the vacuum state is a scalar under the ordinary SU(3), the chiral SU(3) subgroup of SW(3) will be realized as a Goldstone symmetry, in contrast to the ordinary SU(3) subgroup, and our theory presented here would go through unchanged. These points will be discussed in greater detail elsewhere.

Lastly, we may make the following remark: We showed that \( b \) as a function of \( a \) is discontinuous at \( a = -1 \) and \( a = 2 \). From this, we concluded that we may have essential singularities at these points, provided that \( b \) is an analytic function of \( a \) except for a few isolated points in the complex plane of the variable \( a \). However, there is another possibility that \( b \) may have branch cuts, instead of the essential singularities, passing through points \( a = -1 \) and 2, since these will also give the desired discontinuity. An interesting possibility is the conjecture that the Kuo transformation \( e \rightarrow (2 - e)(1 + 4e)^{-1}, e_0 \rightarrow -\frac{1}{4}(1 + 4e) e_0 \) may transform physical quantities on the first Riemann sheet to those on the unphysical second sheet.

---

**PHYSICAL REVIEW D**

**VOLUME 1, NUMBER 12**

**15 JUNE 1970**

**Inclusion of Toller-Angle Dependence in the Multi-Regge Integral Equation**

**DENNIS SILVERMAN AND CHUNG-I TAN**

*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540*

(Received 15 January 1970; revised manuscript received 11 March 1970)

The nonforward multiperipheral integral equation for the Reggeon-particle absorptive amplitude is generalized to include complete dependence on the Toller-angle variable.

**I. INTRODUCTION**

Much progress has been made in formulating the multiperipheral bootstrap equation using a multi-Regge production model.\(^1\)\(^2\) In a recent publication,\(^3\) Goldberger, Tan, and Wang have constructed a simplified integral equation for the Reggeon-particle absorptive amplitude \( \alpha(p, p_0; Q) \) in a formulation of the multi-Regge model. Their construction seemed to depend on the assumption that the double Regge coupling is independent of the Toller angle \( \omega \), and an approximate justification for this assumption was suggested by Tan and Wang.\(^4\) It is our purpose to show that an integral equation which includes the complete dependence on the Toller angle can be written for the absorptive amplitude \( \alpha \). This establishes the full generality of the integral-equation approach through the \( \alpha \) amplitude.

In the process of formulating this equation, we elaborate the relation between the \( \omega \) angle and the other invariants. We then express the integration of the loop momentum in terms of a particular set of invariants


\(^3\) M. L. Goldberger, C.-I Tan, and J. M. Wang, Phys. Rev. 184, 1920 (1969). We use a slightly different notation, \( \delta(p, p_0; Q) \), for the absorptive amplitude of the reaction Reggeon \( (p + iQ) \) + particle \( (p_0 + iQ) \) \rightarrow Reggeon \( (p - iQ) \) + particle \( (p + iQ) \), while reserving \( A(p, p_0; Q) \) for the physical on-shell absorptive amplitude.

Fig. 1. Kinematic notation for nonforward multiperipheral diagram.

and

\[ B_0(p', p''; Q) = \pi g(t', \mu^2; t'', \mu^2) \times (s'/s^2) \delta^4((p' - p'')^2 - \mu^2). \]  

(3)

We use the invariants (see Fig. 1) \( t' = (p' - p')^2, t'' = (p' - p'')^2, s' = (p' + p')^2, s'' = (p' + p'')^2, \) and \( \Sigma = (p - p'' - p')^2. \) Toller angles \( \omega_{\pm}, \omega'_{\pm}, \omega''_{\pm} \) are spatial angles between planes formed by \((p'\pm p''\pm p, p - p')\) and \((p_0\mp\frac{1}{2}Q, p' + p'')\) in the rest frame of the 4-vector \((p' - p'').\)

