UNIVERSITY OF CALIFORNIA,
IRVINE

Overconvergent Families of p-adic Representations

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

James Upton

Dissertation Committee:
Professor Daqing Wan, Chair
Associate Professor Nathan Kaplan
Assistant Professor Jesse Wolfson

2020

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to thank my advisor, Daqing Wan, for his support and guidance throughout this project, as well as for introducing me to this fascinating subset of number theory. I owe much of my knowledge of the subject to our many meetings and conversations, without which this dissertation would not have been possible. I would like to extend my thanks to Nathan Kaplan and Jesse Wolfson, for their participation in my defense committee and their helpful comments in finalizing this dissertation. I am indebted to Joe Kramer-Miller, for our many conversations which have formed this inspiration for this project. I would also like to express my gratitude to Michiel Kosters, for his guidance when I was first beginning my research in this field.

I am very grateful for the financial support afforded to me over the past six years, including the GAANN Fellowship in Fall 2014, Spring 2015, and Summer 2015, as well as the UC Irvine Mathematics Department Fellowship in Fall 2016.

On a personal note, I would like to thank David and Dustin for the love and support they have provided over the past five years. I am grateful to all of my friends—Graham, Greg, Amir, and many others—who have made my graduate school experience so much more enjoyable. Lastly, I would like to acknowledge my parents, Sara and Kelly Upton, for encouraging me to take this path from the outset.

VITA

James Upton

EDUCATION

Doctor of Philosophy in Mathematics **2020**
University of California, Irvine *Irvine, California*

Master of Science in Mathematics **2016**
University of California, Irvine *Irvine, California*

Bachelor of Science in Mathematics **2014**
Arizona State University *Tempe, Arizona*

Bachelor of Science in Physics **2014**
Arizona State University *Tempe, Arizona*

RESEARCH EXPERIENCE

Graduate Research Assistant **2014–2020**
University of California, Irvine *Irvine, California*

TEACHING EXPERIENCE

Teaching Assistant **2014–2020**
University of California, Irvine *Irvine, California*

ABSTRACT OF THE DISSERTATION

Overconvergent Families of p-adic Representations

By

James Upton

Doctor of Philosophy in Mathematics

University of California, Irvine, 2020

Professor Daqing Wan, Chair

Let X be a variety over a finite field of characteristic p. The purpose of this dissertation is to extend many known results about p-adic representations of the fundamental group $\pi_1(X)$ to families of p-adic representations. The notion of a family of p-adic representations parameterized by a rigid analytic space arises naturally in many contexts, including geometric Iwasawa theory and the theory of p-adic modular forms. In either context there is significant interest in understanding the variation of the *p-adic L-functions* $L(\rho, s)$ as ρ moves through a given family. It seems unlikely that we can say much in general, as there are far too many p-adic representations of the group $\pi_1(X)$. In this dissertation we restrict our attention to so-called *overconvergent representations*, which have the property that the L-function $L(\rho, s)$ is always a p-adic meromorphic function in s. Thus for overconvergent families of representations, the question of understanding the p-adic variation of the $L(\rho, s)$ reduces to the understanding of the variation of their zeroes and poles. Our main theorem is a relative version of the Dwork-Monsky trace formula, which says that these zeroes and poles are are naturally interpolated by rigid analytic objects which we call *spectral varieties*. In general, the geometry of these spectral varieties is quite mysterious: in the context of p-adic modular

forms, the question is the subject of Coleman's well known *spectral halo conjecture*. For a few specific examples of overconvergent families, analogues of Coleman's conjecture have been proven by studying suitable *integral models* of the parameter space. For this reason, we choose to work primarily with formal schemes and their overconvergent analogues. We hope that this paves the way to a greater understanding of the arithmetic of these overconvergent families.

Chapter 1

Introduction

Fix a prime number $p > 0$ and a finite field $\mathbb{F}_q$ of characteristic p. Let X be a smooth affine curve over $\mathbb{F}_q$ and let $\overline{X}$ denote its projective completion. Choose a geometric point $\bar{x}$ of X, and consider the étale fundamental group $\pi_1(X, \bar{x})$. It is known that the maximal tame quotient of this group is topologically finitely generated, and moreover admits an explicit presentation ([11], XIII 2.12). Thus at least theoretically, the finite étale coverings of X which are tamely ramified in $\overline{X}$ are well understood. In contrast, a pro-p quotient of the fundamental group need not be finitely generated in general. Let us give some indication of the complicated nature of this group: Consider the affine line $X = \mathrm{Spec}(\mathbb{F}_q[t])$ over $\mathbb{F}_q$. Choose a separable closure $\mathbb{F}_q^s/\mathbb{F}_q$ and let X^s denote the fiber of X at $\mathbb{F}_q^s$. For any $c \in \mathbb{F}_q^s$, the Artin-Schreier equation

$$x^p - x = ct \tag{1.1}$$

determines an étale covering of X^s of degree p. Given a second $c' \in \mathbb{F}_q^s$, the corresponding coverings are isomorphic if and only if $c - c' \in \mathbb{F}_p$ ([11], 1.10). It follows that the geometric fundamental group $\pi_1(X^s, \bar{x})$ is a group of *infinite* p-rank.

If we let $S = \mathrm{Spec}(\mathbb{F}_q[c])$, then the Artin-Schreier equations (1.1) may be regarded as a family of "geometric coverings" of X parameterized by the geometric points $\mathbb{F}_q$-scheme S. This strongly suggests the approach of studying representations of $\pi_1(X^s, \bar{x})$ in families. As the above example indicates, if the "parameter space" S is a scheme of characteristic p, then the resulting theory can only describe the *mod* p representations of this group. One natural solution to this problem is to *lift* S to a scheme of characteristic p^n, or more generally to a *formal scheme* $\mathfrak{S}$ of characteristic 0. We thus obtain the notion of a *family of p-adic representations* of $\pi_1(X^s, \bar{x})$ parameterized by the $\mathbb{C}_p$-valued points of $\mathfrak{S}$. Such families of representations will be our main objects of study.

Suppose for example that $\mathfrak{S}$ is the formal spectrum of a Noetherian local ring R with residue field $\mathbb{F}_q$. Then a family of p-adic representations parameterized by $\mathfrak{S}$ is nothing more than a continuous representation

$$\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}_n(R).$$

For any continuous map $P : R \to \mathbb{C}_p$, we obtain a corresponding p-adic representation ρ_P of the fundamental group of X. One natural question is to understand how the *p-adic L-functions* $L(\rho_P, s)$ vary in the parameter P. For example, if ρ_P is a representation of finite order, then it is known that $L(\rho_P, s)$ is a rational function in s. The p-adic properties of its zeroes and poles can be understood, at least theoretically, by the action of Frobenius on a corresponding sequence of *rigid cohomology* groups. In contrast, Wan ([21], 1.2) has constructed examples of p-adic representations whose Artin L-functions are not even meromorphic. It seems that we cannot say much about the *variation* of these L-functions unless we impose some additional restrictions on ρ.

If ρ is a family for which $L(\rho_P, s)$ is p-adic meromorphic for each p, then our question regarding the p-adic variation of these functions *reduces* by Weierstrass factorization to understanding the p-adic variation of their zeroes and poles. The Monsky trace formula [18] guarantees this fact for a class of p-adic representations which we will call *overconvergent representations*. Let us call the family ρ *overconvergent* if each ρ_P is an overconvergent p-adic representation. We now arrive at our main question:

Question 1.0.1. *Suppose that ρ is an overconvergent family of p-adic representations. How does the p-adic distribution of zeroes and poles of $L(\rho_P, s)$ vary with P?*

There is at least one non-trivial example of an overconvergent family of p-adic representations for which Question 1.0.1 has a complete answer: Let $X = \mathrm{Spec}(\mathbb{F}_q[t, t^{-1}])$ be the punctured affine line over $\mathbb{F}_q$. Let $\mathbb{Z}_q$ denote the Witt ring of $\mathbb{F}_q$, and let $A = \mathbb{Z}_q[t, t^{-1}]$ be the natural lifting of X to characteristic 0. Fix a Laurent polynomial $f(t) \in A$. There is an obvious lifting of the absolute Frobenius to an endomorphism $\sigma : t \mapsto t^p$ of A. For every $n \geq 0$, the Artin-Schreier-Witt equation

$$\sigma(x) - x \equiv f(t) \pmod{p^n}$$

defines an étale $\mathbb{Z}/p^n\mathbb{Z}$-covering of A/p^nA, which *reduces* to a finite étale covering X_n of X. Thus we obtain an étale $\mathbb{Z}_p$-covering

$$\cdots \to X_1 \to X_0 = X$$

which corresponds in a unique way to a continuous map $\pi_1(X, \bar{x}) \to \mathbb{Z}_p$. Every $\mathbb{C}_p$-valued character of $\mathbb{Z}_p$ thus pulls back a continuous p-adic character of $\pi_1(X, \bar{x})$.

The assignment $\chi \mapsto \pi_\chi = \chi(1) - 1$ gives a one-to-one correspondence between the $\mathbb{C}_p$-valued characters of $\mathbb{Z}_p$ and the points in the open unit disk in $\mathbb{C}_p$. Since the latter are precisely the $\mathbb{C}_p$-valued points of the ring $R = \mathbb{Z}_p[[T]]$, we see that the above construction gives a family p-adic representations of $\pi_1(X, \bar{x})$ which we call the *Artin-Schreier-Witt family* associated to f. For this family, the main theorem of [6] gives the following answer to Question 1.0.1:

Theorem 1.0.2. *(Davis, Wan, Xiao). There exists a power series $C(T,s) \in 1 + R[[s]]$ with the following properties:*

1. *For every $\mathbb{C}_p$-valued character of $\mathbb{Z}_p$, the p-adic power series $C(\pi_\chi, s)$ is entire in s. Moreover, we have the meromorphic continuation*

$$L(\chi, s) = \frac{C(\pi_\chi, s)}{C(\pi_\chi, qs)}$$

2. *In some annulus $r < |\pi_\chi| < 1$, the π_χ-adic Newton slopes of $C(\pi_\chi, s)$ form a finite union of arithmetic progressions which are* independent *of the character χ.*

Statement (1) of this theorem may be regarded as a *global* meromorphic continuation for the L-functions of the Artin-Schreier-Witt family. Statement (2) is a consequence of certain precise estimates for the T-*adic* Newton polygon of the power series $C(T,s)$. Our main theorem indicates that Statement (1) holds in great generality: any overconvergent family of p-adic representations over a smooth affine variety over $\mathbb{F}_q$ admits a similar global meromorphic continuation. We hope that this lays the foundation for future work on Question 1.0.1. In particular, it would be interesting to know to what extent the remarkable slope uniformity displayed by the Artin-Schreier-Witt family holds for more general families, or for more general varieties.

We now give an overview of the contents of this paper, and state our main results.

1.1 Families of Representations

Suppose now that X is any variety over $\mathbb{F}_q$. For the sake of this introduction, we define a *family of p-adic representations* of $\pi_1(X, \bar{x})$ to be a continuous representation

$$\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}_n(R), \tag{1.2}$$

where R is a complete Noetherian local ring with residue field $\mathbb{F}_q$. There is a canonical way to attach a rigid analytic space to R, let us call this space $\mathcal{S}$. For now, we will think of $\mathcal{S}$ in terms of its underlying set of "rigid points" (2.4). Informally, a *rigid point* of R is a continuous map

$$P : R \to R_P.$$

where R_P is a discrete valuation ring. The rigid points of R are analogous to closed points in the rigid analytic space $\mathcal{S}$. Let us write $\langle R \rangle$ or $\langle \mathcal{S} \rangle$ for the set of all such points.

Given a family ρ as above, we can specialize along a rigid point $P \in \langle R \rangle$ to obtain a representation

$$\rho_P : \pi_1(X, \bar{x}) \to \mathrm{GL}_n(R_P).$$

Thus ρ is truly a family of representations parameterized by the set $\langle R \rangle$. We remark that in general, R may have rigid points of characteristic p, and so not every representation in this family is "p-adic." We can attach to ρ an Artin L-function in the usual way:

$$L(\rho, s) = \prod_{x \in |X|} \frac{1}{\det\left(I - \rho(\mathrm{Frob}_x) s^{\deg(x)}\right)} \in 1 + sR[[s]].$$

The essential property of this L-function is that its specialization $L(\rho, s)_P \in 1 + sR_P[[s]]$ at a rigid point of R *agrees* with the L-function $L(\rho_P, s)$ of the corresponding representation. Thus $L(\rho, s)$ becomes a convenient object for studying the variation of the $L(\rho_P, s)$ as P moves through the family of all rigid points of R.

Let's assume that there is a "nice" lifting of X to a formal scheme $\mathfrak{X}/R$. In particular, we assume that there is a lifting σ of the absolute Frobenius to an R-linear endomorphism of $\mathfrak{X}$. In this case, there is a well known classification of our families of representations:

Theorem 1.1.1. *(Katz [14]). The category of R-valued representations of the fundamental group $\pi_1(X, \bar{x})$ is equivalent to the category of pairs (M, ϕ), where M is a locally free $\mathcal{O}_{\mathfrak{X}}$-module, and $\phi : \sigma^* M \to M$ is an isomorphism.*

We refer to such a pair (M, ϕ) as a σ-*module* over $\mathfrak{X}$. In 4.2, we introduce a more general category of σ-modules with *relative* Frobenius structure. These objects capture the notion of a family of representations parameterized by a *formal scheme* $\mathfrak{S}$, including for example the family of Artin-Schreier equations (1.1).

1.2 Weak Formal Geometry

Previously, we have said that the family ρ is *overconvergent* if, for every $\mathbb{C}_p$-valued point P of R, the corresponding p-adic representation ρ_P is overconvergent. One downside of this definition is that it ignores the rigid points of R of characteristic p. More importantly, it is not at all obvious that this fiber-by-fiber condition is sufficient to guarantee good *global* behavior of the family ρ. For example, we would like to say that the fiber-wise meromorphic continuations of $L(\rho_P, s)$ glue to give a global meromorphic continuation of the L-function $L(\rho, s)$. For this purpose we will now introduce a more precise and slightly stronger notion of overconvergence.

Our theory is essentially a relative version of that introduced by Monsky and Washnitzer [19] to construct their *formal cohomology*. The building blocks of this theory are certain "weakly complete" algebras over a local ring R which they call *w.c.f.g. algebras*. Meredith [17] has shown that w.c.f.g. algebras can be glued to construct a larger category of *weak formal schemes* over R, which are the natural setting for our theory of overconvergent families of representations. Recall that such a family is truly parameterized by rigid points of the rigid analytic space $\mathcal{S}$ associated to R. In order to prove our analogue of the Monsky trace formula, it will be necessary to work with other formal models of this space which are *not* local in general. The constructions of Meredith no longer extend to this case, for the simple reason that the property of being a w.c.f.g. algebra is not a local property on the base.

In Chapter 3, we introduce a much more general notion of *weak formal schemes*, which agrees with that of Meredith in the case that R is a discrete valuation ring. The idea of our theory is as follows: we replace the base ring R by a formal scheme $\mathfrak{S}$. A weak formal scheme in our sense is a morphism of ringed spaces

$$\mathfrak{X} \to \mathfrak{S}$$

with the property that, for every "rigid point" P of the base $\mathfrak{S}$, the fiber $\mathfrak{X}_P$ is a weak formal scheme in the sense of Meredith. Using a slight modification of the Monsky-Washnitzer weak completion, we can construct these objects as gluings of w.c.f.g. algebras over a (not necessarily local) topological ring.

Let's assume now that there is a "nice" lifting of X to a *weak* formal scheme $\mathfrak{X}/R$. In particular, we assume that there is a lifting σ of the absolute Frobenius to an R-linear endomorphism of $\mathfrak{X}$. In light of Theorem 1.1.1, we define:

Definition 1.2.1. An *overconvergent family of representations* of $\pi_1(X, \bar{x})$ is a σ-module defined over the weak formal scheme $\mathfrak{X}/R$.

Note that the lifting of Frobenius σ prolongs uniquely to the formal completion $\mathfrak{X}^\infty$ of $\mathfrak{X}$.

The functor sending a σ-module (M, ϕ) over $\mathfrak{X}$ to the completion (M^∞, ϕ) over $\mathfrak{X}^\infty$ is fully faithful. Thus every such σ-module corresponds to a family of representations in the previous sense.

1.3 The Trace Formula

Suppose now that X is a smooth affine variety over $\mathbb{F}_q$. In this case, there always exists a "nice" lifting $\mathfrak{X}/R$ of X to a weak formal scheme over R. Let $\rho : \pi_1(X, \bar{x}) \to \mathrm{GL}_n(R)$ be an *overconvergent* family of representations, corresponding to a σ-module (M, ϕ) over $\mathfrak{X}$. We will now formulate our main result on the meromorphic continuation of the L-function $L(\rho, s)$.

Let $M^\vee = \mathrm{Hom}_{\mathcal{O}_\mathfrak{X}}(M, \mathcal{O}_\mathfrak{X})$ denote the dual module of M. For each $i \geq 0$, we define

$$\Omega^i M^\vee = \Omega^1_{\mathfrak{X}/R} \otimes_{\mathcal{O}_\mathfrak{X}} M^\vee.$$

In 4.3, we show that the operator ϕ induces R-linear operators $\theta_i : \Omega^i M^\vee \to \Omega^i M^\vee$ which are σ^{-1}-linear, in the sense that $\theta_i(\sigma(a)m) = a\theta_i(m)$ for any sections a of $\mathcal{O}_\mathfrak{X}$ and m of $\Omega^i M^\vee$. We will call such an operator a *Dwork operator*. In the classical case that R is a discrete valuation ring of characteristic 0, Monsky [18] proves that every Dwork operator Θ on a finite $\mathcal{O}_\mathfrak{X}$-module is an operator of *trace class*. The essential feature of these operators is the existence of the *Fredholm series*

$$C(\Theta, s) = \det(I - s\Theta) \in 1 + sR[[s]]$$

which is a p-adic power series with infinite radius of convergence. In 5.2, we prove the a relative version of Monsky's result:

Theorem 1.3.1. *Let Θ be a Dwork operator on a finite $\mathcal{O}_{\mathfrak{X}}$-module. There is a unique power series $C(\Theta, s) \in 1 + sR[[s]]$ with the interpolation property*

$$C(\Theta, s)_P = C(\Theta_P, s)$$

for every rigid point P of R.

In other words, the Fredholm series of the classical Dwork operators Θ_P can be glued to give a power series which is entire, in a relative sense, over the rigid analytic space $\mathcal{S}$ associated to R. Let us regard the power series $C(\Theta, s)$ as an analytic function on the *relative* multiplicative group $\mathbb{G}_{m,\mathcal{S}}$.

Definition 1.3.2. Let Θ be a Dwork operator on a finite $\mathcal{O}_{\mathfrak{X}}$-module. The *spectral variety* of Θ is the hypersurface $\mathcal{E}(\Theta)$ cut out by the Fredholm series $C(\Theta, s)$.

