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Measurable Brooks's Theorem for Directed Graphs and the Complexity of Finite Borel Asymptotic Dimension

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Measurable Brooks's Theorem for Directed Graphs and the Complexity of Finite Borel Asymptotic Dimension

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Cecelia Higgins

2025

ABSTRACT OF THE DISSERTATION

Measurable Brooks's Theorem for Directed Graphs
and the Complexity of Finite Borel Asymptotic Dimension

by

Cecelia Higgins
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2025
Professor Andrew Marks, Co-Chair
Professor Anton Bernshteyn, Co-Chair

This dissertation concentrates on two prominent lines of research in modern descriptive set theory: Descriptive combinatorics and the theory of definable equivalence relations.

Chapter 2 is focused on a body of results related to Brooks's theorem, a fundamental graph theory result that characterizes the graphs of maximum degree d having chromatic number at most d. Measurable versions of Brooks's theorem, as well as definable versions of a similar result on list coloring known as Gallai's theorem, are surveyed in Section 2.2. Classical results that extend Brooks's theorem and Gallai's theorem to directed graphs are also discussed in Section 2.3. The chapter culminates in the proofs of both a definable version of Gallai's theorem for directed graphs and a measurable version of Brooks's theorem for directed graphs in Section 2.4. In the final two sections, potential connections with Johansson's theorem and the theory of LOCAL algorithms are explored.

Chapter 3 contains joint work with Jan Grebík on the projective complexity of finite Borel asymptotic dimension. The chapter begins with an overview of the study of countable Borel equivalence relations, followed by a survey of open problems related to the well-known question of whether the set of Borel codes of hyperfinite equivalence relations is $\boldsymbol{\Sigma}^1_2$-complete. An overview of Borel asymptotic dimension and the role it plays in the study of hyperfiniteness is provided in Sections 3.3 and 3.4. The main theorem that the

set of Borel codes of locally finite Borel graphs having finite Borel asymptotic dimension is $\mathbf{\Sigma}^1_2$-complete is described in Section 3.5. The chapter concludes with a brief section concerning potential extensions to the study of Borel semigroup actions.

The dissertation of Cecelia Higgins is approved.

Pavel Galashin

Itay Neeman

Anton Bernshteyn, Committee Co-Chair

Andrew Marks, Committee Co-Chair

University of California, Los Angeles

2025

To my mom,

my superheroine

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CHAPTER 1

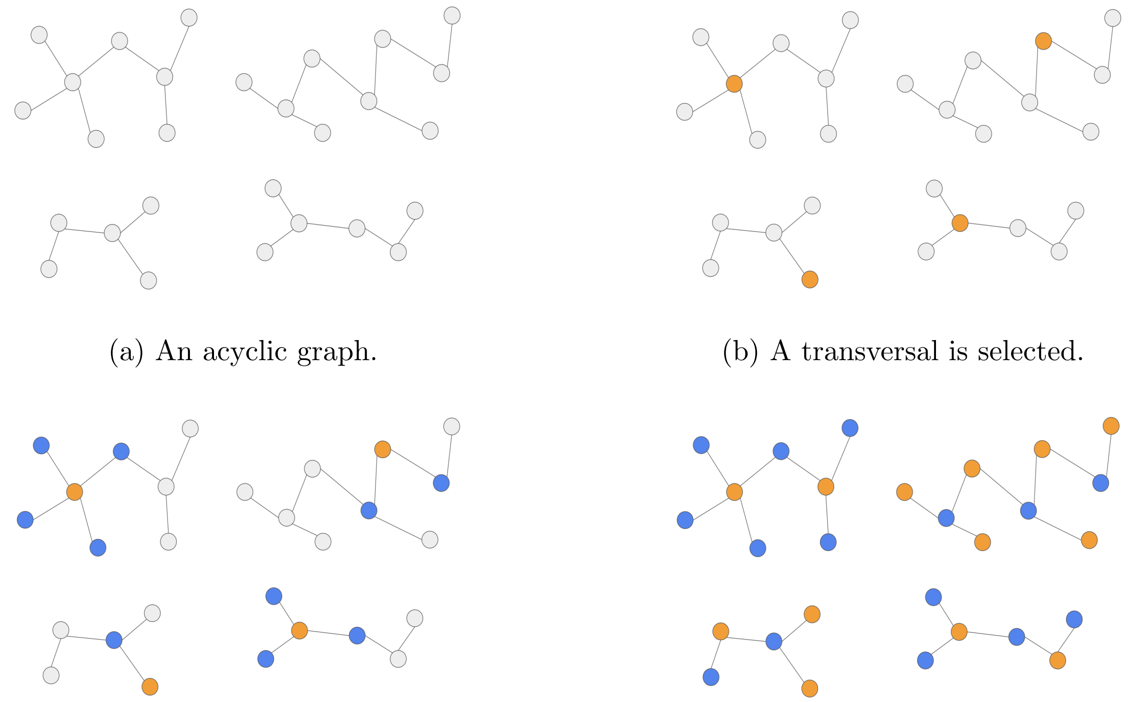
Introduction

1.1 Background and Motivation

This dissertation is concerned with two prominent threads through modern research in descriptive set theory. The first is descriptive combinatorics; the second is the theory of definable equivalence relations.

Descriptive combinatorics is the study of combinatorial problems under specific definability or measurability constraints. A keystone example is the problem of coloring the vertices of a graph. A *proper coloring* of a graph is an assignment of colors to the vertices of the graph so that adjacent vertices are assigned different colors; the *chromatic number* of the graph is the minimum number of colors needed to properly color it (see Section 2.1 for rigorous definitions). It is a basic result of classical graph theory that the chromatic number of any acyclic graph is 2: To begin the coloring, select one starting vertex in each connected component of the graph and color it orange. Then, given any vertex v in the graph, color v orange if the (unique) path from v to the starting vertex in its connected component has even length, and color v blue otherwise. See Figure 1.1 for an illustration.

Notice that, in the initial step of selecting a starting vertex in each component (that is, of selecting a *transversal* for the connectedness relation of the graph), the axiom of choice is invoked. Therefore, when a graph has uncountably many connected components, this initial step may be "non-constructive". One perspective on descriptive combinatorics is that it is an attempt to address the following body of questions: Suppose we choose to limit, or even disallow completely, the use of non-constructive tools to solve combinatorial



(a) An acyclic graph.

(b) A transversal is selected.

(c) The adjacent vertices are colored.

(d) All remaining vertices are colored.

Figure 1.1: A 2-coloring of an acyclic graph.

problems. Under these constraints, which problems remain solvable? And, what new proof techniques are needed to solve them?

The formal study of descriptive combinatorics was initiated in the 1990s with the seminal paper [KST99] of Kechris, Solecki, and Todorčević on Borel graph colorings. In a topological space, a coloring – or more generally, a set, function, or relation – that is Borel is nicely definable; each Borel set is expressible as the result of applying a countable sequence of countable operations, namely, complementation, countable union, and countable intersection, to the open sets in the space (see Section 1.2). While various important, more general notions of definability exist, many typical definable constructions are already Borel, and Borel combinatorial constructions are a major focus of descriptive set theory. A *Borel graph* is a graph on a topological space whose edge relation is Borel; a *Borel coloring* of a Borel graph is a *Borel* assignment of colors to vertices so that adjacent vertices are assigned different colors, and the *Borel chromatic number* of the graph is the minimum number of colors needed to give it a Borel coloring. The classical and Borel chromatic numbers of an individual Borel graph may differ strikingly.

Theorem 1.1.1. *[KST99] There is an acyclic Borel graph whose Borel chromatic number is uncountable.*

It is therefore perhaps unsurprising that many of the theorems of classical combinatorics fail in the definable setting. For instance, by a celebrated theorem of Marks [Mar16] whose proof depends crucially on Martin's Borel determinacy theorem [Mar75], various classical results on proper coloring, edge coloring, and perfect matching do not go through in the Borel context. However, several classical theorems can be recovered by allowing the use of non-constructive tools on "small" (for instance, null or meager) sets, or equivalently by relaxing the requirement that solutions to the combinatorial problem be Borel to the requirement that solutions be measurable in a weaker sense; see, for instance, [CMT16].[1]

[1]One major goal, which has seen increased attention in the past several years, is to determine how these different measurability settings relate to one another and to complexity classes originating in

Over the past three decades, descriptive combinatorics has evolved into a sophisticated and exciting field (see, for instance, the survey papers [KM20], [Pik21], and [Mar23]). In Chapter 2, we contribute to the descriptive combinatorics literature with the study of a particular combinatorial coloring problem for *directed graphs*, graphs whose edge relations need not be symmetric, under certain measurability constraints. Chapter 2 culminates in a proof of a measurable version of a classical combinatorial theorem known as Brooks's theorem for dicolorings. The material for this chapter is based on the author's paper [Hig24].

The second focus of this dissertation is on the theory of *countable Borel equivalence relations* (or *CBERs*). An equivalence relation on a topological space is *countable* if its classes are countable and *Borel* if it is Borel as a set of ordered pairs with respect to the product topology. Countable Borel equivalence relations are frequently encountered, for instance, as the orbit equivalence relations of Borel actions of countable discrete groups, or as the connectedness relations of locally countable Borel graphs (see Section 3.1 for further details).

Because of the diversity of the class of CBERs, it is useful to have a method for comparing different CBERs with one another. The typical definition for this purpose is *Borel reducibility* (see Section 3.1 for the definition), which is similar to the notion of many-one reducibility in computability theory. A prominent goal in descriptive set theory is to uncover the global structure of the hierarchy of countable Borel equivalence relations ordered by Borel reducibility. Of particular interest is the structure of this hierarchy at the level of the *hyperfinite* equivalence relations, which may be viewed as the simplest non-trivial CBERs [HKL90]. Hyperfiniteness, insofar as it reflects the idea of "minimal non-triviality", occupies a privileged place in various disciplines, including operator algebras [Con76; CFW81], combinatorics [Lov12], and ergodic theory and dynamical systems [KM04; Mat12; Ele18]. The term "hyperfinite" as it is used in the theory of countable Borel equivalence relations is defined in Section 3.1.

probability theory and theoretical computer science (see the survey papers [Ber22] and [GV25] and the introduction to [BBLW25]).

A major open problem originating in the work of Kechris [Kec93] concerns the *projective complexity* of hyperfinite equivalence relations. Similarly to how the NP-completeness of a problem in, say, computability theory implies that there is no "simple" characterization of the problem, $\mathbf{\Sigma}^1_2$-completeness, which is defined in Section 3.2, of a descriptive set theoretic problem implies that the problem cannot be rephrased in a "simple" way (for instance, as a dichotomy). It is immediate from the definition of 'hyperfinite' that the set of (Borel codes of) hyperfinite CBERs is $\mathbf{\Sigma}^1_2$, but the question of whether this set is $\mathbf{\Sigma}^1_2$-complete – that is, maximally complex amongst all $\mathbf{\Sigma}^1_2$ sets – remains open. Whether or not hyperfiniteness is $\mathbf{\Sigma}^1_2$-complete, an answer to this question would have significant consequences for our understanding of the structure of the Borel reducibility hierarchy (see Sections 3.1 and 3.2 for further details).

In Chapter 3, we study a strong form of hyperfiniteness known as *finite Borel asymptotic dimension*, which was first introduced in the paper [CJM+23] of Conley, Jackson, Marks, Seward, and Tucker-Drob. The main result of this chapter demonstrates that the set of (Borel codes of) locally finite Borel graphs having finite Borel asymptotic dimension is $\mathbf{\Sigma}^1_2$-complete. The material for this chapter is based on the author's joint work [GH24] with Jan Grebík.

1.2 Preliminaries

In this section, we present the preliminary material needed throughout the rest of the dissertation. Standard references for the results in this section are [Kec95] and [Tse22]. In Sections 2.1, 3.1, and 3.2, we provide additional chapter-specific preliminaries.

Let X be a topological space. The *Borel σ-algebra* of X, denoted $\mathbf{B}(X)$, is the σ-algebra generated by the open sets in X. A set $A \subseteq X$ is a *Borel set* if $A \in \mathbf{B}(X)$. The set $\mathbf{B}(X)$ is closed under the operations of countable union, countable intersection, and complementation. For a topological space Y, a map $f : X \to Y$ is a *Borel function* if $f^{-1}(V)$ is Borel in X for all Borel (equivalently, open) $V \subseteq Y$. The map f is a *Borel isomorphism* if f is bijective and both f and f^{-1} are Borel functions. Note that a function is Borel if and only if its graph is Borel.

The space X is *Polish* if it is separable and completely metrizable. Given a σ-algebra $\mathcal{A}$ on X, the space $(X, \mathcal{A})$ is a *standard Borel space* if there is a topology τ on X such that (X, τ) is a Polish space and $\mathcal{A}$ is the Borel σ-algebra generated by τ. Note that, if X is Polish, then $(X, \mathbf{B}(X))$ is a standard Borel space.

Proposition 1.2.1. *Let X be a Polish space, and let $B \subseteq X$ be a Borel set. Then equipped with the relative topology, B is a standard Borel space.*

The following theorem is frequently used.

Theorem 1.2.2. *All uncountable standard Borel spaces are Borel isomorphic.*

Given sets X and Y and a relation $R \subseteq X \times Y$, a *uniformization* of R is a function $f : \mathrm{dom}(R) \to Y$ such that, for all $x \in \mathrm{dom}(R)$, $(x, f(x)) \in R$. By the axiom of choice, any relation admits a uniformization. However, uniformizations are not in general definable. The next theorem, often referred to as the Lusin–Novikov uniformization theorem, provides sufficient conditions for the existence of a definable uniformization. Before stating the theorem, we summarize additional terminology: For $x \in X$, the *x-section* of R, denoted R_x, is the set $R_x = \{y \in Y : (x, y) \in R\}$, and the *projection* of R onto X, denoted $\mathrm{proj}_X(R)$, is the set $\mathrm{proj}_X(R) = \{x \in X : R_x \neq \emptyset\}$. Note also that, in general, whenever $R \subseteq X \times Y$ is Borel, then for each $x \in X$, R_x is Borel; indeed, consider the map $f_x : Y \to X \times Y$ defined by $y \mapsto (x, y)$. Note that f_x is Borel, since if $U \subseteq X, V \subseteq Y$ are open, then $f_x^{-1}(U \times V) = V$ if $x \in U$ and $\emptyset$ otherwise. Then since $f_x^{-1}(R) = R_x$, R_x is Borel. It is not true in general that $\mathrm{proj}_X(R)$ is Borel, but the Lusin-Novikov uniformization theorem gives a sufficient condition for this as well.

Theorem 1.2.3. *Let X and Y be standard Borel spaces, and let $R \subseteq X \times Y$ be a Borel relation such that, for each $x \in X$, the section R_x is countable. Then R admits a Borel uniformization, and $\mathrm{proj}_X(R)$ is Borel. Furthermore, there is a countable sequence $(f_n)_{n \in \mathbb{N}}$ of Borel functions $\mathrm{proj}_X(R) \to Y$ such that, for all $x \in \mathrm{proj}_X(R)$, $R_x = \{f_n(x) : n \in \mathbb{N}\}$.*

Given a set X, let $[X]^{<\infty}$ denote the set of finite subsets of X. The following fact will be useful in the sequel.

Proposition 1.2.4. *Let X be a Polish space. Then $[X]^{<\infty}$ is standard Borel with respect to the topology inherited from $\bigsqcup_{n \in \mathbb{N}} X^n$.*

We also frequently use the following.

Proposition 1.2.5. *Let X be a standard Borel space. Then there is a Borel linear ordering on X.*

Now let X be a(n uncountable) standard Borel space. Below we briefly summarize the terminology for Borel group actions on X and Borel graphs on X.

Let Γ be a countable discrete group. An action $\cdot : \Gamma \curvearrowright X$ is a *Borel action* if the map $\Gamma \times X \to X$ defined by $(\gamma, x) \mapsto \gamma \cdot x$ is Borel. The orbit equivalence relation of $\cdot$ is denoted E_Γ^X or merely E_Γ when X is understood.

Example 1.2.6. *Let $\alpha \in (0,1)$ be an irrational number, and let $\mathbb{Z}$ act on the standard Borel space $[0,1)$ by $1 \cdot x = x + \alpha \mod 1$. Then $\cdot$ is a Borel action.*

Suppose $\mathcal{G}$ is a graph with vertex set X (see Section 2.1 for further details on graphs). Then $\mathcal{G}$ is a *Borel graph* if the edge relation of $\mathcal{G}$ is Borel as a subset of X^2 with the product topology. The connectedness relation of $\mathcal{G}$ is denoted $E_\mathcal{G}$.

Example 1.2.7. *Consider the $\mathbb{Z}$-action from Example 1.2.6. Let $\mathcal{G}$ be the graph on $[0,1)$ in which vertices x and y are adjacent if and only if $1 \cdot x = y$ or $1 \cdot y = x$. Then $\mathcal{G}$ is a Borel graph. The graph $\mathcal{G}$ is often referred to as the* Schreier graph *of the $\mathbb{Z}$-action with generating set $\{-1, 1\}$.*

CHAPTER 2

Measurable Brooks's Theorem for Directed Graphs

In this chapter, we present a theorem of [Hig24] on the existence of measurable dicolorings for Borel directed graphs.

2.1 Graphs and Their Colorings

We begin with a brief overview of terminology and notation from graph theory. For a comprehensive source, see [Die25].

A *(simple, undirected) graph* is a pair $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, where $V(\mathcal{G})$ is the *vertex set* of $\mathcal{G}$ and $E(\mathcal{G}) \subseteq (V(\mathcal{G}))^2$ is the symmetric and irreflexive *edge relation* of $\mathcal{G}$. If $x, y \in V(\mathcal{G})$ are such that $(x, y) \in E(\mathcal{G})$ (equivalently, $(y, x) \in E(\mathcal{G})$), then x and y are *adjacent*, or y is a *neighbor* of x (and vice versa). The set of neighbors of x is denoted $N_{\mathcal{G}}(x)$ or simply $N(x)$ when $\mathcal{G}$ is understood. For a set $S \subseteq V(\mathcal{G})$, the set of neighbors of S, denoted $N_{\mathcal{G}}(S)$ or merely $N(S)$, is given by

$$N_{\mathcal{G}}(S) = \big\{y \in V(\mathcal{G}) : \text{for some } x \in S,\, y \in N_{\mathcal{G}}(x)\big\} = \bigcup_{x \in S} N_{\mathcal{G}}(x).$$

The *degree* of a vertex x in a graph $\mathcal{G}$, denoted $\deg_{\mathcal{G}}(x)$ or simply $\deg(x)$ if $\mathcal{G}$ is understood, is the number of vertices adjacent to x. The graph $\mathcal{G}$ is *locally finite* if each vertex has finite degree and *locally countable* if each vertex has countable degree. A locally finite graph $\mathcal{G}$ is of *bounded degree* if $\max\{\deg(x) : x \in V(\mathcal{G})\}$ exists and has *maximum degree* d if $\max\{\deg(x) : x \in V(\mathcal{G})\}$ (exists and) is equal to d.

For a graph $\mathcal{G}$, a *subgraph* $\mathcal{H}$ of $\mathcal{G}$ is a graph whose vertex set $V(\mathcal{H})$ is a subset of $V(\mathcal{G})$ and whose edge relation $E(\mathcal{H})$ is a subset of $\{(x, y) \in E(\mathcal{G}) : x, y \in V(\mathcal{H})\}$. If

$X \subseteq V(\mathcal{G})$, then the *subgraph induced by* X, denoted $\mathcal{G}[X]$, is the subgraph of $\mathcal{G}$ whose vertex set is X and whose edge relation is $\{(x, y) \in E(\mathcal{G}) : x, y \in X\}$.

For vertices $x, y \in V(\mathcal{G})$, a *path* from x to y is a finite sequence $(x_0, \ldots, x_k)$ of vertices of $\mathcal{G}$ such that $x = x_0$, $y = x_k$, and x_i is adjacent to x_{i+1} for all $i < k$. A path from x to itself is a *cycle*. Without loss of generality, we may assume that, for any path $(x_0, \ldots, x_k)$ with $k \geq 2$, the elements $x_0, \ldots, x_{k-1}$ are pairwise distinct. A graph is *acyclic* if it has no cycles. The relation E on $V(\mathcal{G})$ defined by $x \mathrel{E} y$ if and only if there is a path from x to y is an equivalence relation, known as the *connectedness relation* of $\mathcal{G}$ and denoted $E_{\mathcal{G}}$. The equivalence classes of $E_{\mathcal{G}}$ are the *connected components* of $\mathcal{G}$. The graph $\mathcal{G}$ is *connected* if $E_{\mathcal{G}}$ has just one class.

For graphs $\mathcal{G}$ and $\mathcal{H}$, a function $\varphi : V(\mathcal{G}) \to V(\mathcal{H})$ is a *graph homomorphism* if, whenever $x, y \in V(\mathcal{G})$ are adjacent in $\mathcal{G}$, $\varphi(x)$ and $\varphi(y)$ are adjacent in $\mathcal{H}$. The function φ is a *graph isomorphism* if φ is bijective and has the property that, whenever $x, y \in V(\mathcal{G})$ are not adjacent in $\mathcal{G}$, $\varphi(x)$ and $\varphi(y)$ are not adjacent in $\mathcal{H}$.

Let κ be a (potentially infinite) cardinal. We frequently identify κ with the set $\{\alpha : \alpha < \kappa\}$. A *(proper) κ-coloring* of $\mathcal{G}$ is a map $c : V(\mathcal{G}) \to \kappa$ such that, whenever $x, y \in V(\mathcal{G})$ are adjacent, $c(x) \neq c(y)$. When $\kappa = \aleph_0$, we frequently write $\mathbb{N}$ in place of κ. The *chromatic number* of $\mathcal{G}$, denoted $\chi(\mathcal{G})$, is the least cardinal κ for which there is a κ-coloring of $\mathcal{G}$.

Example 2.1.1. *Let $n \geq 3$. The* cyclic graph *on n vertices, denoted $\mathcal{C}_n$, is the (unique up to graph isomorphism) graph on n in which vertices i and j are adjacent if and only if $|i - j| = 1 \pmod{n}$. The chromatic number of $\mathcal{C}_n$ is 2 if n is even and 3 if n is odd.*

Example 2.1.2. *Let $d \in \mathbb{N}$. The* complete graph *on d vertices, denoted $\mathcal{K}_d$, is the (unique up to graph isomorphism) graph on d in which any two distinct vertices are adjacent. The chromatic number of $\mathcal{K}_d$ is d.*

The following is a basic proposition; the proof uses a simple greedy algorithm argument.

Proposition 2.1.3. *Let $\mathcal{G}$ be a graph of maximum degree $d \in \mathbb{N}$. Then $\chi(\mathcal{G}) \leq d+1$.*

Proof. Let $\kappa = |V(\mathcal{G})|$, and enumerate $V(\mathcal{G})$ as $\{x_\alpha : \alpha < \kappa\}$. We define a map $c : V(\mathcal{G}) \to d+1$ by induction on α. First, set $c(x_0) = 0$. If $\alpha < \kappa$ and $c(x_\beta)$ has been defined for all $\beta < \alpha$, define $c(x_\alpha)$ to be the least $i < d+1$ for which $i \notin \{c(y) : y \in N(x)\}$; this is possible since $|N(x)| \leq d$. Then c is a $(d+1)$-coloring of $\mathcal{G}$. □

Note that this greedy algorithm upper bound of $d+1$ on the chromatic number of $\mathcal{G}$ in terms of the maximum degree d cannot be improved. For instance, $\mathcal{K}_{d+1}$ has maximum degree d but does not admit a d-coloring. Similarly, if n is odd, then the maximum degree of $\mathcal{C}_n$ is 2, but $\mathcal{C}_n$ does not admit a 2-coloring.

2.2 Brooks's Theorem and Measurability

The two obstructions to an improved greedy algorithm bound described in the previous section – the odd cycles and the complete graphs – in fact constitute the only such obstructions, by a fundamental theorem of Brooks.

Theorem 2.2.1. *[Bro41] Let $\mathcal{G}$ be a graph of maximum degree $d \in \mathbb{N}$. If $d = 2$, suppose $\mathcal{G}$ has no odd cycles as subgraphs; if $d \geq 3$, suppose $\mathcal{G}$ does not have $\mathcal{K}_{d+1}$ as a subgraph. Then $\chi(\mathcal{G}) \leq d$.*

It is natural to ask whether this important result has a definable analogue: If the colorings of $\mathcal{G}$ are required to be Borel, can Theorem 2.2.1 be recovered?

Recall that a *Borel graph* is a graph whose vertex set is a standard Borel space X and whose edge relation is Borel with respect to the product topology on X^2, and the *Borel chromatic number* of a Borel graph is the minimum number of colors needed to give it a Borel proper coloring. Because of results such as Theorem 1.1.1, one should not expect in general for classical combinatorics to resemble Borel combinatorics. However, the appropriate analogue of Proposition 2.1.3 holds.

Theorem 2.2.2. *[KST99] Let $\mathcal{G}$ be a Borel graph of maximum degree $d \in \mathbb{N}$. Then $\chi_B(\mathcal{G}) \leq d+1$.*

Note that the proof of Proposition 2.1.3 involves an implicit well-ordering of $V(\mathcal{G})$. Such a well-ordering cannot typically be produced using only constructive or definable tools. Below, we provide the proof of Theorem 2.2.2 to emphasize the differences in allowable techniques between the classical context and the definable one.

Proof of Theorem 2.2.2. Let $X = V(\mathcal{G})$, and let τ be a topology such that (X, τ) is Polish. Then in particular, (X, τ) is second-countable, and so there is a countable basis $(U_n)_{n\in\mathbb{N}}$ for τ. Define $c : X \to \mathbb{N}$ as follows: For each $x \in X$, let $c(x)$ be the minimum $n \in \mathbb{N}$ such that $x \in U_n$ but $N(x) \cap U_n = \emptyset$. Then c is well-defined since $\mathcal{G}$ is locally finite, and it is easy to see that c is a proper coloring of $\mathcal{G}$. To see that c is Borel, note first that, when $U \subseteq X$ is Borel, the set $N(U)$ is also Borel:

$$\begin{aligned} N(U) &= \big\{y \in X : \text{for some } x \in U,\, (y, x) \in E(\mathcal{G})\big\} \\ &= \operatorname{proj}_X\big(E(\mathcal{G}) \cap (X \times U)\big), \end{aligned}$$

which is Borel by Theorem 1.2.3, which is applicable since $\mathcal{G}$ is locally finite. Now, for each $n \in \mathbb{N}$,

$$\begin{aligned} c^{-1}(\{n\}) &= \big\{x \in U_n : N(x) \cap U_n = \emptyset\big\} \cap \bigcap_{m<n} \big(V(\mathcal{G}) \setminus U_m \cup \{x \in U_m : N(x) \cap U_m \neq \emptyset\}\big) \\ &= U_n \setminus N(U_n) \cap \bigcap_{m<n} \big(V(\mathcal{G}) \setminus U_m \cup (U_m \cap N(U_m))\big), \end{aligned}$$

which is Borel. $\square$

However, Brooks's theorem has no Borel analogue, as made precise by the following theorem of Marks.

