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Section 1. Introduction

The outstanding feature in the development of nucleon-nucleon scattering experiments since 1953 is the use of polarized beams. Initial nucleon-nucleon polarization experiments were reported in 1954 by Oxley, Cartwright and Rouvina and two years later triple scattering experiments were performed by Chamberlain, Segrè, Tripp, Wiegand and Ypsilantis. These, and subsequent similar experiments, have been the decisive factors in the comparative successes of recent phenomenological analyses and will accordingly receive a major emphasis in this article.

In typical polarization experiments the internal proton beam of a cyclotron is scattered from a target such as carbon and the scattered protons are found to have their spins partially aligned in a direction normal to the scattering plane, with generally more having spin up than down for a left scattering. A beam such as this, in which the spins are partially aligned, is said to be polarized, and if it is caused to strike a (second) target of, say, liquid hydrogen the second scattering cross section generally exhibits an azimuthal asymmetry. The measurement of this asymmetry is the object of the so-called double-scattering polarization experiments. In the more

complicated triple-scattering experiments the protons emerging from the second collision are allowed to scatter still again and the asymmetry after this third scattering process is measured. In variations of the experiment magnetic fields may be interposed between the various scatterings.

In these experiments the final measured asymmetry depends upon the properties of the various interactions involved and upon the geometric configuration. It is the object of the formalism discussed in this part to exhibit in a simple way the dependence of the observed quantities on the characteristic parameters describing these two factors. A nonrelativistic treatment is given first, but this is later extended to the relativistic case. At the outset the two nucleons are considered distinguishable with the effects of undistinguishability being brought in later.

The discussion is based upon the use of the density matrix, which was introduced in this connection by Wolfenstein & Ashkin (1) and by Dalitz (2). This device greatly simplifies the analysis, both mathematically and conceptually, and is the basis of all contemporary work in the field. The next section is devoted to a description of the density matrix formalism in nucleon-nucleon scattering.

Section 2. Statistical Mixtures and the Density Matrix

The spin vector of a Pauli particle is defined here as the expectation value of the Pauli spin operator $\underline{\sigma} = i\underline{\sigma}_x + j\underline{\sigma}_y + k\underline{\sigma}_z$. Thus if an arbitrary normalized spin wave function is written in the form $[\cos \frac{\theta}{2} \exp(i\alpha/2), \sin \frac{\theta}{2} \exp(i\beta/2)]$ the spin vector is a unit vector with polar angle θ and

azimuthal angle $\phi = (\alpha - \beta)/2$. It is important, however, that the particles injected into or emerging from a cyclotron are generally not all in the same quantum state. The various individual spin vectors therefore have different directions, generally, and the expectation value of the spin operator averaged over the particles of the beam will be a vector of length less than unity. This average over the particles of the beam of the individual unit spin vectors is called the polarization vector of the beam.

In polarization experiments this beam polarization is the central object; the measured quantities are directly related to it. Although it is possible to carry out calculations for each individual quantum state and then to perform (classical)averages over the various particles, it is much easier to deal directly with statistical averages over the particles of the beam. The method is described in this section.

If the fraction of beam particles in the pure quantum state $|\psi_i\rangle$ is f_i , the beam expectation value of an operator A is

$$\langle A \rangle = \sum f_i \langle \psi_i | A | \psi_i \rangle . \qquad (2.1)$$

Often a bar is placed over the quantity on the left-hand side to signify that it is an average over individual ~~expectation values~~ quantum states. It is convenient to re-express 2.1 in the form

$$\langle A \rangle = \mathrm{Tr}\, \rho A , \qquad 2.2.$$

where Tr means trace and ρ is a Hermitian matrix called the density matrix.

As is apparent from 2.1 and 2.2, the density matrix ρ can be represented in the form

$$\rho = \sum f_i P_i ,$$

where P_i is a projection operator which is unity when acting on the state $|\psi_i\rangle$ and zero when acting on states orthogonal to $|\psi_i\rangle$.

From a more abstract point of view the fact that 2.1 can be cast in the form 2.2 can be deduced as follows: The physical state of a system is determined by the expectation values of a complete set of Hermitian operators. In an N-dimensional space there are N^2 independent Hermitian operators. The corresponding N^2 equations of the form 2.2 determine completely the Hermitian matrix ρ, and by linearity this equation then determines the expectation values of all other operators.

In treating scattering experiments it is important that the density matrix can be used to characterize not only a collection of particles but also a single particle. For example, in a system of two particles the wave function may take the form

$$\psi(x_1, x_2) = a_1 \phi_1(x_1)\psi_1(x_2) + a_2 \phi_2(x_1)\psi_2(x_2) + \cdots .$$

If the states $\psi_i(x_2)$ are nonoverlapping (i.e., orthogonal) then the contributions from the various $\phi_i(x_1)$ cannot interfere and must be combined according to classical statistics. The density matrix is therefore appropriate for describing the first particle alone if information regarding the second particle is unavailable. If the states of the second particle are partially overlapping the situation is more

complex, but the first particle is still characterized by some density matrix, as the abstract argument shows.

The density matrix defined above characterizes a system at a given time. As time proceeds the density matrix changes--at least in the Schroedinger representation, which is used throughout. In particular, if the states before and after a scattering process are related by $\psi_f = S\psi_i$, then the density matrices characterizing the system before and after the scattering are related by

$$\rho_f = S\rho_i S^* , \qquad 2.3$$

as is seen from Eqs. 2.1 and 2.2. For scattering experiments in which the final particles are counted outside the unscattered beam, the incident beam must be subtracted out and one uses, instead of 2.3,

$$\rho_{sc} = R\rho_i R^* , \qquad 2.4$$

where $R = S - 1$. The ρ_{sc} defined in this way describes the scattered wave. Equations 2.2 and 2.4 are the two fundamental equations for treating the scattering of statistical mixtures; 2.2 applied to both ρ_i and ρ_{sc} determines the connection to experiment, and 2.4 gives the dynamical content.

In scattering experiments the momentum can usually be considered well defined. Consequently, the operators A used in Eq. 2.2 can be considered the product of an operator A_s in spin space times an operator $\underline{P}(\underline{k})$ that projects onto a state of relative momentum $\underline{k}$. According to 2.2 the expectation value of a spin operator, A_s, in this momentum state is

$$\langle A_s \rangle_{\underline{k}} \equiv \frac{\langle P(\underline{k})\, A_s \rangle}{\langle P(\underline{k}) \rangle} = \frac{\mathrm{Tr}\, \rho(\underline{k})\, A_s}{\mathrm{Tr}\, \rho(\underline{k})} , \qquad 2.2'$$

where the denominator $\langle P(\underline{k}) \rangle$ is the probability that the particle is in the momentum state $\underline{k}$, and the spin-space density matrix $\rho(\underline{k})$ is some as yet arbitrary multiple of $\langle \underline{k} \,|\, \rho \,|\, \underline{k} \rangle$, the diagonal momentum-space matrix element of ρ. In 2.2' the trace is over spin states only, the sum over momentum states present in 2.2 having been cancelled by the momentum-state projection operator $P(\underline{k})$.

Aside from the undetermined normalization Eq. 2.2' can be considered the definition of the spin density matrix $\rho(\underline{k})$. Indeed, if one introduces in the four-dimensional spin space of the two Pauli particles the sixteen independent matrixes $\sigma_\mu^{(1)}\, \sigma_\nu^{(2)}$ defined by

$$\langle m, n \,|\, \sigma_\mu^{(1)}\, \sigma_\nu^{(2)} \,|\, m', n' \rangle = (\sigma_\mu)_{mm'}\, (\sigma_\nu)_{nn'} , \qquad 2.5.$$

where m and n specify the spin states of the first and second particles respectively, and σ_1, σ_2, σ_3, and σ_0 are the Pauli spin matrices and the Pauli unit matrix, one obtains, using the orthogonality relations

$$\frac{1}{4}\, \mathrm{Tr}\, \sigma_\mu^{(1)}\, \sigma_\nu^{(2)}\, \sigma_\lambda^{(1)}\, \sigma_\rho^{(2)} = \delta_{\mu\lambda}\, \delta_{\nu\rho} , \qquad 2.6.$$

the inverse of 2.2' :

$$\rho(\underline{k}) = \frac{1}{4}\, \mathrm{Tr}\, \rho(\underline{k}) \sum_{\mu,\nu} \left\langle \sigma_\mu^{(1)}\, \sigma_\nu^{(2)} \right\rangle_{\underline{k}} \sigma_\mu^{(1)}\, \sigma_\nu^{(2)} . \qquad 2.7.$$

In this form the spin density matrix is expressed directly in terms of $\mathrm{Tr}\ \rho\ (\underline{k})$ and the sixteen expectation values $\left\langle \sigma_\mu^{(1)}\ \sigma_\nu^{(2)} \right\rangle_{\underline{k}}$. Six of these,

$$\left\langle \sigma_i^{(1)}\ \sigma_0^{(2)} \right\rangle_{\underline{k}} \equiv \left\langle \sigma_i^{(1)} \right\rangle_{\underline{k}} \quad \text{and} \quad \left\langle \sigma_0^{(1)}\ \sigma_j^{(2)} \right\rangle_{\underline{k}} \equiv \left\langle \sigma_j^{(2)} \right\rangle_{\underline{k}},$$

are the components of the polarization vectors for the first and second particles respectively, and $\left\langle \sigma_i^{(1)}\ \sigma_j^{(2)} \right\rangle_{\underline{k}} \equiv C_{ij}\ (\underline{k})$ are nine parameters related to a correlation between spin expectations in the two beams. These fifteen parameters can be considered as specifying the spin state of the combined two-particle system.

