A

Numerics and Details

A.1 Explicit Bounds on RD(n)

The values of $G'(m)$ for various m were first established at different times and we consolidated these variations into eight different groups. Group 1 is when $m = 2$ and the bounds were proved by the Babylonians and Egyptians. Group 2 is when $m = 3$ and the bounds were proved by Ferrari. Group 3 is when $m = 4$ and the bounds were proved by Bring in [Bri1786]. Group 4 is when $m = 5$ and the bounds were proved by Segre in [Seg1945]. Group 5 is when $m = 6$ and the bounds are Theorem 4.1.1, whose proof fixes the gaps in the argument found in [Che1954]. Group 6 is for $m \in [7, 12]$ and the bounds are Theorem 4.1.4. Group 7 is when $m \in [13, 17] \cup [22, 25]$, which was proved as Theorem 4.2.3. Group 8 is for $m \in [18, 21] \cup [26, \infty]$ and the bounds are Theorem 4.3.10.

Table A.1: Upper Bounds on RD(n), I

m	$G'(m)$	$G(m)$	$F(m)$	$\sim F(m)/G'(m)$	Group
2	3	3	3	1	1
3	4	4	4	1	2
4	5	5	5	1	3
5	9	9	9	1	4
6	21	21	41	1.952	5
7	109	109	121	1.175	
8	325	325	841	2.645	
9	1681	1681	6721	3.998	6
10	15121	15121	60481	4.000	
11	151,201	151,201	604,801	4.000	
12	1,663,201	1,663,201	6,652,801	4.000	
13	5,250,198	19,958,401	78,485,043	14.949	
14	51,891,841	259,459,201	320,082,459	6.168	
15	726,485,761	3,632,428,801	3,632,428,801	5.000	7
16	10,897,286,401	54,486,432,001	54,486,432,001	5.000	
17	174,356,582,401	348,489,068,134	871,782,912,001	5.000	

Table A.2: Upper Bounds on RD(n), II

m	$\sim G'(m)$	$\sim G(m)$	$\sim F(m)$	$\sim F(m)/G'(m)$	Group
18	$2.964 \cdot 10^{12}$	$2.964 \cdot 10^{12}$	$1.482 \cdot 10^{13}$	5.000	
19	$5.335 \cdot 10^{13}$	$5.335 \cdot 10^{13}$	$2.668 \cdot 10^{14}$	5.000	8
20	$1.014 \cdot 10^{15}$	$1.014 \cdot 10^{15}$	$5.069 \cdot 10^{15}$	5.000	
21	$2.027 \cdot 10^{16}$	$2.027 \cdot 10^{16}$	$1.014 \cdot 10^{17}$	5.000	
22	$3.819 \cdot 10^{17}$	$4.258 \cdot 10^{17}$	$2.129 \cdot 10^{18}$	5.574	
23	$1.561 \cdot 10^{18}$	$9.367 \cdot 10^{18}$	$4.683 \cdot 10^{19}$	30.000	7
24	$3.591 \cdot 10^{19}$	$2.154 \cdot 10^{20}$	$1.077 \cdot 10^{21}$	30.000	
25	$8.617 \cdot 10^{20}$	$9.235 \cdot 10^{20}$	$2.585 \cdot 10^{22}$	30.000	
26	$2.154 \cdot 10^{22}$	$2.154 \cdot 10^{22}$	$6.463 \cdot 10^{23}$	30.000	8

A.2 Explicit Approximations of $F(m)/G'(m)$

For $m \geq 26$, the values of $G'(m)$ are established by Theorem 4.3.10. We now provide data about $\frac{F(m)}{G'(m)}$ and note that the transition from $m = 58$ to $m = 59$ exhibits why the ratio $\frac{F(m)}{G'(m)}$ is not always non-decreasing.

Table A.3: $F(m)/G'(m)$ for $m \in [27, 43]$

m	$F(m)/G'(m)$	$G'(m)$ given by determining an	$F(m)$ given by determining an
27	30.000		
28	30.000		
29	30.000		
30	30.000	$(m-7)$-plane on $\tau_{1,\ldots,6}$	$(m-5)$-plane on $\tau_{1,2,3,4}$
31	30.000		
32	30.000		
33	30.000		
34	146.129		
35	210.000		
36	210.000		
37	210.000		
38	210.000	$(m-8)$-plane on $\tau_{1,\ldots,7}$	$(m-5)$-plane on $\tau_{1,2,3,4}$
39	210.000		
40	210.000		
41	210.000		
42	210.000		
43	210.000		

Table A.4: $F(m)/G'(m)$ for $m \in [44, 59]$

44	294.103		
45	1680.000		
46	1680.000		
47	1680.000		
48	1680.000		
49	1680.000	$(m-9)$-plane on $\tau_{1,\ldots,8}$	$(m-5)$-plane on $\tau_{1,2,3,4}$
50	1680.000		
51	1680.000		
52	1680.000		
53	1680.000		
54	1680.000		
55	1680.000		
56	2613.173		
57	15120.000	$(m-10)$-plane on $\tau_{1,\ldots,9}$	$(m-5)$-plane on $\tau_{1,2,3,4}$
58	15120.000		
59	3024.000	$(m-10)$-plane on $\tau_{1,\ldots,9}$	$(m-6)$-plane on $\tau_{1,\ldots,5}$

A.3 Python Implementations of the Obliteration Algorithm and Related Phenomena

In Appendix A.3.1, we provide an implementation (Algorithm A.3.1) of the geometric obliteration algorithm. In Appendix A.3.2, we prove several lemmata which make the computations for the proof of Theorem 4.2.3 feasible. Algorithm A.3.5 takes the same input and provides the same output as Algorithm A.3.1, but uses the lemmata of Subsection A.3.2 to decrease computation time. Finally, Algorithm A.3.6 in Appendix A.3.4 computes the information necessary for Theorem 4.2.3.

A.3.1 The Geometric Obliteration Algorithm

Algorithm A.3.1. ***(The Geometric Obliteration Algorithm)***

- Input: An intersection of hypersurfaces V of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ with $d \geq 2$, encoded as the list DegreeList $= [\ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1]$.
- Output: The geometric dimension bound $g(d; \ell_d, \ldots, \ell_1)$.

The function ComputePolarCone inputs a list which contains the type of an intersection of hypersurfaces W. It then returns a list which contains the type of a polar cone $\mathcal{C}(W; P)$. In particular, recall that for each $d' < d$, each hypersurface H with $\deg(H) > d'$ defining W contributes exactly one new degree d' hypersurface defining $\mathcal{C}(W; P)$ and each hypersurface defining $\mathcal{C}(W; P)$ arises in this manner.

```
function ComputePolarCone(List):
    counter = List[0]
    ReturnList = [counter]
    for index in range(1,len(List)): do
        counter += List[index]
        ReturnList.append(counter)
    end for
    return ReturnList
end function
```

The function ObliterateLargestDegreeHypersurfaces inputs a list which contains the type of an intersection of hypersurfaces W whose largest degree hypersurface has degree $d \geq 3$. It identifies the number of hypersurfaces of largest degree and proceeds to iteratively remove a hypersurface H of largest degree and compute a polar cone of the remaining intersection of hypersurfaces W' (with an additional hyperplane included).

Note that an additional hyperplane is added each time to avoid repeated polar cone points, i.e. if P was the cone point of the previous polar cone point, we pass to a hyperplane which does not contain P to ensure that the cone point Q of the next polar cone satisfies $Q \neq P$. Also, the polar cone of a hyperplane plane at any point is just the hyperplane itself, so to compute the combinatorics, it suffices to add one after computing the polar cone instead of doing it beforehand.

As taking the polar cone of a hypersurface H introduces only hypersurfaces of strictly smaller degree, this process terminates and ObliterateLargestDegreeHypersurfaces returns a list whose data is the multi-degree of an intersection of hypersurfaces V' whose largest degree hypersurface has degree $d-1$.

```
function ObliterateLargestDegreeHypersurfaces(List):
    while List[0] > 0: do
        List[0] -= 1
        TempList = ComputePolarCone(List)
        List = TempList
        List[len(List)-1] += 1
    end while
    ReturnList = []
    for index in range(1,len(List): do
        ReturnList.append(List[index])
    end for
    return ReturnList
end function
```

The function ObliterateQuadricsViaLoops works similarly to ObliterateLargestDegreeHypersurfaces, but the input is the multi-degree of an intersection of hypersurfaces of type $\begin{bmatrix} 2 & 1 \\ \ell_2 & \ell_1 \end{bmatrix}$ and the loop ends with a single quadric remaining instead of zero quadrics remaining.

```
function ObliterateQuadricsViaLoops(List):
    while List[0] > 1: do
        List[0] -= 1
        TempList = ComputePolarCone(List)
        List = TempList
        List[len(List)-1] += 1
    end while
    return [List[0],List[1]]
end function
```

The procedure Main inputs the multi-degree of an intersection of hypersurfaces V as the list DegreeList and proceeds to successively "obliterate" the hypersurfaces of largest degree. The final step of the procedure is to return a list of the form $[1, \alpha]$, which is the requisite intersection of a single quadric and α hyperplanes.

```
procedure Main(DegreeList):
    for index in range(1,len(DegreeList)-1): do
        TempDegreeList = ObliterateLargestDegreeHypersurfaces(DegreeList)
        DegreeList = TempDegreeList
    end for
    FinalList = ObliterateQuadricsViaLoops(DegreeList)
    Sum = FinalList[0] + FinalList[1]
    return Sum
end procedure
```

A.3.2 Lemmata for Computational Improvements

In this subsection, we give explicit numerics for Proposition 3.3.10 when $d = 2, 3, 4$.

Lemma A.3.2. ***(Obliterating Quadrics)***

Consider an intersection of hypersurfaces V of type $\begin{bmatrix} 2 & 1 \\ \ell_2 & \ell_1 \end{bmatrix}$. *Then,*

$$g(V) = 1 + \ell_1 + \frac{1}{2}(\ell_2 - 1)(\ell_2 + 2).$$

Proof. First, observe that $V^{\mathrm{Syl}}(2; 1)$ has type

$$\begin{bmatrix} 2 & 1 \\ \ell_2 - 1 & \ell_1 + \ell_2 \end{bmatrix},$$

by Definition 3.3.9. Similarly, $V^{\mathrm{Syl}}(2; 2)$ has type

$$\begin{bmatrix} 2 & 1 \\ \ell_2 - 2 & \ell_1 + \ell_2 + \ell_2 - 1 \end{bmatrix}.$$

Proceeding in this manner yields that $V^{\mathrm{Syl}}(2; \lambda_2 - 1)$ has type

$$\begin{bmatrix} 2 & 1 \\ 1 & \ell_1 + \sum_{j=1}^{\ell_2 - 1} (\ell_2 - j + 1) \end{bmatrix},$$

and we note that

$$\sum_{j=1}^{\ell_2 - 1} (\ell_2 - j + 1) = \frac{1}{2}\left(\ell_2 - 1\right)\left(\ell_2 + 2\right).$$

From Lemma 3.3.5 and Definition 3.3.9, we see that

$$g(V) = g\left(V^{\mathrm{Syl}}(2; \lambda_2 - 1)\right) = 1 + \ell_1 + \frac{1}{2}\left(\ell_2 - 1\right)\left(\ell_2 + 2\right).$$

$\square$

Lemma A.3.3. ***(Obliterating Cubics)***

Consider an intersection of hypersurfaces V of type $\begin{bmatrix} 3 & 2 & 1 \\ \ell_3 & \ell_2 & \ell_1 \end{bmatrix}$. *Then, V_1^{Syl} is of type* $\begin{bmatrix} 2 & 1 \\ \beta_3 & \alpha_3 \end{bmatrix}$*, where*

$$\begin{aligned} \beta_3 &= \ell_2 + \frac{1}{2}(\ell_3 - 1)\ell_3, \\ \alpha_3 &= \ell_1 + \ell_2\ell_3 + \frac{1}{2}\ell_3(\ell_3 + 1) + \frac{1}{6}\ell_3\left(2\ell_3^2 - 3\ell_3 + 1\right). \end{aligned}$$

Proof. An argument analogous to the proof of Lemma A.3.2 yields that

$$\beta_3 = \ell_2 + \sum_{j=1}^{\ell_3}(\ell_3 - j) = \ell_2 + \frac{1}{2}(\ell_3 - 1)\ell_3.$$

Next, observe that $V^{\mathrm{Syl}}(3; j)$ has type

$$\begin{bmatrix} 3 & 2 & 1 \\ \ell_3 - j & \ell_2 + \sum_{k=1}^{j}(\ell_3 - k) & \lambda_j \end{bmatrix}.$$

Consequently,

$$\lambda_{j+1} = \lambda_j + (\ell_3 - j - 1) + \left(\ell_2 + \sum_{k=1}^{j}(\ell_3 - k)\right) + 1.$$

Combined with the initial condition $\lambda_0 = \ell_1$, we obtain that

$$\begin{aligned}
\alpha_3 &= \ell_1 + \left(\sum_{j_1=1}^{\ell_3} (\ell_3 - j_1 + 1) \right) + \left(\sum_{j_2=1}^{\ell_3} \ell_2 + \sum_{j_3=2}^{\ell_3} \sum_{j_4=1}^{j_4-1} \ell_3 - j_4 \right), \\
&= \ell_1 + \frac{1}{2}\ell_3(\ell_3+1) + \left(\ell_2\ell_3 + \sum_{j_3=2}^{\ell_3} \sum_{j_4=1}^{j_3-1} \ell_3 - j_4 \right), \\
&= \ell_1 + \ell_2\ell_3 + \frac{1}{2}\ell_3(\ell_3+1) + \sum_{j_3=2}^{\ell_3} \sum_{j_4=1}^{j_3-1} (\ell_3 - j_2), \\
&= \ell_1 + \ell_2\ell_3 + \frac{1}{2}\ell_3(\ell_3+1) + \frac{1}{6}\ell_3 \left(2\ell_3^2 - 3\ell_3 + 1\right).
\end{aligned}$$