Theorem 2.2.3. *[Mar16] For each $d \in \mathbb{N}$, there is an acyclic Borel graph of maximum degree d with no Borel d-coloring.*

The proof of this result relies on a novel application of Martin's Borel determinacy theorem [Mar75]. Marks's technique has since been adapted to prove a variety of other impossibility results in descriptive combinatorics [CJM+20; BCG+24].

Several further questions arise naturally. First: If additional restrictions are placed on the class of Borel graphs under consideration, can a version of Brooks's theorem for Borel colorings be obtained for this smaller class? In order for this question to be of interest, the restriction should be both natural and not so reductive as to trivialize the problem. For instance, all finite graphs, and all colorings of finite graphs, are Borel, and so restricting attention to the class of finite graphs does not result in a meaningful question.[1]

One naturally arising class of graphs is the class of *(Borel) hyperfinite* graphs. For the definition, see Section 3.1. It is known that, unlike with finite graphs, the descriptive combinatorics of hyperfinite graphs is non-trivial [KST99]; however, not all Borel graphs are hyperfinite, and so one may ask whether a Borel version of Brooks's theorem can be proved for the class of hyperfinite graphs. Results of Conley, Jackson, Marks, Seward, and Tucker-Drob demonstrate that this is still impossible.

Theorem 2.2.4. *[CJM+20] For each $d \in \mathbb{N}$, there is an acyclic hyperfinite Borel graph of maximum degree d with no Borel d-coloring.*

We remark that further restricting the class of Borel graphs to the class of Borel graphs of *finite Borel asymptotic separation index* yields a recovery of Brooks's theorem for Borel colorings (see Section 3.3 and [BW25]).

Second: Is it possible to prove a version of Brooks's theorem for Borel colorings if additional subgraph configurations beyond complete graphs and odd cycles are excluded? For instance, perhaps by forbidding as subgraphs the graphs from Theorem 2.2.3, one could obtain a positive result. More broadly, it could be possible to prove a countable basis theorem – a result guaranteeing the existence of a Borel d-coloring for any Borel

[1] In fact, all *smooth* graphs exhibit trivial definable combinatorics by a theorem of [CM16]; see also Section 3.1.

graph of maximum degree d not containing $\mathcal{B}_n$ as a subgraph for any $n \in \mathbb{N}$, where $\{\mathcal{B}_n : n \in \mathbb{N}\}$ is a countable set of obstructions to a d-coloring. However, the following theorem of Brandt, Chang, Grebík, Grunau, Rozhoň, and Vidnyánszky rules out any such countable basis theorem.

Theorem 2.2.5. *[BCG+24] For each $d \in \mathbb{N}$, the set of (Borel codes of) acyclic Borel graphs of maximum degree d admitting Borel d-colorings is $\mathbf{\Sigma}^1_2$-complete.*

See Section 3.2 for further details.

Third: Suppose the condition imposed on the colorings is weakened somewhat, from a requirement that the colorings be Borel to a requirement that the colorings satisfy a less stringent measurability condition. Is it then possible to prove a version of Brooks's theorem for these measurable colorings? As it happens, there is a positive answer to this question, which we proceed to explicate.

From this point on, we consider only finite or countable graph colorings. Therefore, each coloring is implicitly viewed as having $\mathbb{N}$ (or an initial segment of $\mathbb{N}$) as its codomain.

Let $\mathcal{G}$ be a Borel graph on a standard Borel space X. A *Borel probability measure* μ on X is a probability measure on the Borel σ-algebra $\mathbf{B}(X)$. A coloring c of $\mathcal{G}$ is *μ-measurable* if $c^{-1}(\{n\})$ is μ-measurable for each $n \in \mathbb{N}$. The *μ-measurable chromatic number* of $\mathcal{G}$, denoted $\chi_\mu(\mathcal{G})$, is the minimum number of colors needed for a μ-measurable coloring. Similarly, if τ is a Polish topology compatible with the Borel structure of X, a coloring c is *τ-Baire-measurable* if $c^{-1}(\{n\})$ is τ-Baire-measurable for each $n \in \mathbb{N}$. The *τ-Baire-measurable chromatic number* of $\mathcal{G}$, denoted $\chi_\tau(\mathcal{G})$, is the minimum number of colors needed for a Baire-measurable coloring. Note that any Borel coloring is both μ-measurable and Baire-measurable. Therefore, for any Borel graph $\mathcal{G}$,

$$\chi_B(\mathcal{G}) \geq \max\{\chi_\mu(\mathcal{G}), \chi_\tau(\mathcal{G})\}.$$

Example 2.2.6. *Consider the graph $\mathcal{G}$ from Example 1.2.7. Note that $\mathcal{G}$ is acyclic, and so $\chi(\mathcal{G}) = 2$. However, if μ is the Lebesgue measure on $[0, 1)$, then $\chi_\mu(\mathcal{G}) = 3$: Assume for contradiction that there is a μ-measurable 2-coloring c of $\mathcal{G}$. Let $T : [0, 1) \to [0, 1)$*

be the translation defined by $x \mapsto 1 \cdot x$. Let $A = c^{-1}(\{0\}), B = c^{-1}(\{1\})$. Then A is μ-measurable, and $T^{-1}(A) = B$, so that $\mu(A) = \mu(B) = \frac{1}{2}$ since A and B are disjoint and T preserves μ. However, $T^{-2}(A) = A$, and since T^2 is ergodic, either $\mu(A) = 0$ or $\mu(A) = 1$, a contradiction. Thus, $\chi_\mu(\mathcal{G}) > 2$; by Theorem 2.2.2, $\chi_\mu(\mathcal{G}) = 3$.

Conley, Marks, and Tucker-Drob demonstrated that, when the requirement for Borel colorings is relaxed to a requirement for μ-measurable or τ-Baire-measurable colorings, a version of Brooks's theorem holds true.

Theorem 2.2.7. *[CMT16] Let $\mathcal{G}$ be a Borel graph on a standard Borel space X of maximum degree $d \geq 3$. Suppose $\mathcal{G}$ does not have $\mathcal{K}_{d+1}$ as a subgraph. Then $\chi_\mu(\mathcal{G}) \leq d$ for any Borel probability measure μ on X, and $\chi_\tau(\mathcal{G}) \leq d$ for any Polish topology τ compatible with the Borel structure of X.*

Note that, by Example 2.2.6, the $d = 2$ case for Brooks's theorem cannot be recovered for measurable colorings.

It is also instructive to consider a more general coloring problem known as *list coloring*. Given a graph $\mathcal{G}$ and a set Y, let $L : V(\mathcal{G}) \to [Y]^{<\infty}$ be a *list assignment*, that is, an assignment of finite subsets (lists) of Y to the vertices of $\mathcal{G}$. An *L-list-coloring* of $\mathcal{G}$ is a proper coloring c of $\mathcal{G}$ such that, for each $x \in V(\mathcal{G})$, $c(x) \in L(x)$. The graph $\mathcal{G}$ is *degree-list-colorable* if, whenever L is a list assignment such that $|L(x)| \geq \deg(x)$ for all $x \in V(\mathcal{G})$, then $\mathcal{G}$ has an L-list-coloring. Similarly, $\mathcal{G}$ is *degree-plus-one-list-colorable* if, whenever L is a list assignment such that $|L(x)| > \deg(x)$ for all $x \in V(\mathcal{G})$, then $\mathcal{G}$ has an L-list-coloring. Note that, for any $d \in \mathbb{N}$, the problem of producing a proper d-coloring of $\mathcal{G}$ can be construed as a list-coloring problem; set $L(x) = \{0, \ldots, d-1\}$ for each $x \in V(\mathcal{G})$.

A proof similar to that of Proposition 2.1.3 demonstrates that, classically, any graph of maximum degree $d \in \mathbb{N}$ is degree-plus-one-list-colorable. A classical "Brooks-like" theorem, typically known as Gallai's theorem, characterizes the graphs that are degree-list-colorable. First, additional terminology is required: For a graph $\mathcal{G}$, a subgraph $\mathcal{H}$ of $\mathcal{G}$ is *biconnected* if $\mathcal{H}$ is connected and has no cut vertices. A *block* in $\mathcal{G}$ is a connected subgraph that is a maximal biconnected subgraph, that is, a connected subgraph that is not a proper[2] subgraph of any biconnected subgraph. A connected component of $\mathcal{G}$ is a *Gallai tree* if each of its blocks is either an odd cycle or a complete graph.

Theorem 2.2.8. *[Bor77; ERT79] Let $\mathcal{G}$ be a graph. Suppose no connected component of $\mathcal{G}$ is a Gallai tree. Then $\mathcal{G}$ is degree-list-colorable.*

The relevant definitions are imported to the definable setting as follows: Given a Borel graph $\mathcal{G}$ and a standard Borel space Y, let $L : V(\mathcal{G}) \to [Y]^{<\infty}$ be a *Borel list assignment*, that is, a list assignment that is Borel. A *Borel L-list-coloring* of $\mathcal{G}$ is an L-list-coloring of $\mathcal{G}$ that is Borel. The graph $\mathcal{G}$ is *Borel degree-list-colorable* if, whenever L is a Borel list assignment such that $|L(x)| \geq \deg(x)$ for all $x \in V(\mathcal{G})$, then $\mathcal{G}$ has a Borel L-list-coloring. Similarly, $\mathcal{G}$ is *Borel degree-plus-one-list-colorable* if, whenever L is a Borel list assignment such that $|L(x)| > \deg(x)$ for all $x \in V(\mathcal{G})$, then $\mathcal{G}$ has a Borel L-list-coloring. The following is a definable version of Gallai's theorem due to Conley, Marks, and Tucker-Drob.

Theorem 2.2.9. *[CMT16] Let $\mathcal{G}$ be a locally finite Borel graph on a standard Borel space X. Suppose no connected component of $\mathcal{G}$ is a Gallai tree. Then $\mathcal{G}$ is Borel degree-list-colorable.*

2.3 Directed Graphs and Their Dicolorings

While undirected graphs enjoy a rich theory in both the classical and descriptive settings, they do not always suffice to capture or model various real-world phenomena. Therefore, it is useful to consider certain generalizations of undirected graphs.

One generalization involves removing the requirement that the edge relation of the graph be symmetric. A *directed graph* (or *digraph*) is a pair $\mathcal{D} = (V(\mathcal{D}), \overrightarrow{E}(\mathcal{D}))$, where

[2]Given subgraphs $\mathcal{H}, \mathcal{H}'$, $\mathcal{H}$ is a proper subgraph of $\mathcal{H}'$ if either $V(\mathcal{H}')$ is strictly larger than $V(\mathcal{H})$ *or* $E(\mathcal{H}')$ is strictly larger than $E(\mathcal{H})$. In particular, it is possible for $\mathcal{H}$ to be a proper subgraph of $\mathcal{H}'$ even when $V(\mathcal{H}) = V(\mathcal{H}')$.

$V(\mathcal{D})$ is the vertex set of $\mathcal{D}$ and $\overrightarrow{E}(\mathcal{D}) \subseteq (V(\mathcal{D}))^2$ is the irreflexive (but not necessarily symmetric) *directed edge relation* of $\mathcal{D}$. Directed graphs arise in a great many applications and are capable of modeling situations for which undirected graphs are too restrictive (see [BG09] for an extensive overview).

If $x, y \in V(\mathcal{D})$ are such that $(x, y) \in \overrightarrow{E}(\mathcal{D})$, then y is an *out-neighbor* of x and x is an *in-neighbor* of y. The set of out-neighbors of x is denoted $N^+_{\mathcal{D}}(x)$ or simply $N^+(x)$ when $\mathcal{D}$ is understood; similarly, the set of in-neighbors of x is denoted $N^-_{\mathcal{D}}(x)$ or $N^-(x)$. For a set $S \subseteq V(\mathcal{D})$, the set of out-neighbors of S, denoted $N^+_{\mathcal{D}}(S)$ or $N^+(S)$, is given by

$$N^+_{\mathcal{D}}(S) = \big\{y \in V(\mathcal{D}) : \text{for some } x \in S,\ (x, y) \in \overrightarrow{E}(\mathcal{D})\big\},$$

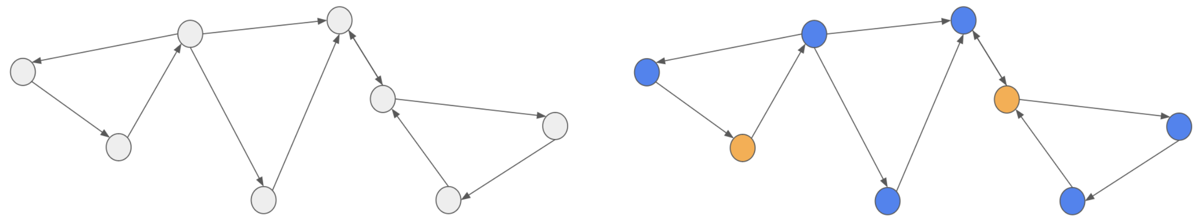
and the set of in-neighbors of S, denoted $N^-_{\mathcal{D}}(S)$ or $N^-(S)$, is given by

$$N^-_{\mathcal{D}}(S) = \big\{y \in V(\mathcal{D}) : \text{for some } x \in S,\ (y, x) \in \overrightarrow{E}(\mathcal{D})\big\}.$$

The *out-degree* of a vertex x in a digraph $\mathcal{D}$, denoted $\deg^+_{\mathcal{D}}(x)$ or simply $\deg^+(x)$, is equal to $|N^+_{\mathcal{D}}(x)|$, and the *in-degree* of x, denoted $\deg^-_{\mathcal{D}}(x)$, is equal to $|N^-_{\mathcal{D}}(x)|$. The *maximum degree* of x, denoted $\deg^{\max}_{\mathcal{D}}(x)$ or $\deg^{\max}(x)$, is $\max\{\deg^+(x), \deg^-(x)\}$, and the *minimum degree* of x, denoted $\deg^{\min}_{\mathcal{D}}(x)$ or $\deg^{\min}(x)$, is $\min\{\deg^+(x), \deg^-(x)\}$. The *maximum side* of x, denoted $N^{\max}_{\mathcal{D}}(x)$ or $N^{\max}(x)$, is $N^+(x)$ if $\deg^+(x) \geq \deg^{\min}(x)$ and $N^-(x)$ otherwise, and the *minimum side* of x, denoted $N^{\min}_{\mathcal{D}}(x)$ or $N^{\min}(x)$, is $N^+(x)$ if $\deg^+(x) \leq \deg^-(x)$ and $N^-(x)$ otherwise.

For a digraph $\mathcal{D}$, a *subdigraph* $\mathcal{D}'$ of $\mathcal{D}$ is a digraph whose vertex set $V(\mathcal{D}')$ is a subset of $V(\mathcal{D})$ and whose directed edge relation $\overrightarrow{E}(\mathcal{D}')$ is a subset of $\{(x, y) \in \overrightarrow{E}(\mathcal{D}) : x, y \in V(\mathcal{D}')\}$. If $X \subseteq V(\mathcal{D})$, then the *subdigraph induced by* X, denoted $\mathcal{D}[X]$, is the subdigraph of $\mathcal{D}$ whose vertex set is X and whose directed edge relation is $\{(x, y) \in \overrightarrow{E}(\mathcal{D}) : x, y \in X\}$.

For vertices $x, y \in V(\mathcal{D})$, a *directed path* from x to y is a finite sequence $(x_0, \ldots, x_k)$ of vertices of $\mathcal{D}$ such that $x = x_0$, $y = x_k$, and $(x_i, x_{i+1}) \in \overrightarrow{E}(\mathcal{D})$ for all $i < k$. A directed path from x to itself is a *directed cycle* (or *dicycle*). A directed cycle of the form (x, y, x) for some vertices $x \neq y$ is sometimes called a *digon*. Without loss of generality, we may



(a) A directed graph. (b) A 2-dicoloring.

Figure 2.1: A digraph of dichromatic number 2. Note that the central triangle is monochromatic as the cycle is not directed. Note also that the vertices in a digon must have distinct colors.

assume that, for any directed path $(x_0, \dots, x_k)$ with $k \geq 1$, the elements $x_0, \dots, x_{k-1}$ are pairwise distinct.

The *underlying (undirected) graph* of $\mathcal{D}$, denoted $\widetilde{\mathcal{D}}$, is the (undirected) graph whose vertex set is $V(\mathcal{D})$ and whose edge relation is $\overrightarrow{E}(\mathcal{D}) \cup \left(\overrightarrow{E}(\mathcal{D})\right)^{-1}$. The *connectedness relation* of $\mathcal{D}$, denoted $E_{\mathcal{D}}$, is the connectedness relation of $\widetilde{\mathcal{D}}$, and the *connected components* of $\mathcal{D}$ are the connected components of $\widetilde{\mathcal{D}}$. Conversely, given an undirected graph $\mathcal{G}$, the *symmetrization* of $\mathcal{G}$, denoted $\overleftrightarrow{\mathcal{G}}$, is the directed graph whose vertex set is $V(\mathcal{G})$ and whose directed edge relation is $E(\mathcal{G})$.[3] The graph $\mathcal{D}$ is *locally finite* (respectively, *locally countable*, of *bounded degree*) if $\tilde{\mathcal{D}}$ is locally finite (respectively, locally countable, of bounded degree).

For digraphs $\mathcal{D}$ and $\mathcal{D}'$, a function $\varphi : V(\mathcal{D}) \to V(\mathcal{D}')$ is a *digraph homomorphism* if, whenever $x, y \in V(\mathcal{D})$ are such that $(x, y) \in \overrightarrow{E}(\mathcal{D})$, $(\varphi(x), \varphi(y)) \in \overrightarrow{E}(\mathcal{D}')$. The function φ is a *digraph isomorphism* if φ is bijective and has the property that, whenever $x, y \in V(\mathcal{D})$ are such that $(x, y) \notin \overrightarrow{E}(\mathcal{D})$, $(\varphi(x), \varphi(y)) \notin \overrightarrow{E}(\mathcal{D}')$.

For a cardinal κ, a *κ-dicoloring* of $\mathcal{D}$ is a map $c : V(\mathcal{D}) \to \kappa$ such that, for any vertex x and any directed cycle $(x_0, \dots, x_k = x_0)$, there are $i, j < k$ such that $c(x_i) \neq c(x_j)$. The *dichromatic number* of $\mathcal{D}$, denoted $\overrightarrow{\chi}(\mathcal{D})$, is the least cardinal κ for which there is a κ-dicoloring of $\mathcal{D}$. See Figure 2.1. Note that, for any undirected graph $\mathcal{G}$, $\chi(\mathcal{G}) = \overrightarrow{\chi}(\overleftrightarrow{\mathcal{G}})$.

[3]Note that an undirected graph and its symmetrization are the same object.

The notion of dicoloring was introduced in work of Neumann-Lara [Neu82] and rediscovered independently in more recent work of Mohar [Moh03]. One compelling motivation for studying dicolorings originates in spectral graph theory. For an undirected graph $\mathcal{G}$ with vertex set $\{x_0, \ldots, x_n\}$, the *adjacency matrix* $A(\mathcal{G})$ of $\mathcal{G}$ is the $(n+1)\times(n+1)$ real matrix such that, for all $i, j \leq n$, the (i, j)-entry of $A(\mathcal{G})$ is 1 if x_i and x_j are adjacent and 0 otherwise. The *spectral radius* $\rho(\mathcal{G})$ of $\mathcal{G}$ is the maximum of the moduli of the eigenvalues of $A(\mathcal{G})$. A classical theorem of Wilf relates the chromatic number of $\mathcal{G}$ to its spectral radius.

Theorem 2.3.1. *[Wil67] Let $\mathcal{G}$ be a finite undirected graph. Then $\chi(\mathcal{G}) \leq \rho(\mathcal{G}) + 1$.*

For a directed graph $\mathcal{D}$ with vertex set $\{x_0, \ldots, x_n\}$, the *directed adjacency matrix* $\overrightarrow{A}(\mathcal{D})$ of $\mathcal{D}$ is the $(n + 1) \times (n + 1)$ real matrix such that, for all $i, j \leq n$, the (i, j)-entry of $\overrightarrow{A}(\mathcal{D})$ is 1 if $(x_i, x_j) \in \overrightarrow{E}(\mathcal{D})$ and 0 otherwise. The *spectral radius* $\overrightarrow{\rho}(\mathcal{D})$ of $\mathcal{D}$ is the maximum of the moduli of the eigenvalues of $\overrightarrow{A}(\mathcal{D})$.

Example 2.3.2. *Let $\mathcal{D}$ be the digraph on $\{1, 2, 3\}$ with directed edge relation*

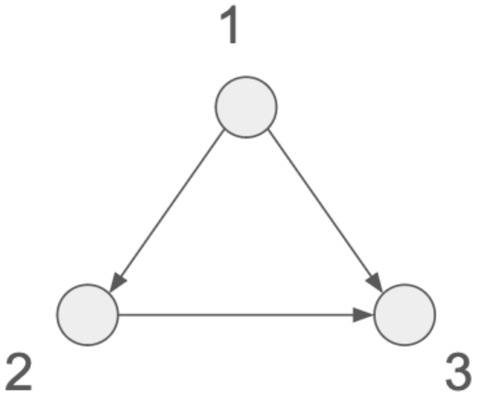
$$\{(1, 2), (1, 3), (2, 3)\}.$$

Then $\overrightarrow{A}(\mathcal{D})$ has 0 as its only eigenvalue; see Figure 2.2.

The digraph $\mathcal{D}$ in Example 2.3.2 is a *transitive tournament.* If one ignores the edge orientations in $\mathcal{D}$ and instead considers $\widetilde{\mathcal{D}}$, then the (ordinary) chromatic number of $\widetilde{\mathcal{D}}$ is 3. Therefore, $\chi(\widetilde{\mathcal{D}}) > \overrightarrow{\rho}(\mathcal{D}) + 1$. By passing to the dichromatic number, however, one obtains a version of Theorem 2.3.1 for directed graphs due to Mohar.

Theorem 2.3.3. *[Moh10] Let $\mathcal{D}$ be a finite digraph. Then $\overrightarrow{\chi}(\mathcal{D}) \leq \overrightarrow{\rho}(\mathcal{D}) + 1$.*

It is also possible to generalize the definitions for list colorings to digraphs. Given a digraph $\mathcal{D}$, a set Y, and a list assignment $L : V(\mathcal{G}) \to [Y]^{<\infty}$, an *L-list-dicoloring* of $\mathcal{D}$ is a dicoloring c of $\mathcal{D}$ such that, for each $x \in V(\mathcal{D})$, $c(x) \in L(x)$. The graph $\mathcal{D}$ is *degree-list-dicolorable* if, whenever L is a list assignment such that $|L(x)| \geq \deg^{\max}(x)$ for all



(a) The digraph from Example 2.3.2.

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) The directed adjacency matrix.

Figure 2.2: A digraph that serves as a counterexample to Theorem 2.3.1 when the ordinary chromatic number is used.

$x \in V(\mathcal{D})$, then $\mathcal{D}$ has an L-list-dicoloring. Similarly, $\mathcal{D}$ is *degree-plus-one-list-dicolorable* if, whenever L is a list assignment such that $|L(x)| \geq \deg^{\max}(x)+1$ for all $x \in V(\mathcal{D})$, then $\mathcal{D}$ has an L-list-dicoloring. A subdigraph $\mathcal{D}'$ of $\mathcal{D}$ is *biconnected* if $\widetilde{\mathcal{D}'}$ has no cut vertices. A *block* in $\mathcal{D}$ is a connected subdigraph that is a maximal biconnected subdigraph, that is, a connected subdigraph that is not a proper subdigraph of any biconnected subdigraph. A connected component of $\mathcal{D}$ is a *(directed) Gallai tree* if each of its blocks is a dicycle, the symmetrization of an odd cycle, or the symmetrization of a complete graph.

In the next section, we present results of Harutyunyan and Mohar that give versions of Brooks's theorem and Gallai's theorem for directed graphs and dicolorings, and we extend these results to the measurable setting.

2.4 Brooks's Theorem for Directed Graphs and Measurability

We proceed to describe versions of Brooks's theorem and Gallai's theorem for dicolorings before considering the descriptive context. A *symmetric cycle* is the symmetrization of a cyclic graph, and a *complete symmetric digraph* is the symmetrization of a complete graph.

Theorem 2.4.1. *[HM11] Let $\mathcal{D}$ be a directed graph such that each vertex has maximum degree at most $d \in \mathbb{N}$. If $d = 1$, suppose $\mathcal{D}$ has no directed cycles as subdigraphs; if $d = 2$, suppose $\mathcal{D}$ has no odd symmetric cycles as subdigraphs; and if $d \geq 3$, suppose $\mathcal{D}$ does not have $\overleftrightarrow{\mathcal{K}}_{d+1}$ as a subdigraph. Then $\overrightarrow{\chi}(\mathcal{D}) \leq d$.*

See also [AA23] for several proofs of the above theorem.

It is clear from Theorem 2.2.3 that there is no Borel version of Theorem 2.4.1: Take $d \in \mathbb{N}$ and an acyclic Borel graph $\mathcal{G}$ of maximum degree d having no Borel d-coloring, and notice that $\chi_B(\mathcal{G}) = \overrightarrow{\chi}_B(\overleftrightarrow{\mathcal{G}})$. However, as Marks observed in a personal communication with the author, even digon-free digraphs provide obstructions to a Borel version of Theorem 2.4.1.

Proposition 2.4.2. *Let $d \geq 3$. Then there is a digon-free Borel digraph $\mathcal{D}$ on a standard Borel space X such that $\deg^+(x) = \deg^-(x) = d$ for all $x \in X$, $\widetilde{\mathcal{D}}$ has no odd cycles as subgraphs, and $\mathcal{D}$ has no Borel d-dicoloring.*

Proof. For each $i < d$, let $\Gamma_i = \mathbb{Z}/4\mathbb{Z} = \langle \gamma_i \mid \gamma_i^4 = 1 \rangle$. Let $X = \left(2^{\mathbb{N}}\right)^{\Gamma_0 * \cdots * \Gamma_{d-1}}$, and consider the left shift action $\Gamma_0 * \cdots * \Gamma_{d-1} \curvearrowright X$ defined by

$$(\alpha \cdot x)(\beta) = x(\alpha^{-1}\beta)$$

for all $\alpha, \beta \in \Gamma_0 * \cdots * \Gamma_{d-1}, x \in X$. Define a Borel digraph $\mathcal{D}$ on $\mathrm{Free}(X)$ by placing a directed edge from x to y if $x \neq y$ and $\gamma_i \cdot x = y$ for some $i < d$. Then it is easy to see that $\deg^+(x) = \deg^-(x) = d$ for all $x \in \mathrm{Free}(X)$, $\mathcal{D}$ is digon-free, and $\widetilde{\mathcal{D}}$ has no odd cycles as subgraphs.