The spectral variety of Θ is naturally equipped with a morphism $\mathcal{E}(\Theta) \to \mathcal{S}$. If Q is a rigid point of $\mathcal{E}(\Theta)$ lying over $P \in \langle \mathcal{S} \rangle$, then Q corresponds uniquely to a non-zero $\mathrm{Gal}(k(P)^s/k(P))$-orbit of eigenvalues of the trace class operator Θ_P. Thus $\mathcal{E}(\Theta)$ becomes the natural object for studying the *variation* of the spectral theory of Θ_P with the parameter P. We can now state our main theorem on families of overconvergent representations, which relates the problem of understanding the L-function $L(\rho, s)$ to the understanding of the associated sequence of spectral varieties:

Theorem 1.3.3. *The L-function $L(\rho, s)$ is meromorphic over $\mathcal{S}$. More precisely, we have the relation*

$$L(\rho, s) = \prod_i C(\theta_i, s)^{(-1)^{i-1}}. \tag{1.3}$$

1.4 Further Questions

Let Γ be a compact abelian p-adic Lie group, and let $\Lambda = \mathbb{Z}_p[[\Gamma]]$. It would seem that the natural source of families of p-adic representations of $\pi_1(X, \bar{x})$ are the étale Γ-coverings of X studied in geometric Iwasawa theory. Let $f : \pi_1(X, \bar{x}) \to \Gamma$ be such a covering, and let

$$\rho_f : \pi_1(X, \bar{x}) \to \Lambda^\times$$

be the corresponding rank-1 family of representations. When is such a family overconvergent? The question is not entirely well posed, as it depends on our chosen lifting of X and its Frobenius structure to characteristic 0. For now, suppose that we have a fixed lifting of $\mathfrak{X}$ to a weak formal scheme over $\mathbb{Z}_p$, and a $\mathbb{Z}_p$-linear endomorphism $\sigma : \mathfrak{X} \to \mathfrak{X}$ lifting the absolute Frobenius. Let $\mathfrak{X}_\Lambda$ denote the weak base change (3.5) of $\mathfrak{X}$ along $\mathbb{Z}_p \to \Lambda$. Then the representation ρ_f corresponds uniquely to a σ-module (M_f, ϕ_f) over the formal completion $\mathfrak{X}_\Lambda^\infty$.

It would be interesting to characterize the overconvergence of the pair (M_f, ϕ_f) in terms of the "convergence" properties of the map f. Suppose for example that X is affine, and $\mathfrak{X}$ is the weak formal spectrum of a weakly complete algebra A. When $\Gamma = \mathbb{Z}_p$, the map f corresponds via Artin-Schreier-Witt theory to a section of A^∞ [15]. In 5.4, we show in a special case that if this section is "overconvergent" (i.e. it is an element of A), then so is the corresponding σ-module. More generally, suppose that G is a commutative formal group over $\mathbb{Z}_p$ and that $\Gamma = G(\mathbb{Z}_p)$ is its group of $\mathbb{Z}_p$-points. Then an analogue of Katz' correspondence shows that f corresponds to a pair (P, ϕ), where P is an étale G-torsor over X^∞ and ϕ is a "Frobenius structure" on P. In this case we conjecture the following:

Conjecture 1.4.1. *Suppose that (P, ϕ) is overconvergent, i.e. that this pair can be defined over $\mathfrak{X}$. Then ρ_f is an overconvergent family of representations.*

Is there a more natural class of "well behaved" families of representations than the overconvergent families? In particular, such a class should not depend on our chosen lifting of X and its Frobenius structure to characteristic 0. Let us describe one possible candidate: Suppose for simplicity that X is a curve, and let

$$\eta : \pi_1(X, \bar{x}) \to \mathrm{GL}_n(\mathbb{Z}_p)$$

be an abelian p-adic representation. Then $\Gamma = \mathrm{im}(\eta)$ is a compact abelian p-adic Lie group, and therefore corresponds to a rank-1 family of p-adic representations ρ as above. Every open subgroup $H \leq \Gamma$ corresponds uniquely to a finite étale covering $X_H \to X$ with Galois group Γ/H. Our question regarding the p-adic variation of the L-functions $L(\rho_P, s)$ is closely connected with the question of understanding the p-adic variation of the *zeta functions* $Z(X_H, s)$ as $[\Gamma : H] \to \infty$. First, let us mention that without hypotheses on ρ, we have the following theorem regarding the analytic continuation of $L(\rho, s)$:

Theorem 1.4.2. *(Crew [5]). The L-function $L(\rho, s) \in \Lambda[[s]]$ continues analytically to the closed disk $|s| \leq 1$ over Λ.*

In a recent article [23], Wan has shown that this analytic continuation guarantees a p-adic formula for the class numbers in Γ-extensions of the function field of X, generalizing the celebrated theorem of Iwasawa. Wan has posed a series of conjectures describing the variation of the $Z(X_H, s)$ under the additional hypothesis that η *comes from algebraic geometry.* More precisely, this means that η is the unit-root part of an ordinary overconvergent F-crystal on X. In particular, Theorem 1.0.2 says precisely that the Artin-Schreier-Witt families of [6] satisfy Wan's *slope-stability* conjecture.

It would seem that the trace formula (1.3) is the natural means by which to approach Wan's conjectures. Thus we are led to ask: what is the relationship between overconvergent families and families coming from algebraic geometry? A partial answer is provided Große-Klönne

[10], who proves that all *rank*-1 families coming from algebraic geometry admit a global meromorphic continuation analogous to Theorem 1.3.3.

We should remark that a fundamental example of an overconvergent family is provided by the theory of p-adic modular forms: Suppose that $N \geq 3$ is prime to p, and let $q \equiv 1 \pmod{N}$. Let X denote the ordinary locus of the (compactified) modular curve of level N over $\mathbb{F}_q$. Let $f : E \to X$ denote the universal generalized elliptic curve of level N. The first p-adic étale cohomology $R^1 f_* \mathbb{Z}_p$ affords a continuous representation

$$\eta : \pi_1(X, \bar{x}) \to \mathbb{Z}_p^\times.$$

By a theorem of Igusa [12], this map is surjective. As above, this corresponds uniquely to a family of representations ρ which we call the *Igusa family.*

In this setting, the curve X and its Frobenius structure admit a *canonical lifting* $(\mathfrak{X}, \sigma)$ to characteristic 0. As above, let (M, ϕ) be the σ-module over $\mathfrak{X}_\Lambda$ corresponding to the Igusa family. Let $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ be a continuous character, which corresponds uniquely to a rigid point $P : \Lambda \to \mathbb{C}_p$. It has been known for some time that if χ is *algebraic*, then the σ-module (M_P, ϕ_P) is overconvergent with respect to the canonical lifting of Frobenius [14]. The overconvergent sections of the module $\mathbb{Q} \otimes M_P$ are precisely the *overconvergent modular forms* of level N and weight χ. This result was extended to *all* p-adic characters by the work of Coleman [4], who showed that these modules are naturally interpolated by families of modular forms of arbitrary p-adic weight.

More recently, Andreatta, Iovita, and Pilloni [2] have extended these results to the rigid points of characteristic p. Using the σ-module (M, ϕ), those authors construct integral models of Coleman's families of overconvergent modular forms. This strongly suggests that the Igusa family is overconvergent in our sense. For this family, our trace formula specializes to a theorem of Coleman ([4], I2) relating the L-function $L(\rho, s)$ to the spectral theory of the compact operator U_p acting on overconvergent modular forms:

$$\det(I - sU_p) = \prod_{r\geq 0} L(\rho \otimes \eta^{\otimes(-2r-2)}, p^r s).$$

The spectral theory of U_p is the subject of a number of conjectures, perhaps the most well known being Coleman's *spectral halo conjecture*. One weaker form of this conjecture states that the (suitably normalized) Newton polygon of $L(\rho_P, s)$ is independent of P near the boundary of p-adic weight space. Theorem 1.0.2 can be regarded as an analogue of this statement for certain Artin-Schreier-Witt families. The methods of Davis *et al.* have since been adapted by Liu, Wan, and Xiao to prove an analogue of Coleman's conjecture for overconvergent automorphic forms for a definite quaternion algebra over $\mathbb{Q}$ [16]. We hope that a suitable understanding of overconvergent families of representations will shed some additional light on Coleman's conjecture.

Chapter 2

Preliminaries

In this chapter we outline our conventions on topological rings, formal schemes, and their associated rigid analytic spaces. We use [1] as a reference throughout.

2.1 Topological Algebra

Definition 2.1.1. We say that a topological ring R is an *adic ring* if there is an ideal $I \subset R$ such that $\{I^n : n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of 0. We refer to such an I as an *ideal of definition* of R.

Our definition of an "adic ring" coincides with that of a "pre-adic ring" in [7], in particular we do not assume *a priori* that an adic ring is separated and complete. For the purposes of weak formal geometry, the category of adic rings is far too large, but we cannot in general restrict our attention to Noetherian adic rings. We will instead work with the following class of topological rings, which share many good properties of Noetherian adic rings:

Definition 2.1.2. We will say that an adic ring R is *admissible* if R admits an ideal of definition $I \subset R$ such that:

1. I is finitely generated
2. R is I-separated, i.e. $\bigcap_n I^n = 0$
3. The quotient R/I is a Noetherian Jacobson ring.

Let R be an admissible ring. We will denote by $\mathfrak{N}(R)$ the ideal of topologically nilpotent elements of R. By ([7], 7.1.6), $\mathfrak{N}(R)$ is the maximal ideal of definition of R. In general, we will write $R^n = R/\mathfrak{N}(R)^{n+1}$. We will often refer to R^0 as the *reduction* of R.

Let M be a topological R-module. We will say that M is *adic* if for some (hence any) ideal of definition $I \subset R$, the set $\{I^n M : n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of 0 in M. If M is adic, then we define its *completion* to be the inverse limit

$$M^\infty = \varprojlim_n M/I^n M.$$

There is a natural map $M \to M^\infty$, which is injective if and only if M is I-separated. We will say that M is *complete* if $M \to M^\infty$ is an isomorphism. Since R admits a finitely generated ideal of definition, M^∞ is always complete. In general, completion defines a functor from the category of adic R-modules to the category of adic R^∞-modules.

Let us write AdRing_+ for the category of admissible rings, the morphisms being continuous maps. We say that a morphism $h : R \to R'$ in AdRing_+ is *adic* if R' is adic when viewed as an R-module. Consider the corresponding diagram of reductions:

$$\begin{array}{ccc} R & \xrightarrow{h} & R' \\ \downarrow & & \downarrow \\ R^0 & \xrightarrow{h_0} & (R')^0 \end{array}$$

We say that h is of *finite presentation* if h_0 is of finite presentation. Let us write AdRing for the subcategory of AdRing_+ with the same objects, but whose morphisms are adic morphisms of finite presentation. For any object R of AdRing, we write AdRing_R for the category of admissible rings over R, i.e. the category of maps $R \to R'$ which are adic of finite presentation.

Definition 2.1.3. Let R be a Noetherian adic ring, and M an adic R-module. We define the *topological torsion* of M the submodule $M_{\mathrm{tor}} \subseteq M$ consisting of elements $m \in M$ which are annihilated by an ideal of definition of M.

Let R be an admissible ring, and let $\mathbb{S} = \mathrm{Spec}(R)$. If M is an R-module, let $\tilde{M}$ denote the sheaf on $\mathbb{S}$ associated to M. Now let $\mathbb{U} \subseteq \mathbb{S}$ be the complement of the closed subscheme defined by some ideal of definition. By ([7], I 6.8.4), M_{tor} is the *kernel* of the natural map

$$M \to \Gamma(\mathbb{U}, \tilde{M}).$$

Lemma 2.1.4. *In the above setting, suppose that R is Noetherian and that M is a finite R-module. Let I be an ideal of definition of R. The natural map*

$$\varinjlim_n \mathrm{Hom}_R(I^n, M) \to \Gamma(\mathbb{U}, \tilde{M})$$

is an isomorphism.

Proof. This is ([1], 1.8.34). □

2.2 Formal Schemes

Let $\mathfrak{S}$ be a topological space. We define a *sheaf of admissible rings* on $\mathfrak{S}$ to be a sheaf $\mathcal{O}$ valued in the category AdRing. A *morphism* of sheaves of admissible rings is defined in the evident manner. We will say that an ideal $\mathcal{I} \subset \mathcal{O}$ is an *ideal of definition* if, for every open $\mathfrak{U} \subseteq \mathfrak{S}$, $\Gamma(\mathfrak{U}, \mathcal{I})$ is an ideal of definition of the admissible ring $\Gamma(\mathfrak{U}, \mathcal{O})$. Note that the ideal $\mathfrak{N}(\mathcal{O})$ of topologically nilpotent sections is the maximal ideal of definition of $\mathcal{O}$.

We say that a topological $\mathcal{O}$-module $\mathcal{F}$ is *adic* if for each open $\mathfrak{U} \subseteq \mathfrak{S}$, $\mathcal{F}(\mathfrak{U})$ is an adic $\mathcal{O}(\mathfrak{U})$-module. Given an adic $\mathcal{O}$-module $\mathcal{F}$, we write $\mathcal{F}^n$ for the reduction of $\mathcal{F}$ mod $\mathfrak{N}(\mathcal{O})^{n+1}$. If $\mathcal{F}$ is adic, then we define its *completion* to be the inverse limit

$$\mathcal{F}^\infty = \varprojlim_n \mathcal{F}/\mathfrak{N}(\mathcal{O})^n \mathcal{F}.$$

As before, there is a natural map $\mathcal{F} \to \mathcal{F}^\infty$, which is injective if and only if $\mathcal{F}$ is separated. We will say that $\mathcal{F}$ is *complete* if $\mathcal{F} \to \mathcal{F}^\infty$ is an isomorphism. Note that $\mathcal{F}^\infty$ is always complete. In general, completion defines a functor from the category of $\mathcal{O}$-modules to the category of $\mathcal{O}^\infty$-modules.

Definition 2.2.1. Let $(\mathfrak{S}, \mathcal{O}_{\mathfrak{S}})$ be a locally ringed space. We say that $\mathfrak{S}$ is an *admissibly ringed space* if $\mathcal{O}_{\mathfrak{S}}$ is a sheaf of admissible rings.

We define a morphism of admissibly ringed spaces to be a morphism $h : \mathfrak{S}' \to \mathfrak{S}$ of locally ringed spaces for which the map $h^{-1}\mathcal{O}_{\mathfrak{S}} \to \mathcal{O}_{\mathfrak{S}'}$ is a morphism of sheaves of admissible rings. We can associate to any Noetherian adic ring R an adic ringed space as follows: Let $\mathbb{S} = \mathrm{Spec}(R)$, and let $\mathfrak{S} \subset \mathbb{S}$ be the closed subset cut out by the ideal $\mathfrak{N}(R)$. The structure sheaf $\mathcal{O}_{\mathbb{S}}$ is naturally a sheaf of adic rings, and we define

$$\mathcal{O}_{\mathfrak{S}} = \mathcal{O}_{\mathbb{S}}^\infty.$$

The sheaf $\mathcal{O}_{\mathfrak{S}}$ is supported in the closed subset $\mathfrak{S}$. We refer to the pair $(\mathfrak{S}, \mathcal{O}_{\mathfrak{S}})$ as the *formal spectrum* of R, and denote we will denote it by $\mathrm{Spf}(R)$. Note that there is a natural isomorphism $\mathrm{Spf}(R) \cong \mathrm{Spf}(R^\infty)$. It will be convenient nonetheless to assume that R is not complete in general.

Definition 2.2.2. A *formal scheme* is an admissibly ringed space locally of the form $\mathrm{Spf}(R)$ for some Noetherian adic ring R.

We will generally use Gothic font to denote formal schemes. It will be convenient to denote the reduction of $\mathfrak{S}$ by the same letter in Latin font. For example, if $\mathfrak{S} = \mathrm{Spf}(R)$, then $S = \mathrm{Spec}(R^0)$. Let us write FS_+ for the category of *quasi-compact* and *quasi-separated* formal schemes. We say that a morphism $h : \mathfrak{S}' \to \mathfrak{S}$ is *adic* if $\mathcal{O}_{\mathfrak{S}'}$ is adic when regarded as a $h^{-1}\mathcal{O}_{\mathfrak{S}}$-module. Consider the corresponding diagram of reductions:

$$\begin{array}{ccc} \mathfrak{S}' & \xrightarrow{h} & \mathfrak{S} \\ \uparrow & & \uparrow \\ S' & \xrightarrow{h_0} & S \end{array}$$

We say that h is *locally of finite presentation* if h_0 is locally of finite presentation. Let us write FS for the subcategory of FS_+ with the same objects, but whose morphisms are adic morphisms locally of finite presentation. For any object $\mathfrak{S}$ of FS, we write $\mathrm{FS}_{\mathfrak{S}}$ for the category of formal schemes over $\mathfrak{S}$, i.e. the category of morphisms $\mathfrak{S}' \to \mathfrak{S}$ which are adic of finite presentation.

Let R be a Noetherian admissible ring, and M be a finite R-module. Let $\mathbb{S} = \mathrm{Spec}(R)$ and $\mathfrak{S} = \mathrm{Spf}(R)$. The sheaf $\tilde{M}^\infty$ is supported on the closed subset $\mathfrak{S}$, and is naturally equipped with the structure of a $\mathcal{O}_{\mathfrak{S}}$-module.

Lemma 2.2.3. *Suppose that R is complete. The functor $M \mapsto \tilde{M}^\infty$ defines an equivalence between the category of finite R-modules and the category of coherent $\mathcal{O}_{\mathfrak{S}}$-modules.*

Proof. This is ([1], 2.7.2). □

Now let $\mathfrak{S}$ be any object of FS, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathfrak{S}}$-module. If $\mathcal{I}$ is an ideal of definition of $\mathcal{O}_{\mathfrak{S}}$, we define the *rig-closure* of $\mathcal{F}$ to be the sheaf

$$\mathcal{H}^0(\mathcal{F}) = \varinjlim_n \mathcal{H}om_{\mathcal{O}_{\mathfrak{S}}}(\mathcal{I}^n, \mathcal{F}).$$

Note that $\mathcal{H}^0$ is independent of the choice of ideal of definition $\mathcal{I}$. The sheaf $\mathcal{H}^0(\mathcal{O}_{\mathfrak{S}})$ is naturally a sheaf of $\mathcal{O}_{\mathfrak{S}}$-algebras, and we may regard $\mathcal{F} \mapsto \mathcal{H}^0(\mathcal{F})$ as a functor from the category of $\mathcal{O}_{\mathfrak{S}}$-algebras to the category of $\mathcal{H}^0(\mathcal{O}_{\mathfrak{S}})$-algebras. In light of Lemmas 2.1.4 and 2.2.3, we make the following definition:

Definition 2.2.4. Let $\mathfrak{S}$ be a formal scheme, and $\mathcal{F}$ a coherent $\mathcal{O}_{\mathfrak{S}}$-module. We define the *topological torsion* of $\mathcal{F}$ the kernel $\mathcal{F}_{\text{tor}}$ of the natural map

$$\mathcal{F} \to \mathcal{H}^0(\mathcal{F}).$$

2.3 Rigid Analytic Spaces

For our purposes we will work with rigid analytic spaces in the sense of Raynaud [20]. Here we give a brief overview of the general theory.

Let $\mathfrak{S}$ be a formal scheme, and let $\mathcal{I} \subseteq \mathcal{O}_{\mathfrak{S}}$ be a coherent open ideal. There is a universal morphism in FS

$$\varphi : \mathfrak{S}_{\mathcal{I}} \to \mathfrak{S}$$

with the property that $\varphi^{-1}\mathcal{I}\mathcal{O}_{\mathfrak{S}_{\mathcal{I}}}$ is an effective Cartier divisor on $\mathfrak{S}_{\mathcal{I}}$ ([1], 3.1.9). We refer to the formal scheme $\mathfrak{S}_{\mathcal{I}}$ as the *admissible blow-up* of $\mathfrak{S}$ in $\mathcal{I}$. Admissible blow-ups in the category of formal schemes are well behaved with respect to a number of operations, for example the composition of two admissible blow-ups is again an admissible blow-up.