□

Lemma A.3.4. ***(Obliterating Quartics)***

Consider an intersection of hypersurfaces $V \subseteq \mathbb{P}^r_K$ of type $\begin{bmatrix} 4 & 3 & 2 & 1 \\ \ell_4 & \ell_3 & \ell_2 & \ell_1 \end{bmatrix}$. Then, V_1^{Syl} is of type $\begin{bmatrix} 3 & 2 & 1 \\ \gamma_4 & \beta_4 & \alpha_4 \end{bmatrix}$, where

$$\begin{aligned}
\gamma_4 &= \ell_3 + \frac{1}{2}(\ell_4 - 1)\ell_4, \\
\beta_4 &= \ell_2 + \ell_3\ell_4 + \frac{1}{2}(\ell_4 - 1)\ell_4 + \frac{1}{6}\ell_4 \left(2\ell_4^2 - 3\ell_4 + 1\right), \\
\alpha_4 &= \ell_1 + \ell_4 \left(\ell_2 + \ell_3 + \frac{1}{2}(\ell_4 + 1) \right) + \ell_4 \left(\frac{1}{2}\ell_3(\ell_4+1) + \frac{1}{3}\left(2\ell_4^2 - 3\ell_4 + 1\right) \right) \\
&\quad + \frac{1}{24}(\ell_4 - 2)(\ell_4 - 1)\ell_4(3\ell_4 - 1).
\end{aligned}$$

Proof. The proofs of Lemmata A.3.2 and A.3.3 generalize to determine γ_4 and β_4 in a straightforward manner. It remains to determine α_4. Note that $V^{\mathrm{Syl}}(4; j)$ has type

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ \ell_4 - j & \ell_3 + \sum\limits_{k_1=1}^{j} (\ell_4 - k_1) & \ell_2 + \left(\sum\limits_{k_2=1}^{j} \ell_4 - k_2 \right) + \sum\limits_{k_3=1}^{j} \left(\ell_3 + \sum\limits_{k_4=1}^{j-1} (\ell_4 - k_4) \right) & \lambda_j \end{bmatrix}.$$

As a result,

$$\lambda_{j+1} = \lambda_j + (\ell_4 - j - 1) + \left(\ell_3 + \sum_{k=1}^{j} (\ell_4 - k)\right) + \left(\ell_2 + \left(\sum_{k_1=1}^{j} \ell_4 - k_1\right)\right)$$
$$+ \left(\sum_{k_2=1}^{j} \left(\ell_3 + \sum_{k_3=1}^{j-1} (\ell_4 - k_3)\right)\right) + 1.$$

Given the initial condition $\lambda_0 = \ell_1$, it follows that

$$\alpha_4 = \ell_1 + \left(\sum_{j_1=1}^{\ell_4} \ell_4 - j_1 + 1\right) + \left(\sum_{j_2=1}^{\ell_4} \ell_3 + \sum_{j_3=2}^{\ell_4} \sum_{j_4=1}^{j_3-1} (\ell_4 - j_4)\right)$$
$$+ \left(\sum_{j_5=1}^{\ell_4} \ell_2 + \sum_{j_6=2}^{\ell_4} \sum_{j_7=1}^{j_6-1} (\ell_4 - j_7) + \sum_{j_8=2}^{\ell_4} \sum_{j_9=1}^{j_8-1} \ell_3 + \sum_{j_{10}=3}^{\ell_4} \sum_{j_{11}=2}^{j_{10}-1} \sum_{j_{12}=1}^{j_{11}-1} (\ell_4 - j_{12})\right),$$

$$= \ell_1 + \left(\frac{1}{2}\ell_4(\ell_4 + 1)\right) + \left(\ell_3\ell_4 + \frac{1}{6}\ell_4\left(2\ell_4^2 - 3\ell_4 + 1\right)\right)$$
$$+ \left(\ell_2\ell_4 + \frac{1}{6}\ell_4\left(2\ell_4^2 - 3\ell_4 + 1\right) + \frac{1}{2}(\ell_4 - 1)\ell_4\ell_3 + \sum_{j_{10}=3}^{\ell_4} \sum_{j_{11}=2}^{j_{10}-1} \sum_{j_{12}=1}^{j_{11}-1} (\ell_4 - j_{12})\right),$$

$$= \ell_1 + \ell_4\left(\ell_2 + \ell_3 + \frac{1}{2}(\ell_4 + 1)\right) + \ell_4\left(\frac{1}{2}\ell_3(\ell_4 - 1) + \frac{1}{3}\left(2\ell_4^2 - 3\ell_4 + 1\right)\right),$$
$$+ \sum_{j_{10}=3}^{\ell_4} \sum_{j_{11}=2}^{j_{10}-1} \sum_{j_{12}=1}^{j_{11}-1} (\ell_4 - j_{12}),$$

$$= \ell_1 + \ell_4\left(\ell_2 + \ell_3 + \frac{1}{2}(\ell_4 + 1)\right) + \ell_4\left(\frac{1}{2}\ell_3(\ell_4 + 1) + \frac{1}{3}\left(2\ell_4^2 - 3\ell_4 + 1\right)\right)$$
$$+ \frac{1}{24}(\ell_4 - 2)(\ell_4 - 1)\ell_4(3\ell_4 - 1).$$

$\square$

A.3.3 The Geometric Obliteration Algorithm with Computational Improvements

Algorithm A.3.5. ***(The Geometric Obliteration Algorithm with Computational Improvements)***

- Input: An intersection of hypersurfaces V of type $\begin{bmatrix} d & d-1 & \cdots & 2 & 1 \\ \ell_d & \ell_{d-1} & \cdots & \ell_2 & \ell_1 \end{bmatrix}$ with $d \geq 2$, encoded as the list DegreeList $= [\ell_d, \ell_{d-1}, \ldots, \ell_2, \ell_1]$.
- Output: The geometric dimension bound $g(d; \ell_d, \ldots, \ell_1)$.

We will use the same functions ComputePolarCone and ObliterateLargestDegreeHypersurfaces which were originally defined in Algorithm A.3.1.

We now implement Lemma A.3.4 (respectively, Lemmata A.3.3 and A.3.2) via the following three functions.

```
function ObliterateQuartics(List):
    a = List[0]
    b = List[1]
    c = List[2]
    d = List[3]
    gammafour = b + (1/2)*(a-1)*a
    betafour = c + a*b + (1/2)*a*(a+1) + (1/6)*(a-1)*a*(2*a-1)
    alphafour = d + a*(b+c+(1/2)*(a+1)) + a*((1/2)*b*(a-1)+(1/3)*((2*(a**2))-
(3*a)+1))
              + (1/24)*(a-2)*(a-1)*a*(3*a-1)
    return [gammafour,betafour,alphafour]
end function
```

```
function ObliterateCubics(List):
    a = List[0]
    b = List[1]
    c = List[2]
    betathree = b + (1/2)*(a-1)*a
    alphathree = c + a*b + (1/2)*a*(a+1) + (1/6)*a*((2*(a**2))-(3*a)+1)
    return [betathree,alphathree]
end function

function ObliterateQuadrics(List):
    a = List[0]
    b = List[1]
    alphatwo = b + (1/2)*a*(a+1)
    return [1,alphatwo]
end function
```

The Main procedure works very similarly to its counterpart in Algorithm A.3.1, with the only differences being the use of specialized functions to obliterate quartic, cubic, and quadric hypersurfaces.

```
procedure Main(DegreeList):
    if len(DegreeList) == 2: then
        FinalDegreeList = ObliterateQuadrics(DegreeList)
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    else if len(DegreeList) == 3: then
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateQuadrics(DegreeList)
        FinalDegreeList = TempDegreeList
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    else if len(DegreeList == 4: then
        TempDegreeList = ObliterateQuartics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateQuadrics(DegreeList)
        FinalDegreeList = TempDegreeList
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
```

```
    else:
        for index in range(1,len(DegreeList)-3): do
            TempDegreeList = ObliterateLargestDegreeHypersurfaces(DegreeList)
            DegreeList = TempDegreeList
        end for
        TempDegreeList = ObliterateQuartics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateQuadrics(DegreeList)
        FinalDegreeList = TempDegreeList
        Sum = FinalDegreeList[0] = FinalDegreeList[1]
        return Sum
    end if
end procedure
```

A.3.4 The Geometric Obliteration Algorithm for Iterated Polar Cones of Tschirnhaus Complete Intersections

Algorithm A.3.6. ***(The Geometric Obliteration Algorithm for Iterated Polar Cones of Tschirnhaus Complete Intersections)***

- Imported Packages: scipy.special, math
- Input: A positive integer d and and another positive integer $m \geq d + 2$.
- Output: The optimal reduction bound of $\tau_{1,\ldots,d}$ for m, $\Xi(m, d)$.

We will use the same functions ComputePolarCone and ObliterateLargestDegreeHypersurfaces which were originally defined in Algorithm A.3.1, as well as the functions ObliterateQuartics and ObliterateCubics which originally defined in Algorithm A.3.5.

We first implement a closed form for the type of an $(m - d - 1)^{st}$ polar cone of $\tau_{1,\ldots,d}$, which is Proposition 2.26 of [Sut2021C].

```
function PolarConeOfTschirnhausType(Type,Level):
    ReturnList = [1]
    for counter in range(1,Type): do
        NewTerm = scipy.special.comb((Level+counter), counter, exact=True)
        OutputList.append(NewTerm)
    end for
    return ReturnList
end function
```

This function takes the type of an $(m-d-1)^{st}$ polar cone of $\tau_{1,\ldots,d}$ as an input and outputs $\Xi(m,d)$.

```
function ObliterateAMinimalNumberOfQuadrics(List):
    a = List[0]
    b = List[1]
    Dimension = b + (1/2)*(a**2 + a - 2)
    NumberOfQuadrics = 1
    DimensionList = [Dimension]
    while 2**NumberOfQuadrics < Dimension: do
        NumberOfQuadrics += 1
        Dimension = NumberOfQuadrics
        + (1/2)*(a**2 + a - NumberOfQuadrics**2 - NumberOfQuadrics)
        DimensionList.append(Dimension)
    end while
    MaxList1 = [2**(NumberOfQuadrics-1)+1, DimensionList[NumberOfQuadrics-2]+m-d+1]
    MaxList2 = [2**NumberOfQuadrics+1, DimensionList[NumberOfQuadrics-1]+m-d+1]
    Max1 = max(MaxList1[0], MaxList1[1])
    Max2 = max(MaxList2[0], MaxList2[1])
    if Max2 < Max1: then
        if MaxList2[1] < MaxList2[0]: then
            return MaxList2[0]
        else:
            return MaxList2[1]
        end if
    else:
        if MaxList1[1] < MaxList1[0]: then
            return MaxList1[0]
        else:
            return MaxList1[1]
        end if
    end if
end function
```

The Main procedure functions similarly to its counterpart in Algorithm A.3.5. The two differences are that the degree list is computed based on m and d and the use of ObliterateAMinimalNumberQuadrics instead of ObliterateQuadrics.

```
procedure Main(m,d):
    PolarConeLevel = m-d-1
    DegreeList = PolarConeOfTschirnhausType(d,PolarConeLevel)
    if len(DegreeList) == 2: then
        return ObliterateAMinimalNumberQuadrics(DegreeList)
    else if len(DegreeList) == 3: then
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        return ObliterateAMinimalNumberQuadrics(DegreeList)
    else if len(DegreeList == 4: then
        TempDegreeList = ObliterateQuartics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        return ObliterateAMinimalNumberQuadrics(DegreeList)
    else:
        for index in range(1,len(DegreeList)-3): do
            TempDegreeList = ObliterateLargestDegreeHypersurfaces(DegreeList)
            DegreeList = TempDegreeList
        end for
        TempDegreeList = ObliterateQuartics(DegreeList)
        DegreeList = TempDegreeList
        TempDegreeList = ObliterateCubics(DegreeList)
        DegreeList = TempDegreeList
        return ObliterateAMinimalNumberQuadrics(DegreeList)
    end if
end procedure
```

Appendix B

Translations

In this appendix, we include three papers translated by the author, namely

1. "Über die Auflösung der allgemeinen Gleichung fünften und sechsten Grades (Auszug aus einem Schreiben an Herrn K. Hensel)" by Felix Klein,
2. "Über die Anwendung der Tschirnhausen-Transformation auf die Reduktion algebraicher Gleichungen" by Anders Wiman, and
3. "К Проблеме Резольвент" by G.N. Chebotarev.

In each case, we translate the original mathematics and mathematical errors are not corrected. In particular, there are errors which arise from considering intersections in affine spaces instead of projective spaces in the works of Wiman and Chebotarev.

In the translation of Klein's work, there are additional footnotes with the identifier "Translator's Footnote:" which provide additional context. We have also included a more formal bibliography and citations throughout the work.

In the translation of Wiman's works, there are additional footnotes with the identifier "Translator's Footnote:." These footnotes refer to remarks by the author which are consolidated after the translation which provide additional insight.

B.1 About the Solution of the General Equations of Fifth and Sixth Degree (Excerpt from a letter to Mr. K. Hensel)

B.1.1 Main Text

From,

Felix Klein in Göttingen.

By responding to your earnest request to contribute to the journal's book dedicated to the memory of Dirichlet, I refer to a note I published six years ago in the *Rendiconti dell'Accademia dei Lincei* [12] and in which I outlined a general solution of equations of sixth degree.

I set myself the goal of explaining in more detail and in more concrete terms what was suggested there. In fact, even an expert of the relevant literature (such as Mr. Lachtin) has not taken the approach in question in its simplicity (as I will explain more below). [1]

Moreover, I act under the impulses of my old friend Mr. Gordan, who has recently turned his great algebraic ability to the problem in question. Mr. Gordan will soon publish a first relevant treatise in the Math. Annalen [7] [2]. But this is only a beginning; I hope that his continued efforts will succeed in clarifying the subject in every respect as fully as we have been able to do in the past with the theory of equations of the fifth degree.