III. INTEGRAL EQUATION FOR NONFORWARD REGGEON-PARTICLE ABSORPTIVE AMPLITUDE

The basic approximations leading from the CGL equation to the simplified equation for the Reggeon-particle absorptive amplitude are the kinematic relations

\[ \Sigma s' = s = f(t', \omega'_{\pm}, \mu^2) \]

and

\[ \Sigma s'/s = f(t'', \omega''_{\pm}, \mu^2), \]

where

\[ f(t', \omega', \mu^2) = \frac{\Delta(t', \omega', \mu^2)}{\mu^2 - t' - t'' + 2(t' t'')^{1/2} \cos \theta} \]

and

\[ \Delta(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx. \]

Equation (4) is valid in the multiperipheral region of interest where \( s, s', \) and \( \Sigma \) are large compared to masses and \( t'_{s}, \) and where by strong ordering \( s', s, \Sigma \ll s. \) Substituting these relations into the CGL equation, one finds

\[ B(p, p', p''; Q) = B_0(p, p', p''; Q) + (s/s') \times f(t', \omega', \mu^2) \times [f(t'', \omega'', \mu^2)]^2 \]  

(5)

where

\[ \mathcal{A}_0 = \int \frac{d^4p''}{(2\pi)^4} \delta^4((p' - p'')^2 - \mu^2) G(t'_{\pm}, \omega'_{\pm}, t'_{\pm}) \times f(t', \omega', \mu^2) \]  

\[ \times [f(t'', \omega'', \mu^2)]^2 \times \left( [f(t', \omega', \mu^2)]^2 \right)^{t'_{\pm}} \]  

\( \times B(p', p''; Q). \]  

Because of the presence of \( \omega_{\pm}, \) which are functions of \( p, p', p''; p_0, \) and \( Q, \) this quantity \( \mathcal{A}_0 \) will in general depend on all vectors \( p, p', p''; p_0, \) and \( Q. \) We will show in Sec. IV that under the same conditions leading to Eq. (4) the Toller angles \( \omega_{\pm}, \omega', \omega'', \) and \( t'_{\pm}, t'', t'_{\pm}, t''_{\pm}, \) and \( t. \) Because of this, \( \omega_{\pm} \) will not depend on \( p \) and will also be true of the functions \( f(t'_{\pm}, \omega'_{\pm}, t'_{\pm}'), \) and \( G(t'_{\pm}, \omega'_{\pm}, t'_{\pm}'). \) Hence we can rewrite Eq. (6) as

\[ \mathcal{A}_0(p', p''; Q) = \int \frac{d^4p'}{(2\pi)^4} \delta^4((p' - p'')^2 - \mu^2) \]  

\[ \times G(t'_{\pm}, \omega'_{\pm}, t'_{\pm}') B(p', p''; Q), \]  

(7)

where

\[ G(t'_{\pm} t''_{\pm}, \gamma) = G(t'_{\pm}, \omega'_{\pm}, t'_{\pm}') \times [f(t'', \omega'', \mu'')^2] \times [f(t', \omega', \mu')] \]

(8)

This will now allow us to derive an integral equation for the nonforward Reggeon-particle absorptive amplitude with general double Regge coupling \( \beta(t', \omega', \omega''). \) Defining the single-particle intermediate-state contribution,

\[ \mathcal{A}_1(p, p'; Q) = \pi g(t, \mu^2; t', \mu^2) \times \left[ f(t', \omega', \mu')^2 \right] \times \left[ f(t'', \omega'', \mu'')^2 \right]. \]

The general integral equation for the \( \alpha \) amplitude is

\[ \mathcal{A}(p', p''; Q) = \mathcal{A}_1(p', p''; Q) \times \left[ f(t', \omega', \mu')^2 \right] \times \left[ f(t'', \omega'', \mu'')^2 \right] \times \left[ f(t', \omega', \mu')^2 \right]. \]

(9)

The loop integration will be expressed in terms of invariant variables after our discussion of the Toller angle.