Turning to the dynamical side, the momentum-space matrix element $\left\langle \underline{k} \mid R \mid \underline{k}' \right\rangle$ is a matrix in spin space whose elements, aside from a normalization factor, are the scattering amplitudes for individual initial and final spin states. In the description of polarization phenomena it is convenient to incorporate this normalization factor and deal also with the spin matrix $M\ (\underline{k}\ ,\ \underline{k}')$ whose matrix elements are exactly the scattering amplitudes in various final spin states for fixed initial spin states. In terms of $M(\underline{k}\ ,\ \underline{k}')$ and the $\rho\ (\underline{k})$ defined in 2.2' Eq. 2.4 becomes

$$\rho_{sc}\ (\underline{k}) = M(\underline{k}, \underline{k}')\ \rho_i\ (\underline{k}')\ M^*\ (\underline{k}\ ,\ \underline{k}')\ , \qquad 2.4'$$

where the freedom in the normalization of the $\rho(\underline{k})$ has been exploited. With this choice of normalization the differential cross section becomes simply

$$\frac{d\sigma}{d\Omega} \equiv I\ (\theta, \phi\ ) = \frac{\mathrm{Tr}\ \rho_{sc}}{\mathrm{Tr}\ \rho_i} = \frac{\mathrm{Tr}\ M(\underline{k}\ ,\ \underline{k}')\ \rho_i\ M^*(\underline{k}\ ,\ \underline{k}')}{\mathrm{Tr}\ \rho_i} \qquad 2.8$$

which is the straightforward generalization to the case of polarized initial particles of the usual rule of summing over the final states and averaging over the initial states.

The operator $M(\underline{k}, \underline{k}')$ is a matrix in the spin space of the two particles and, like $\rho(\underline{k})$, it can be expanded in terms of the $\sigma_\mu^{(1)} \sigma_\nu^{(2)}$:

$$M(\underline{k}, \underline{k}') = \sum M^{\mu\nu}(\underline{k}, \underline{k}') \sigma_\mu^{(1)} \sigma_\nu^{(2)} . \qquad 2.9$$

In the center-of-mass frame, where $\underline{k}$ and $\underline{k}'$ are the only vectors upon which the $M^{\mu, \nu}(k, k')$ can depend, the most general function of the form 2.9 invariant under rotations and spatial reflections is

$$\begin{aligned} M(\underline{k}, \underline{k}') = {} & \\ & a + b\,(\underline{\sigma}^{(1)} \cdot \underline{N} - \underline{\sigma}^{(2)} \cdot \underline{N}) + c\,(\underline{\sigma}^{(1)} \cdot \underline{N} + \underline{\sigma}^{(2)} \cdot \underline{N}) \\ & + m\,(\underline{\sigma}^{(1)} \cdot \underline{N}\,\underline{\sigma}^{(2)} \cdot \underline{N}) + g\,(\underline{\sigma}^{(1)} \cdot \underline{P}\,\underline{\sigma}^{(2)} \cdot \underline{\sigma}^{(2)} \cdot \underline{P} + \underline{\sigma}^{(1)} \cdot \underline{K}\,\underline{\sigma}^{(2)} \cdot \underline{K}) \\ & + h\,(\underline{\sigma}^{(1)} \cdot \underline{P}\,\underline{\sigma}^{(2)} \cdot \underline{P} - \underline{\sigma}^{(1)} \cdot \underline{K}\,\underline{\sigma}^{(2)} \cdot \underline{K}) \\ & + j\,(\underline{\sigma}^{(1)} \cdot \underline{P}\,\underline{\sigma}^{(2)} \cdot \underline{K}) + \ell\,(\underline{\sigma}^{(1)} \cdot \underline{K}\,\underline{\sigma}^{(2)} \cdot \underline{P}) \end{aligned} \qquad 2.10$$

Here $\underline{N}$, $\underline{P}$ are unit orthogonal vectors in the directions $\underline{k} \times \underline{k}$, $\underline{k}' + \underline{k}$, and $\underline{k} - \underline{k}'$ respectively, and the coefficients a, b, $\cdots$, ℓ are scalar functions of the vectors $\underline{k}$ and $\underline{k}'$.

For proton-proton scattering, in which the two particles are identical, $b = 0$ and $j = \ell$, since M must be symmetric under interchange of the two particles. The same conditions would be satisfied also for n-p scattering if charge independence were strictly maintained.

It was pointed out by Wolfenstein & Ashkin[1] that the requirement of time-reversal invariance implies the vanishing of j and ℓ. As is shown later, this requirement directly implies the invariance of $M(\underline{k}, \underline{k}')$ under the simultaneous substitutions $\underline{k} \longleftrightarrow -\underline{k}'$ and $\underline{\sigma} \longleftrightarrow -\underline{\sigma}$. Since the terms multiplying j and ℓ change sign under this transformation while j and ℓ, like the available scalars $\underline{k}\cdot\underline{k}$, $\underline{k}'\cdot\underline{k}'$ and $\underline{k}\cdot\underline{k}'$, are invariant, j and ℓ must vanish. Thus the combined requirements of invariance under rotation, spactial reflection, and time reversal imply that the scattering matrix M can be expressed in terms of the six (five for p-p) remaining complex scalar parameters in Eq. 2.10. These parameters are called the Wolfenstein parameters. For fixed energy, they are functions of the single scalar variable $\underline{k}\cdot\underline{k}'$ or, equivalently, of the polar scattering angle θ. They are, of course, independent of the initial and final polarizations; these latter quantities enter the theory only through the initial and final density matrices $\rho_i(\underline{k}')$ and $\rho_{sc}(\underline{k})$.

An alternative way of writing the M matrix has been given by Wolfenstein (3):

$$M(\underline{k}, \underline{k}') = BS + C(\underline{\sigma}^{(1)} + \underline{\sigma}^{(2)})\cdot\underline{N}$$

$$+ \tfrac{1}{2} G(\underline{\sigma}^{(1)}\cdot\underline{K}\,\underline{\sigma}^{(2)}\cdot\underline{K} + \underline{\sigma}^{(1)}\cdot\underline{P}\,\underline{\sigma}^{(2)}\cdot\underline{\rho})\,T$$

$$+ \tfrac{1}{2} H(\underline{\sigma}^{(1)}\cdot\underline{K}\,\underline{\sigma}^{(2)}\cdot\underline{K} - \underline{\sigma}^{(1)}\cdot\underline{P}\,\underline{\sigma}^{(2)}\cdot\underline{\rho})\,T$$

$$+ N(\underline{\sigma}^{(1)}\cdot\underline{N}\,\underline{\sigma}^{(2)}\cdot\underline{N})\,T\,. \qquad 2.10'$$

The quantities S and T in this equation are the singlet and triplet projection operators, $\frac{1}{4}(1 - \underline{\sigma}^{(1)} \cdot \underline{\sigma}^{(2)})$ and $\frac{1}{4}(3 + \underline{\sigma}^{(1)} \cdot \underline{\sigma}^{(2)})$ respectively and the coefficients B, C, G, H, and N are again functions of the scattering angle θ, but now with the symmetry properties $B(\theta) = B(\pi - \theta)$, $C(\theta) = C(\pi - \theta)$, $G(\theta) = -G(\pi - \theta)$, $H(\theta) = H(\pi - \theta)$, and $N(\theta) = -N(\pi - \theta)$ for the isotopic triplet case. The isotopic singlet amplitudes have the opposite symmetries.

The formalism described above is the basis of the discussions in the following sections and, indeed, of all contemporary discussions of polarization phenomena.

Section 3. Possible Experiments

In principle 256 experiments can be performed on the nucleon-nucleon system at a single scattering angle. The final spin-space density matrix depends on sixteen independent real scalar parameters and each of these depends linearly through Eq. 2.4' on each of the 16 real scalar parameters that determine the initial density matrix. In terms of the 256 scalar coefficients

$$Z_{\mu,\nu;\lambda\rho}(\underline{k}, \underline{k}') \qquad 3.1$$

$$= \frac{1}{4} \mathrm{Tr}\, \sigma_\mu^{(1)} \sigma_\nu^{(2)} M(\underline{k}, \underline{k}') \sigma_\lambda^{(1)} \sigma_\rho^{(2)} M^*(\underline{k}, \underline{k}')$$

the relation between initial and final expectation values is simply

$$I(\theta,\phi) \left\langle \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right\rangle_{\underline{k}} = Z_{\mu,\nu;\lambda\rho}(\underline{k}, \underline{k}') \left\langle \sigma_\lambda^{(1)} \sigma_\rho^{(2)} \right\rangle_{\underline{k}'} . \qquad 3.2$$

The experimental problem is to fix the initial expectation values and measure the final ones. This determines the Z's, which in turn give information on the nucleon-nucleon interaction through Eq. 3.1 .