I would like to discuss this theory of the equations of the fifth degree in advance, as I

[1] 1901, Moscow Mathematical Collection, Vol. XXLI, pp. 181-218 (Russian)

[2] A contribution to the solution of the general equations of the sixth degree. Compare with a message to the Heidelberg International Congress of Mathematics

summarized them in my "Lectures on the Icosahedron" [13], in such a way that I bring forth those moments which are generalized when dealing with the considerations of the sixth degree. In Chapter V of these Lectures, I have dealt with two methods for solving equations of the fifth degree (which, incidentally, differ only by the order in which the steps are carried out) and the second of these methods will prove to be the natural continuation of Kronecker's (and Brioschi's) method. This method, like the first one, is developed in geometric form, with special relations that emerge only in equations of the fifth degree. Instead, I refer here to the algebraic justification developed in Volume 15 of the Math. Annalen [10] and accompanied with reflections on the solution of arbitrary higher equations. [3]

The icosahedral theory of the equations of the fifth degree and the general considerations connected with it have since been portrayed several times by others, especially in the second volume of the excellent textbook on algebra by Mr. Weber [19], as well as in the detailed report that Mr. Wiman has made in Volume I of the Encyclopedia of Mathematical Sciences [22]. Nevertheless, it seems that the basic meaning of the whole approach in the mathematical audience is still often not understood. It is not a matter of considerations which are to the sides of the earlier investigations on the solution of equations of the fifth degree, but of those which claim to constitute the very core of these earlier investigations. Accordingly, in the following report, I will try to describe the main points of the theory (which will later be found *mutatis mutandis* in the approach to the equations of the sixth degree) as accurately as possible while maintaining brevity.

The first is that we have the icosahedral equation, i.e. the equation of the sixtieth degree, which is written in the above Lectures as follows:

$$\frac{H^3(x)}{1728 f^5(x)} = X \tag{B.1.1}$$

[3] In particular, see Section 4 - "The formulas of Kronecker and Brioschi for the fifth degree"

as a *Normalgleichung sui generis* (Normal General Equation) which, by virtue of their excellent qualities, is the next generalization of the "pure" equations:

$$x^n = X. \tag{B.1.2}$$

In fact, given any root of (B.1.1), the 60 roots of (B.1.1) can be calculated by the 60 linear equations that are already known (the icosahedral substitutions), just as the n roots of (B.1.2) can be found from any one of them by the n substitutions given by $x' = e^{\frac{2\pi ik}{n}}x$. Now, the group of icosahedral substitutions proves to be isomorphic with the group of 60 alternating permutations on five letters (i.e. A_5). In this way, it is impossible to trace the solution of the general equations of the fifth degree back to a sequence of pure equations (B.1.2). The task is thus *to solve the equations of the fifth degree with the help of an icosahedral equation.* Here we distinguish an algebraic and a transcendental part of the investigation. The first part will deal with the algebraic construction of a root x of an icosahedral equation (B.1.1) from the roots $z_1, \ldots, z_5$ of a given fifth degree equation - the parameter X of (B.1.1) is determined by the coefficients of the fifth degree equation. We first calculate the square root of the discriminant of the fifth degree equation in terms of the $z_1, \ldots, z_5$. The transcendental part is to calculate the root x of the icosahedral equation from the parameter X by infinite processes. This is due to the hypergeometric series, as well as the transcendental solution of equation (B.1.2) by the binomial series. In the "Lectures on the Icosahedron," it has been shown, in particular, that all algebraic investigations which have been made for the purpose of solving the general equations of the fifth degree are reenactments of the aforementioned algebraic problem. The transcendental part of the task is only barely touched. It is clearly stated, however, what the connection is with the so-called 'solution of the equations of the fifth degree by elliptic functions.' I refer here to my other detailed explanations in the "Lectures on the Theory of Elliptic Modular Functions" [14], edited by Fricke and myself. There is a necessary connection between the fifth order transformation of the elliptic functions and the theory of the icosahedron. In (B.1.1), substituting J (the absolute invariant of an elliptic modular function for X), the variable x gets the meaning of the "principal modular function of the principal congruence group of the fifth degree."[4]

All modes of relating the solution of the fifth degree equations to the elliptic functions are based on this fundamental theorem. In particular, x can be represented by elliptic theta functions; it is a formula of principled simplicity, you have (if I may use Jacobi notation for the sake of brevity):

$$x = q^{\frac{2}{5}} \frac{\theta_1\left(\frac{2iK'\pi}{K}, q^5\right)}{\theta_1\left(\frac{iK'\pi}{K}, q^5\right)} \tag{B.1.3}$$

However, the use of this formula to solve the icosahedral equation (or similar formulas for solving any resolvents of the Icosahedral equation) is just as much a detour as the solution of the pure equation (B.1.2) by logarithms:

$$x = e^{\frac{1}{n}\log(X)} \tag{B.1.4}$$

You have to first calculate $\frac{K'}{K}$, respectively, by calculating $\log(X)$ from X before applying formulas (B.1.3),(B.1.4). The meaning of the formulas for the solution is at most a practical one, namely if one has a logarithm table of elliptic periods K, K'. *Thus, we can finally realize that the use of elliptic functions is not the essence of the theory of equations of the fifth degree.* This mode of expression via elliptic functions is only a residue of accidental historical development: the transformation theory of functions has given the first approach to establishing certain simple equations closely related to the icosahedral equation, namely the modular equations and multiplier equations for the fifth degree transformation.

[4]Translator's Note: The terms "principal congruence [sub]group of the fifth degree" [of the modular group] and "principal modular function" are defined in [14]. See pages 388 and 591 in the original German books, or pages 323 and 475 in the English translation.

So much for the introduction of the icosahedron into the theory of the fifth degree in general. I now have to limit myself to the algebraic side of the task. And here, above all else, I have to mention a fundamental proposition about the icosahedral substitutions, which becomes particularly important in what follows. One can pass from the icosahedral substitutions of the variable x appearing in (B.1.1) to homogeneous substitution formulas (by replacing x in the substitution formulas everywhere by $x_1 : x_2$ and separating numerator and denominator in an appropriate manner). If one chooses the determinant of the resulting binary substitutions equal to 1, one has 120 binary substitutions; specifically, the identity substitution $x' = x$ corresponds to the two homogeneous substitution formulas

$$x_1' = x_1, x_2' = x_2 \quad \text{and} \quad x_1' = -x_1, x_2' = -x_2. \tag{B.1.5}$$

It is not possible in any way (even if you change the value of the determinant), *to assemble from such homogeneous substitutions a group which is isomorphic with the non-homogeneous substitution group (which contains fewer than 120 substitutions).* The surjective homomorphism from the substitution group of the $x_1 : x_2$ to the substitution group of the x therefore has a non-trivial kernel. This fundamental proposition, which is somewhat abstract, gives the algebraic theory of equations of the fifth degree its peculiar form, as we shall have to explain at once. Let us note in advance that it is not difficult to prove it. On pages 46 and 47 of my book on the icosahedron, it is traced back to the fact that the group of non-homogeneous icosahedral substitutions contains the Klein four-group and the corresponding proposition already applies to the Klein four-group. Let us take the following to be the simplest representation of the Klein four-group as (non-homogeneous) substitutions; it is given by

$$I : \xi' = \xi \qquad II : \xi' = -\xi \qquad III : \xi' = \frac{1}{\xi} \qquad IV : \xi' = -\frac{1}{\xi} \tag{B.1.6}$$

Here, II, III, and IV are substitutions of order 2 and, at the same time,

$$II \cdot III \cdot IV = I. \tag{B.1.7}$$

If one now wishes to create an isomorphic group of homogeneous substitutions, one certainly has a group with

$$I' : \xi_1' = \xi_1, \xi_2' = \xi_2$$

and replacing II, III, and IV by

$$\begin{aligned} II' &: \xi_1' = \mp\xi_1, \xi_2' = \pm\xi_2 \\ III' &: \xi_1' = \pm\xi_2, \xi_2' = \pm\xi_1 \\ IV' &: \xi_1' = \mp\xi_2, \xi_2' = \pm\xi_1 \end{aligned}$$

(where, in the individual horizontal rows, the upper or lower signs are to be taken as desired). But, as one must also choose the signs here, *the substitutions II', III', and IV' each have determinant* -1 and thus it is impossible that

$$II' \cdot III' \cdot IV' = I'$$

However, this contradicts (B.1.7) and no such isomorphism can exist. From now on, we will understand the homogeneous icosahedral substitutions as the 120 binary substitutions with determinant $+1$, corresponding to the 60 non-homogeneous substitutions of x.

I will now formulate the central problem for which we are responsible:

From the five independent variables $z_1, \ldots, z_5$ (the roots of the equation of the fifth degree), one has to compose a function $x(z_1, \ldots, z_5)$ which gives an isomorphism from the 60 icosahedral substitutions to the 60 even permutations of $z_1, \ldots, z_5$.

From our fundamental proposition, it immediately follows that there is no such *rational* function of the five free variables (Lectures on the Theory of Elliptic Modular Functions [14], p.255). Namely, by dividing x into coprime polynomials corresponding to the numerator and denominator; i.e. writing

$$x(z_1, \ldots, z_5) = \frac{\phi(z_1, \ldots, z_5)}{\psi(z_1, \ldots, z_5)}$$

with ϕ and ψ coprime, the ϕ, ψ thus introduced would necessarily be homogeneously linear in the 60 permutations of the $z_1, \ldots, z_5$. These homogeneous substitutions would correspond individually to the icosahedral substitutions of x. So, one would have a group of binary homogeneous substitutions that is isomorphic with the group of inhomogeneous icosahedral substitutions and such an isomorphism does not exist, as we have seen previously.

The required function $x(z_1, \ldots, z_5)$ must therefore depend on its argument algebraically [5]. With this, we are led into the domain of those irrationalities of the theory of equations which I call *accessory* in my Lectures ([13], p. 158,159), because they are added as something new to the immediately existing irrationalities of the rational functions of the $z_1, \ldots, z_5$. [6] The usual Galois theory of equations can only deal with the immediately existing irrationalities and not the accessory irrationalities, as usually happens when something new comes forward.

We do not know anything about the efficacy of these accessory irrationalities in general. Rather, we are dependent on tentative experiments in individual cases. Certainly, in the solution of any higher equation, the only allowable accessory irrationalities are those calculated

[5]Translator's Note: In particular, this dependence is not rational.

[6]Translator's Note: We refer to these in modern language as natural irrationalities.

from the symmetric functions of the roots (possibly the predetermined rational functions) by means of *lower* equations. [7] In the equations of the fifth degree, which we treat here, the symmetric functions of the $z_1, \ldots, z_5$ and its difference product (the square root of its discriminant) are known.

We can successfully construct (in many ways) one of the icosahedrally-dependent $x(z_1, \ldots, z_5)$ *as soon as we adjoin the square root of a suitable rational function of the* $z_1, \ldots, z_5$. [8] The two methods of solving equations of the fifth degree, which I give in my lectures, differ only by where they adjoin this accessory square root. In the first method, the accessory square root (transforming the fifth-degree equation by a Tschirnhaus transformation into a so-called fifth degree 'principal' equation - that is an equation in which the sum of the roots and the sum of the square roots vanishes) comes first. In the second method, we first take a step towards the icosahedral problem and then adjoin the accessory square root. As we said in the introduction, I give preference to the second method here, by introducing its individual steps in such a way that the whole approach can analogously be transferred to the sixth degree.

Here, in numbered order, are the main considerations (of the second method):

1. When $x_1 : x_2$ undergo the 120 homogeneous binary icosahedral substitutions, the squares and the product

   $$x_1^2, x_1x_2, x_2^2.$$

   undergo only 60 homogeneous ternary substitutions of determinant 1 (whose group is

[7] Translator's Note: This is Klein's statement of resolvent degree.

[8] The fifth root of unity $\epsilon = e^{\frac{2\pi i}{5}}$ occurs in the icosahedral substitutions and will be useful in the construction of a suitable function x. If we count the accessory irrationalities rigorously, then one has accessory irrationalities in the theory of equations from the beginning - namely, in the reduction of the cyclic equations to pure equations.

isomorphic to the 60 non-homogeneous icosahedral substitutions of $\frac{x_1}{x_2}$, and thus to the 60 even permutations of the five quantities $z_1, \ldots, z_5$).

2. The same is true, according to the general principles of invariant theory, of the coefficients of the quadratic binary form of $x_1 : x_2$. In order to have an immediate connection to the style of Kronecker and Brioschi (also used in my Lectures), I shall hereby designate such a form as follows:

$$A_1 x_1^2 + 2A_0 x_1 x_2 - A_2 x_2^2 \tag{B.1.8}$$

   The A_1, $2A_0$, A_2 depend contragrediently on x_1^2, x_1x_2, x_2^2, according to the notions of invariant theory. [9]

3. We readily conclude that it is possible to form (from any given five variables $z_1, \ldots, z_5$) rational functions such that the alternating permutations of the $z_1, \ldots, z_5$ also permute the A_0, A_1, A_2. In fact, in the work already mentioned in the introduction to Vol. 15 of the Math. Annalen, I have given a general approach which implies that whenever two given sets of variables (here the $z_1, \ldots, z_5$ and the A_0, A_1, A_2) undergo isomorphic homogeneous linear substitutions, we can simply construct rational functions from which we can translate the first set of variables (like the $z_1, \ldots, z_5$) into the second set (like the A_0, A_1, A_2).

4. We do not reproduce the general approach here (which would be unnecessarily lengthy), but instead give the abbreviated form relevant to our particular problem, which deals directly with the developments of Kronecker and Brioschi on equations of fifth degree [1]. These are the following points:

[9] Translator's Note: Recall that given a group G and linear G-representations V and W, a W-valued contragredient of V is a G-equivariant regular map $\mathbb{A}(V^\vee) \to \mathbb{A}(W)$.

(a) We form six quadratic expressions in $x_1 : x_2$:

$$\sqrt{5}x_1x_2, \quad \epsilon^\nu x_1^2 + x_1x_2 - \epsilon^{4\nu}x_2^2 \quad \text{where } \epsilon = e^{\frac{2\pi i}{5}}, 0 \leq \nu \leq 4 \tag{B.1.9}$$

which are permuted when $x_1 : x_2$ undergo the icosahedral substitutions.