Definition 2.3.1. The category Rig of *rigid analytic spaces* is defined to be the localization of FS at the class of admissible blow-ups.

We will denote the canonical localization functor FS $\to$ Rig by $\mathfrak{S} \mapsto \mathfrak{S}^{\mathrm{rig}}$. If $\mathcal{S}$ is an object of Rig, then we will say that $\mathfrak{S}$ is a *formal model* of $\mathcal{S}$ if there is an isomorphism $\mathfrak{S}^{\mathrm{rig}} \cong \mathcal{S}$. If $h : \mathcal{S}' \to \mathcal{S}$ is a morphism in Rig, then we will say that a morphism $\mathfrak{S}' \to \mathfrak{S}$ is a *model* of h if there is a commutative diagram

$$\begin{array}{ccc} (\mathfrak{S}')^{\mathrm{rig}} & \longrightarrow & \mathfrak{S}^{\mathrm{rig}} \\ \downarrow & & \downarrow \\ \mathcal{S}' & \xrightarrow{h} & \mathcal{S} \end{array}$$

where the vertical arrows are isomorphisms.

The category Rig has a natural topology defined as follows: a morphism $\mathcal{U} \to \mathcal{S}$ is an *open immersion* if it admits a model which is an open immersion in FS. A family of open immersions $\{h_i : \mathcal{U}_i \to \mathcal{S}\}_i$ is an *admissible covering* if there is a model $\mathfrak{S}$ of $\mathcal{S}$ and models $\mathfrak{U}_i \to \mathfrak{S}$ of h_i which form a Zariski covering of $\mathfrak{S}$. Thus by construction, every object of Rig is a quasi-compact and quasi-separated with respect to the admissible topology.

Let $\mathfrak{S}$ be a formal scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathfrak{S}}$-module. Let $\varphi : \mathfrak{S}' \to \mathfrak{S}$ be an admissible blow-up. Then there is a natural isomorphism ([1], 3.5.5)

$$\varphi^*\mathcal{H}^0(\mathcal{F}) \to \mathcal{H}^0(\varphi^*\mathcal{F}).$$

Let $\mathcal{S} = \mathfrak{S}^{\mathrm{rig}}$, and let $\mathcal{U} \to \mathcal{S}$ be an open immersion. Then $\mathfrak{U}$ admits a model of the form $\mathfrak{U} \to \mathfrak{X}'$ for some admissible blow-up $\varphi : \mathfrak{X}' \to \mathfrak{X}$. The above isomorphism guarantees that the assignment $\mathcal{U} \mapsto \mathcal{H}^0(\varphi^*\mathcal{F})$ is *independent* of the choice of model of $\mathcal{U} \to \mathcal{S}$. This assignment defines a sheaf on $\mathcal{S}$ which we denote by $\mathcal{F}^{\mathrm{rig}}$. In particular, we set $\mathcal{O}_{\mathcal{S}} = \mathcal{O}_{\mathfrak{S}}^{\mathrm{rig}}$, which is independent of the choice of model $\mathfrak{S}$. The pair $(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ is a locally ringed topos

([1], 4.8.6), which we will call a (quasi-compact, quasi-separated) *rigid analytic space*. If we identify $\mathfrak{S}$ with its Zariski topos, then there is a natural morphism of locally ringed topoi $\mathcal{S} \to \mathfrak{S}$.

Example 2.3.2. *Suppose that R is a Noetherian admissible ring and $I \subseteq R$ is a coherent open ideal. Let $\mathbb{S} = \mathrm{Spec}(R)$ and $\mathfrak{S} = \mathrm{Spf}(R)$. The admissible blow-up $\mathfrak{S}_I$ is the formal completion of the ordinary blow-up $\mathbb{S}_I$, viewed as an adic ringed space via the I-adic topology. More explicitly, choose a set of generators $I = (r_1, ..., r_d)$. Then $\mathfrak{S}_I$ admits a covering by spaces of the form $\mathrm{Spf}(R_i)$, where*

$$R_i = R\left[\tfrac{r_j}{r_i} : j \neq i\right] \Big/ (r_i\text{–tor})$$

If $\mathcal{S} = \mathfrak{S}^{\mathrm{rig}}$, then it follows that $\mathcal{S}$ admits a covering by rigid analytic spaces of the form $\mathcal{S}_i$, where the topology on $\mathcal{O}_{\mathcal{S}_i}$ coincides with the π_i-adic topology. If M is a finite R-module, then we have the following explicit description of the functor $M \mapsto M^{\mathrm{rig}}$:

$$\Gamma(\mathcal{S}_i, M^{\mathrm{rig}}) = R_i[\tfrac{1}{\pi_i}] \otimes_R M.$$

In the classical case that R is equipped with the p-adic topology, this functor is given globally by $M \mapsto \mathbb{Q} \otimes M$

Occasionally we will need to work with rigid analytic spaces which are not necessarily quasi-compact. For this purpose, we define more generally a (quasi-separated) *rigid analytic space* to be an ind-object in Rig, where the transition maps are all open immersions. For our purposes, the fundamental example of such a space is the following:

Example 2.3.3. *Let R be a Noetherian adic ring admitting a principal ideal of definition $\pi R \subset R$. Let $\mathcal{S} = \mathrm{Spf}(R)^{\mathrm{rig}}$. For each $n \geq 0$, consider the adic R-algebra*

$$A_n = R[\pi^n s, \pi^n s^{-1}],$$

and let $\mathcal{X}_n = \mathrm{Spf}(A_n)^{\mathrm{rig}}$. *Then the natural maps* $\mathcal{X}_n \to \mathcal{X}_{n+1}$ *are open immersions in* Rig. *We define the relative* multiplicative group *over* $\mathcal{S}$ *to be the ind-object*

$$\mathbb{G}_{m,\mathcal{S}} : \mathcal{X}_0 \to \mathcal{X}_1 \to \cdots .$$

A global section of the structure sheaf of $\mathbb{G}_{m,\mathcal{S}}$ *is a power series*

$$\sum_{j=-\infty}^{\infty} r_j s^j \in R[\tfrac{1}{\pi}][[s, s^{-1}]]$$

satisfying the convergence property

$$\frac{v_\pi(r_j)}{|j|} \to \infty \text{ as } j \to \pm\infty.$$

In general, a rigid analytic space admits an admissible covering by spaces of the above form. By a natural gluing construction we may speak more generally of the multiplicative group $\mathbb{G}_{m,\mathcal{S}}$ *for any rigid analytic space* $\mathcal{S}$.

2.4 Rigid Points

Definition 2.4.1. Let P be an object of FS. We say that P is a *rigid point* if $P = \mathrm{Spf}(\Omega)$, where Ω is a complete Noetherian 1-dimensional local domain such that $\Omega_{\mathrm{tor}} = 0$.

If $P = \mathrm{Spf}(\Omega)$ is a rigid point, then we define the *residue field* of P to be the field of fractions $k(P) = Q(\Omega)$. The integral closure of Ω in $k(P)$ is a discrete valuation ring ([1], 1.11.4). We will write v_P for the valuation, normalized so that the value group of $k(P)$ equals $\mathbb{Z}$.

Definition 2.4.2. Let $\mathfrak{S}$ be an object of FS. A *rigid point* of $\mathfrak{S}$ is an isomorphism class of immersions $P \to \mathfrak{S}$, where P is a rigid point.

We denote the set of rigid points of $\mathfrak{S}$ by $\langle\mathfrak{S}\rangle$. Since the reduction S is a Jacobson scheme, every rigid point of $\mathfrak{S}$ is a closed immersion ([1], 3.3.1). Suppose that $\mathfrak{S} = \mathrm{Spf}(R)$, where R is an admissible ring ring. Then we may write $\langle R\rangle = \langle\mathfrak{S}\rangle$, and speak of the set of *rigid points* of R. Note in particular that $\langle R\rangle = \langle R^\infty\rangle$. In this setting, it will be convenient to identify P with the corresponding ring map, and let R_P denote the image of P.

Many of the rings we encounter are naturally Zariski rings. In this setting, we have the following interpretation of the rigid points of R:

Lemma 2.4.3. *Let R be a Zariski ring. Let I be an ideal of definition, and let U be the complement of* $\mathrm{Spec}(R/I)$ *in* $\mathrm{Spec}(R)$. *For a prime ideal $\mathfrak{p} \subset R$, the following are equivalent:*

1. $\mathrm{Spf}(R/\mathfrak{p})$ *is a rigid point*
2. $I \not\subseteq \mathfrak{p}$ *and* $\dim(R/\mathfrak{p}) = 1$
3. $\mathfrak{p}$ *determines a closed point of U.*

Proof. If $\mathrm{Spf}(R/\mathfrak{p})$ is a rigid point, then $(R/\mathfrak{p})_{\mathrm{tor}} = 0$. It follows that $(I + \mathfrak{p})/\mathfrak{p} = J(R/\mathfrak{p}) \neq 0$, so $I \not\subseteq \mathfrak{p}$. Since R is a Zariski ring, $R/\mathfrak{p}$ and $(R/\mathfrak{p})^\infty$ have the same dimension, so $\dim(R/\mathfrak{p}) = 1$. Thus $1 \implies 2$. To see $2 \implies 3$, note if $\dim(A/\mathfrak{p}) = 1$ and $f \in I$ is not in $\mathfrak{p}$, then $(R/\mathfrak{p})_f$ is a zero-dimensional domain, hence a field. Thus $\mathfrak{p}$ determines a closed point of U.

Assume then that $\mathfrak{p}$ determines a closed point of U. Since U is covered by affine open schemes of the form $\mathrm{Spec}(R_f)$, where $f \in I$, it follows that $(R/\mathfrak{p})_f$ is a field. By ([7], IV.0.16.3.3), $R/\mathfrak{p}$ is a semi-local domain of dimension ≤ 1. Since the Jacobson radical of $R/\mathfrak{p}$ is non-zero, we have $\dim(R/\mathfrak{p}) = 1$. It remains only to show that $R/\mathfrak{p}$ is local. Let $\mathfrak{m}_1, ..., \mathfrak{m}_n$ be the distinct maximal ideals of $R/\mathfrak{p}$. By ([3], III. Cor. to 2.3.19), we have an isomorphism

$$(R/\mathfrak{p})^\infty \cong \prod_i (R/\mathfrak{p})_i^\infty,$$

where $(R/\mathfrak{p})_i$ denotes a copy of $R/\mathfrak{p}$ equipped with the $\mathfrak{m}_i$-adic topology. But $(R/\mathfrak{p})^\infty$ is a domain, so we must have that $n = 1$, i.e. $R/\mathfrak{p}$ is local. □

Proposition 2.4.4. *Let $f : P \to \mathfrak{S}$ be a morphism in* FS, *where P is a rigid point. Then f factors uniquely through a rigid point of $\mathfrak{S}$.*

Proof. See ([1], 3.3.4). □

As a consequence of Proposition 2.4.4, we see that for every morphism $\mathfrak{S}' \to \mathfrak{S}$ in FS, there is a corresponding map $\langle \mathfrak{S}' \rangle \to \langle \mathfrak{S} \rangle$. In particular, taking rigid points defines a functor FS $\to$ Set. By ([1], 3.3.8), if $\mathfrak{S}' \to \mathfrak{S}$ is an admissible blow-up, then the induced map on rigid points is an isomorphism. By the universal property of localization, we see that the functor $\mathfrak{S} \mapsto \langle \mathfrak{S} \rangle$ factors uniquely through FS $\to$ Rig. Thus is makes sense to speak of the *rigid points* of a rigid analytic space $\mathcal{S}$ in Rig. Again, we will denote this set by $\langle \mathcal{S} \rangle$. The rigid points of a rigid analytic space are analogous to the closed points of a variety, as the following results indicate:

Proposition 2.4.5. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathfrak{S}}$-module. If $\mathcal{F}_P = 0$ for all $P \in \langle \mathfrak{S} \rangle$, then $\mathcal{F} = \mathcal{F}_{\mathrm{tor}}$.*

Proof. The problem is local on $\mathcal{S}$, so we may assume that $\mathfrak{S} = \mathrm{Spf}(R)$, where R is some complete Noetherian adic ring. Let $M = \Gamma(\mathfrak{S}, \mathcal{F})$, and let U be as in Lemma 2.4.3. Then $\tilde{M}|_U$ is a coherent $\mathcal{O}_U$-module, which by assumption vanishes at each closed point of U. By Nakayama's lemma, $\tilde{M}|_U = 0$, and by the definition of U we see that $M = M_{\mathrm{tor}}$, as desired. □

Corollary 2.4.6. *Let $\mathcal{S}$ be an object of* Rig, *and let $\mathcal{F}$ be a coherent $\mathcal{O}_{\mathfrak{S}}$-module. If $\mathcal{F}_P = 0$ for all $P \in \langle \mathcal{S} \rangle$, then $\mathcal{F} = 0$.*

Chapter 3

Weak Formal Geometry

3.1 Theory of Monsky and Washnitzer

Let R be an admissible ring, and let A be an adic R-algebra. For any ideal of definition $I \subset R$, we define a pseudo-valuation on R by letting

$$v_I(r) = \sup\{n \in \mathbb{N} : r \in I^n\}.$$

Let $R\langle X_1, ..., X_n\rangle$ denote the ring of *convergent power series* in n variables with coefficients in R. In multi-index notation, this is the R-algebra of all power series

$$f(X_1, ..., X_n) = \sum_u r_u X^u$$

such that $v_I(r_u) \to \infty$ as $|u| \to \infty$. We say that f is *overconvergent* if moreover

$$\lim_{k\to\infty} \inf_{|u|>k} \frac{v_I(r_u)}{|u|} > 0.$$

Note that this condition depends only on the topology of R, and not on the choice of ideal of definition. The overconvergent power series in n variables form an R-algebra which we denote by $R[X_1, ..., X_n]^{\mathrm{mw}}$. It is well known that this ring is Noetherian whenever R is Noetherian [9].

Definition 3.1.1. Let A be an adic R-algebra. The *Monsky-Washnitzer weak completion* of A is the R-subalgebra $A^{\mathrm{mw}} \subseteq A^{\infty}$ consisting of elements of the form

$$f(a_1, ..., a_n) \tag{3.1}$$

where f is an overconvergent power series and $a_1, ..., a_n \in A$. We say that A is mw-*weakly complete* if the natural map $A \to A^{\mathrm{mw}}$ is an isomorphism

The Monsky-Washnitzer weak completion of an adic R-algebra is always mw-weakly complete ([19], 1.2). If A is an admissible adic R-algebra, then there are natural inclusions

$$A \subseteq A^{\dagger} \subseteq A^{\infty}.$$

It follows easily that $A^{\dagger}$ is separated and that the reduction of $A^{\dagger}$ agrees with A^0. In particular, $A^{\dagger}$ is an admissible adic R-algebra which is of finite presentation if and only if A is of finite presentation. If $f : A \to B$ is a continuous map of adic R-algebras, then f prolongs uniquely to a map of Monsky-Washnitzer weak completions $A^{\mathrm{mw}} \to B^{\mathrm{mw}}$ ([19], 1.5). Note in particular that $R^{\mathrm{mw}} = R^{\infty}$. In general, we will regard the Monsky-Washnitzer weak completion as a functor

$$\mathrm{AdRing}_R \to \mathrm{AdRing}_{R^{\infty}}.$$

Let $h : R \to R'$ be morphism in AdRing. Let A' be an adic R'-algebra, and let A denote A' regarded as an adic algebra over R. Since h is an adic ring map, we see that h induces a map

$$R[X_1, ..., X_n]^{\mathrm{mw}} \to R'[X_1, ..., X_n]^{\mathrm{mw}}.$$

It follows that there is an injective ring map

$$A^{\mathrm{mw}} \to (A')^{\mathrm{mw}} \tag{3.2}$$

which fails to be surjective in general. In particular, the failure of this map to be surjective when h is a *localization* tells us that the property of being mw-weakly complete is not a local property on the base.

3.2 Weak Completion

Let R be an admissible ring, and let A be an adic R-algebra. In this section we introduce a modification of the Monsky-Washnitzer weak completion, which has better local properties on the "base" ring R. Informally, the weak completion of A includes all of those elements of A^∞ which overconverge at every rigid point of R.

For any rigid point P of R, let A_P denote the base change of A along the map $R \to R_P$. The completion of the natural map $A \to A_P$ gives a commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A^\infty \\ \downarrow & & \downarrow \\ A_P & \longrightarrow & A_P^\infty \end{array}$$

Recall that the Monsky-Washnitzer weak completion of A_P is by definition an R_P-subalgebra of A_P^∞. For every element $a \in A^\infty$, let a_p denote the image of a in A_P^∞.

Definition 3.2.1. The *weak completion* of A is the R-algebra

$$A^\dagger = \{a \in A^\infty : a_P \in A_P^{\mathrm{mw}} \text{ for all } P \in \langle R \rangle\}.$$

We say that A is *weakly complete* if the natural map $A \to A^\dagger$ is an isomorphism.

Remark 3.2.2. In general, we have a containment $A^{\mathrm{mw}} \subseteq A^\dagger$. The two weak completions evidently coincide when R is a rigid point, but for a general base R the containment may be strict: consider for example the localization R_f at some $f \in R$. Then $R_f^\dagger$ coincides with the full completion R_f^∞.