The initial polarisation is fixed and the final ones measured, generally, by making auxiliary scattering before and after the principal nucleon-nucleon scattering. These auxiliary scatterings can be treated by using a formalism analogous to the one described above. If, for simplicity, the initial auxiliary scatterer is taken to have spin zero and if invariance under spatial reflection is assumed, this initial scattering is described by a two-by-two spin matrix of the form

$$M_1 = a_1 + b_1 \, \underline{\sigma} \cdot N \,, \qquad 3.3$$

as discussed by Wolfenstein (4) in an earlier volume. Inserting this into 2.4' and using 2.2' one obtains for the polarization vector after the scattering of an initially unpolarized beam the expression

$$\langle \underline{\sigma} \rangle_{\underline{k}} = \left[2\mathrm{Re}\, a_1 \, b_1^{*} / (|a_1|^2 + |b_1|^2) \right] \underline{N}_1 \equiv P_1 \, \underline{N}_1 \,. \qquad 3.4$$

Here $\underline{N}_1$ is the normal to the first scattering plane and the quantity P_1, which gives the magnitude of the polarization vector, is called the polarizing power of the reaction. The subscripts 1 identify the quantities as pertaining to the initial scattering.

A possible third scattering acts as an analyzer. If the final target is also taken to have spin zero it will be represented by a matrix like that in Eq. 3.3, but now with subscripts 3 to denote the third scattering. Inserting this matrix into 2.4', one obtains, for the differential cross section 2.8,

$$I_3 = I_{03} \left[1 + a_3 \, \underline{N}_3 \cdot \langle \underline{\sigma} \rangle_{\underline{k}} \,. \right] \qquad 3.5$$

where

$$I_{03} = |a_3|^2 + |b_3|^2 \qquad 3.6$$

and

$$a_3 I_{03} = 2 \operatorname{Re} a_3 b_3^* \,. \qquad 3.7$$

Here I_{03} is the differential cross section for the third scattering when the incident beam is unpolarized, and a_3 is called the analyzing power of the reaction. According to 3.5 the deviation from azimuthial symmetry after the final scattering is a measure of $\langle \underline{\sigma} \rangle_{\underline{k}} \cdot \underline{N}_3$, the component perpendicular to the final scattering plane of the polarization vector of the incident beam of the final reaction.

The fact that the vectors appearing in Eqs. 3.4 and 3.5 are the normals to the scattering plane is a consequence of the assumed invariance of the interaction under spatial reflection; the normal is the only axial vector that can be formed from the initial and final relative momentum vectors. The equality of polarizing and analyzing powers for a given reaction, which is seen from Eqs. 3.4 to 3.7 , was shown by Wolfenstein & Ashkin (1) to be a general consequence of the assumed invarances under rotation, reflection, and time reversal, true even for targets of nonzero spin. This equality is often tacitly assumed and the polarizing and analyzing powers, undifferentiated, are called the polarization function of the reaction.

According to Eq. 3.4 the polarization vector after the scattering of an initially unpolarized beam is normal to the line of flight of the nucleon. To obtain polarization components along the line of flight the beam can be passed through a magnetic field. Relativistic formulas for the precession

rate are given by Bargmann, Michel & Telegdi (5). Magnetic fields may also be used to rotate a longitudinal component of the final polarization vector into a measureable transverse component.

At present no experiments in which the target nucleon is polarized have been performed. When polarized targets become available the dependence on the initial correlation parameters can be investigated. Since there can be no correlation between the orientations of individual target and beam nucleons, the initial correlation parameter is simply the product of the individual polarizations:

$$\left\langle \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right\rangle_{\underline{k}'} = \left\langle \sigma_\mu^{(1)} \right\rangle_{\underline{k}'} \left\langle \sigma_\nu^{(2)} \right\rangle_{\underline{k}'} .$$

The dependence of final expectation values on this term may be isolated by performing experiments with various combinations of the signs of $\left\langle \sigma_\mu^{(1)} \right\rangle_{\underline{k}'}$ and $\left\langle \sigma_\nu^{(2)} \right\rangle_{\underline{k}'}$ and then averaging with an appropriately signed weighting factor to eliminate the unwanted terms. [As in many of these experiments it must be remembered that the differential cross section appearing on the left of 3.2 generally depends on the initial spin expectation values, so that the relation between the initial and final spin expectation values, unlike the relationship between the two density matrices, is not linear.]

In order to measure the final correlation parameters, $\left\langle \sigma_i \sigma_j \right\rangle_{\underline{k}}$, one can rescatter both outgoing nucleons. Contrary to the case for the initial state, the orientations of the two nucleons can now possess correlations, and the correlation parameter is expected to differ from the product of the individual polarizations. It is therefore necessary to consider simultaneously the scattering of both particles.

If the two particles are considered a single system the final scattering process is described by the product scattering matrix

$$M_3 = (a_3^{(1)} + b_3^{(1)}\ \underline{\sigma}^{(1)} \cdot \underline{N}^{(1)})\ (a_3^{(2)} + b_3^{(2)}\ \underline{\sigma}^{(2)} \cdot \underline{N}^{(2)})$$

where for simplicity the final targets are again assumed spinless. Substituting this expression into 2.8 and identifying the ρ_i of that equation with $\rho(\underline{k})$, the final density matrix of the nucleon nucleon collision, one obtains (6) after some rearrangement the coincidence cross section

$$I_3^{(1)(2)} = I_{03}^{(1)}\ I_{03}^{(2)} \Big[1 + a_3^{(1)} \left\langle \underline{\sigma}^{(1)} \right\rangle_{\underline{k}} \cdot \underline{N}_3^{(1)} + a_3^{(2)} \left\langle \underline{\sigma}^{(2)} \right\rangle_{\underline{k}} \cdot N_3^{(2)}$$

$$+ a_3^{(1)}\ a_3^{(2)} \left\langle \underline{\sigma}^{(1)} \cdot N_3^{(1)}\ \underline{\sigma}^{(2)} \cdot \underline{N}_3^{(2)} \right\rangle_{\underline{k}} \Big] . \qquad 3.8.$$

Here the analyzing powers, $a_3^{(i)}$, are $2\mathrm{Re}\ a_3^{(i)}\ b_3^{(i)*} / (|a_3^{(i)}|^2 + |b_3^{(i)}|^2)$, as before. The final correlation parameter $\left\langle \sigma_i^{(1)}\ \sigma_j^{(2)} \right\rangle_{\underline{k}} = C_{ij}(\underline{k})$ can again be isolated by averaging over appropriate combinations of the senses of $\underline{N}_3^{(1)}$ and $\underline{N}_3^{(2)}$. For example,

$$C_{NN} = \frac{(L, L) + (R, R) - (L, R) - (R, L)}{(L, L) + (R, R) + (L, R) + (R, L)} \times \left[a_3^{(1)}\ a_3^{(2)} \right] - 1 ,$$

where (L, R) represents the number of times that the first of the nucleons from the nucleon-nucleon collision scatters left at its subsequent final scattering and the other nucleon scatters right, etc. In order to establish that the two nucleons come from a single nucleon-nucleon scattering event high-speed coincidence circuits are used.

Certain of the more common experimental quantities have been given names which are now fairly standard in the field. In terms of the z's defined in 3.1 and 4.1 the simplest of the 256 observables is $Z(\underset{\sim}{k}, \underset{\sim}{k}';0,0;0,0) \equiv I_0(\theta)$, which is the differential cross section for an initially unpolarized beam. If Particle 2 is considered to be the target particle, the quantities next in order of experimental simplicity are $Z(\underset{\sim}{k}, \underset{\sim}{k}';\underset{\sim}{N}, 0, 0, 0)/I_0(\theta)$ and $Z(\underset{\sim}{k}, \underset{\sim}{k};0, 0;\underset{\sim}{N}, 0)/I_0(\theta)$, the polarizing and analyzing powers respectively. As mentioned before, these are the magnitude of the polarization vector after the scattering of an initially unpolarized beam and the coefficient of azimuthal asymmetry after the scattering of a completely polarized beam. The simplest of the triple scattering experiment measures $Z(\underset{\sim}{k}, \underset{\sim}{k}';\underset{\sim}{N}, 0;\underset{\sim}{N}, 0)I_0(\theta)$, which determines the dependence of the normal component of the polarization vector after the nucleon-nucleon scattering on the normal component before the scattering, where the normal is defined relative to the nucleon-nucleon scattering plane. This observable is denoted by D and called depolarization. In the D experiment all three scattering planes are evidently coplanar. The quantities $Z(\underset{\sim}{k}, \underset{\sim}{k}';\underset{\sim}{K}, 0;N \times \underset{\sim}{K}_{in}, 0)I_0(\theta)$ and $Z(\underset{\sim}{k}, \underset{\sim}{k}';\underset{\sim}{K}, 0;\underset{\sim}{K}_{in}, 0)I_0(\theta)$ are denoted by R, for rotation, and A respectively, where $\underset{\sim}{K}_{in}$ is a unit vector in the direction $\underset{\sim}{k}'$. In the experiment that measures R, the nucleon-nucleon scattering plane is perpendicular to the first scattering plane and hence contains the initial polarization vector. The third scattering plane contains the final laboratory momentum, of course, and is perpendicular to the nucleon-nucleon scattering plane. The normal to this third plane is then $\underset{\sim}{K}$, the unit vector along $\underset{\sim}{k} - \underset{\sim}{k}'$, provided small relativistic corrections are neglected. In the "A" triple scattering experiment a magnetic field is used to precess the spin of the incident particle in order to give it a component the line of flight. The final scattering plane is perpendicular to the nucleon-nucleon scattering plane as in the R experiment.