(b) Further, let $v(z_1, \ldots, z_5)$ be a rational function of $z_1, \ldots, z_5$, which remains invariant by the cyclic permutation of $z_1, \ldots, z_5$ taken in natural order. We form the difference

$$v(z_1, \ldots, z_5) - v(z_5, \ldots, z_1)$$

and square it. Then, we have a "metacyclic" function,[10] which should be called u_∞^2, while the five other values that result from it by the alternating permutations of $z_1, \ldots, z_5$ may be labeled u_ν^2 in a proper order ($\nu = 0, 1, 2, 3, 4$). One can then choose the signs of the $u_\infty, u_0, \ldots, u_4$ *such that the alternating permutations of the $z_1, \ldots, z_5$ permute the*

$$u_\infty, u_0, \ldots, u_4 \tag{B.1.10}$$

isomorphically by the corresponding icosahedral substitutions of x_1, x_2; namely, they undergo the same sign changes as (B.1.9).

(c) We conclude that the following form

$$\Omega(z_1, \ldots, z_5 | x_1 : x_2) = \sqrt{5}u_\infty x_1x_2 + \sum_{\nu=0}^{4} u_\nu \left(\epsilon^\nu x_1^2 + x_1x_2 - \epsilon^{4\nu}x_2^2\right) \tag{B.1.11}$$

remains invariant if one simultaneously applies an alternating permutation to the

[10]Translator's Note: This means that the corresponding stabilizer is a metacyclic group. Recall that a group is metacyclic if it is an extension of a cyclic group by a cyclic group. In this case, the corresponding stabilizer is D_{10}, the dihedral group with 10 elements, which is indeed metacyclic.

$z_1, \ldots, z_5$ and the corresponding icosahedral substitution to the $x_1 : x_2$.

(d) We now put, in accordance with (B.1.8):

$$\Omega(z_1, \ldots, z_5 | x_1, x_2) = A_1 x_1^2 + 2A_0 x_1 x_2 - A_2 x_2^2$$

and find by comparison

$$\begin{cases} 2A_0 & = \sqrt{5} u_\infty + \sum_{\nu=0}^{4} u_\nu \\ A_1 & = \sum_{\nu=0}^{4} \epsilon^\nu u_\nu \\ A_2 & = \sum_{\nu=0}^{4} \epsilon^{4\nu} u_\nu \end{cases} \tag{B.1.12}$$

Thus, we have constructed from $z_1, \ldots, z_5$ the quantities A_0, A_1, A_2 which are permuted in the desired way when the $z_1, \ldots, z_5$ undergo an alternating permutation.

5. We refer to the above result by saying that we have assigned a covariant quadratic binary form (B.1.8) to the $z_1, \ldots, z_5$. [11] The discriminant of (B.1.8) is

$$A = A_0^2 + A_1 A_2, \tag{B.1.13}$$

which is a binary function of $z_1, \ldots, z_5$ that is invariant under alternating permutations of $z_1, \ldots, z_5$; it is thus a rational function of the coefficients of the given fifth degree equation and the square root of its discriminant. The goal is not to assign to $z_1, \ldots, z_5$ a covariant binary quadratic form or a "pair of points" of the binary form

$$A_1 x_1^2 + 2A_0 x_1 x_2 - A_2 x_2^2 = 0, \tag{B.1.14}$$

[11] Translator's Note: Given a group G and linear G-representations V and W, a W-valued covariant of V is a G-equivariant regular map $\mathbb{A}(V) \to \mathbb{A}(W)$.

but to assign a quotient $\frac{x_1}{x_2}$, i.e. a point. We do this in the simplest way by solving the quadratic equation (B.1.14) and accordingly writing

$$\frac{x_1}{x_2} = x = \frac{-A_0 + \sqrt{A_0^2 + A_1 A_2}}{A_1}. \tag{B.1.15}$$

6. *Thus, we have solved our central task: to compose such an x from the $z_1, \ldots, z_5$ which undergoes the icosahedral substitutions corresponding to the alternating permutations of the $z_1, \ldots, z_5$.* Note that the A_0, A_1, A_2 of item 4d are rational functions of $z_1, \ldots, z_5$ and their construction uses only one irrationality - the fifth root of unity ϵ. According to item 5, we see that the expression of A_0, A_1, A_2 under the square root[12] is invariant under alternating permutations of the $z_1, \ldots, z_5$. *We have thus achieved the goal with the aid of such accessory irrationalities, which in the theory of the fifth degree, will suitably be called lower irrationalities.*

7. We now further investigate the parameter X of the icosahedral equation, which satisfies our x (B.1.15) as a function of the coefficients of the equation of the fifth degree whose roots are the $z_1, \ldots, z_5$, respectively. We will say that the equation is solved if we calculate the square root of the discriminant and how the $z_1, \ldots, z_5$ are rationally represented by x from the coefficients [of the equation of the fifth degree] and the adjoined square root [i.e. we do not concern ourselves with the transcendental portion].

8. In summary, let us emphasize why one can justifiably speak of such a solution of equations of the fifth degree. Not only is there a sequence of steps that could be numerically traversed (in the given case) so that one actually obtains the numerical values of $z_1, \ldots, z_5$, but it is also a full theoretical insight into the internal nature of the problem of solution. [13] After all, the $z_1, \ldots, z_5$ are the different branches of a finite-valued algebraic function, which depends on the coefficients of the fifth degree

[12] Translator's note: e.g. the $A_0^2 + A_1 A_2$ in equation (15)

[13] Translator's note: By "problem of solution", Klein means the problem of solving generic polynomials.

equation and is initially of a very confusing design. *These five branches* $z_1, \ldots, z_5$ *are defined over the field of rationality determined by the icosahedral irrationality.* [14] *The field of rationality is given by the coefficients of the fifth degree equation, the square root of the discriminant, and the adjoined accessory irrationalities. The icosahedral irrationality is of a more transparent construction and is a higher irrationality that depends only on a single parameter from the field of rationality.*

I would like to take up at this point a more personal remark about the relationship between my papers on the equations of the fifth degree and those of Kronecker - as you, dear colleague, are in the position of being able to see the manuscripts of Kronecker and thus can complete my information in an authentic way. As is well known, Kronecker and Brioschi [1] used the same quantities A_0, A_1, A_2 in their first papers on equations of the fifth degree (from 1858) which I quoted earlier (in list item 4b); they then constructed the sixth-degree equation satisfying $\zeta = 5A_0^2$, and which Brioschi calls a "Jacobi equation" because of its close connection with certain equations established by Jacobi for the transformation of elliptic functions; finally, they state that by adjoining a root, one can arrive at an equation with only one parameter. This square root defines an accessory irrationality equivalent to that used in formula (B.1.15). Furthermore, Kronecker [15] set up the fundamental theorem, which I designate as Kronecker's theorem in my Lectures. Moreover, the exposition and proof of Kronecker's theorem are the crowning achievement of my Lectures; the theorem is that *it is impossible to form a resolvent to the general equation of the fifth degree with only one parameter and without resorting to accessory irrationalities.* As in the 12th volume of the Math. Annalen [9], I prove this proposition by invoking the property of the icosahedral group discussed above, namely the doubling (at least) of its substitutions in the transition to homogeneous substitutions. My first proof, which I gave in 1877 in the reports of the Er-

[14] Translator's Note: For more on fields of rationality, see Ackerman's English translation of "Development of Mathematics in the 19th Century" by Felix Klein. In particular, see Chapter VII - Deeper Insight into the Nature of Algebraic Varieties and Structures, Section 3: The Theory of Algebraic Integers and Its Interaction with the Theory of Algebraic Functions, p.312-314.

langen physical-medical school (meeting of January 13), was considerably more complicated. Twenty-four years ago (Easter, 1881), I had the opportunity to talk in detail with Kronecker about these things. It turned out that in his investigations, Kronecker was unaware of the icosahedral substitutions to which he had come so close and, accordingly, did not have sufficient proof for his main claim! I think this is a very remarkable fact, but also very common, for it confirms in a particularly interesting case what Gauss so often emphasizes: that the discovery of the most important mathematical theorems is more a matter of intuition than of deduction, and the production of the proof is a very different business from the discovery of the theorems. I did not return to the subject later with Kronecker, but some years ago, I heard that after the publication of my Lectures in a college, Kronecker has commented on the solution of the fifth degree equations and the theory of the icosahedron. I would be very interested (as certainly would other mathematicians) in finding out what may be contained in Kronecker's papers on these matters, and I would like to ask you to review the relevant material and publish it soon.

A new proof of Kronecker's theorem has been given by Mr. Gordan in Volume 29 of the Math. Annalen [6]. [15] It is easier to read than mine, in that it does not refer to an explicit knowledge of the icosahedral substitutions anywhere. Nevertheless, as I shall point out, it is most closely related to the basic idea of my proof. Following a development by Mr. Lüroth [18], we both use a proposition which can be formulated as follows: [16]

Suppose an equation of n^{th} degree, whose roots are the independent variables $z_1, \ldots, z_n$, has a rational resolvent with only one parameter. Then, the n^{th} degree equation must have a rational function x of the $z_1, \ldots, z_n$ such that when the $z_1, \ldots, z_n$ are permuted by the Galois group, x undergoes a linear transformation. We have the same understanding that,

[15] Compare this with the presentation of Gordan's proof in the textbook of Weber and Netto's "Algebra"

[16] Translator's Note: The theorem of Lüroth that Klein and Kronecker use can be stated as follows - Let $\Bbbk$ be an algebraically closed field of characteristic 0. Then, any unirational function field of transcendence degree 1 is isomorphic to $\Bbbk(t)$, where t is transcendental over $\Bbbk$.

when changing to homogeneous coordinates $x_1 : x_2$, we must go from our group of linear substitutions to an isomorphic groups of homogeneous linear substitutions. On the other hand, of course, this group must be isomorphic to the Galois group of the equation modulo a specified subgroup.

Now, on p.44-47 of my Lectures, I have given the proposition that only the following groups of linear substitutions of a single variable can be converted isomorphically to their corresponding binary forms:

1. The cyclic groups

2. The dihedral groups of odd n.

It follows that *an equation of the* n^{th} *degree (whose roots are the independent variables* $z_1, \ldots, z_n$*) only admits a rational resolvent with only one parameter (which can be immediately transformed into a pure equation, or dihedral equation of odd* n*) if its Galois group is (isomorphic to) a normal subgroup of a cyclic group or a dihedral group of odd* n*.* [17] An associated resolvent with only one parameter can then be set up immediately according to the principles in the Math. Annalen Vol 15 [10]. The general statement above is included in both Gordan's proof and my proof of Kronecker's theorem. Indeed, my proof is done by pointing out that the group of a fifth degree equation with the square root of the discriminant adjoined is simple; however, it is isomorphic to the group of linear substitutions of the icosahedron and thus the previous theorem yields the claim. Gordan's proof, on the other hand (if I understand it correctly), uses the obvious fact that the group in question, like every group, contains itself as a normal subgroup. The quotient of the group by itself is isomorphic to the identity group. And the identical substitution falls under the premise of our theorem. Thus, there are in fact resolvents with one parameter, but they are completely useless for the solution of the equations of the fifth degree! Namely, there are linear resolvents whose square

[17] Translator's Note: In modern language, this is the classification of groups of essential dimension 1.

root is a function of the $z_1, \ldots, z_5$ that is invariant under the alternating permutations of the $z_1, \ldots, z_5$. But there are no other (rational) resolvents with just one parameter, or better: every rational resolvent of our fifth degree equation with just one parameter is linear and therefore useless.

So much for the icosahedral substitutions and the solution of fifth degree equations mediated by them. Instead of the "unitary" substitutions $x' = e^{\frac{2\pi i k}{n}}$, which link the roots of a *pure* equation to one another, "binary" linear substitutions of two homogeneous variables $x_1 : x_2$ have entered. At the same time, the way to new generalizations has opened up - one simply has to use groups of linear substitutions of several homogeneous variables! I cannot possibly repeat here the reflections which I gave in this regard first in the 15th volume of the Math. Annalen [10] or recall the elaborations which have later been concluded. It suffices to refer to Weber's textbook [19] and to the already cited encyclopedia article by Wiman [22]. [18] In this sequence, we consider *an equation of sixth degree along with the square root of its discriminant*, whose Galois group consists of the 360 alternating permutations of the roots $z_1, \ldots, z_6$. It will be necessary to use the smallest number of homogeneous $x_1 : \ldots : x_\mu$ for which there exists a surjective homomorphism from the group of linear substitutions of the $x_1 : \cdots : x_\mu$ to the group of the 360 alternating permutations. If this surjective homomorphism were to prove to be an isomorphism, we would be able to write down rational functions of $z_1, \ldots, z_6$ (according to the prescriptions of [10]) which are linear in the 360 alternating permutations of $z_1, \ldots, z_6$ and yield $x_1 : \ldots : x_\mu$ as a result. However, it turns out that here, as in the equations of the fifth degree, the homomorphism must have a non-trivial kernel, so that we are asked the question whether (or respectively, how) we can get by with the help of lower accessory irrationalities.