Proposition 3.2.3. *If A is admissible, then $A^\dagger$ is admissible and weakly complete.*

Proof. If A is admissible then there are natural inclusions $A \subseteq A^\dagger \subseteq A^\infty$, and the first statement follows easily. If P is a rigid point of R, there is commutative diagram:

$$\begin{array}{ccc} A^\dagger & \longrightarrow & A^\infty \\ \downarrow & & \downarrow \\ (A^\dagger)_P & \longrightarrow & A_P^\infty \end{array}$$

By the definition of weak completion, the elements of $A^\dagger$ are precisely those elements of A^∞ for which $a_P \in A_P^\dagger$ for every rigid point P of R. It follows that the image of the bottom arrow lies in the weakly complete algebra $A_P^\dagger$, and consequently $(A^\dagger)_P^\dagger = A_P^\dagger$. □

Proposition 3.2.4. *Every map $f : A \to B$ of adic R-algebras prolongs uniquely to a map $A^\dagger \to B^\dagger$.*

Proof. Note that f prolongs uniquely to a map $A^\infty \to B^\infty$. Let $a \in A^\dagger$. By definition, $a_P \in A_P^\dagger$ for every rigid point P of R. Thus a_P has an expression of the form

$$\bar{F}(\bar{a}_1, ..., \bar{a_n}),$$

where $\bar{F} \in R_P[X_1, ..., X_n]^\dagger$ is an overconvergent power series, and $\bar{a}_1, ..., \bar{a}_n \in A_P$. Clearly the fiber f_P of f maps $A_P \to B_P$, so

$$f_P(a) = \bar{F}(f_P(\bar{a}_1), ..., f_P(\bar{a}_n)) \in B_P^\dagger.$$

□

Applying Proposition 3.2.4 to the map $R \to A$, we see that there is a natural map $R^\infty \to A^\dagger$. If A is admissible of finite presentation over R, then the same is true of $A^\dagger$. In general, we will regard weak completion as a functor

$$\mathrm{AdRing}_R \to \mathrm{AdRing}_{R^\infty}.$$

Unlike the Monsky-Washnitzer weak completion, it is not true that the homomorphic image of a weakly complete R-algebra is weakly complete in general. However, we do have the following:

Lemma 3.2.5. *If A is weakly complete, then A_P is weakly complete for every rigid point P of R.*

Proof. Let P be a rigid point of R. There are inclusions

$$A_P \subseteq A_P^\dagger \subseteq A_P^\infty.$$

It suffices to prove that the map $A \to A_P^\dagger$ is surjective. Let $\pi \in R$ be any element for which $\pi_P R_P$ is an ideal of definition of R_P. Let $a_P \in A_P$. By definition, there are elements $a_{1,P}, ..., a_{n,P} \in A$ such that $a_P = f_P(a_{1,P}, ..., a_{n,P})$, where

$$f_P = \sum_u r_{u,P} \pi_P^{n(u)} X^u,$$

with $r_{u,P} \in R_P$ and

$$\lim_{k\to\infty} \inf_{|u|>k} \frac{n(u)}{|u|} > 0.$$

Choose elements $r_u \in R$ and $a_i \in A$ lifting $r_{u,P}$ and $a_{i,P}$, respectively. Define

$$f = \sum_u r_u \pi^{n(u)} X^u \in R[X_1, ..., X_n]^{\text{mw}}.$$

Then $a = f(a_1, ..., a_n)$ is an element of A lifting a_P. □

Let $h : R \to R'$ be morphism in AdRing. Let A' be an adic R'-algebra, and let A denote A' regarded as an adic algebra over R. Recall that there is an injective map (3.2)

$$A^{\text{mw}} \to (A')^{\text{mw}}.$$

Now, suppose that P' is a rigid point of R', which factors uniquely through a rigid point P of R. If $a \in A^\dagger$, then by definition a is congruent modulo $\ker(P)$ to an element of A^{mw}. But $h(\ker(P)) \subseteq \ker(P')$, and thus we see that $a_P \in (A')^\dagger_P$. It follows that the above map extends to an injective ring map

$$A^\dagger \to (A')^\dagger. \tag{3.3}$$

In contrast to the Monsky-Washnitzer weak completion, we have:

Proposition 3.2.6. *Suppose that $h : R \to R'$ induces an injection $\langle R' \rangle \to \langle R \rangle$. Then the map (3.3) is an isomorphism.*

Proof. Let $a \in (A')^\dagger$. We must show that for every rigid point $P : R \to R_P$, the image a_P of

a in A_P^∞ lies in $A_P^\dagger$. Let us write $R'_P = R_P \otimes_R R'$. We consider the fiber

$$A_P = R_P \otimes_R A = R'_P \otimes_{R'} A'. \tag{3.4}$$

First, if $\langle R'_P \rangle = \emptyset$, then $R'_P = (R'_P)_{\text{tor}}$. In this case, $A_P = A_P^\dagger = A_P^\infty$ and the result is immediate. Suppose then that R'_P has a rigid point $Q : R'_P \to \Omega$. Then the composition

$$P' : R' \to R'_P \xrightarrow{Q} \Omega$$

is surjective, and therefore is a rigid point of R. By construction, P' is a rigid point lying above P, and by assumption it is the unique such point.

We claim that Q is an isomorphism, in which case it follows from (3.4) that $a_P \in A_P$, since A' is weakly complete. Since Q is surjective, the result will follow from ([1], 1.11.5(ii)) if we can show that R'_P is a one-dimensional local domain. Since R'_P has a unique rigid point, it follows from ([1], 3.3.10) that $\mathrm{Spf}(R'_P)$ is a one-point formal scheme, and consequently that R'_P is a Noetherian local domain. The natural map $R_P \to R'_P$ is adic, and therefore R'_P admits a principal ideal of definition. It follows that R'_P is one-dimensional, completing the proof. □

3.3 w.c.f.g. Algebras

In this section we fix Noetherian admissible ring R. We will now introduce a suitable class of admissible R-algebras for constructing our weak formal schemes over R.

Definition 3.3.1. Let A be an adic R-algebra, and let $S \subseteq A$. We say that S *weakly generates* A if:

1. The R-subalgebra of A generated by S is dense in A

2. For every $a \in A$ and every rigid point P of R, a_P is of the form

$$F(s_1, ..., s_n)$$

where $s_i \in S$, and $F \in R_P[X_1, ..., X_n]^\dagger$ for some n.

We say that A is *weakly finitely generated* if A admits a finite set of weak generators. We say that A is a *w.c.f.g. algebra* if A is weakly complete and weakly finitely generated.

Let A be a weakly finitely generated algebra over R. Then A is an adic R-algebra of finite presentation, and the weak completion $A^\dagger$ is a w.c.f.g. algebra over R. In particular, we see that A is admissible. Every homomorphic image of A is weakly finitely generated over R, although we remark again that the homomorphic image of a w.c.f.g. algebra need not be weakly complete in general.

Lemma 3.3.2. *Let A be a weakly finitely generated algebra over R. For every morphism $h : R \to R'$ in* AdRing*, $A_h = R' \otimes_R A$ is weakly finitely generated over R'.*

Proof. Choose a finite set $s_1, ..., s_n$ of weak generators for A. Recall that every element of A_h is a finite R'-linear combination elements of the form $r \otimes a$, where $r \in R'$ and $a \in A$. Follows easily that the R' algebra generated by $s_1, ..., s_n$ is dense in A_h. Now let P' be a rigid point of R'. Then P' factors uniquely through a rigid point P of R. By assumption, for any $a \in A$ there is an overconvergent power series $f = \sum_u r_u X^u \in R_P[X_1, ..., X_n]^\dagger$ such that

$$a_P = f(s_1, ..., s_n).$$

Then for any $r \in R'$ we have $(r \otimes a)_{P'} = f'(s_1, ..., s_n)$, where

$$f' = \sum_u (r \otimes r_u) X^u \in R_{P'}[X_1, ..., X_n].$$

$\square$

For the remainder of this section, we fix a weakly finitely generated algebra A/R. Let $X = \mathrm{Spec}(A^0)$. For any $f \in A$, let X_f denote the distinguished open subset of X on which $f_0 \in A^0$ is invertible.

Lemma 3.3.3. *Every inclusion $X_g \to X_f$ lifts to a unique A-algebra homomorphism $A_f^\dagger \to A_g^\dagger$.*

Proof. The proof of ([17], 2.3) adapts verbatim to our case. $\square$

Definition 3.3.4. Let M be a finite A-module. We define the *weak completion* of M to be the A-module

$$M^\dagger = \{m \in M^\infty : m_P \in M_P \text{ for all } P \in \langle R \rangle\}.$$

To any finite A-module M, we associate a presheaf $\tilde{M}^\dagger$ on the distinguished open sets of X via

$$\tilde{M}^\dagger(X_f) = M_f^\dagger$$

Note that $\tilde{M}^\dagger$ is well defined by Lemma 3.3.3.

Theorem 3.3.5. *$\tilde{M}^\dagger$ is a sheaf on the distinguished open sets of X.*

Proof. Let $f_1, ..., f_n \in A$, $U_i = X_{f_i}$, and $U = U_1 \cup \cdots \cup U_n$. For each i, let $m_i \in M_{f_i}^\dagger$ be such that

$$m_i|_{U_i \cap U_j} - m_j|_{U_i \cap U_j} = 0. \tag{3.5}$$

Note that the presheaf $\tilde{M}^\infty$ defined by

$$X_f \mapsto M_f^\infty$$

is a *sheaf* on the distinguished open subsets of X. Indeed it is the coherent sheaf associated to M on the formal scheme $\mathrm{Spf}(A)$ ([1], 2.7.2). In particular, there is a unique section $m \in \tilde{M}^\infty(U)$ such that $m|_{U_i} = m_i$ for each i. The restriction $\tilde{M}^\dagger|_U$ is the presheaf associated to the finite $\Gamma(U, \mathcal{O}_U)$-module $M_{f_1,\ldots,f_n}$. Therefore we only need to show that $m \in M^\dagger_{f_1,\ldots,f_n}$, i.e. that for every rigid point P of R, m_P lies in $(M_{f_1,\ldots,f_n})_P$.

Fix a rigid point P of R and let $X_P = \mathrm{Spec}(A_P)$. Note that the distinguished open subsets of X_P are all of the form $X_f \cap X_P$, where f is some element of A. Consider the presheaf $\tilde{M}_P^\dagger$ associated to the finite A_P-module M_P. By ([17], 2.8), $\tilde{M}_P^\dagger$ is a sheaf on the distinguished open subsets of X_P. For each i, there is a restriction map

$$\tilde{M}^\dagger(U_i) \to \tilde{M}_P^\dagger(U_i \cap X_P).$$

The cocycle condition (3.5) implies that

$$m_i|_{U_i \cap U_j \cap X_P} - m_j|_{U_i \cap U_j \cap X_P} = 0.$$

Therefore these sections glue to a unique section $m' \in \Gamma(U \cap X_P, \tilde{M}_P^\dagger) = (M_{f_1,\ldots,f_n})_P$. Now $\tilde{M}_P^\dagger$ is a subsheaf of the sheaf $\tilde{M}_P^\infty$, and clearly we have

$$m|_{U_i \cap X_P} = m_i|_{U_i \cap X_P},$$

so necessarily $m = m'$. □

Definition 3.3.6. The *weak formal spectrum* of A is the admissibly ringed space with underlying topological space X, and whose sheaf of adic rings is $\tilde{A}^\dagger$.

We will denote the weak formal spectrum of A by $\mathrm{Spwf}(A)$. If $B \to A$ is a continuous map of weakly finitely generated R-algebras, then there is an induced map $\mathrm{Spwf}(B) \to \mathrm{Spwf}(A)$ of admissibly ringed spaces. In particular, the map $A \to A^\dagger$ induces an isomorphism $\mathrm{Spwf}(A^\dagger) \to \mathrm{Spwf}(A)$, but in general it will be convenient not to require that A be weakly complete. Note that $\mathrm{Spwf}(R) = \mathrm{Spf}(R)$, and consequently there is a natural *structure map*

$$p : \mathrm{Spwf}(A) \to \mathrm{Spf}(R).$$

For every finite A-module M, the sheaf $\tilde{M}$ is naturally an $\mathcal{O}_{\mathrm{Spwf}(A)}$-module.

Question 3.3.7. *Let M be a finite A-module. By a theorem of Meredith [17], for every rigid point P of R the sheaf $\tilde{M}_P^\dagger$ is acyclic. It would be interesting to know if more generally, the modules*

$$R^i p_* \tilde{M}^\dagger$$

vanish for $i > 0$. In contrast to the classical case, the sheaf $\tilde{M}^\dagger$ need not be coherent. Can coherent sheaves on $\mathrm{Spwf}(A)$ *be characterized in terms of their global sections?*

3.4 Weak Formal Schemes

We are now ready to introduce weak formal schemes, which will be our main objects of study. Informally, these are admissibly ringed spaces which can be obtained by gluing weak formal spectra along the base. Throughout, we fix a formal scheme $\mathfrak{S}$ in FS_+.

Definition 3.4.1. Let $p : \mathfrak{X} \to \mathfrak{S}$ be a morphism of admissibly ringed spaces.

1. We say that p is an *affine weak formal scheme* if $\mathfrak{S}$ admits a covering $\{\mathfrak{S}_i \to \mathfrak{S}\}$ with $\mathfrak{S}_i = \mathrm{Spf}(R_i)$ such that for each i, $\mathfrak{X}_i = p^{-1}(\mathfrak{S}_i)$ is of the form $\mathrm{Spwf}(A_i)$ for some w.c.f.g. algebra A_i over R_i.

2. We say that p is a *weak formal scheme* if $\mathfrak{X}$ admits a $\mathfrak{S}$-covering by affine weak formal schemes over $\mathfrak{S}$.

When the base is understood, it will be convenient to refer to a weak formal scheme $p : \mathfrak{X} \to \mathfrak{S}$ simply by $\mathfrak{X}$. A *morphism* of weak formal schemes over $\mathfrak{S}$ is defined to be a morphism in the category of admissibly ringed spaces over $\mathfrak{S}$. Note that p is always adic and locally of finite presentation, in the sense that $p_0 : X \to S$ is a morphism of schemes locally of finite presentation. We say that p is *of finite presentation* if in addition it is quasi-compact and quasi-separated. If $\mathfrak{X}$ is affine over $\mathfrak{S}$, then p_0 is an affine morphism and so p is always a morphism of finite presentation. Let us write $\mathrm{FS}^{\dagger}_{\mathfrak{S}}$ for the category of weak formal schemes of finite presentation over $\mathfrak{S}$

The following result indicates that the property of being a weak formal scheme is local on the base. This fact was our main motivation for modifying the weak completion of Monsky and Washnitzer:

Proposition 3.4.2. *Let $\mathfrak{S}' \to \mathfrak{S}$ be an open immersion. For every weak formal scheme $p : \mathfrak{X} \to \mathfrak{S}$, $\mathfrak{X}' = p^{-1}(\mathfrak{S}')$ is a weak formal scheme over $\mathfrak{S}'$. If $\mathfrak{X}$ is affine over $\mathfrak{S}$, then $\mathfrak{X}'$ is affine over $\mathfrak{S}'$.*

Proof. The problem is local on $\mathfrak{X}$, so we may assume that $\mathfrak{X}$ is affine over $\mathfrak{S}$. Choose a covering $\{\mathfrak{S}_i \to \mathfrak{S}\}$ with $\mathfrak{S}_i = \mathrm{Spf}(R_i)$ such that for each i, $\mathfrak{X}_i = p^{-1}(\mathfrak{S}_i)$ is a space of the form $\mathrm{Spwf}(A_i)$ for some w.c.f.g. algebra A_i over R_i. Let $\mathfrak{S}'_i = \mathfrak{S}_i \cap \mathfrak{S}'$, and $\mathfrak{X}'_i = p^{-1}(\mathfrak{S}'_i)$.

Then $\{\mathfrak{X}'_i \to \mathfrak{X}'\}$ is a covering in the category of adic ringed spaces over $\mathfrak{S}'$. Since $\mathfrak{S}'_i$ is an open subset of $\mathrm{Spf}(R_i)$, there exist $f_{i,j} \in R_i$ such that

$$\mathfrak{S}'_i = \bigcup_j \mathrm{Spf}(R_{i,j}),$$

where $R_{i,j} = (R_i)_{f_{i,j}}$. Let $A_{i,j} = (A_i)^{\dagger}_{f_{i,j}}$. Then each $A_{i,j}$ is a w.c.f.g. algebra over $R_{i,j}$. But $\{\mathrm{Spwf}(A_{i,j}) \to \mathfrak{X}'\}_{(i,j)}$ is a covering of $\mathfrak{X}'$ and consequently $\mathfrak{X}'$ is a weak formal scheme over $\mathfrak{S}'$. □

Our next goal is to characterize the affine weak formal schemes over $\mathfrak{S} = \mathrm{Spf}(R)$, where R is a Noetherian adic ring. First we require the following:

Lemma 3.4.3. *Suppose that $\mathfrak{S} = \mathrm{Spf}(R)$ is affine, and let $p : \mathfrak{X} \to \mathfrak{S}$ be a weak formal scheme of finite presentation. For every $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, the canonical map*

$$\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})^{\dagger}_f \to \Gamma(\mathfrak{X}_f, \mathcal{O}_{\mathfrak{X}})$$

is an isomorphism.

Proof. By the definition of a weak formal scheme, $\mathfrak{S}$ admits a covering by open immersions $\mathrm{Spf}(R') \to \mathfrak{S}$ such that $p^{-1}(\mathrm{Spf}(R'))$ admits a covering by spaces of the form $\mathrm{Spwf}(A')$, where A' is a w.c.f.g. algebra over R'. By assumption, $\mathfrak{X}$ is quasi-compact over $\mathfrak{S}$ and hence quasi-compact. Thus we may choose a *finite* number of diagrams

$$\begin{array}{ccc} \mathrm{Spwf}(A_i) & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathrm{Spwf}(R_i) & \longrightarrow & \mathfrak{S} \end{array}$$

where the horizontal arrows are open immersions, A_i is a w.c.f.g. algebra A_i/R_i, and the $\mathrm{Spwf}(A_i)$ jointly cover $\mathfrak{X}$. We will write $\mathfrak{S}_i = \mathrm{Spf}(R_i)$, $\mathfrak{X}_i = \mathrm{Spwf}(A_i)$.

Let $\mathfrak{X}_{i,j} = \mathfrak{X}_i \cap \mathfrak{X}_j$. Then $\mathfrak{X}_{i,j}$ is an affine weak formal scheme over $\mathfrak{S}_{i,j} = \mathfrak{S}_i \cap \mathfrak{S}_j$. By definition, $\mathfrak{S}_{i,j}$ admits a finite covering of the form $\mathfrak{S}_{i,j,k} = \mathrm{Spf}(R_{i,j,k})$ such that $p^{-1}(\mathfrak{S}_{i,j,k})$ is of the form $\mathrm{Spwf}(A_{i,j,k})$ for some w.c.f.g. algebra $A_{i,j,k}/R_{i,j,k}$. Then $A = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is the equalizer of the diagram

$$\prod_i A_i \rightrightarrows \prod_{i,j} \Gamma(\mathfrak{X}_{i,j}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow \prod_{i,j,k} A_{i,j,k} \ ,$$

where the last map is given by restriction and all products are finite. Since localization is exact and commutes with finite direct products, it follows that A_f is the equalizer of the diagram

$$\prod_i (A_i)_f \rightrightarrows \prod_{i,j,k} (A_{i,j,k})_f \ .$$

Now, each $A_{i,j,k}$ may be regarded as a w.c.f.g. algebra over R, by Proposition 3.2.6. Thus we may pass to weak completions to obtain a diagram

$$\prod_i (A_i)_f^\dagger \rightrightarrows \prod_{i,j,k} (A_{i,j,k})_f^\dagger \ .$$

whose equalizer is $A_f^\dagger$. On the other hand, $\mathfrak{X}_f$ admits a covering by the open subsets $(\mathfrak{X}_i)_f = \mathrm{Spwf}((A_i)_f^\dagger)$, and each of these is covered by the $\mathrm{Spwf}((A_{i,j,k})_f^\dagger)$. The sheaf exact sequence shows $\Gamma(\mathfrak{X}_f, \mathcal{O}_{\mathfrak{X}})$ is the equalizer of the above diagram, and consequently $A_f^\dagger \to \Gamma(\mathfrak{X}_f, \mathcal{O}_{\mathfrak{X}})$ is an isomorphism. □

Proposition 3.4.4. *Let $\pi : \mathfrak{X} \to \mathfrak{S}$ be a morphism of admissibly ringed spaces. The following are equivalent:*

1. *$\mathfrak{X} \to \mathfrak{S}$ is an affine weak formal scheme*

2. *For every affine open $\mathfrak{S}' = \mathrm{Spf}(R)$ of $\mathfrak{S}$, $\mathfrak{X}' = \pi^{-1}(\mathfrak{S}')$ is of the form* Spwf(A) *for*

some w.c.f.g. algebra A over R.

Proof. Evidently (2) implies (1). Conversely, if $\mathfrak{X} \to \mathfrak{S}$ is affine then by Lemma 3.4.2, $\mathfrak{X}' \to \mathfrak{S}'$ is affine. Let $A = \Gamma(\mathfrak{X}', \mathcal{O}_{\mathfrak{X}'})$. Then by Lemma 3.4.3, $\mathfrak{X}' = \mathrm{Spwf}(A)$, completing the proof. □

It follows that every affine weak formal scheme over $\mathfrak{S} = \mathrm{Spf}(R)$ is of the form $\mathrm{Spwf}(A)$, where A is a w.c.f.g. algebra over R. In A.1, we show more generally that every affine weak formal scheme $\mathfrak{X} \to \mathfrak{S}$ is the *weak formal spectrum* of a sheaf of w.c.f.g. algebras over $\mathfrak{S}$.