The only other experiments yet performed are the correlation experiments which measure $Z(\underline{k}, \underline{k}'; \underline{N}, \underline{N}' 0, 0)/I_0(\theta) = C_{NN}$ and $Z(\underline{k}, \underline{k}'; \underline{K}, \underline{P}; \underline{0}, \underline{0})/I_0(\theta) = C_{KP}$. In the former the two final scattering planes are parallel to the nucleon-nucleon scattering plane and in the latter they are perpendicular to it. The vector $\underline{P}$ appearing in C_{KP} specifies the direction perpendicular to the laboratory velocity of the recoil target nucleon, again neglecting relativistic corrections. For identical particles the final momentum $\underline{k}$ refers, of course, to the particle identified as the scattered incident particle.

Correlation experiments in which the initial nucleon is polarized appear feasible although none have yet been carried out. The symbol C_{ABC} has been used for the relevant quantity $Z(\underline{k}, \underline{k}'; \underline{B}, \underline{C}; \underline{A}, 0)/I_0(\theta)$.

The quantities obtained by replacing in the above expressions for A and R the vector $\underline{K}$ by $\underline{P}$ are called A' and R' respectively. The vector $\underline{P}$ is a unit vector along the laboratory velocity of the scattered nucleon, again neglecting relativistic corrections. Thus in order to measure A' or R' the scattered beam from the nucleon-nucleon reaction can be passed through a magnetic field.

The experimental quantities defined above can be expressed in terms of the Wolfenstein parameters by inserting 2.10 or 2.10' into 3.1 and performing the matrix multiplication. Results have been given by many authors (References 2 and 6 through 12). A list is given in Table I.

Section 4. Theoretical Relations Between Experiments

Although the 256 experiments are experimentally independent there are theoretical relationship between them. The results of presumably related experiments would be useful as checks on either the theoretical assumptions or the experimental results.

The theoretical assumptions, aside from basic quantum mechanics, are the invariances under rotations, reflections, and time reversal. The basic principles of quantum mechanics already imply 224 relations among the 256 observables, for the 256 observables are functions of the 16 complex M-matrix elements, as one sees by Eq. 3.1. The specific consequences following from these relations alone have apparently not been examined.

The requirement of invariance under space reflection implies the vanishing of half of the 256 Z coefficients, since half are pseudoscalars. A nonzero value of any one of the 128 pseudoscalar Z coefficients would constitute unambiguous proof that parity is not conserved in the interaction that produces the scattering.

The consequences of time-reversal invariance may be expressed in a simple form if one introduces the definition

$$Z(\underline{k}, \underline{k}'; \underline{A}, \underline{B}; \underline{C}, \underline{D}) \equiv \sum_{\substack{\mu\nu \\ \lambda,\rho}} Z_{\mu,\nu;\lambda\rho}(\underline{k},\underline{k}') A_\mu B_\nu C_\lambda D_\rho . \qquad 4.1.$$

In this expression if $\underline{A}$, $\underline{B}$, $\underline{C}$ or $\underline{D}$ is replaced by a zero the corresponding index of $Z_{\mu,\nu;\lambda,\rho}$ is to be taken to be zero. Then invariance under time reversal implies

$$Z(\underline{k}, \underline{k}'; \underline{A}, \underline{B}; \underline{C}, \underline{D}) = A(-\underline{k}', -\underline{k}; -\underline{C}; -\underline{D}; -\underline{A}, -\underline{B}), \qquad 4.2$$

where plus and minus zero are considered equivalent. Equation 4.2 gives 128 conditions, the simplest of which is the equality of polarizing and analyzing power mentioned before. To derive this equality notice first that the analyzing

power $A(\underline{k}, \underline{k}'; 0, 0; \underline{N}\, 0)/I_0$, being a scalar linear in $\underline{N}$, must, by rotation and space-reflection invariance, be of the form $\underline{k} \times \underline{k}' \cdot \underline{N}$ times a function of the scalar products of $\underline{k}'$ and $\underline{k}'$. It is therefore equal to $Z(-\underline{k}', -\underline{k}; 0, 0; -\underline{N}, 0)/I_0$ which, by 4.2, is equal to the polarizing power $Z(\underline{k}, \underline{k}'; \underline{N}, \underline{0}; 0, 0)$. Another consequence of the same three invariances is a relationship between A, R, A', and R', also first deduced by Wolfenstein (3). To derive this relation one takes from 4.1 the relation

$$Z(\underline{k}, \underline{k}'; \underline{K}, 0; \underline{P}; 0) = Z(-\underline{k}', -\underline{k}; \underline{P}, 0; \underline{K}\, 0), \qquad 4.3$$

where the linearity in the last four variables has been used. The right-hand side, being a scalar linear in both $\underline{P}$ and $\underline{K}$, must change sign when the first two arguments are changed to $\underline{k}$ and $\underline{k}'$ respectively, as one sees by enumerating the possible forms. Both sides of 4.3 can then be expressed in terms of A, R, A', and R', and the relation $(A+R')/(A'-R) = \tan(\theta/2)$ follows.

If rotation, space-reflection, and time-reversal invariances are all maintained there are for the p-p system at one scattering angle only nine independent scattering experiments; the M matrix and hence also the observables are determined by the five complex Wolfenstein parameters, and the over-all phase of M is irrelevant. For the n-p system there are also nine independent experiments if charge independence is assumed--otherwise eleven.

At a scattering angle of 90 deg two of the Wolfenstein parameters of 2.10' vanish for p-p scattering and five experiments are sufficient to fix M up to the over-all phase. Also at this angle the combination $I_0 (1 - C_{nn})$ is determined completely

by the absolute value of singlet scattering amplitude and therefore measures this quantity. Details are given in Reference (6). For n-p scattering at 90 deg the situation is also very favorable, since there are only two non-vanishing isotopic singlet amplitudes.

By virtue of the symmetry properties of the Wolfenstein parameters appearing in Eq. 2.10', experimental observables at θ and $\pi-\theta$ can both be expressed in terms of the scattering amplitudes at θ. For some types of experiments--such as p-p differential cross sections and polarizations--the two experiments do not give independent information, but for others--such as p-p triple scattering experiments--the observables at the two scattering angles are given by different expressions in terms of the Wolfenstein parameters and furnish independent conditions on them. For n-p scattering the symmetries involve the isotopic singlet and triplet parts of the Wolfenstein parameters separately which in effect doubles the number of unknowns and renders the experiments at θ and $\pi-\theta$ essentially independent unless isotopic spin invariance is assumed. In this latter case n-p and p-p experiments can be analyzed simultaneously, and nineteen experiments at the two angles are sufficient to determine the five isotopic singlet and five isotopic triplet parts of the Wolfenstein parameters up to an over-all phase. This is discussed in detail by Golovin, Dzhelepov, Nadezhden & Satarov (11).

Some relations between experimental observables imposed by unitarity are discussed in the next section.

Section 5. Unitarity and Phase Shifts

The important fact that the S matrix is unitary has not been incorporated into the formalism developed above. A standard way of including unitarity is to

decompose the scattering amplitudes into partial waves. Unitarity is then easily expressed by using phase shifts, as is discussed below. The partial-wave expansion is also useful because only the lower partial waves contribute significantly to the scattering and the reaction can be approximately described by a small finite number of parameters.

The S matrix, by its definition, depends only on the asymptotic form of the wave function; it is a transformation in the spin-angle variables, the radial dependence being essentially known. To simplify the writing it is convenient, therefore, to suppress the radial factor $i^L j_L(kr)$, where $j_L(kr)$ is the spherical Bessel function. Specifically, $|L, m; S_1, m_1; S_2, m_2\rangle$ will represent the state having spin quantum numbers (S_1, m_1) and (S_2, m_2) for the first and second particles, respectively, and a spatial dependence $Y_L^m(\theta, \phi)\,(i^L j_L(kr))$, where $Y_L^m(\theta,\phi)$ is the spherical Bessel function as defined in Blatt and Weisskopf (13). The symbol $\langle L, m; S_1, m_1; S_2, m_2 | \psi\rangle$ will represent the amplitude of this state. Since the asyptotic outgoing part of $(i^L j_L(kr))$ is $e^{ikr}/2ikr$ the scattering amplitude becomes, using these conventions,

$$f_{S_1 S_2}{}^{m_1 m_2}(\theta,\phi) = \frac{1}{2ik} \sum_{L,m} Y_L^m(\theta,\phi) \left\langle L, m; S_1, m_1; S_2, m_2 \,|\, R \,|\, in \right\rangle$$

$$\equiv \frac{1}{2ik} \left\langle \theta, \phi \,|\, L, m \right\rangle \left\langle L, m; S_1, m_1; S_2, m_2 \,|\, R \,|\, in \right\rangle$$

$$\equiv \frac{1}{2ik} \left\langle \theta, \phi ; S_1, m_1; S_2, m_2 \,|\, R \,|\, in \right\rangle , \qquad 5.1.$$

where the symbol $\left\langle \theta, \phi \,|\, L, m \right\rangle = Y_L^m(\theta, \phi)$, the summation convention,

$\langle L, m; S_1, m_1$

and the completeness relation $|L, m\rangle\langle L, m| = 1$ have been used.

