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ABSTRACT

We develop a variational principle for finding approximate eigenvalues and eigenfunctions of the kernel of the Lippmann-Schwinger equation. Regge trajectories are then easily found from the solution to the eigenvalue equation. We apply the variational principle to other potential theory calculations with very good results in both accuracy and simplicity.

I. INTRODUCTION

This paper is concerned with two main points. First we develop a method of simplifying and improving potential theory calculations along the lines suggested by Weinberg,[1] and second we apply this method to the problem of calculating Regge trajectories. Our approach is to expand the scattering matrix, at least partially, in eigenfunctions of the Lippmann-Schwinger kernel. The approximate eigenfunctions and eigenvalues are found by a variational method.

The singularities of the T matrix, which determine the Regge poles, can in principle by found from the zeros of the Fredholm determinant. However, in practice the method converges slowly and works well only for weak potentials.[2] Since the zeros of the determinant are related to the eigenvalues, it seems more practical to calculate the eigenvalues directly. Once the eigenvalues are known for complex ℓ and s, it is relatively easy to find the trajectory.

Most of the existing methods for finding approximate eigenvalues and eigenfunctions apply only to Hermitian operators. Unfortunately, the kernel of the Lippmann-Schwinger equation is not Hermitian; however it has similar properties and the variational principle for the eigenvalues takes a form analogous to the one for Hermitian operators. In Section II we discuss this relationship and establish two forms of a variational principle for finding eigenvalues.

In Section III we discuss the form the approximate trial wave functions should take, and we extend the results of Section II to partial waves. In Section IV we present the results of the Regge trajectory calculation. In Section V we consider other possible applications of our results. The use of the methods of Sections II and III can greatly simplify and improve

calculations using Weinberg's quasiparticle method.[1] We compare the form of the quasiparticle equation to the eigenfunction expansions of the T matrix.

This analysis can be extended to field theory, that is to the Bethe-Salpeter equation. For the $\lambda\phi^3$ theory below threshold, our theory is exactly the same except for the complications of the relative time or relative energy variable. Other theories need a cutoff since they are not Fredholm.

II. THE EIGENVALUE PROBLEM

In this section we introduce a variational principle for finding approximate eigenfunctions and eigenvalues of the kernel for the Lippmann-Schwinger equation

$$T(s) = V + T(s)\, G_0(s)\, V. \tag{2.1}$$

The formal solution to this equation can be written[3,4,5]

$$T(s) = V^{1/2} \left[1 - \tilde{K}(s)\right]^{-1} V^{1/2} \tag{2.2}$$

where

$$\tilde{K}(s) = V^{1/2}\, G_0(s)\, V^{1/2}. \tag{2.3}$$

For well-behaved potentials[3] $\tilde{K}$ is a square integrable kernel for all complex and real s, and therefore the only singularities of the T matrix occur when an eigenvalue of $\tilde{K}$ is equal to one. We write the eigenvalue equation as

$$\tilde{K}(s)\, V^{1/2} \left|\psi(s)\right\rangle = \eta(s)\, V^{1/2} \left|\psi(s)\right\rangle \tag{2.4}$$

where we have factored out $V^{1/2}$ since the $\left|\psi(s)\right\rangle$ is a solution of the unsymmetrized equation

$$G_0(s)\, V \left|\psi(s)\right\rangle = \eta(s) \left|\psi(s)\right\rangle . \tag{2.5}$$

Although $\tilde{K}(s)$ is not Hermitian for complex s, its adjoint is simply related to it by

$$\tilde{K}^{\dagger}(s) = \tilde{K}(s^*) . \tag{2.6}$$

This condition is quite important as it implies that if $V^{1/2}\left|\psi(s)\right\rangle$ is a right eigenvector of $\tilde{K}(s)$, then $\left\langle\psi(s^*)\right| V^{1/2}$ is a left eigenvector, and furthermore

$$\eta_\nu^*(s^*) = \eta_\nu(s) . \tag{2.7}$$

With the property (2.6) and the resulting relationship between the left and right eigenfunctions, the variational principle takes a particularly simple form. Recall that as Hermitian operators have left eigenfunctions which are the adjoints of the right eigenfunctions, only one set of parameters need be

varied. Our operator has this property except that s is replaced by s^* in the left eigenvector.

