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Discovering independent parameters in complex dynamical systems

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Abstract

The transformation of a nonlinear dynamical system into a standard form by using one of its variables and its successive derivatives can be used to identify the relationships that may exist between the parameters of the original system such as the subset of the parameter space over which the dynamics is left invariant. We show how the size of the attractor or the time scale (the pseudo-period) can be varied without affecting the underlying dynamics. This is demonstrated for the Rössler and the Lorenz systems. We also consider the case when two Rössler systems are unidirectionally coupled and when a Lorenz system is driven by a Rössler system. In both cases, the dynamics of the coupled system is affected.

I. INTRODUCTION

Coupling dynamical systems is a very popular approach for understanding how synchronization and control can occur in the natural world. Network of coupled dynamical systems are ubiquitous throughout nature including biological systems (see [1–4] and references therein). It is therefore essential to better understand how coupled non-linear systems interact, that is, drive each other, and how such interactions affect the collective dynamics. This is particularly relevant for complex networks when selecting for the terms of greatest influence on the collective dynamics and, consequently, which parameters most efficiently affect the network dynamics. This is dual to the observability problem of complex networks [5, 6].

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In the context of global modeling dynamical systems can be rewritten in a “standard form” [7] (now also called a jerk form [8–12]), that is, by using one of the variables of the original system and its Lie derivatives. When the Ansatz library is used for rewriting a dynamical system in a standard form [13, 14], it is possible to identify sets of parameter values or, equivalently, domains in the parameter space, for which the dynamics is invariant; the dynamics is thus left unchanged under a variation of well chosen parameter values as shown for the Rössler system when variable y and its successive Lie derivatives are used [15] or for the Lorenz-like systems when variable x is used [16].

In the present paper, our aim is to determine how many independent parameters can be identified in the Rössler and the Lorenz systems and how they can be affected by the variable retained for the analysis. Moreover, in both cases, we determine how it is possible to vary the pseudo-period of the system by modifying a subset of the system parameters, a consequence previously not considered [13].

We will then consider the case where two Rössler systems are coupled and the case where a Rössler system is used for driving a Lorenz system without affecting the standard function when the driven system is still considered in a three-dimensional space. The number of independent parameters of the resulting six-dimensional system is investigated and it is shown how the algebraic structure of the two systems can no longer be considered as independent.

We show here the sets of connected parameters in the Lorenz and the Rössler systems, respectively. We also investigate how connections are modified in the case of two systems unidirectionally coupled. Only parameters that are connected in the individual uncoupled systems can be further connected in the coupled system. Furthermore, parameters that are not connected within the individual systems remain independent in the coupled system.

The paper is organized as follows. Section II starts with some background on the interplay between the original system and its standard form. The numbers of independent parameters in the Rössler and in the Lorenz systems are then extensively investigated. Section III is devoted to the determination of the parameter to adjust for varying the pseudo-period of these two systems. The case of two coupled Rössler systems is discussed in Section IV and the case of a Lorenz system driven by a Rössler system is discussed in Section V. Section VI gives some conclusions.

II. LORENZ AND RÖSSLER SYSTEMS

A. General background

Consider a three-dimensional dynamical system of ordinary differential equations (ODEs) in the form

$$\dot{x}=f_x(x,y,z)$$

$$\dot{y}=f_y(x,y,z) \quad (1)$$

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$$\dot{z}=f_z(x,y,z)$$

with

$$f_i(x,y,z)=a_{i,0}+a_{i,1}x+a_{i,2}y+a_{i,3}z+a_{i,4}x^2+ a_{i,5}xy+a_{i,6}xz+a_{i,7}y^2+a_{i,8}yz+ a_{i,9}z^2$$

where $i \in \{x, y, z\}$ and monomials are of a polynomial form up to the second-order. It was shown in [7, 13, 14] that when a small subset of coefficients $a_{i,j}$ is nonzero, system (1) can be transformed into a standard form whose structure is

$$\dot{X}_1=X_2$$

$$\dot{X}_2=X_3 \quad (2)$$

$$\dot{X}_3=F_s(X_1,X_2,X_3)$$

where the first variable $X_1 = h(x, y, z) = s$ is some function of the state variables, h being the so-called measurement function. The standard function F_s has a multivariate polynomial form depending on some parameters α_k. Map $\Phi : \mathbb{R}^3(x, y, z) \rightarrow \mathbb{R}^3(X_1, X_2, X_3)$ between the original state space and the differential embedding allows one to obtain the standard form (2) from the original system (1). There is an associated map $\varphi : \mathbb{R}^p(a_{i,j}) \rightarrow \mathbb{R}^q(\alpha_k)$ that expresses parameters α_k's of the standard function F_s in terms of parameters $a_{i,j}$ from the original system.

Lainscsek [15, 16] showed that there are algebraically in-equivalent dynamical systems in the form (1) which can be transformed to the same standard form (2): it is therefore possible to define a class of equivalent dynamical systems in the sense that from the time series point of view they have an identical variable. Moreover, there is a subspace of the parameter space in which the dynamics is unchanged: therefore, there exists an infinite number of global models corresponding to a single time series.

