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The analysis of exclusive decay of the B-meson to the charmonium and a kaon by means of effective field theory

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The Analysis of Exclusive Decay of the B-meson to the Charmonium and a Kaon by Means of Effective Field Theory.

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy
in
Physics

by

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2006

The dissertation of Mikhail A. Savrov is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

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TABLE OF CONTENTS

LIST OF FIGURES

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ABSTRACT OF THE DISSERTATION

The Analysis of Exclusive Decay of the B-meson to the Charmonium and a Kaon by Means of Effective Field Theory

by

Mikhail A. Savrov
Doctor of Philosophy in Physics
University of California, San Diego, 2006
Professor Benjamín Grinstein, Chair

In this dissertation the matrix element (ME) for the weak decay of the B-meson into J/ψ-particle and a kaon K is analysed from the first principles using the method of effective field theory based on exploiting the decay kinematics. Namely, the large masses of the b- and c-quarks, the large energy of the s-quark in the rest frame of the B-meson makes it reasonable to use HQET, NRQCD, and SCET to express the ME of the decay in terms of the MEs of these theories. The covariant version of NRQCD necessary to describe the J/ψ in the B-meson rest frame is proposed in the dissertation. The tree-level expression for the decay Lagrangian in the effective theory is derived to all orders in g_s. The Wilson coefficients for the decay Lagrangian at the leading logarithmic approximation and their initial values at $\mu = m_b$ are calculated. It is shown that the singlet piece of the decay Lagrangian factorizes into the product of two currents at the leading order in power expansion in the effective theory but the octet piece doesn't. It is shown that the contribution due to the non-factorizable octet piece is of the same order of magnitude as the singlet contribution at $\mu = m_b$ and it is argued that the octet contribution although suppressed by one power of α_s compared to the singlet contribution cannot be ignored.

1

Introduction

A picture of the Universe has emerged so far that is based on several remarkably simple concepts. The simplicity of these concepts is illusory and it actually takes years of study to acquire even a basic understanding of them. Nevertheless they provide a logical guide into the vast world of well studied and documented physical phenomena. The world as we know it today can be thought of as a space-time continuum where elementary particles (matter) propagate and interact. There is a widespread belief that this picture is not final and actually matter and space represent different sides of the same physical entity. However we don't know it yet and a search for the so-called "Theory of Everything" (with a string theory as a favorite candidate) as exciting as it is has not produced a solid theory so far that could be tested by experiment.

The subject of this dissertation is the study of one particular decay mode of the B-meson, namely, when it decays into charmonium aka J/ψ particle and a kaon. Like a small piece of a hologram is able to give a crude reproduction of the whole image the study of a particular process alows one to obtain some first-hand understanding of the basic principles of particle physics. Before going into the details of this particular process I would like to give a brief overview of the principles in the foundation of particle physics.

In Chapter 2 the concepts of elementary particles and their interactions are discussed. The basic theoretical objects and principles used to actually build testable theories, namely, quantum fields, gauge invariance, and renormalizability are introduced. For a full and detailed discussion the reader may want to consult the classical book by S. Weinberg [1].

In Chapter 3 the reader is introduced to the Standard Model (SM) of Particles and Interactions that incorporates the modern knowledge of particle physics. For a careful and complete survey of the logics behind the SM (and beyond) see the elegant book by P. Ramond [4]. The book by J. Donoghue et.al. [5] provides a thorough introduction into the phenomenology of the SM.

However consistent and exhaustive the SM is it doesn't explain the origin of masses of elementary particles. The latter have to be introduced into the SM more or less by hand by means of the Higgs mechanism. The so-called Cabibbo-Kobayashi-Maskawa (CKM) matrix whose elements are directly related to decay rates of hadrons provides an insight into the physics of the flavor changing interactions. A large amount of concerted effort has been devoted to the extraction of the numerical values of the elements of the CKM matrix from experiment.

There is a branch of particle physics called B-physics that studies decay modes of heavy B-mesons consisting of one heavy b-quark and one light quark, typically u or d, and s. The large mass of the b-quark and large energies of decay products in some kinematic regions of the phase space allows one to implement reliable theoretical methods (perturbative expansions, typically in several parameters) when extracting predictions from the SM and thereby making possible the experimental measurement of the entries of CKM matrix related to the b-quark (mostly V_{ub} and V_{cb}). For an excellent introduction into the field of B-physics and the detailed description of theoretical tools used see the book by A. Manohar and M. Wise [6].

In this dissertation the effective theory for the particular decay $B \to J/\psi K$ is discussed. In developing a theoretical description of this decay it was necessary to combine the three existing effective theories: soft-collinear effective theory (SCET), heavy quark effective theory (HQET), and non-relativistic quantum chromodynamics (NRQCD) and to develop a covariant formulation of the latter. This project has been done in collaboration with Christoph Bobeth under the close supervision of professor Benjamín Grinstein who has suggested it in the first place.

The main results of the dissertation are:

- the effective tree-level Lagrangian for the decay $B \to J/\psi K$ at the leading order in power expansion in the effective theory and to all orders in g_s, Chapter 4;
- the Wilson coefficients of the effective theory Lagrangian calculated at the order α_s, Chapter 5;
- a covariant version of NRQCD, section 7.2.

In Chapters 6 and 7 the actual derivations supporting the results of Chapter 4 are given. Technical details like LSZ and renormalization factors and typical Feynman diagrams supporting the results of Chapters 4 and 5 are presented in Chapter 8.

2

Particles and Fields

In this section the concepts of elementary particles and particle interactions are introduced. The discussion follows the presentation given in Chapter 2 of [1].

2.1 Particles

To introduce elementary particles one should begin with the Galileo principle, which in its modern form states that fundamental laws of physics which we know nowadays must be the same in all inertial frames related by transformations from the Poincaré group:

- *Time-translations.* The fundamental equations don't change if one makes a shift $t \to t + t_0$.
- *Translations in space.* The equations remain unchanged under the transformations $\vec{x} \to \vec{x} + \vec{x}_0$, where $\vec{x}_0$ is an arbitrary vector.
- *Lorentz Transformations.* Laws of physics should be the same in all inertial frames related by Lorentz boosts and rotations.

The only known limitations on the validity of Galileo's principle come from gravity. Poincaré transformations (PT) are symmetry transformations of

Minkowski space and hold far away from large gravitating masses where space-time is flat. According to Einstein's equivalence principle any space-time is Minkowski space locally. So, if a space-time is curved only slightly, i.e. essentially everywhere not very close to the black holes, the transformations of the Poincaré group are almost exact. PT could possibly become invalid also at very small time intervals and distances. But currently we don't know of any scale where deviations from PT start showing up, except for the Planck scale $M_{Pl} = 1.22 \cdot 10^{19}$ GeV which is far beyond the reach of modern accelerators including the Large Hadron Collider (LHC).

PT form a group. Laws of physics (with a remarkable exception of gravity) remain unchanged under the group transformations. Therefore physical states must form representations of the Poincaré group. Symbolically this statement can be written as follows.

Let O be some combination of observables (operators) and $|\Phi\rangle$ be a vector in the space of physical states. Let $T = T(\Lambda, a)$ be a transformation of the Poincaré group specified by three boost parameters, three rotation angles (called collectively Λ), and by a four-vector $a = (t_0, \vec{x}_0)$. Then the statement that the law holds in some frame of reference can be written as

$$O\ |\Phi\rangle = 0. \tag{2.1}$$

Now the law is invariant under transformations of the Poincaré group. In a new frame obtained from the original one by the transformation, the law holds as well:

$$O\ |\Phi\rangle = 0 \quad \Longrightarrow \quad T\,O\ |\Phi\rangle = 0, \tag{2.2}$$

$$T\,O\,T^{-1}\,T\ |\Phi\rangle = 0, \tag{2.3}$$

$$O'\ |\Phi'\rangle = 0. \tag{2.4}$$

Then $O' = TOT^{-1}$ and $|\Phi'\rangle = T\,|\Phi\rangle$ must be identified with the combination of observables and the state vector as being viewed from the transformed frame of reference.

The state $|\Phi'\rangle$ is obtained from state $|\Phi\rangle$ by a symmetry transformation, therefore it must belong to the same representation of the Poincaré group as the vector $|\Phi\rangle$. According to group theory any group representation is a linear combination of tensor products of irreducible representations of the group. Now we are ready to come up with the first (but not the last) specification of an elementary particle, namely, *a set of physical states corresponding to the states of elementary particles forms one of the simplest irreducible representations of the Poincaré group.*

2.1.1 Poincaré Algebra

According to group theory a group of continuous transformations T can be represented by a set of operators of $U[T]$. These operators are unitary if the group manifold M is compact and the unitary operators of the representation according to Lie's theorem can be written as exponents of hermitian operators J (generators) which form a Lie algebra:

$$U[T(a)] = \exp(iJ \cdot a), \qquad \text{where} \qquad [J_i,\, J_j] = i f_{ijk} J_k. \tag{2.5}$$

Here a stands for the coordinates of group element on M and f_{ijk} are the group constants.

The Poincaré group manifold is not compact, it is the direct product of Minkowski space M^4, the group manifold of boosts R^3, and the group manifold of rotations RP^3. Only the latter is compact but it is double-connected and therefore topologically non-trivial. Nevertheless, it is customary to represent elements of the Poincaré group using the set of generators, corresponding to the Hamiltonian, momentum, boost, and angular momentum operators of the physical system whose states form the group representation:

$$J = \{H,\, \vec{P},\, \vec{K}, \vec{J}\}. \tag{2.6}$$

Operators $H, \vec{P}$, and $\vec{J}$ are hermitian and the boost operators $\vec{K}$ are anti-hermitian. Operators (2.6) form a Lie algebra, and from it one can see that the angular momentum operators $\vec{J}$ and the momentum operators $\vec{P}$ commute with the Hamiltonian H while the boost operators $\vec{K}$ do not:

$$[\vec{J},\, H] = 0, \qquad [\vec{P},\, H] = 0, \quad \text{and} \quad [\vec{K},\, H] \neq 0. \tag{2.7}$$

The Hamiltonian H is the generator of translations in time, therefore angular momentum $\vec{J}$ and momentum $\vec{P}$ correspond to conserved quantities. The components of $\vec{J}$ and $\vec{P}$ along the same axis in position space commute; therefore the Poincaré group has rank 2 and its irreducible representations can be classified by two Casimir operators.

2.1.2 Irreducible Representations

One of the Casimir operators is the mass operator $M^2 = H^2 - \vec{P}^2$, so every irreducible representation is specified by its mass squared M^2, which can be both positive or negative. To specify the representation for non-negative M^2 it is necessary to choose the sign of the energy E, the eigenvalue of H, which is Lorentz invariant. Then the classification of states according to the signs of M^2 and E gives six types of different representations, only three of those are known to correspond to physical states. These states have non-negative M^2, and non-negative energy E. Further analysis of each of these three states is simplified in the frame where the energy-momentum vector of the state has the so-called standard form. A specific subgroup of Lorentz transformations that leaves the standard vector invariant is called little group of the representation. The physical representations with their standard vectors and little groups are listed below.

- $|M^2 > 0,\, E > 0\rangle$, massive state, $(M, 0, 0, 0)$, $SO(3)$;
- $|M^2 = 0,\, E > 0\rangle$, massless state, $(E, 0, 0, E)$, $ISO(2)$;
- $|M^2 = 0,\, E = 0\rangle$, vacuum state, $(0, 0, 0, 0)$, $SO(3, 1)$.

Irreducible representations with positive mass are then classified according to the irreducible representations of the rotation group $SO(3)$. Such representations have $2J+1$ states according to their spin $J = 0, 1/2, 1, 3/2 \ldots$. Massless representations are classified according to their helicity $\sigma = 0, 1/2, 1, 3/2, \ldots$ which like spin takes both integer and half-integer values. Massless representations contain either one state, or two states of opposite helicities $\pm\sigma$ if the states form parity multiplets.

2.1.3 Elementary Particles

Finally we can list the irreducible representations of the Poincaré group known to correspond to elementary particles.

- Spin 1/2 massive states (fermions of the SM: leptons and quarks);
- Spin 1 massive states (vector bosons of the SM: mediators of weak interaction);
- Spin 1 parity multiplets of massless states (photons and gluons, mediators of electromagnetic and strong interactions);

The LHC is expected to answer the question whether the Higgs boson, a hypothetic massive particle with spin 0, is also a physical state actually existing in nature.

A huge theoretical effort has been devoted to the study of the idea that at high energies the Poincaré group should be extended to include the so-called supersymmetry transformations. Such an extended Poincaré group is usually called superPoincaré group. Its irreducible representations are composed of doublets of representations of Poincaré group whose spin differs by 1/2: supersymmetry transformations combine bosons and fermions into supermultiplets. Despite of the elegancy of this construction no traces of supersymmetry have been found/observed experimentally so far.

2.2 Fields and Interactions

Elementary particles interact and we need a theoretical description of interactions. In this section we discuss some basic principles leading to such a description. Then we will be able to complete the task of defining elementary particles started in the previous section. Namely, in addition of being the simplest irreducible representations of the Poincaré group *interactions of elementary particles are described by renormalizable gauge field theories.* In this section we introduce the concepts of field operators, gauge invariance, and renormalizability. The discussion is based on Chapters 3-7 of [1].

2.2.1 Cluster Decomposition Principle

Interactions of elementary particles are described by the so-called S-matrix. Suppose that the initial state of a physical system (it could be the incoming beam of an accelerator) at $t \to -\infty$ is given by vector $|\Phi_\alpha\rangle$ and the final state (detector events) at $t \to +\infty$ by $|\Phi_\beta\rangle$. Here α and β are quantum numbers specifying the states which are essentially wave packets localized in space. Between initial and final moments particles interact, so that the initial vector $|\Phi_\alpha\rangle$ becomes transformed by the evolution operator which in perturbation theory is customarily written as $U(t_1, t_2) = e^{iH_0 t_1} e^{-iH(t_1 - t_2)} e^{-iH_0 t_2}$ to extract a large phase due to the Hamiltonian H_0 of non-interacting particles. Then the probability amplitude to detect particles in the final state $|\Phi_\beta\rangle$ is given by the inner product of the final state vector and the evolved vector of the initial state. This inner product is called S-matrix:

$$S_{\beta\alpha} = \langle\Phi_\beta|\, S\, |\Phi_\alpha\rangle = \langle\Phi_\beta|\, U(+\infty, -\infty)\, |\Phi_\alpha\rangle\,. \tag{2.8}$$

The S-matrix must reproduce the experimentally observed fact that experiments in spatially separated laboratories do not interfere. To write this requirement formally let's assume that $\alpha = \alpha_1, \ldots, \alpha_n$ and $\beta = \beta_1, \ldots, \beta_n$ are the sets of quantum numbers specifying initial and final states, so that the states with different subindexes are spatially separated. Then the Cluster Decomposition Principle (CDP) states that in the limit of large spatial separation between the states with different indexes the S-matrix factorizes:

$$S_{\beta,\alpha} \to S_{\beta_1,\alpha_1} \cdots S_{\beta_n,\alpha_n}. \tag{2.9}$$

CDP can be reformulated in terms of the so-called connected parts of the S-matrix, the latter term coming from perturbation theory where connected parts of the S-matrix are represented by connected graphs of Feynman diagrams. In terms of connected parts a generic S-matrix element becomes

$$S_{\beta,\alpha} = \sum_{part} (\pm) S^C_{\beta_1,\alpha_1} S^C_{\beta_2,\alpha_2} \ldots, \tag{2.10}$$

where the sum is taken over all possible ways of partitioning the particles into clusters $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ and the sign takes into account permutations of fermion states. Then CDP is equivalent to the requirement that connected parts of the S-matrix vanish when spatial separation between initial and final states goes to infinity:

$$S^C_{\beta_i,\alpha_i} \overset{|\vec{x}_{\beta_i} - \vec{x}_{\alpha_i}| \to \infty}{\longrightarrow} 0 \tag{2.11}$$

Using the Riemann-Lebesgue theorem and the translational invariance of the theory which says that the S-matrix depends only on the differences between coordinates $\vec{x}_{\beta_i} - \vec{x}_{\alpha_i}$ it is possible to show that the Fourier transform of the S-matrix should be

$$S^C_{\beta_i,\alpha_i} = \delta^4(p_{\beta_i} - p_{\alpha_i}) C(p_{\beta_i}, p_{\alpha_i}), \tag{2.12}$$

where the function $C(p_{\beta_i}, p_{\alpha_i})$ although in general not regular doesn't contain a δ-function singularity (see section 4.3 in [1]).