Theorem

The functional forms

$$\eta[\psi(s)] = \frac{\langle \psi(s^*) | V\, G_0(s)\, V | \psi(s) \rangle}{\langle \psi(s^*) | V | \psi(s) \rangle} \tag{2.8}$$

and

$$\eta[\psi(s)] = \frac{\langle \psi(s^*) | V | \psi(s) \rangle}{\langle (s^*) | G_0^{-1}(s) | \psi(s) \rangle} \tag{2.9}$$

are stationary, i.e., $\delta\eta = 0$ if and only if the trial state $|\psi(s)\rangle$ is an eigenstate of $\tilde{K}(s)$, or if the equivalent condition

$$\eta\, G_0^{-1} |\psi\rangle = V |\psi\rangle \tag{2.10}$$

is satisfied.

The proof is very similar to the proof for Hermitian operators, so we will just sketch the proof for (2.9).

$\langle \psi(s^*) | G_0^{-1}(s) | \psi(s) \rangle\, \delta\eta = \langle \delta\psi(s^*) | V + G_0^{-1}(s)\eta | \psi(s) \rangle + \langle \psi(s^*) | V - G_0^{-1}(s)\eta | \delta\psi(s) \rangle$.

With $|\delta\psi(s^*)\rangle$ and $|\delta\psi(s)\rangle$ taken to be independent, $\delta\eta = 0$ implies

$$\eta(s)\, G_0^{-1}(s) |\psi(s)\rangle = V |\psi(s)\rangle$$

and

$$\eta^*(s)\, G_0^{-1}(s^*) |\psi(s^*)\rangle = V |\psi(s^*)\rangle .$$

Equation (2.7) completes the proof. It should be noted that for negative s, (2.8) and (2.9) are maximum principles. Special care must be taken for positive energies, but as usual a proper limiting procedure removes the difficulty. It is most convenient to handle the quantity $\langle \psi(s^*) | G_0^{-1} | \psi(s) \rangle$ in momentum space and $\langle \psi(s^*) | V | \psi(s) \rangle$ in coordinate space.

In the next section we consider the partial-wave problem and all of the formulas in this section apply equally well there.

III. PARTIAL-WAVE EIGENFUNCTIONS

When we consider spherically symmetric potentials, and the Yukawa potential in particular, the eigenvalues $\eta_\nu(s)$ and the eigenfunctions $|\psi_\nu(s)\rangle$ become parameterized by the angular momentum ℓ,

$$\eta_\nu(s) = \eta_\ell^{(n)}(s); \quad \{\nu\} \to \{n, \ell\} \tag{3.1}$$

$$\langle \vec{r} | \psi_\nu(s) \rangle = \frac{1}{r} \langle r | \psi_\ell^{(n)}(s) \rangle \, Y_\ell^m(\hat{r}). \tag{3.2}$$

The partial-wave Schrodinger equation with the complex potential $\frac{V}{\eta_\ell(s)}$ is

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{V(r)}{\eta_\ell^{(n)}(s)} \right] \langle r | \psi_\ell^{(n)}(s) \rangle = 0 \tag{3.3}$$

where

$$k^2 = s, \quad \operatorname{Im} k \geq 0 \tag{3.4}$$

and

$$k^*(s) = -k(s^*). \tag{3.5}$$

We shall use the Riccatti Bessel functions

$$u_\ell(z) = \sqrt{\frac{\pi z}{2}} \, J_{\ell+1/2}(z) = z \, j_\ell(z) \tag{3.6}$$

where

$$\frac{2}{\pi} \int_0^\infty u_\ell(qr) \, u_\ell(q'r) = \delta(q - q') \tag{3.7}$$

so that the partial-wave Green's function is

$$G_\ell(s; r', r) = \frac{2}{\pi} \int_0^\infty \frac{dq}{s - q^2} \, u_\ell(qr) \, u_\ell(qr'), \quad \operatorname{Im} s > 0 \tag{3.8}$$

and the partial-wave scattering kernel is

$$\tilde{K}_\ell(s; r', r) = V^{1/2}(r') \, G_\ell(s; r', r) \, V^{1/2}(r). \tag{3.9}$$