Parameters involved in relationships defining parameter subspaces in which dynamics remains unchanged are said to be *connected*. This property is important for global modeling techniques [17] since there is an infinite number of dynamical systems of the form (1) that share the same time series; the global model which can be obtained from a given time series is thus not unique. Map φ further reveals the connections between parameters from the original form (1) and those from the standard form (2), and the number of non-connected and independent parameters in the original system. Since not all parameters of the original system (1) are independent, it was shown [15] that its pseudo-period can be varied by using specific relationships between some of these parameters.

B. Lorenz equations

The Lorenz equations [18]

$$\dot{x}_L = a_{1,1}x_L + a_{1,2}y_L$$

$$\dot{y}_L = a_{2,1}x_L + a_{2,2}y_L + a_{2,6}x_L z_L \quad (3)$$

$$\dot{z}_L = a_{3,3}z_L + a_{3,5}x_L y_L$$

can be transformed into a standard form (2) whose function F_s is more or less complicated depending on the "measured" variable [7]. With the choice $X_1 = h(x, y, z) = x_L$ the standard function

$$F_{x_L} = \alpha_1 X_1 + \alpha_2 X_1^3 + \alpha_3 X_2 + \alpha_4 X_1^2 X_2 + \alpha_5 \frac{X_2^2}{X_1} + \alpha_6 Z + \alpha_7 \frac{X_2 X_3}{X_1} \quad (4)$$

is made of $N_d = 7$ terms of which two are rational. The standard function F_{x_L} has therefore two singular terms (corresponding to parameters α_5 and α_7) in $X_1 = 0$. Parameters α_k of this standard function are related to parameters $a_{i,j}$ of the Lorenz system (3) as follows

$$\phi_{x_L} = \left| \begin{array}{l} \alpha_1 = (a_{1,1}a_{2,2} - a_{1,2}a_{2,1})a_{3,3} \\ \alpha_2 = -a_{1,1}a_{2,6}a_{3,5} \\ \alpha_3 = -(a_{1,1} + a_{2,2})a_{3,3} \\ \alpha_4 = a_{2,6}a_{3,5} \\ \alpha_5 = -a_{1,1} - a_{2,2} \\ \alpha_6 = a_{1,1} + a_{2,2} + a_{3,3} \\ \alpha_7 = 1. \end{array} \right. \quad (5)$$

By introducing the scaling transformation [16]

$$a_{i,j} \to \tilde{a}_{i,j} = \lambda^{p(i,j)} a_{i,j} \quad (6)$$

and leaving the values of coefficients α_k unchanged, it was shown that there is a class of Lorenz-like systems sharing the same x_L-time series which can be defined.

The simplest way to determine these scaling relations, in particular, the set of exponents $p(i, j)$, is to note that each coefficient α_k is a sum of products of powers of parameters $a_{i,j}$. Taking the logarithms of these nonlinear product functions, we can construct the appropriate coefficient matrix and determine the null space.

The scaled version of the inverse transformation (5) with $a_{i,j} \to \lambda_{i,j}a_{i,j}$ is

$$\alpha_1 = a_{1,1}a_{2,2}a_{3,3}\lambda_{1,1}\lambda_{2,2}\lambda_{3,3} - a_{1,2}a_{2,1}a_{3,3}\lambda_{1,2}\lambda_{2,1}\lambda_{3,3}$$

$$\alpha_2 = -a_{1,1}a_{2,6}a_{3,5}\lambda_{1,1}\lambda_{2,6}\lambda_{3,5}$$

$$\alpha_3 = -a_{1,1}a_{3,3}\lambda_{1,1}\lambda_{3,3} - a_{2,2}a_{3,3}\lambda_{2,2}\lambda_{3,3} \quad (7)$$

$$\alpha_4 = a_{2,6}a_{3,5}\lambda_{2,6}\lambda_{3,5}$$

$$\alpha_5 = -a_{1,1}\lambda_{1,1} - a_{2,2}\lambda_{2,2}$$

$$\alpha_6 = a_{1,1}\lambda_{1,1} + a_{2,2}\lambda_{2,2} + a_{3,3}\lambda_{3,3}$$

$$\alpha_7 = 1.$$

To leave α_i unchanged in (5), the scale factors have to be one (e.g. for α_1: $\lambda_{1,1}\lambda_{2,2}\lambda_{3,3} = 1$ and $\lambda_{1,2}\lambda_{2,1}\lambda_{3,3} = 1$). Taking the logarithm leads to linear relations as, for instance,

$$\log(\lambda_{1,1}) + \log(\lambda_{2,2}) + \log(\lambda_{3,3}) = 0 \quad (8)$$

and

$$\log(\lambda_{1,2}) + \log(\lambda_{2,1}) + \log(\lambda_{3,3}) = 0. \quad (9)$$

The set of linear relations derived from the 12 terms of Eq. (7) is summarized in the matrix formulation

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \log(\lambda_{1,1}) \\ \log(\lambda_{1,2}) \\ \log(\lambda_{2,1}) \\ \log(\lambda_{2,2}) \\ \log(\lambda_{2,6}) \\ \log(\lambda_{3,3}) \\ \log(\lambda_{3,5}) \end{pmatrix} = 0. \quad (10)$$