My first approach to the formulation of this question is in Volume 28 of the Math. Annalen

[18] A first clear introduction is also the Lecture IX of my Evanston Colloquium, held at the World's Fair in Chicago (Macmillan, New York, 1894)

(1886, On the Theory of General Equations of the Sixth and Seventh Degrees)[19] [9]. At the time, it seemed like the preliminary work of Mr. C. Jordan would not allow a ternary group of linear substitutions with the requisite surjective homomorphism to the group of the 360 alternating permutations on six letters; such a group was first discovered by Mr. Valentiner in 1889 (Volume 6 of Series V of the Danish Academy's papers: The Definitions of the Final Transformation Groups) [20] and examined by structural and related fundamental invariants for the first time by Mr. Wiman in 1895 [20] (Math. Annalen 47: About the Simple Group of 360 Plane Collineations) [21]. At that time, I constructed a group of quarternary collineations that is isomorphic to the group for the general equation of degree six - and also for the general equation of degree seven - and showed that the original problems rest on the corresponding problems for the groups of quarternary collineations, which can be obtained from the general equations of degree six (respectively, seven) with at most two accessory square roots. [21]

As far as the equations of the sixth degree are concerned (to which we confine ourselves here), this approach is currently unnecessary, because of the discovery of the Valentiner group. [22] I note this expressly, because this is where Mr. Lachtin makes an unnecessary detour (as mentioned at the beginning of this letter). In order to connect the equation of degree six with the Valentiner group, Mr. Lachtin goes through the development given in Volume 28 of the Math. Annalen [16]. [23]

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[19] Translator's Note: Klein gives the correct volume for the article, but the incorrect year. This article is from December 1887 in Issue 4 of Volume 28.

[20] Translator's Note: While Klein gives the correct volume, this work was actually published in December of 1896.

[21] The group which I put forward for the sixth degree equation contains as many as 720 collineations, so that it is not necessary to use all of them; adjoin the square root of the discriminant of the sixth degree equation beforehand. In contrast, the group corresponding to the equations of the seventh degree contains only $\frac{7!}{2} = 2520$ collineations.

[22] For the equations of seventh degree, the quaternary approach persists; but it is impossible to pursue the interesting questions in this text.

[23] Translator's Note: Lachtin only has two articles in the Mathematische Annalen. The one Klein is

This is not uninteresting, [24] but is by no means necessary for what we do next. The transition from the equations of the sixth degree to the Valentiner group (as I suggested in my Roman note of 1899) and which I now wish to introduce in more detail, does not require any reference to the quarternary substitution group. For the sake of clarity, I shall again divide the considerations in question into a numbered list below which makes clear the analogy with the train of thought followed for the equations of the fifth degree:

1. The task is to compose three functions x_1, x_2, x_3 from the six free variables $z_1, \ldots, z_6$ such that the homogeneous $x_1 : x_2 : x_3$ undergo the corresponding collineations of the Valentiner group when the $z_1, \ldots, z_6$ undergo one of the 360 alternating permutations.

2. Now, Mr. Wiman has already noticed that the number of collineations from the Valentiner group at least triples when one goes from substitutions of the plane to the corresponding ternary linear substitutions. Therefore, it is not possible for the required x_1, x_2, x_3 to be rational functions of $z_1, \ldots, z_6$.

3. From now on, we want to fix the homogeneous linear Valentiner substitutions so that their determinant is always 1. Thus, there are exactly $3(360) = 1080$ of them and the

referring to from September 1898 in Issue 3 of Volume 51. Lachtin's other article is on the septic and is from September 1902 in Issue 3 of Volume 56.

[24]Mr. Lachtin notes that in the quaternary group, the degree two surfaces in space interchange in much the same way as the degree three curves of the plane. From here, as is noted in passing, it is possible without any great difficulty to arrive at the same Σ, which I communicate below under (19). One only has to keep in mind that the roots $z_1, \ldots, z_6$ of the sixth degree equation, and also their squares $z_1^2, \ldots, z_6^2$ define a linear complex in space according to the developments of Volume 28, and that these two complexes together with the "unitary complex" introduced there determine a degree two surface through their common lines. No accessory irrationality occurs here. It is then in no way necessary, in the transition from space to plane, to refer to the comparatively complicated formulas, as Mr. Lachtin does, by which in Volume 28 I have assigned a point of space to the roots $z_1, \ldots, z_6$, so also in this regard, the approach of Mr. Lachtin can be shortened.

three substitutions corresponding to the identity substitution [of the plane] are:

$$\begin{cases} x_1' = j^\nu x_1, \\ x_2' = j^\nu x_2, \qquad j = e^{\frac{2\pi i}{3}}, \quad 0 \le \nu \le 2 \\ x_3' = j^\nu x_3, \end{cases} \tag{B.1.16}$$

4. We now note that in these 1080 homogeneous substitutions, the ten degree three terms coming from the x_1, x_2, x_3 are:

$$x_1^3,\ x_1^2 x_2, \ldots$$

which only undergo 360 homogeneous linear substitutions (and whose group will be isomorphic with the group of alternating permutations of $z_1, \ldots, z_6$).

5. We now consider any cubic ternary form

$$a_{1,1,1}x_1^3 + 3a_{1,1,2}x_1^2 x_2 + \cdots$$

(which, if set equal to 0, represents a "degree three curve" in the plane of x_1, x_2, x_3). The coefficients $a_{1,1,1}, 3a_{1,1,2}, \ldots$ for any homogeneous linear substitutions of $x_1 : x_2 : x_3$ are related to the $x_1^3, x_1^2 x_2, \ldots$ contravariantly. Thus, in the substitutions of the Valentiner group, they also undergo exactly 360 homogeneous linear substitutions, which can be uniquely identified with the 360 alternating permutations of $z_1, \ldots, z_6$.

6. We readily conclude that it is possible to form ten rational functions of the free variables $z_1, \ldots, z_6$ denoted by:

$$\phi_{1,1,1}, \phi_{1,1,2}, \ldots$$

which, in the case of the alternating substitutions of the $z_1, \ldots, z_6$, are substituted just as the

$$a_{1,1,1}, a_{1,1,2}, \ldots$$

are by the corresponding substitutions of the Valentiner group; i.e., the roots $z_1, \ldots, z_6$ rationally and covariantly assign a degree three curve.

7. To put it another way, *one can construct* (in many different ways and without the use of accessory irrationalities [25]), *a cubic form depending on the* $z_1, \ldots, z_6$ *and* x_1, x_2, x_3

$$\Omega(z_1, \ldots, z_6 | x_1, x_2, x_3) = \phi_{1,1,1} x_1^3 + 3\phi_{1,1,2} x_1^2 x_2 + \cdots \tag{B.1.17}$$

which remains invariant if one simultaneously performs an alternating permutation on the $z_1, \ldots, z_6$ *and its corresponding Valentiner substitution on the* x_1, x_2, x_3.

8. As far as the actual construction of such a form Ω is concerned, I do not give the general and extensive process which I provided in [10], but develop it in an abbreviated manner that emerged from my correspondence with Mr. Gordan (last winter), just as with equations of the fifth degree. One has to combine the following relationships:

   (a) The 360 collineations of the Valentiner group play an important role in two systems of six conic sections, as Mr. Wiman proved first. *The six conic sections of each of the two systems are permuted by the corresponding 360 collineations in 360 ways.*

   (b) The equations of these $2{\cdot}6 = 12$ conic sections were first proposed by Mr. Gerbaldi [5] (Rendiconti del Circolo Matematico di Palermo, t. XII, 1898: "Sul gruppo semplice di 360 collineazioni piane," I; already published in 1882 in the Atti di

[25] Apart, of course, from the numerical irrationalities that occur in the substitutions of the Valentiner group. These are (in accordance with the following formulas (18), etc.) the square roots $\sqrt{-3}$ and $\sqrt{5}$.

Torino, Vol. XV, p. 358 ff., Note: "Sui gruppi di sei coniche in involuzione")[26]. Here we equate the corresponding ternary quadratic forms of determinant 1 by restricting to the one system of six conic sections, according to the procedure of Mr. Gordan. We can then write: [27]

$$\begin{cases} k_1 &= x_1^2 + jx_2^2 + j^2x_3^2 \\ k_2 &= x_1^2 + j^2x_2^2 + jx_3^2 \\ k_3 &= \alpha\left(x_1^2 + x_2^2 + x_3^2\right) + \beta\left(x_2x_3 + x_3x_1 + x_1x_2\right) \\ k_4 &= \alpha\left(x_1^2 + x_2^2 + x_3^2\right) + \beta\left(x_2x_3 - x_3x_1 - x_1x_2\right) \\ k_5 &= \alpha\left(x_1^2 + x_2^2 + x_3^2\right) + \beta\left(-x_2x_3 + x_3x_1 - x_1x_2\right) \\ k_6 &= \alpha\left(x_1^2 + x_2^2 + x_3^2\right) + \beta\left(-x_2x_3 - x_3x_1 + x_1x_2\right) \end{cases} \tag{B.1.18}$$

where $\alpha = \frac{1-\sqrt{-15}}{8}$ and $\beta = \frac{-3+\sqrt{-15}}{4}$.

(c) The $k_1, \ldots, k_6$ are determined, up to a third root of unity, by the requirement that their determinant be 1. In fact, the 1080 Valentiner substitutions permute the $k_1, \ldots, k_6$ by multiplication by certain third roots of unity.

(d) We want, now, from any three of the k:

$$k',\ k'',\ k'''$$

one whose coefficients form a trilinear covariant and a similar invariant. For the former, we choose the functional determinant $|k'k''k'''|$, which changes its sign when two of the k', k'', k''' are exchanged. As an invariant, we take a symmetric combination of the coefficients of the k', k'', k''' (namely the expression used in

[26]Translator's Note: The modern reference for this article is the one appearing in the Mathematische Annalen, which is what we give. Thus, we have left the two previous journal references Klein gave in the main text.

[27]Translator's note: As in equation (16), j is the primitive 3rd root of unit $e^{\frac{2\pi i}{3}}$.

the development of the coefficient-determinant of the form $\lambda' k' + \lambda'' k'' + \lambda''' k'''$ in which $\lambda'\lambda''\lambda'''$ appears). I will temporarily call it $(k'k''k''')$ here; this is a simple numerical quantity in the present case.

(e) For all possible triples k', k'', k''', we now form the quotient

$$\frac{|k'k''k'''|}{(k'k''k''')}.$$

One can show *that the* $\binom{6}{3} = 20$ *quotients obtained above undergo the same sign changes from the 1080 substitutions of the Valentiner groups that the 20 difference products*

$$(z'' - z''')(z''' - z')(z' - z'')$$

undergo from the corresponding alternating permutations of the $z_1, \ldots, z_6$.

(f) Therefore, the sum of all triples

$$\sum (z'' - z''')(z''' - z')(z' - z'') \cdot \frac{|k'k''k'''|}{(k'k''k''')} \tag{B.1.19}$$

is a simple example of a form

$$\Omega\,(z_1, \ldots, z_6, x_1, x_2, x_3)$$

as we were looking for in item number 7 above.

(g) More general examples (which we do not need in the following) are obtained by substituting the determinant

$$\begin{vmatrix} z'^{\alpha} & z''^{\alpha} & z'''^{\alpha} \\ z'^{\beta} & z''^{\beta} & z'''^{\beta} \\ z'^{\gamma} & z''^{\gamma} & z'''^{\gamma} \end{vmatrix}$$

for the difference product of z', z'', z''' in (19).

(h) Let us now assign the sum (19) to the successive terms $x_1^3, x_1^2 x_2, \ldots$ by writing, as in formula (17):

$$\sum (z'' - z''')(z''' - z')(z' - z'') \cdot \frac{|k'k''k'''|}{(k'k''k''')} = \phi_{1,1,1} x_1^3 + 3\phi_{1,1,2} x_1^2 x_2 + \cdots \, , \tag{B.1.20}$$

so that the $\phi_{1,1,1}, \phi_{1,1,2}, \ldots,$ are just such the rational functions of $z_1, \ldots, z_6$ that we looked for in item number 6 above.

9. The degree three curve

$$\sum (z'' - z''')(z''' - z')(z' - z'') \cdot \frac{|k'k''k'''|}{(k'k''k''')} = 0 \tag{B.1.21}$$

(whose coefficients depend rationally on the $z_1, \ldots, z_6$) covariantly assigns a point $x_1 : x_2 : x_3$ using accessory irrationalities that are as low as possible.

10. The theory of degree three plane curves offers various possibilities. For the sake of brevity, as I did in my Roman note, I want to choose an inflection point in third-order curve here.

11. According to the well-known theory of Hesse, the determination of such an inflection point requires only square roots and cube roots; for the solution of the equations of the sixth degree, these are indeed lower irrationalities. The details [of the process of

determining an inflection point] are not discussed here.

12. On the other hand, the inflection point is certainly connected in a covariant manner with the degree three curve: if any collineation is performed on a curve with a chosen inflection point P, then P is taken to an inflection point P' on the new curve and each of the nine inflection points can be realized in this manner. In particular, this applies to the 360 collineations of the Valentiner group.

13. We now think of the coordinates $x_1 : x_2 : x_3$ of our chosen inflection point, instead of the coefficients $\phi_{1,1,1}, \phi_{1,1,2}, \ldots$ of the curve of third order (whose values come from (B.1.20) and the $z_1, \ldots, z_6$).

14. If the $z_1, \ldots, z_6$ undergo an alternating permutation, the $x_1 : x_2 : x_3$ undergo the corresponding collineation of the Valentiner group.

    We conclude that the rational functions of $z_1, \ldots, z_6$ that remain invariant after the reductions in the expressions of $x_1 : x_2 : x_3$ from the occurring square roots and cubic roots also remain invariant under the alternating permutations of the $z_1, \ldots, z_6$. Thus, they can be represented as rational functions of the coefficients of the presented sixth degree equation and the square root of their discriminant.

15. Therefore, we will are justified in designating the irrationalities required in the calculation of the inflection point as *lower* accessory irrationalities.

16. By computing the coordinates $x_1 : x_2 : x_3$ of an inflection point of our C_3 [our degree three curve], we have accomplished the goal: *we form the functions x_1, x_2, x_3 from the free variables $z_1, \ldots, z_6$ and lower accessory irrationalities such that when the the $z_1, \ldots, z_6$ undergo an alternating permutation, the x_1, x_2, x_3 undergo the corresponding collineation of the Valentiner group.*

This is the explanation of the particular content of my Roman note, which I thought to give here. [28]

One might wish for a closer examination of the inference used in list item number 14. The simplest thing would be to calculate all the known equations leading to the determination of an inflection point of the degree three curve (B.1.21) and thus actually confirm the assertion. Moreover, as Mr. Gordan remarks, the whole of the conclusion can be dealt with in the following way. Just consider the ninth degree equation, which satisfies the nine values that an absolute invariant of the Valentiner group (e.g. the ν to be named immediately) assumes in the nine inflection points! This equation must be the same for all 360 third-order curves (which results from the substitutions of the Valentiner group and thus by the alternating substitutions of the two). After disposing of indifferent factors, the coefficients are rational functions of the $z_1, \ldots, z_6$ that are invariant under the alternating permutations of the $z_1, \ldots, z_6$. The affect of this ninth degree equation can be none other than that of the original inflection point equation. It is thus solved by square roots and cubic roots of rational functions of $z_1, \ldots, z_6$ that are invariant under the alternating permutations of $z_1, \ldots, z_6$.