Assume for contradiction that there is a Borel d-dicoloring c of $\mathcal{D}$. For each $i < d$, let $A_i = \{x \in X : c(x) = i\}$. Then the sets $A_0, A_1, \ldots, A_{d-1}$ form a Borel partition of $\mathrm{Free}(X)$. By Theorem 1.2 in [Mar17], there are $i < d$ and a continuous injection $f : \mathrm{Free}\big((2^{\mathbb{N}})^{\Gamma_i}\big) \to \mathrm{Free}(X)$ that is equivariant under the left shift actions of Γ_i on $\mathrm{Free}\big((2^{\mathbb{N}})^{\Gamma_i}\big)$ and $\mathrm{Free}(X)$ with $\mathrm{range}(f) \subseteq A_i$.

Now let $y \in \mathrm{range}(f)$; then for some $x \in \mathrm{Free}\big((2^{\mathbb{N}})^{\Gamma_i}\big)$, $f(x) = y$. Notice that $(f(x), \gamma_i \cdot f(x), \gamma_i^2 \cdot f(x), \gamma_i^3 \cdot f(x), f(x))$ is a directed cycle in $\mathcal{D}$. However, since f is

Γ_i-equivariant, we have $\gamma_i \cdot f(x) = f(\gamma_i \cdot x)$, $\gamma_i^2 \cdot f(x) = f(\gamma_i^2 \cdot x)$, and $\gamma_i^3 \cdot f(x) = f(\gamma_i^3 \cdot x)$. Then $f(x), \gamma_i \cdot f(x), \gamma_i^2 \cdot f(x), \gamma_i^3 \cdot f(x)$ all belong to range(f) and hence to A_i, contradicting that A_i contains no dicycles. $\square$

See also Section 2.5.

Harutyunyan and Mohar also proved (classically) a digraph version of Gallai's theorem.

Theorem 2.4.3. *Let $\mathcal{D}$ be a directed graph. Suppose no connected component of $\mathcal{D}$ is a Gallai tree. Then $\mathcal{D}$ is degree-list-dicolorable.*

Our next goal is to prove a Borel version of Theorem 2.4.3.

Theorem 2.4.4. *[Hig24] Let $\mathcal{D}$ be a Borel directed graph. Suppose no connected component of $\mathcal{D}$ is a Gallai tree. Then $\mathcal{D}$ is Borel degree-list-dicolorable.*

This result will be used in the proof of the following main theorem.

Theorem 2.4.5. *[Hig24] Let $\mathcal{D}$ be a Borel directed graph on a standard Borel space X such that each vertex has maximum degree at most $d \geq 3$. Suppose $\mathcal{D}$ does not have the symmetrization of $\mathcal{K}_{d+1}$ as a subdigraph. Then if μ is a Borel probability measure on X, $\vec{\chi}_\mu(\mathcal{D}) \leq d$, and if τ is a Polish topology compatible with the Borel structure of X, $\vec{\chi}_\tau(\mathcal{D}) \leq d$.*

2.4.1 Preliminaries and Initial Results on Descriptive Dicolorings

We begin with several basic propositions on the definable and measurable combinatorics of digraphs. The first is a digraph analogue of Theorem 2.2.2. Note that the definitions of "Borel list dicoloring", "Borel degree-list-dicolorable", and so on are as in Section 2.2. It is typically left to the reader to verify that certain constructions are Borel.

Proposition 2.4.6. *Let $d \in \mathbb{N}$, and let $\mathcal{D}$ be a locally finite Borel digraph on a standard Borel space X. Suppose $\deg^{\min}(x) \leq d$ for all $x \in X$. Then $\vec{\chi}_B(\mathcal{D}) \leq d + 1$.*

Proof. We proceed by induction on d. Suppose $d = 0$; then $\mathcal{D}$ has no dicycles, and so clearly there is a Borel dicoloring of $\mathcal{D}$ with just one color.

Now let $d \geq 0$. Assume that, for any locally finite Borel digraph $\mathcal{D}'$ on a standard Borel space X', if $\deg^{\min}(x) \leq d$ for all $x \in X'$, then $\vec{\chi}_B(\mathcal{D}') \leq d + 1$. Suppose now $\deg^{\min}(x) \leq d + 1$ for all $x \in X$. Since the underlying graph $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is locally finite, it follows from Proposition 4.3 in [KST99] that there is a countable Borel proper coloring c of $\widetilde{\mathcal{D}}$. Now define $A_0 = \{x \in X : c(x) = 0\}$, and, having defined A_n, let

$$A_{n+1} = \big\{x \in X : c(x) = n + 1 \text{ and } N^{\min}(x) \cap (A_0 \cup \cdots \cup A_n) = \emptyset\big\}.$$

Write $A = \bigcup_{n \in \mathbb{N}} A_n$. Then A is Borel. We claim A contains no dicycles. Otherwise, let $(x_0, \ldots, x_k)$ be a dicycle in A. If $k = 2$, then assume without loss of generality that $c(x_0) > c(x_1)$. Then $N^{\min}(x_0) \cap A_{c(x_1)} \neq \emptyset$, so that $x_0 \notin A_{c(x_0)}$, a contradiction. If $k > 2$, then let $\alpha = \max\{c(x_0), c(x_1), \ldots, c(x_{k-1})\}$. Then for some $i < k$, $c(x_i) = \alpha$. If $N^{\min}(x_i) = N^+(x_i)$, then since $c(x_i) > c(x_{i+1})$ and since $x_{i+1} \in A_{c(x_{i+1})}$, we have $x_i \notin A_{c(x_i)}$, a contradiction. Similarly, if $N^{\min}(x_i) = N^-(x_i)$, then since $c(x_i) > c(x_{i-1})$ and since $x_{i-1} \in A_{c(x_{i-1})}$, we have $x_i \notin A_{c(x_i)}$.

We claim also that, for each $x \in X$, either $x \in A$ or there is $y \in N^{\min}(x) \cap A$. If $x \notin A$, then $x \notin A_{c(x)}$, so $c(x) > 0$ and $N^{\min}(x) \cap (A_0 \cup \cdots \cup A_{c(x)-1}) \neq \emptyset$. Therefore, $\deg^{\min}_{\mathcal{D}[X \setminus A]}(x) \leq d$. So by the inductive hypothesis, there is a Borel dicoloring $c' : (X \setminus A) \to \{0, \ldots, d\}$ of $\mathcal{D}[X \setminus A]$. Now define a Borel dicoloring c_0 of $\mathcal{D}$ by $c_0(x) = c'(x)$ if $x \in X \setminus A$ and $c_0(x) = d + 1$ if $x \in A$. $\square$

We remark that, if the maximum degree of each vertex of $\mathcal{D}$ is finite, then $\widetilde{\mathcal{D}}$ is locally finite, and by Proposition 4.3 of [KST99], there is a countable Borel proper coloring of $\widetilde{\mathcal{D}}$, so clearly there is a countable Borel dicoloring of $\mathcal{D}$. This was also observed by Raghavan and Xiao in [RX24]. Their work provides a dichotomy theorem characterizing which Borel digraphs have countable Borel dicolorings [RX24]. In particular, Raghavan and Xiao define a canonical digraph – an analogue of the graph $\mathcal{G}_0$ [KST99] – and prove that the only digraphs that do not have countable Borel dicolorings are those digraphs that admit continuous homomorphisms from this canonical digraph.

Next, we show that any locally finite Borel digraph is Borel degree-plus-one-list-dicolorable. For the proof, we first separate the vertex set of the underlying graph into countably many independent sets $A_0, A_1, \ldots$. Then we list-dicolor the independent sets in order, beginning with A_0. After we have list-dicolored $A_0 \cup \cdots \cup A_n$, we update the list assignments of each vertex $x \in A_{n+1}$ by removing the colors which appear among both the out-neighbors and the in-neighbors of x.

Theorem 2.4.7. *Let $\mathcal{D}$ be a locally finite Borel digraph on a standard Borel space X. Let Y be Polish, and let $L : X \to [Y]^{<\infty}$ be a Borel list assignment such that $|L(x)| >$ $\deg^{\max}(x)$ for all $x \in X$. Then $\mathcal{D}$ has a Borel L-dicoloring. In particular, $\mathcal{D}$ is Borel degree-plus-one-list-dicolorable.*

Proof. Since $\widetilde{\mathcal{D}}$ is locally finite, by Proposition 4.3 in [KST99], there is a countable Borel proper coloring c of $\widetilde{\mathcal{D}}$. For each $n \in \mathbb{N}$, define $A_n = \{x \in X : c(x) = n\}$. Now let $<_Y$ be a Borel linear ordering on Y, and define $c_0 : A_0 \to Y$ by letting $c_0(x)$ be the $<_Y$-least element of $L(x)$ for each $x \in A_0$.

For each $i \in \mathbb{N}$, write $B_i = \bigcup_{j \leq i} A_j$. Let $n > 0$, and assume that, for each $i \leq n$, there are a list assignment $L_i : A_i \to [Y]^{<\infty}$ and a Borel function $c_i : B_i \to Y$ such that the following conditions obtain:

- $L_0(x) = L(x)$ for all $x \in A_0$;
- $c_i \restriction B_j = c_j$ for all $j < i$;
- c_i is a Borel L-dicoloring of $\mathcal{D}[B_i]$;
- $L_{i+1}(x)$ is equal to $L(x) \setminus \{\alpha \in Y :$ there exist $y, z \in B_i$ such that $y \in N^+(x), z \in N^-(x)$, and $c_i(y) = c_i(z) = \alpha\}$ for all $x \in A_{i+1}$; and
- $|L_{i+1}(x)| > \deg^{\max}_{\mathcal{D}[X \setminus B_i]}(x)$ for all $x \in A_{i+1}$.

We proceed to define a list assignment $L_{n+1} : A_{n+1} \to [Y]^{<\infty}$ and a Borel function $c_{n+1} : B_{n+1} \to Y$ such that:

1. $c_{n+1} \restriction B_i = c_i$ for all $i \leq n$;

2. c_{n+1} is a Borel L-dicoloring of $\mathcal{D}[B_{n+1}]$;

3. $L_{n+1}(x)$ is equal to $L(x) \setminus \{\alpha \in Y : \text{there exist } y, z \in B_n \text{ such that } y \in N^+(x), z \in N^-(x), \text{ and } c_n(y) = c_n(z) = \alpha\}$ for all $x \in A_{n+1}$; and

4. $|L_{n+1}(x)| > \deg^{\max}_{\mathcal{D}[X \setminus B_n]}(x)$ for all $x \in A_{n+1}$.

Let $x \in A_{n+1}$, and define $L_{n+1}(x)$ as in (3) above. Note that

$$\deg^{\max}_{\mathcal{D}[X \setminus B_n]}(x) = \max\{\deg^+(x) - |N^+(x) \cap B_n|, \deg^-(x) - |N^-(x) \cap B_n|\}$$

and that

$$|L_{n+1}(x)| \geq |L(x)| - \min\{|N^+(x) \cap B_n|, |N^-(x) \cap B_n|\}.$$

Without loss of generality, assume $\max\{\deg^+(x) - |N^+(x) \cap B_n|, \deg^-(x) - |N^-(x) \cap B_n|\} = \deg^+(x) - |N^+(x) \cap B_n|$. We have

$$\begin{aligned}
\deg^+(x) - |N^+(x) \cap B_n| &\leq \deg^{\max}(x) - |N^+(x) \cap B_n| \\
&\leq |L(x)| - 1 - |N^+(x) \cap B_n| \\
&\leq |L(x)| - 1 - \min\{|N^+(x) \cap B_n|, |N^-(x) \cap B_n|\} \\
&\leq |L_{n+1}(x)| - 1.
\end{aligned}$$

So, $\deg^{\max}_{\mathcal{D}[X \setminus B_n]}(x) < |L_{n+1}(x)|$. In particular, $|L_{n+1}(x)| \geq 1$. Then for any $x \in B_{n+1}$, we may define $c_{n+1}(x) = c_n(x)$ if $x \notin A_{n+1}$ and $c_{n+1}(x) = \alpha$, where α is the $<_Y$-least element of $L_{n+1}(x)$, if $x \in A_{n+1}$. It is then easy to check that conditions (1), (3), and (4) above are satisfied by L_{n+1} and c_{n+1}. To see that condition (2) is satisfied, assume for contradiction that there is a c_{n+1}-monochromatic dicycle $(x_0, x_1, \ldots, x_k = x_0)$ in B_{n+1}. If $k = 2$, then without loss of generality, assume $c(x_0) > c(x_1)$. Then $x_1 \in N^+(x_0) \cap N^-(x_0) \cap A_{c(x_1)}$, so that $L_{c(x_0)}(x_0)$ does not contain $c_{n+1}(x_1)$, which is a contradiction since $c_{n+1}(x_0) = c_{n+1}(x_1)$. If $k > 2$, then there is $i < k$ such that $c(x_i) > c(x_{i-1})$ and $c(x_i) > c(x_{i+1})$. Then if we set $\alpha = c_{n+1}(x_{i-1}) = c_{n+1}(x_{i+1})$, we have $\alpha \notin L_{c(x_i)}(x_i)$, which is a contradiction since $c_{n+1}(x_i) = \alpha$.

Now define $c' : X \to Y$ by $c' = \bigcup_{n\in\mathbb{N}} c_n$. Then c' is a Borel L-dicoloring of $\mathcal{D}$. $\square$

Now we turn to the measurable setting. The proofs of Theorems 2.4.5 and 2.4.4 rely heavily on the one-ended spanning forest technique developed by Conley, Marks, and Tucker-Drob in [CMT16]. If f is a function on a set X, then f is *one-ended* if there is no sequence $(x_n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, $f(x_{n+1}) = x_n$. Note that one-ended functions do not have fixed points.

We first show that, if a digraph $\mathcal{D}$ of bounded degree admits a Borel one-ended function, then $\mathcal{D}$ is Borel degree-list-dicolorable. The proof combines the proof of Theorem 2.4.7 with the proof of Lemma 3.9 in [CMT16]: First, given a Borel one-ended function f, we separate the graph into layers using the ranks provided by f. Within each layer, we then construct a finite sequence of sets that are independent in the underlying graph and that cover the entire layer. Then we list-dicolor the independent sets in order, removing from the list-assignment of each vertex in the subsequent independent set the colors which appear already among both its out-neighbors and its in-neighbors.

Theorem 2.4.8. *Let $\mathcal{D}$ be a Borel digraph of bounded degree on a standard Borel space X with no isolated vertices, let $B \subseteq X$ be Borel, and let $f : B \to X$ be a one-ended Borel function whose graph is contained in $\widetilde{\mathcal{D}}$. Let $d \in \mathbb{N}$ be such that $\deg^{\max}(x) \leq d$ for all $x \in X$. Let Y be standard Borel, and let $L : X \to [Y]^{<\infty}$ be a Borel list assignment such that $|L(x)| \geq \deg^{\max}(x)$ for all $x \in B$. Then $\mathcal{D}[B]$ has a Borel L-dicoloring. In particular, $\mathcal{D}[B]$ is Borel degree-list-dicolorable.*

Proof. For each $n \in \mathbb{N}$, write

$$f^n[B] = \{x \in X : \text{there exist } x_1, x_2, \ldots, x_n \in B \text{ such that}$$
$$f(x_1) = x \text{ and } f(x_{i+1}) = x_i \text{ for all } i < n\}$$

Let $B_n = B \cap (f^n[B] \setminus f^{n+1}[B])$. Note that, if $n \neq m$, then $B_n \cap B_m = \emptyset$. We claim that $B = \bigcup_{n\in\mathbb{N}} B_n$. Assume for contradiction that there is $x \in B \setminus \bigcup_{n\in\mathbb{N}} B_n$. Then $x \in f^n[B]$ for all $n \in \mathbb{N}$. So the set $f^{-\mathbb{N}}(x) = \{y \in B : \text{for some } n \geq 1, \text{ there are } x_1, x_2, \ldots, x_{n-1} \in B \text{ such that } f(x_1) = x, f(x_{i+1}) = x_i \text{ for all } i < n,\ f(y) = x_{n-1}\}$ is an infinite, finitely branching tree with root x. By König's lemma, there is an infinite branch $(y_n)_{n\in\mathbb{N}}$ through $f^{-\mathbb{N}}(x)$. Then for each $n \in \mathbb{N}$, $f(y_{n+1}) = y_n$, contradicting that f is one-ended.

We proceed to L-dicolor $\mathcal{D}[B]$ one layer at a time. First, we handle B_0. Note that $\widetilde{\mathcal{D}}[B_0]$ is an undirected graph in which each vertex has degree at most $2d$, since $\deg^{\max}(x) \leq d$ for all $x \in B$. So, by Proposition 4.2 in [KST99], there is a Borel maximal $\widetilde{\mathcal{D}}$-independent set $B_0^0 \subseteq B_0$. Now, let $0 < i < 2d-1$, and suppose $B_0^j \subseteq B_0$ has been defined for each $j \leq i$ such that B_0^j is a Borel maximal $\widetilde{\mathcal{D}}$-independent set in $B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{j-1})$. Then let B_0^{i+1} be a Borel maximal $\widetilde{\mathcal{D}}$-independent set in $B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^i)$. Finally, note that, if $x \in B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d-1})$, then $\deg^{\max}_{\mathcal{D}[B_0\setminus(B_0^0\cup B_0^1\cup\cdots\cup B_0^{2d-1})]}(x) = 0$. Then $B_0^{2d} = B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d-1})$ is a Borel $\tilde{D}$-independent set. For each $i \leq 2d$, write $B_0^{<i} = \emptyset$ if $i = 0$ and $B_0^{<i} = \bigcup_{j<i} B_0^j$ if $i > 0$, and write $B_0^{\leq i} = \bigcup_{j\leq i} B_0^j$.

Now note that, for each $x \in B_0^0$, $\deg^{\max}_{\mathcal{D}[B_0^0]}(x) = 0$. Since $\mathcal{D}$ has no isolated vertices, and since $|L(x)| \geq \deg^{\max}_{\mathcal{D}}(x)$ for each $x \in X$, it follows that $|L(x)| \geq 1$ for each $x \in B_0^0$. So, by Theorem 2.4.7, there is a Borel L-dicoloring c_0^0 of $\mathcal{D}[B_0^0]$. Now let $1 \leq i < 2d$, and suppose Borel L-dicolorings $c_0^j : B_0^{\leq j} \to Y$ have been defined for all $j \leq i$ so that the following conditions hold:

- If $j' < j$, then $c_0^j \restriction (B_0^{\leq j'}) = c_0^{j'}$; and
- For any $1 \leq j \leq i$ and for any $x \in B_0^j$, if x has an out-neighbor $y \in B_0^{<j}$ and an in-neighbor $z \in B_0^{<j}$ with $\alpha = c_0^{j-1}(y) = c_0^{j-1}(z)$, then $c_0^j(x) \neq \alpha$.

Then for each $x \in B_0^{i+1}$, let

$$L_0^{i+1}(x) = L(x) \setminus \{\alpha \in Y : \text{there are } y, z \in B_0^{\leq i} \text{ such that } c_0^i(y) = c_0^i(z) = \alpha\}.$$

Then, since x has at least one neighbor, namely $f(x)$, not contained in B_0, we have

$$\begin{aligned} |L_0^{i+1}(x)| &\geq |L(x)| - (\deg^{\max}(x) - 1) \\ &\geq \deg^{\max}(x) - (\deg^{\max}(x) - 1) \\ &= 1, \end{aligned}$$

so that $0 = \deg^{\max}_{\mathcal{D}[B_0^{i+1}]}(x) < |L_0^{i+1}(x)|$. Therefore, again by Theorem 2.4.7, there is a Borel L_0^{i+1}-dicoloring c of $\mathcal{D}[B_0^{i+1}]$. Define $c_0^{i+1} : B_0^{\leq i+1} \to Y$ by $c_0^{i+1}(x) = c(x)$ if $x \in B_0^{i+1}$ and $c_0^{i+1}(x) = c_0^i(x)$ otherwise. Finally, define $c_0 = c_0^{2d}$.

We now proceed to handle B_1. The procedure is similar to that for dicoloring B_0; we construct $\widetilde{\mathcal{D}}$-independent sets $B_1^0, B_1^1, \dots, B_1^{2d}$ as above and color one set at a time to obtain a coloring c_1 of $\mathcal{D}[B_1]$. However, in this case, if $x \in B_1^0$, then

$$L_1^0(x) = L(x) \setminus \{\alpha \in Y : \text{there exist } y, z \in B_0 \text{ such that } y \in N^+(x),\\ z \in N^-(x) \text{ and } c_0(y) = c_0(z) = \alpha\}.$$

Then since x has at least one neighbor, namely $f(x)$, not in B_1, we have $0 = \deg^{\max}_{\mathcal{D}[B_1^0]}(x) < |L_1^0(x)|$, so that $\mathcal{D}[B_1^0]$ may be L_1^0-dicolored. The sets $B_2, B_3, \dots$ are handled similarly by functions $c_2, c_3, \dots$.

Now let $c = \bigcup_{n \in \mathbb{N}} c_n$. Then c is Borel. Assume for contradiction that there is a c-monochromatic cycle $C = (x_0, x_1, \dots, x_k)$ in $\mathcal{D}$. If $k = 2$, then let $n \in \mathbb{N}$ be maximal with $C \cap B_n \neq \emptyset$, and let $j \leq 2d$ be maximal with $C \cap B_n^j \neq \emptyset$. Without loss of generality, suppose $x_0 \in B_n^j$. Then $x_1 \notin B_n^j$. So, $c(x_1) \notin L_n^j(x_0)$, a contradiction since $c(x_0) = c(x_1)$. If $k > 2$, then again let $n \in \mathbb{N}$ be maximal with $C \cap B_n \neq \emptyset$, and let $j \leq 2d$ be maximal with $C \cap B_n^j \neq \emptyset$. Then there is $i \leq k$ such that $x_i \in B_n^j$. It follows that $x_{i-1}, x_{i+1} \notin B_n^j$. Then either $x_{i-1} \in B_m$ for some $m < n$, or $x_{i-1} \in B_n^{j'}$ for some $j' < j$, and similarly for x_{i+1}. In any case, it follows that $\alpha = c(x_{i-1}) = c(x_{i+1})$ is not an element of $L_n^j(x_i)$, a contradiction since $c(x_i) = \alpha$. Thus, c is a Borel L-dicoloring of $\mathcal{D}[B]$. $\square$

We will use the following proposition to construct one-ended functions. For a graph $\mathcal{G}$ on a set X and for $A \subseteq X$, the notation $[A]_{\mathcal{G}}$ is used to denote the set of all points $x \in X$ such that x has a path through $\mathcal{G}$ to some point of A. For a vertex x of $\mathcal{G}$, we write $[x]_{\mathcal{G}}$ for $[\{x\}]_{\mathcal{G}}$, that is, the connected component of x in $\mathcal{G}$. Also, if $\mathcal{D}$ is a digraph, then we write $[A]_{\mathcal{D}}$ for $[A]_{\widetilde{\mathcal{D}}}$.

Proposition 2.4.9. *[CMT16] Let $\mathcal{G}$ be a locally finite Borel graph on a standard Borel space X, and let $A \subseteq X$ be Borel. Then there is a one-ended Borel function $f : ([A]_{\mathcal{G}} \setminus$*

$A) \to [A]_{\mathcal{G}}$ whose graph is contained in $E(\mathcal{G})$.

2.4.2 The Proof of Theorem 2.4.4

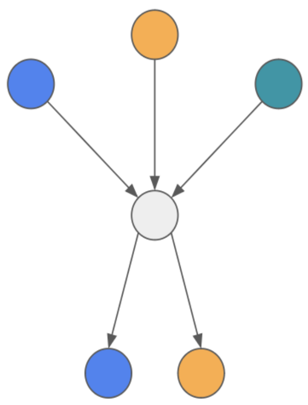
Our first step toward Theorem 2.4.4 is to show that digraphs of bounded degree are Borel degree-list-dicolorable on connected components that contain vertices whose minimum degree is smaller than the maximum degree. To construct the dicoloring, we first reserve a set of small-minimum-degree vertices that is independent in the underlying graph and dicolor the vertices outside of this reserved set using a one-ended function. Then, since each reserved vertex has small minimum degree, there is at least one color in its list of available colors that does not appear among both its out-neighbors and its in-neighbors; then the initial dicoloring can be extended to this vertex. The proof resembles the first part of the proof of Theorem 1.2 in [CMT16]. For an illustration, see Figure 2.3.

Proposition 2.4.10. *Let $\mathcal{D}$ be a Borel digraph of bounded degree on a standard Borel space X, and let $B = \{x \in X : \deg^{\min}(x) < \deg^{\max}(x)\}$. Then $\mathcal{D}[[B]_{\mathcal{D}}]$ is Borel degree-list-dicolorable.*

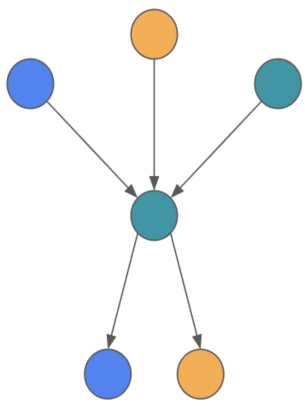
Proof. Let Y be standard Borel, and let $L : X \to [Y]^{<\infty}$ be a Borel list assignment such that $|L(x)| \geq \deg^{\max}(x)$ for each $x \in X$. Since $\widetilde{\mathcal{D}}$ is of bounded degree, by Proposition 4.2 in [KST99], there is a Borel maximal $\widetilde{\mathcal{D}}$-independent set $B' \subseteq B$. Since $[B']_{\mathcal{D}} = [B]_{\mathcal{D}}$, by Proposition 2.4.9, there is a one-ended Borel function $f : ([B]_{\mathcal{D}} \setminus B') \to [B]_{\mathcal{D}}$ whose graph is contained in $E(\widetilde{\mathcal{D}})$. Then by Theorem 2.4.8, $\mathcal{D}[[B]_{\mathcal{D}} \setminus B']$ has a Borel L-dicoloring.

Now we extend c to a function $c' : [B]_{\mathcal{D}} \to Y$ by letting $c'(x) = c(x)$ if $x \notin B'$; if $x \in B'$, then let $<_Y$ be a Borel linear ordering of Y, and define $c'(x)$ to be the $<_Y$-least color $\alpha \in L(x)$ such that there are no $y \in N^+(x) \cap ([B]_{\mathcal{D}} \setminus B')$ and $z \in N^-(x) \cap ([B]_{\mathcal{D}} \setminus B')$ with $c(y) = c(z) = \alpha$. Such an α exists since $\deg^{\min}(x) < \deg^{\max}(x)$, so that at most $\deg^{\max}(x) - 1$ colors from $L(x)$ appear in $c[N^+(x)] \cap c[N^-(x)]$. Then c' is a Borel L-dicoloring of $\mathcal{D}[[B]_{\mathcal{D}}]$. $\square$

From now on, we say that $\mathcal{D}$ is *Eulerian* if, for each vertex, its in-degree and its



(a) A vertex with minimum degree 2.