IV. RELATION OF TOLLER ANGLES TO KINEMATIC INVARIANTS

We now show that the angles \( \omega_{\pm}, \) depend only on \( t'_{\pm}, t''_{\pm}, \) and \( t, \) in the kinematic region in which Eq. (4) is valid. Because the angles \( \omega_{\pm}' \) may be found from Eq. (4) in terms of \( t'_{\pm}, t''_{\pm}, \) and \( \Sigma s'/s, \) we need another relation between \( \Sigma s'/s \) and \( t', t''_{\pm}, t'. \) The proof of our assertion depends crucially on the fact that \( t = (2\pi)^{n-1}. \) In this case, there are five momentum vectors in problem: \( p, p', \) and \( Q, \) and since they are 4-vectors, they cannot all be linearly independent. This may be expressed by the vanishing of the determinant of the \( 5 \times 5 \) matrix of their scalar products. In the regions of interest where \( s, s', \) and \( \Sigma \) are large compared with masses and \( t's, \) the vanishing of the determinant to leading order gives us the relation for \( \Sigma s'/s: \)

\[ (\Sigma s'/s)^2 + 2b(\Sigma s'/s) + c = 0, \]

(11)
where coefficients $b$ and $c$ are functions of $t_+', t_-, t_+''$, $L_-$, and $t$. Equation (11) yields two solutions:

$$
\Sigma' = \frac{\alpha^2}{2} \left( t_+' + \frac{L_-' - \frac{1}{2}(L_-' - \frac{1}{2}) \pm (t_+'' - \frac{1}{2}) \sqrt{1 + \frac{4}{\alpha^2}} \right),
$$

Equation (12), together with Eq. (4), allows us to solve for $\omega_+$ and $\omega_-$ in terms of $t_+', t_+''$, and $t$ and this completes our proof. The existence of two solutions indicates that special care has to be taken in transforming the loop-momentum integration in Eqs. (2) or (9) to invariant variables.7

V. TRANSFORMATION TO INVARIANT VARIABLES

The two roots of Eq. (12) indicate that a given set of values for the variables $t_+', L_-$, $s'$ corresponds to two values of $\beta''$. This results in two possible values of $\Sigma = \beta''$. We will now show how this nonuniqueness can be removed by making use of our knowledge that the physical ranges of $\omega_+$ and $\omega_-$ are $[0, 2\pi]$. An alternative method is to exhibit the components of $\beta''$ in the center-of-mass frame in terms of the invariants; this is carried out in the Appendix.

We introduce the invariant variables,

$$
\beta'' = \frac{\beta'' \cdot Q}{(L_-' - \frac{1}{2})^{1/2}} = \frac{t_+' - \frac{1}{2}L_-' - \frac{1}{2}}{2(\frac{1}{2} - \frac{1}{2})^{1/2}},
$$

$$
\alpha'' = (-\beta'' - \beta'^{-2})^{1/2} = \frac{(L_+' + \frac{1}{2}L_-' - \frac{1}{2} (L_-' - \frac{1}{2})^{1/2})}{4L_-'},
$$

$$
\alpha'' = (-\beta'' - \beta'^{-2})^{1/2} = \frac{(L_+' + \frac{1}{2}L_-' - \frac{1}{2} (L_-' - \frac{1}{2})^{1/2})}{4L_-'},
$$

where $\alpha'$ and $\alpha''$ are taken positive. We note immediately that

$$
\frac{L_+' - \frac{1}{2}L_-' - \frac{1}{2}}{2(\frac{1}{2} - \frac{1}{2})^{1/2}} = \frac{L_+' + \frac{1}{2}L_-' - \frac{1}{2} (L_-' - \frac{1}{2})^{1/2}}{4L_-'},
$$

and they are even functions of $\alpha'$, $\alpha''$. We work in the region $\Delta(t_+', L_') < 0$, $L_-' < 0$, which guarantees in the integral equation that $\Delta(t_+', L_') < 0$ and $\Delta(t_+', L_-') < 0$. In terms of our new variables, Eq. (12) reads

$$
\Sigma/s = \frac{\alpha^2}{2} \left( (t_+' - \frac{1}{2})^2 + (\beta'' - \beta'^{-2})^2 \right).
$$

In order to cover the entire physical range in $\cos \phi_+'$, with $\alpha'$ and $\alpha''$ positive, we must use both branches of Eq. (15). This is, however, equivalent to choosing, say, the branch with the minus sign, and allowing $\alpha'$ and $\alpha''$ to take negative as well as positive values.