This 12 × 7 matrix has a two-dimensional null space spanned by the null vectors (0, 0, 0, 0, −1, 0, 1) and (0, −1, 1, 0, 0, 0, 0). This means that, among the seven parameter $a_{i,j}$ of the Lorenz system (3), there are two sets of connected parameters made of i) $a_{1,2}$ and $a_{2,1}$, and

ii) $a_{2,6}$ and $a_{3,5}$. Consequently, until the products $a_{1,2} \cdot a_{2,1}$ and $a_{2,6} \cdot a_{3,5}$ are constant while varying appropriately the corresponding parameters, the standard function F_{x_L} is left unchanged, resulting in an unchanged dynamics. In each pair, one of these parameters can be set to 1, the others being set to the value of their product. Consequently, each set of connected parameters represents a single independent parameter. Obviously, non-connected parameters are independent parameters. In the case of the Lorenz system (3), there are only five independent parameters, the three non-connected ones plus one for each of the two non-overlapping sets of connected parameters. In fact, keeping constant the product $a_{1,2} \cdot a_{2,1}$ or $a_{2,6} \cdot a_{3,5}$ does not change the dynamics, that is, for instance, the population of periodic orbits embedded within the attractor remain the same; nevertheless, the amplitude of variable oscillations or, equivalently, the size of the attractor, can be changed. It is thus possible to rescale the size of the attractor by varying these two products. This can be expressed explicitly by rewriting the system (3) as

$$\begin{aligned} \dot{x}_L &= a_{1,1} x_L + \frac{1}{\nu} a_{1,2} y_L \\ \dot{y}_L &= \nu a_{2,1} x_L + a_{2,2} y_L + \frac{1}{\mu} a_{2,6} x_L z_L \\ \dot{z}_L &= a_{3,3} z_L + \mu a_{3,5} x_L y_L \end{aligned} \quad (11)$$

where μ and ν are the so-called scaling parameters. We did not investigate the standard function induced by variable y_L or z_L due to their complexity. Thus it is possible that other connected parameters may exist in the Lorenz system.

C. Rössler equations

Starting from variable y_{R_1} as the measured variable, the Rössler system [19]

$$\begin{aligned} \dot{x}_R &= b_{1,2} y_R + b_{1,3} z_R \\ \dot{y}_R &= b_{2,1} x_R + b_{2,2} y_R \\ \dot{z}_R &= b_{3,0} + b_{3,3} z_R + b_{3,6} x_R z_R \end{aligned} \quad (12)$$

can be rewritten into the standard form in the differential embedding $\mathbb{R}^3(X_1 = y_R, X_2, X_3)$. The corresponding standard function $F_{y_R}(X_1, X_2, X_3)$ is explicitly

$$F_{y_R} = \alpha_1 + X_1\alpha_2 + X_1^2\alpha_3 + X_2\alpha_4 + X_1X_2\alpha_5 + X_2^2\alpha_6 + X_3\alpha_7 + X_1X_3\alpha_8 + X_2X_3\alpha_9 \quad (13)$$

whose parameters α_k are related to parameters $b_{i,j}$ of the Rössler system (12) by

$$\phi_{y_R} = \left| \begin{array}{l} \alpha_1 = b_{1,3} b_{2,1} b_{3,0} \\ \alpha_2 = -b_{1,2} b_{2,1} b_{3,3} \\ \alpha_3 = b_{1,2} b_{2,2} b_{3,6} \\ \alpha_4 = b_{1,2} b_{2,1} - b_{2,2} b_{3,3} \\ \alpha_5 = \frac{b_{2,2}^2 b_{3,6}}{b_{2,1}} - b_{1,2} b_{3,6} \\ \alpha_6 = -\frac{b_{2,2} b_{3,6}}{b_{2,1}} \\ \alpha_7 = b_{2,2} + b_{3,3} \\ \alpha_8 = -\frac{b_{2,2} b_{3,6}}{b_{2,1}} \\ \alpha_9 = \frac{b_{3,6}}{b_{2,1}}. \end{array} \right. \quad (14)$$

Thus, the standard function has $N_d = 9$ nonzero parameters α_k. By using the scaling relations (6), the set of linear relations after removing repeated entries is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \log(\lambda_{1,2}) \\ \log(\lambda_{1,3}) \\ \log(\lambda_{2,1}) \\ \log(\lambda_{2,2}) \\ \log(\lambda_{3,0}) \\ \log(\lambda_{3,3}) \\ \log(\lambda_{3,6}) \end{pmatrix} = 0 \quad (15)$$

with the null vectors (−1, −1, 1, 0, 0, 0, 1) and (0, −1, 0, 0, 1, 0, 0). This means, that there are two sets of connected parameters: since parameter $b_{1,3}$ belongs to the two sets, the five connected parameters can be reduced to a single independent parameter. Let $b_{3,0} = b$. The two remaining parameters that are non-connected, are $b_{2,2}$ and $b_{3,3}$. Setting $b_{2,2} = a$, and $b_{3,3} = -c$ recovers the Rössler system as originally proposed [19]. In fact, Rössler only retained these three parameters because they affected the dynamics most [20].