(b) The dicoloring is extended.

Figure 2.3: Completing the dicoloring in the proof of Proposition 2.4.10 on components with vertices of low minimum degree for the $d = 3$ case. In this example, the central uncolored vertex has no green out-neighbor. Therefore, it is possible to extend the dicoloring by assigning the color green to the central vertex.

out-degree are the same. So, the proposition above demonstrates that, if $\mathcal{D}$ is a bounded-degree Borel digraph that is not Eulerian, then $\mathcal{D}$ is Borel degree-list-dicolorable.

As a corollary to the proof of Proposition 2.4.10, we obtain the following.

Corollary 2.4.11. *Let $\mathcal{D}$ be a Borel digraph on a standard Borel space X, and let d be such that $\deg^{\max}(x) \leq d$ for all $x \in X$. Let $B = \{x \in X : \deg^{\min}(x) < d\}$. Then $\mathcal{D}[[B]_{\mathcal{D}}]$ is Borel d-dicolorable.*

Proof. Take B' and c as in the proof of the previous proposition, and let $L(x) = \{0, \ldots, d-1\}$ for all $x \in X$. Then extend c to a function c' on X by setting $c'(x) = c(x)$ for all $x \notin B'$; for each $x \in B'$, let $c'(x) = \alpha$, where α is the least color such that α does not belong to both $N^+(x) \cap ([B]_{\mathcal{D}} \setminus B')$ and $N^-(x) \cap ([B]_{\mathcal{D}} \setminus B')$. Such an α exists since $\deg^{\min}(x) < d$. $\square$

Next we work towards proving Theorem 2.4.4. In particular, we show that digraphs of bounded degree are Borel degree-list-dicolorable on connected components that are not

Gallai trees. The proof is similar to the proof of Theorem 4.1 in [CMT16]. The argument again involves reserving a set B'' of points. These points will belong to biconnected subdigraphs that are not dicycles, odd symmetric cycles, or complete symmetric digraphs and that are sufficiently well-separated in the underlying graph. Once a degree-list-dicoloring a of the points outside B'' has been constructed, the points of B'' will be colored in a two-step procedure: First, the values of a are changed on certain neighborhoods of the points in B''; then, the dicoloring that results from changing a on these neighborhoods will be extended until all points are colored. The separation between the points of B'' will guarantee that the alterations made to a on different regions of the digraph do not interfere with one another.

The biconnected subdigraphs to which the points of B'' will belong will be required to have the following form.

Definition 2.4.1. *Let $\mathcal{D}$ be a digraph on a set X. Let $G = (x_0, \ldots, x_k)$ be a cycle in $\widetilde{\mathcal{D}}$. Then G is a* good cycle *if the following two conditions hold:*

1. *There is some orientation of the edges in G that does not yield a dicycle; and*
2. *$\mathcal{D}[G]$ is not an odd symmetric cycle or a complete symmetric digraph.*

The following proposition can be deduced from the proof of Lemma 2.1 in [HM11]. It shows that any "good" biconnected subdigraph of size at least 3 contains a good cycle.

Proposition 2.4.12. *[HM11] Let $\mathcal{D}$ be an Eulerian digraph on a set X, and let $M \subseteq X$ be a finite set such that $\widetilde{\mathcal{D}}[M]$ is biconnected, $|M| \geq 3$, and $\widetilde{\mathcal{D}}[M]$ is not a dicycle, an odd symmetric cycle, or a complete symmetric digraph. Then there is a good cycle contained in M.*

Consider an Eulerian digraph $\mathcal{D}$. Let L be a list assignment such that $|L(x)| \geq \deg^{\max}(x)$ for all vertices x, and assume that all vertices of $\mathcal{D}$ except one vertex x_0 have been L-dicolored according to a function c. If $|L(x_0)| > d$, or if $|L(x_0)| = d$ and the number of colors in $L(x_0)$ that appear among both the in-neighbors and the out-neighbors of x_0 is less than d, then the coloring c may be extended to x_0 by defining

$c(x_0)$ to be the least color in $L(x_0)$ which does not appear among both the out-neighbors and the in-neighbors of x_0. Otherwise, the following proposition of Harutyunyan and Mohar shows that, by uncoloring any neighbor y of x_0 and then coloring x_0 with the color that y previously had, we obtain a new L-dicoloring c' of $\mathcal{D}[X \setminus \{y\}]$.

Proposition 2.4.13. *[HM11] Let $\mathcal{D}$ be an Eulerian digraph on a set X, and let L be a list assignment such that $|L(x)| \geq \deg^{\max}(x)$ for all $x \in X$. Let $x_0 \in X$, and suppose c is an L-dicoloring of $\mathcal{D}[X \setminus \{x_0\}]$ such that, for each $\alpha \in L(x_0)$, x_0 has both an out-neighbor and an in-neighbor of color α. Let y be a neighbor of x_0, and define c' on $X \setminus \{y\}$ by $c'(x_0) = c(y)$ and $c'(z) = c(z)$ if $z \neq x_0$. Then c' is an L-dicoloring of $\mathcal{D}[X \setminus \{y\}]$.*

The final result that we need before proving Theorem 2.4.4 can be deduced from Lemmas 2.4 and 2.5 in [HM11]. It shows that, if G is a good cycle and all vertices of $\mathcal{D}$ except one vertex in G have been L-dicolored, then by repeatedly uncoloring and then coloring pairwise adjacent vertices in G as in the statement of Proposition 2.4.13, we reach a vertex $y \in G$ for which some element of $L(y)$ does not appear among both the out-neighbors and the in-neighbors of y. At this stage, the L-dicoloring may be extended to y.

Proposition 2.4.14. *[HM11] Let $\mathcal{D}$ be an Eulerian digraph on a set X, and let L be a list assignment for $\mathcal{D}$ such that $|L(x)| \geq \deg^{\max}(x)$ for all $x \in X$. Let $G \subseteq X$ be a good cycle. Let $x \in G$, and suppose that there is an L-dicoloring a of $\mathcal{D}[X \setminus \{x\}]$. Then there are a sequence $(x = x_0, x_1, \ldots, x_k)$ of vertices in G and a sequence $(a = a_0, a_1, \ldots, a_k)$ of functions such that, for each $i < k$:*

1. *x_i is $\widetilde{\mathcal{D}}$-adjacent to x_{i+1};*
2. *a_i is an L-dicoloring of $\mathcal{D}[X \setminus \{x_i\}]$, and a_k is an L-dicoloring of $\mathcal{D}[X \setminus \{x_k\}]$;*
3. *For each $\alpha \in L(x_i)$, $\alpha \in a_i[N^-(x_i)] \cap a_i[N^+(x_i)]$;*
4. *a_{i+1} is obtained from a_i by uncoloring x_{i+1} and coloring x_i with $a_i(x_{i+1})$ (that is, through an application of Proposition 2.4.13); and*

5. *There is some color* $\alpha \in L(x_k)$ *such that either* $\alpha \notin a_k[N^-(x_k)]$ *or* $\alpha \notin a_k[N^+(x_k)]$.

Finally, the *boundary* of a set S in the digraph $\mathcal{D}$, denoted ∂S, is the set $\partial S = \{x \in X : x \notin S$ but there is $y \in S$ with $x \, \widetilde{\mathcal{D}} \, y\}$.

Now we prove Theorem 2.4.4. For an illustration, see Figure 2.4.

Proof of Theorem 2.4.4. Note that, by Theorem 2.4.10, we may assume that $\deg^{\max}(x) = \deg^{\min}(x)$ for all $x \in X$, that is, that $\mathcal{D}$ is Eulerian.

Throughout the proof, fix $d \in \mathbb{N}$ such that $\deg^{\max}(x) \leq d$ for all $x \in X$. Also, we call a set S *bad* if $\widetilde{\mathcal{D}}[S]$ is a dicycle, an odd symmetric cycle, or a complete symmetric digraph. For technical reasons, we assume first that each block of $\mathcal{D}[B]$ that is not bad has cardinality at least 3. We will explain at the end of the proof that there is a Borel L-dicoloring of the connected components of $\mathcal{D}[B]$ that have non-digon blocks of cardinality 2.

Let

$$[E_{\mathcal{D}}]^{<\infty} = \{S \in [X]^{<\infty} : S \text{ is contained in a connected component of } \mathcal{D}\},$$

and let

$$A = \{S \in [E_{\mathcal{D}}]^{<\infty} : \text{each connected component of } \mathcal{D}[S] \text{ contains a block of cardinality at least 3 that is not bad}\}.$$

Define G_I to be the intersection graph on $[E_{\mathcal{D}}]^{<\infty}$, so that $S, T \in [E_{\mathcal{D}}]^{<\infty}$ are G_I-adjacent if and only if $S \neq T$ and $S \cap T \neq \emptyset$. Then by Proposition 2 of [CM16], there is a countable Borel proper coloring c_I of G_I. Let now

$$A' = \{S \in A : c_I(S \cup \partial S) \leq c_I(T \cup \partial T) \text{ for all } T \in A \text{ in the same } \mathcal{D}\text{-component as } S\},$$

and define $B' = \bigcup A'$.

We next prove several claims about B'. First, we need the following.

Claim 2.4.15. *Let $x \in X$. Then $\mathcal{D}[[x]_{\mathcal{D}}]$ is not a Gallai tree if and only if there is a finite set $T \subseteq [x]_{\mathcal{D}}$ such that the induced subdigraph $\mathcal{D}[T]$ is connected and contains a block of cardinality at least 3 that is not bad.*

Proof of Claim 2.4.15. ($\Rightarrow$) Suppose $\mathcal{D}[[x]_{\mathcal{D}}]$ is not a Gallai tree. Then there is a block in $[x]_{\mathcal{D}}$ that is not a dicycle, an odd symmetric cycle, or a complete symmetric digraph. If there is a finite such block $\mathcal{M}$ in $[x]_{\mathcal{D}}$, then since $\mathcal{M}$ has cardinality at least 3 by the assumption at the beginning of the proof, we may take $T = V(\mathcal{M})$ in the statement of the claim.

Otherwise, there is an infinite block $\mathcal{M}$ in $[x]_{\mathcal{D}}$. Now we take inspiration from the proof of Lemma 2.1 in [HM11]. Write $M = V(\mathcal{M})$, and let $y, z \in M$ be such that $d(y, z) \geq d + 2$; such y, z exist because $\mathcal{M}$ is an infinite connected subdigraph of a locally finite graph. Since $\mathcal{M}$ is biconnected, there are two internally-vertex-disjoint paths P_0, P_1 from y to z. Note that each of P_0, P_1 has length at least $d + 2$. Again since the $(d + 2)$-neighborhood of $P_0 \cup P_1$ is a finite set in a locally finite graph, there is some $v \in M$ such that $d(v, P_0 \cup P_1) \geq d + 2$. Since $\mathcal{M}$ is connected, there is a path $Q = (y = q_0, q_1, \dots, q_k = v)$ through $\mathcal{M}$ from y to v.

Case 1: Q contains some point of $P_0 \cup P_1$ besides y. Let $i < k$ be maximal such that $q_i \neq y$ and $q_i \in P_0 \cup P_1$, and write $y' = q_i$. Since $\mathcal{M}$ is biconnected, there is a path $Q' = (y = r_0, r_1, \dots, r_l = v)$ through $\mathcal{M}$ from y to v that is internally-vertex-disjoint from Q. Let $j < l$ be maximal such that $r_j \in P_0 \cup P_1$, and write $y'' = r_j$. Then y', y'' are distinct points of $P_0 \cup P_1$, and there are three internally-vertex-disjoint paths from y' to y'': These are the two paths S_0, S_1 arising from the fact that y', y'' are distinct points of the cycle $P_0 \cup P_1$, together with the concatenation S_2 of the sub-path of Q from y' to v and the sub-path of Q' from v to y''. Notice that the length of S_2 and the length of at least one of S_0, S_1 are at least $d + 2$. So, two of the three paths S_0, S_1, S_2 form a cycle C of even length with cardinality at least $d + 2$. Since each vertex of $\mathcal{D}$ has maximum degree at most d, the induced sub-digraph $\mathcal{D}[C]$ is not a complete symmetric digraph. So, if $\mathcal{D}[C]$ is not a dicycle, then we may take $T = C$ in the statement of Claim 2.4.15.

If $\mathcal{D}[C]$ is a dicycle, then we may take $T = S_0 \cup S_1 \cup S_2$.

Case 2: The only point of $P_0 \cup P_1$ in Q is y. Again since $\mathcal{M}$ is connected, there is a path R through $\mathcal{M}$ from z to v. If R contains some point of $P_0 \cup P_1$ besides z, then we may proceed as in Case 1 by replacing y with z and Q with R. Otherwise, there are three internally-vertex-disjoint paths P_0, P_1, and $R \cup Q$ (with repetitions removed) between y and z. Note that each of these paths has length at least $d+2$. Then we may again proceed as in Case 1 by replacing S_0, S_1, S_2 with $P_0, P_1, R \cup Q$.

($\Leftarrow$) Suppose there is a finite set $T \subseteq [x]_{\mathcal{D}}$ such that $\mathcal{D}[T]$ is connected and contains a block $\mathcal{M}$ of size at least 3 that is not bad. If there is an infinite block in $[x]_{\mathcal{D}}$ which contains $\mathcal{M}$, then clearly $\mathcal{D}[[x]_{\mathcal{D}}]$ is not a Gallai tree, and the proof is complete. If the only blocks in $[x]_{\mathcal{D}}$ that contain $\mathcal{M}$ are finite, then since $\mathcal{M}$ has cardinality at least 3, there is no block in $\mathcal{D}$ containing $\mathcal{M}$ that is a dicycle, an odd symmetric cycle, or a complete symmetric digraph. So $\mathcal{D}[[x]_{\mathcal{D}}]$ is again not a Gallai tree. $\square$

Now we claim $B' \subseteq B$. Let $x \in B'$. Then there is $S \in A'$ such that $x \in S$. Because each connected component of $\mathcal{D}[S]$ contains a block of size at least 3 that is not bad, it follows from Claim 2.4.15 that $\mathcal{D}[[x]_{\mathcal{D}}]$ is not a Gallai tree. So $x \in B$. Furthermore, Claim 2.4.15 implies that each connected component of $\mathcal{D}[B]$ meets B'.

Next, we show that each connected component of $\mathcal{D}[B']$ is finite. Let $x \in B'$; then there is some $S \in A'$ such that $x \in S$. We claim $[x]_{\mathcal{D}[B']} \subseteq (S \cup \partial S)$, which is a finite set. Assume for contradiction that there is some $y \in B'$ such that $y \notin S \cup \partial S$, but there is a path $(x = x_0, x_1, \ldots, x_k = y)$ through $\widetilde{\mathcal{D}}[B']$ from x to y. Since $y \notin S \cup \partial S$ and $x \in S$, there is $i < k$ such that $x_i \in \partial S$ and $x_{i+1} \notin S \cup \partial S$. Let $T \in A'$ be such that $x_{i+1} \in T$. Then $(S \cup \partial S) \cap (T \cup \partial T)$ contains x_i, so that $c_I(S \cup \partial S) \neq c_I(T \cup \partial T)$. This is a contradiction since $S, T \in A'$ belong to the same connected component of $\mathcal{D}$.

Note now that each connected component C of $\mathcal{D}[B']$ contains a set M of size at least 3 such that $\mathcal{D}[M]$ is biconnected and not bad. Let $<_X$ be a Borel linear ordering of X; then we define M_C to be the lexicographically least such M, assuming without loss of generality that the elements of any finite subset of X are listed in increasing order according to $<_X$. Further, we may assume that, if C, C' are distinct connected components of $\mathcal{D}[B']$, then M_C and $M_{C'}$ are such that $(M_C \cup \partial M_C) \cap (M_{C'} \cup \partial M_{C'}) = \emptyset$; this is because, if $S \subseteq C$ and $T \subseteq C'$ are elements of A', then $S \cup \partial S \neq T \cup \partial T$, so that $(S \cup \partial S) \cap (T \cup \partial T) = \emptyset$.

For each connected component C of $\mathcal{D}[B']$, by Proposition 2.4.12, there is a good cycle G in M_C; let G_C be the lexicographically least such cycle according to $<_X$, and let x_C be the $<_X$-least element of G_C. Finally, set

$$B'' = \{x \in X : \text{there is a connected component } C \text{ of } \mathcal{D}[B'] \text{ such that } x = x_C\}.$$

Then B'' is Borel, and $[B'']_{\mathcal{D}} = [B']_{\mathcal{D}} = B$.

So, by Proposition 2.4.9, there is a one-ended Borel function from $(B \setminus B'')$ to B. Then by Theorem 2.4.8, there is a Borel L-dicoloring a of $\mathcal{D}[B \setminus B'']$.

We proceed to define a Borel L-dicoloring a' of $\mathcal{D}[B]$. For each $x \in B''$, let C be the connected component of $\mathcal{D}[B']$ witnessing $x \in B''$, and let $(x = x_0, x_1, \ldots, x_k), (a = a_0, a_1, \ldots, a_k)$ be as in Proposition 2.4.14; this proposition may be applied since G_C is a good cycle and $x = x_C \in G_C$. Then the L-dicoloring a_k may be extended to x_k, since there is some color $\alpha \in L(x_k)$ such that either $\alpha \notin a_k[N^-(x_k)]$ or $\alpha \notin a_k[N^+(x_k)]$; write a_C for this extension of a_k. Then define $a' : B \to Y$ by $a'(x) = a_C(x)$ if $x \in G_C$ and $a'(x) = a(x)$ if $x \notin G_C$ for any C.

We claim that a' is a dicoloring of $\mathcal{D}[B]$. It is enough to show the following.

Claim 2.4.16. *Let $x, y \in B''$ be distinct. Let c be a dicoloring of $\mathcal{D}[X \setminus \{x, y\}]$ such that, for each $\alpha \in L(x)$, $\alpha \in c[N^-(x)] \cap c[N^+(x)]$, and for each $\beta \in L(y)$, $\beta \in c[N^-(y)] \cap c[N^+(y)]$. Let x' be a neighbor of x, and let y' be a neighbor of y. Then the function c' defined on $X \setminus \{x', y'\}$ by $c'(z) = c(z)$ if $z \neq x, y$, $c'(x) = c(x')$, and $c'(y) = c(y')$ is an L-dicoloring of $\mathcal{D}[X \setminus \{x', y'\}]$.*

Proof of Claim 2.4.16. Note first that, since $x, y \in B''$ are distinct, then if C, C' are the connected components of B' witnessing that $x, y \in B''$, respectively, then $(G_C \cup \partial G_C) \cap (G_{C'} \cup \partial G_{C'}) = \emptyset$. Now assume for contradiction that there is a c'-monochromatic dicycle

K. Then K cannot contain x' or y', since these vertices are uncolored. Further, K must contain either x or y; otherwise, there is a c-monochromatic dicycle, contradicting that c is a dicoloring. Suppose $x \in K$. Then $c'(x) = c(x')$. Since, for each $\alpha \in L(x)$, $\alpha \in c[N^-(x)] \cap c[N^+(x)]$, and since $(G_C \cup \partial G_C) \cap (G_{C'} \cup \partial G_{C'}) = \emptyset$ so that y and y' are not neighbors of x, it follows that there is exactly one neighbor z of x such that $c'(x) = c'(z)$. Therefore, any dicycle passing through x either contains points of different colors or contains an uncolored point. So, no dicycle containing x is monochromatic. A similar argument shows that no dicycle containing y is monochromatic. □

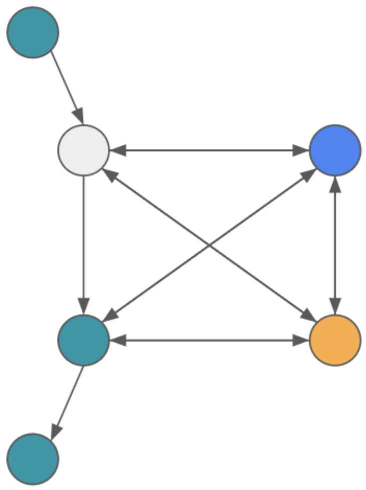
This completes the proof in the case where each block of $\mathcal{D}[B]$ that is not bad has cardinality at least 3. For the other case, let

$$B_0 = \{x \in X : [x]_{\mathcal{D}} \text{ has a non-digon block of cardinality } 2\}.$$

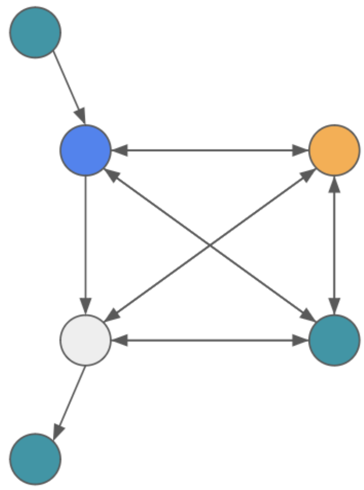
As before, take a Borel set A' of non-digon blocks in B_0 such that, if M, M' are elements of A' with $M \cup \partial M \neq M' \cup \partial M'$, then $(M \cup \partial M) \cap (M' \cup \partial M') = \emptyset$ and such that A' meets each connected component of $\mathcal{D}[B_0]$. Then set $B' = \bigcup A'$. Use a Borel linear ordering to select from each connected component C of B' a point x_C that belongs to a non-digon block of cardinality 2, and collect these points x_C into a set B''. Then use Theorem 2.4.8 to produce a Borel L-dicoloring a of $\mathcal{D}[B_0 \setminus B'']$. To color the points of B'', note that, if $x \in B''$, then there is some neighbor y of x such that $\{x, y\}$ is a maximal biconnected set that is not a digon. In particular, no dicycle contains both x and y. So, if $L(x)$ contains $a(y)$, then a may be extended to x by setting $a(x) = a(y)$. Otherwise, there is some color in $L(x)$ which does not appear among both the out-neighbors and the in-neighbors of x, and so again a can be extended to x by setting $a(x)$ to be the least such color. □

2.4.3 The Proof of Theorem 2.4.5

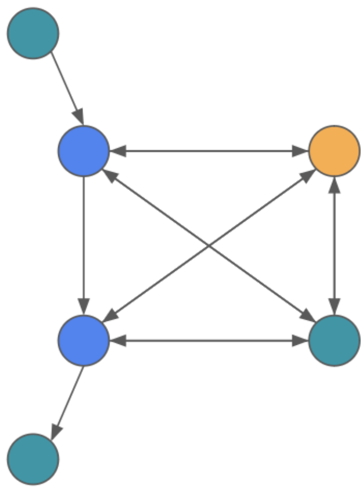
Now we have nearly all the tools required to prove Theorem 2.4.5. We next need a result of [CMT16] that shows that, modulo a null set or a meager set, an undirected acyclic graph in which no connected component has 0 or 2 ends admits a one-ended



(a) A block that is not a dicycle, an odd symmetric cycle, or a complete symmetric digraph.



(b) Repeatedly uncoloring and recoloring adjacent vertices yields a new configuration.



(c) The dicoloring is extended.

Figure 2.4: Completing the dicoloring in the proof of Theorem 2.4.4 on components that are not Gallai trees for the $d = 3$ case. Here the connected component includes a subdigraph having the form of $\overleftrightarrow{\mathcal{K}}_4$ with one directed edge removed. The uncolored vertex in (a) has, for each of the three colors, both an in-neighbor and an out-neighbor of that color. By repeatedly uncoloring and recoloring adjacent vertices, we obtain the configuration in (b). Now the uncolored vertex has no blue out-neighbor, so the dicoloring may be extended.

Borel function. If $\mathcal{G}$ is a locally finite graph on a set X, then a set $B \subseteq X$ is *$\mathcal{G}$-invariant* if, whenever $x \in B$ and there is a path from x to y through $\mathcal{G}$, then $y \in B$. If $\mathcal{D}$ is a digraph on X, then we say $B \subseteq X$ is *$\mathcal{D}$-invariant* if B is $\widetilde{\mathcal{D}}$-invariant. A *ray* in $\mathcal{G}$ is an infinite sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise-adjacent vertices in $\mathcal{G}$ such that $x_n \neq x_m$ whenever $n \neq m$. Two rays r_0, r_1 in $\mathcal{G}$ are *end-equivalent* if, whenever $S \subseteq X$ is finite, r_0 and r_1 eventually lie in the same connected component of $\mathcal{G}[X \setminus S]$. End-equivalence is an equivalence relation on the set of rays; the equivalence classes are called *ends*.

Theorem 2.4.17. *[CMT16] Let $\mathcal{G}$ be a locally finite acyclic graph on a standard Borel space X. Assume no connected component of $\mathcal{G}$ has either 0 or 2 ends.*

1. *For any Borel probability measure μ on X, there are a μ-conull, $\mathcal{G}$-invariant Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in $E(\mathcal{G})$.*

2. *For any Polish topology τ compatible with the Borel structure on X, there are a τ-comeager, $\mathcal{G}$-invariant Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in $E(\mathcal{G})$.*

Next, we show that any bounded-degree Borel digraph $\mathcal{D}$ having no connected components that are 0-ended or 2-ended Gallai trees is Borel degree-list-dicolorable modulo a small set. For the proof, we apply Theorem 2.4.17 to an acyclic sub-digraph $\mathcal{D}'$ of $\mathcal{D}$. The digraph $\mathcal{D}'$ is constructed by removing edges from the blocks of $\widetilde{\mathcal{D}}$. Each such block is either a cycle or a complete graph; from each complete graph, edges are removed until only a maximal-length cycle remains, and then from each cycle, the least (according to some Borel linear ordering on the set of edges) edge is removed.

Theorem 2.4.18. *Let $\mathcal{D}$ be a Borel digraph of bounded degree on a standard Borel space X. Assume that $\mathcal{D}$ has no finite connected components that are Gallai trees and no infinite connected components that are 2-ended Gallai trees. Then:*

1. *For any Borel probability measure μ on X, there is a μ-conull, $\mathcal{D}$-invariant Borel set B such that $\mathcal{D}[B]$ is Borel degree-list-dicolorable.*

2. For any Polish topology τ compatible with the Borel structure on X, there is a τ-comeager, $\mathcal{D}$-invariant Borel set B such that $\mathcal{D}[B]$ is Borel degree-list-dicolorable.

Proof. We prove (1); the proof of (2) proceeds in the same way.