We will now prove general theorem on representable Set-valued functors on $\mathrm{FS}^{\dagger}_{\mathfrak{S}}$, which is useful for giving basic constructions of weak formal schemes. Let us embed the category $\mathrm{FS}^{\dagger}_{\mathfrak{S}}$ into its category of presheaves in the usual way. We will identify each weak formal scheme over $\mathfrak{S}$ with its associated representable functor.

Lemma 3.4.5. *Every representable functor on $\mathrm{FS}^{\dagger}_{\mathfrak{S}}$ is a sheaf.*

Proof. The statement means that morphisms $\mathfrak{Y} \to \mathfrak{X}$ can be glued locally on the source, which follows from the analogous statement in the category of admissibly ringed spaces. □

Definition 3.4.6. Let F be a presheaf on $\mathrm{FS}^{\dagger}_{\mathfrak{S}}$. We say that a sub-presheaf $H \subseteq F$ is *open* if for every weak formal scheme $\mathfrak{X}$ over $\mathfrak{S}$, there is an open immersion $\mathfrak{Y} \to \mathfrak{X}$ for which the following diagram is Cartesian:

$$\begin{array}{ccc} \mathfrak{Y} & \longrightarrow & H \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & F \end{array}$$

Theorem 3.4.7. *Let F be a sheaf on $\mathrm{FS}^{\dagger}_{\mathfrak{S}}$. Suppose that F admits an open covering by weak formal schemes $\mathfrak{X}_i$. Then F is representable.*

Proof. By definition, for each i, j there is an open immersion $\mathfrak{X}_{i,j} \to \mathfrak{X}_i$ for which the diagram

$$\begin{array}{ccc} \mathfrak{X}_{i,j} & \longrightarrow & \mathfrak{X}_j \\ \downarrow & & \downarrow \\ \mathfrak{X}_i & \longrightarrow & F \end{array}$$

is Cartesian. Swapping the roles of i, j, we see that there are unique isomorphisms $\varphi_{i,j} : \mathfrak{X}_{i,j} \to \mathfrak{X}_{j,i}$ compatible with the respective diagrams. By forming the evident cube involving $\mathfrak{X}_i$, $\mathfrak{X}_j$, and $\mathfrak{X}_k$, it follows that $\varphi_{i,j}$ restricts to an isomorphism

$$\mathfrak{X}_{i,j} \cap \mathfrak{X}_{i,k} \to \mathfrak{X}_{j,i} \cap \mathfrak{X}_{j,k}.$$

Moreover, by uniqueness of the $\varphi_{i,j}$, we see that these maps satisfy the cocycle condition

$$\varphi_{i,k}|_{\mathfrak{X}_{i,j}\cap\mathfrak{X}_{i,k}} = \varphi_{j,k}|_{\mathfrak{X}_{j,i}\cap\mathfrak{X}_{j,k}} \circ \varphi_{i,k}|_{\mathfrak{X}_{i,j}\cap\mathfrak{X}_{i,k}}.$$

By descent, we see that there is a unique admissibly ringed space $\mathfrak{X}$ obtained by gluing the $\mathfrak{X}_i$. Since this space admits an $\mathfrak{S}$-covering by weak formal schemes over $\mathfrak{S}$, it follows that $\mathfrak{X}$ is also a weak formal scheme over $\mathfrak{S}$.

It remains only to show that $\mathfrak{X}$ represents the functor F. Let $\mathfrak{X}'$ be a weak formal scheme over $\mathfrak{S}$. Recall that a section of $F(\mathfrak{X}')$ is naturally identified with a morphism of sheaves $\mathfrak{X}' \to F$. By definition, for each i there is an open immersion $\mathfrak{X}'_i \to \mathfrak{X}'$ for which the diagram

$$\begin{array}{ccc} \mathfrak{X}'_i & \longrightarrow & \mathfrak{X}_i \\ \downarrow & & \downarrow \\ \mathfrak{X}' & \longrightarrow & F \end{array}$$

is Cartesian. By considering the analogous diagram with $\mathfrak{X}_{i,j}$ in place of $\mathfrak{X}_i$, we see that the maps $\mathfrak{X}'_i \to \mathfrak{X}_i$ agree on double intersections. By Lemma 3.4.5, these morphisms glue to a unique morphism $\mathfrak{X}' \to \mathfrak{X}$. □

3.5 Weak Base Change

Let us define an "absolute" category $\mathrm{FS}_+^\dagger$ of weak formal schemes as follows: the objects of $\mathrm{FS}_+^\dagger$ are weak formal schemes $\mathfrak{X} \to \mathfrak{S}$ of finite presentation, where $\mathfrak{S}$ is an object of FS_+. A morphism f from $\mathfrak{X}' \to \mathfrak{S}'$ to $\mathfrak{X} \to \mathfrak{S}$ is defined to be a commutative square

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f} & \mathfrak{X} \\ \downarrow & & \downarrow \\ \mathfrak{S}' & \xrightarrow{\bar{f}} & \mathfrak{S} \end{array} \tag{3.6}$$

where the horizontal arrows are morphisms of admissibly ringed spaces. Occasionally, we will say that f is a morphism *covering* the morphism $\bar{f}$. We say that f is *adic* if $\bar{f}$ is adic. In this case, consider the induced diagram of reduced schemes:

$$\begin{array}{ccc} X' & \xrightarrow{f_0} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\bar{f}_0} & S \end{array}$$

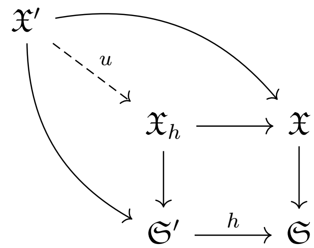
We say that f is of *locally of finite presentation* if f_0 and $\bar{f}_0$ are locally of finite presentation. As usual, we say that f is of *finite presentation* if it is locally of finite presentation, quasi-compact, and quasi-separated. Let us write $\mathrm{FS}^\dagger$ for the subcategory of $\mathrm{FS}_+^\dagger$ with the same objects, but whose morphisms are adic morphisms of finite presentation.

There is a natural forgetful functor

$$\mathrm{FS}_+^\dagger \to \mathrm{FS}_+ \tag{3.7}$$

which sends each weak formal scheme to its base. The fiber of this functor over any formal scheme $\mathfrak{S}$ is naturally identified with the category $\mathrm{FS}_{\mathfrak{S}}^\dagger$ introduced in the previous section. We will now apply Theorem 3.4.7 to show that the functor (3.7) is a fibration:

Proposition 3.5.1. *Let* $\mathfrak{X} \to \mathfrak{S}$ *be an object of* $\mathrm{FS}^{\dagger}_{+}$*, and let* $h : \mathfrak{S}' \to \mathfrak{S}$ *be a morphism in* FS_{+}*. There is a unique weak formal scheme* $\mathfrak{X}_h$ *over* $\mathfrak{S}'$ *equipped with a morphism* $\mathfrak{X}_h \to \mathfrak{X}$ *covering* h *with the following universal property: For every solid commutative diagram*



where $\mathfrak{X}'$ *is a weak formal scheme over* $\mathfrak{S}'$*, there exists a unique dashed morphism* $u : \mathfrak{X}' \to \mathfrak{X}_h$ *of weak formal schemes making the diagram commute.*

Proof. Let F be the functor which sends a formal $\mathfrak{S}'$-scheme $\mathfrak{X}'$ to the set of morphisms $f : \mathfrak{X}' \to \mathfrak{X}$ covering h. Choose an open cover $\{\mathfrak{S}_i \to \mathfrak{S}\}_i$ with $\mathfrak{S}_i = \mathrm{Spf}(R_i)$. For each i, choose open covers $\{\mathfrak{S}'_{i,j} \to h^{-1}(\mathfrak{S}_i)\}_j$ and $\{\mathfrak{X}_{i,k} \to \pi^{-1}(\mathfrak{S}_i)\}_k$ with $\mathfrak{S}'_{i,j} = \mathrm{Spf}(R'_{i,j})$ and $\mathfrak{X}_{i,k} = \mathrm{Spwf}(A_{i,k})$ where each $A_{i,k}$ is a w.c.f.g. algebra over R_i. Define $\mathfrak{X}_{i,j,k} = \mathrm{Spwf}(A_{i,j,k})$, where

$$A_{i,j,k} = (R'_{i,j} \otimes_{R_i} A_{i,k})^{\dagger}.$$

is a w.c.f.g. algebra over $R'_{i,j}$. Regarded as a weak formal scheme over $\mathfrak{S}'_{i,j}$, $\mathfrak{X}_{i,j,k}$ represents the functor which sends $\mathfrak{X}'$ to the set of $f \in F(\mathfrak{X}')$ whose image lies in $\mathfrak{X}_{i,k}$. Regarded as Set-valued functors on $\mathrm{FS}^{\dagger}_{\mathfrak{S}'}$, we see that the $\mathfrak{X}_{i,j,k}$ collectively cover F.

To see that each $\mathfrak{X}_{i,j,k}$ is open, suppose that we have a map $\mathfrak{X}' \to F$. Let $\mathfrak{X}'_{i,j}$ be the fiber of $\mathfrak{X}'$ over $\mathfrak{S}'_{i,j}$. Composing with the natural map $F \to \mathfrak{X}$, we obtain a map $f : \mathfrak{X}'_{i,j} \to \mathfrak{X}$

covering h. Let $\mathfrak{X}'_{i,j,k} = f^{-1}(\mathfrak{X}_{i,k})$. Then there is a Cartesian diagram:

$$\begin{array}{ccc} \mathfrak{X}'_{i,j,k} & \longrightarrow & \mathfrak{X}_{i,j,k} \\ \downarrow & & \downarrow \\ \mathfrak{X}' & \longrightarrow & F \end{array}$$

By Theorem 3.4.7, it follows that F is representable. □

Definition 3.5.2. In the notion of Proposition 3.5.1, we refer to $\mathfrak{X}_h$ as the *weak base change* of $\mathfrak{X}$ along h.

From the universal property it follows that (3.7) is a fibration, with the Cartesian arrows being given by weak base change. By construction, if $\mathfrak{X}$ is an affine weak formal scheme over $\mathfrak{S}$, then for every $h : \mathfrak{S}' \to \mathfrak{S}$, the weak base change $\mathfrak{X}_h$ is an affine weak formal scheme over $\mathfrak{S}'$.

Chapter 4

Frobenius Structures

We now turn to arithmetic applications of the theory developed above. Let p be a prime number, and fix a power $q = p^a$. Let us call a formal scheme $\mathfrak{S}$ *q-typical* if its reduction S is a scheme over $\mathbb{F}_q$. In this case, we let F denote the absolute q-Frobenius endomorphism of S. Recall that there is a canonical lifting of F to an endomorphism of the ring $\mathbb{Z}_q = W(\mathbb{F}_q)$.

In this chapter we will work with weak formal schemes *and* ordinary formal schemes over a formal base $\mathfrak{S}$. To parallel our notation for weak formal schemes, let FS_+^∞ denote the category of adic morphisms of finite presentation $\mathfrak{X} \to \mathfrak{S}$ in FS. A morphism in this category is defined exactly as for $\mathrm{FS}_+^\dagger$, and as always we write FS^∞ for the subcategory of $\mathrm{FS}_+^\dagger$ with the same objects, but whose *morphisms* are adic morphisms of finite presentation. The superscript is meant to indicate that FS_+^∞ is the "completion" of $\mathrm{FS}_+^\dagger$. Often, we will write FS_+^* for either FS_+^∞ or $\mathrm{FS}_+^\dagger$.

4.1 Formal F-Schemes

Let $\mathfrak{S}'$, $\mathfrak{S}$ be q-typical formal schemes. Given an $\mathbb{F}_q$-morphism $h_0 : S' \to S$, we define a *lifting* of h_0 to be a $\mathbb{Z}_q$-linear map $h : \mathfrak{S}' \to \mathfrak{S}$ whose reduction is h_0.

Definition 4.1.1. A formal F-scheme is a pair $(\mathfrak{S}, \sigma)$, where $\mathfrak{S}$ is a q-typical formal scheme, and $\sigma : \mathfrak{S} \to \mathfrak{S}$ is a lifting of the q-Frobenius endomorphism $F : S \to S$.

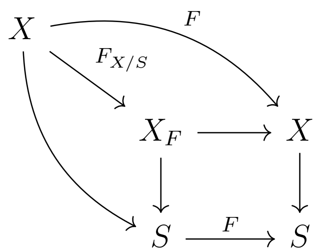
When the lifting of Frobenius σ is understood, we may refer to a formal F-scheme $(\mathfrak{S}, \sigma)$ simply by $\mathfrak{S}$. A *morphism* of formal F-schemes $(\mathfrak{S}', \sigma') \to (\mathfrak{S}, \sigma)$ is defined to be a morphism $h : \mathfrak{S}' \to \mathfrak{S}$ for which the following square commutes:

$$\begin{array}{ccc} \mathfrak{S}' & \xrightarrow{h} & \mathfrak{S} \\ {\scriptstyle\sigma'}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \mathfrak{S}' & \xrightarrow{h} & \mathfrak{S} \end{array}$$

Let us write F-FS_+ for the category of formal F-schemes. As usual, we will denote by F-FS the subcategory with the same objects, but whose morphisms are adic morphisms of finite presentation.

We will mainly be interested in *relative* Frobenius structures, in both the convergent and overconvergent settings. For now, we will work over a fixed formal F-scheme $(\mathfrak{S}_0, \sigma_0)$. Let $g : \mathfrak{S} \to \mathfrak{S}_0$ be any morphism in FS, and let $\mathfrak{X}_0$ be a (weak) formal scheme over $\mathfrak{S}_0$. In this setting, we will write $\mathfrak{X} = (\mathfrak{X}_0)_g$ for the (weak) base change of $\mathfrak{X}_0$ along g. Let $\mathfrak{X}_\sigma$ denote the (weak) base change of $(\mathfrak{X}_0)_{\sigma_0}$ along g. Then the reduction of $\mathfrak{X}_\sigma$ is precisely the base change X_F of X along the q-Frobenius $F : S \to S$. Recall that the endomorphism $F : X \to X$

factors uniquely through an S-morphism $F_{X/S} : X \to X_F$:



Definition 4.1.2. Let $(\mathfrak{S}_0, \sigma_0)$ be a formal F-scheme. A *(weak) formal F-scheme* is a triple $(\mathfrak{S}, \mathfrak{X}_0, \sigma)$, where

1. $\mathfrak{S} \to \mathfrak{S}_0$ is a morphism in FS
2. $\mathfrak{X}_0$ is an object of $\mathrm{FS}^*_{\mathfrak{S}_0}$
3. $\sigma : \mathfrak{X} \to \mathfrak{X}_\sigma$ is a lifting of the *relative Frobenius* $F_{X/S} : X \to X_F$.

We will be most interested in the case $\mathfrak{S} = \mathfrak{S}_0$. In this case, to give $\mathfrak{X}$ the structure of a (weak) formal F-scheme is *equivalent* to giving an extension of the endomorphism σ_0 to an endomorphism of $\mathfrak{X}$. Our general definition of a (weak) formal F-scheme is to allow for base change along a morphism which is not necessarily a morphism of formal F-schemes. If the base $\mathfrak{S}$ is understood, we may refer to the triple $(\mathfrak{S}, \mathfrak{X}_0, \sigma)$ simply by $(\mathfrak{X}, \sigma)$ or by $\mathfrak{X}$, and say that $\mathfrak{X}$ is a *(weak) formal F-scheme over* $\mathfrak{S}$ with *Frobenius structure* σ.

Leaving $(\mathfrak{S}_0, \sigma_0)$ fixed, we define a *morphism* of (weak) formal F-schemes

$$(\mathfrak{S}', \mathfrak{X}'_0, \sigma') \to (\mathfrak{S}, \mathfrak{X}_0, \sigma)$$

to be a morphism $f : \mathfrak{X}'/\mathfrak{S}' \to \mathfrak{X}/\mathfrak{S}$ in FS^* for which $\bar{f}$ is an $\mathfrak{S}_0$-morphism, and which is compatible with the Frobenius structures σ', σ. In particular, we define a morphism $(\mathfrak{X}', \sigma') \to (\mathfrak{X}, \sigma)$ of (weak) formal F-schemes over $\mathfrak{S}$ to be an $\mathfrak{S}$-morphism $f : \mathfrak{X}' \to \mathfrak{X}$

making the diagram

$$\begin{array}{ccc} \mathfrak{X}' & \xrightarrow{f} & \mathfrak{X} \\ \sigma' \downarrow & & \downarrow \sigma \\ \mathfrak{X}'_{\sigma'} & \xrightarrow{f} & \mathfrak{X}_\sigma \end{array}$$

commute. Let us write $F\text{-FS}^*_{\mathfrak{S}}$ for the resulting category of (weak) formal F-schemes over $\mathfrak{S}$.

Suppose that $(\mathfrak{S}'_0, \sigma')$ is a second formal F-scheme, and $\bar{h} : \mathfrak{S}'_0 \to \mathfrak{S}_0$ is a morphism in $F\text{-FS}_+$. Let $\mathfrak{S}'$ be a formal scheme over $\mathfrak{S}'_0$, and suppose we are given a morphism $h : \mathfrak{S}' \to \mathfrak{S}$ covering $\bar{h}$. Then there is a *(weak) base change* functor

$$F\text{-FS}^*_{\mathfrak{S}} \to F\text{-FS}^*_{\mathfrak{S}'}$$

sending $(\mathfrak{X}_0, \sigma) \mapsto (\mathfrak{X}'_0, \sigma_h)$, where $\mathfrak{X}'_0$ is the base change of $\mathfrak{X}_0$ along $\bar{h}$, and σ_h is the inverse image of σ along h.

Situation 4.1.3. *In applications, we will almost always restrict our attention to the following special case:* $\mathfrak{S}_0 = \mathrm{Spf}(R)$*, where R is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}_q$. In this setting, if $(\mathfrak{X}, \sigma)$ is a (weak) formal F-scheme over $\mathfrak{S}$, then σ is an $\mathcal{O}_{\mathfrak{S}}$-linear* endomorphism *of $\mathfrak{X}$.*

For the remainder of this section, we restrict our attention to Situation 4.1.3 and suppose moreover that $\mathfrak{X}_0$ is a *formal* F-scheme over $\mathfrak{S}_0$. Let x be a closed point of X_0, with residue field $k(x)$. Recall that the q-Frobenius endomorphism of $k(x)$ lifts canonically to a $\mathbb{Z}_q$-endomorphism of the Witt ring $W(k(x))$. Define

$$\mathfrak{S}_0(x) = \mathrm{Spf}(R(x)),$$

where $R(x) = W(k(x)) \otimes_{\mathbb{Z}_q} R$. Then $\mathfrak{S}_0(x)$ is a formal F-scheme over $\mathfrak{S}_0$. We define $\mathfrak{S}(x)$

to be the base change of $\mathfrak{S}_0(x)$ to a formal F-scheme over $\mathfrak{S}$. Observe that we have a map of reductions

$$x : S(x) \to X$$

given by base change of the map $x : \mathrm{Spec}(k(x)) \to X_0$.