According to the above conventions the state represented by $|\theta', \phi'\rangle$ is $(4\pi)^{-1}$ times a plane wave moving in the direction (θ', ϕ') as one sees from the Gegenbauer expansion (Ref. 13),

$$\sum_{L,m} i^L j_L(kr)\, Y_L^m(\theta,\phi)\,(Y_L^m(\theta',\phi'))^* = \exp i\underline{k}' \cdot \underline{r}/4\pi . \qquad \text{S.2.}$$

Combining this fact with Eq. S.1 and the definition of the M matrix, one obtains the important relationship

$$M(\underline{k}, \underline{k}') = \frac{4\pi}{2ik} \left\langle \theta, \phi \,|R|\, \theta', \phi' \right\rangle . \qquad \text{S.3.}$$

The most convenient phase shifts are related to the matrix elements of R in the representation where J, the total angular momentum, is diagonal. These states are characterized by quantum numbers J, L, S, M, where S is 0 or 1 for the singlet or triplet states respectively and M is the Z component of the total angular momentum J. Transforming from this representation, one obtains for the matrix elements of $M(\underline{k}, \underline{k}')$ in the singlet-triplet representation

$$\begin{aligned}
&\left\langle S, m_S \,|M(k;k')|\, S', m'_S \right\rangle \\
&= \frac{4\pi}{2ik} \left\langle \theta,\phi; S, m_S \,|R|\, \theta', \phi; S', m'_S \right\rangle \\
&= \frac{4\pi}{2ik} \left\langle \theta,\phi; S, m_S \,|L, S'', J, M\right\rangle \left\langle L, S'', J, M|R|L'S'''J'M'\right\rangle \\
&\quad \left\langle L'S'''J'M' \,|\theta'\phi'; S' m'_S\right\rangle . \qquad \text{S.4.}
\end{aligned}$$

The transformation functions occurring in §4 are sums of products of spherical harmonics and Clebsch-Gordan coefficients:

$$\left\langle \theta, \phi ; S, m_S \mid L, S'', J, M \right\rangle$$

$$= \left\langle \theta, \phi \mid L, m \right\rangle \left\langle L, m; S\, m_S \mid L, S'', J, M \right\rangle$$

$$\equiv \sum_{m, L} Y_L^m (\theta, \phi)\, C_{LS} (J, M; m, m_S)\, \delta_{SS''} , \qquad \text{§5.}$$

where $C_{LS}(J, M; m, m_S) \equiv \left\langle L, m; S, m_S \mid L, S, J, M \right\rangle$ is the Clebsch-Gordan coefficient as defined in Blatt and Weisskopf (13).

Equations §4 and §5 allow the M-matrix elements to be expressed in terms of the R-matrix element $\left\langle LSJM \mid R \mid L' S' J' M' \right\rangle$. Because total angular momentum and its Z component are conserved these R-matrix elements must vanish unless $J = J'$ and $M = M'$. Moreover, because of rotational invariance the matrix elements are independent of M, as follows from Schur's lemma. For a fixed J the possible values of L are $L = J$ and $L = J \pm 1$, where the second class can occur only in the triplet case. Also the two classes cannot be coupled because of conservation of parity.

For brevity we introduce for the nonvanishing matrix elements of the class $L = J \pm 1$ the definitions

$$\left\langle J \pm 1, 1, J, M \mid R \mid J \pm 1, 1, J, M \right\rangle = R_{J \pm 1, J} , \qquad \text{§6.}$$

$$\left\langle J \pm 1, 1, J, M \mid R \mid J \mp 1, 1, J, M \right\rangle = R_{\pm}^{J} = R^{J} , \qquad \text{§7.}$$

and, for the class $L = J$,

$$\left\langle J,\ 1,\ J,\ M \mid R \mid J,\ 1,\ J,\ M \right\rangle = R_{JJ},$$

$$\left\langle J,\ 0,\ J,\ M \mid R \mid J,\ 0,\ J,\ M \right\rangle = R_{J}, \qquad \text{§.8.}$$

$$\left\langle J,\ \tfrac{1}{2} \pm \tfrac{1}{2},\ J,\ M \mid R \mid J,\ \tfrac{1}{2} \mp \tfrac{1}{2},\ J,\ M \right\rangle = R'^{J}_{\pm} = R'^{J}. \qquad \text{§.9.}$$

The equalities $R_{+}^{J} = R_{-}^{J} = R^{J}$ and $R'^{J}_{+} = R'^{J}_{-} = R'^{J}$ stated here express the fact that the R matrix is symmetric in this representation, This which is a consequence of time-reversal invariance discussed in the next section. The off-diagonal element R'^{J} vanishes for the p-p system, as the antisymmetry of the wave function precludes states having the same L but different total spin. If isotopic spin is conserved the R'^{J} also vanishes for the n-p system, since for the same L the two spin states have different isotopic spin.

By carrying out the arithematic implied in §.4 and §.5 and using the abbreviations §.6 through §.9, one obtains the expressions for the M-matrix element given in Table II. Table II refers specifically to the n-p case. For the p-p case only the antisymmetric states contribute, and in these a factor of two must be added, as is discussed in Section 7. For the p-p case the Coulomb effects must also be included. Explicit p-p formulas are given in Ref. 14. An extensive discussion of relativistic Coulomb corrections is given by Breit (15).

Equation §.4 and §.5 give the decomposition of R into partial waves. However, the unitarity condition on S = R + 1 has not yet been invoked. By our choice of representation the only non-diagonal, nonvanishing matrix elements of R are the R^J and R'^J. Thus by grouping in pairs the two states $L = J \pm 1$ for $S = 1$ and the two states $S = \frac{1}{2} \pm \frac{1}{2}$ for $L = J$, the R matrix breaks into a series of two-by-two matrices, all other element vanishing. The matrix S = R + 1 also must have this form, and the condition that S be unitary is equivalent to the condition that each of the two-by-two submatrices be unitary.

A symmetric, unitary two by two matrix has three degrees of freedom and can be expressed in terms of three real parameters in either of the forms

$$\begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \quad \begin{pmatrix} e^{2i\delta_-} & 0 \\ 0 & e^{2i\delta_+} \end{pmatrix} \quad \begin{pmatrix} \cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon \end{pmatrix}$$

or

$$\begin{pmatrix} e^{i\bar{\delta}_-} & 0 \\ 0 & e^{i\bar{\delta}_+} \end{pmatrix} \quad \begin{pmatrix} \cos 2\bar{\epsilon} & i \sin 2\bar{\epsilon} \\ i \sin 2\bar{\epsilon} & \cos 2\bar{\epsilon} \end{pmatrix} \quad \begin{pmatrix} e^{i\bar{\delta}_-} & 0 \\ 0 & e^{i\bar{\delta}_+} \end{pmatrix}.$$

Here the individual matrices are all unitary and the forms are obviously symmetric.

The first form has been used by Blatt and Biedenharn and the real parameters δ_-, δ_+ and ϵ are often called the Blatt and Biedenharn (type) phase shifts, or mixing parameter for the case of ϵ. The two matrices on the outside can be considered the transformation to the representation where S is diagonal. As the elements of a diagonal unitary matrix must be pure phase factors, they may be defined to be $e^{2i\delta_\pm}$.

The parameters of the second form are called "bar" phase shifts and are useful for several reasons (14). First, these phases are proportional to the R-matrix elements in lowest order and approach zero as R goes to zero. Second, the parameter $\bar{\epsilon}$ gives a measure of the amount by which orbital angular momentum is not conserved in the sense that a particle entering in one orbital state has a probability $\sin^2 2\bar{\epsilon}$ of being in the other state when it emerges. Third, in the Born approximation the phases are given by simple matrix elements of the interaction energy and hence obey simple interval rules. Fourth, Coulomb effects can be subtratted to lowest order by subtracting the Coulomb phases from the total bar phases. The essential difference between the two types of phases is that the mixing is in the asymptotic region for the Blatt and Biedenharn phases and at the core for bar phases. Equations relating the two types are given in References (6) and (14).

The equations given above and in Table II allow the matrix elements of M in the single-triplet representation to be expressed in terms of phase shifts. Since the observables are expressed in terms of the Wolfenstein parameters, the relationship between the M-matrix elements and the Wolfenstein parameters are still needed. These may be obtained by taking traces of the expression for M given in 2.10 to obtain explicit formulas for the Wolfenstein parameters. Carrying out the trace operations and using the well-known connection between the single-particle and the singlet-triplet representations, one obtains the results given in Table III. Intermediate steps are given in Ref. (6).

An alternative way of obtaining the connection between the Wolfenstein parameters and the phase shifts has been given by Wright (16) and by Bakke and Steck (17). The general phase-shift formalism for the n-p system is developed in a review article by Blatt and Biedenharn (18). General phase-shift formulas are also given by Breit and Hull (19).