Let L be a list assignment such that $|L(x)| \geq \deg^{\max}(x)$ for all $x \in X$. By the assumptions in the theorem statement together with Theorem 2.4.4, we may assume without loss of generality that each connected component of $\mathcal{D}$ is an infinite Gallai tree that does not have 2 ends. Then each block in $\mathcal{D}$ is a dicycle, a finite odd symmetric cycle, or a finite complete symmetric digraph. So, in the underlying graph $\widetilde{\mathcal{D}}$, each block is either a cyclic graph or a complete graph.

Now let E denote the set of edges of $\widetilde{\mathcal{D}}$. Note that each element of E is contained in a unique block. We now create a subset $E' \subseteq E$ as follows. Let $<_X$ be a Borel linear ordering of X, and let $<_E$ be a Borel linear ordering of E. From each block of $\widetilde{\mathcal{D}}$ that is a cyclic graph, remove the $<_E$-least edge; from each block $\mathcal{M}$ of $\widetilde{\mathcal{D}}$ that is a complete graph of cardinality at least 3, write $V(\mathcal{M}) = \{x_0, x_1, \ldots, x_k\}$ in ascending order according to $<_X$, and then delete all edges of $\mathcal{M}$ except $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, x_0\}$ before also deleting the $<_E$-least edge in the list $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, x_0\}$. Now define a subdigraph $\mathcal{D}'$ of $\mathcal{D}$ whose vertex set is X with $(x, y) \in \overrightarrow{E}(\mathcal{D}')$ if and only if $(x, y) \in \overrightarrow{E}(\mathcal{D})$ and $(x, y) \in E'$.

We prove several claims about $\mathcal{D}'$. Note first that, if x, y are adjacent in $\mathcal{D}$, then there is a path from x to y through $\widetilde{\mathcal{D}'}$; indeed, the edge between x and y in the underlying graph $\widetilde{\mathcal{D}}$ determines a unique block $\mathcal{M}$. If $\mathcal{M}$ has cardinality 2, then the edge $\{x, y\}$ belongs to E'. Otherwise, if $\widetilde{\mathcal{D}}[V(\mathcal{M})]$ is a cycle, then there is a path through $\mathcal{M}$ from x to y which does not contain the edge $\{x, y\}$; if $\tilde{D}[V(\mathcal{M})]$ is a complete graph, then $\widetilde{\mathcal{D}'}[V(\mathcal{M})]$ is a cycle missing just one edge, so that again there is a path through $\mathcal{M}$ from x to y which does not contain the edge $\{x, y\}$. This implies that no connected component of $\mathcal{D}'$ is 0-ended, since no connected component of $\mathcal{D}$ is 0-ended. This also implies that, if $B \subseteq X$ is $\mathcal{D}'$-invariant, then B is also $\mathcal{D}$-invariant.

Next, to see that $\mathcal{D}'$ is acyclic, note that any cycle C in $\widetilde{\mathcal{D}}$ induces either a cyclic graph or a complete graph. If C induces a cyclic graph, then some edge of C is deleted in passing from $\mathcal{D}$ to $\mathcal{D}'$. If C induces a complete graph, then let $\mathcal{M}$ be the maximal complete graph containing C. Then at least one of the edges of $\mathcal{M}$ that is removed in passing to $\mathcal{D}'$ is an edge of C.

Finally, to see that no connected component of $\mathcal{D}'$ is 2-ended, note first that the number of ends in a connected component of $\mathcal{D}$ is less than or equal to the number of ends in the corresponding connected component of $\mathcal{D}'$. Suppose now that there are two rays r, r' in the same connected component of $\mathcal{D}'$ such that, for some finite set $S \subseteq X$, $r \setminus S$ and $r' \setminus S$ do not eventually lie in the same connected component of $\mathcal{D}'[X \setminus S]$. Let $T = S \cup \bigcup\{M \in [E_D]^{<\infty} : M \text{ is a block containing a point of } S\}$. Then it is easy to prove that $r \setminus T$ and $r' \setminus T$ do not eventually lie in the same connected component of $\mathcal{D}[X \setminus T]$. So, since no connected component of $\mathcal{D}$ is 2-ended, no connected component of $\mathcal{D}'$ is 2-ended.

It now follows from Theorem 2.4.17 that there are a μ-conull, $\mathcal{D}'$-invariant Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in $\widetilde{\mathcal{D}'}$. Note that B is $\mathcal{D}$-invariant by the claim above and that the graph of f is contained in $E(\widetilde{\mathcal{D}})$. By Theorem 2.4.8, $\mathcal{D}[B]$ has a Borel L-dicoloring. $\square$

Finally, we prove Theorem 2.4.5.

Proof of Theorem 2.4.5. By Corollary 2.4.11, we may assume that $\deg^{\max}(x) = \deg^{\min}(x) = d$ for all $x \in X$. Note that, if all the vertices in a finite Gallai tree have out-degree and in-degree equal to d, then the Gallai tree is the complete symmetric digraph on d vertices. Also, if all the vertices in an infinite 2-ended Gallai tree have out-degree and in-degree equal to d, then either $d = 1$ and the tree is a one-directional bi-infinite line, or $d = 2$ and the tree is a symmetric bi-infinite line. Since $d \geq 3$, it follows from Theorem 2.4.18 that $\mathcal{D}$ has a (μ- or τ-)measurable d-dicoloring. $\square$

2.5 Remarks on Johansson's Theorem

In this section, we discuss a classical theorem of Johansson that strengthens Brooks's theorem for the class of *triangle-free* (undirected) graphs, that is, graphs having no cycles of length 3, and explore potential extensions to directed graphs and dicolorings.

Theorem 2.5.1. *[Joh96] Let $\mathcal{G}$ be a graph of maximum degree $d \in \mathbb{N}$. Suppose $\mathcal{G}$ is triangle-free. Then $\chi(\mathcal{G}) \sim O(d/\log d)$.*

Currently, it is unknown whether there is a directed graph analogue of Theorem 2.5.1. In fact, even the following question, which dates back to work of Erdős and Neumann-Lara from the 1970s, remains open.

Question 2.5.2. *[Erd79] Let $\mathcal{D}$ be a directed graph such that each vertex has maximum degree at most $d \in \mathbb{N}$. Suppose $\mathcal{D}$ is digon-free. Then is $\vec{\chi}(\mathcal{D}) \sim o(d)$?*

We remark that, by Proposition 2.4.2, even if classically the answer to Question 2.5.2 is "yes", there is no possibility of obtaining a positive result in the Borel setting, even for triangle-free directed graphs. However, Bernshteyn proved a measurable version of Theorem 2.5.1 in [Ber23], leaving open the possibility of a measurable solution to Question 2.5.2. If such a measurable approach could be ruled out, this would demonstrate a meaningful contrast between measurable proper coloring and measurable dicoloring.

2.6 Remarks on LOCAL Colorings

Recently, deep connections between procedures in theoretical computer science known as *distributed graph algorithms* and descriptive combinatorics have recently been uncovered (see [Ber19; GV25; BBLW25] for surveys). In this section, we first briefly outline the relevance of these connections to Brooks's theorem for undirected graphs, and we then discuss how dicoloring may or may not fit into this theoretical framework.

We begin with several definitions, following the presentation in [BW25]. The first definition is intended to capture the intuitive idea of a graph theoretic problem for which the correctness of a proposed solution is "locally checkable", in the sense that determining whether the proposed solution is correct amounts to determining correctness on only a small neighborhood of each vertex in the graph; see [NS95]. Formally, a *locally checkable labeling problem* (or *LCL problem*) Π is a pair $\Pi = (R, \mathcal{P})$, where $R \in \mathbb{N}$ is the *radius* of Π and $\mathcal{P} : \mathsf{FLG}_\bullet \to \{0, 1\}$ is a function. Here, the notation $\mathsf{FLG}_\bullet$ refers to the set of isomorphism classes of *finite rooted labeled graphs*, that is, finite graphs $\mathcal{G}$ equipped with a distinguished vertex called a *root* and a function $V(\mathcal{G}) \to \mathbb{N}$ called a *labeling* (here, the isomorphisms must preserve roots and labelings). The function $\mathcal{P}$ appearing in the definition of the problem Π may be viewed as specifying the allowed configurations of radius-R neighborhoods of the vertices in $\mathcal{G}$.

A *solution* to Π on a graph $\mathcal{G}$ is a function $f : V(\mathcal{G}) \to \mathbb{N}$ such that, for each $x \in V(\mathcal{G})$, the output of $\mathcal{P}$ on the isomorphism class of $\mathcal{G}[N_R(x)]$, viewed as being rooted at x and labeled by $f \restriction N_R(x)$, is 1.

Example 2.6.1. *For $n \in \mathbb{N}$, the problem of giving a graph $\mathcal{G}$ a proper n-coloring is an LCL problem. Indeed, let $R = 1$, and let $\mathcal{P}$ be the function sending the isomorphism class of a graph rooted at x with labeling f to 1 if $f(x) < n$ and $f(x) \neq f(y)$ for all $y \neq x$. Then a solution c to $\Pi = (R, \mathcal{P})$ on $\mathcal{G}$ is a proper n-coloring of $\mathcal{G}$; for each $x \in V(\mathcal{G})$, the (isomorphism class of the) graph $\mathcal{G}[N_1(x)]$ rooted at x and labeled by $c \restriction N_1(x)$ is sent by $\mathcal{P}$ to 1, so that $c(x) < n$ and $c(x) \neq c(y)$ for all $y \neq x$ with $y \in N_1(x)$, that is, for all neighbors y of x.*

Instrumental in solving LCL problems are LOCAL algorithms, which were first introduced in the seminal work of Linial [Lin92] (see [BE13] for a survey of results on LOCAL algorithms for graph coloring problems). A *LOCAL algorithm* $\mathcal{A}$ is a function $\mathsf{FSG}_\bullet \to \mathbb{N}$; for each $T \in \mathbb{N}$, the *output of $\mathcal{A}$ after time T* is the function $\mathcal{A}_T$ sending a finite graph $\mathcal{G}$ with labeling f to the function $\mathcal{A}_T(\mathcal{G}, f) : V(\mathcal{G}) \to \mathbb{N}$ defined by mapping $x \in V(\mathcal{G})$ to the output of $\mathcal{A}$ on the (isomorphism class of the) graph $\mathcal{G}[N_T(x)]$ rooted at x and labeled by $f \restriction N_T(x)$. For an LCL problem Π and a class $\mathbf{G}$ of finite graphs, the *deterministic LOCAL complexity* of Π is the function $\mathrm{Det}_{\Pi,\mathbf{G}} : \mathbb{N}^+ \to \mathbb{N} \cup \{\infty\}$ defined by mapping

$n \in \mathbb{N}^+$ to the least $T \in \mathbb{N}$ such that there is a LOCAL algorithm $\mathcal{A}$ for which, for each $\mathcal{G} \in \mathbf{G}$ of size n labeled by a bijection $f : V(\mathcal{G}) \to n$, $\mathcal{A}_T(\mathcal{G}, f)$ is a solution to Π on $\mathcal{G}$, and to ∞ if no such T exists; the *randomized* LOCAL *complexity* of Π is the function $\mathrm{Rand}_{\Pi,\mathbf{G}} : \mathbb{N}^+ \to \mathbb{N} \cup \{\infty\}$ defined by mapping $n \in \mathbb{N}^+$ to the least $T \in \mathbb{N}$ such that there are a LOCAL algorithm $\mathcal{A}$ and a number ℓ for which, for each $\mathcal{G} \in \mathbf{G}$ of size n labeled by a function $f : V(\mathcal{G}) \to \ell$ selected uniformly at random, the probability that $\mathcal{A}_T(\mathcal{G}, f)$ is a solution to Π on $\mathcal{G}$ is at least $1 - \frac{1}{n}$, and to ∞ if no such T exists. Intuitively, the LOCAL complexity of Π specifies the size of the neighborhood each vertex in a graph $\mathcal{G}$ in $\mathbf{G}$ needs to access in order to correctly determine its label in a solution to Π.

Example 2.6.2. *[Ber22] Let Π be the LCL problem encoding proper 2-coloring, and let $\mathbf{G}$ be the class of cyclic graphs of even size. Then $Det_{\Pi,\mathbf{G}}(n) \sim O(n)$. Indeed, assume for contradiction that $Det_{\Pi,\mathbf{G}}(n) \sim o(n)$. Then for a sufficiently large even $n \in \mathbb{N}$, $Det_{\Pi,\mathbf{G}}(n) = T < \frac{n}{100}$. Let $\mathcal{A}$ be a deterministic LOCAL algorithm as described in the previous paragraph, and let $\mathcal{G}$ be the cyclic graph of size n equipped with a bijection $f : V(\mathcal{G}) \to n$. Let x, y, z be vertices of $\mathcal{G}$ such that the distance between x and y is even, the distance between x and z is odd, and the pairwise distances among x, y, z are all greater than $2T$. Then after running the algorithm $\mathcal{A}$ on $\mathcal{G}$, without loss of generality, x and y have color 0 and z has color 1. Define a new bijection $\tilde{f} : V(\mathcal{G}) \to n$ by exchanging the labels on the T-neighborhoods of x and z. Then once $\mathcal{A}$ is run on $\mathcal{G}$ with the new labeling $\tilde{f}$, x now receives the color 1 and z receives the color 0, with y's color remaining 0. However, since the distance between x and y is even, this is a contradiction.*

A celebrated theorem of Bernshteyn relates the LOCAL complexity of an LCL problem Π with the descriptive set theoretic measurability of the possible solutions to Π. The essence of the theorem is the following: LCL problems that can be solved with efficient deterministic LOCAL algorithms (that is, deterministic LOCAL algorithms of low computational complexity) admit Borel (in fact, continuous) solutions, and LCL problems that can be solved with efficient randomized LOCAL algorithms (that is, randomized LOCAL algorithms of low computational complexity) admit μ-measurable and Baire-measurable solutions.

Theorem 2.6.3. *[Ber23] Let Π be an LCL problem, and let $\mathbf{G}$ be a class of finite graphs admitting solutions to Π.*

1. *If $Det_{\Pi,\mathbf{G}}(n) \sim O(\log^*(n))$, then for any Borel graph $\mathcal{G}$ of bounded degree whose finite induced subgraphs all belong to $\mathbf{G}$, there is a continuous solution to Π on $\mathcal{G}$.*

2. *If $Det_{\Pi,\mathbf{G}}(n) \sim O(\log^*(n))$, then for any Borel graph $\mathcal{G}$ of bounded degree whose finite induced subgraphs all belong to $\mathbf{G}$, there are, for any Borel probability measure μ on $V(\mathcal{G})$, a μ-measurable solution to Π on $\mathcal{G}$, and for any Polish topology τ compatible with the Borel structure on $V(\mathcal{G})$, a τ-Baire-measurable solution to Π on $\mathcal{G}$.*

An impressive battery of novel results in descriptive combinatorics has been imported from the distributed algorithms literature via applications of Theorem 2.6.3; conversely, extensions of the theorem have enabled the transference of results from descriptive combinatorics to distributed graph coloring (see, for instance, [GV25]). Furthermore, new theorems of previously established results have been discovered. For instance, Ghaffari, Hirvonen, Kuhn, and Maus describe in [GHKM21] a randomized LOCAL algorithm of complexity $O(\log^*(n))$ for properly d-coloring the class of finite graphs of maximum degree d having no odd cycles or complete graphs as subgraphs, thereby proving an "efficient randomized LOCAL algorithm" version of Brooks's theorem. By combining this result with Theorem 2.6.3, we obtain an alternative proof of Theorem 2.2.7. Furthermore, Marks's impossibility result (Theorem 2.2.3), when combined with Theorem 2.6.3, also demonstrates that no "efficient deterministic LOCAL algorithm" version of Brooks's theorem is possible.

One may therefore consider whether similar connections hold between distributed algorithms for dicoloring and measurable dicolorings. However, dicoloring is not an LCL problem; directed cycles need not in general be bounded in size. This presents a potentially interesting challenge; we explore several potential approaches below.

First, is there a class of digraphs for which dicoloring can be construed as an LCL problem? Trivially, for each $k \in \mathbb{N}$, dicoloring is an LCL problem for the class of digraphs having (directed) circumference at most k; however, this is not a particularly natural class of digraphs to consider.

Second, is there a class of problems broader than the class of LCL problems that both includes dicoloring and allows for a result similar to Theorem 2.6.3? Several approaches to generalizing the class of LCLs so as to capture additional commonly encountered combinatorial problems exist (see, for instance, the class of BPLD problems [FKP13]). It is possible that some such naturally occurring class is suitable for transferring algorithmic results on dicoloring to descriptive combinatorics.

What would likely be needed for the broader class is an appropriate version of the Lovász local lemma [EL75]. Indeed, the main tool in the proof of Theorem 2.6.3 is a measurable version of the symmetric Lovász local lemma for solving certain LCL problems. We provide a brief overview of the ideas here: Let $p < 1$, and consider a set $\mathcal{S}$ of events, each of which has probability at most p. Suppose that each event in $\mathcal{S}$ depends on only a small number d of other events in $\mathcal{S}$. If p and d satisfy a certain inequality, then the local lemma ensures that the probability that none of the events in $\mathcal{S}$ occurs is nonzero. For graph coloring, we may take $\mathcal{S}$ to be the set of events in which some vertex shares a color with one of its neighbors; for a graph of bounded degree, each such event has probability bounded away from 1. Furthermore, due to the local nature of proper colorings, each event in $\mathcal{S}$ depends on only a small number of other events in $\mathcal{S}$. Bernshteyn's measurable symmetric local lemma then ensures the existence of a measurable coloring in which adjacent vertices do not share colors.

While the symmetric local lemma cannot be directly applied to obtain dicolorings, there are other versions of the local lemma that may be of use. For instance, generalizations of the Lovász local lemma that appear in the book of Alon and Spencer [AS08] allow for dependencies of events in $\mathcal{S}$ on arbitrarily many other elements of $\mathcal{S}$ under certain additional constraints. Work of Kun from 2013 appearing in [Kun13] further extends these generalizations to infinitary settings.

CHAPTER 3

Borel Asymptotic Dimension and Complexity

In this chapter, we present a theorem of [GH24] on the $\mathbf{\Sigma}^1_2$-completeness of finite Borel asymptotic dimension. Borel asymptotic dimension, first introduced in [CJM+23], is a definable version of Gromov's notion of asymptotic dimension for metric spaces [Gro93]. The complete definition of the term "Borel asymptotic dimension" is deferred until Section 3.4.

3.1 Hyperfiniteness

Since the early 1990s, a major goal of research in descriptive set theory has been to determine the global structure of the class of definable equivalence relations. Here we provide a brief overview of this line of study; we refer the reader to [Kec24] for further information.

Let X be a standard Borel space equipped with an equivalence relation E. Then E is *countable* if each equivalence class of E is countable, and E is *Borel* if it is Borel as a subset of X^2 with the product topology. The abbreviation "(C)BER" is commonly used in place of the term "(countable) Borel equivalence relation".

CBERs arise in many different settings. Particularly important are the CBERs arising from group actions.

Example 3.1.1. *Let Γ be a countable discrete group, and let $\cdot$ be a Borel action of Γ on a standard Borel space X. Then the orbit equivalence relation E_Γ of $\cdot$ is a CBER: Since Γ is countable, E_Γ is countable. To see that E_Γ is Borel, note first that, for each $\gamma \in \Gamma$,* *the function $f_\gamma : X \to X$ defined by $x \mapsto \gamma \cdot x$ is Borel since it is a section of f. Then*

$$E_\Gamma = \bigcup_{\gamma \in \Gamma} \left(f_\gamma \cup (f_\gamma)^{-1} \right)$$

is Borel.

In fact, by a well-known theorem of Feldman and Moore, every CBER arises from a Borel action of a countable discrete group.

Theorem 3.1.2. *[FM77] Let E be a CBER on a standard Borel space X. Then there are a countable discrete group Γ and a Borel action of Γ on X such that $E = E_\Gamma$.*

Example 3.1.3. *Let $\mathcal{G}$ be a locally countable Borel graph on a standard Borel space X. Then the connectedness relation $E_\mathcal{G}$ of $\mathcal{G}$ is a CBER: Since $\mathcal{G}$ is locally countable, $E_\mathcal{G}$ is countable. For $k \in \mathbb{N}$, define $E^{\leq k}(\mathcal{G})$ recursively by $E^{\leq 1}(\mathcal{G}) = \{(x, y) \in X^2 : x = y\} \cup E(\mathcal{G})$ and*

$$E^{\leq (k+1)}(\mathcal{G}) = \{(x, y) \in X^2 : \textit{there is } z \in X \textit{ such that } (x, z) \in E^{\leq k}(\mathcal{G}) \\ \textit{and } (z, y) \in E^{\leq 1}(\mathcal{G})\}.$$

Then $E^{\leq 1}(\mathcal{G})$ is Borel. If $E^{\leq k}(\mathcal{G})$ is Borel, then by Theorem 1.2.3, which applies since $\mathcal{G}$ is locally countable, there is a countable sequence $(f_n)_{n \in \mathbb{N}}$ of Borel functions such that, for each $x \in X$, $\{y \in X : (x, y) \in E^{\leq k}(\mathcal{G})\} = \{f_n(x) : n \in \mathbb{N}\}$. Then

$$E^{\leq (k+1)}(\mathcal{G}) = \bigcup_{n \in \mathbb{N}} \{(x, y) \in X^2 : (f_n(x), y) \in E^{\leq 1}(\mathcal{G})\},$$

which is Borel. So

$$E_\mathcal{G} = \bigcup_{k \in \mathbb{N}} E^{\leq k}(\mathcal{G})$$

is Borel.

For standard Borel spaces X and Y and CBERs E on X and F on Y, E is *Borel reducible* to F, written $(X, E) \leq_B (Y, F)$, or simply $E \leq_B F$ if X and Y are understood, if there is a Borel map $f : X \to Y$ such that $x \; E \; y$ if and only if $f(x) \; F \; f(y)$ (or, equivalently, such that the induced map $\tilde{f} : X/E \to Y/F$ defined by $\tilde{f}([x]_E) = [f(x)]_F$ is injective). The intuition is that, when $E \leq_B F$, then from the perspective of descriptive complexity, E is "no more complicated than" F. In particular, Borel reducibility may be viewed as the descriptive set theoretic analogue of many-one reducibility in computability theory. The relations E and F are *Borel bireducible* if $E \leq_B F$ and $F \leq_B E$. When two CBERs are Borel bireducible, they are "equally complicated".

Borel reducibility can be used to stratify the class of CBERs into a structure known as the *Borel reducibility hierarchy.* At the bottom of the hierarchy are the smooth CBERs. A CBER E is *smooth* if there is an uncountable standard Borel space Y such that $E \leq_B =_Y$, where $=_Y$ denotes the equality relation on the space Y.[1] For $n \in \mathbb{N}$, the notation $=_n$ denotes the equality relation on the set $\{0, \ldots, n-1\}$. Note that, given any standard Borel space Y, the relation $=_Y$ is smooth. In fact, all smooth BERs arise in this way.

Theorem 3.1.4. *[Kec24] Up to Borel bireducibility, the smooth BERs are* $=_1, =_2, =_3, \ldots,$ $=_{\mathbb{N}}, =_{\mathbb{R}}$.

The theory of smooth BERs is by now fully understood (see [Kec24]). Above the smooth BERs in the Borel reducibility hierarchy are the hyperfinite BERs, which are the simplest non-trivial BERs [HKL90]. A CBER E is *hyperfinite* if it can be expressed as a countable increasing union of finite BERs, that is, if there is a sequence $(F_n)_{n\in\mathbb{N}}$ of BERs with finite classes such that $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$ and $E = \bigcup_{n\in\mathbb{N}} F_n$.

Example 3.1.5. *The* eventual equality relation E_0 *on* $2^{\mathbb{N}}$ *is defined by* $x\ E_0\ y$ *if and only if there exists* $n \in \mathbb{N}$ *such that, for all* $m \geq n$, $x(m) = y(m)$. *It is easy to see that* E_0 *is hyperfinite: For each* $n \in \mathbb{N}$, *define* F_n *on* $2^{\mathbb{N}}$ *by* $x\ F_n\ y$ *if and only if* $x(m) = y(m)$ *for all* $m \geq n$. *Then* $E_0 = \bigcup_{n\in\mathbb{N}} F_n$.

However, E_0 *is not smooth. We follow the proof in [TS12]. Assume for contradiction that there is a Borel reduction* $f : E_0 \leq_B =_{[0,1]}$. *Then the sets* $f^{-1}([0, \frac{1}{2}])$ *and* $f^{-1}([\frac{1}{2}, 1])$ *are Borel. Note that* $f^{-1}([0, \frac{1}{2}])$ *is a tail set; if* $(x_n)_{n\in\mathbb{N}} \in f^{-1}([0, \frac{1}{2}])$ *and* $(y_n)_{n\in\mathbb{N}} \in 2^{\mathbb{N}}$ *is such that there is* $n \in \mathbb{N}$ *with* $x_m = y_m$ *for all* $m \geq n$, *then* $(x_n)_{n\in\mathbb{N}}\ E_0\ (y_n)_{n\in\mathbb{N}}$,

[1] Since any two uncountable standard Borel spaces are Borel isomorphic (see Theorem 1.2.2), E is smooth if and only if it is Borel-reducible to $=_{\mathbb{R}}$.

and so $f((x_n)_{n\in\mathbb{N}}) = f((y_n)_{n\in\mathbb{N}})$*, so that* $(y_n)_{n\in\mathbb{N}} \in f^{-1}([0,\frac{1}{2}])$*. The same argument demonstrates that* $f^{-1}([\frac{1}{2},1])$ *is a tail set. Let* μ *be the product measure on* $2^{\mathbb{N}}$ *arising from the probability measure* μ_0 *on* 2 *with* $\mu_0(\{0\}) = \mu_0(\{1\}) = \frac{1}{2}$*. By the zero-one law (see [Kec95]), at least one of* $f^{-1}([0,\frac{1}{2}])$ *and* $f^{-1}([\frac{1}{2},1])$ *has* μ*-measure* 1*.*

Without loss of generality, assume $\mu(f^{-1}([0,\frac{1}{2}])) = 1$*. Then repeat the argument in the previous paragraph to show that at least one of* $f^{-1}([0,\frac{1}{4}])$ *and* $f^{-1}([\frac{1}{4},\frac{1}{2}])$ *has* μ*-measure* 1*. Continuing in this way, we obtain a sequence* $([a_n,b_n])_{n\in\mathbb{N}}$ *of closed intervals in* $[0,1]$ *such that* $a_0 = 0, b_0 = \frac{1}{2}$ *and, for each* $n \in \mathbb{N}$*, the length of* $[a_n,b_n]$ *is* $\frac{1}{2^{n+1}}$ *and* $\mu(f^{-1}([a_n,b_n])) = 1$*. Let* $a = \lim_{n\to\infty} a_n$*. Then* $\mu(f^{-1}(\{a\})) = 1$*, which implies that there is a* μ*-measure-*1 *set of* E_0*-equivalent points. This is a contradiction since* E_0 *is countable.*

Common in many mathematical contexts are dichotomy theorems. In descriptive set theory, loosely speaking, a *dichotomy theorem* for a definable property P typically has the following form: If a Borel structure fails to have P, then it is because the structure contains a canonical definable obstruction to P as a sub-structure. In the early 1990s, Harrington, Kechris, and Louveau proved a dichotomy theorem for smoothness known as the Glimm-Effros dichotomy. The theorem demonstrates that, if a CBER E fails to be smooth, then it is because E admits a Borel reduction from the canonical non-smooth relation E_0.