The Rössler system (12) can thus be rewritten as follows

$$\dot{x}_R = \frac{1}{\rho} b_{1,2} y_R + \frac{1}{\rho} \frac{1}{\kappa} b_{1,3} z_R$$

$$\dot{y}_R = \rho b_{2,1} x_R + b_{2,2} y_R \quad (16)$$

$$\dot{z}_R = \kappa b_{3,0} + b_{3,3} z_R + \rho b_{3,6} x_R z_R$$

where ρ and κ are the so-called scaling factors since they may only affect the amplitude of the oscillations while leaving the population of unstable periodic orbits embedded within the attractor unchanged. They explain two important features frequently observed in global modeling [21]: i) global models are not unique and ii) a successful global model can produce an attractor that is topologically equivalent to the attractor underlying measured data but of a different size, e.g. smaller or larger in size than the original attractor.

In the case of the Rössler system, the standard function F_i can be easily computed when variable x_R or z_R is "measured" [7]: it has some monomials $\alpha_k X_1^l X_2^m X_3^n$ whose exponents l, m and n may also be negative yielding rational terms [7, 14]. Note that, as pointed out in [14], the standard function F_{x_R} induced by variable x_R of the Rössler system is not in a polynomial form involving terms $X_1^l X_2^m X_3^n$ but can be brought into such a form by shifting variable x_R according to

$$x_R \rightarrow x_R + \frac{b_{2,2} - b_{3,3}}{b_{3,6}}. \quad (17)$$

Such a shift induces a rigid displacement of the attractor in the state space $\mathbb{R}^3(x_R, y_R, z_R)$ but leaves it unchanged, that is, the population of unstable periodic orbits embedded within remains the same.

Applying the same procedure as used for determining the null vectors among parameters, we obtain two such vectors for each of these variables as reported in Tab. I. These two null vectors are associated with the two subsets of the relationships between parameters $b_{i,j}$ already identified from those obtained using y_R as the "measured" variable. This means that all existing relationships between parameters $b_{i,j}$ of the Rössler system can be determined from the sole y_R-variable; in contrast only a subset of the systems interconnections are obtained from the two other variables.

We conjecture that there is a strong connection between these features and the observability provided by each of the variables of the Rössler system. Observability is related to the ability provided by some measurements to distinguish the different states of the system [22]. There is a full observability of the Rössler system when it is investigated by measuring variable y_R, meaning that any state of the system can be distinguished from the sole measurements of this variable. In other words, all the information related to the dynamics of the Rössler system is contained in variable y_R. This is not the case for the two other variables for which there is a singular observability manifold; that is, a domain of the original state space $\mathbb{R}^3(x_R, y_R, z_R)$ which cannot be oberved by only measuring x_R or z_R [23, 24]. The lack of information contained in these two variables also prevents us from identifying all possible relationships between coefficients $b_{i,j}$ solely from variables x_R or z_R. Consequently, since there is not a full observability from variable x_L of the Lorenz system [25], we cannot be ensured to have determined all possible relationships between parameters $a_{i,j}$.

From the null vectors listed in Tab. I, it is possible to extract the four different null vectors by introducing the four scaling factors κ, ν, ϕ and ρ in system (12) as

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$$\dot{x}_R = \frac{1}{\rho}\frac{1}{\nu} b_{1,2} y_R + \frac{1}{\nu}\frac{1}{\varphi}\frac{1}{\kappa} b_{1,3} z_R$$

$$\dot{y}_R = \nu\rho b_{2,1} x_R + b_{2,2} y_R \quad (18)$$

$$\dot{z}_R = \kappa b_{3,0} + b_{3,3} z_R + \nu\varphi b_{3,6} x_R z_R.$$

Varying these factors does not affect the dynamics from the unstable periodic orbits point of view: only the scale of the attractor is affected.

III. TIME SCALING

As introduced in [15], it is possible to extend the list of connections between parameters $b_{i,j}$ by introducing the time scaling transformation

$$\tilde{t} = Dt \quad (19)$$

which affects the inverse transformation (14) as

$$\phi'_{y_R} = \left| \begin{array}{l} \alpha_1 = D^3 b_{1,3} b_{2,1} b_{3,0} \\ \alpha_2 = -D^3 b_{1,2} b_{2,1} b_{3,3} \\ \alpha_3 = D^3 b_{1,2} b_{2,2} b_{3,6} \\ \alpha_4 = D^2 (b_{1,2} b_{2,1} - b_{2,2} b_{3,3}) \\ \alpha_5 = D^2 \left(\frac{b_{2,2}^2 b_{3,6}}{b_{2,1}} - b_{1,2} b_{3,6} \right) \\ \alpha_6 = -D \frac{b_{2,2} b_{3,6}}{b_{2,1}} \\ \alpha_7 = D (b_{2,2} + b_{3,3}) \\ \alpha_8 = -D \frac{b_{2,2} b_{3,6}}{b_{2,1}} \\ \alpha_9 = \frac{b_{3,6}}{b_{2,1}}. \end{array} \right. \quad (20)$$

Using scaling relations (6) with an additional scaling factor $D \to \delta D$, the set of linear relations after removing repeated entries is