Now, if we adjoin one of the resulting nine values of our absolute invariant (ν), then it and the equation (B.1.21) of the third order curve, (respectively, of the equation of its Hessian curve), becomes the corresponding single inflection point $x_1 : x_2 : x_3$ and is calculated rationally. Consequently, the assertion of list item number 14 concerning the irrationalities required in the calculation of the inflection point, is self-evident.

The further treatment of equations of the sixth degree will have to be done, in any case, by calculating the absolute invariants of the Valentiner group of the selected inflection point of our third order curve. According to Mr. Wiman, the Valentiner group has three lowest

[28]The final sentence of the note has become incomprehensible when printed in the Rendiconti of the Accademia dei Lincei by a strange change. It should read "And with the aid of accessory irrationalities, which we usually regard as elementary, we come to the goal." Instead, what is printed is "And so, with the aid of ancillary irrationalities, we come to the goal, which usually regarded as elementary."

invariants:

$$F,\ H,\ \Phi \tag{B.1.22}$$

of degrees 6, 12, and 30 in the x_1, x_2, x_3. From them, the two fundamental absolute invariants come together, which I call v and w here in connection with the work of Mr. Lachtin to be mentioned immediately:

$$v = \frac{\Phi}{F^3}, \quad w = \frac{H}{F^2}. \tag{B.1.23}$$

If we enter the coordinates of our point of inflection for $x_1 : x_2 : x_3$, then the v, w are rational functions of the coefficients of the sixth degree equation and the square root of the discriminant (respectively, the occasionally introduced accessory irrationalities). *The* Normalproblem [29] *of solving equations of the sixth degree is thus the reduction to calculating* $x_1 : x_2 : x_3$ *from the known* v, w. As just stated, this is now a problem with two arbitrary parameters and is distinguished by the fact that all of its 360 solutions $x_1 : x_2 : x_3$ can be determined from a given solution by the 360 collineations of the Valentiner group. We do not currently have a method to reduce the number of parameters to one by means of further lower irrationalities. For example, if we try to assign a point $x_1' : x_2' : x_3'$ to the degree six curve $F = 0$ in a covariant manner and thus set the following (instead of the Normalproblem (B.1.23) to the inflection point $x_1 : x_2 : x_3$)

$$F' = 0, \quad t' = \frac{\Phi'^2}{H'^5}, \tag{B.1.24}$$

in the usual approach (intersection of the curve $F = 0$ with a straight line covariantly

[29] In German, this was indeed one word - *Normalproblem.*

dependent on the point $x_1 : x_2 : x_3 :$), one encounters an auxiliary equation which itself is of the sixth degree!

For the sake of completeness, we finally ask for a reverse form, i.e. to rationally calculate the quantities assumed to be known $[v, w]$ from the roots $z_1, \ldots, z_6$ of the presented sixth order equation and a single solution system $x_1 : x_2 : x_3$ of (B.1.23). *Thus, we have completely sketched the algebraic part of the solution of the equations of the sixth degree.*

The transcendental part will require infinite processes to actually compute the $x_1 : x_2 : x_3$ from the equations (B.1.23). A first approach to this is done by Mr. Lachtin in a voluminuous work, which was first published in Russian (1901) in the 22nd volume of the Moscow Mathematics Collection and then in 1902 in the German edition of Volume 56 of the Math. Annalen [17]. [30] Writing

$$y_1 = \frac{x_1}{\sqrt[6]{F}}, \quad y_2 = \frac{x_2}{\sqrt[6]{F}}, \quad y_3 = \frac{x_3}{\sqrt[6]{F}}, \tag{B.1.25}$$

the y_1, y_2, y_3 are a system of solutions to the three simultaneous linear partial differential equations expressing the second derivatives

$$\frac{\partial^2 y}{\partial v^2}, \quad \frac{\partial^2 y}{\partial v \partial w}, \quad \frac{\partial^2 y}{\partial w^2}$$

linearly in the $\left(\frac{\partial y}{\partial v} \text{ and } \frac{\partial y}{\partial w}\right)$ and the y. Mr. Lachtin has shown that the coefficients of these equations are rational functions of the absolute invariants v, w, which do not exceed certain definable degrees. However, he did not calculate the numerical coefficients of these polynomials. The remaining gap is now being filled by the work of Mr. Gordan, which I

[30] "The differential resolvent of an algebraic equation of the sixth degree of a general kind." (Math. Ann., Vol. 56, p. 445-481.)

referred to in the beginning of this letter. [31] *In fact, Mr. Gordan succeeded in making explicit the partial differential equations in question.* It is thus possible to develop the y_1, y_2, y_3 in powers of v and w, or even in any series of linear functions of v and linear functions of w; thus, it is no longer an issue to determine the regions in which the various series thus formed converge - *in other words, we can solve the transcendental problem in a direct way.*

Again, for the sake of completeness, it must be added that the special problem presented by equation (B.1.24) is already discussed in detail in terms of function theory. In 1896, at the Frankfurter Scientific Congress, Mr. Fricke [32] dealt with the decomposition of the Riemann surface (of genus 10) corresponding to the Valentiner group into fundamental domains and a closed relation of the same with the decomposition of the half plane in the semicircular triangles from the angles

$$\frac{\pi}{2}, \quad \frac{\pi}{4}, \quad \frac{\pi}{5}$$

Mr. Lachtin then confirms this information in [16] and established the third-order linear differential equation, for which - in the case of equations (B.1.24) - the parameter t is satisfied by the variables x_ν multiplied by a suitable factor.

I am at the end of my presentation. I hope the analogy of the proposed sixth degree equations with the solution of the equations of the fifth degree by the icosahedral equation appears convincing. A finer examination of the details given by Mr. Gordan and myself for the equations of the fifth degree, as well as a geometric presentation, are given in my "Lectures on the Icosahedron."

Göttingen, March 22, 1905.

[31] Translator's Note: We believe that Klein is referring to the work that became [7]

[32] Translator's Note: For more information, see [4]

B.1.2 Ending Footnote

Mr. Gordan has been able to devote only one introductory essay to the questions raised above [8]. There he makes a substantial simplification of the necessary accessory irrationality of x_1, x_2, x_3. Instead of the degree three curve of the x_1, x_2, x_3-plane, which I rationally assigned to the value system $z_1, \ldots, z_6$, it uses a (1,1)-connection; i.e. a bilinear form in the x and u (whose coefficients must be assumed to be whole rational functions of $z_1, \ldots, z_6$ and that the form remains invariant under the 360 permutations of the $z_1, \ldots, z_6$ and the corresponding linear substitutions of the x and u). Then, to find a covariant point x_1, x_2, x_3 for one of the permutations of the $z_1, \ldots, z_6$, one only has to determine one more root of an easy cubic equation, namely to go to a fixed point of the connection.

In particular, Gordan succeeds in setting up a linear form of the desired kind, which is of degree 6 in the $z_1, \ldots, z_6$. In the meantime, Mr. Coble showed by a systematic process that one need only go to the fourth degree [3]. He sets up the associated cubic equation and then further sketches the course of the required algebraic calculation to determine the $z_1, \ldots, z_6$. K.

I will conclude by mentioning the explanations given on p.491, footnote 10, referring to Kronecker's theorem.

First of all, for the sake of completeness, a few hints about my original proof of January, 1877. At that time, I had operated on the fact that all icosahedral forms, and also the tetrahedral form, have a direct degree. The circumstance is, of course, in turn, a consequence of the doubling of the number of homogeneous substitutions which I have emphasized in Abh. LIV, which was the actual reason for the proof.

Incidentally, I must go into more detail about the reference between Kronecker and myself from Easter 1881 [see p.10]. At that time, I asked Kronecker for a manuscript from 1861, from which I could copy the part that was suitable for me (the copy bears the date of March 23). In the proof of his theorem, Kronecker uses it exactly as I did later by anticipating Lüroth's theorem on rational curves [18] and, from there on, the task is to form a rational function $\frac{\phi}{\psi}$ out of five free variables $x_0, \ldots, x_4$, which is a linear transformation in the 60 alternating permutations of the $x_0, \ldots, x_4$. He then encounters a strange lapse. Since Kronecker had not yet familiarized himself with the general notion of a group of linear substitutions (of one variable), he erroneously concludes that the 60 linear transformations in question ought to arise from the repetition of the same linear substitution; that is, from a cyclic equation of 60th degree, which (according to Galois theory) is of course impossible. At that time, Kronecker went quiet [on the subject].

In the lectures of 1885-86, this mistake is corrected. It is concluded that in the case of the alternating permutations of the free variables $x_0, \ldots, x_4$, the polynomials ϕ and ψ would have to be substituted in a linearly binary manner, and further, that such a binary behavior is already impossible if one of the permutations fixes an x_i and cyclically permutes the others x_j's. Even without the evidence of this impossibility, I still find an unnecessary complication. I showed above (p. 485) that even in the Klein four group, the impossibility in question arises. In order to come to a contradiction, Kronecker instead combines an operation of the Klein four group with the cyclic permutation of the x_1, x_2, x_3 - this is less transparent.

Aside from these secondary points, a complete consensus exists. There remains only a subjective difference, which I already discussed in detail on pages p.158-159 in the book on the icosahedron, but which I do not want to leave untouched here because of its importance. For the first time in his investigations into the solution of equations of the fifth degree, Kronecker begged to have a clear distinction between the natural irrationalities (which are rational functions of $x_0, \ldots, x_4$) and the other irrationalities (which I call accessory). Incidentally, in his first communication of 1858 [2], he himself makes an unobjectionable use of an accessory square root. Is is only in the later work of 1861 [15] that he believes that he should forbid the use of accessory irrationalities in the theory of equations altogether. In his 1885-86 lectures, he maintains this verdict:

> ...the use of accessory irrationalities is "algebraically worthless," because it "tears apart" the type.

In order to emphasize this demand, he calls it the "Abelian postulate." In contrast to other authors of similar thinking, I have explored as far as possible in my papers printed here above, as in the book on the icosahedron, the efficacy of using naturally occurring accessory irrationalities.

There is a principled difference in this thinking. I do not want to further emphasize that Abel continues to use the roots of unity ϵ in his investigations into the solution of the equations by radicals [which in the context of his considerations are also accessory irrationalities (see above, p. 486 footnote 8)], which incidentally Kronecker continues to do himself, because otherwise he would not be able to act on the connection of the equation of the fifth degree with the Jacobian equations of the sixth degree. Nor do I want to argue that it [the use of accessory irrationalities] is as advantageous in the theory of numbers as it is in function theory, because of the simplicity of the higher-order algebraic relations in transcendental fields. I only want to emphasize the fundamentals.

When presented with new phenomena (such as the efficacy of the accessory irrationalities), should we we stop developing these ideas to align with our current conceptions, or rather push back against our narrow, systematic ways of thinking and pursue the new ideas in an unbiased way? Should one be a dogmatist or a natural scientist, and endeavor to keep learning from new ideas?

There is nothing special to be inferred from Kronecker's original notes, which Mr. Hensel sent to me. These are mainly 23 unilaterally described folios, of which 1-10 refer to the work of 1858 and 11-23 to those of 1861. It is worth noting that the passages I copied in 1881 are missing. On the back of the pages 17-18, there are bills with fifth roots of unity, by virtue of which Kronecker has evidently been convinced that the G_{60} of broken icosahedral substitutions really do exist.

The criticism which I then apply to the transmitted material is intended to reflect the high position which I have given to Kronecker's investigations on the equations of the fifth degree in the above reprinted essays, especially in the historical account of the book on the icosahedron ([13], see p.141-161). Kronecker first found the path which leads into the fundamental questions of the theory, only he did not finish it at first and later, at least formally, refused to accompany others on the way forward. K.

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B.2 On the Application of Tschirnhaus Transformations to the Reduction of Algebraic Equations

B.2.1 Main Text

1. We consider a general equation of n^{th} degree:

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0 \tag{B.2.1}$$

with roots $x_1, \ldots, x_n$. We then apply a Tschirnhaus transformation, which has the general form

$$y = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}. \tag{B.2.2}$$

This will transform (B.2.1) into an equation

$$y^n + C_1 y^{n-1} + \cdots + C_n = 0, \tag{B.2.3}$$

where the coefficients C_i are all homogeneous functions of degree i in the variables a_ν and include all such functions of weight up to i in the coefficents c_i. Tschirnhaus hoped to use this type of transformation to convert equation (B.2.1) to the binomial form in such a way that the determination of parameters a_ν should require the solution of equations of degree at most $n-1$. As is well known, this is not the case, even though there have many attempts for the degree 5; this will never be the case, just like for [the problem of] trisecting an angle. However, the situation is completely different if the problem is formulated in the following way: Is it possible to satisfy the conditions

$$C_i(a_0, a_1, \ldots, a_{n-1}) = 0, \quad i = 1, \ldots, m \tag{B.2.4}$$

in such a way that determining the parameters a_ν only requires equations of degree up to m, *when n is sufficiently large?* As a result of the following treatment for the case $m = 4$, it should not appear doubtful that question should also be decided in the affirmative for larger m. However, the general problem of determining the lower bound on n associated to each m appears to be very complex.