The partial-wave momentum-space wave functions can be found from the solutions of (3.3) by

$$\langle q|\psi_\ell(s)\rangle = \sqrt{\frac{2}{\pi}} \int_0^\infty dr\, u_\ell(qr) \langle r|\psi_\ell(s)\rangle \tag{3.10}$$

so that the functional (2.9) becomes

$$\eta_\ell[\psi] = \frac{\langle \psi_\ell(s^*)|V|\psi_\ell(s)\rangle}{\langle \psi_\ell(s^*)|G_\ell^{-1}(s)|\psi_\ell(s)\rangle} \tag{3.11}$$

$$= \frac{\int_0^\infty dr \langle \psi_\ell(s^*)|r\rangle V(r) \langle r|\psi_\ell(s)\rangle}{\int_0^\infty dq \langle \psi_\ell(s^*)|q\rangle (s-q^2) \langle q|\psi_\ell(s)\rangle}. \tag{3.12}$$

At this point we restrict ourselves to the Yukawa potential

$$V(r) = -\lambda \frac{e^{-r}}{r} \tag{3.13}$$

for which

$$\langle p|V|q\rangle = \frac{-\lambda}{\pi} Q_\ell \left(\frac{q^2+p^{2+1}}{2qp}\right). \tag{3.14}$$

Our trial wave functions are chosen in as simple a form as possible consistent with the boundary conditions they must satisfy. These can be found directly from the defining equation

$$\eta(s) \langle q|\psi_\ell(s)\rangle = \frac{1}{s-q^2} \int_0^\infty Q_\ell \left(\frac{q^2+p^2+1}{2qp}\right) \langle p|\psi_\ell(s)\rangle\, dp \tag{3.15}$$

from which we see that

$$\langle q|\psi_\ell(s)\rangle \xrightarrow[q\to 0]{} q^{\ell+1}, \tag{3.16}$$

$$\langle q|\psi_\ell(s)\rangle \xrightarrow[q\to\infty]{} q^{-\ell-3}, \tag{3.17}$$

and

$$\langle q|\psi_\ell(s)\rangle \propto \frac{1}{s-q^2}. \tag{3.18}$$

We have assumed the integral in Eq. (3.15) converges uniformly, and the results (3.16) and (3.17) indicate that this is correct.

We now construct a trial wave function corresponding to the lowest eigenfunction, with one arbitrary parameter, μ.

$$\langle q|\psi_\ell(s)\rangle = \frac{q^{\ell+1}}{(q^2-s)\,[q^2+(\mu-ik)^2]^{\ell+1}}, \tag{3.19}$$

where $k^2 = s$. In the limit $\mu \to 0$, this wave function goes over to the exact lowest Coulomb wave functions,

$$\langle q|\psi_\ell(s)\rangle_{\text{Coulomb}} = \frac{q^{\ell+1}}{(q^2-s)^{\ell+2}}. \tag{3.20}$$

With these trial wave functions, we can use the variational principle to find good approximate eigenvalues; we do so in the next section to find Regge trajectories.

IV. EIGENVALUE REGGE TRAJECTORIES

We now let ℓ become complex and look for solutions of the equation

$$\eta(\ell, s) = 1 . \tag{4.1}$$

This solution is then inverted to give the trajectory

$$\ell = \alpha(s) . \tag{4.2}$$

The reflection property of equations (2.6) and (2.8) becomes

$$\tilde{K}^{\dagger}(\ell, s) = \tilde{K}(\ell^*, s^*) \tag{4.3}$$

and

$$\eta^*(\ell^*, s^*) = \eta(\ell, s) . \tag{4.4}$$

As an illustration we consider the Coulomb potential $V(r) = \frac{-\lambda}{r}$. If the wave function (3.20) is inserted in (3.12), the result is of course the exact lowest Coulomb eigenvalue

$$\eta(\ell, s) = \frac{i\lambda}{2k(\ell + 1)} , \tag{4.5}$$

which gives the exact leading trajectory,[6]

$$\ell(s) = -1 + \frac{i\lambda}{2k} . \tag{4.6}$$

For the Yukawa potential (3.13) we were forced to do the integrals in (3.12) numerically. Using the wave functions (3.19), we have the result,

$$\eta(\ell, s) = \frac{\frac{2}{\pi}\int_0^\infty dr\, \frac{\lambda e^{-r}}{r} \left\{ \int_0^\infty \frac{dq\, q^{\ell+1} u_\ell(qr)}{(q^2 - s)[q^2 + (\mu - ik)^2]^{\ell+1}} \right\}^2}{\int_0^\infty \frac{q^{2\ell+2}\, dq}{(q^2 - s)\,[q^2 + (\mu - ik)^2]^{2\ell+2}}} . \tag{4.7}$$