Theorem 4.1.4. *Every closed point $x \in |X_0|$ lifts uniquely to a closed immersion of formal F-schemes over $\mathfrak{S}$*

$$\hat{x} : \mathfrak{S}(x) \to \mathfrak{X}.$$

Proof. Recall that the relative Frobenius σ induces the identity map on the topological space $\mathfrak{X}$, and therefore may be regarded as an endomorphism of $\mathcal{O}_{\mathfrak{X}}$. Let $\mathcal{I}$ denote the ideal of $\mathcal{O}_{\mathfrak{X}}$ generated by sections of the form $\sigma(a) - a$, where a is a section of $\mathcal{O}_{\mathfrak{X}}$. We define $\mathfrak{X}(1)$ to be the closed (weak) formal subscheme cut out by the ideal $\mathcal{I}$. Note that the reduction $X(1)$ is affine over S: explicitly, it is the disjoint union of sections

$$x : S \to X$$

given by base change of the $\mathbb{F}_q$-valued points $x : \mathbb{F}_q \to X_0$. It follows that $\mathfrak{X}(1)$ is affine over $\mathfrak{S}$, and is is a disjoint union of sections

$$\hat{x} : \mathfrak{S} \to \mathfrak{X}.$$

This gives the lifting for $\mathbb{F}_q$-rational points of X_0. For points of degree $d > 1$, we base change along $R \to \mathbb{Z}_{q^d} \otimes_{\mathbb{Z}_q} R$ and apply the above construction to the iterate σ^d. $\square$

Definition 4.1.5. For every closed point $x \in |X_0|$, we refer to the closed immersion $\hat{x}$ as the *Teichmüller lifting* of x.

4.2 Frobenius Modules

In this section, we fix a formal F-scheme $(\mathfrak{S}_0, \sigma_0)$, a formal scheme $\mathfrak{S}$ over $\mathfrak{S}_0$, and a (weak) formal F-scheme $(\mathfrak{X}, \sigma)$ over $\mathfrak{S}$. Suppose that M_0 is a coherent $\mathcal{O}_{\mathfrak{X}_0}$-module, and let M denote its inverse image along $\mathfrak{X} \to \mathfrak{X}_0$. We write M_σ for the inverse image of M along the natural map $\mathfrak{X}_\sigma \to \mathfrak{X}$.

Definition 4.2.1. A *σ-module* over $\mathfrak{X}/\mathfrak{S}$ is a pair (M_0, ϕ), where M_0 is a locally free $\mathcal{O}_{\mathfrak{X}_0}$-module, and $\phi : \sigma^* M_\sigma \to M$ is an $\mathcal{O}_{\mathfrak{X}}$-linear map. We say that (M_0, ϕ) is a *unit-root* σ-module if ϕ is an isomorphism.

Often, we will refer to a σ-module (M_0, ϕ) simply by (M, ϕ), leaving the module M_0 implicit. A *morphism* $(M, \phi) \to (M', \phi')$ of σ-modules over $\mathfrak{X}/\mathfrak{S}$ is defined to an $\mathcal{O}_{\mathfrak{X}}$-linear map $f : M \to M'$ for which the diagram

$$\begin{array}{ccc} \sigma^* M & \xrightarrow{\phi} & M \\ {\scriptstyle \sigma^* f}\downarrow & & \downarrow{\scriptstyle f} \\ \sigma^* M' & \xrightarrow{\phi'} & M' \end{array}$$

commutes. We will write $\sigma\text{-Mod}(\mathfrak{X}/\mathfrak{S})$ for the the category of σ-modules over $\mathfrak{X}$, and $\sigma\text{-Mod}_0(\mathfrak{X}/\mathfrak{S})$ for the full subcategory consisting of unit-root σ-modules.

Suppose that $f : (\mathfrak{S}', \mathfrak{X}'_0, \sigma') \to (\mathfrak{S}, \mathfrak{X}_0, \sigma)$ is a morphism of (weak) formal F-schemes. If (M, ϕ) is a σ-module over $\mathfrak{X}/\mathfrak{S}$, then $f^* M$ is naturally a σ'-module over $\mathfrak{X}'/\mathfrak{S}'$ with Frobenius structure $\sigma' \otimes \phi$. Thus we have an *inverse image* functor

$$f^* : \sigma\text{-Mod}(\mathfrak{X}/\mathfrak{S}) \to \sigma\text{-Mod}(\mathfrak{X}'/\mathfrak{S}')$$

which restricts to a functor of the unit-root subcategories.

For the remainder of this section, suppose that we are in Situation 4.1.3. Let $\bar{x}$ be a geometric point of the $\mathbb{F}_q$-scheme X_0. Our primary interest in σ-modules is their connection to (families of) representations of $\pi_1(X_0, \bar{x})$. In this direction we recall a theorem of Katz:

Theorem 4.2.2. *(Katz [14], 4.1.1) The category* $\sigma\text{-Mod}(\mathfrak{X}_0/\mathfrak{S}_0)$ *is equivalent to the category of continuous representations in a finite free R-module*

$$\rho : \pi_1(X_0, \bar{x}) \to \mathrm{GL}(V). \tag{4.1}$$

The equivalence of Theorem 4.2.2 is functorial in the following sense: Suppose that $f : \mathfrak{Y}_0 \to \mathfrak{X}_0$ is a morphism of (weak) formal F-schemes over $\mathfrak{S}_0$, and that $\bar{y}$ is a geometric point of Y_0 lying over $\bar{x}$. Let ρ (4.1) be a representation of $\pi_1(X, \bar{x})$ corresponding to a unit-root σ-module (M, ϕ). Then $f^*(M, \phi)$ is the unit-root σ-module over $\mathfrak{Y}_0/\mathfrak{S}_0$ corresponding to the pullback

$$f^*\rho : \pi_1(Y_0, \bar{y}) \to \mathrm{GL}(V).$$

Assume that $\mathfrak{X}_0$ is a *formal* F-scheme over $\mathfrak{S}_0$. We will now define the L-function of a σ-module (M, ϕ) over $\mathfrak{X}/\mathfrak{S}$. We will be most interested in the case $\mathfrak{S} = \mathfrak{S}_0$, but it will be convenient to define L-functions more abstractly. For each closed point $x \in |X|$, let (M_x, ϕ_x) denote the inverse image of (M, ϕ) along the Teichmüller lifting $\hat{x} : \mathfrak{S}(x) \to \mathfrak{X}$. Then by definition, the module M_x is a free $\mathcal{O}_{\mathfrak{S}(x)}$-module of finite rank. Choose a basis of global sections, and let $E(x)$ denote the "matrix" of ϕ_x regarded as a global section of $\mathrm{GL}_n\mathcal{O}_{\mathfrak{S}(x)}$. The map ϕ_x is only $\mathcal{O}_{\mathfrak{S}}$-linear, but the iterate $\phi_x^{\deg(x)}$ is $\mathcal{O}_{\mathfrak{S}(x)}$-linear, with "matrix"

$$\mathrm{N}_{k(x)/\mathbb{F}_q} E(x) = E(x)E(x)^{\sigma} \cdots E(x)^{\sigma^{\deg(x)}}.$$

In particular, the characteristic polynomial of $\phi_x^{\deg(x)}$ is σ-invariant, and therefore has coefficients in $\Gamma(\mathfrak{S}, \mathcal{O}_{\mathfrak{S}})$.

Definition 4.2.3. The L-function of the σ-module (M,ϕ) is defined to be

$$L(\phi,s)=\prod_{x\in|X|}\frac{1}{\det\left(I-\phi_x^{\deg(x)}s^{\deg(x)}\right)}\in\Gamma(\mathfrak{S},\mathcal{O}_{\mathfrak{S}})[[s]].$$

The important point is that our abstract L-functions are compatible with base change of σ-modules. Concretely, these L-functions generalize the Artin L-functions of (families of) representations: Suppose that (M,ϕ) is a unit-root σ-module over $\mathfrak{X}_0/\mathfrak{S}_0$. Let $x\in|X_0|$ and let $\bar{x}\to x$ be a geometric point. Then (M,ϕ) corresponds to a representation ρ (4.1), and the fiber (M_x,ϕ_x) corresponds to the representation

$$\rho_x:\pi_1(x,\bar{x})\to\pi_1(X_0,\bar{x})\to\mathrm{GL}(V).$$

Note that ρ_x sends the canonical generator $F^{\deg(x)}$ to the action of Frob_x. It follows that $\rho(\mathrm{Frob}_x)=\phi_x^{\deg(x)}$, and consequently $L(\phi,s)$ agrees with the L-function $L(\rho,s)$.

Question 4.2.4. *Since they appear naturally in our theory, it would be interesting to have an interpretation of σ-modules over $\mathfrak{X}/\mathfrak{S}$ in terms of families of representations. Is there a characterization of these objects in terms of a "relative fundamental group" of X/S, as in ([11], XIII)?*

4.3 Differentials

In this section, we work over a fixed Noetherian local ring R. Let A be a w.c.f.g. algebra over R, and fix a finite set $s_1,...,s_n$ of weak generators for A. There is a corresponding "presentation"

$$p:R[X_1,...,X_n]^{\mathrm{mw}}\to A$$

which sends $X_i \mapsto s_i$. Let A_0 denote the image of the above map, which is a Noetherian R-algebra for which $A = A_0^\dagger$. In [19], Monsky and Washnitzer define an A_0-module of *continuous differentials* $\Omega_{A_0/R}$ of the map $R \to A_0$. There is a universal continuous derivation

$$d : A_0 \to \Omega_{A_0/R}.$$

The module $\Omega_{A_0/R}$ is finite and generated by the $ds_1, ..., ds_n$ ([19], 4.5).

Lemma 4.3.1. *Let $M(A_0) = A \otimes_{A_0} \Omega_{A_0/R}$. Then $M(A_0)/M(A_0)_{\text{tor}}$ is independent of the choice of generators $s_1, ..., s_n$.*

Proof. Choose another finite set of generators $s'_1, ..., s'_m$ of A. Without loss of generality, we may assume that $n \leq m$ and that $s_i = s'_i$ for $1 \leq i \leq n$. Define A'_0 to be the image of the map

$$R[X_1, ..., X_m]^{\text{mw}} \to A$$

which sends $X_i \mapsto s'_i$. Then we have an inclusion $A_0 \to A'_0$, and there is a corresponding inclusion of A_0-modules.

$$\Omega_{A_0/R} \to \Omega_{A'_0/R}$$

which sends $ds_i \mapsto ds'_i$. Now for any ideal of definition $I \subset R$, we have an isomorphism $A_0/IA_0 \to A'_0/IA'_0$. By ([19], 2.1), we see that $A_0 \to A'_0$ is flat. Therefore we have an inclusion of A'_0-modules

$$A'_0 \otimes_{A_0} \Omega_{A_0/R} \to \Omega_{A'_0/R}$$

It suffices to show that this map is surjective. We recall that for every rigid point P of R, the map $(A_0)_P \to (A'_0)_P$ is an isomorphism. The relative differentials $\Omega_{A_0/R}$ are compatible with base change along *any* quotient map $R \to R'$. Thus we see that the fibers of f at the rigid points of R are surjective. Since both modules are finite A'_0-modules, and A'_0 is a Zariski ring, the result follows. □

Definition 4.3.2. The module of *continuous differentials* of A/R is defined to be the finite A-module $\Omega_{A/R} = M(A_0)/M(A_0)_{\mathrm{tor}}$.

We will denote by $\Omega^\bullet_{A/R}$ the free exterior algebra on $\Omega_{A/R}$. We remark that for every rigid point P of R, the fiber of $\Omega^\bullet_{A/R}$ over P agrees with the complex $\Omega^\bullet_{A_P/R_P}$ of continuous differentials of A_P/R_P, as defined by Monsky and Washnitzer. If A is formally smooth over R, then we see that for every ideal of definition $I \subset R$, $A_0/IA_0 = A/IA$ is smooth over R/I. It follows from ([19], 2.5) that A_0 is formally smooth over R, and from ([19], 4.6) that $\Omega_{A/R}$ is *locally free* of rank $d = \dim(A^0)$.

We now restrict our attention to a special case: Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}_q$. We assume moreover that A/R is formally smooth, that $A^0 = A/\mathfrak{m}A$ is integral, and that A_0 is equipped with a lifting $\sigma : A_0 \to A_0$ of the q-Frobenius endomorphism of A^0. Our next goal will be to construct canonical Frobenius operators on the exterior algebra $\Omega^\bullet_{A/R}$.

Lemma 4.3.3. *σ is injective.*

Proof. This is ([19], 3.2). □

For brevity, let us write $\Omega^i A = \Omega^i_{A/R}$. Let $B = \sigma(A)$, which is a formally smooth w.c.f.g. subalgebra of A, and let $B_0 = \sigma(A_0) \subseteq A_0$.

Lemma 4.3.4. *The canonical map $\Omega^\bullet B_0 \to \Omega^\bullet A_0$ induces an isomorphism*

$$\Omega^\bullet B_0 \otimes_{B_0} Q(A_0) \to \Omega^\bullet A_0 \otimes_{A_0} Q(A_0)$$

Proof. This is ([19], 8.1). □

Let $\alpha_\bullet$ denote the inverse of the isomorphism (4.3.4). We define a *trace map*

$$\mathrm{Tr}_\bullet : \Omega^\bullet A_0 \otimes_{A_0} Q(A_0) \xrightarrow{\alpha_\bullet} \Omega^\bullet B_0 \otimes_{B_0} Q(A_0) \xrightarrow{1\otimes \mathrm{Tr}} \Omega^\bullet B_0 \otimes_{B_0} Q(A_0). \tag{4.2}$$

This map extends to a map

$$\Omega^\bullet A \otimes_B Q(A) \to \Omega^\bullet B \otimes_B Q(A) \tag{4.3}$$

Lemma 4.3.5. *The map* $\mathrm{Tr}_\bullet$ *sends* $\Omega^\bullet A \to \Omega^\bullet B$.

Proof. By ([19], 8.1) the map $\mathrm{Tr}_\bullet$ restricts to a map $\Omega^\bullet A_0 \to \Omega^\bullet B_0$, from which the result follows. □

Definition 4.3.6. The *canonical Dwork operators* on $\Omega^\bullet A$ are defined via

$$\theta_\bullet = \sigma^{-1} \circ \mathrm{Tr}_\bullet : \Omega^\bullet A \to \Omega^\bullet A.$$

Now, let $\mathfrak{S} = \mathrm{Spf}(R)$ and $\mathfrak{X}_0 = \mathrm{Spwf}(A)$. Suppose that $g : \mathfrak{S} \to \mathfrak{S}_0$ is a morphism in FS, and let $\mathfrak{X}$ denote the weak base change of $\mathfrak{X}_0$ along g. Then σ pulls back to an $\mathcal{O}_{\mathfrak{S}}$-linear endomorphism of $\mathfrak{X}$, and the triple $(\mathfrak{S}, \mathfrak{X}, \sigma)$ is a weak formal F-scheme. We define $\Omega_{\mathfrak{X}/\mathfrak{S}}$ to be the inverse image of $\tilde{\Omega}_{A/R}$, and let $\Omega^\bullet_{\mathfrak{X}/\mathfrak{S}}$ denote the free exterior algebra on $\Omega_{\mathfrak{X}/\mathfrak{S}}$. Then by inverse image we have *canonical Dwork operators*

$$\theta_\bullet : \Omega^\bullet_{\mathfrak{X}/\mathfrak{S}} \to \Omega^\bullet_{\mathfrak{X}/\mathfrak{S}}.$$

Suppose that we have a σ-module (M, ϕ) on $\mathfrak{X}/\mathfrak{S}$. Let $M^\vee$ denote the $\mathcal{O}_{\mathfrak{X}}$-dual of M, and let us abbreviate $\Omega^i M^\vee = \Omega^i_{\mathfrak{X}/\mathfrak{S}} \otimes_{\mathcal{O}_\mathfrak{X}} M^\vee$. The exterior product defines a perfect pairing

$$\Omega^i_{\mathfrak{X}/\mathfrak{S}} \times \Omega^{d-i}_{\mathfrak{X}/\mathfrak{S}} \to \Omega^d_{\mathfrak{X}/\mathfrak{S}},$$

via which we identify the dual $(\Omega^i_{\mathfrak{X}/\mathfrak{S}})^\vee$ with $\Omega^{d-i}_{\mathfrak{X}/\mathfrak{S}}$. This gives an identification

$$\Omega^{d-i} M^\vee = \mathrm{Hom}(\Omega^i M, \Omega^d_{\mathfrak{X}/\mathfrak{S}}).$$

Definition 4.3.7. Given (M, ϕ) as above, we define the *associated Dwork operators* on $\Omega^{d-i} M^\vee$ by sending $f : \Omega^i M \to \Omega^d_{\mathfrak{X}/\mathfrak{S}}$ to the composition

$$\Omega^i M \xrightarrow{\sigma \otimes \phi} \Omega^i M \xrightarrow{f} \Omega^d_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\theta_d} \Omega^d_{\mathfrak{X}/\mathfrak{S}}.$$

Chapter 5

Spectral Varieties

Let K be a field equipped with a discrete valuation, and let V be a K-vector space. Fix a separable closure K^s of K. Following Monsky [18], we say that a K-linear operator $\psi : V \to V$ is *nuclear* if

1. The set $\overline{E}(\psi)$ of non-zero eigenvalues of ψ forms a summable sequence in K^s.
2. For every $\lambda \in \overline{E}(\psi)$, the generalized eigenspace V_λ is finite-dimensional.

For each $n > 0$, let V_n denote the sum of the V_λ for all $v(\lambda) < n$. It follows easily from the definition that the limit

$$C(\psi, s) = \lim_{n\to\infty} \det(I - \psi s | V_n)$$

exists, and moreover is an entire function on K. We refer to $C(\psi, s)$ as the *Fredholm determinant* of ψ. We regard $C(\psi, s)$ as an analytic function on the rigid analytic space $\mathbb{G}_{m,K}$. If $E(\psi)$ denotes its zero-locus, then the set of rigid points of $E(\psi)$ is naturally identified with the set of $\mathrm{Gal}(K^s/K)$-orbits in $\overline{E}(\psi)$.

Our goal in this final chapter is to construct an analogous spectral theory for *families* of nuclear operators, parameterized by some rigid analytic space $\mathcal{S}$. To each such operator ψ we associate an analytic function $C(\psi, s)$ on the relative multiplicative group $\mathbb{G}_{m,\mathcal{S}}$ (2.3.3). Its zero locus $\mathcal{E}(\psi)$ is naturally a rigid analytic variety over $\mathcal{S}$, whose fiber over a rigid point $P \in \langle \mathcal{S} \rangle$ becomes the set $E(\psi_P)$ described above. Our nuclear operators are a mild generalization of Coleman's families of completely continuous operators acting on a Banach module [4].

In 5.2, we show that if M is a coherent module over an affine weak formal F-scheme, and if Θ is a Dwork operator on M, then Θ is a nuclear operator. In 5.3 we prove our main theorem, relating the L-function of an overconvergent σ-module to the spectral theory of the associated Dwork operators. Finally, in 5.4 we give a concrete examples of overconvergent families over the punctured affine line.

5.1 Nuclear Operators

Let $\mathfrak{S}$ be a formal scheme, and let $\mathcal{S} = \mathfrak{S}^{\mathrm{rig}}$ denote the rigid analytic space associated to $\mathfrak{S}$. Throughout, we fix a $\mathcal{O}_{\mathfrak{S}}$-module M.