2. Observe that

$$C_1(a_0, \ldots, a_{n-1}) = na_0 + \cdots .$$

If $C_1 = 0$, then a_0 is expressed linearly in the other parameters. The coefficients $C_i, (i = 2, \ldots, n-1)$ are then homogeneous functions of degree i in the parameters $a_1, \ldots, a_{n-1}$. In order to [find a point that will] satisfy a single condition

$$C_x = 0, \quad (x > 1)$$

it is evident that it is only necessary to find an intersection of the hypersurface $C_x = 0$ with an arbitrary straight line to solve an equation of degree x. If all the roots of (1) are real, then you cannot get a real solution for $x = 2$ because the hypersurface

$$\sum_{i=1}^{n} y_i^2 = C_2 = 0 \tag{B.2.5}$$

has only the trivial 0. In contrast, there are always real points on the surface $C_3 = 0$. Indeed, as you can easily see, the same is the case for all surfaces $C_x = 0$ $(x > 2)$ if x is an even number.

If one has $n \geq 5$, then one obtains the solution of (B.2.4) for $m = 3$ by the well-known Bring-Jerrard transformation, which is illustrated geometrically by F. Klein [33] in the following way.[34] First, a point P is obtained on the surface $C_2 = 0$, which, as noted above, can be done using a square root. If $n > 5$, we then consider a three-dimensional space R_3 which is tangent [to $C_2 = 0$] at P, which then has a hypersuface of degree 2 in common with $C_2 = 0$. A second square root is now required in order to select one of the two generators of this surface going through P. Determining an intersection of one of these generators with $F_3 = 0$ requires the solution of a degree 3 equation. Although only one pair of imaginary roots need to occur in the real equations $C_1 = C_2 = C_3 = 0$, at least two pairs of imaginary roots must exist to actually execute this transformation. Otherwise, there is no real line at $C_2 = 0$. This is due to the fact that if one converts

$$C_2(a_1, \ldots, a_{n-1}) = \sum_{i=1}^{n} y_i^2$$

to a sum of $n-1$ real squares, one gets the λ with the sign $-$, where 2λ denotes the number of imaginary roots.

3. Now, let $n > 5$. We assume that a point P that lies on both $C_2 = 0$ and $C_3 = 0$ has been determined by the procedure given above. The associated coordinates are $a_1^{(0)}, \ldots, a_{n-1}^{(0)}$. We write

$$\alpha_i - \alpha_i^{(0)} = \beta_i, \quad i = 1, \ldots, n-2$$

[33]We refer to the in-depth treatment of F. Klein, *Lectures on the Icosahedron and the Solution of the Equation of Fifth Degree*, Leipzig, 1884.

[34]Translator's footnote: See Remark B.2.1 for the translator's summary of this argument.

and reconstruct C_2 and C_3 in terms of $\beta_1, \ldots, \beta_{n-2}$. [35] In this manner, by summing the terms with same total degree in the β_i, we get:

$$\begin{cases} C_2 & = \phi_1 + \phi_2 \\ C_3 & = \psi_1 + \psi_2 + \psi_3. \end{cases} \tag{B.2.6}$$

We want to solve the present problem in such a way that we determine a straight line going through P that lies on both the surfaces $C_2 = 0$ and $C_3 = 0$. If this is successful, then the further condition $C_4 = 0$ only requires the solution of an equation of the fourth degree. We denote the space with homogeneous coordinates $a_1, \ldots, a_{n-1}$ as a R_{n-2}, in accordance with the number that is its dimension.

The first conditions to be introduced are

$$\phi_1 = \psi_1 = 0. \tag{B.2.7}$$

The space R_{n-2} is then reduced to a R_{n-4}.[36] If we consider the straight lines through P as elements of the space, the point space R_{n-4} has only $n-5$ dimensions. From this perspective, we refer to it as $\mathcal{L}_{n-5}$ and consider the subvarieties $\phi_2 = 0$, $\psi_2 = 0$, and $\psi_3 = 0$ inside it. If one takes any plane in this line space $\mathcal{L}_{n-5}$, then $\phi_2 = 0$ and $\psi_2 = 0$ have four elements in common - that is, four straight line generators. If $n > 7$, then a common straight line of $\phi_2 = 0$ and $\psi_2 = 0$ can be determined by solving a fourth degree equation. We denote such a straight line by ℓ_1.

If we can complete the task in such a way that we get a plane tangent to $\phi_2 = 0$ and $\psi_2 = 0$,

[35]One can assume that $a_{n-1}^{(0)} \neq 0$, after possibly changing the indices, and then write $a_{n-1} = 1$

[36]Translator's footnote: See Remark B.2.2 for the translator's summary of this argument.

then this result will be solved, as this plane intersects $\psi_3 = 0$ in three straight lines, so that everything else comes down to the solution of a degree three equation. The equations of the hyperplanes, which meet the subvarieties $\phi_2 = 0$ and $\psi_2 = 0$ along the straight line ℓ_1 are given by

$$\phi_1^{(1)} = 0, \quad \psi_1^{(1)} = 0. \tag{B.2.8}$$

We assume the relations (B.2.8) are satisfied. The R_{n-4} discussed above then reduces to an R_{n-6}. If we restrict to this $\mathbb{P}^{n-6}$, then elements of $\phi_2 = 0$ and $\psi_2 = 0$ appear as planes through ℓ_1. However, there are exceptions for when $n = 7$, as R_{n-6} coincides with ℓ_1, and for when $n = 8$, as $\phi_2 = 0$ and $\psi_2 = 0$ are reduced to the double-counted straight line ℓ_1. We now replace the point space R_{n-6} with a space whose elements are the planes that contain the line ℓ_1. This plane space obviously has dimension $n-8$ and is denoted by $\mathcal{E}_{n-8}$. We now have the intersection of the quadratic [algebraic] manifolds $\phi_2 = 0$ and $\psi_2 = 0$ in R_{n-6} in the plane space $\mathcal{E}_{n-8}$. If $n \geq 10$, one concludes that only a fourth degree equation has to be solved to determine a common plane of $\phi_2 = 0$ and $\psi_2 = 0$ in $\mathcal{E}_{n-8}$. Therefore, we have the theorem:

If $n \geq 10$, the general equation (1) can be reduced to the form

$$y^n + C_5 y^{n-5} + \cdots + C_n = 0 \tag{B.2.9}$$

by a transformation (B.2.2) in such a way that determining the parameters a_i requires only the solution of a finitely many quartic, cubic, and quadratic irrationalities.

Although an equation of the form (B.2.9) can be achieved with only two pairs of imaginary roots, at least three pairs of imaginary roots of (B.2.1) must exist for this transformation to be completed using only real numbers. It is only under this condition that the hypersurface $C_2 = 0$ contains real planes. However, if one wishes for not only necessary, but also sufficient conditions here, this does seem to be possible without fairly in-depth discussions.

4. The case $n = 9$ was examined by D. Hilbert in a recently published work.[37] It is first demonstrated how one can determine a three-dimensional space R_3 which is completely contained in the hypersurface $C_2 = 0$ and then one considers the degree three surface F_3 in this R_3 which is cut out by the hypersurface $C_3 = 0$. As is known, this surface F_3 only depends on four fundamental parameters. This is also particularly evident when one transforms only the left term of the equation to a sum of five cubes, for which it is necessary to solve a fifth degree equation. Hilbert now puts the general equation of ninth degree in the form

$$y^9 + C_5 y^4 + C_6 y^3 + C_7 y^2 + C_8 y + C_9 = 0 \tag{B.2.10}$$

by first determining one of the 27 straight lines on the surface F_3 and then intersecting one of these straight lines with the hypersurface $C_4 = 0$. Both the equation (B.2.10), where one can easily set $C_9 = 1$, as well as the equation of degree 27 are functions of only four parameters. The result of this is that the solution of the general equation of ninth degree only requires algebraic functions of four arguments in such a way that "one can get by with functions of one argument, sums, and two special functions of four arguments".

It can be shown that, if the general equation of ninth degree is reduced to the form (B.2.10), then there is no need to solve an equation of degree higher than five, so that one of the special functions of four arguments above is unnecessary. At the end, we generalize our task set at (B.2.4) by using auxiliary equations larger than m, but with the restriction that each degree will still always be $< n$.

[37] Über die Gleichungen neuten Grades, Math. Ann. 97, S. 243 (1926)

As in the previous case, we determine a point P on both $C_2 = 0$ and $C_3 = 0$ and still suppose the conditions (B.2.8) hold. Since $n = 9$ here, the left terms of the subcone $\phi_2 = 0$ and $\psi_2 = 0$ can be written in five homogeneous coordinates, such as z_1, z_2, z_3, z_4, z_5. We now look for the self-conjugate pentahedron common to both ϕ_2 and ψ_2, which corresponds to solving a fifth degree equation.[38] If this pentahedron is assumed to be a coordinate pentahedron, we have expressions for ϕ_2 and ψ_2 of the form:

$$\begin{cases} \phi_2 &= a_1 z_1^2 + a_2 z_2^2 + a_3 z_3^2 + a_4 z_4^2 + a_5 z_5^2 \\ \psi_2 &= b_1 z_1^2 + b_2 z_2^2 + b_3 z_3^2 + b_4 z_4^2 + b_5 z_5^2. \end{cases} \tag{B.2.11}$$

One can now eliminate any of the five variables from $\phi_2 = 0$ and $\psi_2 = 0$ and thus obtain five relations, one of which we write in the form

$$c_1 z_1^2 + c_2 z_2^2 + c_3 z_3^2 + c_4 z_4^2 = 0. \tag{B.2.12}$$

We take (B.2.12) as the equation of a degree two hypersurface and try to determine the corresponding straight line generators, which only requires taking square roots. The straight lines in this generating set are assigned to the values of a a parameter λ and, likewise, the points of a specified generator are assigned to the values of another parameter t_1 and the their coordinates z_1, z_2, z_3, z_4 can be expressed linearly in both λ and t. According to (B.2.11), we get the relation for z_5:

$$z_5^2 = a_2(\lambda)t^2 + b_2(\lambda)t + c_2(\lambda). \tag{B.2.13}$$

[38] Translator's footnote: See Remark B.2.3 for more exposition.

Since the elements of (B.2.11) are straight lines through P, it can be seen that a generator of (B.2.12) corresponds to a two-dimensional cone whose apex is P. Since this cone must have six generators in common with the hypersurface $\psi_3 = 0$, it follows that one need not use auxiliary equations of degree more than six when reducing the general equation of ninth degree to the form (B.2.10).

5. This matter can be simplified by looking for a value of λ such that the two-dimensional cone splits into two planes. We need only solve the degree four equation

$$[b_2(\lambda)]^2 - 4a_2(\lambda)c_2(\lambda) = 0. \tag{B.2.14}$$

The generating system of (B.2.12) has four other conjugate pairs of planes, so that the total number of planes common to the subcones $\phi_2 = 0$ and $\psi_2 = 0$ is 16. According to the five different relations in four variables of the form (B.2.12), each of these planes can be paired with five others. Two planes that can be paired together intersect each other in a straight line. If not, they only intersect at P.

The theory of the intersection of the two cones $\phi_2 = 0$ and $\psi_2 = 0$ is indeed well known.[39] They system of equations (B.2.11) is often used to study properties of a degree four surface with a double conic section. The 16 common planes of the two subcones correspond to the 16 straight lines lying on such a surface.

We have now demonstrated *that the general equation of ninth degree can also be converted to the form (9) without having to presuppose the solution of auxiliary equations which each depend on more than one parameter.* Among the auxiliary equations, however, there is one of fifth-degree, so we cannot get by with square roots and cube roots, as we can for $n > 9$. However, it seems to be difficult to prove strictly that the latter is not possible at all for $n = 9$.

[39]Translator's footnote: See Remark B.2.4 for the translator's summary of this argument.

B.2.2 Translator's Notes

Remark B.2.1. (Main Argument of Section 2)

Let $\mathbb{A}^n$ be the affine space of Tschirnhaus transformations and $\mathbb{P}^{n-1}$ its projectivization. $\mathbb{V}(C_1)$ is a hyperplane and thus $\mathbb{V}(C_1) \cong \mathbb{P}^{n-2}$. A rational point P of $\mathbb{V}(C_1) \cap \mathbb{V}(C_2)$ can be determined over a quadratic extension of the base field. Then, the tangent hyperplane T at P can be computed rationally and will have dimension at least 4 in $\mathbb{V}(C_1) = \mathbb{P}^{n-2}$ if $n > 6$. Hence $\mathbb{V}(C_1) \cap \mathbb{V}(C_2) \cap T$ is a quadric that is singular at P in $\mathbb{V}(C_1) \cap T \cong \mathbb{P}^{n-3}$. Consequently, a line in this cone can be determined by solving a quadratic polynomial and it suffices to intersect this line with $F_3 = 0$.

Remark B.2.2. (Main Argument of Section 3)

To re-state Wiman's approach, this $\mathbb{P}^{n-4}$ is obtained by projectivizing and then considering $\mathbb{V}(C_1) \cap \mathbb{V}(\phi_1) \cap \mathbb{V}(\psi_1)$ inside $\mathbb{P}^{n-1}$. By shifting P to the origin (e.g. $[0 : \cdots : 0 : 1]$), Wiman then uses a classical correspondence to consider the pencil of lines through P and identifies it as $\mathcal{L}_{n-5} \cong \mathbb{P}^{n-5}$. Note that ϕ_2, ψ_2, and ψ_3 induce hypersurfaces of the same degree in $\mathcal{L}_{n-5}$. If $n \geq 7$, a point Q of $\mathbb{V}(\phi_2) \cap \mathbb{V}(\psi_2) \subseteq \mathcal{L}_{n-5}$ can be determined by solving a quartic equation. Moreover, by construction, the line ℓ_1 determined by P and Q lies in $\mathbb{V}(C_1) \cap \mathbb{V}(C_2)$ in the ambient space.