2.2.2 Field Operators

The interaction Hamiltonian $H - H_0 = \int d^3\vec{x}\, \mathcal{H}(x)$ generates the Lorentz-invariant S-matrix if the interaction density is a Lorentz-scalar and satisfies the causality condition:

$$U(\Lambda, a)\mathcal{H}(x)U^{-1}(\Lambda, a) = \mathcal{H}(\Lambda x + a),$$
$$[\mathcal{H}(x), \mathcal{H}(x')] = 0 \qquad \text{for} \qquad (x - x')^2 \leq 0. \tag{2.13}$$

As it is shown in Ch. 5 in [1], Eqs. (2.11) and (2.13) together result in a requirement that the interaction density $\mathcal{H}(x)$ must be a polynomial,

$$\mathcal{H}(x) = \sum_{1\cdots N} g_{l_i\cdots l_N}\phi_{l_1}(x)\cdots\phi_{l_N}(x), \tag{2.14}$$

built out of field operators $\phi_l(x)$ (including their derivatives). The index l_i stands for the internal quantum numbers of the field operator including spin. The field operators must meet the following requirements:

- be linear combinations of creation/annihilation operators of one-particle states;
- transform according to some representation D of the Lorentz group (in general reducible):

$$U(\Lambda, a)\phi_l(x)U^{-1}(\Lambda, a) = D_{ll'}(\Lambda^{-1})\phi_{l'}(\Lambda x + a); \tag{2.15}$$

- be causal:

$$[\phi_l(x), \phi_{l'}(x')]_\pm = 0 \qquad \text{for} \qquad (x - x')^2 \leq 0. \tag{2.16}$$

Here $\pm$ stands for anticommutator/commutator corresponding to fermion/boson operators.

The properties of field operators listed above define field operators completely and using their explicit form it is possible:

- to prove the famous *spin-statistics theorem* which says that particles with integer spin are bosons and particles with half-integer spin are fermions;
- to show that field operators provide a system of quantum operators $q_n(x)$ and their conjugates $p_n(x)$.

For example, for a scalar field $\phi(x)$ corresponding to a spin-0 particle the following equal-time commutation relations hold:

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}), \tag{2.17}$$

and equal-time fields and field derivatives separately commute. As it has been mentioned before Eq. (2.15) the field operators are linear combinations of the creation/annihilation operators of one-particle states. The commutation relations between the creation/annihilation operators simply follow from the definition of the Fock space of physical states as a tensor product of one-particle states. Eq. (2.17) allows one to define canonical variables

$$q(\vec{x},t) = \phi(\vec{x},t) \qquad \text{and} \qquad p(\vec{x},t) = \dot{\phi}(\vec{x},t) \tag{2.18}$$

and then use the canonical formalism of classical field theory for quantization.

2.2.3 Gauge Fields and Renormalizability

One can see that the Lagrange formalism follows logically from the set of well established principles: Poincaré invariance of physical laws, cluster decomposition, and causality. The explicit form of field operators makes the free-field Lagrangian (describing non-interacting particles) unambiguous. So, the next task would be to try to find restrictions on possible interaction Lagrangians. It is not obvious that such restrictions should exist but it has been found that the Lagrangians which describe the interactions in nature must satisfy the principles of *gauge invariance* and *renormalizability*.

As an example let's discuss the spin-1/2 Dirac particle. The free-field Lagrangian density for this system is

$$\mathcal{L}(x) = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi, \qquad \text{where} \qquad \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{2.19}$$

The Noether's theorem applied to the Lagrangian shows that this system conserves a quantity that can be identified with the electric charge after the standard quantization procedure is applied to (2.19). This quantity is the difference between the number of particles and antiparticles and when written in terms of fields becomes

$$Q = \int d^3\vec{x}\bar{\psi}\gamma^0\psi. \tag{2.20}$$

The volume of integration here is infinite because from the start we've assumed validity of Poincaré transformations. Locally, the Noether's theorem states that there is a conserved current associated with the charge (2.20):

$$\frac{d}{dt}\int_V d^3\vec{x}(\bar{\psi}\gamma^0\psi) + \int_S(\bar{\psi}\vec{\gamma}\psi)d\vec{A} = 0 \qquad \Longrightarrow \qquad \partial^\mu(\bar{\psi}\gamma_\mu\psi) = 0. \tag{2.21}$$

To introduce the interaction between the fermions we couple the conserved current to the so-called *gauge field*. The modified Lagrangian,

$$\mathcal{L}'(x) = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi \qquad \text{where} \qquad D = \partial + igA, \tag{2.22}$$

is invariant under the *local gauge transformations*:

$$\psi \rightarrow e^{igf}\psi, \qquad \text{and} \qquad A \rightarrow A - \partial f. \tag{2.23}$$

It is straightforward to show that the newly introduced gauge field A should be identified with the two helicity states of a massless spin-1 particle (see Ch. 5.9 in [1]). The gauge field is ambiguous because of the gauge transformation (2.23) but this ambiguity matches perfectly the ambiguity arising when one tries to write down the field operator for massless spin-1 particle in terms of creation and annihilaton operators. However the second rank tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant and unambigous and can be used to write down the Lagrangian for the gauge field, which in general is an arbitrary function of $F_{\mu\nu}$.

The last principle that we have to use to keep our particles elementary (structureless, point-like) is the requirement that the Lagrangian describing particle interactions contains as few parameters as possible. In the classical theory this requirement is not really restrictive because one can write down an arbitrary number of terms as long as gauge invariance is preserved. In quantum field theory quantum corrections will generate new interactions even if the original interaction has the simplest form. The only exception is a *renormalizable* theory where the quantum corrections do not generate new interactions but reproduce those present in the theory from the beginning. So, our last requirement is that the Lagrangian must be renormalizable.

The renormalizabilty requirement restricts the dimension of the interactions in the Lagrangian to be less or equal to four. The minimal Lagrangian for a spin-1/2 elementary particle interacting with the gauge field will then contain only three terms [1], all other possible gauge invariant terms have the dimensions greater than four and are forbidden:

$$\mathcal{L}_{QED}(x) = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi + \frac{1}{4}F^2_{\mu\nu}. \tag{2.24}$$

This is the Lagrangian of quantum electrodynamics (QED) that describes electrons and photons.

It is remarkable that the imposed restrictions are so severe that Eq. (2.24) exhausts almost all possibilites for writing down Lagrangians of interacting elementary spin-1/2 particles in four dimensions. There are only two possibilities known to date. The first one is to introduce a new internal degree of freedom for the fermion (isospin, color), so that the symmetry group of the new theory is a direct product of the Poincaré group and an internal symmetry group (non-abelian gauge group). The famous Coleman-Mandula theorem (see Chapter 24 in [3]) states that under reasonable assumptions[2] this is the only way to introduce internal symmetry into Poincaré invariant theories. The second possibility is to include supersymmetry in the SM which is a subject of the ongoing research.

[1] There is one more term of dimension four but it is a total derivative and doesn't contribute in the abelian case.

[2] Finiteness of the number of particles below any given mass, the existence of scattering at almost all energies, and the analyticity of the S-matrix.

3

Standard Model

It is amazing but all the information we have today about elementary particles and their interactions can be written down in a single formula: the Standard Model Lagrangian. The current state of affairs is such that wherever we're able to extract reliable predictions from the SM we don't see statistically significant deviations from experimental data [7]. Why is it so? Do we know already all the elementary particles and their interactions or is the SM just the first step-stone on the long road to the Planck scale?

There is still one danger to the SM looming in the Higgs sector. In spite of extensive experimental searches the Higgs boson has not been discovered yet. The LHC should be able to give a definite answer to the question of whether the Higgs boson exists or it is a theoretical contrivance that makes possible the formulation of the SM in the framework of gauge invariant renormalizable quantum field theory. If the latter is the case the principal foundations of the SM should be reconsidered, so that the presence of the Higgs field in the SM (at least in its perturbative formulation) is explained.

In this chapter the SM is introduced. The discussion follows the logics of the presentation given in Chapters 2&3 of [4] to which the interested reader is referred for details.

3.1 The SM Lagrangian

In this section we describe the Lagrangian of the SM that includes the degrees of freedom which are more elementary than the particle spectrum observed in experiment. The SM Lagrangian includes only massless particles (both bosons and fermions) and possesses a higher degree of symmetry than what is actually observed. We obtain a realistic Lagrangian after the so-called symmetry breaking of the original Lagrangian. According to the modern interpretation of the symmetry breaking mechanism, the vacuum of the SM is less symmetric than its Lagrangian. Although symmetry breaking is a well understood phenomenon (e.g. in solid state physics) we haven't observed yet the particle dynamics corresponding to the *restored* symmetry (above the electro-weak transition) and should keep in mind that the physics underlying the SM could be very different.

3.1.1 Gauge Symmetries

The gauge symmetry group of the SM is a direct product: $U(1)\times SU(2)\times SU(3)$, reminding of the series of integers $1, 2, 3, \ldots$. The $U(1)$ group corresponds to the quantum number called hypercharge, the $SU(2)$ quantum number is known as weak isospin, and $SU(3)$ corresponds to quark color (red, green, and blue). Each gauge symmetry has a massless (at the level of Lagrangian) spin-1 gauge particle as a mediator. The Lagrangians of the gauge fields are given by the only possible gauge invariant dimension-four operators that are CP even:

- hypercharge $-\frac{1}{4g_1^2}B_{\mu\nu}^2$, where $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$;
- weak isospin $-\frac{1}{2g_2^2}\mathrm{Tr}[F_{\mu\nu}^2]$ where $F_{\mu\nu} = [D_\mu, D_\nu]$ and $D = \partial + iW^a\tau^a$;
- color $-\frac{1}{2g_3^2}\mathrm{Tr}[G_{\mu\nu}^2]$ where $G_{\mu\nu} = [D_\mu, D_\nu]$ and $D = \partial + iA^a t^a$.

Here g_1, g_2, and g_3 are dimensionless coupling constants and τ^a and t^a are generators of fundamental representations of $SU(2)$ and $SU(3)$, respectively.

3.1.2 Fermion Sector

Fermion Content

The original fermions are massless and described by the left-handed Weyl fields which transform according to the $(2, 1)$ representation of the Poincaré group. Particles and antiparticles fall into different internal symmetry multiplets, so we should distinguish between them, the symbol for the antiparticle has a bar over it. There are three generations of fermions, two leptons and two quarks per generation, and they form exactly the same symmetry multiplets under the gauge transformations:

- electron e and electron neutrino ν_e, u-quark and d-quark;
- muon μ and muon neutrino ν_μ, c-quark and s-quark;
- taon τ and tau neutrino ν_τ, t-quark and b-quark.

The symmetry multiplets are listed below according to the notation $(SU(2), SU(3)^c)_Y$, where the first entry is the dimension of $SU(2)$ representation (singlet 1 or doublet 2), and $SU(3)^c$ is the dimension of the color representation (triplet 3^c, antitriplet $\bar{3}^c$, or singlet 1^c). The subindex Y stands for hypercharge. Index i indicates the generation.

- lepton weak doublet $L_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix} \sim (2, 1^c)_{-1}$,
- antilepton weak singlet $\bar{e}_i \sim (1, 1^c)_2$,
- quark weak doublet $Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix} \sim (2, 3^c)_{1/3}$,
- antiquark weak singlet $\bar{u}_i \sim (1, \bar{3}^c)_{-4/3}$,
- antiquark weak singlet $\bar{d}_i \sim (1, \bar{3}^c)_{2/3}$.

Recently discovered neutrino oscillations have shown that the neutrinos have mass [7] and therefore we have to extend the SM to account for this fact. With the particle content given above it is not possible to make neutrinos massive without violating the renormalizability of the SM. We'll address this issue in section 3.1.4.

Fermion Lagrangian

Fermion Lagrangians include covariant derivatives that follow from the fermion representations listed above:

$$\begin{aligned}
\mathcal{D}L_i &= (\partial + iW - \frac{i}{2}B)L_i, \\
\mathcal{D}\bar{e}_i &= (\partial + iB)\bar{e}_i, \\
\mathcal{D}Q_i &= (\partial + iA + iW + \frac{i}{6}B)Q_i, \\
\mathcal{D}\bar{u}_i &= (\partial - iA^* - \frac{2i}{3}B)\bar{u}_i, \\
\mathcal{D}\bar{d}_i &= (\partial - iA^* + \frac{i}{3}B)\bar{d}_i.
\end{aligned} \tag{3.1}$$

Then the fermion Lagrangian density that includes all three generations is:

$$\mathcal{L}_f = \sum_{i=1}^{3} \left(L_i^\dagger \sigma^\mu \mathcal{D}_\mu L_i + \bar{e}_i^\dagger \sigma^\mu \mathcal{D}_\mu \bar{e}_i + Q_i^\dagger \sigma^\mu \mathcal{D}_\mu Q_i + \bar{u}_i^\dagger \sigma^\mu \mathcal{D}_\mu \bar{u}_i + \bar{d}_i^\dagger \sigma^\mu \mathcal{D}_\mu \bar{d}_i \right). \tag{3.2}$$

Here $\sigma^\mu = (1, \vec{\sigma})$ where $\vec{\sigma}$ is Pauli matrix arising when free-field Lagrangian is constructed for the massless spin-1/2 particle according to the general algortihm described in Chapter 2.