Combining the denominators in (4.7) by Feynman's method, we obtain the simplification

$$\eta(\ell, s) = \lambda 4^{\ell} \frac{I_2}{I_1}, \tag{4.8}$$

where

$$I_1 = \int_0^1 \frac{x^{2\ell+1}\, dx}{[b(x)]^{2\ell+3}} \tag{4.9}$$

and

$$I_2 = \int_0^\infty \frac{x^{\ell}}{b(x)}\, dx \int_0^\infty \frac{y^{\ell}\, dy}{b(y)\,[b(x)+b(y)+1]^{2\ell+2}} \tag{4.10}$$

with

$$b(x) = [x(\mu^2 - 2i\mu k) - k^2]^{1/2}. \tag{4.11}$$

In the above form the integrals are easily evaluated numerically.

In Figs. 1 and 2 we compare our results with the Regge trajectories calculated by Lovelace and Masson.[7] We used the same value of μ for an entire trajectory, with μ being chosen at some average value.[8] In a more accurate calculation μ would be determined at each point on the trajectory from (3.12).

V. THE VARIATIONAL METHOD AND QUASIPARTICLES

The methods developed in Section II are ideally suited to the quasiparticle[1] method of calculating scattering amplitudes. The two improvements we have to offer are the use of the variational principle to improve the choice of the subtraction dyad and the use of (2.9) rather than (2.8) to calculate the eigenvalues. To illustrate the considerable difference in computational labor and the increase in accuracy obtained by the suggested modifications, we perform two simple calculations.

The introduction of a quasiparticle into the theory is just the separation from $\tilde{K}$ of a dyad term, which we take to be formed from our approximate eigenfunctions.

$$\tilde{K} = \tilde{K}_Q + V^{1/2} |\psi(s)\rangle \langle\psi(s^*)| V^{1/2} . \tag{5.1}$$

A little algebra[3] yields the result for the T matrix,

$$T(s) = T_Q(s) + T_Q(s) |\psi(s)\rangle \Delta(s) \langle\psi(s^*)| T_Q(s) \tag{5.2}$$

where

$$\Delta(s) = \frac{1}{1 - \langle\psi(s^*)| T_Q(s) |\psi(s)\rangle} \tag{5.3}$$

and

$$T_Q(s) = V^{1/2} [1 - \tilde{K}_Q(s)]^{-1} V^{1/2} . \tag{5.4}$$

The normalization and phases of $\langle\psi(s^*)|$ and $|\psi(s)\rangle$ are chosen so that

$$\langle\psi(s^*)| V |\psi(s)\rangle = \eta(s) , \tag{5.5}$$

where $\eta(s)$ is given by (2.8) or (2.9). For potentials weak enough[3] so that

$$T_Q(s) \approx V , \tag{5.6}$$

Equation (5.2) simplifies to

$$T(s) \approx V + \frac{V|\psi(s)\rangle\, \eta\, \langle\psi(s^*)|V}{\langle\psi(s)|V|\psi(s^*)\rangle\,(1-\eta)}. \tag{5.7}$$

It is probably worth comparing this expression to the eigenfunction expansion for T,[9]

$$T(s) = V + \sum \frac{V|\psi_\nu(s)\rangle\, \eta_\nu(s)\, \langle\psi_\nu(s^*)|V}{\langle\psi_\nu(s)|V|\psi_\nu(s^*)\rangle\,[1-\eta_\nu(s)]}. \tag{5.8}$$