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \log(\lambda_{1,2}) \\ \log(\lambda_{1,3}) \\ \log(\lambda_{2,1}) \\ \log(\lambda_{2,2}) \\ \log(\lambda_{3,0}) \\ \log(\lambda_{3,3}) \\ \log(\lambda_{3,6}) \\ \log(d) \end{pmatrix} = 0 \quad (21)$$

with the null vectors $N_1 = (2, 3, 0, 1, 0, 1, 0, -1)$, $N_2 = (-1, -1, 1, 0, 0, 0, 1, 0)$, and $N_3 = (0, -1, 0, 0, 1, 0, 0, 0)$. Therefore Eq. (16) extends to

$$\dot{x}_R = \delta^2 \frac{1}{\rho} b_{1,2} y_R + \delta^3 \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} z_R$$

$$\dot{y}_R = \rho b_{2,1} x_R + \delta b_{2,2} y_R \quad (22)$$

$$\dot{z}_R = \kappa b_{3,0} + \delta b_{3,3} z_R + \rho b_{3,6} x_R z_R$$

where the additional parameter δ is a time scaling factor of the dynamics: this means that the pseudo-period of oscillations (revolutions in the attractor) is varied without any change in the population of unstable periodic orbits embedded within the attractor.

Eq. (22) can be rewritten in a different form by changing the null vector $N_1 \mapsto N_1 + 2N_2 + N_3 = (0, 0, 2, 1, 1, 1, 2, -1)$:

$$\dot{x}_R = \frac{1}{\rho} b_{1,2} y_R + \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} z_R$$

$$\dot{y}_R = \delta^2 \rho b_{2,1} x_R + \delta b_{2,2} y_R \quad (23)$$

$$\dot{z}_R = \delta\kappa b_{3,0} + \delta b_{3,3} z_R + \delta^2 \rho b_{3,6} x_R z_R$$

To be more explicit, Eq. (22) can be rewritten in the form (23) as follows: parameters $b_{1,2}$, $b_{1,3}$, $b_{2,1}$, and $b_{3,6}$ are connected by the scaling factor ρ. Therefore, we can apply the transformation $\rho \mapsto \rho\, \delta^2\ (N_1 + 2N_2)$

$$\left| \begin{array}{rcl} \delta^2 \frac{1}{\rho} b_{1,2} & \mapsto & \frac{1}{\rho} b_{1,2} \\ \delta^3 \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} & \mapsto & \delta \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} \\ \rho b_{2,1} & \mapsto & \delta^2 \rho b_{2,1} \\ \rho b_{3,6} & \mapsto & \delta^2 \rho b_{3,6} \end{array} \right. \quad (24)$$

and leaving the dynamics unchanged. In a similar way, parameters $b_{1,3}$ and $b_{3,0}$ being connected, the transformation $\kappa \mapsto \kappa\, \delta\ (N_1 + N_3)$

$$\left| \begin{array}{rcl} \delta \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} & \mapsto & \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} \\ \kappa b_{3,0} & \mapsto & \delta\kappa b_{3,0} \end{array} \right. \quad (25)$$

also leaves the dynamics unchanged. The Rössler system can thus be rewritten as Eq. (23). Consequently, the time can be rescaled in the Rössler system only by varying five parameters ($b_{2,1}$, $b_{2,2}$, $b_{3,0}$, $b_{3,3}$ and $b_{3,6}$) in the appropriate way: note that the first equation is left unchanged.

Applying such a time scaling to the Lorenz system does not leave any of the three equations unchanged. Indeed, the Lorenz system rewritten with the time scaling factor is

$$\dot{x}_L = \delta a_{1,1} x_L + \delta^2 \frac{1}{\nu} a_{1,2} y_L$$

$$\dot{y}_L = \nu a_{2,1} x_L + \delta a_{2,2} y_L + \delta^2 \frac{1}{\mu} a_{2,6} x_L z_L \quad (26)$$

$$\dot{z}_L = \delta a_{3,3} z_L + \mu a_{3,5} x_L y_L,$$

meaning that the time scaling factor δ appears in the three equations and that there is no connected parameter which would allow to remove factor δ in one of them.

IV. COUPLED RÖSSLER SYSTEMS

A general original system obtained from the Ansatz Library [13, 14] provides the same three standard functions as the Rössler system (12):

$$\dot{x}_R = b_{1,2} y_R + b_{1,3} z_R + \boldsymbol{b_{1,0}} + \boldsymbol{b_{1,1} x_R}$$

$$\dot{y}_R = b_{2,1} x_R + b_{2,2} y_R + \boldsymbol{b_{2,0}} \quad (27)$$

$$\dot{z}_R = b_{3,0} + b_{3,3} z_R + b_{3,6} x_R z_R + \boldsymbol{b_{3,1} x_R} + \boldsymbol{b_{3,2} y_R}.$$

This has five additional terms (in bold) compared to the original Rössler system. Our purpose is now to unidirectionally couple two Rössler systems: in order to do this, we choose one of the five additional terms (one of those in bold in Eq. (27)). Since the quality of the coupling depends on the quality of the information transmitted by the variable used for the coupling [26], we choose to use information carried by variable y_R which provides a full observability of the Rössler dynamics [23, 25]. The single possibility not affecting the algebraic structure of the standard function is therefore to use term $b_{3,2}\, y_R$ in the third equation of general system (27); the connection between the parameters in each Rössler system are thus as discussed in the previous section.