The fermions we observe in Nature have masses. However massive spin-1/2 representations of Poincaré group are parity doublets and that contradicts the experimental fact that weak interaction violates parity. Only left-handed particles exchange W bosons. We cannot couple left-handed weak doublets to right-handed weak singlets (left-handed anifermions) through a mass term because that would violate $SU(2)$ gauge invariance. Fermion masses in the SM are provided by coupling the left-handed doublets and right-handed singlets to a scalar field, the Higgs field H, which is a weak doublet and which acquires then a vacuum expectation value (VEV), so that only one component of the left-handed doublet couples to the right-handed singlet. The contrivance works so well that it provides not only fermion masses but also gives self-consistency constraints on the relative strength of fermion couplings to gauge fields that fixes them completely (see Eq. (3.1)) leaving only g_1, g_2, and g_3 introduced in 3.1.1 to be determined from experiment.

Yukawa Sector

This is the only term in the SM Lagrangian that mixes all three generations:[1]

$$\mathcal{L}_Y = i\hat{L}_i\bar{e}_j H^* Y^e_{ij} + i\hat{Q}_i\bar{d}_j H^* Y^d_{ij} + i\hat{Q}_i\bar{u}_j\tau_2 H Y^u_{ij} + \text{c.c.} \tag{3.3}$$

Here $\hat{\psi} = \psi^T\sigma_2$, so that $\hat{\psi}\eta$ is a Lorentz scalar built out of left-handed fields. The symbol H stands for the Higgs field which is a weak doublet that transforms according to $(2, 1^c)_1$ under the gauge group. Matrices Y_{ij} where subindices indicate generations are called Yukawa matrices and are completely arbitrary complex. The Yukawa Lagrangian (3.3) must be invariant under gauge symmetry transformations so that the sum of U(1) charges of different fields in each term in the Lagrangian is zero.

The values of hypercharge Y are unambiguously determined by the requirement that the SM is anomaly free. The SM is a chiral theory and in chiral theories the classical global $U(1)$ symmetry of the Lagrangian associated with chiral rotation, $q_L \to e^{i\alpha}q_L$, is broken by quantum corrections. After renormalization the measure of the path integral is not invariant under the chiral rotation. The anomaly violates consistency of the theory, at least, on the perturbative level. Technically, the anomaly is due to the contribution of a single *anomalous triangle diagram* (which would be zero if the chiral symmetry were not broken, therefore the name).

[1] It is also possible to write down the kinetic term (3.2) which is not generation-diagonal but then field redefinitions will cast it into the diagonal form. So, the only non-trivial generation mixing comes from Eq. (3.3).

The SM has several types of fermions that contribute to the same anomaly amplitude. When contributions due to different fermions to the same anomaly amplitude are summed up, the result is proportional to a combination of hypercharges that can be set to zero. It turns out that there are three different anomalies that give three equations for five values of hypercharge and two more equations follow from invariance of Eq. (3.3) under $U(1)$ hypercharge transformations. The hypercharge of the Higgs field is set to 1 without loss of generality.

3.1.3 CKM Matrix

The Yukawa matrices introduced in Eq. (3.3) do not bring $3\times(2\times3\times3) = 54$ new parameters into the SM. By doing field redefinitions it is possible to reduce the number of parameters to only 13. Nine of them are fermion masses, and four are the entries of the so-called CKM matrix. Three entries are the real rotation angles and one is a complex phase responsible for CP violation. Since the topic of the dissertation is closely related to the determination of one entry of the CKM matrix (in fact a product of two!) let's discuss how the CKM matrix comes into the SM.

Any matrix can be written as a product of two (different) unitary matrices and one real diagonal matrix. So, we can write the Yukawa matrix for leptons as $Y^e = U_e^T M^e V_e$ where U_e and V_e are unitary matrices and M_e is a real diagonal matrix. The matrices U_e and V_e are then absorbed by field redefinitions $L \to U_e^{-1} L$ and $\bar{e} \to V_e^{-1}\bar{e}$ which does not affect the fermion Lagrangian Eq. (3.2). After the field redefinition the first term in (3.3) simplifies to

$$\mathcal{L}_Y^e = \sum_{i=1}^{3} i\hat{L}_i \bar{e}_i H^* y_{ii}^e + \text{c.c.}, \tag{3.4}$$

where y_{ii}^e are the diagonal elements of M^e which after Higgs field H acquires its VEV give rise to lepton masses.

Writing Yukawa matrices for quarks in the form $Y^u = U_u^T M^u V_u$ and $Y^d = U_d^T M^d V_d$ and doing field redefinitions $\bar{u} \to V_u^{-1}\bar{u}$, $\bar{d} \to V_d^{-1}\bar{d}$, and $Q \to U_d^{-1} Q$

reduces the quark part of the Yukawa Lagrangian to

$$\mathcal{L}_Y^q = \sum_{i=1}^{3} i\hat{Q}_i \bar{d}_i H^* y_{ii}^d + i\hat{Q}_i (U_u U_d^\dagger)_{ji} \bar{u}_j \tau_2 H y_{jj}^u + \text{c.c.}. \tag{3.5}$$

The last simplification can be done if one uses unitarity of the matrix $U_u U_d^\dagger$. A unitary matrix can be written as $\mathcal{P}^T \mathcal{U} \mathcal{P}'$, where the matrices $\mathcal{P}$ and $\mathcal{P}'$ are diagonal phase matrices generated by the elements of the Cartan subalgebra, so that the matrix $\mathcal{U}$ is specified by the rest. For a unitary (3×3) matrix there are nine parameters. One of them is the overall phase and there are two Cartan generators. Redefining quark fields allows one to get rid of five parameters in the diagonal matrices $\mathcal{P}$ and $\mathcal{P}'$. The matrix $\mathcal{U}$ then depends on four parameters, three of them are rotation angles of $SO(3)$ and are real and one must be a complex phase. The matrix $\mathcal{U}$ is called Cabibbo-Kobayashi-Maskawa (CKM) matrix.

After the field redefinitions are done the Yukawa Lagrangian is given by the sum of Eqs. (3.4) and (3.5), which contains only 13 parameters: nine fermion masses and four parameters of the CKM matrix. Note that Eq. (3.5) gives Yukawa Lagrangian before the spontaneous symmetry breaking (SSB) described in the next section. After the SSB the last field redefiniton makes Yukawa Lagrangian diagonal and moves the CKM matrix into the kinetic term.

3.1.4 Neutrino Mass

Recently measured differences of squares of neutrino masses (see [7]) indicate that neutrinos are massive. This fact requires including at least one more term in the Lagrangian of the SM. There are two different ways to do it and it could be that Nature actually chooses both ways.

Dirac Neutrino

The first option is to add to the Yukawa Lagrangian (3.3) one more renormalizable term where the weak lepton doublet couples to the weak singlet ν, the new lepton field:

$$\mathcal{L}_\nu = i\hat{L}_i\bar{\nu}_j\tau_2 H Y^\nu_{ij} + \text{c.c.}. \tag{3.6}$$

To be consistent with the symmetries of the SM and in order to reproduce the values of hypercharges for the rest of the fields following from equations for anomalies the new field must transform under $(1, 1^c)_0$ representation, i.e. be completely neutral. Therefore the corresponding kinetic term for the new lepton field is simply the kinetic term of a free field:

$$\mathcal{L}_\nu = \sum_{i=1}^{3} \bar{\nu}_i^\dagger \sigma^\mu \partial_\mu \bar{\nu}_i. \tag{3.7}$$

The matrix Y^ν_{ij} will then introduce a new lepton flavor mixing matrix into the theory after removing the redundant parameters, the same way it is done for quarks. It is called $\mathcal{U}^{MNS}_{ij}$, or Maki-Nakagawa-Sakata (MNS) matrix, the analog of the CKM-matrix. Therefore Lagrangian (3.6) introduces seven more parameters into the SM: three neutrino masses, three lepton mixing angles, and one more CP violating parameter.

This is probably the most conservative way to introduce neutrino mass into the SM. Eqs. (3.6) and (3.7) are the only terms one can possibly add without violating renormalizability of the theory and adding new Higgs fields. In this sense including these terms *completes building the SM.*

Majorana Neutrino

The second option to give masses to neutrinos is to assume that the SM is only a low energy effective theory. If so, the Lagrangian of the SM will include an infinite number of *non-renormalizable* terms in addition to the set of renormalizable ones that we already know. The non-renormalizable terms have dimension greater than four and come into the Lagrangian suppressed by inverse powers of a high energy scale Λ. From this perspective the SM Lagrangian is viewed as an expansion in inverse powers of Λ:

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \ldots \tag{3.8}$$

It turns out there is only one dimension five operator that is consistent with the symmetries of the SM. This term couples the left-handed weak doublet of leptons to the Higgs field and after spontaneous symmetry breaking gives masses to left-handed neutrinos:

$$\mathcal{L}_5 = c_5(\hat{L}_i \tau_2 H)(L_j \tau_2 H) Y_{ij}^M. \tag{3.9}$$

Coupling the left-handed neutrino to itself means that lepton number is not conserved. The Majorana neutrino is neutral in the sense that it is its own antiparticle. Violation of lepton number can be observed in the so-called neutrinoless double beta decay, when a light nucleus simultaneously emits two electrons and nothing else. The amplitude of this process is proportional to the neutrino mass and therefore small (see [7]).

3.1.5 Higgs Sector

The last part of the SM Lagrangian is the Lagrangian of the Higgs field. The latter transforms according to $(2, 1^c)_1$ and therefore its covariant derivative is

$$\mathcal{D}H = (\partial + iW + \frac{i}{2}B)H. \tag{3.10}$$

The most general Lagrangian for the field is given by the sum of three terms which are weak singlets of dimension four or less:

$$\mathcal{L}_H = (\mathcal{D}_\mu H)^\dagger(\mathcal{D}^\mu H) + \mu^2 H^\dagger H - \lambda(H^\dagger H)^2. \tag{3.11}$$

Here μ and λ are positive parameters to be determined from experiment. The sign of the third term must be negative to ensure stability of the vacuum at least at the classical level. The sign of the second term is specifically chosen to give the Higgs field a vacuum expectation value at tree level.

3.2 Spontaneous Symmetry Breaking

The paradigm used to extract predictions from the SM is a perturbative expansion around the classical vacuum of the model. We do not know yet whether perturbative expansion is an intrinsic feature of the SM itself or the SM as it is can be understood as a field theory with a cutoff and the use of perturbation theory simply reflects the lack of more sophisticated computational methods. At least for the QCD sector of the SM lattice calculations show convincingly enough that QCD could and should be treated non-perturbatively in order to extract the spectrum of light hadrons from QCD Lagrangian. In this dissertation we are concerned with phenomena that happen at energies greater than ~ 1 GeV, so for our purposes even for QCD perturbation theory should suffice.

3.2.1 Classical Vacuum of the SM

The lowest energy configurations of the SM fields as required by gauge invariance are:

- fermion fields should be set to zero due to Lorentz invariance of the vacuum: $\langle 0|\,\psi\,|0\rangle = 0$;
- gauge fields give positive definite contributions to the Hamiltonian and therefore should be set to pure gauge configurations, $(A, B, W) \to i\Omega^\dagger \partial \Omega$;
- Higgs field should be set to a constant value to minimize the value of the kinetic term and Higgs potential, $V(H) = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2$.

The last condition is non-trivial because the minimum of the Higgs potential corresponds to a field configuration that singles out a direction in the weak isospin space:

$$H_{vac} = \frac{e^{i\theta_0}}{\sqrt{2}} U \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad \text{where} \qquad v = \sqrt{\frac{\mu^2}{\lambda}} \quad \text{and} \quad U \in SU(2). \tag{3.12}$$

The vacuum configurations form a manifold on which one can move by doing $SU(2) \times U(1)$ transformations and in this sense the symmetry is preserved. However, a tunneling amplitude between different configurations is zero in the limit when the volume of the system goes to infinity. So, once a direction in the isospin space is chosen the system sits there. This phenomenon is known as spontaneous symmetry breaking (SSB).

The classical vacuum configuration breaks three quantum numbers associated with isospin and hypercharge transformations and leaves invariant only one linear combination $Q = I_3 + Y/2$ that corresponds to the electric charge of QED. The broken symmetries give rise to massless Goldstone bosons which become the longitudinal components of the massless gauge fields associated with the broken symmetries. After "eating up" Goldstone modes the gauge fields become massive and are identified with massive vector bosons, the mediators of the weak interaction. The linear combination of weak isospin and hypercharge gauge fields corresponding to the unbroken symmetry (electric charge conservation) becomes the massless photon.

3.2.2 The SM Lagrangian after SSB

Writing down the Lagrangian of the SM after SSB, imposing appropriate gauge fixing conditions in the broken phase, and making sure that the Lagrangian remains renormalizable after SSB is a difficult and non-trivial task. When completed it gives the spectrum of elementary particles and their interactions observed in Nature. Here we write only the particle spectrum and the currents coupled to gauge fields following [4] with a modification because of massive neutrinos.

Fermions

There are three generations of fermions in the SM listed in 3.1.2. Each generation contains a massive neutrino of zero electric charge (either left-handed Majorana neutrino, or Dirac neutrino, or both), one Dirac lepton of charge -1, and two Dirac quarks of charge $2/3$ and $-1/3$.

Gauge Particles

The SM after SSB contains the following gauge particles:

- Massless photon mediating electromagnetic interactions, the unit for electric charge (on tree-level) is given by:

$$e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}}; \tag{3.13}$$

- Eight massless gluons mediating strong interactions;

- Massive neutral spin-1 boson, called Z-boson, with tree-level mass

$$M_Z = \sqrt{g_1^2 + g_2^2}\,\frac{v}{2}; \tag{3.14}$$

- Massive spin-1 boson of unit charge, called W^+ (W^- for antiparticle), with mass

$$M_W = g_2 \frac{v}{2}. \tag{3.15}$$

Currents

Here we write down fermion currents coupled to gauge bosons of the SM and the interaction Lagrangian. This Lagrangian is relevant for the process studied in this dissertation. Index $i = 1, 2, 3$ stands for generation. Currents are written in Dirac form because after SSB left-handed fermion and antifermion fields are combined into massive CP pairs, so in the equations below $\bar{\psi} = \psi^\dagger \gamma^0$ and $P_L = \frac{1-\gamma_5}{2}$.

- Electromagnetic current:

$$J^\mu_{em} = \frac{2}{3}\bar{u}_i \gamma^\mu u_i - \frac{1}{3}\bar{d}_i \gamma^\mu d_i - \bar{e}_i \gamma^\mu e_i; \tag{3.16}$$

- Strong current:

$$J^a_\mu = \bar{u}_i \gamma^\mu t^a u_i + \bar{d}_i \gamma^\mu t^a d_i; \tag{3.17}$$

- Charged weak currents (flavor changing currents):

$$\begin{aligned} J^-_\mu &= \mathcal{U}_{ij} \bar{u}_i \gamma_\mu P_L d_j + \mathcal{U}^{MNS}_{ij} \bar{\nu}_i \gamma_\mu P_L e_j \\ J^+_\mu &= \mathcal{U}^\dagger_{ij} \bar{d}_i \gamma_\mu P_L u_j + \mathcal{U}^{\dagger MNS}_{ij} \bar{e}_i \gamma_\mu P_L \nu_j; \end{aligned} \tag{3.18}$$

- Neutral weak current:

$$J_\mu^3 = \frac{1}{2}\bar{u}_i\gamma_\mu P_L u_i - \frac{1}{2}\bar{d}_i\gamma_\mu P_L d_i + \frac{1}{2}\bar{\nu}_i\gamma_\mu\nu_i - \frac{1}{2}\bar{e}_i\gamma_\mu e_i. \tag{3.19}$$

The interaction between the weak currents and gauge fields is given by the Lagrangian:

$$i\frac{g_2}{\sqrt{2}}(W_\mu^+ J^{-\mu} + W^{-\mu}J_\mu^+) + i\frac{e}{\sin\theta_w\cos\theta_w}Z^\mu(J_\mu^3 - \sin^2\theta_w J_{em\,\mu}), \tag{3.20}$$

where $\sin\theta_w = \frac{g_1}{\sqrt{g_1^2+g_2^2}}$ represents the so-called weak mixing angle (at tree-level).