In our calculations we treat the Yukawa potential $V = \frac{-\lambda e^{-r}}{r}$ for $\ell = 0$. The trial wave function (3.19) becomes

$$\langle q|\psi(s)\rangle = \frac{q}{(q^2 - s)\,[q^2 + (\mu - ik)^2]} \tag{5.9}$$

and

$$\langle r|\psi(s)\rangle = \sqrt{\frac{\pi}{2}}\, \frac{e^{ikr}\,(1 - e^{-\mu r})}{\mu(\mu - 2ik)}. \tag{5.10}$$

Using (3.12), we obtain the eigenvalue

$$\eta(s) = \frac{2(\lambda - ik)}{\mu^2} \log\left[\frac{(1 + \mu - 2ik)^2}{(1 + 2\mu - 2ik)\,(1 - 2ik)}\right]. \tag{5.11}$$

Using (2.8) and the wave function (5.10) with $\mu = 1$, Scadron and Weinberg[10] obtained

$$\eta(s) = \frac{i\lambda}{k \log\left[\frac{(1-2ik)\,(3-2ik)}{(2-2ik)^2}\right]} \left\{ \frac{1}{2} \log^2\left(\frac{2-2ik}{1-2ik}\right) + L_{i_2}\left(\frac{-2}{1-2ik}\right) + L_{i_2}\left(\frac{-1}{2-2ik}\right) - L_{i_2}\left(\frac{-1}{1-2ik}\right) - L_{i_2}\left(\frac{-1}{1-ik}\right) \right\}. \tag{5.12}$$

There is considerable difference in complexity between (5.11) and (5.12), which represent, approximately, the same eigenvalue. For sample

calculations to compare with known results, we choose first to obtain an upper limit on the radius of convergence of the Born series at zero energy. This gives the coupling strength necessary to yield a zero-energy bound state and is obtained by setting $\eta = 1$ in (5.11) or (5.12) for $s = 0$.

Without benefit of a variational principle, Scadron and Weinberg[10] obtain $\lambda = 1.693$. We obtain with much less effort $\lambda = 1.6804$ (with $\mu = 1.5350$). The best value in the literature is $\lambda = 1.6798$.[11]

For our second example we perform a scattering-length calculation and compare results with those obtained from Blatt and Jackson's interpolation formula.[12] The scattering length is given by

$$a_s = -\lambda \left[1 + \frac{\eta(0) \left(\frac{\mu}{1+\mu} \right)^2}{[1-\eta(0)] \, \ell n \left[\frac{(1+\mu)^2}{1+2\mu} \right]} \right]. \tag{5.13}$$

We wish to calculate the value of λ necessary to give a scattering length of $a_s = 5$. We obtain

$$\lambda = 2.234 \tag{5.14}$$

as compared with Scadron and Weinberg's value of 2.342 and Blatt and Jackson's value of 2.222.

In both examples our results are more accurate and were obtained with less effort. Calculations at $s > 0$ will show up the difference even more, as a glance at (5.12) will readily verify.

In regard to bounds on the radius of convergence of the Born series, we were able to obtain[3] a lower bound from the inequality

$$\lambda^2 \geq \frac{1}{TR \, \tilde{K}^\dagger \tilde{K}}. \tag{5.15}$$

We thus have upper and lower bounds on the radius of convergence (for $s < 0$), which for the Yukawa potential are

$$1.6091 < \lambda < 1.6804 . \tag{5.16}$$

We mention this latter result because we were able to obtain similar results for the Bethe Salpeter equation for a $\lambda\phi^3$ theory. There we obtained upper and lower bounds for the coupling constant to give a zero-energy bound state of

$$0.4 \leqslant \lambda_{s=0} \leqslant 0.5 . \tag{5.17}$$

VII. FURTHER REMARKS

We have seen that it is possible to get quite good approximations to the lowest eigenfunction and eigenvalue. Whether it is possible to get a reasonable approximation to higher eigenfunctions we don't know; however, if more terms can be calculated, it is probably simpler to work with them than it is to iterate the modified T matrix in Weinberg's[1] quasiparticle method. The reason is that even if all the eigenvalues are less than one the Born series may converge quite slowly, whereas if the first few terms are expanded in eigenfunctions it may be possible to lump the rest into the potential.