The unitarity condition can also be introduced without using phase shifts. In particular the relation

$$SS^* = 1 = (R + 1)(R^* + 1) = RR^* + R + R^* + 1$$

implies for all θ, ϕ and all $\theta' \phi'$ the equation

$$-\left\langle \theta, \phi \mid R + R^* \mid \theta' \phi' \right\rangle = \left\langle \theta', \phi' \mid R \mid \theta'' \phi'' \right\rangle \left\langle \theta'' \phi'' \mid R^* \mid \theta' \phi' \right\rangle .$$

Using the relation 5.3 between R and M, one obtains immediately

$$\left(\frac{2k}{4\pi}\right) 2 \operatorname{Im} M(\Omega, \Omega') = \left(\frac{2k}{4\pi}\right)^2 \int d\Omega'' \, M(\Omega, \Omega'') \, M^*(\Omega'', \Omega') .$$

Multiplying this by $\sigma_\mu^{(1)} \sigma_\nu^{(2)}$ and taking one-quarter the trace, one finds

$$\operatorname{Im} M^{\mu\nu}(\Omega, \Omega') = \frac{2k}{4\pi} \int d\Omega'' \, \frac{1}{4} \operatorname{Tr} \left(M(\Omega\Omega'') \, M^*(\Omega''\Omega') \, \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right) .$$

5.10.

Although 5.10 contains sixteen equations, only six of these (five for p-p) are independent. The same invariance conditions that allow M_3 and

hence the left-hand side to contain only six (five)degrees of freedom also constrain the right-hand side correspondingly. Since the phase enters in 5.10, and hence is no longer arbitrary, the six (five) unitary conditions reduce the number of arbitrary angle-dependent functions required to specify the M matrix to six (five) and thus, as pointed out by Puzikov, Ryndin, and Smorodinsky (9), six (five) independent experiments, performed at all angles, are sufficient to determine M at all angles. If inelastic processes are considered the unitarity condition involves the additional states, and the argument is no longer complete.

This result regarding the number of experiments needed to determine the scattering matrix, though often quoted, is rather academic. In practice, if data at many angles are to be analyzed simultaneously a phase-shift analysis is used. A maximum angular momentum is usually chosen, and this fixes the number of powers of cos θ in the various observables, and hence the number of parameters to be experimentally determined. The number of phases is also fixed. An analysis by Ypsilantis (10) shows that for L max > 0 three types of experiments--cross section, polarization, and one triple-scattering experiment--are more than sufficient to determine the phases in principle. For $L_{max} = 0$ one experiment at one angle is evidently sufficient for the p-p or n-n case, since there is only one phase shift. For the p-p case cross-section measurements involve interference with the approximately known Coulomb amplitude, and this also tends to reduce the number of types of experiments needed.

Section 6. Consequences of Time-Reversal Invariance

The fundamental consequence of time-reversal invariance is (Reference 13, page 529)

$$\langle a \,|S|\, b \rangle = \langle b_T \,|S|\, a_T \rangle \ , \qquad 6.1.$$

where a and b are time-independent (t = 0) basis vectors and a_T b_T are their respective time-inverse states. In the notation of Section 5 the time inverse of the state $|L, m\rangle$ is $(-1)^{L-m} |L, -m\rangle$. The $(-1)^L$ comes from the required complex conjugation of the radial part $i^L j_L(kr)$ and the remaining changes come from the relation (Ref. 13) $Y_L^{m*}(\theta, \phi) = (-1)^m Y_L^{-m}(\theta, \phi)$. For the case with spin the time inverse of the state $|J, M\rangle$ is defined to be $(-1)^{J-M} |J, -M\rangle$. The spin is hereby reversed, as one requires for a time-reversed state, and the consistency of the definition with the composition law of angular momenta is guaranteed by the symmetry law for Clebsch-Gordan coefficients

$$C_{LS}(J, -M; -m, -m_s) = (-1)^{(J-M+L-m+S-m_s)} C_{LS}(J, M; m, m_s) \ .$$

In some earlier treatments of time reversal this consistency condition was not maintained, as was pointed out by Huby (20). Inserting the definition for the time-reversed state into Equation 6.1 , one obtains

$$\langle L, S, J, M \,|S|\, L', S', J, M \rangle$$

$$= \langle L' S' J, -M \,|S|\, L S J, -M \rangle \ (-1)^{2J-2M} \ . \qquad 6.2.$$

Since the matrix element is independent of M and $(-1)^{2J-2M}$ is unity, the symmetry of the matrix follows.

It should be noted that a unitary transformation does not generally leave a symmetric matrix symmetric, and a change of basis, even by phase factors only, generally destroys the symmetry of S.

The time inverse of a Pauli spin state, $\left| \frac{1}{2} , \pm \frac{1}{2} \right\rangle$, is $\pm \left| \frac{1}{2} , \mp \frac{1}{2} \right\rangle$; that is, $\left| \chi_T \right\rangle = i\sigma_2 \left| \chi \right\rangle$. The condition of time-reversal invariance applied to the R matrix can therefore be written in the form

$$\left\langle \underline{k}\chi \left| R(\underline{\sigma}^{(1)}, \underline{\sigma}^{(2)}) \right| \underline{k}' \chi \right\rangle =$$

$$\left\langle -\underline{k}', \chi' \left| \sigma_2^{(1)} \sigma_2^{(2)} R(\underline{\sigma}^{(1)}, \underline{\sigma}^{(2)}) \sigma_2^{(1)} \sigma_2^{(2)} \right| -\underline{k}, \chi \right\rangle$$

$$= \left\langle -\underline{k}', \chi' \left| R(\sigma_2^{(1)} \underline{\sigma}^{(1)} \sigma_2^{(1)}, \sigma_2^{(2)} \underline{\sigma}^{(2)} \sigma_2^{(2)}) \right| -\underline{k}, \chi \right\rangle$$

$$= \left\langle -\underline{k}, \chi' \left| R(-\tilde{\underline{\sigma}}^{(1)}, \tilde{\underline{\sigma}}^{(2)}) \right| -\underline{k}, \chi \right\rangle$$

$$= \left\langle -\underline{k}' \chi \left| R(-\underline{\sigma}^{(1)}, -\underline{\sigma}^{(2)}) \right| -\underline{k}, \chi' \right\rangle$$

Thus time-reversal invariance implies

$$\left\langle \underline{k} \left| R(\underline{\sigma}^{(1)}, \underline{\sigma}^{(2)}) \right| \underline{k}' \right\rangle = \left\langle -\underline{k}' \left| R(-\underline{\sigma}^{(1)}, -\sigma^{(2)}) \right| -\underline{k} \right\rangle, \quad 6.3.$$

as was stated earlier. Equation 4.2 can be obtained by using similar methods. A discussion of time-reversal invariance and its consequences is given by Shirokov (21). Consequences of time irreversibility are discussed by Phillips (22).

Section 7. Symmetrization

In the above the two particles have been assumed distinguishable. If, as in p-p scattering, the Fermi particles are identical, the amplitudes in the symmetric states are zero and those in the antisymmetric states are twice those for nonidentical particles.

It is perhaps not immediately clear that this factor two should not be rather the square root of two. If one simply writes the familiar antisymmetrized wave function $\frac{1}{\sqrt{2}} [\psi(\chi) - \psi(-\chi)]$, or its generalized version with spins, it is $\sqrt{2}$ that occurs. As the question involved here seems to arise often in practical work a discussion of the problem seems warranted.

If identical particles are considered to differ from distinguishable particles only in that they happen to start in an antisymmetrized state and remain so because the interaction is symmetrical, then in an incident plane wave the amplitudes in the symmetrical and antisymmetrical spin-orbit partial wave states are 0 or $\sqrt{2}$ times their normal values respectively, as indicated above. However, the detecting apparatus measures both particles, and this introduces an additional and independent factor of two in the measured particle densities. The wave function can be normalized to the measured densities if an additional factor of $\sqrt{2}$ is supplied.

In a somewhat more sophisticated view the identical nature of indistinguishable particles is ingrained more deeply in the mode of description. A wave function and a second wave function obtained from it by an interchange of particle coordinates are considered to represent not different states, but rather the same state multipled by minus one. The

effect of this is that in summing over a complete set of states care must be taken to include each state only once. For instance, the normalization condition becomes

$$\int |\psi(x_1, x_2)|^2 \, \theta(x_1, x_2) \, dx_1 \, dx_2 = 1 ,$$

where $\theta(x_1, x_2)$ is zero for x_1 less than x_2 and one for x_1 not less than x_2. This factor $\theta(x_1, x_2)$ ensures that a state contributes only once to the sum over states. The normalized plane wave is than $\exp(ikx) - \exp(-ikx)$, and the totally antisymmetric partial-wave amplitudes are twice normal. Since the particles are considered identical there is no additional factor due to the counter's being able to detect either one. This second approach is the wave-function transcription of what happens in field theory.

The most satisfactory method of treating identical particles is to use quantum field theory. The formalism is completely defined and there is no question of interpretation or viewpoint. For $\underline{k} \neq 0$ the normalized plane wave state is given by $|\psi\rangle = a^*(\underline{k}) \, a^*(-\underline{k}) \, |0\rangle$, where $a^*(\underline{k})$ and $a^*(-\underline{k})$ are operators that create particles in states $\underline{k}$ and $-\underline{k}$ respectively. The wave function is $\langle 0 | \psi(x_2) \psi(x_1) | \psi \rangle = \exp(ikx) - \exp(-ikx)$, where $x = x_1 - x_2$, and where the quantization volume is taken to be unity. When the Dyson-Wick (23) expansion of the S matrix is applied to this initial state the scattering amplitude is found to be $f(\theta, \phi) = f_u(\theta, \phi) - f_u(\pi - \theta, \phi + \pi)$, where the two terms on the right-hand side come from the forward and backward scattered particle respectively, for $\theta \; L\pi/2$.