To verify these statements, we look, for example, in a two-dimensional plane of $\alpha''$, $\beta'' - \frac{1}{2}(t_-' - \frac{1}{2})^{1/2}$. From Eq. (4) we find that the values of $\cos \phi_+' = 1$ occur along the line through the origin $\alpha''/(\beta'' - \frac{1}{2}(t_-' - \frac{1}{2})^{1/2}) = \alpha''/(\beta'' - \frac{1}{2}(t_-' - \frac{1}{2})^{1/2})$, with $\cos \phi_+' = 1$ for the segment with $\cos \phi_+' = 1$ for the segment with $\cos \phi_+' = 1$ when $\cos \phi_+' = 1$ when $\cos \phi_+' = 1$. In this plane, fixed $t_+'$ corresponds to a circle about the origin, and to cover all values of $\omega_-$ clearly requires $\alpha''$ to be both positive and negative.

We now rewrite Eq. (10) in terms of the newly defined variables:

$$
G(p', p_0; Q) = G_t(p', p_0; Q)
$$

$$
+ \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\beta'' \int_{-\infty}^{\infty} d\beta' \int_{-\infty}^{\infty} d\beta'''
$$

$$
\times J(s', s''', \alpha''', \beta', \beta'', \beta'''; J) G_t(p', \alpha', \beta', \beta'', \beta'''),
$$

$$
\times (s'/s'''')^{(\alpha''')^2 + (\alpha'')^2} \alpha(p', p_0; Q),
$$

where

$$
J(s', s''', \alpha''', \beta', \beta'', \beta''')
$$

$$
= \frac{1}{(\alpha''')^{1/2}}(\alpha''')^{1/2} + (\alpha''')^{1/2} + \mu^2,
$$

and

$$
\frac{s'''}{s'} = \frac{\rho - (\alpha''')^{1/2}}{2\alpha'''}.
$$

Despite the double valuedness of the variables $\alpha''$, $\beta''$ with respect to $t_+', L_-'$, one can write the equation with variables $t_+', L_-'$ because the solution $G(p', p_0; Q)$ is a function of $\alpha''$. This follows from the fact that $G_t$ and $J$ depend on $\alpha''$ and the residue has the property

$$
G_0(\alpha''', \beta', \beta'', \beta''') = G_0(-\alpha' - \alpha'', \beta', \beta'', \beta''').
$$

Using the symmetrized residue

$$
G_s(\alpha''', \beta', \beta'', \beta''')
$$

$$
= \frac{1}{2} \left[ G_t(\alpha'', \beta', \beta'', \beta''') + G_t(-\alpha'', \alpha'', \beta', \beta'', \beta''') \right],
$$

the equation is written in the conventional form$^5$:

$$
G_t(s', s''; \beta', \beta'', \beta''') = G_t(s', s''; \beta', \beta'', \beta''') + G_t(s', s''; \beta', \beta'', \beta''').
$$

$$
\times \left[ (s'/s''')^{(\alpha''')^2 + (\alpha'')^2} \alpha(p', p_0; Q) \right],
$$

(18)

$$
(19)
$$

$$
(20)
$$

7 We wish to thank Carlton DeTar for emphasizing this point. This question also arose in the group-theoretical treatment of the nonforward multi-Regge integral equation. (See the last two articles of Ref. 2.)
where
\[ L_{1n} = [-(\alpha''/s')^{1/2}/2s'] \times J^{-1}. \] (21)

VI. FORWARD INTEGRAL EQUATION

The continuation of the integral equation to the forward direction, \( t=0 \), can be made directly in terms of the new variables. This is because the variables \( \alpha'', \beta'' \) are just components of momenta which do not vanish at \( t=0 \) (see Appendix). In the forward direction the variables become
\[ t'' = t' = -\alpha'^2 - \beta'^2 = t''', \]
\[ t' = -\alpha'^2 - \beta'^2 = t', \]
\[ \cos \omega' = \cos \omega_0 = \cos \omega', \]
where \( \cos \omega' \) is given by
\[ \frac{\Sigma s'}{s} = \frac{\mu^2 + (\alpha' - \alpha'')^2 + (\beta' - \beta'')^2}{\mu^2 - t'' - t'' + 2(t't'')^{1/2} \cos \omega'}. \] (22)

In terms of the \( \alpha', \beta'' \) variables, the Jacobian and integration limits of the integral equation (16) are not singular at \( t=0 \).