An additional motivation for the eigenfunction approach is given by Lovelace.[13] He suggests that one approximation scheme for solving the 3-body problem is to approximate the 2-body Green's function by a series of eigenfunctions.

FOOTNOTES AND REFERENCES

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

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8. The values used for the parameter μ were:

   $\mu = 1.8$ for $\lambda = 5$.

   $\mu = 2.5$ for $\lambda = 15$.
9. The completeness of the eigenfunction expansion for complex s has not been conclusively proven; however we believe it to be true. See Meetz, reference 4.
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FIGURE CAPTIONS

Fig. 1. First Regge trajectory of Yukawa potential $V(r) = -G \frac{\exp[-r]}{r}$. Real part of α as a function of s. —— Lovelace and Masson, Ref. 7; ● our results.

Fig. 2. First Regge trajectory of Yukawa potential $V(r) = -G \frac{\exp[-r]}{r}$. Plotted in the complex ℓ plane. —— Lovelace and Masson, Ref. 7; ● our results.

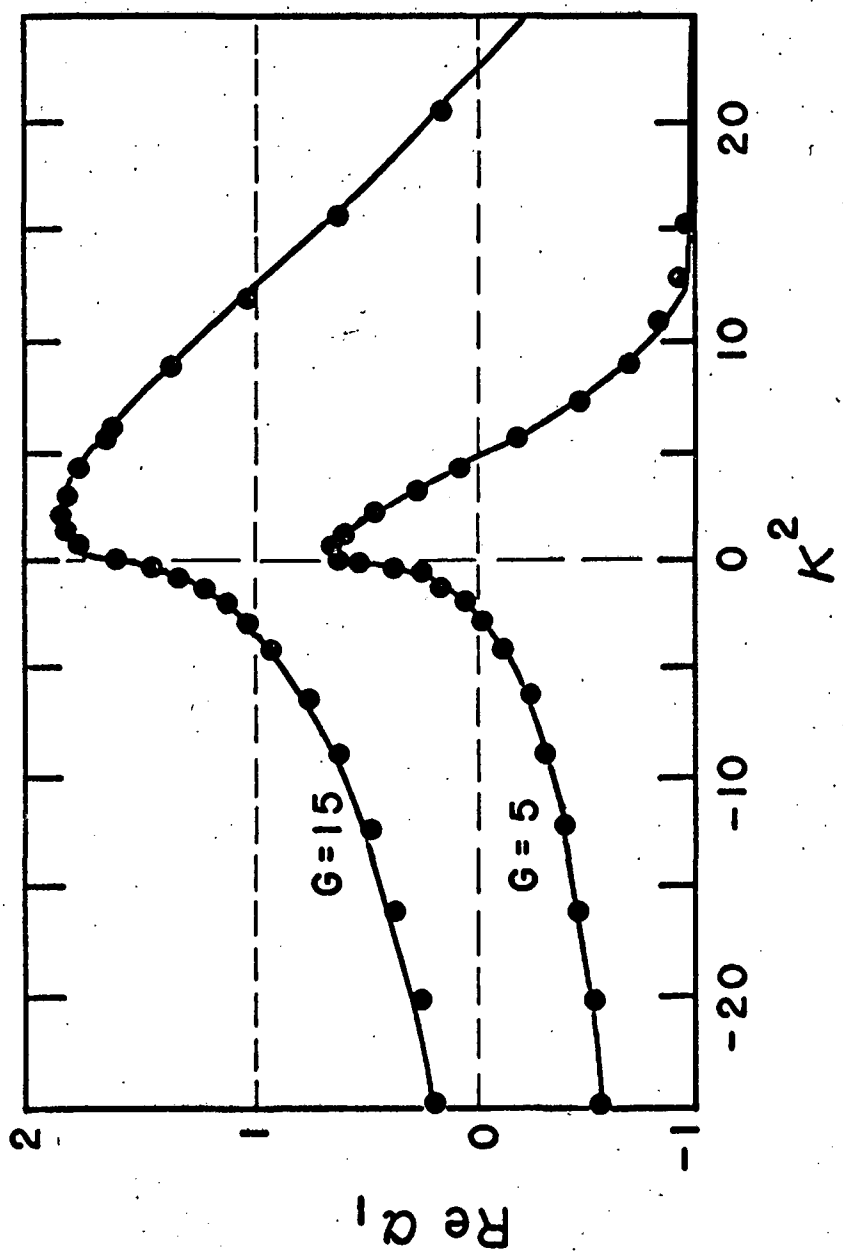

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Fig. 1.