Theorem 3.1.6. *[HKL90] Let* E *be a CBER. If* E *is non-smooth, then* $E_0 \leq_B E$*.*

This theorem gives insight into the global structure of the Borel reducibility hierarchy in the following way; it reveals that E_0 is the unique immediate Borel-reducibility successor to the top smooth CBER $=_{\mathbb{R}}$. See Figure 3.1 for an illustration.

It is a major open problem in descriptive set theory, dating back to work of Kechris [Kec93], to determine whether there is also a dichotomy theorem for hyperfiniteness. One precise formulation of the problem is the following.

Question 3.1.7. *Is there a countable basis theorem for non-hyperfiniteness? That is, is*

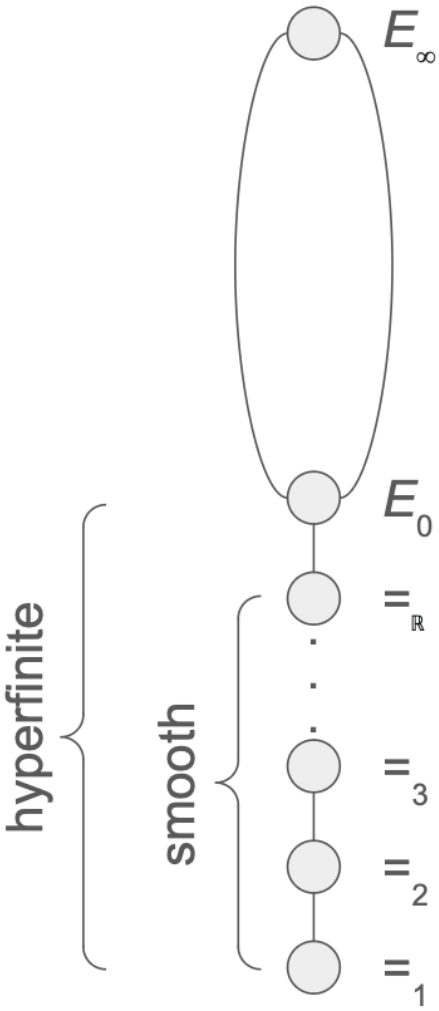


Figure 3.1: The Borel reducibility hierarchy. At the bottom of the hierarchy are the smooth CBERs, which are (up to Borel bireducibility) $=_1, - =_2, \ldots, =_\mathbb{N}$, and $= \mathbb{R}$. Next is the unique (up to Borel bireducibility) non-smooth hyperfinite CBER, E_0. At the top of the hierarchy is the universal CBER E_∞, which has the property that $E \leq_B E_\infty$ for all CBERs E.

there a countable set $\{F_n\}_{n\in\mathbb{N}}$ of CBERs such that, for any CBER E, E is non-hyperfinite if and only if $F_n \leq_B E$ for some $n \in \mathbb{N}$?

Intuitively, if there is a countable basis theorem for hyperfiniteness as in Question 3.1.7, then the question of whether a CBER E is non-hyperfinite can be stated "in a simple way": Is there $n \in \mathbb{N}$ such that there exists a Borel reduction $F_n \leq_B E$? We expand on this idea in the next section.

3.2 Projective Complexity

In this section, we make precise the intuitive ideas around complexity described in the previous section. Our primary references are [Kec95; Mos09].

Throughout, we use the notation $\mathbb{N}^{\mathbb{N}}$ to refer to the space of sequences of natural numbers indexed by $\mathbb{N}$ with the product topology induced by the discrete topology on $\mathbb{N}$. Fix a Polish space X. Write $\mathbf{\Sigma}^0_1(X)$ for the set of open subsets of X, $\mathbf{\Pi}^0_1(X)$ for the set of closed subsets of X, and $\mathbf{\Delta}^1_1(X)$ for the set of Borel subsets of X. A set $A \subseteq X$ is *analytic* if it has the form $A = \mathrm{proj}_X(F)$ for some closed set $F \subseteq X \times \mathbb{N}^{\mathbb{N}}$ and *coanalytic* if it is the complement of an analytic set. Write $\mathbf{\Sigma}^1_1(X)$ for the set of analytic subsets of X and $\mathbf{\Pi}^1_1(X)$ for the set of coanalytic subsets of X; note that $\mathbf{\Delta}^1_1(X) = \mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X)$. A set $P \subseteq X$ is $\mathbf{\Sigma}^1_2$ if it has the form $P = \mathrm{proj}_X(C)$ for some coanalytic set $C \subseteq X \times \mathbb{N}^{\mathbb{N}}$ and $\mathbf{\Pi}^1_2$ if it is the complement of a $\mathbf{\Sigma}^1_2$ set. Write $\mathbf{\Sigma}^1_2(X)$ for the set of $\mathbf{\Sigma}^1_2$ subsets of X and $\mathbf{\Pi}^1_2(X)$ for the set of $\mathbf{\Pi}^1_2$ subsets of X. The notation $\mathbf{\Delta}^1_2(X)$ denotes the intersection $\mathbf{\Sigma}^1_2(X) \cap \mathbf{\Pi}^1_2(X)$; it is a fact that $\mathbf{\Delta}^1_2(X)$ is a proper subset of each of $\mathbf{\Sigma}^1_2(X)$ and $\mathbf{\Pi}^1_2(X)$.

A *pointclass* is a class of sets in Polish spaces. The pointclasses $\mathbf{\Sigma}^0_1$,$\mathbf{\Pi}^0_1$,$\mathbf{\Delta}^1_1$, $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$, $\mathbf{\Delta}^1_2$ are defined by $\mathbf{\Sigma}^0_1 = \bigcup_{X \text{ Polish}} \mathbf{\Sigma}^0_1(X)$, $\mathbf{\Pi}^0_1 = \bigcup_{X \text{ Polish}} \mathbf{\Pi}^0_1(X)$, and so on. The *projective complexity* of a set in a pointclass is defined relative to the other sets in the same pointclass. More precisely, let $\mathbf{\Gamma}$ be a pointclass, let X be a Polish space, and let $A \subseteq X$ belong to $\mathbf{\Gamma}(X)$. Then A is Γ-*complete* if, for any Polish space Y and any $B \subseteq Y$ with $B \in \mathbf{\Gamma}(Y)$, there is a continuous $f : Y \to X$ such that $B = f^{-1}(A)$. Intuitively, a

$\mathbf{\Gamma}$-complete set is "maximally complex" among all sets in $\mathbf{\Gamma}$.

Often, it is straightforward to determine that a given set belongs to a certain projective pointclass $\mathbf{\Gamma}$, but deciding whether the set is $\mathbf{\Gamma}$-complete is highly difficult. Therefore, it is a significant priority in descriptive set theory to compute the projective complexities of various sets.

We begin by considering the important example of the projective complexity of the collection of locally finite Borel graphs having finite Borel chromatic number. (While the collection of all locally finite Borel graphs with finite Borel chromatic number is a proper class, we may without loss of generality consider only the set of locally finite Borel graphs on $\mathbb{N}^{\mathbb{N}}$; see [Mos09] for further details.) We first need a well-known fact.

Proposition 3.2.1. *[Mos09] Let X be a Polish space. Then there are sets $\mathbf{BC}(X) \in \mathbf{\Pi}^1_1(\mathbb{N}^{\mathbb{N}})$, $\mathbf{A}(X) \in \mathbf{\Sigma}^1_1(\mathbb{N}^{\mathbb{N}} \times X)$, and $\mathbf{C}(X) \in \mathbf{\Pi}^1_1(\mathbb{N}^{\mathbb{N}} \times X)$ such that the following hold.*

1. *If $c \in \mathbf{BC}(X)$, then for any $x \in X$, $(c, x) \in \mathbf{A}(X)$ if and only if $(c, x) \in \mathbf{C}(X)$.*

2. *For any Borel set $B \subseteq X$, there is $c \in \mathbf{BC}(X)$ such that $\mathbf{A}(X)_c = \mathbf{C}(X)_c = B$.*

In the above proposition, the set $\mathbf{BC}(X)$ is a set of *Borel codes* for Borel subsets of X; the sections of the sets $\mathbf{A}(X)$ and $\mathbf{C}(X)$ are, respectively, the analytic subsets of X and the coanalytic subsets of X. When $c \in \mathbf{BC}(X)$, the sections $\mathbf{A}(X)_c$ and $\mathbf{C}(X)_c$ are equal and therefore give a set that is both analytic and coanalytic, and hence Borel; furthermore, any Borel subset of X arises as some section of $\mathbf{A}(X)$ (equivalently, $\mathbf{C}(X)$).

Now let $X = (\mathbb{N}^{\mathbb{N}})^2$. Consider the set $\mathcal{C}$ of Borel codes for Borel subsets R of X such that R is the edge relation of a locally finite Borel graph on $\mathbb{N}^{\mathbb{N}}$ having countable Borel chromatic number, that is, the set of all $c \in \mathbf{BC}(X)$ such that $\mathbf{A}(X)_c$ (equivalently, $\mathbf{C}(X)_c$) is the edge relation of a locally finite Borel graph on $\mathbb{N}^{\mathbb{N}}$ having countable Borel chromatic number. It is easy to see that $\mathcal{C}$ is $\mathbf{\Sigma}^1_2$; a simple heuristic depending on (1) in Proposition 3.2.1 is that any set defined extensionally by an existential quantification over a Borel object is $\mathbf{\Sigma}^1_2$. More precisely, note that $c \in \mathcal{C}$ if and only if there exists $d \in \mathbb{N}^{\mathbb{N}}$ such that:

- $\mathbf{C}(X)_c$ is symmetric and irreflexive;

- $c \in \mathbf{BC}(X)$; and

- $d = \langle d_n \rangle_{n\in\mathbb{N}}$[2] is such that for each $n \in \mathbb{N}$, $d_n \in \mathbf{BC}(\mathbb{N}^{\mathbb{N}})$ and for any $x, y \in \mathbb{N}^{\mathbb{N}}$, if (x, y) belongs to $\mathbf{A}(X)_c$, then either $x \notin \mathbf{A}(\mathbb{N}^{\mathbb{N}})_{d_n}$ or $y \notin \mathbf{A}(\mathbb{N}^{\mathbb{N}})_{d_n}$, and the vertex set of the graph whose edge relation is $\mathbf{C}(X)_c$ is contained in $\bigcup_{n\in\mathbb{N}} \mathbf{C}(\mathbb{N}^{\mathbb{N}})_{d_n}$,

so that $\mathcal{C}$ is the projection of a $\mathbf{\Pi}^1_1$ subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.

Now, it is natural to ask whether $\mathcal{C}$ is $\mathbf{\Sigma}^1_2$-complete. By a landmark theorem of Kechris, Solecki, and Todorčević known as the $\mathcal{G}_0$ *dichotomy theorem* [KST99], there is a Borel graph $\mathcal{G}_0$ such that, for any Borel graph $\mathcal{G}$ whose Borel chromatic number is uncountable, there is a Borel homomorphism from $\mathcal{G}_0$ to $\mathcal{G}$. Therefore, the set of Borel codes for Borel subsets R of X such that R is the edge relation of a locally finite Borel graph on $\mathbb{N}^{\mathbb{N}}$ having uncountable Borel chromatic number is also $\mathbf{\Sigma}^1_2$, since the question of whether a Borel graph has uncountable Borel chromatic number can be reduced to a question about the existence of a certain Borel homomorphism. Then $\mathcal{C}$ is not $\mathbf{\Sigma}^1_2$-complete since it is both $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$. A similar result known as the $\mathcal{L}_0$ *dichotomy theorem*, due to Carroy, Miller, Schrittesser, and Vidnyánszky [CMSV21], demonstrates that the set of Borel codes of locally finite Borel graphs having Borel chromatic number at most 2 also fails to be $\mathbf{\Sigma}^1_2$-complete.

However, by a celebrated theorem of Todorčević and Vidnyánszky, the problem of determining whether a Borel graph admits a finite Borel coloring is $\mathbf{\Sigma}^1_2$-complete.

Theorem 3.2.2. *[TV21] The set of Borel codes of locally finite Borel graphs having finite Borel chromatic number is $\mathbf{\Sigma}^1_2$-complete.*

Among the most important open problems in descriptive set theory is to determine the projective complexity of the set $\mathcal{H}$ of CBERs on $\mathbb{N}^{\mathbb{N}}$ that are hyperfinite. It is again clear that $\mathcal{H}$ is $\mathbf{\Sigma}^1_2$, since the definition of hyperfiniteness involves the assertion of the existence of certain Borel objects. However, the following is open.

[2] Here, we implicitly use the fact that there is a homeomorphism between $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

Question 3.2.3. *[DJK94] Is the set of Borel codes of hyperfinite CBERs $\mathbf{\Sigma}^1_2$-complete?*

If this question has a positive answer, then Question 3.1.7 must have a negative answer; indeed, a positive answer to Question 3.1.7 would demonstrate that both hyperfiniteness and non-hyperfiniteness are $\mathbf{\Sigma}^1_2$, thereby ruling out the possibility that $\mathcal{H}$ is $\mathbf{\Sigma}^1_2$-complete. Therefore, a positive answer to Question 3.2.3 would have significant consequences for our current understanding of the structure of the Borel reducibility hierarchy.

Recent work of Frisch, Shinko, and Vidnyánszky has revealed an insightful connection between Question 3.2.3 and another notorious open problem known as the *increasing union problem*, which asks whether a countable increasing union of hyperfinite CBERs is again hyperfinite.

Theorem 3.2.4. *[FSV24] If the increasing union problem has a negative answer, then the set of Borel codes of hyperfinite CBERs is $\mathbf{\Sigma}^1_2$-complete.*

3.3 Motivation for Borel Asymptotic Dimension

Among the primary reasons for studying Borel asymptotic dimension is its potential to clarify and progress the study of definable group actions. Indeed, a primary goal of descriptive set theory is to determine the extent to which the properties of groups influence the properties of their definable actions. Amenability in particular has been a central property in this line of study. It is known that, if Γ is a countable non-amenable group, then for some standard Borel space X and some Borel action $\Gamma \curvearrowright X$, the orbit equivalence relation E^X_Γ is non-hyperfinite. Whether the converse is true is a major open problem dating back to work of Weiss from the 1980s.

Question 3.3.1. *[Wei84] Let Γ be a countable amenable group with a Borel action $\Gamma \curvearrowright X$ on a standard Borel space X. Is E^X_Γ hyperfinite?*

Over the past four decades, gradual progress has been made towards answering this question. It is now known that, for any of the following groups, their Borel actions on standard Borel spaces are hyperfinite[3]: $\mathbb{Z}^n$ for each $n \in \mathbb{N}$ [SS88; Wei84]; polynomial-growth groups [JKL02]; countable abelian groups [GJ15]; and locally nilpotent groups [SS24]. In [CJM+23], using a Borel asymptotic dimension argument, the authors prove that Borel actions of polycyclic groups are hyperfinite, marking significant progress.

Theorem 3.3.2. *[CJM+23] Let Γ be a polycyclic group with a Borel action $\Gamma \curvearrowright X$ on a standard Borel space X. Then E_Γ^X is hyperfinite.*

Other Borel asymptotic dimension arguments have since been used to demonstrate the hyperfiniteness of various group actions, including the Borel action of a polynomial-growth group on a standard Borel space [BY25], the action of a finitely generated hyperbolic group on its Gromov boundary [NV25], and the generic continuous action of Cantor space [IS25].

Borel asymptotic dimension is also of interest due to its applications in descriptive combinatorics. In particular, by restricting attention to the class of Borel graphs of finite Borel asymptotic dimension, one can recover several classical combinatorics results that fail for larger classes of Borel graphs. Of course, it is important to ensure first that the class of Borel graphs of finite Borel asymptotic dimension is non-trivial from the perspective of descriptive combinatorics. A locally countable Borel graph $\mathcal{G}$ is *smooth* if its connectedness relation $E_\mathcal{G}$ is smooth as a CBER. The descriptive combinatorics of a smooth graph exactly matches its classical combinatorics.

Theorem 3.3.3. *[CM16] Let $\mathcal{G}$ be a smooth locally countable Borel graph. Then $\chi_B(\mathcal{G}) = \chi(\mathcal{G})$.*

However, not all Borel graphs of finite Borel asymptotic dimension are smooth, and indeed, Borel graphs of finite Borel asymptotic dimension are capable of exhibiting non-classical combinatorial behavior. For instance, the (acyclic) Schreier graph of a free Borel action of $\mathbb{Z}$ (see Example 1.2.7) has Borel asymptotic dimension equal to 1, but its Borel chromatic number is 3.

[3]It is standard in descriptive set theory to abuse terminology and refer to the action of a group as hyperfinite if the associated orbit equivalence relation is hyperfinite.

In general, even acyclic locally finite Borel graphs need not have finite Borel chromatic number [KST99]. However, the following theorem holds for graphs of finite Borel asymptotic dimension.

Theorem 3.3.4. *[CJM+23] Let $\mathcal{G}$ be a locally finite Borel graph. Suppose $asdim_B(\mathcal{G}) < \infty$. Then $\chi_B(\mathcal{G}) \leq 2\chi(\mathcal{G}) - 1$. In particular, if $\mathcal{G}$ is acyclic, then $\chi_B(\mathcal{G}) \leq 3$.*

There is also an interesting connection between Borel asymptotic dimension and Brooks's theorem. Marks's determinacy argument demonstrates the failure of Brooks's theorem in the definable context [Mar16] (refer to Section 2.2 for further details). This failure extends even to hyperfinite Borel graphs; that is, for each $d \in \mathbb{N}$, there is an acyclic Borel graph of maximum degree d *that is hyperfinite* but has no Borel d-coloring [CJM+20]. However, by a result of Bernshteyn and Weilacher, a definable version of Brooks's theorem can be recovered for Borel graphs of finite Borel asymptotic dimension.

Theorem 3.3.5. *[BW25] Let $\mathcal{G}$ be a Borel graph of maximum degree $d \geq 3$. Suppose $\mathcal{G}$ does not have $\mathcal{K}_{d+1}$ as a subgraph and is of finite Borel asymptotic dimension. Then $\chi_B(\mathcal{G}) \leq d$.*

3.4 Definition and Examples for Borel Asymptotic Dimension

We now present the definition of "Borel asymptotic dimension" from [CJM+23] and give some examples.

A *Borel extended metric space* is a standard Borel space X with a Borel extended metric $\rho : X^2 \to [0, +\infty]$. The *equivalence relation generated by* ρ, denoted E_ρ, is the relation on X such that $x \; E_\rho \; y$ if and only if $\rho(x, y) < +\infty$. The metric ρ is *proper* if any set $A \subseteq X$ with finite ρ-diameter is finite. Note that, if each E_ρ-class is countable, then E_ρ is a CBER, since $E_\rho = \rho^{-1}([0, +\infty))$. For a set $U \subseteq X$ and a number $r > 0$, the *r-hop relation generated by* U, denoted $\mathcal{F}_r(U)$, is the equivalence relation on U generated by the relation $\{(x, y) \in U^2 : \rho(x, y) \leq r\}$. An equivalence relation E on X is *ρ-bounded*, or merely *bounded* if ρ is understood, if each E-class has finite ρ-diameter. The relation E is *uniformly ρ-bounded*, or merely *uniformly bounded* if ρ is understood, if there is a number s such that each E-class has ρ-diameter at most s.

Proposition 3.4.1. *[CJM+23] Let (X, ρ) be a Borel extended metric space such that E_ρ is countable. For each $d \in \mathbb{N}$, the following are equivalent:*

1. *For each $r > 0$, there is a uniformly ρ-bounded Borel equivalence relation E on X such that, for each $x \in X$, $B_r(x)$ meets at most $d + 1$ classes of E.*
2. *For each $r > 0$, there are Borel sets $U_0, \ldots, U_d$ covering X such that, for each $i \leq d$, $\mathcal{F}_r(U_i)$ is uniformly ρ-bounded.*

If there is $d \in \mathbb{N}$ such that (1) (equivalently, (2)) in the above proposition is satisfied, then the least such d is the *Borel asymptotic dimension* of (X, ρ), denoted $\mathrm{asdim}_B(X, \rho)$. If no such d exists, then $\mathrm{asdim}_B(X, \rho) = \infty$. The *(standard) asymptotic dimension* of (X, ρ), denoted $\mathrm{asdim}(X, \rho)$, is defined in the same way as the Borel asymptotic dimension, but the definability requirements in (1) and (2) are removed. Under certain assumptions, the standard and Borel asymptotic dimensions are the same.

Theorem 3.4.2. *[CJM+23] Let (X, ρ) be a Borel extended metric space such that ρ is proper. If $\mathrm{asdim}_B(X, \rho) < \infty$, then $\mathrm{asdim}_B(X, \rho) = \mathrm{asdim}(X, \rho)$.*

Example 3.4.3. *Consider the space $\mathbb{Z}^2$ with the usual Euclidean metric ρ. The standard asymptotic dimension of $(\mathbb{Z}^2, \rho)$ is equal to 2; see Figure 3.2 for an illustration. Since the Borel asymptotic dimension of $(\mathbb{Z}^2, \rho)$ is finite, $\mathrm{asdim}_B(\mathbb{Z}^2, \rho) = 2$.*

If the conditions on the uniformity of the relations in (1) and (2) in Proposition 3.4.1 are removed to obtain (1') and (2'), then (1') and (2') remain equivalent. If there is $d \in \mathbb{N}$ such that (1') (equivalently, (2')) is satisfied, then the least such d is the *Borel asymptotic separation index* of (X, ρ), denoted $\mathrm{asi}_B(X, \rho)$. If no such d exists, then $\mathrm{asi}_B(X, \rho) = \infty$. The following proposition relates Borel asymptotic dimension and Borel asymptotic separation index.

Proposition 3.4.4. *[CJM+23] Let (X, ρ) be a Borel extended metric space. Then if $asdim_B(X, \rho)$ is finite, then $asi_B(X, \rho) \leq 1$.*

For spaces equipped with a proper metric, finite Borel asymptotic dimension implies hyperfiniteness.

Theorem 3.4.5. *[CJM+23] Let (X, ρ) be a Borel extended metric space such that ρ is proper. If $asdim_B(X, \rho) < \infty$, then E_ρ is hyperfinite.*

It is open whether $\mathrm{asi}_B(X, \rho) < \infty$ implies that E_ρ is hyperfinite; see [CJM+23; Wei21].

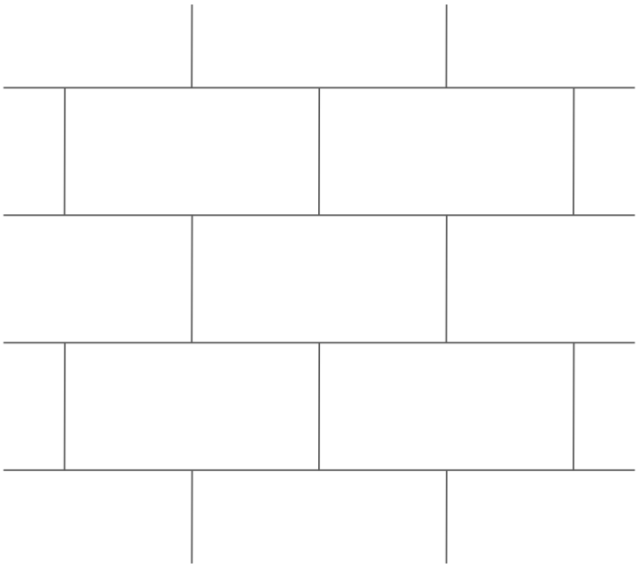
Frequently, we work with Borel asymptotic dimension in the context of Borel graphs. For a Borel graph $\mathcal{G}$ on a standard Borel space X, the Borel asymptotic dimension of $\mathcal{G}$, denoted $\mathrm{asdim}_B(\mathcal{G})$, is defined to be the Borel asymptotic dimension of (X, ρ), where ρ is the metric on X induced by $\mathcal{G}$. Note that, if $\mathcal{G}$ is locally finite, then ρ is proper.

3.5 $\boldsymbol{\Sigma}^1_2$-Completeness of Finite Borel Asymptotic Dimension

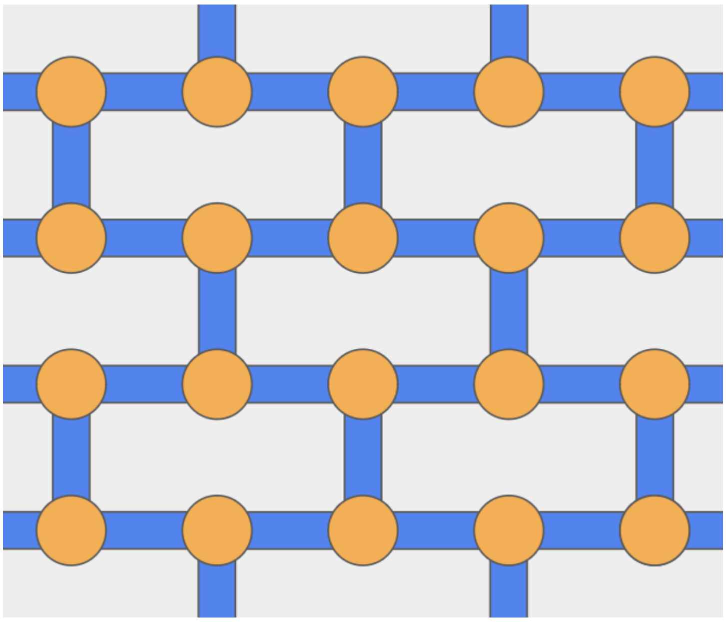
The goal of this section is to present the proof of the following theorem.

Theorem 3.5.1. *[GH24] The set of Borel codes of locally finite Borel graphs having finite Borel asymptotic dimension is $\boldsymbol{\Sigma}^1_2$-complete.*

The main tool we use to prove Theorem 3.5.1 involves transforming the geometric question of whether a graph generated by a single Borel function has finite Borel asymptotic dimension into a purely combinatorial question concerning the existence of certain forward-recurrent sets. A Borel function f on a standard Borel space X is *acyclic* if $f^k(x) \neq x$ for every $x \in X$ and $k \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. We write $\mathcal{G}_f$ for the *Borel graph generated by f*, that is, the graph on X where vertices $x, y \in X$ are adjacent if and only if $x \neq y$ and either $f(x) = y$ or $f(y) = x$, and $\overrightarrow{\mathcal{G}}_f$ for the *Borel digraph generated by f*, that is, the digraph on X where there is a directed edge from a vertex $x \in X$ to a vertex $y \in X$ if $x \neq y$ and $f(x) = y$. A set $H \subseteq X$ is *hitting for f* if, for each $x \in X$, there



(a) An equivalence relation witnessing that $\operatorname{asdim}(\mathbb{Z}^2) = 2$ for a single value of r; each rectangle represents an equivalence class.