The two coupled Rössler systems are:

$$\dot{x}_{R_1} = c_{1,2} y_{R_1} + c_{1,3} z_{R_1}$$

$$\dot{y}_{R_1}=c_{2,1}x_{R_1}+c_{2,2}y_{R_1}$$

$$\dot{z}_{R_1}=c_{3,0}+c_{3,3}z_{R_1}+c_{3,9}x_{R_1}z_{R_1}+\boldsymbol{Cy}_{R_2} \quad (28)$$

$$\dot{x}_{R_2}=c_{4,5}y_{R_2}+c_{4,6}z_{R_2}$$

$$\dot{y}_{R_2}=c_{5,4}x_{R_2}+c_{5,5}y_{R_2}$$

$$\dot{z}_{R_2}=c_{6,0}+c_{6,6}z_{R_2}+c_{6,24}x_{R_2}z_{R_2}.$$

Starting from variable $X_1 = y_R$, this six-dimensional system can be rewritten in the standard form

$$\dot{X}_1=X_2$$

$$\dot{X}_2=X_3$$

$$\dot{X}_3=X_4 \quad (29)$$

$$\dot{X}_4=X_5$$

$$\dot{X}_5=X_6$$

$$\dot{X}_6=F_{X_1}(X_1,X_2,X_3,X_4,X_5,X_6)$$

where $F_{y_{R1}}$ is a polynomial function depending on the six successive Lie derivatives of y_{R1}, the first one being the variable x_{R1} itself. This function has 107 terms $\alpha_k X_1^{l_1} X_2^{l_2} X_3^{l_3} X_4^{l_4} X_5^{l_5} X_6^{l_6}$, where $l_1+l_2+l_3+l_4+l_5+l_6 \leq 4$. There is no rational term in this standard function.

Applying the scaling transformation (6) to parameters $c_{i,j}$, we found from the null vectors (not reported) that there are four non-connected parameters ($c_{2,2}$, $c_{3,3}$, $c_{5,5}$ and $c_{6,6}$), all the others being connected: Therefore this six-dimensional system has only five independent parameters. The six-dimensional system (29) can be rewritten with scaling factors as

$$\begin{aligned}
\dot{x}_{R_1} &= \frac{1}{\rho}\,c_{1,2}\,y_{R_1} + \frac{1}{\rho}\frac{1}{\mu}\frac{1}{\nu}\pi\,c_{1,3}\,z_{R_1} \\
\dot{y}_{R_1} &= \rho\,c_{2,1}\,x_{R_1} + \boxed{c_{2,2}\,y_{R_1}} \\
\dot{z}_{R_1} &= \frac{1}{\pi}\mu\nu\,c_{3,0} + \boxed{c_{3,3}\,z_{R_1}} + \rho\,c_{3,9}\,x_{R_1}\,z_{R_1} \\
&\quad + \nu\,\boldsymbol{C}\,\boldsymbol{y}_{R_2} \\
\dot{x}_{R_2} &= \frac{1}{\mu}\,c_{4,5}\,y_{R_2} + \frac{1}{\pi}\frac{1}{\kappa}\,c_{4,6}\,z_{R_2} \\
\dot{y}_{R_2} &= \mu\,c_{5,4}\,x_{R_2} + \boxed{c_{5,5}\,y_{R_2}} \\
\dot{z}_{R_2} &= \kappa\,c_{6,0} + \boxed{c_{6,6}\,z_{R_2}} + \pi\,c_{6,24}\,x_{R_2}\,z_{R_2}
\end{aligned} \tag{30}$$

where boxed terms correspond to non-connected parameters. These four non-connected parameters are exactly those that were non-connected in the autonomous Rössler systems (12). All the others are connected. We checked that the change of co-ordinates Φ_{yR1} : $\mathbb{R}^6(x_{R_1}, y_{R_1}, z_{R_1}, x_{R_2}, y_{R_2}, z_{R_2}) \to (X_1, X_2, X_3, X_4, X_5, X_6)$ is a global diffeormorphism by computing the determinant of its jacobian matrix $\mathcal{J}_\Phi = -C^3$ [22] which consequently never vanishes. Φ_{yR1} therefore defines a global diffeormorphism, such that y_{R_1} provides a full observability of the six-dimensional system (28). According to our conjecture, all relationships between parameters are thus obtained by investigating this single case.

The fact that the five independent parameters are related to the coupling factor C between the two Rössler systems clearly shows that the resulting dynamics directly depends on this coupling factor. Thus, while investigating this six-dimensional system, the dynamics of each of the two Rössler systems is chosen by the pairs ($c_{2,2}$; $c_{3,3}$) and ($c_{5,5}$; $c_{6,6}$), respectively. The dynamics produced by the coupled systems is said to be tuned by parameter C.