3.2.3 Higgs Boson

After three degrees of freedom associated with the Higgs field have been absorbed by the $SU(2)$ gauge fields there is still one degree of freedom left. It corresponds to a scalar neutral particle with mass $m_H = \mu\sqrt{2} = v\sqrt{2\lambda}$ called the Higgs boson. Although the Higgs boson has not been observed yet the boundaries on its mass are already fairly well known from fits to precision measurements to the SM observables. (The SM is a renormalizable theory and beyond tree-level the Higgs boson will contribute through loops even if it is not observed in the external states.) The current best fit value is 95^{+45}_{-32} GeV at 95% confidence level (see [7]).

There is a firm hope that the LHC with its ATLAS and CMS detectors optimized for Higgs searches will either find the Higgs boson in the predicted mass range or rule it out. In the latter case that would bring back a question of the foundations of the SM: what does the non-observable degree of freedom correspond to? Integrating out the Higgs field most likely would not help because that would result in a non-renormalizable theory whose terms are organized as a series in inverse powers of an energy scale associated with Higgs sector ($\sim v \approx 246$ GeV). The latter would mean that elementary particles have a structure at this scale which is not supported by data [7].

4

Effective Field Theory for $B \to J/\psi K$. Analysis

Precision calculations in the SM involving hadrons are complicated due to the confinement of quarks and each particular process requires computational tools developed specifically for that purpose. In this section we outline the basic steps of the calculation of the matrix element for the process $B \to J/\psi K$ by means of an effective field theory leaving technical details to the next chapters.

4.1 What Can be Done Despite Complications Brought by Confinement of Quarks

One of the decay modes of the B-meson is a decay into charmonium J/ψ and a kaon K. On the quark level the decay can be described as the b-quark decaying into the $c\bar{c}$-pair and s-quark due to the weak interaction. The spectator quark in the B-meson becomes the second quark in the kaon, the force binding the spectator quark is due to quantum chromodynamics (QCD). The decay amplitude is proportional to the Fermi constant G_F and small, so it can be calculated by means of perturbation theory. At the first order in perturbation theory in the electroweak coupling it is given by the matrix element (ME)

$$\langle J/\psi,\, K|\, H_W \,|B\rangle\,, \tag{4.1}$$

where H_W is a flavor changing operator, it is essentially a product of two quark currents introduced in Eq. (3.18). In the next section we discuss how to calculate H_W from the SM Lagrangian. The initial and final states are hadronic states. Currently our understanding of hadronic states is far from being complete and we don't know how to compute hadronic matrix elements using approximate analytic methods[1]. The best we can do with a matrix element like (4.1) is to write it down as a combination of other matrix elements which we cannot compute as well but hope that reducing a variety of matrix elements to just a few basic ones we can extract the values of the latter from a restricted set of observables and thereby gain some control over other observables. Mathematically the idea is to use Wigner-Eckart theorem by exploiting approximate dynamical symmetries of QCD.

The approach outlined above is called effective field theory and works only under certain kinematic conditions. In general the wavelengths of the particles involved in the decay must be much less than the wavelength associated with the confinement scale $\Lambda_c \sim 350\ \mathrm{MeV}^{-1}$, the effective mass of the constituent quarks. In practice "much less" often means only several times less. When this condition is met we can use asymptotic freedom of QCD, the fact that the strong coupling constant α_s goes to zero when the transferred momentum goes to infinity, so that the strong interaction between short wavelength quarks is well described by perturbation theory.

In our case we use the fact that the b-quark undergoing the weak decay is heavy, its mass is $m_b \sim 4.2$ GeV, one order of magnitude larger than the confinement scale. The decay products are a $c\bar{c}$-pair made of quark and antiquark whose masses are $m_c \sim 1.25$ GeV each and a light s-quark with $m_s \sim 0.1$ GeV. The relative velocity of quarks of the $c\bar{c}$-pair must be small, so that the quarks could

[1] The only known general method is numerical evaluation of path integrals on the lattice but it takes a lot of computational effort. We don't discuss lattice calculations in the dissertation.

form a J/ψ bound-state. Therefore the process is essentially a two-body decay and we can estimate the energy/momentum of the outgoing s-quark to be $E_s \sim 1.4$ GeV in the b-quark rest frame. So, the wavelengths of all quarks involved are reasonably large except for the so-called spectator quark (one of the light quarks: u, d, or s) that participates in the decay indirectly. Before the decay the spectator quark belongs to the B-meson and after the decay it must become the constituent quark of the kaon.

Temporarily neglecting the spectator quark we'll develop an effective theory for the ME (4.1) in the limit where $m_b \to \infty$, $m_c \to \infty$, and $E_s \to \infty$. The corresponding effective theories are known as heavy quark effective theory (HQET) [17], [18], and [19] non-relativistic quantum chromodynamics (NRQCD) [16], and soft-collinear effective theory (SCET) [11]. To describe this particular process we have to combine them all.

4.2 From M_W Scale to m_b Scale

Before dealing with hadronic states we discuss how to derive the effective Hamiltonian of electroweak interactions H_W from the SM. At tree-level (in the lowest order of the electroweak coupling) the decay amplitude is given by the Feynman diagram in Fig. 4.1. The amplitude is proportional to the propagator of

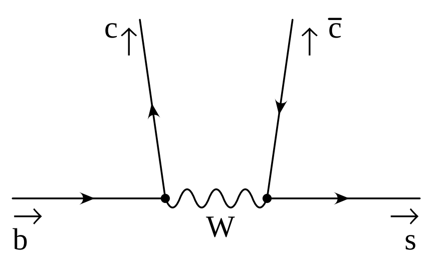


Figure 4.1: b-quark decaying into c-, $\bar{c}$-, and s -quarks.

W-boson, $[(p_b - p_c)^2 - M_W^2]^{-1}$. The external momenta of b and c quarks are much smaller than $M_W \approx 80$ GeV, so the first approximation is to expand the propagator in powers of quark momenta and keep only the leading term in the expansion, often referred to as an operator product expansion (OPE). The decay amplitude given by the leading term is the same as that produced by a local four-fermion interaction aka the Fermi interaction:

$$\mathcal{L}_W = -H_W = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{cs}^* [\bar{s}\gamma_\mu P_L c][\bar{c}\gamma^\mu P_L b], \tag{4.2}$$

where $G_f = \frac{g_2^2}{4\sqrt{2}M_W^2}$ is the Fermi constant.

This Lagrangian would be a good zero order approximation if one could make sure that the coupling constants g_1, g_2, and g_3 are infinitesimally small. In reality they are small but finite and when one calculates loop corrections in perturbation theory they become multiplied by a large coefficient that gives a substantial contribution to the decay amplitude. For example, gluon exchange between two quarks, as in Fig. 4.2, gives a contribution proportional to $\frac{\alpha_s}{4\pi} \ln \frac{M_W^2}{m^2}$, where m is a heavy quark mass. This number is of order one. In fact there is an

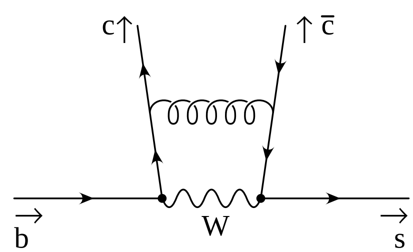


Figure 4.2: One of six diagrams with one gluon exchange in the full theory.

infinite subset of Feynman diagrams at higher orders in perturbation theory that give contributions proportional to $(\frac{\alpha_s}{4\pi})^n \ln^n \frac{M_W^2}{m^2}$ and we must sum them all to make sure that all these leading logarithmic contributions have been taken into account. This is a general property of perturbation theory to produce large logarithms for problems with several largely separated scales - that is why one has to use effective theories in the first place - to sum such logarithms.

Besides there is also an important qualitative effect associated with gluon exchange. Gluon emission changes the quark color and that would change the order in which color indices are contracted in the original interaction (4.2). Explicitly, with only the color indices shown:

$$[\bar{s}_i c_i][\bar{c}_j b_j] \to c_1[\bar{s}_i c_i][\bar{c}_j b_j] + c_2[\bar{s}_i c_j][\bar{c}_j b_i]. \tag{4.3}$$

The coefficients c_1 and c_2 are called Wilson coefficients.

Evaluating the leading order (LO) contributions, often also referred to as leading logarithmic approximation, (i.e. $\sim \alpha_s^n \ln^n \frac{M_W^2}{m^2}$) is not difficult thanks to the powerful technique of the renormalization group (RG). Nowadays this is a textbook calculation [8]. Firstly, one has to evaluate the decay amplitude at one loop in QCD in the effective theory of electroweak interactions which is given by a sum of six diagrams of the type shown in Fig. 4.3 and which is a correction to the operator (4.2). Without the W-boson propagator the loop diagrams are

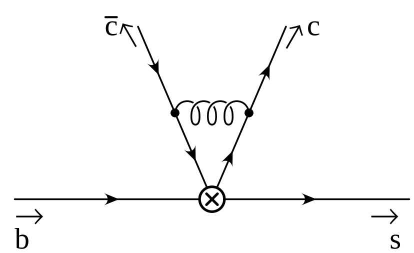


Figure 4.3: One of six diagrams with one gluon exchange in the effective theory.

UV-divergent and after renormalization the amplitude becomes dependent on the renormalization scale μ. Secondly, setting $\mu = M_W$ and equating this effective theory decay amplitude to the amplitude of the full theory calculated at tree-level allows one to obtain the initial Wilson coefficients in (4.3) at $\mu = M_W$[2]. Thirdly, from the renormalization constants of the effective theory operators one extracts the so-called anomalous dimension matrix (ADM). Finally, using the ADM and the initial values for Wilson coefficients one solves the RG equation of the effective theory and finds the dependence of Wilson coefficients on the renormalization scale μ. Setting $\mu = m_b$ reproduces the result of summing up all LO contributions

[2]This procedure is called matching and it is discussed later in the Chapter 8.

($\sim \alpha_s^n \ln^n \frac{M_W^2}{m^2}$) coming from the diagrams of the type shown in Fig. 4.2 without actually writing down the corresponding multi-loop diagrams.

After setting $\mu = m_b$ the result is given by two operators:

$$L_W = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{cs}^* \Big(C_0(m_b)[\bar{s}\gamma^\mu P_L b][\bar{c}\gamma_\mu P_L c] + C_8(m_b)[\bar{s}\gamma^\mu P_L T^a b][\bar{c}\gamma_\mu P_L T^a c] \Big) \tag{4.4}$$

with the LO Wilson coefficients

$$\begin{aligned} C_0^{LO}(\mu) &= \frac{2}{3}\left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right)^{\frac{2}{\beta_0}} - \frac{1}{3}\left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right)^{-\frac{4}{\beta_0}}, \\ C_8^{LO}(\mu) &= \left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right)^{\frac{2}{\beta_0}} + \left(\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right)^{-\frac{4}{\beta_0}}, \end{aligned} \tag{4.5}$$

where

$$\beta_0 = 11 - \frac{2}{3} n_f \qquad \text{and} \qquad \alpha_s(\mu) = \frac{4\pi}{\beta_0 \ln(\mu^2/\Lambda_{QCD}^2)}. \tag{4.6}$$

(For $n_f = 5$, $\Lambda_{QCD} = 225$ MeV.) Numerically,

$$C_0^{LO}(m_b) = 0.076 \qquad \text{and} \qquad C_8^{LO}(m_b) = 2.273. \tag{4.7}$$

The operator basis in Eq. (4.4) is more convenient for our purposes than (4.3) because the $c\bar{c}$-pair eventually forms bound state, J/ψ. The two operator bases are related by Fierz transformations and are linearly dependent, although beyond one loop working out the transformation matrix between the two bases is technically complicated[3].

Unfortunately the LO approximation for the Wilson coefficients is not sufficient. The bound state of the $c\bar{c}$-pair is a color singlet and therefore the singlet operator in Eq. (4.4) has the largest overlap with the ground state of J/ψ. The coefficient of the singlet operator C_0 is given by a difference of two approximately equal numbers and is small. It turns out that the dominant contribution comes from the NLO correction which requires a two-loop calculation of the ADM. The Wilson coefficients for the operators in (4.3) have been calculated at three loops (see [9]) and we can use those results. In that paper Wilson coefficients are given in

[3] In dimensional regularization the complications are due to the so-called evanescent operators

the basis (4.3) which is different from the basis of Eq. (4.4), so we have to work out coefficients of linear transformation at two loops (see section 8.1 for the results). Then the Wilson coefficients at the NLO are given by

$$C_0^{NLO}(m_b) = 0.206\,(0.218) \qquad \text{and} \qquad C_8^{NLO}(m_b) = 2.237\,(2.210). \tag{4.8}$$

The values in parentheses have been obtained using $\alpha_s(m_Z) = 0.119$ as an initial value for the RGE of the QCD-coupling instead of using Eq. (4.6) where α_s is defined in terms of $\Lambda_{QCD} = 0.225$ MeV. The coefficient $C_0^{NLO}(m_b)$ is three times larger than $C_0^{LO}(m_b)$ while the coefficient C_8 stays approximately the same and still it is an order of magnitude larger than C_0. Therefore we should be careful when estimating its contribution to the ME (4.1) because even a small admixture of the state $|(c\bar{c})^8, \text{gluon}\rangle$, where $(c\bar{c})^8$ stands for the octet state of quark-antiquark pair, can give a significant contribution.

4.3 From m_b Scale to Confinement Scale

The Lagrangian (4.4) is different from the corresponding Lagrangian of the SM. Degrees of freedom associated with the mediator of the weak interaction, W-boson, whose wavelength is two orders of magnitude smaller than the wavelengths of particles participating in the decay have been "integrated out". The resulting interaction is *local* and the functional dependence associated with the large mass scale M_W now comes into the theory through the Wilson coefficient and G_F. In other words in the effective Lagrangian (4.4) the dependence of the decay amplitude on M_W has been factorized.

This is essentially what effective theory is about: separation of scales. Going down the energy scale we are not able to resolve fine structure of interactions at higher energies and instead work with *effective interactions* in which the details due to large energy scales are hidden in the Wilson coefficients. We may want to ask if we can go further down the energy scale and integrate out more degrees of freedom, so that the remaining ones would describe only the essential physics. This approach is especially relevant for QCD where the so-called soft degrees of freedom associated with the confinement scale, $\Lambda_c \sim 350$ MeV^{-1}, are not well understood, so it is important to separate them from the degrees of freedom that can be treated perturbatively.