Now, consider the tangent hyperplanes of $\mathbb{V}(\phi_2), \mathbb{V}(\psi_2) \subseteq \mathbb{P}^{n-4}$ defined by the polynomials $\phi_1^{(1)}$ and $\psi_1^{(1)}$. Consider the $\mathbb{P}^{n-6}$ given by $\mathbb{V}(C_1) \cap \mathbb{V}(\phi_1) \cap \mathbb{V}(\psi_1) \cap \mathbb{V}(\phi_1^{(1)}) \cap \mathbb{V}(\psi_1^{(1)})$. Every point not on ℓ_1 in $\mathcal{E}_{n-8} = \mathbb{V}(\phi_2) \cap \mathbb{V}(\psi_2) \subseteq \mathbb{P}^{n-6}$ determines a plane on $\mathbb{V}(\phi_2) \cap \mathbb{V}(\psi_2)$ in the ambient space; this is possible when $n \geq 9$. Determining such a point Q' in E_{n-8} determines a line $\ell_2 \subseteq \mathbb{V}(\phi_2) \cap \mathbb{V}(\psi_2) \subseteq \mathcal{L}_{n-5}$ and thus a point Q' of $\ell_2 \cap \mathbb{V}(\psi_3) \subseteq \mathcal{L}_{n-5}$ can be determined by solving a cubic equation. However, the line determined by P and Q' lies inside $\mathbb{V}(C_1) \cap \mathbb{V}(C_2) \cap \mathbb{V}(C_3)$ in the ambient space and thus a point of $\mathbb{V}(C_1) \cap \mathbb{V}(C_2) \cap \mathbb{V}(C_3) \cap \mathbb{V}(C_4)$ can be determined by solving a quartic equation.

Remark B.2.3. (Pencils of Quadrics)

The forms ϕ_2 and ψ_2 define a pencil of quadratic forms in the five variables $z_1, \ldots, z_5$. The singular fibers of the pencil are given by the roots of the discriminant, which is a polynomial of degree 5. Hence, determining a singular quadric in the pencil corresponds to solving a degree 5 polynomial. Wiman then again uses the observation that a singular quadric is a cone.

Remark B.2.4. (Degree 4 del Pezzo Surfaces)

Here Wiman is observing the fact that the intersection of two quadrics in $\mathbb{P}^4$ is a degree 4 del Pezzo surface.

B.3 On the Problem of Resolvents

B.3.1 Main Text

Consider an equation of n^{th} degree

$$f(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{B.3.1}$$

whose coefficients $a_1, \ldots, a_n$ are indeterminates. Substituting

$$y = t_0 + t_1 x + \cdots + t_{n-2} x^{n-2} + x^{n-1}$$

into $f(x) = 0$ yields an n^{th} degree equation

$$y^n + C_1 y^{n-1} + C_2 y^{n-2} + \cdots + C_n = 0 \tag{B.3.2}$$

where the coefficients $C_1, \ldots, C_n$ depend rationally on $a_1, \ldots, a_n$ and are polynomials in $t_0, \ldots, t_{n-2}$ of respective degrees $1, \ldots, n$.

Equation (B.3.2) is called an *s-parameter resolvent* of equation (B.3.1) if its coefficients $C_1, \ldots, C_n$ are rational functions of s parameters $v_1, \ldots, v_s$ and the coefficients $t_0, \ldots, t_{n-2}$ of the Tschirnhaus transformation depend rationally on $a_1, \ldots, a_n$ and the roots of some auxiliary equations (secondary resolvents), which themselves admit s-parameter resolvents.

It is easy to show that if one does not limit the degree of the secondary resolvents, the s-parameter resolvent of equation (B.3.1) can be put in the particular form

$$y^n + C_{n-s} y^s + C_{n-s+1} y^{s-1} + \cdots + C_{n-1} y + 1 = 0. \tag{B.3.3}$$

In fact, let

$$g(z) = z^n + B_1 z^{n-1} + \cdots + B_n = 0$$

be an s-parameter resolvent of equation (B.3.1), i.e. $B_1, \dots, B_n$ are rational functions s of parameters $v_1, \dots, v_s$.

Take the new Tschirnhaus transformation

$$y = \tau_0 + \tau_1 z + \cdots + \tau_{n-2} z^{n-2} + z^{n-1}.$$

The coefficients $C_1, \dots, C_n$ of the equation that y satisfies are polynomials of the corresponding degree in the $\tau_0, \dots, \tau_{n-2}$ and, moreover, depend rationally on the coefficients $B_1, \dots, B_n$ and therefore on the parameters $v_1, \dots, v_s$. Setting $C_1, \dots, C_{n-s-1}$ equal to zero and C_n equal to 1 and composing the results of these equations (in which $\tau_0, \dots, \tau_{n-2}$ are unknown), we obtain a chain of auxiliary equations whose coefficients depend on s parameters.

D. Hilbert [1] showed in his article "On Equations of the Ninth Degree" that an equation of the ninth degree admits a resolvent that depends on 4 parameters. His method of obtaining this resolvent is as follows.

The coefficients $t_0, \dots, t_{n-2}$ of the Tschirnhaus transformation are considered as the coordinates of a point in the space T taking values from the field of rational functions in $a_1, \dots, a_n$ and its algebraic extensions. The equations $C_1 = 0, C_2 = 0, C_3 = 0$ and $C_4 = 0$ determine hypersurfaces in the space T of degrees 1, 2, 3, and 4, respectively. Finding a 4-parameter resolvent of an equation of 9^{th} degree reduces to finding a common point of these hypersurfaces by solving a chain of algebraic equations that admit $\leq$ 4-parameter resolvents. Substituting $y = \sqrt[n]{C_n} z$ makes the final term a unit.

This problem is solved by Hilbert as follows. A three-dimensional hyperplane is found that entirely belongs to the hypersurfaces $C_1 = 0, C_2 = 0$. On it, the surface $C_3 = 0$ cuts out a cubic surface, which, as you know, always contains lines that lie entirely on it; to find these lines, one has to solve an equation of the 27th degree which depends only on four parameters, since the equation of the cubic surface allows a special technique based on the subtle properties of cubic quarternary forms (by summing up five cubes that are the roots of one equation of fifth-degree) which leads to a form that depends on 4 parameters. The intersection of the line just found with the hypersurface $C_4 = 0$ determines the desired point. Thus, to construct a 4-parameter resolvent of an equation of the 9^{th} degree, in addition to a series of equations of degree between 2 and 5, it is necessary to solve an equation of the 27^{th} degree, which is greater than the degree of the original equation.

In the work "On the Application of Tschirnhaus Transformations to the Reduction of Algebraic Equations" [2], A. Wiman shows that to obtain an $(n-5)$-parameter resolvent of an equation of degree $n \geq 10$, it is sufficient to solve several auxiliary one-parameter equations of degree no higher than 4. To do this, he moves the origin to an intersection point of the hypersurfaces $C_1 = 0, C_2 = 0, C_3 = 0$ and reduces the last two equations to the form

$$0 = C_2 = \phi_2$$
$$0 = C_3 = \psi_2 + \psi_3$$

where ϕ_2 and ψ_2 are quadratic homogeneous forms and ψ_3 is a cubic homogeneous form, and using elegant geometric considerations, searches for a two-dimensional plane that lies entirely in both the hypercones $\phi_2 = 0$ and $\psi_2 = 0$. The intersection of this plane with the cubic cone $\psi_3 = 0$ gives a straight line belonging to the surfaces $C_1 = 0$, $C_2 = 0$, $C_3 = 0$.

For the case $n = 9$, Wiman proves that in order to obtain a 4-parameter resolvent, it is sufficient to only solve one auxiliary equation of 5th degree (which has a one-parameter resolvent) in addition to the the equations of degrees 1-4. To do this, he performs a linear transformation (which is determined by solving an equation of fifth degree) which simultaneously diagonalizes the forms $\phi_2 = 0$ and $\psi_2 = 0$ and defines a one-parameter family of two-dimensional cones, all points of which belong to both cones $\phi_2 = 0$ and $\psi_2 = 0$. By solving an equation of fourth degree, we find the value of the parameter at which the cone of the family splits into a pair of planes.

Applying the method of Wiman, we can find an $(n-6)$-parameter resolvent of an equation of degree $n \geq 77$. In this article, an attempt is made to slightly modify this method, as a result of which, the $(n-6)$-parameter resolvent can be constructed for equations of degree $n \geq 21$.

Following Wiman, we consider the space T_{n-1} of the parameters $t_0, \ldots, t_{n-2}$ and the hypersurfaces $C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0, C_5 = 0$ in this space. We move the origin to a point common to the hypersurfaces $C_1 = 0, C_2 = 0, C_3 = 0$, which can be determined by solving auxiliary equations of the second and third degree. Now, in the equations of the first three surfaces, the free terms disappear and these equations can be written as follows:

$$
\begin{aligned}
0 &= C_1 \\
0 &= C_2 = \phi_1 + \phi_2 \\
0 &= C_3 = \psi_1 + \psi_2 + \psi_3
\end{aligned}
$$

where C_1, ϕ_1, ψ_1 are linear forms of the parameters t_i, ψ_2, ϕ_2 are quadratic [forms], and ψ_3 is a cubic. The equations $\phi_1 = 0$ and $\psi_1 = 0$ determine the hyperplanes tangent to the hypersurfaces $C_2 = 0$ and $C_3 = 0$ at the origin. The intersection of these hyperplanes with the hyperplane $C_1 = 0$ determines the space T_{n-4}, in which the equations of the hypersurfaces cut out by $C_2 = 0$ and $C_3 = 0$ will have the form

$$0 = C_2' = \phi_2'$$
$$0 = C_3' = \psi_2' + \psi_3'.$$

We show that for $n \geq 19$, there exists a two-dimensional plane belonging entirely to the hypersurfaces $C_2' = 0$ and $C_3' = 0$ in the space T_{n-4}.

Lemma B.3.1. *Two $(3k-1)$-dimensional quadratic cones with a common vertex in $3k$-dimensional space share a whole k-dimensional plane passing through the vertex of the cones.*

Proof. We proceed by induction. Find straight-line generators common to both cones (for this, it is enough to intersect both cones with any two-dimensional plane that does not pass through their vertex, find the intersection point of the two quadrics cut by the cones on the plane - which requires solving an equation of fourth degree - and connect the found point to the vertex of the cones). Take a plane of dimension $3k-1$ that does not pass through vertex of the cones. On this hyperplane, our cones will cut out two hypersurfaces of degree 2, the common point of which is rationally defined as the intersection of the hyperplane and the previously found common [straight-line] generator of the cones.

The intersection of the [original] hyperplane and the two [tangent] hyperplanes touching these hypersurfaces at their common point defines a space of $3k-3$ dimensions, in which out cones cut out a pair of cones of $3k-4$ dimensions with a common vertex, which contains a generic $k-1$ dimensional linear space by the inductive hypothesis. The desired k-dimensional subspace common to both cones is defined as the space passing through the $(k-1)$-dimensional space and the vertex of the cones. □

Lemma B.3.2. *A cubic four-dimensional cone in five-dimensional space contains a two-dimensional plane passing through the top of the cone which lies entirely in the cone.*

Proof. Intersect our cone with a four-dimensional plane that does not pass through its top. The cone will cut out a three-dimensional cubic hypersurface on it. We find a point on this surface (for which it is enough to solve one equation of the third degree) and construct a three-dimensional hyperplane that is tangent to the surface at this point. If, after moving the point to the origin, the equation of the hypersurface has the form

$$\phi_1 + \phi_2 + \phi_3 = 0,$$

then the equation of the tangent hyperplane will be

$$\phi_1 = 0.$$

Consider the intersection of our surface with the given tangent hyperplane. Obviously, the equation of this intersection will have the form

$$\phi_2' + \phi_3' = 0$$

where ϕ_2' and ϕ_3' are forms of second and third degree, respectively, in three variables. We consider the quadratic and cubic cones

$$\phi_2' = 0 \text{ and } \phi_3' = 0$$

with a common vertex in three-dimensional space. These cones have a common straight line generator and it suffices to solve an equation of the sixth degree to find it (determine the intersection point of the quadric and cubic cut out by the cones on any two-dimensional plane not passing through their vertex and connect it to the vertex of the cones). The two-dimensional plane which passes through the [original] vertex and through the straight line just found lies entirely in the original cubic cone. □

Theorem B.3.3. *The general algebraic equation of degree* $n \geq 21$ *admits an* $(n-6)$-*parameter resolvent.*

Proof. As above, consider the hypersurfaces

$$C_1 = 0,\ C_2 = 0,\ C_3 = 0,\ C_4 = 0,\ C_5 = 0$$

in the space T_{n-1}. We move the origin to a common point of the hypersurfaces $C_1 = 0,\ C_2 = 0,\ C_3 = 0$. We construct tangent hyperplanes to the hypersurfaces $C_2 = 0$ and $C_3 = 0$ at the origin and consider the space T_{n-4}, which is the intersection of these hyperplanes with the hyperplane $C_1 = 0$. In T_{n-4}, our surfaces $C_2' = 0$ and $C_3' = 0$ are defined by equations of the form

$$0 = C_2' = \phi_2,$$
$$0 = C_3' = \psi_2 + \psi_3.$$

By virtue of Lemma 1 and as $n - 4 \geq 15$, the two cones

$$\phi_2 = 0, \psi_2 = 0$$

have a common 5-dimensional linear subspace. According to Lemma 2, the cubic cone $\psi_3' = 0$ in this subspace contains a two-dimensional plane and according to the above, we do not need to solve any equations above the sixth degree to find one.

The hypersurfaces

$C_4 = 0$ and $C_5 = 0$

cut out curves of the 4^{th} and 5^{th} degree are cut out on this plane, the intersection point of which can be found by solving an equation of the 20^{th} degree, which, according to Wiman, has a resolvent that depends on no more than 15 parameters. However, $n - 6 \geq 15$, which proves the theorem. □

This technique allows us to state the existence of $(n-7)$-parameter resolvents of a general equation of degree $n \geq 121$.

B.3.2 Literature

1. David Hilbert, "Über die Gleichung neunten Grades," *Ges. Abh.,* Bd. II, S. 393.
2. Anders Wiman, "Über die Anwendung der Tschirnhausentransformationen auf die Reduktion algebraischer Gleichungen," *Nova Acta Regiae Societatis Scientiarum Upsaliensis,* volumen extra ordinem 1927.

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Received January 19, 1953.