In applications of the forward integral equation it is convenient to integrate over \( s'', t', \) and \( \omega' \), so we now transform to these variables. This is facilitated by the transformation to polar coordinates in the \( \alpha'', \beta'' \) plane with the \( \phi' \) angle given by
\[ \cos \phi' = \frac{\alpha' \alpha'' + \beta' \beta''}{(-t'')^{1/2}(-t'')^{1/2}}, \] (23)
and
\[ d\alpha' d\beta'' = \frac{1}{2} d(-t') d\phi'. \]
(The definition of \( \phi' \) as a physical angle is given in the Appendix.) The forward Reggeon-particle absorptive part is given by the integral equation
\[ \alpha(t''; s') = \alpha_0(t''; s') + \frac{1}{(2\pi)^3} \int_{t''}^{t'} dt'' \int_0^{2\pi} d\phi' \int_{s''}^{s'} ds'' \times G(t'', t'; \cos \phi') (t''/s'')^{2n''(t'') \alpha} \alpha(t''; s''), \] (24)
where the Jacobian has been approximated for \( s''/s' \ll 1 \). This form of the forward equation has been discussed by Low.\(^{8}\) To transform from \( d\phi' \) to \( d\alpha' \) we note the reciprocal relation of \( \cos \phi' \) to \( \cos \omega' \) from Eq. (22): \[ d\alpha' d\beta'' = \frac{1}{2} d(-t') d\phi'. \] (25)

\[ \text{The forward integral equation is now rewritten as} \]
\[ \alpha(t''; s') = \alpha_0(t''; s') + \frac{1}{(2\pi)^3} \int_{t''}^{t'} dt'' \int_0^{2\pi} d\phi' \int_{s''}^{s'} ds'' \times G(t'', t'; \cos \phi') (t''/s'')^{2n''(t'') \alpha} \alpha(t''; s''). \] (26)

This version of the forward integral equation has been formulated and its properties studied by Finsky and Weisberger.\(^{9}\)

VII. CONCLUSION

This integral equation for \( \alpha(p', p_0; Q) \) may be regarded as a simplified multi-Regge model or as a large \( s'' \) approximation of the CGL equation. The treatment of the Toller-angle dependence in the \( \alpha \) equation is now equally general as that of the CGL equation. The \( \alpha \) equation seems to have all the essential physical content of the CGL equation and to have the additional advantage of being easier to work with. Following the analysis of Refs. 3 and 5, one finds Regge behavior for \( \alpha(p', p_0; Q) \). Applications of this equation are discussed elsewhere.\(^{10,11}\)

Note added in proof. After the completion of this work we were informed that the nonforward integral equation has also been formulated and studied by S. Finsky and W. Weisberger, Princeton report (unpublished).

ACKNOWLEDGMENTS

We would like to thank M. L. Goldberger, J. M. Wang, and W. Weisberger for several valuable discussions.

APPENDIX

Although the negative values of \( \alpha' \) and \( \alpha'' \) have only been introduced formally, we shall show that they have well-defined meaning as components of the vectors \( p' \) and \( p'' \) at high energy.

We shall discuss the problem in the c.m. frame of \( p+\frac{1}{2}Q \) and \( p_0-\frac{1}{2}Q \). Let \( p=-p_0 \) be in the \( \hat{z} \) direction, and let \( \perp \) denote the projection of a vector on the \( x-y \) plane. It can be shown that
\[ |Q, s| = -t+O(1/s), \quad Q_0=O(1/\sqrt{s}), \quad Q_s=O(1/\sqrt{s}), \quad \text{and} \]
\[ |p| \approx |p_0| \approx p_0 \approx \frac{p_0}{\sqrt{s}}+O(1/\sqrt{s}). \] (A1)