In early days of effective field theory people used to write down full QCD Feynman diagrams and analyze the structure of infrared divergences of the integrals, looking for the regions of momenta contributing the most. Soon it was found that the infrared divergences form a structure that could be reproduced by an effective Lagrangian following from full QCD Lagrangian in a specific kinematic limit.

In general such Lagrangians (in this dissertation we use three of them: HQET, NRQCD, and SCET) consist of infinite series of terms organized by powers of an expansion parameter. In HQET it is the ratio of the confinement scale and the mass of the heavy quark, Λ_c/m, in NRQCD it is the relative velocity of the quark/antiquark pair $v \ll c$ (therefore the name), and in SCET the expansion parameter is the ratio of the confinement scale and the energy of the relativistic collinear quark, λ=(either Λ_c/E_s or $\sqrt{\Lambda_c/E_s}$, depending on the process). A search for such a Lagrangian is not straightforward unlike the more direct approach of analyzing full QCD Feynman diagrams but it is easier because such a Lagrangian can be guessed. Whatever method is used to obtain an effective theory Lagrangian the decay amplitude calculated with its help must have the same infrared divergences as the full QCD amplitude order by order in an α_s expansion.

4.3.1 Degrees of Freedom in the Effective Theory

At the beginning of this chapter we have mentioned that the limit m_b, m_c, and $E_s \to \infty$ is a reasonable approximation for this particular decay mode. Integrating out the corresponding degrees of freedom is similar to integrating out the W-boson. The resulting effective theory depends on m_b, m_c, and E_s only indirectly through Wilson coefficients. The effective theory does not contain dynamical degrees associated with the degrees of freedom of the scales higher than several Λ_c. Nevertheless its dynamics is complicated because we have to separate scales in the gauge sector as well and introduce different sorts of gluons.

Taking the limit m_b, m_c and $E_s \to \infty$ is equivalent to sending wavelengths to zero and is similar to the approximation made in geometrical optics. When the wavelength changes slowly, $d\lambda/dx \leq 1$, it is more convenient to work with rays instead of wavefronts. In our case the rays are vectors in Minkowski space corresponding to four-velocities of the in- and outgoing quarks, β_b, $\beta_{c\bar{c}}$, and n. The first two are unit vectors, the four-velocities of the b-quark and $c\bar{c}$-pair. The vector n is light-like: $n^2 = 0$, and specifies the four-velocity of the collinear s-quark whose mass has been neglected. The fields of the effective theory follow when fields of the full QCD are projected on these directions; the list of the fields is given below.

- **Quark sector.** All quark fields are four-component Dirac spinors obtained from the corresponding full QCD fields by applying projection operators. Only two components of each field are linearly independent.

  1. The heavy quark field $h_{\beta_b}(x)$ carries momentum that scales like the confinement scale.

  2. The heavy c-quark field $\xi_{\beta_{c\bar{c}},\tilde{p}_\perp}(x)$ and the heavy $\bar{c}$ anti-quark field $\eta^C_{\beta_{c\bar{c}},-\tilde{p}_\perp}(x)$ carry momentum that scales like $(m_c v^2, m_c v, m_c v, m_c v)$ where v is the relative velocity of the $c\bar{c}$-pair and the first entry stands for the component of the quark momentum parallel to $\beta_{c\bar{c}}$. Numerically $m_c v^2$ for J/ψ is of the same order as the confinement scale. The label $\tilde{p}_\perp$ corresponds to the large momentum $\sim (0, m_c v, m_c v, m_c v)$. The residual momentum of the order of the confinement scale resides in the coordinate x.

  3. The field $\xi_{n,p}(x)$ of the light s-quark carries momentum $p = \frac{1}{2}(\bar{n}\cdot p_s)n + p_{s\perp} + \frac{1}{2}(n\cdot p_s)\bar{n}$ (the label) which components scale like $\sim E_s(1, \lambda, \lambda, \lambda^2)$ plus the residual momentum of the order of the confinement scale resid-

ing in the x-dependence. The light-like vector $\bar{n}$ corresponds to a wave propagating in the opposite direction: $(\bar{n} \cdot n) = 2$.

- **Gluon sector.** There are three types of gluons.

  1. The ultrasoft gluon field $A_{us}(x)$ has a typical momentum of the confinement scale.
  2. The collinear gluon field $A_{n_s,q}(x)$ corresponds to gluons which are part of the kaon wavefunction. Collinear gluons carry a fraction of the kaon momentum $\sim p_K$ and the components of the field $A_{n_s,q}(x)$ scale like momentum of the collinear s-quark p_s: $A_{n_s,q} = \frac{1}{2}(\bar{n} \cdot A_{n_s,q})n + A_{n_s,q\perp} + \frac{1}{2}(n \cdot A_{n_s,q})\bar{n} \sim E_s(1, \lambda, \lambda, \lambda^2)$.
  3. The potential gluon field $A_p(x)$ corresponds to degrees of freedom binding the $c\bar{c}$-pair into charmonium. This field scales like the momentum of the c-quarks in the bound state: the time-like component parallel to $\beta_{c\bar{c}}$, $A_{\parallel}(x)$ scales like $m_c v^2$ and transfers momentum $(m_c v^2, m_c v, m_c v, m_c v)$. The space-like component perpendicular to $\beta_{c\bar{c}}$, $A_{\perp}(x)$, scales like $m_c v$ and transfers momentum $\sim (m_c v, m_c v, m_c v, m_c v)$.

The effective theory of the decay is considered in this dissertation only at the leading order in the expansion parameters Λ_c/m_b, v, and λ. At this order the only interaction between HQET, NRQCD, and SCET sectors is due to the exchange of ultrasoft gluons at the confinement scale Λ_c and the relation between the expansion parameters is arbitrary.

4.3.2 Tree-Level Effective Theory Weak Decay Operator

We can sum up all tree-level diagrams for this process to all orders in g_s with the help of the procedure called tree-level matching. The details of the calculation are given in Chapter 6, here we present only the result:

$$O_{ij}^{(tree)} = [\bar{\xi}_{n,p} W \Gamma_j \mathbf{C}_i \; h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j \, S \, \mathbf{C}_i \, S^\dagger \, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{4.9}$$

where $\mathbf{C} \otimes \mathbf{C}$ stands for color singlet $\mathbb{1} \otimes \mathbb{1}$ or octet $T^a \otimes T^a$ operators and Γ_j is an effective theory Dirac structure. Symbols W and S denote the so-called Wilson lines, nonlocal operators whose expressions in momentum space are:

$$W = \sum_{\text{perm}} \exp\left(-g_s \frac{1}{\bar{n} \cdot \mathcal{P}_c} \bar{n} \cdot A_{n_s,q}\right),$$
$$S = \sum_{\text{perm}} \exp\left(-g_s \frac{1}{n \cdot \mathcal{P}_\perp} n \cdot A_{p\perp}\right). \tag{4.10}$$

These expressions should be understood as Taylor expansions. The operators $\mathcal{P}_c$ and $\mathcal{P}_\perp$ are the so-called label operators that in each term of the Taylor series should be replaced with the sum of momentum labels on the left. The operator $\mathcal{P}_c$ picks up labels of collinear quarks and gluons and operator $\mathcal{P}_\perp$ picks up the momentum component perpendicular to $\beta_{c\bar{c}}$ for potential gluons and quarks of the $c\bar{c}$-pair.

An important feature of Eq. (4.9) is that collinear and potential degrees of freedom have been factorized: collinear quarks and gluons contribute to the $(\bar{s}b)$ current and potential gluons only to the $(c\bar{c})$-current. This phenomenon is referred to as factorization at tree level. Factorization holds even when loop corrections including collinear and potential gluons are considered. Potential and collinear gluons do not couple to each other at the leading order in v and λ (see the end of section 6.3.3). The interaction between these two types of gluons is given by subleading operators. The next question to address is whether the factorization property holds when loop corrections due to ultrasoft gluons are taken into account.

4.3.3 Factorization at Operator Level

The hallmark of the effective field theories is that their leading order Lagrangians can be made free of the ultrasoft gluon field $A_{us}(x)$ by means of a simple field redefinition of all the fields that carry labels (see [12]). This is the same factorization phenomenon we've already encountered when integrating out the W-boson: short and long wavelengths don't interfere. This observation allows one to simplify greatly the analysis of loop corrections to the tree-level Lagrangian (4.9). The details are given in section 6.3.5 and here we present only the result. After the field redefinition has been made HQET, covariant NRQCD, and SCET Lagrangians become free of ultrasoft fields. The singlet and octet operators in (4.9) acquire explicit ultrasoft field dependence:

$$
\begin{aligned}
O_{\mathbf{0}}^{(tree)} &= \sum_j [\bar{\xi}_{n,p}^{(0)} W^{(0)} \Gamma_j \, Y_n^\dagger Y_{\beta_b} \; h_{\beta_b}^{(0)}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}^{(0)} \Gamma_j \eta_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}^{C\,(0)}], \\
O_{\mathbf{8}}^{(tree)} &= \sum_j [\bar{\xi}_{n,p}^{(0)} W^{(0)} \Gamma_j Y_n^\dagger T^a Y_{\beta_b} \; h_{\beta_b}^{(0)}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}^{(0)} \Gamma_j \, S \, Y_{\beta_{c\bar{c}}}^\dagger T^a Y_{\beta_{c\bar{c}}} \, S^\dagger \, \eta_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}^{C\,(0)}].
\end{aligned}
\tag{4.11}
$$

Here $Y_{n,\beta_b,\beta_{c\bar{c}}}$ are the Wilson exponentials along $n, \beta_b, \beta_{c\bar{c}}$ built of ultrasoft field $A_{us}(x)$ similarly to Eq. (4.10) and the superscript $^{(0)}$ stands for the ultrasoft-free quark fields.

Eq. (4.11) makes it explicit that corrections due to ultrasoft gluons will not mix the currents in the singlet operator, so it factorizes but the octet doesn't. Loop corrections to the singlet operator will modify the currents independently and we can write down the general expression for the singlet which is valid to all orders in α_s:

$$
\mathcal{L}_{\mathbf{0}} = [\bar{\xi}_{n,p} W \, C_{sb,j}^{\mathbf{0}}(\bar{\mathcal{P}}/\mu) \Gamma_j \, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,p_\perp} C_{c\bar{c},j}^{\mathbf{0}}(m_c/\mu) \Gamma_j \eta_{\beta_{c\bar{c}}\,-p_\perp}^{C}]. \tag{4.12}
$$

Here the Wilson coefficients follow from matching with full QCD which has been done at one loop both for the singlet and the octet operators and is discussed later in this chapter. Factorization of the singlet operator has been verified explicitly at one-loop level (see section 8.2.2 for the detailed discussion).

4.3.4 Factorization of the ME

Ultrasoft degrees of freedom are the only ones common to all three effective theories and the fact the ultrasoft gluons decouple at the leading order in power expansion means that the net Hilbert space is simply a direct product of

Hilbert spaces of the theories. This observation allows one to simplify the MEs of the operators $O_{0,8}$ from Eq. (4.11) taken between the states of Eq. (4.1) by replacing the latter by the effective theory states. The subindices β_b, $\beta_{c\bar{c}}$, and n indicate the states of the corresponding effective theories. First we discuss the exclusive decay mode when the final state contains a kaon.

Exclusive Decay $B \to J/\psi K$. Singlet operator

The ME of the singlet operator between the effective theory states is a product of the MEs of the currents:

$$\langle K_n| \, [\bar{\xi}_{n,p} W \, C^{\mathbf{0}}_{sb,j}(\bar{\mathcal{P}}/\mu)\Gamma_j \, h_{\beta_b}] \, |B_{\beta_b}\rangle \left\langle J/\psi_{\beta_{c\bar{c}}} \right| [\bar{\xi}_{\beta_{c\bar{c}}\, p_\perp} C^{\mathbf{0}}_{c\bar{c},j}(m_c/\mu)\Gamma_j \eta^{C}_{\beta_{c\bar{c}}\, -p_\perp}] \, |0_{\beta_{c\bar{c}}}\rangle \, . \tag{4.13}$$

At the leading order in the effective theory the ME for the decay factorizes. The first matrix element between eigenstates of the collinear kaon and the B-meson vanishes at the leading order in λ and all orders in α_s in the effective theory because the B-meson contains a spectator quark that represents the ultrasoft degrees of freedom. The ultrasoft quark in the B-meson must become a collinear quark in the kaon, and so we need an operator insertion that mixes ultrasoft and collinear degrees of freedom. Such an operator is subleading ($\sim \lambda^2$, see [10]) and comes suppressed by at least one power of g_s. The second matrix element is computable because in NQRCD the J/ψ is a hydrogen-like bound state of the $(c\bar{c})$-pair. Alternatively the second matrix element can be expressed in terms of the J/ψ decay constant.

Exclusive Decay $B \to J/\psi K$. Octet operator

The ME of the octet operator between the effective theory states can be reduced to:

$$\begin{aligned} &\langle K_n| \, [\bar{\xi}^{(0)}_{n,p} W^{(0)} \Gamma_j Y^{\dagger}_{n} T^a Y_{\beta_b} \; h^{(0)}_{\beta_b}] \times \\ &\times \left\langle J/\psi_{\beta_{c\bar{c}}} \right| [\bar{\xi}^{(0)}_{\beta_{c\bar{c}}\, \tilde{p}_\perp} \Gamma_j \, S^{(0)} \, Y^{\dagger}_{\beta_{c\bar{c}}} T^a Y_{\beta_{c\bar{c}}} \, S^{(0)\,\dagger} \, \eta^{C\,(0)}_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}] \, |0_{\beta_{c\bar{c}}}\rangle \, |B_{\beta_b}\rangle \, . \end{aligned} \tag{4.14}$$

For the octet, the ultrasoft degrees of freedom are not factorized and so the ME between NRQCD vacuum and charmonium states remains trapped inside the ME between the collinear kaon and the heavy B-meson states. However we can still use the fact that both $|J/\psi_{\beta_{c\bar{c}}}\rangle$ and $|0_{\beta_{c\bar{c}}}\rangle$ are color singlets and therefore the corresponding ME is proportional to the color trace

$$\mathrm{Tr}\left[S\, Y^{\dagger}_{\beta_{c\bar{c}}} T^a Y_{\beta_{c\bar{c}}}\, S^{\dagger}\right]. \tag{4.15}$$

The operators Y and S are unitary and the trace would have vanished if there were no operator insertions due to the interactions that break ordering of the exponents. At the tree-level there are none and the trace vanishes. Therefore we can argue that the ME (4.14) is suppressed compared to the ME (4.13) by a factor of α_s. This suppression has a clear physical meaning: J/ψ is a colorless state invisible for ultrasoft gluons responsible for confinement. The state becomes visible only when the $c\bar{c}$-pair forms a virtual color octet plus potential gluon and according to Eq. (4.14) can emit an ultrasoft gluon.