Definition 5.1.1. Let $\mathcal{U} \to \mathcal{S}$ be an open immersion. We say that an $\mathcal{O}_{\mathfrak{S}}$-linear operator $\psi : M \to M$ is *nuclear* over $\mathcal{U}$ if:

1. For every rigid point P of $\mathcal{U}$, the induced operator ψ_P on the vector space $V_P = k(P) \otimes M_P$ is a nuclear operator, in the sense of Monsky [18].

2. There is an analytic function $C(\psi, s)$ on $\mathbb{G}_{m,\mathcal{U}}$ with the interpolation property

$$C(\psi|\mathcal{U}, s)_P = C(\psi_P, s) \in k(P)[[s]].$$

In this case, the function $C(\psi|\mathcal{U}, s)$ is unique and we refer to it as the *Fredholm determinant* of ψ over $\mathcal{U}$.

The property of being a nuclear operator is evidently local on $\mathcal{S}$. We will say that an operator $\psi : M \to M$ is *nuclear* if ψ is nuclear over all of $\mathcal{S}$. In this case, we will denote by $C(\psi, s)$ its Fredholm determinant over $\mathcal{S}$.

Definition 5.1.2. Let $\psi : M \to M$ be a nuclear operator. The *spectral variety* of ψ is the hypersurface $\mathcal{E}(\psi)$ in $\mathbb{G}_{m,\mathcal{S}}$ cut out by the Fredholm determinant $C(\psi, s)$.

Let $\psi : M \to M$ be a nuclear operator. The spectral variety $\mathcal{E}(\psi)$ is a closed rigid analytic subspace of $\mathbb{G}_{m,\mathcal{S}}$ ([1], 4.8.29). In particular there is a morphism of rigid analytic spaces $\mathcal{E}(\psi) \to \mathcal{S}$. Let Q be a rigid point of $\mathcal{E}(\psi)$, and let P denote its image in $\mathcal{S}$. Then Q corresponds uniquely to an element in $E(\psi_P)$. We define the *slope* of Q to be $-v_P(\lambda)$, where λ is any eigenvalue of ψ_P representing Q. If P is any rigid point of $\mathcal{S}$, let $S_P(\psi)$ denote the set of slopes of the operator ψ_P. Here is our main question:

Question 5.1.3. *How does the slope set $S_P(\psi)$ vary with P?*

Let us conclude this section with an elementary slope estimate. Suppose that $\mathfrak{S} = \mathrm{Spf}(R)$, where R is a Noetherian adic ring admitting a principal ideal of definition πR. Let M be an $\mathcal{O}_{\mathfrak{S}}$-module and $\psi : M \to M$ a nuclear operator. The Fredholm determinant $C(\psi, s)$ has a unique representation as a power series

$$C(\psi, s) = \sum_j r_j s^j \in R[\tfrac{1}{\pi}][[s]].$$

Proposition 5.1.4. *For every rigid point P of R, the v_P-adic Newton polygon of $C(\psi_P, s)$ lies above $v_P(\pi)$ times the π-adic Newton polygon of $C(\psi, s)$.*

5.2 Dwork Operators

Let $(p : \mathfrak{X} \to \mathfrak{S}, \sigma)$ be an affine weak formal F-scheme. Let M be a finite $\mathcal{O}_{\mathfrak{X}}$-module, and let Θ be a Dwork operator on M. Our goal in this section is to prove that Θ induces a nuclear operator on the $\mathcal{O}_{\mathfrak{S}}$-module p_*M. As the question is local on $\mathcal{S} = \mathfrak{S}^{\text{rig}}$, we may assume that $\mathfrak{S} = \text{Spf}(R)$, where R is a Noetherian adic ring admitting a principal ideal of definition $\pi R \subset R$. By Proposition 3.4.4, we have $\mathfrak{X} = \text{Spwf}(A)$ for some w.c.f.g. algebra A/R.

Our first step will be to replace A with a suitable Monsky-Washnitzer algebra. Let $s_1, ..., s_n$ be weak generators for A, and let $B = R[X_1, ..., X_n]$. Then we have a natural map

$$B^\dagger \to A$$

sending $X_i \mapsto s_i$, which is surjective on rigid points. Thus if we regard M as a $B^\dagger$-module, there is a finite $B^\dagger$-submodule $M_0 \subseteq M$ which is dense in M. Moreover, for every rigid point P the map $(M_0)_P \to M_P$ is surjective. Choose a finite free resolution

$$F^\bullet \to M_0.$$

Note that $\sigma : B^\dagger \to B^\dagger$ is affine, and consequently σ_* is an exact functor. It follows that Θ prolongs to a map of complexes $\Theta_\bullet : F^\bullet \to F^\bullet$. In this case, we have:

Lemma 5.2.1. *Suppose that for each i, Θ_i is nuclear. Then Θ is nuclear, and*

$$C(\Theta, s) = \prod_i C(\Theta_i, s)^{(-1)^{i-1}}.$$

Proof. We need only check that for every rigid point P of R, that there is an equality

$$C(\Theta_P, s) = \prod_i C((\Theta_i)_P, s)^{(-1)^{i-1}}.$$

But this follows immediately from ([18], 1.4(2)). $\square$

In light of the lemma, we may assume that $B = A$ and that M is a finite free A-module. The remainder of this section will closely follow Monsky's proof that Dwork operators over a discrete valuation ring are nuclear [18]. Write $M = \bigoplus_i A$, and let $q_i : A \to M$ and $p_j : M \to A$ denote the ith canonical injection and that jth canonical projection, respectively. Note that M is a free R-module, with basis $e_{u,i} = q_i(X^u)$. For each $c > 0$, let $e_{u,i}^{(c)} = \pi^{\lfloor c/|u| \rfloor} e_{u,i}$, and let $M^{(c)}$ denote the free R-submodule of M with basis given by the $e_{u,i}^{(c)}$. Then $M = \bigcup_c M^{(c)}$.

We claim that for $c \gg 0$, Θ restricts to an endomorphism of $M^{(c)}$. Let us first decompose Θ as follows: for each $1 \leq i \leq n$, define

$$\Theta_{i,j} = p_j \circ \Theta \circ q_j.$$

Then $\Theta_{i,j}$ is a Dwork operator on A. For positive integer m and c, define the R-module

$$A^{m,c} = \left\{ \sum_j r_u X^u : |u| \leq m + c v_\pi(r_u) \right\}.$$

Observe that the Frobenius map σ defines an embedding $A^{m,c} \to A^{qm,qc}$. Moreover, we have the relations

$$\begin{aligned} A^{m,c} \cdot A^{m',c} &\subseteq A^{m+m',c} \\ \pi^j A^{m,c} &\subseteq A^{m-cj,c}. \end{aligned}$$

Lemma 5.2.2. *There exist positive integers r and c_0 such that for all $c \geq c_0$, and all i, j,* *and* m, $\Theta_{i,j}(A^{qm,qc}) \subseteq A^{m+r,c}$.

Proof. Recall that A is a finite free $\sigma(A)$-module, a basis being given by the monomials X^v where $0 \leq v_i < q$ for all i. It follows that for such v, we may define Dwork operators Θ_v on A via the relation

$$f = \sum_v \sigma(\Theta_v(f))X^v.$$

Note that for any $f \in A$, we have

$$\Theta_{i,j}(f) = \sum_v \Theta_v(f)\Theta_{i,j}(X^v).$$

Since the sum is finite, there exist r, c such that *all* $\Theta_{i,j}(X^v)$ lie in $A^{r,c}$. It suffices then to prove that for $c \gg 0$, we have $\Theta_v(A^{qm,qc}) \subseteq A^{m,c}$.

Suppose then that $f = \sum_u r_u X^u \in A^{qm,qc}$, so that $|u| \leq qm + qcv_\pi(r_u)$ for all u. For each u, write

$$u = qw(u) + v(u),$$

where $w(u)$ and $v(u)$ are multi-indices and each component of $v(u)$ is less than q. Now

$$X^u = \sigma(X^{w(u)})X^{v(u)} + (X^{qw(u)} - \sigma(X^{w(u)}))X^{v(u)}$$

Let c be such that $\sigma(X_i) \in A^{q,qc}$. Then $\sigma(X^{w(u)}) \in A^{q|w(u)|,qc} \subseteq A^{qm+qcv_\pi(r_u),qc}$, so that $\sigma(r_u X^{w(u)}) \in A^{qm,qc}$ and thus $X^{w(u)} \in A^{m,c}$. Similarly, we have that

$$r_u(X^{qw(u)} - \sigma(X^{w(u)}))X^{v(u)} \in A^{q(m+c),qc}.$$

Thus we obtain a decomposition $f = \sum_v \sigma(f_{v,0}) + \pi f'$, where $f_{v,0} \in A^{m,c}$ and $f' \in A^{q(m+c),qc}$. Replacing m with $m + c$ and applying the above procedure repeatedly, we obtain

$$f = \sum_v \sum_{j=0}^{\infty} \sigma(f_{v,j}) \pi^j X^v,$$

where each $f_{v,j} \in A^{m+cj,c}$. Thus

$$\Theta_v(f) = \sum_{j=0}^{\infty} f_{v,j} \pi^j \in A^{m,c}.$$

□

Lemma 5.2.3. *For all $c > c_0$, Θ restricts to an endomorphism of $V^{(c)}$.*

Proof. For any multi-index u, expand

$$\Theta_{i,j}(X^u) = \sum_v r_v X^v.$$

By Lemma 5.2.2, we have $\Theta_{i,j}(X^u) \in A^{|u|/q+r,d}$ for all $d \geq c_0$. Consequently

$$v_\pi(r_u) \geq \frac{|v| - |u|/q - r}{d}.$$

Now for any $c > c_0$, the coefficient of $e^{(c)}_{v,j}$ in $\Theta(e^{(c)}_{u,i})$ is simply $\pi^{(|u|-|v|)/c} r_v$, which has π-adic valuation

$$x \geq \frac{|v| - |u|/q - r}{d} + \frac{|u| - |v|}{c}.$$

Letting $d = c - 1$, we have $c/q \leq d < c$. Thus we see that

$$x \geq \frac{(q-1)|v| - qr}{q(c-1)}. \tag{5.1}$$

Now $x \geq 0$ for all but finitely many v, and consequently we see that Θ restricts to an endomorphism of $V^{(c)}$. □

Theorem 5.2.4. *Θ is a nuclear operator on M.*

Proof. For each $k > 0$ and each $c_0 < c \leq \infty$, let $M_k^{(c)}$ denote the reduction of $M^{(c)}$ modulo π^k. If Θ_k denotes the operator induced by Θ on $M_k^{(c)}$, then by the estimate (5.1) we see that Θ_k has finite image. If N is any finite direct summand of $M_k^{(c)}$ containing the image of Θ_k, then we may define the *Fredholm determinant* of Θ_k acting on $M_k^{(c)}$ to be the polynomial

$$C(\Theta_k|M_k^{(c)}, s) = \det(I - s\Theta_k|N) \in 1 + sR/\pi^k R[s],$$

which is easily seen to be *independent* of the choice of N. It follows that the polynomials $C(\Theta_k|M_k^{(c)}, s)$ are compatible with the projection maps $M_{k+1}^{(c)} \to M_k^{(c)}$. Taking the inverse limit, we obtain power series

$$C(\Theta|M^{(c)}, s) \in 1 + sR[[s]].$$

By ([13], §5), the above power series has an expansion

$$C(\Theta|M^{(c)}, s) = \sum_j c_j s^j,$$

where the coefficients c_j can be constructed explicitly as follows: Let I denote the set of pairs (i, u), where $1 \leq i \leq \operatorname{rank}_A(M)$, and u is a multi-index. For $c > c_0$, let $Q^{(c)}$ denote the matrix of Θ acting on $V^{(c)}$ with respect to the chosen basis $B^{(c)} = \{e_{i,u}^{(c)} : (i, u) \in I\}$. Then we have

$$c_j = (-1)^j \sum_{\substack{S \subseteq I \\ |S| = j}} \det(Q_S^{(c)})$$

where $Q_S^{(c)}$ is the finite submatrix of $Q^{(c)}$ corresponding to S. But for a fixed S, the $Q_S^{(c)}$ are *similar*, and so the power series

$$C(\Theta, s) = C(\Theta|M^{(c)}, s)$$

is *independent* of the choice of c.

To conclude the proof, note that all of the above holds after base change along a rigid point P of R. Thus for each rigid point P of R, the operator Θ_P acts as a nuclear operator on $V_P = k(P) \otimes_R V$, and the power series $C(\Theta, s)$ interpolates the individual Fredholm series $C(\Theta_P, s)$. $\square$

Let $Q^{(\infty)}$ denote the matrix of Θ acting on M, with respect to the chosen basis $\{e_{i,u} : (i, u) \in I\}$. Then for every finite set $S \subseteq I$, the matrix $Q_S^{(\infty)}$ is similar to $Q_S^{(c)}$ for $c \gg 0$. It follows that the power series $C(\Theta, s)$ lies in $1 + sR[[s]]$. We define the *trace* of Θ to be -1 times the coefficient of s in $C(\Theta, s)$. Note that for each $i \geq 1$, the operator Θ^i is nuclear and that we have the usual relation

$$C(\Theta, s) = \exp\left(-\sum_{i=1}^{\infty} \mathrm{Tr}(\Theta^i)\frac{s^i}{i}\right).$$

5.3 The Trace Formula

Suppose again that we are in Situation 4.1.3 and that $X_0/\mathbb{F}_q$ is a smooth affine variety. Let (M, ϕ) be a σ-module over $\mathfrak{X}/\mathfrak{S}$, and recall that we have constructed a sequence of Dwork operators $\theta_i(\phi)$ on the de Rham complex $\Omega^i M^\vee$ (5.2). According to the results of the previous section, each $\theta_i(\phi)$ is a nuclear operator, and so we may consider the characteristic series $C(\theta_i(\phi), s)$, which is a global analytic function on $\mathbb{G}_{m,\mathcal{S}}$.

Theorem 5.3.1. *For any σ-module as above, we have a meromorphic continuation*

$$L(\phi, s) = \prod_i C(\theta_i(\phi), s)^{(-1)^{i-1}}. \tag{5.2}$$

Proof. Suppose that P is a rigid point of R and that R_P is a ring of characteristic 0. The Monsky trace formula ([22], 3.1) says precisely that we have an equality

$$L(\phi, s)_P = \prod_i C(\theta_i(\phi), s)_P^{(-1)^{i-1}}.$$

Thus the theorem holds over any admissible open subset of $\mathcal{S}$ which does not intersect the closed rigid subspace cut out by the global section p.

Suppose then that P is a rigid point of characteristic p, and choose an admissible open neighborhood $\mathcal{U}$ of P of the form $\mathrm{Spf}(R)^{\mathrm{rig}}$, where R is a Noetherian adic ring admitting a principal ideal of definition $(\pi) \subset R$. Let F denote the *difference* of the two sides of (5.2). Then the restriction of F over $\mathcal{U}$ has an expression of the form

$$F(s) = \sum_{i=-\infty}^{\infty} r_i s^i \in R[\tfrac{1}{\pi}][[s, s^{-1}]].$$

It suffices to show that each coefficient of $F(s)$ vanishes. But we have already shown that each coefficient vanishes on the open subscheme of $\mathrm{Spec}(R[\frac{1}{\pi}])$ on which p is invertible. It follows that each coefficient must be identically zero, completing the proof. □

Suppose for example that R is a rigid point. Then Theorem 5.3.1 reduces simply to Monsky's trace formula. If (M, ϕ) is a unit-root σ-module, then it is known that the completion M^∞ is equipped with a natural convergent connection ∇ commuting with ϕ. Suppose moreover that ∇ descends to a connection on M. Then the cohomology of this connection is the *rigid cohomology* with coefficients in M. In this case, Theorem 5.3.1 reduces to the well known Lefschetz trace formula for the rigid cohomology of an affine variety [8]. In general, we cannot expect that our families of representations are equipped with such an overconvergent connection. However, it would be interesting to know if there exists a similar *chain-level* trace formula generalizing Theorem 5.3.1 to non-affine weak formal schemes. Such a formula should take into account the action of $\theta_i(\phi)$ on the higher cohomology $R^j p_* \Omega^i M^\vee$ and should induce—via the degeneration of the Hodge-de Rham spectral sequence—the Lefschetz trace formula for rigid cohomology in the above special case.

5.4 Example: Artin-Schreier-Witt Families

We now give a concrete application of the theory to the study the Artin-Schreier-Witt families of representations over the punctured line $X = \mathrm{Spec}(\mathbb{F}_q[t, t^{-1}])$. In particular, we construct a much larger class of overconvergent examples than those considered by Davis, Wan, and Xiao [6]. This section may be regarded as a "preview" of forthcoming work with Joe Kramer-Miller, where we study overconvergent Artin-Schreier-Witt families over an arbitrary smooth affine curve.

Let $A = \mathbb{Z}_q[t, t^{-1}]$, and let $\mathbb{X} = \mathrm{Spec}(A)$ denote the "obvious" lifting of X to characteristic 0. We choose the lifting σ of the absolute Frobenius defined by $t \mapsto t^p$. The pair $(\mathbb{X}^\infty, \sigma)$ is therefore a formal F-scheme over $\mathbb{Z}_p$. Define the $\mathbb{Z}_p$-module endomorphism

$$\wp = \sigma - \mathrm{id} : A^\infty \to A^\infty.$$

In this setting, we have the following analogue of Katz' correspondence for étale $\mathbb{Z}_p$-coverings of X:

Theorem 5.4.1. *There is an isomorphism of $\mathbb{Z}_p$-modules*

$$A^\infty / \wp A^\infty \to \mathrm{Hom}(\pi_1(X, \bar{x}), \mathbb{Z}_p).$$

Proof. Let $K = \mathbb{F}_q(t)$ denote the function field of X, and let G be the maximal pro-p quotient of the absolute Galois group of K. Recall that the Witt ring $W(K)$ is equipped with a canonical lifting of Frobenius F. For any element $a \in K$, let $[a]$ denote its Teichmüller representative in $W(K)$. Fix an element $\alpha \in \mathbb{F}_q$ with $\mathrm{N}_{\mathbb{F}_q/\mathbb{F}_p}(\alpha) = 1$. Consider the $\mathbb{Z}_p$-module endomorphism $\wp = F - \mathrm{id}$ of $W(K)$. There is a canonical isomorphism of $\mathbb{Z}_p$-modules ([15], 2.2)

$$W(K)/\wp W(K) \to \mathrm{Hom}(G, \mathbb{Z}_p).$$

Recall that the $\mathbb{Z}_p$-module $\mathrm{Hom}(\pi_1(X, \bar{x}), \mathbb{Z}_p)$ is the submodule of $\mathrm{Hom}(G, \mathbb{Z}_p)$ consisting of those homomorphisms which ramify only at t and t^{-1}. Let H denote the corresponding submodule of $W(K)/\wp W(K)$. By ([15], 4.8), each element of H has a unique representative in $W(K)$ of the form

$$\sum_{i=-\infty}^{\infty} c_i [t]^i,$$

where $c_i \in \mathbb{Z}_q$ tend to 0 as $|i| \to \infty$, and c_0 is of the form $c[\alpha]$ for some $c \in \mathbb{Z}_p$.