The factor of two occurring for identical particles must apparently be included also for n-p states if the isotopic spin formalism is used, for

in this approach the neutron and proton are considered different states of the same particle. However, this two is cancelled by a one-half if the formalism is applied consistently. For instance, the S-matrix takes the form $P_0A_0 + P_1A_1$, where P_0 and P_1 are projection operators for the isotopic singlet and triplet states and A_0 and A_1 are the corresponding amplitudes, which include the factor 2. The matrix element of P_1 between two p-p states is unity but between two n-p states is one half.

The factor of two has been discussed in detail by Breit and Hull (19) and in Ref. (6).

Section 8. Relativistic Formalism

In the formalism described above the nucleons are represented by two-component spin functions instead of the relativistic four-component spinors. The M matrix of the two-component formalism can be obtained from the relativistic formalism, by multiplying the relativistic scattering matrix by the free-particle spinors, provided the center-of-mass frame is used. For a Dirac particle with momentum $\underline{p}$ the spinors for the two possible spin state are $u_{a1}(\underline{p})$ and $u_{a2}(\underline{p})$, where a is the Dirac spinor index and 1 and 2 are indices labeling the two spin states. These $u_{ai}(\underline{p})$ may be expressed as matrix elements of the matrix

$$u(p) = (-i\,\underline{p}\cdot\underline{\gamma} + \beta M)\,\beta / [2M(p_0 + M)]^{1/2}, \qquad 8.1.$$

where the spinor and spin indices run over rows and columns respectively. Stated differently, the $u_i(\underline{p})$ are obtained by operating on the proper spin states $w_1 = (1, 0, 0, 0)$ and $w_2 = (0, 1, 0, 0)$ with the Lorentz transformation

$$\mathcal{L}^{-1}(\underline{p}) = (-i\,\underline{p}\cdot\underline{\gamma} + \beta M)\,\beta / [2M(p_0 + M)]^{1/2}, \qquad 8.2.$$

which transforms a spin state amplitudes from their values in the particle rest frame to their value in the frame where the particle has momentum $\underline{p}$. With these definitions it is not hard to see that if the center-of-mass frame is used the reduced Pauli-type scattering matrix

$$M_{i,j;k,\ell} = \bar{u}_i^{(1)}(\underline{p}_1)\,\bar{u}_j^{(2)}(\underline{p}_2)\,\mathcal{M}\,u_k^{(1)}(\underline{p}'_1)\,u_\ell^{(2)}(\underline{p}'_2), \qquad 8.3.$$

where $\mathcal{M}$ is the relativistic scattering matrix and where spinor indices are suppressed, possesses the same transformation properties under rotations, spatial reflections, and time reversal as it has in the nonrelativistic theory. Thus the same group theoretical arguments may be applied and the nonrelativistic theory is formally unchanged.

According to the above connection the spin vector of the nonrelativistic treatment can be equated to the relativistic spin vector provided the latter is measured in the particle rest frame related to the reaction center-of-mass frame by the inverse of the Lorentz transformation 8.2. Conversely, the relativistic spin vector, which, according to Michel and Wightman (24), is a pseudo vector whose fourth component vanishes in the particle rest frame, can be obtained by applying the Lorentz transformation 8.2 to the spin vector of the nonrelativistic treatment. The relativistic spin vector in the laboratory frame may then be obtained by a second Lorentz transformation.

If the outgoing particle for one reaction is the incident particle for a second reaction, the second reaction must be described in its own rest

frame if the two-component formalism is to be applicable. The relativistic spin vector in this second center-of-mass frame may be obtained by a Lorentz transformation from the laboratory frame. Finally the three-component spin vector needed for the two-component formalism is obtained by a transformation from this second center-of-mass frame back to the particle rest frame. Although this sequence of four Lorentz transformation brings vectors back to their values in the particle rest frame, the final rest frame is rotated relative to the original one. This rotation is discussed in some detail in Reference (25), where explicit formulas are given. Discussions of the connection between the relativistic and nonrelativistic formulations have also been given by Steck (26), Garren (27) and Breit (15). A more formal general approach to the question of the relativistic description of polarized particles is given by Chou Kuang Chao and Shirkov (28).

Table I

Experimental observables in terms of Wolfenstein parameters (The Wolfenstein parameters are as defined in Eqs. 2.10 and 2.10' with b of the former set to zero. The observables are defined in section 3. Complete lists are given in References 9 and 12. Expressions for the general case where space reflection and time-reversal invariances are not imposed are given by the author in the Lawrence Radiation Laboratory Report UCRL-8859. (Unpublished) The consequences of these invariances are discussed there in some detail.)

$$I_0 = \frac{1}{4}|B|^2 + 2|C|^2 + \frac{1}{4}|G-N|^2 + \frac{1}{2}|N|^2 + \frac{1}{2}|H|^2$$

$$= |a|^2 + |m|^2 + 2|c|^2 + 2|g|^2 + 2|h|^2$$

$$I_0 P = 2\,\mathrm{Re}\, C^* N = 2\,\mathrm{Re}\, c^* (a+m)$$

$$I_0(1-D) = \frac{1}{4}|G-N-B|^2 + |H|^2$$

$$= 4|g|^2 + 4|h|^2$$

$$I_0 R = \frac{1}{2}\,\mathrm{Re}\,[(G-N)^*(N+H) + B^*(N-H)]\cos\theta/2$$

$$+ \mathrm{Im}[C^*(B+G-N)]\sin\theta/2$$

$$= [\,|a|^2 - |m|^2 - 4\,\mathrm{Re}\, hg^*\,]\cos\theta/2$$

$$+ 2\,\mathrm{Re}\, ic(a^* - m^*)\sin\theta/2$$

$$I_0 A = -\frac{1}{2}\,\mathrm{Re}\,[(G-N)^*(N+H) + B^*(N-H)]\sin\theta/2$$

$$+ \mathrm{Im}[C^*(B+G-N)]\cos\theta/2$$

$$= -[\,|a|^2 - |m|^2 - 4\,\mathrm{Re}\, hg^*\,]\sin\theta/2$$

$$+ 2\,\mathrm{Re}\, ic(a^* - m^*)\cos\theta/2$$

Table I (continued)

$$I_0 R' = - \mathrm{Im}[C^* (B+G-N)] \cos\theta/2 + \frac{1}{2} \mathrm{Re}\{(G-N)^* (N-H) + B^* (N+H)\} \sin\theta/2$$

$$= - 2 \mathrm{Re}\, ic\,(a-m)^* \cos\theta/2 + (|a|^2 - |m|^2 + 4 \mathrm{Re}\, gh^*) \sin\theta/2$$

$$I_0 A' = \frac{1}{2} \mathrm{Re}\{(G-N)^* (N-H) + B^* (N+H)\} \cos\theta/2 + \mathrm{Im}\, C^* (B+G-N) \sin\theta/2$$

$$= (|a|^2 - |m|^2 + 4 \mathrm{Re}\, gh^*) \cos\theta/2 + 2 \mathrm{Re}\, ic\,(a-m)^* \sin\theta/2$$

$$I_0 C_{KP} = - 2 \mathrm{Re}\, i\, CH^* = 4 \mathrm{Re}\, i\, ch^*$$

$$I_0 (1-C_{NN}) = \frac{1}{4} |G-N+B|^2 + \frac{1}{4} |G-N-B|^2$$

$$= |a-m|^2 + 4|g|^2$$

$$I_0 C_{NNN} = 2 \mathrm{Im}\, iCN^* = 2 \mathrm{Im}\, ic\,(a+m)^*$$

$$I_0 C_{NKP} = - \frac{1}{4} \mathrm{Im}[(G-N+B)(G-N-B)^* - 4 NH^*]$$

$$= - 2 \mathrm{Im}[(a-m) g^* + (a+m) h^*]$$

$$I_0 C_{N \times K_{in}} \begin{matrix} N \\ N \\ P \end{matrix} = \frac{1}{2} \cos(\theta/2) \mathrm{Im}[N(G-N-B)^* - (G-N+B)H^*]$$

$$+ \sin(\theta/2) \mathrm{Im}[iC\,(G-N-B)^*]$$

$$= 2 \cos(\theta/2) \mathrm{Im}[(a+m) g^* + (a-m) h^*]$$

$$+ 4 \sin(\theta/2) \mathrm{Im}[i\, c\, g^*]$$

Table II.