The last observation could render the octet contribution to the decay rate irrelevant but it is difficult to tell quantitatively without explicit numerical evaluation of the MEs. The suppression comes from ultrasoft gluons and $\alpha_s(\mu)$ becomes quite large close to Λ_c. Furthermore, the Wilson coefficient of the octet operator is ten times larger than for the singlet, see Eq. (4.8). So, in spite of the fact that the octet operator is sandwiched between two colorless states $\langle J/\psi_{\beta_{c\bar{c}}}|$ and $|0_{\beta_{c\bar{c}}}\rangle$ its contribution to the ME could be as significant or even greater than that of the singlet operator when loop corrections are taken into account.

In the next section we give the expressions for the Wilson coefficients calculated at the NLO. This calculation shows explicitly that the octet and the singlet mix at the α_s-order and so we can get some idea about the relative strength of the octet contribution to the decay amplitude by comparing numerical values of Wilson coefficients of the singlet and octet operators in the ET at $\mu = m_b$ (see Eqs. (5.2) and (5.4)). The zero order coefficients $C^{ET,(0)}(0.30) \sim 0.5$. The coefficients $C^{ET\,(1)}(1, 0.30)$ which mix the singlet and the octet operators are of the order 10, as it follows from Eq. (5.5). Therefore:

$$C^{ET,(0)}(0.30) \sim 0.5 \qquad \text{versus} \qquad \frac{\alpha_s(m_b)}{4\pi} C^{ET\,(1)}(1, 0.30) \sim 0.2. \tag{4.16}$$

So, at $\mu = m_b$ the LO and the NLO terms are of the same order of magnitude and neglecting the octet contribution is not justified unless it could be argued that at higher orders some cancellation happens.

To conclude: factorization of the singlet operator made explicit in Eq. (4.13) at the leading order of the effective theory doesn't really help in making quantitative predictions about the decay amplitude of the process $B \to J/\psi K$. Rather Eqs. (4.13) and (4.14) and the Wilson coefficients listed in Chapter 5 should be considered as a necessary step towards a quantitative analysis of the decay.

Semi-inclusive Decay $B \to J/\psi X_s$

In the semi-inclusive decay the final state is J/ψ and any hadron state X_s that includes one and only one s-quark. This removes the restriction that the spectator quark in the B-meson must necessarily become collinear but it doesn't help much. There is one collinear quark operator in the $(\bar{s}b)$ current and it is still necessary to boost an ultrasoft quark (e.g. from vacuum) into a collinear meson: all final states must be color singlets.[4] Therefore the insertion of an operator that mixes collinear and ultrasoft degrees of freedom is still required and everything that has been said about the exclusive process remains also valid for the semi-inclusive process.

[4]There is not enough energy to produce a relativistic baryon, so the collinear state(s) must be a meson.

5

Wilson Coefficients

In this chapter we present Wilson coefficients for the effective theory Lagrangian at LO in α_s inclusive summing leading logarithmic terms. Also the NLO initial Wilson coeffcients in α_s of the effective theory at $\mu = m_b$ are given. Evolving Wilson coefficients with the help of the RG at NLO requires the anomalous dimension (AD) at two-loops which is not yet known. The singlet operator factorizes in the product of two currents and therefore its anomalous dimension is just the sum of anomalous dimensions for the currents. The octet operator doesn't factorize as we have seen in the previous Chapter and requires the two-loop calculations in the effective theory.

5.1 Leading Order

At the leading order the RG-improved decay Lagrangian is given by

$$L_W^{LO}(\mu) = -\frac{4G_F}{\sqrt{2}} V_{cb} V_{cs}^* \left[\frac{\alpha_s(m_b)}{\alpha_s(\mu)}\right]^{\frac{4\pi C_F}{\beta_0^2 \alpha_s(m_b)}} \sum_{i=0,8} \sum_{j=1}^{4} \times$$
$$\times C_i(m_b)\, C_j^{ET,(0)}(r)\, [\bar{\xi}_{n,p} W \Gamma_j \mathbf{C}_i\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j S \mathbf{C}_i S^\dagger\, \eta^C_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}]. \tag{5.1}$$

In this equation $C_i(m_b)$ are Wilson coefficients from (4.8) and:

$$\begin{aligned} & \mathbf{C}_i \in \{\mathbb{1}, T^a\}, \qquad r = m_c/m_b, \qquad C_F = \frac{4}{3} \\ & \Gamma_j \otimes \Gamma_j \in \{P_R \otimes \not{n}^*_\perp,\, P_R \otimes \not{\beta}_{b\perp},\, P_R \otimes \gamma_5,\, \not{\epsilon}_- \otimes \not{\epsilon}_{+\perp}\}, \\ \text{and} \qquad & C_j^{ET,(0)}(r) \in \left\{\frac{1}{2},\, \frac{1}{2},\, -\frac{1}{4r},\, -\frac{1}{2}\right\}. \end{aligned} \tag{5.2}$$

For the derivation of the effective theory Dirac structures see Chapter 8. In Eq. (5.1) we keep only the zero order term in the expansion of tree-level operator (4.9) in α_s.

The LO AD used to run down the coefficients for singlet and octet turned out to be the same and equal to the AD for $(\bar{s}b)$-current [11]:

$$\gamma(\mu) = \frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{m_b}{\mu}. \tag{5.3}$$

Its derivation is given in 8.2.3.

5.2 Next to the Leading Order

Here we present the result of calculation at order α_s of the Wilson coefficients.

$$\begin{aligned} L_W^{(1)} = & -\frac{4G_F}{\sqrt{2}} V_{cb} V_{cs}^* \sum_{i,k=0,8} \sum_{j=1}^{4} C_i(m_b) \left\{ C_j^{ET,(0)}(r) \delta_{i\,k} + \frac{\alpha_s(\mu)}{4\pi} C_{i\,k;j}^{ET,(1)}(m_b/\mu,\, r) \right\} \times \\ & \times [\bar{\xi}_{n,p} W \Gamma_j \mathbf{C}_k \; h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\, \tilde{p}_\perp} \Gamma_j \, S \, \mathbf{C}_k \, S^\dagger \, \eta^C_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}]. \end{aligned} \tag{5.4}$$

The matching procedure gives initial values of the Wilson coefficients (the expression in parentheses) when $C_{i\,k;j}^{ET,(1)}(m_b/\mu,\, r)$ are evaluated at $\mu = m_b$. Summing all next-to-leading logarithmic contributions requires anomalous dimension at two loops which is not yet known. Below we give numerical values of Wilson coefficients at $\mu = m_b$ and $r = m_c/m_b = 0.30$.

$$C^{ET,(1)}_{i\,k;1}(1,\,0.30) = \begin{pmatrix} -10.73 & -11.14 - 3.41i \\ -2.48 - 0.76i & 8.86 - 11.55i \end{pmatrix},$$
$$C^{ET,(1)}_{i\,k;2}(1,\,0.30) = \begin{pmatrix} -10.73 & -11.52 - 3.41i \\ -2.56 - 0.76i & 10.81 - 11.55i \end{pmatrix},$$
$$C^{ET,(1)}_{i\,k;3}(1,\,0.30) = \begin{pmatrix} 15.20 & 19.64 + 6.58i \\ 4.36 + 1.46i & -14.47 + 20.29i \end{pmatrix},$$
$$C^{ET,(1)}_{i\,k;4}(1,\,0.30) = \begin{pmatrix} 10.20 & 10.81 + 1.23i \\ 2.40 + 0.27i & -8.42 + 9.00i \end{pmatrix}. \tag{5.5}$$

The coefficients $C^{ET,(1)}_{ij;k}$ are not diagonal, meaning there is a mixing between singlet and octet operators at NLO . The octet operator is not factorizable and so the redirection of the color flow starts at order ${\alpha_s}^1$.[1]

5.2.1 Auxiliary Functions

Here we give the explicit analytic expressions for the coefficients $C^{ET,(1)}_{i\,k;j}(m_b/\mu,\,r)$. To simplify the output we introduce functions F_1, F_2, and F_3:

$$F_1 = -\frac{2}{3}\mathrm{Log}\left[\frac{\mu^2}{m_b^2\,(1-4r^2)^2}\right]^2 - \frac{10}{3}\mathrm{Log}\left[\frac{\mu^2}{m_b^2}\right] + \frac{8}{3}\mathrm{Li}_2\left[1 - \frac{1}{1-4r^2}\right] +$$
$$+\frac{4}{3}\mathrm{Log}\left[1-4r^2\right]^2, \tag{5.6}$$
$$F_2 = \frac{(1-4r^2)}{(1-2r^2)}\left(\mathrm{Log}\left[1-4r^2\right] - i\pi\right), \tag{5.7}$$

[1] We've already discussed this phenomenon in Chapter 4. See Eq. (4.3).

$$
\begin{aligned}
F_3 = \quad & -\frac{2}{3}\mathrm{Log}\left[\frac{\mu^2}{m_b^2}\right]^2 + \left(\frac{8}{3}\mathrm{Log}\left[1-4r^2\right] - \frac{6\mathrm{Log}[2r]}{1-4r^2} - \frac{13}{3}\right)\mathrm{Log}\left[\frac{\mu^2}{m_b^2}\right] \\
& -3i\pi\mathrm{Log}\left[\frac{\left(2-4r^2\right)\mu^2}{\left(1-4r^2\right)^2 m_b^2}\right] + \frac{3\left(1+4r^2\right)}{-1+4r^2}\left(\mathrm{Li}_2\left[\frac{1}{2-4r^2}\right] - \mathrm{Li}_2\left[\frac{2r^2}{1-2r^2}\right]\right) \\
& -3\mathrm{Li}_2\left[\frac{2r^2}{-1+4r^2}\right] - \frac{1}{3}\mathrm{Li}_2\left[\frac{4r^2}{-1+4r^2}\right] + 2\pi^2 - \frac{13}{3}\mathrm{Log}\left[1-4r^2\right]^2 \\
& +3\mathrm{Log}\left[1-4r^2\right]\mathrm{Log}\left[1-2r^2\right] - \frac{12\mathrm{Log}[2r]\mathrm{Log}\left[1-4r^2\right]}{-1+4r^2} \\
& +\frac{12\mathrm{Log}[r]\mathrm{Log}\left[1-2r^2\right]}{-1+4r^2} + \frac{3\left(3+4r^2\right)\mathrm{Log}[2]\mathrm{Log}\left[1-2r^2\right]}{-1+4r^2} + \frac{6\mathrm{Log}[r]^2}{1-4r^2} \\
& +\frac{3\left(3+4r^2\right)}{2\left(-1+4r^2\right)}\mathrm{Log}[2]^2. \qquad (5.8)
\end{aligned}
$$

5.2.2 Coefficients $C^{ET,(1)}_{i\,j;k}$

The coefficients $C^{ET,(1)}_{i\,j;k}$ are listed as elements of (2×2) matrices for every ET Dirac structure $k = 1, 2, 3, 4$.

$$
\begin{aligned}
C^{ET,(1)}_{00;1} &= \frac{1}{2}\left(F_1 - \frac{56}{3} + \frac{8}{3}\mathrm{Log}\left[1-4r^2\right]\right), \\
C^{ET,(1)}_{08;1} &= \frac{1}{2}\left(-6\mathrm{Log}\left[\frac{\mu^2}{m_b^2}\right] + \frac{3-8r^2}{1-2r^2}F_2 - \frac{43-92r^2}{2\left(1-2r^2\right)} + \frac{r^2\left(1-4r^2\right)}{\left(1-2r^2\right)^2}\mathrm{Log}\left[2r^2\right]\right), \\
C^{ET,(1)}_{80;1} &= \frac{1}{2}\left(-\frac{4}{3}\mathrm{Log}\left[\frac{\mu^2}{m_b^2}\right] + \frac{2}{9}\frac{3-8r^2}{1-2r^2}F_2 - \frac{43-92r^2}{9\left(1-2r^2\right)}\right. \\
&\left. +\frac{2r^2\left(1-4r^2\right)\mathrm{Log}\left[2r^2\right]}{9\left(1-2r^2\right)^2}\right), \\
C^{ET,(1)}_{88;1} &= \frac{1}{2}\left(F_3 - \frac{i\pi\left(21-104r^2+152r^4\right)}{6\left(1-2r^2\right)^2} + \frac{31-76r^2}{-4+8r^2}\right. \\
&-\frac{18-79r^2+100r^4}{6\left(1-2r^2\right)^2}\mathrm{Log}[2] + \frac{7r^2\left(1-4r^2\right)}{3\left(1-2r^2\right)^2}\mathrm{Log}[r] \\
&\left. +\frac{19-96r^2+144r^4}{6\left(-1+2r^2\right)^2}\mathrm{Log}\left[1-4r^2\right]\right). \qquad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
C_{00;2}^{ET,(1)} &= C_{00;1}^{ET,(1)}, \\
C_{08;2}^{ET,(1)} &= C_{08;1}^{ET,(1)} - \frac{r^2\text{Log}\left[2r^2\right]}{\left(1-2r^2\right)^2}, \\
C_{80;2}^{ET,(1)} &= C_{80;1}^{ET,(1)} - \frac{1}{9\left(1-2r^2\right)} - \frac{2r^2\text{Log}\left[2r^2\right]}{9\left(1-2r^2\right)^2}, \\
C_{88;2}^{ET,(1)} &= C_{88;1}^{ET,(1)} - \frac{7}{12\left(1-2r^2\right)} - \frac{\left(18-65r^2+44r^4\right)}{6\left(1-2r^2\right)^2\left(1-4r^2\right)}\text{Log}[2] \\
&-\frac{\left(9-29r^2+8r^4\right)}{3\left(1-2r^2\right)^2\left(1-4r^2\right)}\text{Log}[r].
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
C_{00;3}^{ET,(1)} &= -\frac{1}{4r}\left(F_1 - \frac{40}{3} + \frac{2\text{Log}\left[1-4r^2\right]}{3r^2}\right), \\
C_{08;3}^{ET,(1)} &= -\frac{1}{4r}\left(-6\text{Log}\left[\frac{\mu^2}{m_b^2}\right] + \frac{\left(3-4r^2\right)}{\left(1-2r^2\right)}F_2 - \frac{2\left(11-20r^2\right)}{1-2r^2}\right), \\
C_{80;3}^{ET,(1)} &= -\frac{1}{4r}\left(-\frac{4}{3}\text{Log}\left[\frac{\mu^2}{m_b^2}\right] + \frac{2}{9}\left(2+\frac{1}{1-2r^2}\right)F_2 - \frac{4\left(11-20r^2\right)}{9\left(1-2r^2\right)}\right), \\
C_{88;3}^{ET,(1)} &= -\frac{1}{4r}\left(F_3 - \frac{i\pi\left(21-76r^2+40r^4\right)}{6\left(1-2r^2\right)^2} + \frac{27-40r^2}{-3+6r^2} - 3\text{Log}[4r]\right. \\
&\left.+\frac{-1+46r^2-156r^4+80r^6}{12r^2\left(1-2r^2\right)^2}\text{Log}\left[1-4r^2\right]\right).
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
C_{00;4}^{ET,(1)} &= -\frac{1}{2}\left(F1 - \frac{56}{3} + \left(4-\frac{1}{3r^2}\right)\text{Log}\left[1-4r^2\right]\right), \\
C_{08;4}^{ET,(1)} &= -\frac{1}{2}\left(-6\text{Log}\left[\frac{\mu^2}{m_b^2}\right] + F_2 - 20 - \frac{4\text{Log}[2]}{1-2r^2} - \frac{8r^2\text{Log}\left[r^2\right]}{1-2r^2}\right), \\
C_{80;4}^{ET,(1)} &= -\frac{1}{2}\left(-\frac{4}{3}\text{Log}\left[\frac{\mu^2}{m_b^2}\right] + \frac{2}{9}F_2 - \frac{40}{9} - \frac{8\text{Log}[2]}{9\left(1-2r^2\right)} - \frac{16r^2\text{Log}\left[r^2\right]}{9\left(1-2r^2\right)}\right), \\
C_{88;4}^{ET,(1)} &= -\frac{1}{2}\left(F_3 - 6 - \frac{i\pi\left(7+8r^2\right)}{6\left(1-2r^2\right)} - \frac{14\text{Log}[2]}{3-6r^2} + \frac{\left(-9+74r^2\right)}{-3+6r^2}\text{Log}[r]\right. \\
&\left.-\frac{1+14r^2+56r^4}{24r^2\left(-1+2r^2\right)}\text{Log}\left[1-4r^2\right]\right).
\end{aligned}
\tag{5.12}
$$

6

Derivation of ET Operators

In this chapter we give a detailed derivation of Eqs. (4.9) and (4.11).