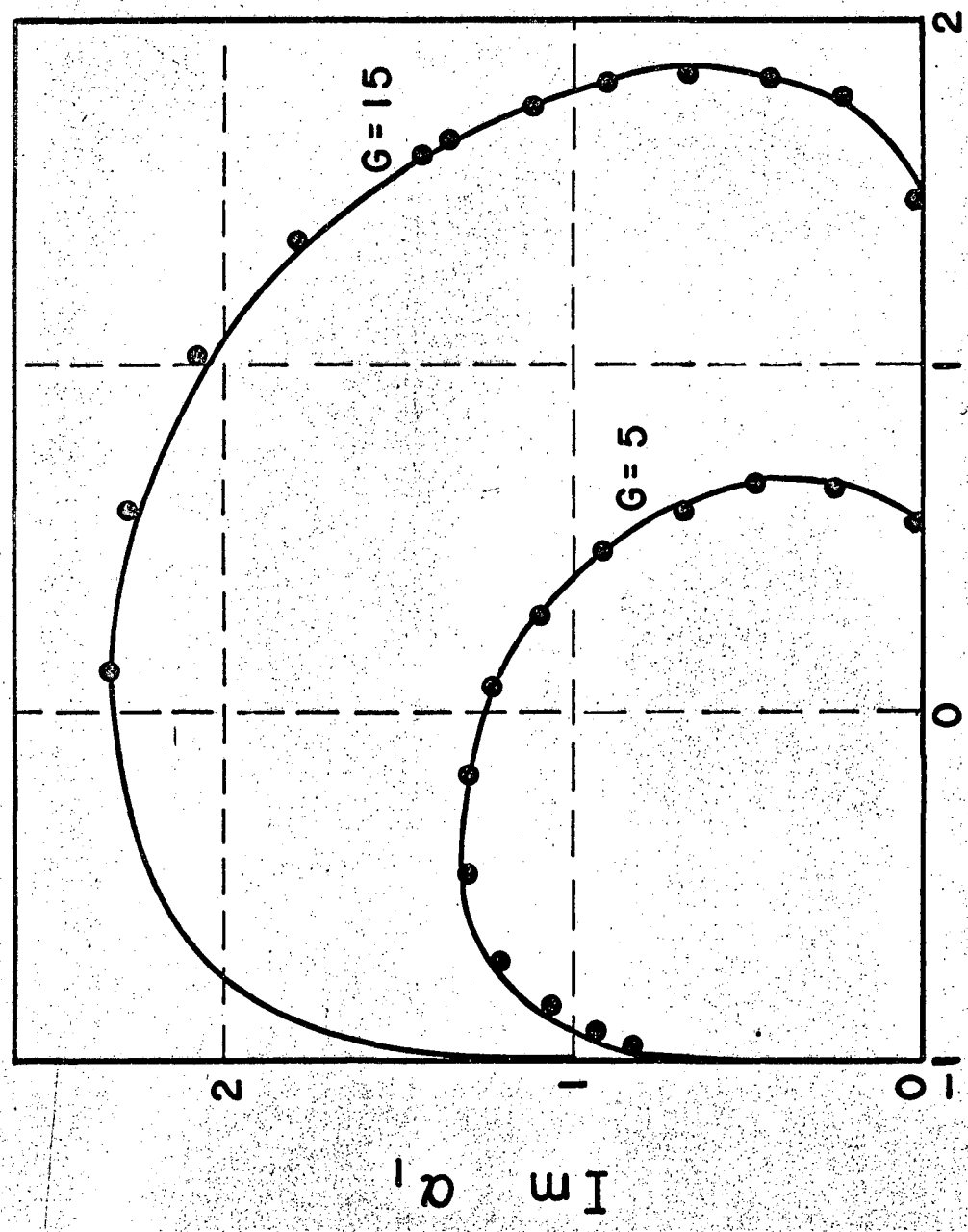

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Fig. 2.