(b) A cover witnessing that $\operatorname{asdim}(\mathbb{Z}^2) = 2$ for a single value of r; each color set represents an element of the cover.

Figure 3.2: Illustrations demonstrating that $\operatorname{asdim}(\mathbb{Z}^2) = 2$. The first image corresponds with (1) in Proposition 3.4.1, and the second image corresponds with (2).

is $k \in \mathbb{N}^+$ such that $f^k(x) \in H$. For $r \in \mathbb{N}^+$, the set H is *r-forward-independent for f* if $f^k(x) \in H$ implies that $k > r$ for any $x \in H$ and $k \in \mathbb{N}^+$. The main lemma is the following.

Theorem 3.5.2. *Let X be a standard Borel space, and let $f : X \to X$ be an acyclic Borel function. Then the following are equivalent:*

(i) $asdim_B(\mathcal{G}_f)$ is finite.

(ii) $asdim_B(\mathcal{G}_f) \leq 1$.

(iii) For each $r \in \mathbb{N}^+$, there is a Borel r-forward-independent hitting set for f.

We need the following remark concerning sufficient conditions under which (iii) in the theorem statement above holds.

Remark 3.5.3. *Condition (iii) in Theorem 3.5.2 is satisfied if, for instance, f is bounded-to-one. This follows from Corollary 5.3 in [KST99]. Furthermore, (iii) is also satisfied when the connectedness relation of $\mathcal{G}_f$ admits a Borel transversal.*

The initial step toward proving Theorem 3.5.1 involves using a general condition for $\mathbf{\Sigma}^1_2$-completeness due to Frisch, Shinko, and Vidnyánszky [FSV24] to deduce the following theorem, which immediately implies that the set of Borel codes of Borel functions admitting Borel forward-independent hitting sets is $\mathbf{\Sigma}^1_2$-complete. The main result, Theorem 3.5.1, is a corollary of Theorem 3.5.4 together with Theorem 3.5.2, whose proof is deferred until later.

Theorem 3.5.4. *There are a standard Borel space X and a finite-to-one acyclic Borel function*

$$f : \mathbb{N}^{\mathbb{N}} \times X \to \mathbb{N}^{\mathbb{N}} \times X$$

such that, for every $\sigma \in \mathbb{N}^{\mathbb{N}}$, there exists $f_\sigma : X \to X$ such that

$$f(\sigma, x) = (\sigma, f_\sigma(x)) \text{ for every } (\sigma, x) \in \mathbb{N}^{\mathbb{N}} \times X,$$

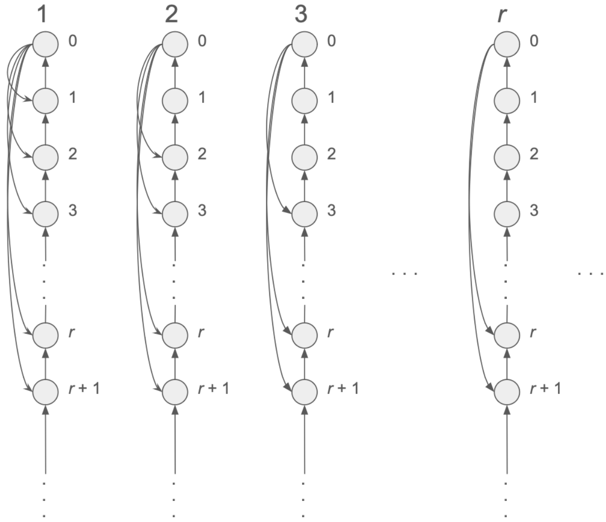


Figure 3.3: The structure $\mathcal{D}_r$ for various values of $r \in \mathbb{N}^+$.

and

$$\left\{\sigma \in \mathbb{N}^{\mathbb{N}} : \forall r \in \mathbb{N}^+ \ f_\sigma \text{ admits a Borel } r\text{-forward-independent hitting set}\right\} \tag{3.1}$$

is $\mathbf{\Sigma}^1_2$*-complete.*

Additionally, for every $r \in \mathbb{N}^+$*, there is a Borel function* f *as above such that*

$$\left\{\sigma \in \mathbb{N}^{\mathbb{N}} : f_\sigma \text{ admits a Borel } r\text{-forward-independent hitting set}\right\} \tag{3.2}$$

is $\mathbf{\Sigma}^1_2$*-complete.*

For each $r \in \mathbb{N}^+$, let $\mathcal{D}_r$ be the digraph on $\mathbb{N}$ such that there is a directed edge from k to ℓ if and only if $k > 0$ and $\ell = k - 1$ or $k = 0$ and $\ell \geq r$; see Figure 3.3.

Lemma 3.5.5. *Let* $f : X \to X$ *be an acyclic Borel function, and let* $r \in \mathbb{N}^+$*. Then the following are equivalent:*

(i) There is a Borel r*-forward-independent hitting set for* f*.*

(ii) There is a Borel map $\varphi : X \to \mathbb{N}$ *that is a homomorphism from* $(X, \overrightarrow{\mathcal{G}}_f)$ *to* $(\mathbb{N}, \mathcal{D}_r)$*.*

Proof. (i) $\implies$ (ii). Let $H \subseteq X$ be a Borel r-forward-independent hitting set for f. Define a Borel map $\varphi : X \to \mathbb{N}$ by letting $\varphi(x)$ be the least $k \in \mathbb{N}$ such that $f^k(x) \in H$.

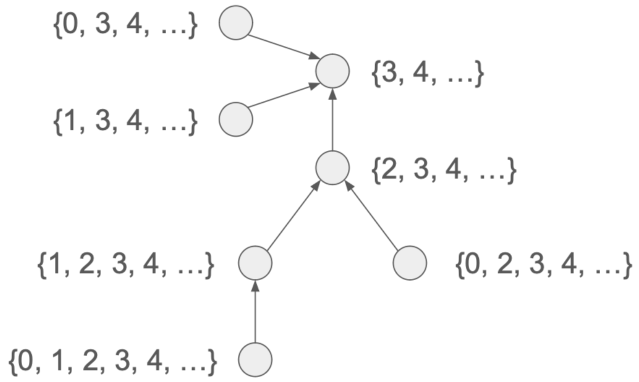


Figure 3.4: The 2-neighborhood of the vertex $\{2, 3, 4, \ldots\}$ in $\overrightarrow{\mathcal{G}}_S$.

Note that $\varphi(x) = 0$ if and only if $x \in H$. To see that φ is a homomorphism, let $x \in X$ and observe that there is a directed edge from $\varphi(x)$ to $\varphi(f(x))$; indeed, if $\varphi(x) > 0$, then $\varphi(f(x)) = \varphi(x) - 1$, and if $\varphi(x) = 0$, then $x \in H$, which implies that $\varphi(f(x)) \geq r$ since H is r-forward-independent.

(ii) $\implies$ (i). Let $\varphi : \overrightarrow{\mathcal{G}}_f \to \mathcal{D}_r$ be a Borel homomorphism. Set $H = \varphi^{-1}(\{0\})$. Then clearly H is Borel. To see that H is r-forward-independent, suppose $x, y \in H$ with $f^k(x) = y$ for some $k > 0$. Then $\varphi(x) = 0$, so $\varphi(f^i(x)) > 0$ whenever $1 \leq i \leq r$. Since $\varphi(f^k(x)) = 0$, we have $k > r$. To see that H is hitting for f, let $x \in X$ and observe that $f^{\varphi(x)}(x) \in H$ as φ is a homomorphism. □

Before proceeding to the proof of Theorem 3.5.4, we briefly introduce some additional terminology. The notation $[\mathbb{N}]^{\mathbb{N}}$ denotes the space of infinite subsets of $\mathbb{N}$ with topology inherited from the product topology on $\mathbb{N}^{\mathbb{N}}$. The *shift function* on $[\mathbb{N}]^{\mathbb{N}}$ is the function $S : [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ defined by $x \mapsto x \setminus \{\min(x)\}$. Note that S is finite-to-one, so that $\mathcal{G}_S$, called the *shift graph*, is locally finite. We write $\overrightarrow{\mathcal{G}}_S$ for the Borel digraph on $[\mathbb{N}]^{\mathbb{N}}$ induced by S. See Figure 3.4.

Now we are ready to prove Theorem 3.5.4. The proof strategy is as follows: We show first that, while there are no Borel forward-independent hitting sets for S on the entirety of $[\mathbb{N}]^{\mathbb{N}}$, such sets can be constructed for S on the *non-dominating* subsets of $[\mathbb{N}]^{\mathbb{N}}$, that is, the sets of the form Dom_x where $x \in [\mathbb{N}]^{\mathbb{N}}$; furthermore, this construction can be done in a uniform Borel way. This then allows us to apply Theorem 3.1 in [FSV24], which gives the $\mathbf{\Sigma}^1_2$-completeness of the sets 3.1 and 3.2.

Proof of Theorem 3.5.4. We first fix $r \in \mathbb{N}^+$ and prove that there are a standard Borel space X and a Borel function $f : X \to X$ as in the statement of Theorem 3.5.4 such that the set 3.1 is $\mathbf{\Sigma}^1_2$-complete. At the end of the proof, we explain how to re-select X and f so that the set 3.2 is $\mathbf{\Sigma}^1_2$-complete.

Note first that there is no Borel map $\varphi : [\mathbb{N}]^{\mathbb{N}} \to \mathbb{N}$ that is a homomorphism from $([\mathbb{N}]^{\mathbb{N}}, \overrightarrow{\mathcal{G}}_S)$ to $(\mathbb{N}, \mathcal{D}_r)$. This follows from Example 3.2 in [KST99], since the existence of a Borel r-forward-independent hitting set for S would imply that the Borel chromatic number of $\mathcal{G}_S$ is finite.

In order to apply Theorem 3.1 in [FSV24], we must show that there is a Borel map $\varphi_r : \mathrm{Dom} \to \mathbb{N}$ such that, for each $x \in [\mathbb{N}]^{\mathbb{N}}$, the restriction $\varphi_r \restriction \mathrm{Dom}_x$ is a homomorphism from $\overrightarrow{\mathcal{G}}_S[\mathrm{Dom}_x]$ to $\mathcal{D}_r$, where $\overrightarrow{\mathcal{G}}_S[\mathrm{Dom}_x]$ denotes the subdigraph of $\overrightarrow{\mathcal{G}}_S$ induced by Dom_x.

Set $x(-r) = 0$ for all $x \in [\mathbb{N}]^{\mathbb{N}}$. Define $\mathcal{H}_r \subseteq ([\mathbb{N}]^{\mathbb{N}})^2$ by $(x, y) \in \mathcal{H}_r$ if and only if $|[x(2rn-r), x(2rn+r)) \cap y| \equiv_{2r} 0$, where $n \in \mathbb{N}$ is minimal such that $[x(2rn-r), x(2rn+r)) \cap y \neq \emptyset$. Observe that $\mathcal{H}_r$ is Borel.

Claim 3.5.6. *If $(x, y) \in$ Dom, then there is $n \in \mathbb{N}$ such that $\big|[x(2rn-r), x(2rn+r) \cap y\big| \geq 2r$.*

Proof. Let $(x, y) \in \mathrm{Dom}$ and assume for contradiction that there is no $n \in \mathbb{N}$ such that $\big|[x(2rn - r), x(2rn + r)) \cap y\big| \geq 2r$. For each $n \in \mathbb{N}$, let $\ell_n \in \mathbb{N}$ be minimal such that $y(\ell_n) \geq x(2rn - r)$; it follows from the assumption that $\ell_{n+1} < \ell_n + 2r$. Hence, $\ell_n \leq 2rn - n$ for each $n \in \mathbb{N}$. So, there is $n_0 \in \mathbb{N}$ such that $\ell_n < 2rn - 3r$ for every $n \geq n_0$. Let $k \geq \ell_{n_0}$, and let $n \geq n_0$ be minimal such that $k \geq \ell_n$. Then we have

$$y(k) \geq y(\ell_n) \geq x(2rn - r) > x(k)$$

since $2rn - r > \ell_n + 2r > \ell_{n+1} \geq k$, which is a contradiction as $y \leq^\infty x$. $\square$

Now for each $(x, y) \in \mathrm{Dom}$, define $\varphi_r(x, y)$ to be the least $k \in \mathbb{N}$ such that $S^k(y) \in (\mathcal{H}_r)_x$. By Claim 3.5.6, we have that φ_r is well-defined and it is easy to check that it is Borel as $\mathcal{H}_r$ is Borel. To see that $\varphi_r \restriction \mathrm{Dom}_x$ is a homomorphism from $\overrightarrow{\mathcal{G}}_S[\mathrm{Dom}_x]$ to $\mathcal{D}_r$ for each $x \in [\mathbb{N}]^{\mathbb{N}}$, note that, if $y, y' \in \mathrm{Dom}_x$ and $S(y) = y'$, then $\varphi_r(x, y) = 0$ implies $\varphi_r(x, S(y)) \geq r$ since

$$\big|[x(2rn - r), x(2rn + r)) \cap S(y)\big| \equiv_{2r} 2r - 1,$$

where $n \in \mathbb{N}$ is minimal such that $[x(2rn - r), x(2rn + r)) \cap y \neq \emptyset$. If $\varphi_r(x, y) > 0$, then $\varphi_r(x, S(y)) = \varphi_r(x, y) - 1$ by the definition.

Now we are ready to finish the first part of the proof. Note that the assumptions of Theorem 3.1 in [FSV24] are satisfied by the construction of φ_r above. Hence, there is a Borel set $B \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$ such that

$$\big\{\sigma \in \mathbb{N}^{\mathbb{N}} : \text{ there is a Borel homomorphism from } (\overrightarrow{\mathcal{G}}'_S[B_\sigma]) \text{ to } \mathcal{D}_r\big\} \tag{3.3}$$

is $\mathbf{\Sigma}^1_2$-complete, where $\overrightarrow{\mathcal{G}}'_S$ is the digraph on $\mathbb{N}^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}$ such that there is a directed edge from (γ, x) to (δ, y) if and only if $\gamma = \delta$ and $S(x) = y$. Let

$$Y = \big\{(\sigma, \gamma, x) \in B : (\sigma, \gamma, S(x)) \notin B\big\},$$

and observe that Y is a Borel set. Set also $Z = \big(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}}\big) \setminus B$. We define

$$X = \bigsqcup_{\ell \in \mathbb{N}} \big(\mathbb{N}^{\mathbb{N}} \times [\mathbb{N}]^{\mathbb{N}} \times \{\ell\}\big)$$

and a function f on $\mathbb{N}^{\mathbb{N}} \times X$ as follows. If $(\sigma, (\gamma, x, \ell)) \in \mathbb{N}^{\mathbb{N}} \times X$ has the property that $(\sigma, \gamma, x) \notin Y \cup Z$, then we set $f(\sigma, (\gamma, x, \ell)) = (\sigma, (\gamma, S(x), \ell))$. Otherwise, we have that $(\sigma, \gamma, x) \in Y \cup Z$ and we let $f(\sigma, (\gamma, x, \ell)) = (\sigma, (\gamma, x, \ell + 1))$. Then $f : \mathbb{N}^{\mathbb{N}} \times X \to \mathbb{N}^{\mathbb{N}} \times X$ is a finite-to-one acyclic Borel function. Fix $\sigma \in \mathbb{N}^{\mathbb{N}}$, and let f_σ be the function on X such that $f(\sigma, (\gamma, x, \ell)) = (\sigma, f_\sigma(\gamma, x, \ell))$ for all $(\gamma, x, \ell) \in X$. Note that the collection $\{f_\sigma\}_{\sigma \in \mathbb{N}^{\mathbb{N}}}$ is well-defined.

To complete this part of the proof, we show that the sets 3.1 and 3.3 coincide. Define a set $T \subseteq X$ by $(\gamma, x, \ell) \in T$ if $\ell = 0$ and $(\sigma, \gamma, x) \in Y \cup Z$, and let $\mathcal{C}$ denote the collection of connected components of the graph $\mathcal{G}_{f_\sigma}$ on X generated by f_σ containing a point $(\gamma, x, \ell) \in X$ such that $(\sigma, \gamma, x) \in Y \cup Z$. Then T is a Borel transversal for the connectedness relation of $\mathcal{G}_{f_\sigma}\left[\bigcup \mathcal{C}\right]$. Therefore, since f_σ is acyclic, we have that $f_\sigma \upharpoonright \bigcup \mathcal{C}$ admits a Borel r-forward-independent hitting set by Remark 3.5.3. It follows that the sets 3.1 and 3.3 coincide, as required.

We now explain how to modify the proof to obtain X and f so that set 3.2 is $\mathbf{\Sigma}^1_2$-complete: Consider a language consisting of countably many binary relation symbols $(\mathcal{R}_r)_{r\in\mathbb{N}^+}$. For each $r \in \mathbb{N}^+$, interpret $\mathcal{R}_r$ in $\mathbb{N}$ as the edge relation of $\mathcal{D}_r$, and apply Theorem 3.1 in [FSV24] to obtain a Borel set $B \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times [\mathbb{N}]^\mathbb{N}$ such that

$$\left\{\sigma \in \mathbb{N}^\mathbb{N} : \forall r \in \mathbb{N}^+ \text{ there is a Borel homomorphism from } (\overrightarrow{\mathcal{G}}'_S[B_\sigma]) \text{ to } \mathcal{D}_r\right\} \tag{3.4}$$

is $\mathbf{\Sigma}^1_2$-complete. The rest of the proof proceeds in the same way. □

Remark 3.5.7. *In personal communication, A. Kastner pointed out that the Borel set $B \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times [\mathbb{N}]^\mathbb{N}$ obtained from the application of Theorem 3.1 in [FSV24] can be chosen so that, for each $\sigma, \gamma \in \mathbb{N}^\mathbb{N}$, the section $B_{\sigma,\gamma}$ is $\mathcal{G}_S$-invariant. This allows one to define $X = \mathbb{N}^\mathbb{N} \times [\mathbb{N}]^\mathbb{N}$ and $f(\sigma, \gamma, x) = (\sigma, \gamma, S(x))$ directly in the proof of Theorem 3.5.4.*

Now we prove Theorems 3.5.2 and 3.5.1. First, recall from Section 3.4 that there are two separate definitions for Borel asymptotic dimension, one in terms of coverings and the other in terms of equivalence relations; these are shown to be equivalent for locally countable Borel graphs in Lemma 3.1 in [CJM+23]. Here we prove that, for graphs generated by a single Borel function f, the two definitions remain equivalent even when f is *uncountable*-to-one.

Fix a standard Borel space X. Since we work exclusively with graphs of the form $\mathcal{G}_f$, where f is a Borel function on X, we denote by ρ_f the graph distance metric of $\mathcal{G}_f$. Given $t \in \mathbb{N}^+$ and $x \in X$, we write $B_t(x)$ for the ρ_f-ball of radius t around $x \in X$. For $U \subseteq X$ and $r \in \mathbb{N}^+$, we define $\mathcal{F}_r(U)$ to be the equivalence relation on U generated by the set of pairs $(x, y) \in U^2$ such that $\rho_f(x, y) \leq r$.

Definition 3.5.1. *Let $f : X \to X$ be a Borel function.*

(i) The Borel asymptotic dimension of $\mathcal{G}_f$ from coverings, *denoted* $asdim_B^{cov}(\mathcal{G}_f)$, *is equal to* $d \in \mathbb{N}$ *if* d *is minimal such that, for every* $r \in \mathbb{N}^+$, *there are Borel sets* $U_0, \ldots, U_d$ *covering* X *such that* $\mathcal{F}_r(U_i)$ *is uniformly* ρ_f*-bounded for every* $i \leq d$.

(ii) The Borel asymptotic dimension of $\mathcal{G}_f$ from equivalence relations, $asdim_B^{eq}(\mathcal{G}_f)$, *is equal to* $d \in \mathbb{N}$ *if* d *is minimal such that, for every* $r \in \mathbb{N}^+$, *there is a uniformly* ρ_f*-bounded Borel equivalence relation* E *on* X *such that, for each* $x \in X$, $B_r(x)$ *meets at most* $(d+1)$ E*-classes.*

In Subsection 3.5.3, we show $\text{asdim}_B^{\text{cov}}(\mathcal{G}_f) = \text{asdim}_B^{\text{eq}}(\mathcal{G}_f)$ for any Borel function f, so that we may define $\text{asdim}_B(\mathcal{G}_f)$ to be $\text{asdim}_B^{\text{cov}}(\mathcal{G}_f)$ (equivalently, $\text{asdim}_B^{\text{eq}}(\mathcal{G}_f)$).

In Subsections 3.5.1 and 3.5.2, we demonstrate, for graphs of the form $\mathcal{G}_f$, a connection between both the equivalence relation and covering definitions of finite Borel asymptotic dimension and the presence of Borel forward-independent hitting sets. In Subsection 3.5.3, we use this hitting-set characterization of finite Borel asymptotic dimension to prove Theorem 3.5.2, and in Subsection 3.5.4, we prove Theorem 3.5.1.

3.5.1 From Finite Asymptotic Dimension to Hitting Sets

Proposition 3.5.8. *Let* $f : X \to X$ *be an acyclic Borel function,* $A \subseteq X$ *be a Borel hitting set for* f *and* $r \in \mathbb{N}^+$ *be such that every equivalence class of* $\mathcal{F}_r(A)$ *has finite diameter. Then there is a Borel* r*-forward-independent hitting set* H *for* f*.*

Proof. Let

$$H = \left\{x \in A : f^j(x) \notin A \text{ for any } 1 \leq j \leq r\right\} = A \setminus \bigcup_{j=1}^{r} f^{-j}(A).$$

Observe that H is a Borel r-forward-independent set. It remains to show that H is hitting for f. As A is hitting for f, we have that for every $x \in X$ there is $k \in \mathbb{N}$ such that $f^k(x) \in A$. Let $\ell \in \mathbb{N}$ be maximal so that $f^{k+\ell}(x)$ and $f^k(x)$ are $\mathcal{F}_r(A)$-equivalent. Note that such $\ell \in \mathbb{N}$ exists as the $\mathcal{F}_r(A)$-equivalence class of $f^k(x)$ has finite diameter. It follows from the definition of $\mathcal{F}_r(A)$ that $f^{k+\ell}(x) \in H$, which finishes the proof. $\square$

Proposition 3.5.9. *Let $t, d \in \mathbb{N}^+$, $f : X \to X$ be an acyclic Borel function and E be a Borel equivalence relation on X such that each E-class has finite diameter and $B_{2t(d+1)}(x)$ meets at most $(d+1)$ E-classes for every $x \in X$. Then there is a Borel t-forward-independent hitting set for f.*

Proof. Define

$$A = \{x \in X : (x, f^k(x)) \notin E \text{ for all } k \in \mathbb{N}^+\}.$$

Then A is a Borel hitting set for f. Indeed, we have that $A = \bigcap_{k \in \mathbb{N}^+} A_k$, where

$$A_k = \mathrm{proj}_0\big(\mathrm{graph}(f^k) \setminus E\big)$$

is Borel for each $k \in \mathbb{N}^+$ by the Lusin–Suslin theorem. Now let $x \in X$. Then since $[x]_E$ has finite diameter, there is $k \in \mathbb{N}$ such that $(x, f^k(x)) \in E$ and $(x, f^{k+j}(x)) \notin E$ for every $j \in \mathbb{N}^+$. Thus $f^k(x) \in A$.

Now set

$$H = \{x \in A : f^j(x) \notin A \text{ for all } 1 \leq j \leq t\}.$$

Then $H = A \setminus \big(\bigcup_{1 \leq j \leq t} f^{-j}(A)\big)$ is Borel and t-forward-independent. We claim that H is hitting. Let $x_0 \in X$. Since A is hitting, there is $x \in A$ that is an f-iterate of x_0. If $x \in H$, then there is nothing to prove. Otherwise, there is $1 \leq j_0 \leq t$ such that $f^{j_0}(x) \in A$. Repeat this procedure $d+1$ times to obtain $1 \leq j_0, \ldots, j_d \leq t$ such that either $f^{j_0+\cdots+j_m}(x) \in H$ for some $m \leq d$ or $f^{j_0+\cdots+j_m}(x) \in A$ for each $m \leq d$. Note that $B_{2t(d+1)}(x)$ contains $f^{j_0+\cdots+j_m}(x) \in A$ for each $m \leq d$. If $m \neq m'$, then

$$\big(f^{j_0+\cdots+j_m}(x), f^{j_0+\cdots+j_{m'}}(x)\big) \notin E,$$

and $\big(x, f^{j_0+\cdots+j_m}(x)\big) \notin E$ for any $m \leq d$. But then $B_{2t(d+1)}(x)$ meets at least $(d+2)$ E-classes, a contradiction. □

Remark 3.5.10. *Note that a* uniform *bound on diameter is not needed to construct the hitting sets in these proofs; boundedness alone is sufficient. Therefore, one can show that, if $\mathcal{G}_f$ has finite Borel asymptotic separation index, denoted $asi_B(\mathcal{G}_f)$, then there is a Borel*

r-forward-independent hitting set for f for each $r \in \mathbb{N}^+$. In particular, for a graph of the form $\mathcal{G}_f$, if $asi_B(\mathcal{G}_f) < \infty$, then $asi_B(\mathcal{G}_f) \leq 1$. It is open whether $asi_B(\mathcal{G}) < \infty$ implies $asi_B(\mathcal{G}) \leq 1$ for general Borel graphs $\mathcal{G}$; see [CJM+23; QW22; BW25].

3.5.2 From Hitting Sets to Finite Asymptotic Dimension

We start with a technical definition of a Borel map $c_H : X \to \{0,1\}$ that will be used to construct the witnesses to finite Borel asymptotic dimension.

Let $t \in \mathbb{N}^+$ and set $s = 6t$ and $r = 4s^2$. Fix a decomposition

$$\{0, 1, \dots, \frac{r}{2} - 1\} = I_0 \sqcup \dots \sqcup I_{2m}, \tag{3.5}$$

where $m \in \mathbb{N}$ and I_k is an interval that contains either s or $s+1$ many elements for every $0 \leq k \leq 2m$. This is possible since $\frac{r}{2} = 2s^2 = s(s+1) + (s-1)s$.

Now let f be a Borel function on a standard Borel space X, and let H be a Borel r-forward-independent hitting set for f. For every $z \in H$, set $T_z = \bigcup_{j=1}^{\frac{r}{2}-1} f^{-j}(z)$, and for every $x \in X$, define $k(x) \in \mathbb{N}^+$ to be minimal such that $f^{k(x)}(x) \in H$. Observe that $T_z \cap T_{z'} = \emptyset$ for every $z, z' \in H$ with $z \neq z'$ as H is r-forward-independent for f.