6.1 Effective Theory Lagrangian at Tree-Level

The tree-Level ET Lagrangian follows from (4.4) if one replaces the FT spinors with the spinors of the ET. In doing so, one obtains the following operator:

$$\sum_{\tilde{p}_\perp, p} [\bar{\xi}_{n,p} \Gamma_j \mathbf{C}\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\, \tilde{p}_\perp} \Gamma_j \mathbf{C}\, \eta^C_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}] \qquad \text{where} \qquad \mathbf{C} = \mathbb{1}, T^a, \tag{6.1}$$

and Γ_j are the Dirac structures (5.2) of the ET operators. The corresponding Feynman diagram in the ET is given in Fig. 6.1. Double line corresponds to the incoming heavy b-quark (labeled by β_b, the 4-velocity), the dashed line corresponds to the outgoing light s-quark (labeled by $\vec{p}$, the momentum), the solid lines correspond to the outoging quarks of $c\bar{c}$-pair (labelad by the label momenta $\pm\tilde{p}_\perp$). For the $\bar{c}$ anti-quark the fermion flow is reversed.

The operator (6.1) is a tree-level operator of zero order in coupling constant g_s. In addition to ultrasoft gluons which couple to all quark fields, the ET has collinear and potential gluons which give rise to local operators of the order g_s. In general terms we have to take into account a possibility that the heavy quark could emit a collinear gluon and become virtual (go off-shell). Likewise, the collinear quark emitting potential gluon goes off-shell. Also, if potential and

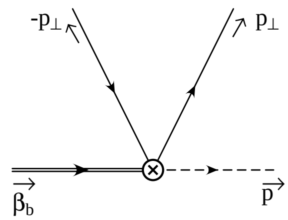


Figure 6.1: The ET amplitude at the tree-level. The heavy b-quark, the heavy quark/antiquark $c\bar{c}$-pair, and the light s-quark are shown. For the anti-quark the fermion flow is reversed like in the FT.

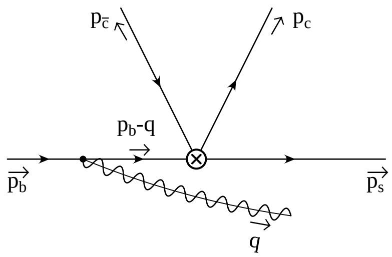
collinear gluons fuse, the result will be an off-shell mode, see Fig. (6.12). Off-shell degrees of freedom are integrated out and do not propagate in the ET: they are not dynamical degrees of freedom. In the ET it neccessary to introduce the *local* operators to account for the effects due to these modes. The local operators generate loops and possibly IR divergencies which should reproduce the corresponding IR divergences of the FT. So, the next step is to find all possible local operators that should be added to operator (6.1) to make sure that all IR divergences of the full theory (FT) amplitude are reproduced at the leading order in v and λ.

6.2 Corrections of the First Order in g_s

Now we are going to discuss tree-level corrections to the ET operator (6.1) at the first order in g_s. There are two types of corrections: due to collinear and potential gluons.

6.2.1 Corrections due to Collinear Gluons

We start by attaching gluons which carry collinear momenta to the heavy quark lines in the FT. Attaching the collinear gluon (wiggly line) to the b-quark line gives the diagram shown in Fig. 6.2. The corresponding matrix element in the ET follows by replacing spinors and momenta in the FT amplitude with their



$$= \quad [\bar{u}(p_s)\gamma^\nu P_L \mathbf{C} \frac{i(\not{p}_b - \not{q} + m_b)}{(p_b - q)^2 - m_b^2 + i0}(-ig_s \not{\epsilon}^a T^a) u(p_b)] \\ \times [\bar{u}(p_c)\gamma_\nu P_L \mathbf{C} v(p_{\bar{c}})]. \tag{6.2}$$

Figure 6.2: The b-quark interacting with a collinear gluon.

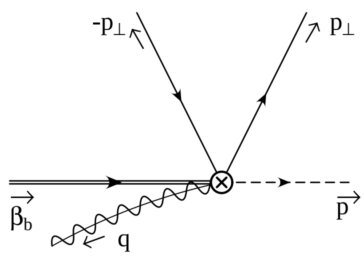
values at the leading order in v and λ:

$$q \to \frac{1}{2}(\bar{n}\cdot q)n, \qquad p_b \to m_b \beta_b, \qquad \epsilon \to \frac{1}{2}(\bar{n}\cdot \epsilon_{n,q})n, \qquad q^2 \sim \lambda^2. \tag{6.3}$$

The result is

$$-g_s \frac{\bar{n}\cdot \epsilon^a_{n,q}}{(\bar{n}\cdot q) - i0} [\bar{\xi}_{n,p} \Gamma_j \mathbf{C} T^a\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j \mathbf{C}\, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]. \tag{6.4}$$

The corresponding Feynman diagram and the Lagrangian in the ET are:



$$\Longrightarrow \qquad \mathcal{L}_1 = -\frac{g_s}{(\bar{n}\cdot q) - i0}[\bar{\xi}_{n,p}\Gamma_j \mathbf{C}(\bar{n}\cdot A_{n,q})\, h_{\beta_b}] \\ [\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j \mathbf{C}\, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]. \tag{6.5}$$

Figure 6.3: Heavy b-quark interacting with collinear gluon in ET.

Attaching collinear gluon lines to heavy c and $\bar{c}$ lines is almost identical. The corresponding ET diagrams are shown in Fig. 6.4. The corresponding ET

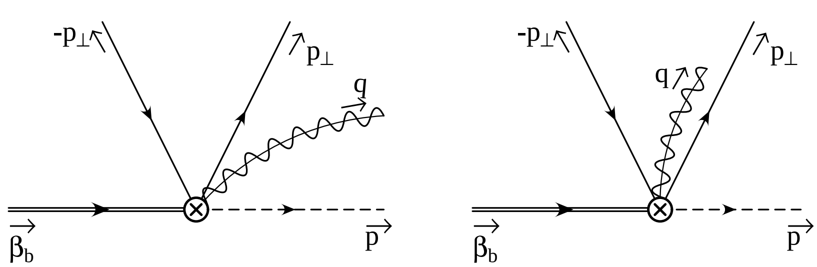


Figure 6.4: Heavy c and $\bar{c}$ quarks interacting with collinear gluons.

Lagrangians are:

$$\mathcal{L}_2 = \frac{g_s}{(\bar{n}\cdot q)+i0}[\bar{\xi}_{n,p}\Gamma_j\mathbf{C}h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}(\bar{n}\cdot A_{n,q})\,\Gamma_j\mathbf{C}\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{6.6}$$

$$\mathcal{L}_3 = -\frac{g_s}{(\bar{n}\cdot q)+i0}[\bar{\xi}_{n,p}\Gamma_j\mathbf{C}h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j\mathbf{C}(\bar{n}\cdot A_{n,q})\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]. \tag{6.7}$$

It is a straightforward exercise to verify that for the singlet and octet operators the sum of all three operators $\mathcal{L}_{1,2,3}$ can be written as:

$$\begin{aligned}
\mathcal{L}_{\mathbf{0}} &= -g_s[\bar{\xi}_{n,p}\frac{(\bar{n}\cdot A_{n,q})}{(\bar{n}\cdot q)-i0}\Gamma_j\,h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}],\\
\mathcal{L}_{\mathbf{8}} &= -g_s[\bar{\xi}_{n,p}\frac{(\bar{n}\cdot A_{n,q})}{(\bar{n}\cdot q)-i0}\Gamma_j\,T^a\,h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j\,T^a\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]\\
&+ 2\pi\delta(\bar{n}\cdot q)\,g_s\,f^{abc}[\bar{\xi}_{n,p}\Gamma_j\,T^b\,h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j(\bar{n}\cdot A^c_{n,q})\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}].
\end{aligned}\tag{6.8}$$

In the last term the delta-function is non-zero only for $\bar{n}\cdot q = 0$, which corresponds to the zero bin in the sum over collinear momenta (see [24]). When calculating loop corrections to $\mathcal{L}_{1,2,3}$ due to collinear gluons the zero bin is excluded and therefore the $\pm i0$ terms (i.e. delta-functions) could be safely dropped. In other words $\bar{n}\cdot q$ could be zero only for the ultrasoft momentum which is taken care of separately. According to this reasoning we can write operators $\mathcal{L}_{\mathbf{0}}$ and $\mathcal{L}_{\mathbf{8}}$ as

$$\mathcal{L}^{(col)} = -g_s[\bar{\xi}_{n,p}\frac{(\bar{n}\cdot A_{n,q})}{(\bar{n}\cdot q)}\Gamma_j\mathbf{C}\,h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j\mathbf{C}\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{6.9}$$

where $\mathbf{C} = \mathbb{1}$, T^a as usual.

6.2.2 Corrections due to Potential Gluons

Attaching potential gluons to b and s-quark lines proceeds through the same steps as for the collinear gluons (see Fig. 6.5). One starts with full QCD diagram and replaces spinors and momenta with their leading values in the ET:

$$q \to q_\perp = q - \beta_{c\bar{c}}(\beta_{c\bar{c}}\cdot q), \qquad q_\perp^2 \sim (mv)^2. \tag{6.10}$$

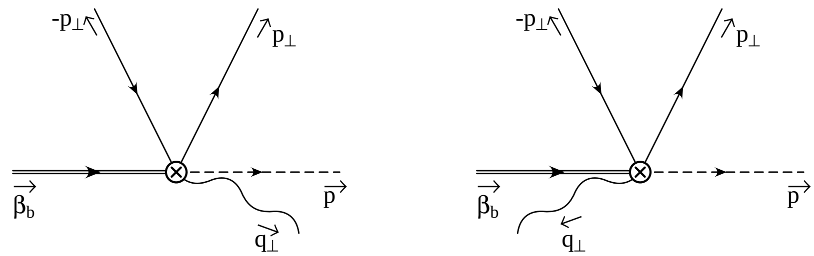


Figure 6.5: Light s and heavy b quarks interacting with potential gluons (the wiggle line).

The corresponding ET Lagrangians are:

$$\mathcal{L}_4 = g_s[\bar{\xi}_{n,p} \frac{n \cdot A_{\tilde{q}}}{n \cdot q_\perp + i0} \Gamma_j \mathbf{C} h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\, \tilde{p}_\perp} \Gamma_j \mathbf{C}\, \eta^C_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}], \tag{6.11}$$

$$\mathcal{L}_5 = -g_s[\bar{\xi}_{n,p} \Gamma_j \mathbf{C} \frac{\beta_b \cdot A_{\tilde{q}}}{\beta_b \cdot q_\perp - i0} h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\, \tilde{p}_\perp} \Gamma_j \mathbf{C} \eta^C_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}]. \tag{6.12}$$

At the leading order in λ and v momentum conservation gives the following relation between four-vectors $\beta_{c\bar{c}}$, β_b, and n:

$$\beta_b = \frac{2m_c}{m_b}\beta_{c\bar{c}} + \frac{E_s}{m_b} n. \tag{6.13}$$

Together with power counting for potential fields (see Chapter 7),

$$\beta_{c\bar{c}} \cdot A_{\tilde{q}} \sim mv^2, \qquad A_{\tilde{q}\perp} \sim mv \tag{6.14}$$

conservation of momentum allows us to write:

$$\frac{\beta_b \cdot A_{\tilde{q}}}{\beta_b \cdot q_\perp} = \frac{n \cdot A_{\tilde{q}}}{n \cdot q_\perp} + o(v). \tag{6.15}$$

Therefore at leading order in v we can combine $\mathcal{L}_4$ and $\mathcal{L}_5$ into a single operator.

After a simple color algebra similar to what we have used when writing down Eq. (6.9) the sum $\mathcal{L}_4 + \mathcal{L}_5$ can be written in a form where potential fields appear only in the $c\bar{c}$-current:

$$\mathcal{L}^{(p)} = -g_s[\bar{\xi}_{n,p} \Gamma_j \mathbf{C} h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\, \tilde{p}_\perp} \Gamma_j [\frac{n \cdot A_{\tilde{q}}}{n \cdot \tilde{q}}\,, \mathbf{C}]\, \eta^C_{\beta_{c\bar{c}}\, -\tilde{p}_\perp}]. \tag{6.16}$$

Here the $\pm i0$ terms have been neglected because they contribute to the zero bin which is considered separately. Also, the label $\perp$ has been dropped because $\tilde{q}$ is the momentum of longitudinal gluon.

6.2.3 ET Lagrangian at the First Order in g_s

Now we can combine operators (6.1), (6.9) and (6.16) into a single operator that suggests how the results of tree-level matching at $o(g_s)$ could be extended to all orders in g_s and at the leading order in v and λ:

$$\mathcal{L}^{(tree)} = [\bar{\xi}_{n,p} W \Gamma_j \mathbf{C} h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j \, S \, \mathbf{C} \, S^\dagger \, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{6.17}$$

where

$$W = 1 - g_s \frac{\bar{n}\cdot A_{n,q}}{\bar{n}\cdot q} + o(g_s^2), \qquad \text{and} \qquad S = 1 - g_s \frac{n\cdot A_{\tilde{q}}}{n\cdot q} + o(g_s^2). \tag{6.18}$$

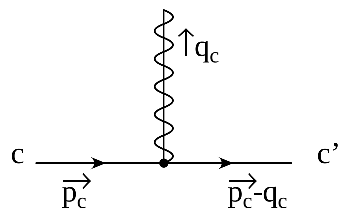
The operators W and S look like the first terms in the expansion of Wilson line operators introduced in Eq. (4.10). In the next section we extend the matching calculation at tree-level to all orders in g_s and show that (6.17), where W and S are the collinear and potential Wilson lines, holds true to all orders in g_s.