The time scaling factors δ_1 and δ_2 can be used to investigate mismatches between the pseudo-periods of these two Rössler systems. Introducing the scaling relations (6) with these two additional time scaling factors for the first and the second Rössler systems as in Eq. (32) yields

$$\dot{x}_{R_1} = \frac{1}{\delta_1^2}\frac{1}{\rho}c_{1,2}y_{R_1} + \frac{1}{\delta_1^2}\frac{1}{\delta_2}\frac{1}{\rho}\frac{1}{\mu}\frac{1}{\nu}\pi c_{1,3}z_{R_1}$$

$$\dot{y}_{R_1} = \rho c_{2,1}x_{R_1} + \frac{1}{\delta_1}c_{2,2}y_{R_1}$$

$$\dot{z}_{R_1} = \frac{1}{\delta_1}\delta_2\frac{1}{\pi}\mu\nu c_{3,0} + \frac{1}{\delta_1}c_{3,3}z_{R_1} + \rho c_{3,9}x_{R_1}z_{R_1} + \nu \boldsymbol{C} y_{R_2} \quad (31)$$

$$\dot{x}_{R_2} = \frac{1}{\delta_2^2}\frac{1}{\mu}c_{4,5}y_{R_2} + \frac{1}{\delta_2^3}\frac{1}{\pi}\frac{1}{\kappa}c_{4,6}z_{R_2}$$

$$\dot{y}_{R_2} = \mu c_{5,4}x_{R_2} + \frac{1}{\delta_2}c_{5,5}y_{R_2}$$

$$\dot{z}_{R_2} = \kappa c_{6,0} + \frac{1}{\delta_2}c_{6,6}z_{R_2} + \pi c_{6,24}x_{R_2}z_{R_2}.$$

These time scaling factors now connect all parameters of the coupled system. There is no longer non-connected parameters. The dynamics of the six-dimensional system (28) will be left unchanged under a time rescaling until $\delta_1 = \delta_2$. Contrary to this, when $\delta_1 \neq \delta_2$, the resulting dynamics will be changed under time rescaling. Thus, of one of the Rössler systems while holding the other system's pseudo-period constant necessarily affects the resulting dynamics by introducing some bifurcations on the population of unstable periodic orbits embedded within the attractor produced by the six-dimensional system.

V. COUPLED LORENZ AND RÖSSLER SYSTEM

We now consider the case where a Lorenz system (3) is coupled with a Rössler system (12). As shown in [16], an additional constant term in the third equation of the Lorenz system does not add any term in the standard function F_{x_L} and, consequently, leaves the resulting dynamics unchanged. This additional parameter is connected with parameter $a_{2,1} = R$. The simplest way to unidirectionally couple a Lorenz system with a Rössler system is thus to link them with the variable y_R — the single variable among the six possible ones providing a full observability — injected into the third equation of the Lorenz system which becomes a six-dimensional system

$$\dot{x}_L = a_{1,1}x_L + a_{1,2}y_L$$

$$\dot{y}_L = a_{2,1}x_L + a_{2,2}y_L + a_{2,6}x_L z_L$$

$$\dot{z}_L = a_{3,3}z_L + a_{3,5}x_L y_L + \boldsymbol{C y}_R \quad (32)$$

$$\dot{x}_R = b_{1,2}y_R + b_{1,3}z_R$$

$$\dot{y}_R = b_{2,1} x_R + b_{2,2} y_R$$

$$\dot{z}_R = b_{3,0} + b_{3,3} z_R + b_{3,6} x_R z_R.$$

It is possible to rewrite this in a standard form (29) using $X_1 = x_L$. The standard function has in this case 131 terms among which some terms are rational, such as $\alpha_{131} \frac{X_5 X_6}{X_1}$.

Applying the scaling relations (6), we found that the three non-connected parameters of the Lorenz system ($a_{1,1}$, $a_{2,2}$ and $a_{3,3}$) and the two non-connected parameters of the Rössler system ($b_{2,2}$ and $b_{3,3}$) remain non-connected in the six-dimensional system (32). Using scaling factors, system (32) can be rewritten as

$$\begin{aligned}
\dot{x}_L &= \boxed{a_{1,1}\, x_L} + \rho\, \nu\, \frac{1}{\mu}\, \frac{1}{\pi}\, a_{1,2}\, y_L \\
\dot{y}_L &= \mu\, \pi\, \frac{1}{\rho}\, \frac{1}{\nu} a_{2,1}\, x_L + \boxed{a_{2,2}\, y_L} \\
&\quad + \frac{1}{\nu}\, a_{2,6}\, x_L\, z_L \\
\dot{z}_L &= \boxed{a_{3,3}\, z_L} + \nu\, a_{3,5}\, x_L\, y_L + \pi\, \boldsymbol{C}\, \boldsymbol{y}_R \\
\dot{x}_R &= \frac{1}{\mu}\, b_{1,2}\, y_R + \frac{1}{\rho}\, \frac{1}{\kappa}\, b_{1,3}\, z_R \\
\dot{y}_R &= \mu\, b_{2,1}\, x_R + \boxed{b_{2,2}\, y_R} \\
\dot{z}_R &= \kappa\, b_{3,0} + \boxed{b_{3,3}\, z_R} + \rho\, b_{3,6}\, x_R\, z_R
\end{aligned} \tag{33}$$

where non-connected parameters are boxed. All the other parameters are connected within each of the two sub-systems or with parameters of the other sub-system: consequently, some parameters of the Lorenz system are connected to some parameters of the Rössler system, and *vice versa*. There are only two scaling factors (ρ and μ) connecting both systems. It is possible to relate any connected parameters to all the other connected ones using these two scaling factors. There is therefore a single independent parameter for the whole set of connected parameters. Parameter C naturally serves as the sixth independent parameter, the five other being the non-connected parameters. As in the preceding case where two Rössler systems were coupled, the dynamics of each subsystem can be chosen using the corresponding non-connected parameters and the effect of the coupling then investigated using parameter C.