It then follows that in the multi-Regge strong-ordering limit of \( \Sigma_s s'' \ll s', s'' < s' \):
\[ t'' = (p''/s')^2 = - (p'/s')^2 + O(2^2/s^2), \] (A2)
\[ t'' = (p'/s')^2 = - (p''/s')^2 + O(2^2/s^2). \]


\(^{11}\) D. Silverman and C.-I Tan, Phys. Rev. D (to be published).
We choose $Q_i$ to lie in the $\hat{z}$ direction so that $\alpha''$, $\beta''$ are related to the components of $p''$ by

$$\beta'' = \frac{p'' \cdot Q}{(-\ell)^{1/2}} = \frac{p_{i''} \cdot Q_i}{(-\ell)^{1/2}} = -p_s''$$  \hspace{1cm} (A3)

and, by using Eqs. (13) and (14),

$$\alpha'' = p_v''$$

Similarly,

$$\beta' = -p_s', \quad \alpha' = p_v'$$

We see that it is necessary to take limits of $(-\infty, +\infty)$ for $\alpha''$, $\beta''$ in order to cover the entire phase space.

Since the variables are related to components of vectors, it is interesting to introduce the angle $\phi'$ in the c.m. frame defined by

$$\cos \phi'_4 = \frac{\varepsilon_{\mu\nu}\varepsilon^0(p' - {1\over 2}Q)(p_0 - {1\over 2}Q)^{\nu}(p' + {1\over 2}Q)^{\nu}}{\varepsilon_{\mu\nu}\varepsilon^0(p + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p' + {1\over 2}Q)^{\nu}}$$ \hspace{1cm} (A4)

At large $s$, $\Sigma$, $s'$ this becomes

$$\cos \phi'_4 = \frac{1}{2(t''_s)^{1/2}} \left[ \frac{s\Sigma}{s} - (\mu^2 - t_4 - t_{i''}) \right]$$

$$\alpha'_{\alpha''} = \left( \beta' - {1\over 2}(-\ell)^{1/2} \right) = \left[ \beta'' - {1\over 2}(-\ell)^{1/2} \right] \left( -t_4'' \right)^{1/2}$$ \hspace{1cm} (A5)

Using (A1) and (A3),

$$\cos \phi'_4 = \frac{\varepsilon_{\mu\nu}\varepsilon^0(p' + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p' + {1\over 2}Q)^{\nu}}{\varepsilon_{\mu\nu}\varepsilon^0(p + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p' + {1\over 2}Q)^{\nu}}$$ \hspace{1cm} (A6)

By changing $Q \rightarrow -Q$ in (A4), we also define $\cos \phi_-'$ and at large $s$, $\Sigma$, $s'$,

$$\cos \phi'_+ = \frac{\varepsilon_{\mu\nu}\varepsilon^0(p_0 - {1\over 2}Q)^{\nu}(p_0 - {1\over 2}Q)^{\nu}}{\varepsilon_{\mu\nu}\varepsilon^0(p + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p' + {1\over 2}Q)^{\nu}}$$

$$\cos \phi_-' = \frac{\varepsilon_{\mu\nu}\varepsilon^0(p' + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p_0 + {1\over 2}Q)^{\nu}}{\varepsilon_{\mu\nu}\varepsilon^0(p + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p' + {1\over 2}Q)^{\nu}}$$

$$\cos \phi_-' = \frac{\varepsilon_{\mu\nu}\varepsilon^0(p_0 - {1\over 2}Q)^{\nu}(p_0 - {1\over 2}Q)^{\nu}}{\varepsilon_{\mu\nu}\varepsilon^0(p + {1\over 2}Q)(p_0 + {1\over 2}Q)^{\nu}(p_0 + {1\over 2}Q)^{\nu}}$$

We find that $\phi_-$ are complementary angles to $\phi_\pm$ by using (A5), (A7), and (4):

$$\mu^2 - t_{s''} - t_{s''}'' - 2(t_{s''} t_{s''}''')^{1/2} \cos \phi_-' = f(t_{s''}, \omega_{s''}, t_{s''}''')$$

$$\Delta(t_{s''} t_{s''}''')^{1/2} \cos \omega_{s''}''$$

\hspace{1cm} (A8)