We define a function $c_H : X \to \{0,1\}$ as follows. For every $x \in X$ such that $k(x) \geq \frac{r}{2}$, define

$$c_H(x) = \lfloor k(x)/s \rfloor \mod 2.$$

Observe that if $z \in H$, then $k(z) \geq r$, so that $c_H(z)$ is defined; also note that, for any x, if $k(x) = \frac{r}{2}$, then $c_H(x) = 0$. For any $z \in H$, if $c_H(z) = 0$, we set

$$c_H(x) = \lfloor k(x)/s \rfloor \mod 2$$

for every $x \in T_z$. If $c_H(z) = 1$, we define

$$c_H(x) = k + 1 \mod 2$$

for every $x \in T_z$, where $0 \leq k \leq 2m$ is the unique integer such that $k(x) \in I_k$. Note that $c_H : X \to \{0,1\}$ is Borel.

Proposition 3.5.11. *Let $t \in \mathbb{N}^+$, $f : X \to X$ be an acyclic Borel function and H be a Borel $4(6t)^2$-forward independent hitting set. Define $c_H^{-1}(i) = U_i$ for each $i \in \{0,1\}$, where c_H is defined as in the previous paragraph. Then U_0 and U_1 are Borel sets such that $X = U_0 \sqcup U_1$ and for each $i \in \{0,1\}$, every equivalence class of $\mathcal{F}_t(U_i)$ has diameter bounded by $28t + 7$.*

Proof. Recall from the definition of c_H that $s = 6t$. Define $\ell(x) \in \mathbb{N}^+$ to be minimal such that $c_H(x) \neq c_H(f^{\ell(x)}(x))$ and set

$$e(x) = f^{\frac{s}{3}+\ell(x)}(x)$$

for every $x \in X$. Note that we have $\ell(x) \leq 2s+2$ for every $x \in X$. Indeed, if $k(x) > s+1$, then by the definition of c_H, we have $\ell(x) \leq s+1$. If $k(x) \leq s+1$, then we have that $k(y) \geq r \geq s+1$, where $y = f^{k(x)}(x) \in H$. Hence, $\ell(f^{k(x)}(x)) \leq s+1$ and consequently $\ell(x) \leq 2s+2$.

Claim 3.5.12. *For every $x \in U_i$, where $i \in \{0,1\}$, we have that*

$$\bigcup_{j=0}^{\frac{s}{3}} f^{-j}\big(e(x)\big) \subseteq U_{1-i}.$$

Proof. As $e(x) = e\big(f^{\ell(x)-1}(x)\big)$, we may assume without loss of generality that $\ell(x) = 1$. First, we claim that $f^j(x) \in U_{1-i} \setminus H$ for every $1 \leq j \leq \frac{s}{3} + 1$.

Indeed, if $f^j(x) \in H$ for any $1 \leq j \leq \frac{s}{3} + 1$, then, by the definition of c_H, we have $x \in U_{1-i}$, a contradiction. Similarly, assume for contradiction that $1 \leq j \leq \frac{s}{3} + 1$ is minimal such that $f^j(x) \notin U_{1-i}$. Then $j \geq 2$, $f^{j-1}(x) \in U_{1-i}$ and we have that $x \in U_{1-i}$ as either $\lfloor k(x)/s \rfloor = \lfloor k(f^{j-1}(x))/s \rfloor \mod 2$ or $k(x), k\big(f^{j-1}(x)\big) \in I_k$ for some $0 \leq k \leq 2\ell$, where I_k is from (3.5).

We finish the proof by showing that for every $0 \leq j \leq \frac{s}{3}$ and $z \in f^{-j}\big(e(x)\big)$, we have that $z \in U_{1-i}$. This clearly holds when $k(z) = k\big(f^{\frac{s}{3}+1-j}(x)\big)$ by the previous paragraph. In particular, as H is r-forward independent, it holds whenever $z \in H$. Now, the claim follows as, by the definition of c_H, we have that $c_H(w) = c_H(z)$ whenever $f^m(w) = z \in H$ and $0 \leq m \leq s-1$. $\square$

Claim 3.5.13. *Let $x, y \in X$ be such that $y \in [x]_{\mathcal{F}_t(U_i)}$ for some $i \in \{0,1\}$. Then there is $k \in \mathbb{N}$ such that $f^k(y) = e(x)$.*

Proof. Assume for contradiction that $f^k(y) \neq e(x)$ for any $k \in \mathbb{N}$. Then, by the definition of $\mathcal{F}_t(U_i)$, there are $z, w \in [x]_{\mathcal{F}_t(U_i)}$ such that $\rho_f(z,w) \leq t$, $f^m(z) = e(x)$ for some $m \in \mathbb{N}$, and $f^k(w) \neq e(x)$ for any $k \in \mathbb{N}$. This is a contradiction, since the shortest path that connects z and w in $\mathcal{G}_f$ must contain $e(x)$ and go through

$$\bigcup_{i=0}^{\frac{s}{3}} f^{-i}\big(e(x)\big),$$

which is a subset of U_{1-i} by Claim 3.5.12. Hence, we have $\rho_f(w,z) \geq \frac{s}{3} > t$, a contradiction. □

Observe that it follows from Claim 3.5.13 together with the assumption that f is acyclic that, if $(x,y) \in \mathcal{F}_r(U_i)$, then

$$\rho_f(x,y) \leq \rho_f(x, e(x)) + \rho_f(y, e(y)) \leq \frac{2s}{3} + \ell(x) + \ell(y).$$

This finishes the proof as we have that $\ell(x) \leq 2s + 2$ holds for every $x \in X$. □

Proposition 3.5.14. *Let $t \in \mathbb{N}^+$, $f : X \to X$ be an acyclic Borel function and H be a Borel $4(6t)^2$-forward independent hitting set. Then there is a Borel equivalence relation E on X such that the ρ_f-diameter of every E-class is bounded by $28t+7$ and $B_t(x)$ meets at most 2 E-classes for any $x \in X$.*

Proof. We continue using the notation developed in the definition of c_H and the proof of Proposition 3.5.11. For each $x \in X$, set

$$L(x) = f^{\ell(x)}(x).$$

Define a relation E on X by $(x,y) \in E$ if and only if $L(f^t(x)) = L(f^t(y))$. Then E is Borel, since

$$E = \left(L \circ f^t, L \circ f^t\right)^{-1}(=_X).$$

We show that E works as required.

To see that E is uniformly ρ_f-bounded, let $x \in X$. Then

$$[x]_E \subseteq \bigcup_{j \leq t+(2s+2)} f^{-j}\big(L(f^t(x))\big).$$

Indeed, if $y \in [x]_E$, then $L\big(f^t(y)\big) = L\big(f^t(x)\big)$. So the distance from y to $L\big(f^t(x)\big)$ is at most $t + (2s+2)$. Therefore, the diameter of $[x]_E$ is at most $2(t + (2s+2))$.

We finish the proof by showing that $B_t(x)$ meets at most 2 E-classes for each $x \in X$. Note that $f^t[B_t(x)] = \big\{x, f(x), \ldots, f^{2t}(x)\big\}$ and that

$$\big\{x, f(x) \ldots, f^{2t}(x)\big\} \subsetneq \big\{x, f(x), \ldots, e(x)\big\}$$

since $s = 6t$. By the definition of $\ell(x)$ and Claim 3.5.12, there is $i \in \{0,1\}$ such that

$$\big\{x, f(x), \ldots, f^{\ell(x)-1}(x)\big\} \subseteq U_i \text{ and } \big\{f^{\ell(x)}(x), \ldots, e(x)\big\} \subseteq U_{1-i}.$$

For each $y \in B_t(x)$, write $0 \leq j_y \leq \frac{s}{3} + \ell(x)$ such that $f^t(y) = f^{j_y}(x)$. Let $y, y' \in B_t(x)$. It follows that if $j_y, j_{y'} < \ell(x)$, then $L\big(f^t(y)\big) = L(x) = L\big(f^t(y')\big)$, and so $(y, y') \in E$. If $j_y, j_{y'} \geq \ell(x)$, then it follows that $L\big(f^t(y)\big) = L\big(L(x)\big) = L\big(f^t(y')\big)$, and so again $(y, y') \in E$. $\square$

3.5.3 The Proof of Theorem 3.5.2

We now show the following, which immediately implies Theorem 3.5.2.

Theorem 3.5.15. *Let X be a standard Borel space, and let $f : X \to X$ be an acyclic Borel function. Then the following are equivalent:*

(i) $asdim_B^{cov}(\mathcal{G}_f)$ is finite.

(i') $asdim_B^{eq}(\mathcal{G}_f)$ is finite.

(ii) $asdim_B^{cov}(\mathcal{G}_f) = 1$.

(ii') $asdim_B^{eq}(\mathcal{G}_f) = 1$.

(iii) For each $r > 0$, there is a Borel r-forward-independent hitting set for f.

Proof. (ii) $\implies$ (i) and (ii') $\implies$ (i') are trivial.

(i) $\implies$ (iii). Suppose that $\mathrm{asdim}_B^{\mathrm{cov}}(\mathcal{G}_f) = d \in \mathbb{N}$. Let $r \in \mathbb{N}^+$ and $U_0, \dots, U_d$ be as in Definition 3.5.1. For each $i \leq d$, define

$$A_i = \big\{x \in X : \forall k \in \mathbb{N} \exists n \in \mathbb{N}\ f^{k+n}(x) \in U_i\big\} = \bigcap_{k \in \mathbb{N}} \bigcup_{n=0}^{\infty} f^{-(k+n)}(U_i).$$

It follows that A_i is a Borel f-invariant set and $X = \bigcup_{i=0}^{d} A_i$. By Proposition 3.5.8, applied to $X = A_i$ and $A = U_i$, we get that $f \restriction A_i$ admits a Borel r-forward-independent hitting set for every $i \leq d$, which gives (iii).

(i') $\implies$ (iii). Suppose that $\mathrm{asdim}_B^{\mathrm{eq}}(\mathcal{G}_f) = d \in \mathbb{N}$. Let $t \in \mathbb{N}^+$, let $r = 2t(d+1)$, and let E be as in Definition 3.5.1. Then by Proposition 3.5.9, there is a Borel t-forward-independent hitting set for f, which gives (iii).

(iii) $\implies$ (ii). Given $t \in \mathbb{N}^+$, apply Proposition 3.5.11 to obtain Borel sets U_0, U_1 that cover X and have the property that $\mathcal{F}_r(U_i)$ is uniformly ρ_f-bounded for each $i \in \{0, 1\}$.

(iii) $\implies$ (ii'). Given $t \in \mathbb{N}^+$, apply Proposition 3.5.14 to obtain a uniformly ρ_f-bounded Borel equivalence relation E on X such that $B_t(x)$ meets at most 2 E-classes for any $x \in X$. $\square$

Remark 3.5.16. *When f is* not *acyclic, define C to be the collection of* cyclic points *of f, that is,*

$$C = \big\{x \in X : \exists k \in \mathbb{N}, \ell \in \mathbb{N}^+\ f^{k+\ell}(x) = f^k(x)\big\}.$$

It is easy to show that C is Borel and f-closed and that the connectedness relation of $\mathcal{G}_f[C]$ admits a Borel transversal T that is hitting for $f \restriction C$. From T, one can construct a witness to $\mathrm{asdim}_B(\mathcal{G}_f[C]) \leq 1$. Note, however, that for any $r \in \mathbb{N}^+$, if there is an r-forward-independent set for f, then $r \leq \ell$, where $\ell = 0$ if f has a fixed point and ℓ is the minimal length of a cycle in $\mathcal{G}_f$ if f has no fixed points.

3.5.4 The Proof of Theorem 3.5.1

By Theorem 3.5.2, it is enough to show that the set of finite-to-one acyclic Borel functions admitting a Borel r-forward-independent hitting set for every $r \in \mathbb{N}^+$ is $\mathbf{\Sigma}^1_2$-complete.

This latter claim follows directly from Theorem 3.5.4.

Remark 3.5.17. *In personal communication, F. Weilacher showed Theorem 3.5.1 also implies that the set of Borel codes of locally finite Borel graphs admitting* toast *is* $\mathbf{\Sigma}^1_2$-*complete. A toast is a strong witness to hyperfiniteness; see the papers of Gao, Jackson, Krohne, and Seward [GJKS25; GJKS24] for a detailed discussion. Toast was used as a critical tool in the proof due to Marks and Unger [MU17] of a Borel solution to Tarski's circle-squaring problem [Tar25].*

3.6 The Classification of $\mathbf{CSP}_B^{\mathbf{function}}(H)$

As an application of the techniques used to prove Theorems 3.5.1 and 3.5.2, we derive a new complexity result for a homomorphism problem for the class of Borel directed graphs generated by a single Borel function. Given a finite digraph H, we follow Thornton[4] [Tho22] and write $\mathrm{CSP}_B(H)$ for the set (of Borel codes) of Borel digraphs that admit a Borel homomorphism to H, and define $\mathrm{CSP}_B^{\mathrm{function}}(H)$ to be the set (of codes) of Borel digraphs *generated by a single Borel function* that admit a Borel homomorphism to H. Note that when investigating $\mathrm{CSP}_B^{\mathrm{function}}(H)$, we can without loss of generality assume that H is *sinkless*, that is, that every vertex of H has at least one out-neighbor.

Recall that a finite digraph H is *ergodic* if there are $v \in V(H)$ and $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$, there is a directed path of length k that starts and ends at v. Note that, in this section, the directed edge relation for a digraph need not be irreflexive. A *loop* is a directed edge of the form (v, v) for some $v \in V(H)$. Note that any digraph containing a loop is ergodic.

Theorem 3.6.1. *Let H be a finite sinkless digraph.*

(i) The digraph H contains a loop if and only if $CSP_B^{function}(H)$ contains all digraphs generated by a single Borel function. In particular, the set $CSP_B^{function}(H)$ is $\mathbf{\Pi}^1_1$.

[4] We refer the reader to [Tho22] for a thorough discussion of the motivation for and connection between classical and Borel *constraint satisfaction problems (CSPs).* We remark that Feder and Vardi [FV99] proved that every fixed template CSP is polynomial-time equivalent to a digraph CSP.

(ii) If H is ergodic and has no loops, then $CSP_B^{function}(H)$ is $\mathbf{\Sigma}^1_2$-complete.

(iii) If H is not ergodic, then $CSP_B^{function}(H)$ is $\mathbf{\Pi}^1_1$.

Theorem 3.6.1 provides a connection with the LOCAL model of distributed computing (see Section 2.6) and also serves as a specific instance of an interesting general phenomenon: For "flexible" problems such as 3-coloring, there is a non-trivial class of graphs for which it is easy to produce solutions in a distributed way, but it is difficult to determine, for general graphs, whether a solution exists; while for "rigid" problems such as 2-coloring, it is impossible to construct solutions on any non-trivial class of graphs, but it is easy to check, for general graphs, whether a solution exists.

To be precise, note first that any LCL problem on oriented paths can be thought of as a *digraph homomorphism problem* Π_H for some finite sinkless digraph H [BHK+17] (see also Section 2.6), that is, the problem of determining whether a given finite digraph admits a digraph homomorphism to H. Let $\mathbf{G}$ be the class of finite directed graphs. The classification of LCL problems on oriented paths in the LOCAL model due to [BHK+17] states that:

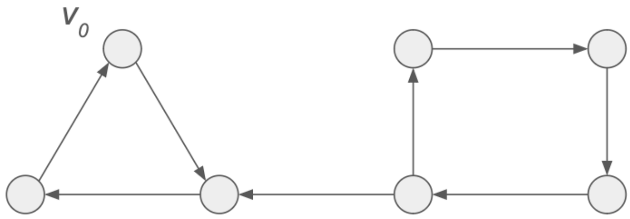
(i) If H contains a loop, then $\mathrm{Det}_{\Pi_H,\mathbf{G}}(n) \sim O(1)$.

(ii) If H is ergodic and has no loops, then $\mathrm{Det}_{\Pi_H,\mathbf{G}}(n) \sim \Theta(\log^*(n))$.

(iii) If H is not ergodic, then $\mathrm{Det}_{\Pi_H,\mathbf{G}}(n) \sim \Theta(n)$.

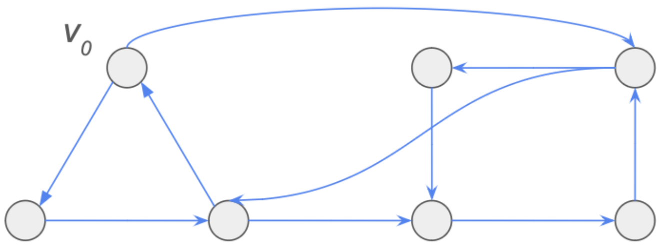
Comparing with Theorem 3.6.1, we see that the existence of an efficient but non-trivial deterministic LOCAL algorithm for solving Π_H is equivalent to $\mathrm{CSP}_B^{\mathrm{function}}(H)$ being a $\mathbf{\Sigma}^1_2$-complete set. If no such LOCAL algorithm exists, then $\mathrm{CSP}_B^{\mathrm{function}}(H)$ is $\mathbf{\Pi}^1_1$.

Remark 3.6.2. *As $\mathbf{\Sigma}^1_2$ is an upper bound on the complexity of $CSP_B(H)$, it follows that if $CSP_B^{function}(H)$ is $\mathbf{\Sigma}^1_2$-complete, then so is $CSP_B(H)$. The reverse implication is easily seen to fail.*



(a) A directed graph H such that $\mathrm{CSP}_B^{\text{function}}(H)$ is $\mathbf{\Pi}^1_1$.

(b) An abstract walk p.



(c) The directed graph H^p, which has the property that $\mathrm{CSP}_B^{\text{function}}(H^p)$ is $\mathbf{\Sigma}^1_2$-complete, showing that $\mathrm{CSP}_B(H)$ is $\mathbf{\Sigma}^1_2$-complete as well.

Figure 3.5: An illustration of a directed graph H for which $\mathrm{CSP}_B^{\text{function}}(H)$ is $\mathbf{\Pi}^1_1$ but $\mathrm{CSP}_B^{\text{function}}(H^p)$ is $\mathbf{\Sigma}^1_2$-complete for some path p.

Suppose now that H is both sinkless and sourceless[5]. Let p be an *abstract walk*, that is, a finite sequence of arrows as in Figure 3.5(b). Define H^p to be the *p-power of* H, that is, the directed graph on $V(H)$ where there is a directed edge from x to y if there is a realization in H of the walk p from x to y (see Figure 3.5(c) for an example). Since H is sinkless and sourceless, it can be easily seen that H^p is also sinkless and sourceless. If $\mathrm{CSP}_B^{\text{function}}(H^p)$ is $\mathbf{\Sigma}^1_2$-complete for some walk p, then so is $\mathrm{CSP}_B(H)$. We leave as an open problem whether the converse holds.

Now, we prove Theorem 3.6.1. As in the proof of Theorem 3.5.4, we utilize Theorem 3.1 in [FSV24]. Note first that the set of codes for Borel functions on a standard Borel space X is $\mathbf{\Pi}^1_1$ by Lemma A.3 in [HKM24].

For the remainder of the section, we fix a finite sinkless digraph H.

Proof of Theorem 3.6.1. (i). Let X be standard Borel, and let f be a Borel function on X. If (v, v) is a loop in H, then the function $\psi : X \to V(H)$ defined by $\psi(x) = v$ for all $x \in X$ is clearly a Borel digraph homomorphism.

Conversely, if $\mathrm{CSP}_B^{\text{function}}(H)$ contains all Borel digraphs, then in particular it contains the shift digraph $\overrightarrow{\mathcal{G}}_S$. Let $\psi : [\mathbb{N}]^{\mathbb{N}} \to V(H)$ be a Borel homomorphism from $\overrightarrow{\mathcal{G}}_S$ to H. By the Galvin-Prikry theorem [GP73], since $V(H)$ is finite, there are $v \in V(H)$ and $y \in [\mathbb{N}]^{\mathbb{N}}$ such that $\psi(S^k(y)) = v$ for every $k \in \mathbb{N}$. Consequently, (v, v) is a loop in H, and this completes the proof.

(ii). Assume that H is an ergodic digraph with no loops. Then by the previous paragraph, we have $\overrightarrow{\mathcal{G}}_S \notin \mathrm{CSP}_B^{\text{function}}(H)$. The next result, Proposition 3.6.3, guarantees that Theorem 3.1 in [FSV24] can be used as in the proof of Theorem 3.5.4 to obtain an acyclic Borel function $f = (f_\sigma)_{\sigma \in \mathbb{N}^{\mathbb{N}}} : \mathbb{N}^{\mathbb{N}} \times X \to \mathbb{N}^{\mathbb{N}} \times X$ such that

$$\left\{\sigma \in \mathbb{N}^{\mathbb{N}} : \overrightarrow{\mathcal{G}}_{f_\sigma} \text{ admits a Borel homomorphism to } H\right\} \tag{3.6}$$

is $\mathbf{\Sigma}^1_2$-complete, thereby giving (ii).

[5] The finite and Borel CSP dichotomy coincide in this case [BKN09; Tho22].

Proposition 3.6.3. *Let H be a finite sinkless digraph that is ergodic and has no loops. Then there is $\ell_0 \in \mathbb{N}$ such that, for every acyclic Borel function f on a standard Borel space X that admits a Borel ℓ_0-forward-independent hitting set A, there is a Borel digraph homomorphism from $\overrightarrow{\mathcal{G}}_f$ to H.*

Proof. Let $v_0 \in V(H)$ and $k_0 \in \mathbb{N}$ be such that for every $k \geq k_0$, there is a directed path from v_0 to v_0 of length exactly k. We may, without loss of generality, and after possibly passing to the strong connectivity component of v_0 in H, assume that H is strongly connected. We claim that there is $\ell_0 \in \mathbb{N}$ such that, for every $\ell \geq \ell_0$ and any $w \in V(H)$, there is a directed path of length ℓ from v_0 to w. Indeed, by strong connectivity of H, for any $w \in V(H)$ there is some path from v_0 to w, and this path can be pre-composed with a path from v_0 to v_0 of any length $k \geq k_0$.

Assign to each $z \in A$ a set $A_z = \bigcup_{j=1}^{\ell_0} f^{-j}(z)$. Note that, if $z, z' \in A$ and $z \neq z'$, then $A_z \cap A_{z'} = \emptyset$; indeed, if $y \in A_z \cap A_{z'}$, then there are $1 \leq j \leq j' \leq \ell_0$ such that $f^j(y), f^{j'}(y) \in A$. In particular, $f^{j'-j}(f^j(y)), f^j(y) \in A$ which can only happen if $j = j'$ as A is ℓ_0-forward-independent, implying that $z = z'$. Now define $Z = \bigsqcup_{z \in A} A_z$ and note that Z is Borel and disjoint from A.

For every $x \in X \setminus Z$, define $k(x) \in \mathbb{N}^+$ to be minimal such that $f^{k(x)}(x) \in Z$. Note that $k(x)$ is well-defined as A is hitting for f. Fix a directed path C from v_0 to v_0; for each $v \in C$, define $m(v)$ to be the number of steps in C from v to v_0. Then, for each $x \in X \setminus Z$, define $\psi(x)$ to be the least $v \in C$ such that $m(v) \equiv_{|C|} k(x)$. Now we define ψ on Z; first, for each $z \in A$, fix a directed path D_z from v_0 to $\psi(z)$ of length ℓ_0. For each $y \in Z$, there is a unique $z \in A$ such that $y \in A_z$. Define $\psi(y)$ to be the unique $w \in D_z$ such that, if $f^j(y) = z$, then j is the number of steps in D_z from w to $\psi(z)$. Then ψ is as required. $\square$

(iii). We start with the following observation.

Claim 3.6.4. *Let $f : X \to X$ be a Borel function (not necessarily acyclic). Then there is a Borel homomorphism ψ from $\overrightarrow{\mathcal{G}}_f$ to H if and only if there is a Borel homomorphism*

ψ' from $\overrightarrow{\mathcal{G}}_f$ to a union of strong connectivity components of H.

Proof. For every strong connectivity component $A \subseteq V(H)$ of H, define

$$X_A = \bigcup_{k\in\mathbb{N}} \bigcap_{j=k}^{\infty} f^{-j}\big(\psi^{-1}(A)\big).$$

Observe that $X = \bigsqcup_A X_A$, where A ranges over all strong connectivity components of H, is a decomposition of X into Borel sets that are f-invariant.

Fix a strong connectivity component A. Assign to each $v \in A$ a sequence $(w(k,v))_{k\in\mathbb{N}}$ contained in A such that $w(0,v) = v$ and $(w(k+1,v), w(k,v)) \in E(H)$ for every $k \in \mathbb{N}$. Define

$$\psi_A(x) = w(k,v) \in A$$

for every $x \in X_A$, where $k \in \mathbb{N}$ is minimal such that $f^k(x) \in \psi^{-1}(A)$ and $\psi(f^k(x)) = v$. Then $\psi' = \bigsqcup_A \psi_A$ is the desired homomorphism. □

By Claim 3.6.4, we may, without loss of generality, assume that H is a disjoint union of strongly connected digraphs. In particular, H is both sinkless and sourceless. By the assumption in (iii), none of the strong connectivity components of H is ergodic; then by Corollary 6.3 in [Tho22], $\mathrm{CSP}_B(H)$ is $\mathbf{\Pi}^1_1$. Consequently, $\mathrm{CSP}_B^{\text{function}}(H)$ is $\mathbf{\Pi}^1_1$ as well. □

3.7 Remarks on Semigroup Actions

Generalizing hyperfinite equivalence relations are *hypersmooth* equivalence relations. A BER E (not necessarily countable) is hypersmooth if it is the countable increasing union of smooth BERs. It is known that the graph generated by a Borel function on a standard Borel space is hypersmooth (see Theorem 8.1 in [DJK94]). The following is a well-known open problem.

Question 3.7.1. *Let X be a standard Borel space, $d \in \mathbb{N}$, and $(f_i)_{i\le d}$ be Borel functions on X such that f_i and f_j commute for each $i, j \le d$. Is the graph $\mathcal{G}_{(f_i)_{i\le d}}$ generated by $(f_i)_{i\le d}$ hypersmooth?*

The results of Section 3.5 indicate that Borel asymptotic dimension may be a useful tool for studying the above question in the case when f_i is bounded-to-one for each $i \leq d$. Furthermore, note that any graph of the form $\mathcal{G}_{(f_i)_{i \leq d}}$ as in Question 3.7.1 can be identified with the Schreier graph of a Borel action of the semigroup $(\mathbb{N}^{d+1}, +)$ on X. An interesting research direction would be to systematically study the properties of Schreier graphs of Borel actions of other semigroups from the perspective of descriptive set theory.

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