It is straightforward to see that every element of A^∞ has a unique representative modulo $\wp A^\infty$ of the form

$$\sum_{i=-\infty}^{\infty} c_i t^i,$$

where $c_i \in \mathbb{Z}_q$ tend to 0 as $|i| \to \infty$, and c_0 is of the form $c[\alpha]$ for some $c \in \mathbb{Z}_p$. Our choice of σ determines a unique Frobenius-compatible homomorphism

$$A^\infty \to W(K),$$

which is given explicitly by $t \mapsto [t]$. But by the above discussion, we see that the induced map

$$A^\infty/\wp A^\infty \to W(K)/\wp W(K)$$

is injective, and its image is precisely H. $\square$

For an element $f \in A^\infty$, we will let $\alpha_f : \pi_1(X, \bar{x}) \to \mathbb{Z}_p$ denote the corresponding map. Let $\Lambda = \mathbb{Z}_p[[T]]$, which is a 2-dimensional local ring with maximal ideal $\mathfrak{m} = (p, T)$ and residue field $\mathbb{F}_p$. We regard Λ as the completed group algebra of $\mathbb{Z}_p$ via the continuous character $\mathbb{Z}_p \to R^\times$ sending $1 \mapsto 1 + T$. The composition

$$\rho_f : \pi_1(X, \bar{x}) \xrightarrow{\alpha_f} \mathbb{Z}_p \to \Lambda^\times$$

is therefore a family of representations which we refer to as the *Artin-Schreier-Witt family* associated to f. Let $\mathcal{W} = \mathrm{Spf}(\Lambda)^{\mathrm{rig}}$ denote the "parameter space" for this family. Our goal in this section is to prove that for any $f \in A^\dagger$, the family ρ_f becomes overconvergent after removing a sufficiently large disk around the rigid point $T = 0$.

Let $\mathbb{X}_\Lambda$ denote the base change of $\mathbb{X}$ along $\mathbb{Z}_p \to \Lambda$, and let (M_f, ϕ_f) denote the σ-module on $\mathbb{X}_\Lambda^\infty$ corresponding to ρ_f. We will begin by determining explicitly the σ-module structure of (M_f, ϕ_f). Recall that the *Artin-Hasse exponential series* is the power series

$$E(s) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right) \in 1 + s + s^2\mathbb{Z}_p[[s]].$$

One sees easily that $E(s)$ defines a bijection $\mathfrak{m} \to 1 + \mathfrak{m}$. In particular, there is a unique $\pi \in \mathfrak{m}$ such that $E(\pi) = 1 + T$.

Definition 5.4.2. The *Artin-Hasse exponential map* is the unique $\mathbb{Z}_p$-module homomorphism

$$A^{\infty} \to 1 + \pi A^{\infty}_{\Lambda}[[\pi]] \subseteq A^{\infty}_{\Lambda}$$

sending $[c]t^i \mapsto E([c]\pi t^i)$.

For any $f \in A^{\infty}$, let E_f denote the Artin-Hasse exponential of f. We regard E_f as an endomorphism of A^{∞}_{Λ} by left multiplication.

Proposition 5.4.3. *There is an isomorphism of σ-modules*

$$(M_f, \phi_f) \cong (A^{\infty}_{\Lambda}, E_f \circ \sigma).$$

Let $f \in A^{\infty}$. Note that f admits a decomposition of the form $f = f_0(t) + f_{\infty}(t^{-1})$, where $f_0, f_{\infty} \in \mathbb{Z}_q\langle t\rangle$. Since the Artin-Hasse exponential is additive, we see that $E_f \in A^{\dagger}_R$ if both $E_{f_0}, E_{f_{\infty}} \in A^{\dagger}_R$. Thus for notational convenience, we will assume that $f = f_0$ is of the form

$$f(t) = \sum_{i=0}^{\infty} f_i t^i.$$

where $f_i \in \mathbb{Z}_q$ tend to 0 as $i \to \infty$. For each i, consider the Teichmüller expansion $f_i = \sum_{j=0}^{\infty}[f_{i,j}]p^j$, where $f_{i,j} \in \mathbb{F}_q$. Since f is a convergent power series in t, we can uniquely write

$$f = \sum_{j=0}^{\infty} F_j(t) p^j,$$

where the coefficients are polynomials

$$F_j(t) = \sum_{i=0}^{d_j} [f_{i,j}] t^i$$

with $f_{d_j,j} \neq 0$. Now suppose that $f \in A^\dagger$. In other words,

$$\delta_f = \lim_{k\to\infty} \inf_{i>k} \frac{v_p(f_i)}{i} > 0.$$

Our first step will be to relate the rate of overconvergence to the integers d_j.

Lemma 5.4.4. *For $j \gg 0$, $d_j \leq j/\delta_j$.*

Proof. Note that (d_j) is bounded above by the increasing sequence

$$q_j = \max\{i : v_p(f_i) < j\}.$$

Clearly we have $v_p(f_{q_j}) < j$ for all j. If (q_j) is bounded, then we are done. Otherwise, choose some $0 < \delta < \delta_f$. By the definition of δ_f, there exists some $i_0 > 0$ such that for all $i > i_0$,

$$v_p(f_i) > i\delta.$$

Since (q_j) is not bounded, there exists some j_0 such that for all $j > j_0$, we have $q_j > i_0$. Thus we have

$$j > v_p(f_{q_j}) > q_j\delta \geq d_j\delta.$$

Letting $\delta \to \delta_f^-$, we obtain the result. □

Now for each $j \geq 0$, there exists a unique $\pi_j \in \mathfrak{m}$ such that $E(\pi_j) = (1+T)^{p^j}$. One easily sees that $\pi_j \in p^j\pi(1+\pi R)$. By definition, we have

$$E_f(t) = \prod_{j=0}^{\infty}\prod_{i=0}^{d_j} E([f_{i,j}]\pi_j t^i).$$

For a fixed value of j, expand

$$\prod_{i=0}^{d_j} E([f_{i,j}]\pi_j t^i) = \sum_{i=0}^{\infty} c_{i,j} t^i.$$

Then by a standard argument, for $i > 0$ we have $v_{\pi_j}(c_{i,j}) \geq i/d_j$.

Theorem 5.4.5. *The family ρ_f is overconvergent.*

Proof. First, observe that the specialization of ρ_f along the rigid point $T = 0$ is the trivial representation, which is overconvergent since σ is an overconvergent lifting of Frobenius. It suffices to prove that ρ_f is overconvergent over the "punctured" space $\mathcal{W}^\circ$ obtained by removing this point. Explicitly, $\mathcal{W}^\circ$ is a quasi-separated rigid analytic space admitting an admissible covering by spaces $\mathcal{W}^n$, where $\mathcal{W}^n$ is the rigid analytic space associated to the adic R-algebra

$$R_n = R[p^n T^{-1}].$$

Note in particular that $\mathfrak{m}R_n = TR_n$, and $v_T(p) = 1/n$ in this ring. We will show that the restriction $\rho_f|_{\mathcal{W}^n}$ is overconvergent for all n. In the above notation, we have for a fixed j that

$$v_T(c_{i,j}) = v_T(\pi_j) v_{\pi_j}(c_{i,j}) \geq (1 + j v_T(p)) \frac{i}{d_j} \geq \left(1 + \frac{j}{n}\right) \frac{i\delta_f}{j} > \frac{i\delta_f}{n}.$$

Taking the product over all j, we see that E_f is overconvergent over $\mathcal{W}^n$, thus proving the theorem. ∎

Bibliography

[1] A. Abbes. *Éléments de Géométrie Rigide*. Springer Science & Business Media, 2010.

[2] F. Andreatta, A. Iovita, and V. Pilloni. Le halo spectral. *Annales Scientifiques de l'École Normale Supérieure*, 54:603–655, 2018.

[3] N. Bourbaki. *Éléments de Mathématique. Algèbre commutative*. Hermann, Paris, 1961.

[4] R. F. Coleman. p-adic banach spaces and families of modular forms. *Inventiones Mathematicae*, 127:417–479, 1997.

[5] R. Crew. L-functions of p-adic characters and geometric Iwasawa theory. *Inventiones Mathematicae*, 88:395–404, 1987.

[6] C. Davis, D. Wan, and L. Xiao. Newton slopes for Artin-Schreier-Witt towers. *Math. Ann.*, 364(3-4):1451–1468, 2016.

[7] J. Dieudonné and A. Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961–1967.

[8] J.-Y. Etesse and B. Le Stum. Fonctions L asociées aux F-isocristaux surconvergents I. *Mathematische Annalen*, 296:557–576, 1993.

[9] W. Fulton. A note on weakly complete algebras. *Bulletin of the American Mathematical Society*, 75(3):591–593, 1969.

[10] E. Große-Klönne. On families of pure slope l-functions. *Documenta Mathematica*, 8, 08 2003.

[11] A. Grothendieck, M. Raynaud, et al. *Revêtements Étales et Groupe Fondamental (SGA 1)*, volume 224 of *Lecture Notes in Mathematics*. Société Mathématique de France, 2003.

[12] J.-I. Igusa. On the algebraic theory of elliptic modular functions. *J. Math. Soc. Japan*, 20(1-2):96–106, 04 1968.

[13] S. Jean-Pierre. Completely continuous endomorphisms of p-adic banach spaces. *Publ. Math. I.H.E.S*, 12:69–85, 1962.

[14] N. M. Katz. p-adic properties of modular schemes and modular forms. In W. Kuijk and J.-P. Serre, editors, *Modular Functions in One Variable III*, volume 350. Springer, Berlin, Heidelberg, 1973.

[15] M. Kosters and D. Wan. Genus growth in $\mathbb{Z}_p$-towers of function fields. *Proceedings of the American Mathematical Society*, 146:1481–1494, 2017.

[16] R. Liu, D. Wan, and L. Xiao. The eigencurve over the boundary of weight space. *Duke Journal of Mathematics*, 166(9):1739–1787, 2017.

[17] D. Meredith. Weak formal schemes. *Nagoya Mathematical Journal*, 45:1–38, 1972.

[18] P. Monsky. Formal cohomology III. *Annals of Mathematics*, 88(2):181–217, 1968.

[19] P. Monsky and G. Washnitzer. Formal cohomology. *Annals of Mathematics*, 93(2):315–343, 1971.

[20] M. Raynaud. Géométrie analytique rigide d'après Tate, Kiehl... *Mémoires de la Société Mathématique de France*, 39-40:319–327, 1974.

[21] D. Wan. Meromorphic continuation of L-functions of p-adic representations. *Annals of Mathematics. Second Series*, 143:469–498, 05 1996.

[22] D. Wan. Higher-rank case of Dwork's conjecture. *Journal of the American Mathematical Society*, 13(4):807–852, 2000.

[23] D. Wan. Class numbers and p-ranks in $\mathbb{Z}_p^d$-towers. *Journal of Number Theory*, 203:139 – 154, 2019.

Appendix A

Constructions for Weak Formal Schemes

In this appendix we give various applications of Theorem 3.4.7 to basic constructions for weak formal schemes.

A.1 Relative Spwf

For applications it will be convenient to have a description of affine weak formal schemes analogous to the relative Spec construction for ordinary schemes. Let R be a Noetherian adic ring, and let A be a Noetherian weakly finitely generated algebra over R. Define a sheaf $\tilde{A}^\dagger$ on the distinguished open subsets of $\mathfrak{S} = \mathrm{Spf}(R)$ via

$$\tilde{A}^\dagger(\mathfrak{S}_f) = A^\dagger_f.$$

Note that $\tilde{A}^\dagger$ is a sheaf: it agrees with the pushforward of the structure sheaf of $\mathrm{Spwf}(A)$ along the natural map $\mathrm{Spwf}(A) \to \mathrm{Spf}(R)$.

Definition A.1.1. Let $\mathfrak{S}$ be a formal scheme, and let $\mathcal{A}$ be an $\mathcal{O}_{\mathfrak{S}}$-algebra. We say that $\mathcal{A}$ is a *w.c.f.g. algebra* if $\mathfrak{S}$ admits a covering $\{\mathfrak{S}_i \to \mathfrak{S}\}$ with $\mathfrak{S}_i = \mathrm{Spf}(R_i)$ such that for each i, $\mathcal{A}|_{\mathfrak{S}_i} = \tilde{A}_i^\dagger$ for some w.c.f.g. algebra A_i over R_i.

By Proposition 3.4.2, if $\pi : \mathfrak{X} \to \mathfrak{S}$ is an affine weak formal scheme then $\pi_* \mathcal{O}_{\mathfrak{X}}$ is a w.c.f.g. algebra. As the following lemma indicates, the property of being a w.c.f.g. algebra is local on the base:

Lemma A.1.2. *Let $\mathcal{A}$ be a w.c.f.g. algebra over $\mathfrak{S}$, and let $\mathfrak{S}' \to \mathfrak{S}$ be an open immersion. Then $\mathcal{A}|_{\mathfrak{S}'}$ is a w.c.f.g. algebra over $\mathfrak{S}'$.*

Proof. We proceed as in Lemma 3.4.2. Choose a covering $\{\mathfrak{S}_i \to \mathfrak{S}\}$ with $\mathfrak{S}_i = \mathrm{Spf}(R_i)$ such that for each i, $\mathcal{A}_i = \mathcal{A}|_{\mathfrak{X}_i}$ is of the form $\tilde{A}_i^\dagger$ for some w.c.f.g. algebra A_i over R_i. Let $\mathfrak{S}_i' = \mathfrak{S}_i \cap \mathfrak{S}'$, and $\mathcal{A}_i' = \mathcal{A}|_{\mathfrak{S}_i'}$. Since $\mathfrak{S}_i'$ is an open subset of $\mathrm{Spf}(R_i)$, there exist $f_j \in R_i$ such that

$$\mathfrak{S}_i' = \bigcup_j \mathrm{Spf}(R_{i,j}),$$

where $R_{i,j} = (R_i)_{f_j}$. Let $A_{i,j} = (A_i)_{f_j}^\dagger$. Then each $A_{i,j}$ is a w.c.f.g. algebra over $R_{i,j}$, and moreover $\mathcal{A}|_{\mathrm{Spf}(R_{i,j})} = \tilde{A}_{i,j}^\dagger$, completing the proof. □

Proposition A.1.3. *Let $\mathcal{A}$ be an $\mathcal{O}_{\mathfrak{S}}$-algebra. The following are equivalent:*

1. *$\mathcal{A}$ is a w.c.f.g. algebra*

2. *For every affine open $\mathfrak{S}' = \mathrm{Spf}(R)$ of $\mathfrak{S}$, $\mathcal{A}|_{\mathfrak{S}'} = \tilde{A}^\dagger$ for some w.c.f.g. algebra A over R.*

Proof. Evidently (2) implies (1). Suppose then that $\mathcal{A}$ is a w.c.f.g. algebra over $\mathfrak{S}$, and let $\mathfrak{S}' = \mathrm{Spf}(R)$ be an open affine in $\mathfrak{S}$. Let $A = \Gamma(\mathfrak{S}', \mathcal{A})$, and define $\mathcal{B} = \tilde{A}^{\dagger}$. Note that there is a natural map $\mathcal{B} \to \mathcal{A}|_{\mathfrak{S}'}$. Arguing as in Lemma A.1, we see that this map is an isomorphism on some affine open cover of $\mathfrak{S}'$, hence $\mathcal{B} \cong \mathcal{A}|_{\mathfrak{S}'}$ as desired. □

Theorem A.1.4. *Let $\mathfrak{S}$ be an object of FS_+, and let $\mathcal{A}$ be a w.c.f.g. algebra on $\mathfrak{S}$. Consider the functor $F : \mathrm{FS}^{\dagger}_{\mathfrak{S}} \to \mathrm{Set}$ which assigns to a weak formal scheme $\pi : \mathfrak{X} \to \mathfrak{S}$ the set of $\mathcal{O}_{\mathfrak{X}}$-algebra maps*

$$\pi^*\mathcal{A} \to \mathcal{O}_{\mathfrak{X}}.$$

Then F is representable by a weak formal scheme over $\mathfrak{S}$ which we denote by $\mathrm{Spwf}(\mathcal{A})$.

Proof. Choose an affine open covering $\{\mathrm{Spf}(R_i) \to \mathfrak{S}\}_i$, so that $\mathcal{A}|_{\mathrm{Spf}(R_i)} = \tilde{A}_i^{\dagger}$ for some w.c.f.g. algebra A_i/R_i. Define $\mathfrak{X}_i = \mathrm{Spwf}(A_i)$. Then we have a natural map $\pi^*A \to \mathcal{O}_{\mathfrak{X}}$, which correspond to maps $\mathfrak{X}_i \to F$ collectively covering F. We must show that $\mathfrak{X}_i \to F$ is open. Suppose then that we have a weak formal scheme $\mathfrak{X}'$ over $\mathfrak{S}$ and a map $\pi^*A \to \mathcal{O}'_{\mathfrak{X}}$. Define $\mathfrak{X}'_i = p^{-1}\mathrm{Spf}(R_i)$. Then we see easily that the diagram

$$\begin{array}{ccc} \mathfrak{X}'_i & \longrightarrow & \mathfrak{X}_i \\ \downarrow & & \downarrow \\ \mathfrak{X}' & \longrightarrow & F \end{array}$$

□

is Cartesian, completing the proof.

A.2 Weak Completion

Let $\mathbb{S}$ be an adic Noetherian scheme, and let $\mathfrak{S} = \mathbb{S}^\infty$. Suppose that $\mathbb{X}$ is a scheme of finite presentation over $\mathbb{S}$. We will now use Theorem 3.4.7 to construct a weak formal scheme $\mathbb{X}^\dagger/\mathfrak{S}$ which we call the *weak completion* of $\mathbb{X}$.

Consider the functor F on $\mathrm{FS}^\dagger_{\mathfrak{S}}$ which sends a weak formal scheme $\mathfrak{X}$ to the set of $\mathbb{S}$-morphisms of adic ringed spaces $\mathfrak{X} \to \mathbb{X}$. Choose a covering $\{\mathbb{X}_i \to \mathbb{X}\}$ where $\mathbb{X}_i = \mathrm{Spec}(\mathcal{A}_i)$ is affine over $\mathbb{S}$. Then $\mathcal{A}^\dagger$ is a w.c.f.g. algebra over $\mathfrak{S}$, and we define $\mathbb{X}_i^\dagger = \mathrm{Spwf}(\mathcal{A}_i)$. Then $\mathbb{X}_i^\dagger$ represents the functor which sends a weak formal scheme $\mathfrak{X}$ to the set of $\mathfrak{S}$-morphisms $\mathfrak{X} \to \mathbb{X}_i$. In particular, each $\mathbb{X}_i^\dagger$ is a subfunctor of F and the $\mathbb{X}_i^\dagger$ collectively cover F.

To see that $\mathbb{X}_i^\dagger$ is open in F, suppose that we have a map $\mathfrak{X} \to F$. Composing with the natural $\mathbb{S}$-morphism $F \to \mathbb{X}$, we obtain an $\mathbb{S}$-morphism $f : \mathfrak{X} \to \mathbb{X}$. Define $\mathfrak{X}_i = f^{-1}(\mathbb{X}_i)$. Then we have a Cartesian square

$$\begin{array}{ccc} \mathfrak{X}_i & \longrightarrow & \mathbb{X}_i^\dagger \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & F \end{array} .$$

Thus $\mathfrak{X}_i$ is open in F, and by Theorem 3.4.7 we obtain the following:

Theorem A.2.1. *There exists a unique weak formal scheme $\mathbb{X}^\dagger/\mathfrak{S}$ with the following universal property: For every $\mathbb{S}$-morphism $\mathfrak{X} \to \mathbb{S}$, where $\mathfrak{X}$ is a weak formal scheme over $\mathfrak{S}$, there exists a unique $\mathfrak{S}$-morphism $\mathfrak{X} \to \mathbb{X}^\dagger$ making the following diagram commute:*

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathbb{X}^\dagger \\ & \searrow & \downarrow \\ & & \mathbb{X} \end{array} .$$