Note that we cannot guarantee that all relationships between parameters of system (32) have been identified since variable x_L does not provide full observability since the determinant of the jacobian matrix of the change of coordinates $\Phi_{x_L} : \mathbb{R}^6(x_L, y_L, z_L, x_R, y_R, z_R) \to \mathbb{R}^6(X_1, X_2, X_3, X_4, X_5, X_6)$ has an order-7 singular observability manifold. Nonetheless, driving the Lorenz system by the Rössler system induces a reduction of the number of independent parameters, since five were found in the autonomous Lorenz system (see Eq. (3) and the corresponding main text) but only three independent parameters were found in the Lorenz sub-system of the six-dimensional system (33): the two non-overlapping sets of connected parameters found in the Lorenz system (3), thus leading to two independent parameters, are now related to the whole set of connected parameters. Parameters of the two sub-systems are now interrelated through the coupling parameter C: the two systems are no longer independent.

Introducing the scaling relation (6) with two time scaling factors δ_L and δ_R for the Lorenz and the Rössler sub-systems of Eq. (32), the coupled system is:

$$\begin{aligned}
\dot{x}_L &= \delta_L a_{1,1} x_L + \delta_R \rho\nu \frac{1}{\mu}\frac{1}{\pi} a_{1,2} y_L \\
\dot{y}_L &= \frac{1}{\delta_R}\delta_L^2 \mu\pi \frac{1}{\rho}\frac{1}{\nu} a_{2,1} x_L + \delta_L a_{2,2} y_L + \delta_L^2 \frac{1}{\nu} a_{2,6} x_L z_L \\
\dot{z}_L &= \delta_L a_{3,3} z_L + \nu a_{3,5} x_L y_L + \pi \boldsymbol{C} \boldsymbol{y}_R \\
\dot{x}_R &= \delta_R^2 \frac{1}{\mu} b_{1,2} y_R + \delta_R^3 \frac{1}{\rho}\frac{1}{\kappa} b_{1,3} z_R \\
\dot{y}_R &= \mu b_{2,1} x_R + \delta_R b_{2,2} y_R \\
\dot{z}_R &= \kappa b_{3,0} + \delta_R b_{3,3} z_R + \rho b_{3,6} x_R z_R.
\end{aligned} \tag{34}$$

The time scaling factor δ_R of the Rössler system is coupled with the common bifurcation parameter $a_{2,1} = R$ of the Lorenz sub-system and can therefore affect the dynamics. Again, varying the pseudo-period of the driving system (here the Rössler system) affects the resulting dynamics.

VI. CONCLUSION

We have determined independent parameters of a dynamical system by investigating the relationships between the parameters of a given system and its corresponding standard forms expressed in terms of one “measured” state variable and its successive Lie derivatives. Some of them correspond to non-connected parameters — individually affecting the dynamics — and some others to the sets of connected parameters, each non overlapping set associated with one independent parameter which can be chosen in various ways.

There is a limited number of parameters to vary to modify the dynamics of a given system (five for the Lorenz system and three for the Rössler system). It is also possible to change the time scale — or the pseudo-period of the system — by modifying a limited set of parameters without acting directly on the time. The map between parameters of the original system and those of its standard form allows us to determine those parameters.

The relationships between the original system and its various standard forms — depending on the “measured” state variable — can be exploited to design networks in which systems are coupled in a way such that the coupled system is characterized by a standard function with the same algebraic structure as the autonomous system; consequently, the resulting dynamics necessarily corresponds to the dynamics observed in the original autonomous system, but with a displacement in the parameter space. By using the time scaling factor, we also showed that varying the pseudo-period of the driving system is similar to varying a bifurcation parameter.

We were surprised to discover that in driving a Lorenz system with a Rössler system, modifying one parameter of the driving system — and therefore its dynamics — can be counter-balanced by modifying one of the parameters related to the modified one. Knowing the connection between parameters of the two sub-system could be quite useful for designing a control technique for a driven system to react to some fluctuations observed in the driving system.

Such a coupling is important when considering the interactions between real-world non-linear systems. For example, in systems biology and neuroscience coupled non-linear dynamical systems are omnipresent and are thus important at all levels of interaction. In these systems, any sort of temporal variance or change in interstimulus interval could dramatically change the dynamics of the coupled systems. Thus, this finding has profound implications for controlling abnormal neurological oscillations in pathological states.

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TABLE I

Components of the null vectors for each of the three state variables x_R, y_R, and z_R of the Rössler system, respectively.

Variable	$b_{1,2}$	$b_{1,3}$	$b_{2,1}$	$b_{2,2}$	$b_{3,0}$	$b_{3,3}$	$b_{3,6}$
x_R	0	−1	0	0	1	0	0
	−1	0	1	0	0	0	0
y_R	−1	−1	1	0	0	0	1
	0	−1	0	0	1	0	0
z_R	0	−1	0	0	0	0	1
	−1	0	1	0	0	0	0