M Matrix Elements In Terms of R-Matrix Elements

$$M_{ss}(\theta,\phi) = (ik)^{-1} \sum P_\ell(\theta) \left(\frac{2\ell+1}{2}\right) R_\ell,$$

$$M_{11}(\theta,\phi) = (ik)^{-1} \sum P_\ell(\theta)$$

$$\times \left\{ \left(\frac{\ell+2}{4}\right) R_{\ell;\,\ell+1} + \left(\frac{2\ell+1}{4}\right) R_{\ell;\,\ell} + \left(\frac{\ell-1}{4}\right) R_{\ell;\,\ell-1} \right.$$

$$\left. + \frac{1}{4}\left[(\ell+1)(\ell+2)\right]^{1/2} R^{\ell+1} + \frac{1}{4}\left[(\ell-1)\ell\right]^{1/2} R^{\ell-1} \right\},$$

$$M_{00}(\theta,\phi) = + (ik)^{-1} \sum P_\ell(\theta)$$

$$\times \left\{ \left(\frac{\ell+1}{2}\right) R_{\ell;\,\ell+1} + \left(\frac{\ell}{2}\right) R_{\ell;\,\ell-1} - \frac{1}{2}\left[(\ell+1)(\ell+2)\right]^{1/2} R^{\ell+1} \right.$$

$$\left. - \frac{1}{2}\left[(\ell-1)(\ell)\right]^{1/2} R^{\ell-1} \right\},$$

$$M_{01}(\theta,\phi) = (ik)^{-1} e^{i\phi} \sum P_\ell^{\,1}(\theta) \left\{ -\frac{\sqrt{2}}{4}\left(\frac{\ell+2}{\ell+1}\right) R_{\ell;\,\ell+1} \right.$$

$$+ \frac{\sqrt{2}}{4}\left(\frac{2\ell+1}{\ell(\ell+1)}\right) R_{\ell;\,\ell} + \frac{\sqrt{2}}{4}\left(\frac{\ell-1}{\ell}\right) R_{\ell;\,\ell-1}$$

$$\left. - \frac{\sqrt{2}}{4}\left(\frac{\ell+1}{\ell+1}\right)^{1/2} R^{\ell+1} + \frac{\sqrt{2}}{4}\left(\frac{\ell-1}{\ell}\right)^{1/2} R^{\ell-1} \right\},$$

$$M_{10}(\theta,\phi) = (ik)^{-1} e^{-i\phi} \sum P_\ell^{\,1}(\theta) \left\{ \left(\frac{\sqrt{2}}{4}\right) R_{\ell;\,\ell+1} - \left(\frac{\sqrt{2}}{4}\right) R_{\ell;\,\ell-1} \right.$$

$$\left. - \frac{\sqrt{2}}{4}\left(\frac{\ell+2}{\ell+1}\right)^{1/2} R^{\ell+1} + \frac{\sqrt{2}}{4}\left(\frac{\ell-1}{\ell}\right)^{1/2} R^{\ell-1} \right\},$$

$$M_{1-1}(\theta,\phi) = (ik)^{-1} e^{-2i\phi} \sum P_\ell^{\,2}(\theta) \left\{ \left(\frac{1}{4(\ell+1)}\right) R_{\ell;\,\ell+1} \right.$$

$$- \left(\frac{2\ell+1}{4\ell(\ell+1)}\right) R_{\ell;\,\ell} + \left(\frac{1}{4\ell}\right) R_{\ell;\,\ell-1}$$

$$\left. + \frac{1}{4}\left[(\ell+1)(\ell+2)\right]^{-1/2} R^{\ell+1} + \frac{1}{4}\left[(\ell-1)\ell\right]^{-1/2} R^{\ell-1} \right\}$$

$$M_{-1-1}(\theta,\phi) = M_{11}(\theta,-\phi), \qquad M_{01}(\theta,\phi) = -M_{0-1}(\theta,-\phi),$$

$$M_{-11}(\theta,\phi) = M_{1-1}(\theta,-\phi), \qquad M_{10}(\theta,\phi) = -M_{-10}(\theta,-\phi).$$

Table II. (continued)

[a] The R-matrix elements are defined in Section 5 with the added constraints that R_{lj} for $j < 0$ and R^j for $j \leq 0$ are zero. The $P_l^m(\theta)$ are the associated Legendre polynomials as defined in Reference 13. The subscripts s and 1, 0, -1 designate the singlet and triplet states respectively. Invariance under time reversal, space reflection and rotations, and isotopic spin rotations is assumed. The sign conventions in this table are in agreement with Eq. 4.4 of the text. In References (14) and (18) the R^j are defined to have the opposite sign from those of this table. In the tables and graphs of Part III the conventions of Reference (14) are used, however.

Table III

Relations Involving Wolfenstein Parameters. (The Wolfenstein Parameters are defined in Eqs. 2.10 and 2.10. The subscripts s and 1, 0, -1 or M_{ij} designate singlet and triplet states respectively. The M - matrix elements are evaluated at $\phi = 0$. Invariance under time reversal, space reflection and rotations, and isotopic spin rotations are assumed.)

$$a = \frac{1}{4} \text{Tr } M = \frac{1}{4}(2M_{11} + M_{00} + M_{ss})$$

$$c = \frac{1}{4} \text{Tr } M \underline{\sigma}^{(1)} \cdot \underline{N} = \frac{1}{4} \text{Tr } M \underline{\sigma}^{(2)} \cdot \underline{N} = \frac{i\sqrt{2}}{4}(M_{10} - M_{01})$$

$$m = \frac{1}{4} \text{Tr } M \underline{\sigma}^{(1)} \cdot \underline{N}\, \underline{\sigma}^{(2)} \cdot \underline{N} = \frac{1}{4}(-2M_{1-1} + M_{00} - M_{ss})$$

$$g = \frac{1}{8} \text{Tr } M \{\underline{\sigma}^{(1)} \cdot \underline{P}\, \underline{\sigma}^{(2)} \cdot \underline{P} + \underline{\sigma}^{(1)} \cdot \underline{K}\, \underline{\sigma}^{(2)} \cdot \underline{K}\} = \frac{1}{4}(M_{11} + M_{1-1} - M_{ss})$$

$$h = \frac{1}{8} \text{Tr } M \{\underline{\sigma}^{(1)} \cdot \underline{P}\, \underline{\sigma}^{(2)} \cdot \underline{P} - \underline{\sigma}^{(1)} \cdot \underline{K}\, \underline{\sigma}^{(2)} \cdot \underline{K}\} = \frac{1}{4\cos\theta}(M_{11} - M_{1-1} - M_{00})$$

$$= \frac{\sqrt{2}}{4\sin\theta}(M_{10} + M_{01})$$

$$B = M_{ss}$$

$$C = \frac{i\sqrt{2}}{4}(M_{10} - M_{01})$$

$$H = \frac{-\sqrt{2}}{2\sin\theta}(M_{10} + M_{01})$$

$$N = \frac{1}{2}(M_{11} + M_{00} - M_{1-1})$$

$$(G\text{-}N) = (M_{11} + M_{1-1})$$

Table III (continued)

$$a = \frac{1}{4}(B + G + N)$$

$$g = \frac{1}{4}(G - B - N)$$

$$m = \frac{1}{4}(3N - B - G)$$

$$c = C$$

$$h = -\frac{1}{2}H$$

Table IV.

Expressions for the Wolfenstein parameters for the n-n system. The conventions for the sign of R^j are the same as in References 10 and 14. See these references for a description of the changes required for p-p scattering.

$$B = +\frac{1}{8ik}\{[8R_0-20R_2+27R_4]$$
$$+[60R_2-270R_4]\cos^2\theta+315R_4\cos^4\theta\},$$

$$C = -\frac{\sin\theta}{64k}\{[16R_{10}+24R_{11}-40R_{12}-40R_{32}-14R_{33}$$
$$+54R_{34}+54R_{54}+11R_{55}-65R_{56}]+[200R_{32}$$
$$+70R_{33}-270R_{34}-756R_{54}-154R_{55}+910R_{56}]\cos^2\theta$$
$$+[1134R_{54}+231R_{55}-1365R_{56}]\cos^4\theta\},$$

$$H = \frac{1}{32ik}\{[16a_{10}-24R_{11}+8R_{12}-8R_{32}+14R_{33}-6R_{34}$$
$$+6R_{54}-11R_{55}+5R_{56}-(32\sqrt{6})R^2+(48\sqrt{5})R^4]$$
$$+[40R_{32}-70R_{33}+30R_{34}-84R_{54}+154R_{55}-70R_{56}$$
$$+(80\sqrt{6})R^2-(456\sqrt{5})R^4]\cos^2\theta+[126R_{54}$$
$$-231R_{55}+105R_{56}+(504\sqrt{5})R^4]\cos^4\theta\},$$

$$N = +\frac{1}{32ik}\{[16R_{10}+24R_{11}+56R_{12}$$
$$-136R_{32}-14R_{33}-186R_{34}+294R_{54}+11R_{55}$$
$$+355R_{56}+(16\sqrt{6})R^2-(24\sqrt{5})R^4]\cos\theta$$
$$+[200R_{32}+70R_{33}+290R_{34}-1316R_{54}-154R_{55}-1610R_{56}+(56\sqrt{5})R^4]\cos^3\theta$$
$$+[1134R_{54}+231R_{55}+1407R_{56}]\cos^5\theta\},$$

$$G-N = +\frac{1}{32ik}\{[48R_{11}+48R_{12}$$
$$+32R_{32}-308R_{33}-60R_{34}-48R_{54}+638R_{55}$$

Table IV. (continued)

$$+70R_{56}-(32\sqrt{6})R^2+(48\sqrt{5})R^4]\cos\theta$$

$$+[420R_{33}+140R_{34}+112R_{54}$$

$$-2772R_{55}-420R_{56}-(112\sqrt{5})R^4]\cos^3\theta$$

$$+[-0R_{54}+2310R_{55}+462R_{56}]\cos^5\theta\}.$$