6.3 Tree-Level ET Lagrangian to All Orders

We can extend the tree-level matching calculation for collinear and potential gluons to all orders in g_s using the technique explained in Appendix A of [12]. The method is to introduce auxiliary fields corresponding to off-shell quarks and gluons, further to write down a Lagrangian for the auxiliary and on-shell fields, and then integrate out the auxiliary fields using EOM. The latter corresponds to tree-level matching to all orders in g_s.

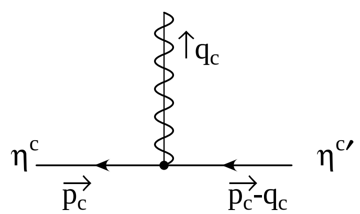
6.3.1 Auxiliary Quark Fields

The first step is to draw all tree-level diagrams that introduce off-shell fields in the FT and then to write down the corresponding analytic expressions at the leading order in the ET power expansion. Figures 6.6, 6.7, and 6.8 show how off-shell modes arise for quarks of $c\bar{c}$-pair and s-quark, respectively.



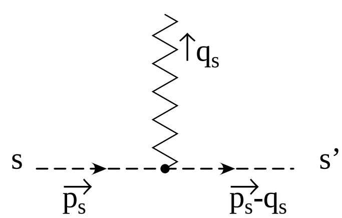
$$\Longrightarrow \qquad i\, g_s \frac{\bar{n}\cdot A_c}{\bar{n}\cdot q_c} i\, P_{\beta_{c\bar{c}}} \qquad (6.19)$$

Figure 6.6: Collinear gluon interacting with c-quark field and off-shell c'-field.



$$\Longrightarrow \qquad i\, g_s \frac{-\bar{n}\cdot A_c}{\bar{n}\cdot q_c} i\, P_{-\beta_{c\bar{c}}} \qquad (6.20)$$

Figure 6.7: Collinear gluon interacting with η^c-field of the antiquark and off-shell $\eta^{c\prime}$-field.



$$\Longrightarrow \qquad i\, g_s \frac{n\cdot A_p}{n\cdot q_p}\, i\, \frac{\not{n}}{2} \qquad (6.21)$$

Figure 6.8: Potential gluon interacting with s-quark field and off-shell s'-field.

For the b-quark we have to match in two steps. Both potential and collinear gluons give the b-quark an off-shell momentum but the corresponding off-shellnesses are of different orders of magnitude. Recall that the potential momentum is $\sim mv$ and collinear momentum is $\sim m$. Once a collinear gluon is attached to a b-quark line any off-shell momentum brought by potential gluons is absorbed by the off-shell momentum delivered by the collinear gluon. In diagrammatic language only the diagrams with potential gluons next to the heavy quark field h_{β_b} and collinear gluons next to the weak current operator shown in Fig. (6.9) will contribute. To account for this we introduce two off-shell fields for the b-quark: b' which is off-shell by $\sim mv$ and b'' which is off-shell by $\sim m$. The

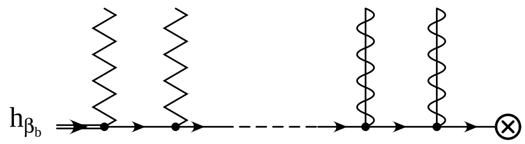
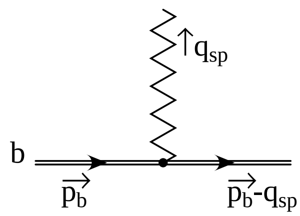


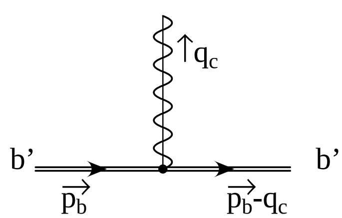
Figure 6.9: Diagrams of the FT which contribute to the tree-level matching of the heavy b-quark field. Potential gluons should matched before matching the collinear gluons.

matching is then performed in two steps, see Figs. 6.10 and 6.11.



$$\Longrightarrow \qquad i\, g_s \frac{\beta_b \cdot A_p}{\beta_b \cdot q_p} i\, P_{\beta_b} \tag{6.22}$$

Figure 6.10: Potential gluon interacting with the b-quark field and the off-shell b'-field.



$$\Longrightarrow \qquad i\, g_s \frac{\bar{n} \cdot A_c}{\bar{n} \cdot q_c} i\, P_{\beta_b} \tag{6.23}$$

Figure 6.11: Collinear gluon interacting with off-shell b'-quark field and off-shell b''-field.

6.3.2 Auxiliary Quark Lagrangian

Below we list the auxiliary Lagrangians that reproduce Feynman rules depicted on Figs. 6.6 - 6.8 and 6.11. Only the leading terms in λ and v are kept in each of the Lagrangians. The spinor structure of the auxiliary fields is the same as of the primary fields and is suppressed for simplicity. Also we don't show explicitly labels on the auxiliary fields.

$$
\begin{aligned}
\mathcal{L}_{cc'} &= \bar{\xi}_{\beta_{c\bar{c}},p}(g_s\,\bar{n}\cdot A_c + g_s\,\bar{n}\cdot A_X)c' + \bar{c}'(\bar{n}\cdot P + g_s\,\bar{n}\cdot A_c + g_s\,\bar{n}\cdot A_X)c', \\
\mathcal{L}_{\eta^c\eta^{c'}} &= -\bar{\eta}^{c\prime}(g_s\,\bar{n}\cdot A_c + g_s\,\bar{n}\cdot A_X)\,\eta^{C}_{\beta_{c\bar{c}}\,-\tilde{p}_\perp} - \bar{\eta}^{c\prime}(\bar{n}\cdot P + g_s\,\bar{n}\cdot A_c + g_s\,\bar{n}\cdot A_X)\eta^{c\prime}, \\
\mathcal{L}_{b'b''} &= \bar{b}''(g_s\,\bar{n}\cdot A_c + g_s\,\bar{n}\cdot A_X)b' + \bar{b}''(\bar{n}\cdot P + g_s\,\bar{n}\cdot A_c + g_s\,\bar{n}\cdot A_X)b'', \\
\mathcal{L}_{ss'} &= \bar{\xi}_{n,p}(g_s\,n\cdot A_p + g_s\,n\cdot A_X)s' + \bar{s}'(n\cdot P_p + g_s\,n\cdot A_p + g_s\,n\cdot A_X)s'. \qquad (6.24)
\end{aligned}
$$

Here P and P_p are label operators for collinear and potential momentum, respectively. An auxiliary gluon field A_X comes about when a potential gluon fuses with a collinear one, see Fig. (6.12). The gluon Lagrangian for this off-shell field is discussed in the next subsection.

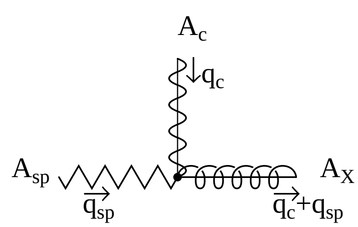


Figure 6.12: The potential gluon fusing with a collinear gluon into an off-shell field A_X.

The ET operator is then given by its tree-level formula (6.1) where quark spinors are replaced by the sum of quark spinors and auxiliary fields:

$$
O^{(tree)} = [(\bar{\xi}_{n,p} + \bar{s}')\Gamma_j\mathbf{C}\,(b'' + b')][(\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} + \bar{s}')\Gamma_j\mathbf{C}\,(\eta^{c\prime} + \eta^{C}_{\beta_{c\bar{c}}\,-\tilde{p}_\perp})] \qquad (6.25)
$$

Here the auxiliary fields should be understood as perturbative expansions of solutions for the corresponding EOM following from (6.24). The latter is equivalent to summing up all tree-level diagrams where auxiliary fields, potential, and collinear gluons couple to each other and to on-shell quark fields. For the b-quark we have to sum up first potential gluons using the auxiliary Lagrangian (see Fig. 6.10):

$$
\mathcal{L}_{bb'} = \bar{b}'(g_s\,\beta_b\cdot A_p)h_{\beta_b} + \bar{b}'(\beta_b\cdot P_p + g_s\,\beta_b\cdot A_p)b' \qquad (6.26)
$$

and then use the off-shell field b' in Eqs. (6.24) and (6.25). Otherwise the subleading contribution due to potential fields will be eaten by collinear off-shellness and we'll not be able to reproduce the first order result in g_s (see (6.17)).

Now the vector β_b in the Lagrangian (6.26) can be replaced by the vector n as it was the case for the first order correction (6.15). This follows from scaling of the potential gluon (6.14) and the observation that the $\beta_{c\bar{c}} \cdot P_p$ component of the potential momentum label must scale like mv^2 in order that potential gluons end up in a charmonium eigenstate. So, at the leading order in v the vector β_b can be replaced with $(E_s/m_b)\, n$ according to momentum conservation (6.13). The kinematic constant E_s/m_b is then absorbed by a redefinition of the auxiliary field $\bar{b}'$ which doesn't enter into the matching Lagrangian (6.25) and therefore is irrelevant. Then the Lagrangian (6.26) becomes:

$$\mathcal{L}_{bb'} = \bar{b}'(g_s\, n \cdot A_p) h_{\beta_b} + \bar{b}'(n \cdot P_p + g_s\, n \cdot A_p) b'. \tag{6.27}$$

The EOM for the field b' following from (6.27) is

$$-g_s\, n \cdot A_p\, h_{\beta_b} = (n \cdot P_p + g_s\, n \cdot A_p) b'. \tag{6.28}$$

Its solution is given by the path ordered exponential

$$b' = (S-1) h_{\beta_b} \qquad \text{where} \qquad S = P \exp\left(i g_s \int_{-\infty}^{y} ds\, n \cdot A_p(s\, n) \right), \tag{6.29}$$

where y is the Fourier transformed label of the potential field momentum label $\tilde{q}$. The field b' includes off-shell fluctuations of the heavy quark field h_{β_b} induced by its coupling to potential gluons. From now on we change the notation by replacing

$$b' \to b' + h_{\beta_b} = S\, h_{\beta_b}, \tag{6.30}$$

so that the redefined b' includes both the heavy field and potential fluctuations.

The EOMs of the auxiliary fields following from the Lagrangian (6.24) are

$$
\begin{aligned}
&-(g_s\,\bar n\cdot A_c+g_s\,\bar n\cdot A_X)\xi_{\beta_{c\bar c},p}=(\bar n\cdot P+g_s\,\bar n\cdot A_c+g_s\,\bar n\cdot A_X)c',\\
&-(g_s\,\bar n\cdot A_c+g_s\,\bar n\cdot A_X)\,\eta^C_{\beta_{c\bar c}\,-\tilde p_\perp}=(\bar n\cdot P+g_s\,\bar n\cdot A_c+g_s\,\bar n\cdot A_X)\eta^{c\prime},\\
&-(g_s\,\bar n\cdot A_c+g_s\,\bar n\cdot A_X)b'=(\bar n\cdot P+g_s\,\bar n\cdot A_c+g_s\,\bar n\cdot A_X)b'',\\
&-(g_s\,n\cdot A_p+g_s\,n\cdot A_X)\xi_{n,p}=(n\cdot P_p+g_s\,n\cdot A_p+g_s\,n\cdot A_X)s'. \qquad (6.31)
\end{aligned}
$$

The solutions to this set are given by

$$
\begin{aligned}
c'&=(W_X-1)\xi_{\beta_{c\bar c}},\\
\eta^{c\prime}&=(W_X-1)\eta^c_{\beta_{c\bar c}},\\
b''&=(W_X-1)b',\\
s'&=(S_X-1)\xi_{n,p}, \qquad (6.32)
\end{aligned}
$$

where W_X and S_X are path ordered exponentials

$$
\begin{aligned}
W_X&=P\exp\left(ig_s\int_{-\infty}^{y}ds\,[\bar n\cdot A_X(s\,\bar n)+\bar n\cdot A_c(s\,\bar n)]\right)\qquad\text{and}\\
S_X&=P\exp\left(ig_s\int_{-\infty}^{y}ds\,[n\cdot A_X(s\,n)+n\cdot A_p(s\,n)]\right). \qquad (6.33)
\end{aligned}
$$

Now we can combine Eqs. (6.25), (6.30), (6.32), and (6.33) to write down the intermediate result for the tree-level ET Lagrangian:

$$
\mathcal{L}^{(tree)}=[\bar\xi_{n,p}\,S_X^\dagger\,\Gamma_j\mathbf{C}\,W_X\,S\,h_{\beta_b}][\bar\xi_{\beta_{c\bar c}\,\tilde p_\perp}\,W_X^\dagger\,\Gamma_j\mathbf{C}\,W_X\,\eta^C_{\beta_{c\bar c}\,-\tilde p_\perp}]. \qquad (6.34)
$$

The last step is to to solve the EOM for auxiliary field A_X and rewrite $\mathcal{L}^{(tree)}$ in terms of A_c and A_p only.

6.3.3 Auxiliary Gluon Lagrangian

In this section we briefly repeat the derivation given in Appendix A of [12]. The Lagrangian $\mathcal{L}[A_X]$ for the auxiliary field A_X is the standard Lagrangian for a non-abelian field in the background field gauge with A_c and A_p taken as (constant) background fields:

$$
\mathcal{L}[A_X]=\frac{1}{2g_s^2}\mathrm{Tr}\left\{[iD_X+g_s\,A_X,\,iD_X+g_s\,A_X]\right\}^2+\frac{1}{\alpha_L}\mathrm{Tr}\left\{[iD_X,\,A_X]\right\}^2, \qquad (6.35)
$$

where

$$iD_X = \frac{n}{2}(\bar{n}\cdot P + g_s\bar{n}\cdot A_c) + \frac{\bar{n}}{2}(n\cdot P_p + g_s n\cdot A_p). \tag{6.36}$$

When writing down the derivative we keep only leading terms in n and $\bar{n}$ directions. Also we don't write $\perp$ components because they don't appear when matching at the leading order in power corrections.

The covariant derivative depends on two linearly independent vectors. We're looking for classical solutions to the EOM following from the Lagrangian, and therefore ignore all possible loop corrections. By this reasoning it would suffice to use the two-dimensional ansatz for the auxiliary field as well:

$$A_X = \frac{n}{2}(\bar{n}\cdot A_X) + \frac{\bar{n}}{2}(n\cdot A_X). \tag{6.37}$$

The approximations (6.36) and (6.37) are valid only at the leading order in power corrections.

Then the Lagrangian (6.35) is symmetric under the interchanges

$$\bar{n}\leftrightarrow n, \qquad \bar{n}\cdot P\leftrightarrow n\cdot P_p \qquad \bar{n}\cdot A_c\leftrightarrow n\cdot A_p. \tag{6.38}$$

The EOM following from this Lagrangian,

$$[iD^\mu_X + g_s\, A^\mu_X,\, [iD^\mu_X + g_s\, A^\mu_X,\, iD_X + g_s\, A_X]] = 0, \tag{6.39}$$

can be rewritten in terms of the path ordered exponentials W_X and S_X by means of the identity

$$iD_X + g_s\, A_X = \frac{n}{2}W_X\bar{n}\cdot PW^\dagger_X + \frac{\bar{n}}{2}S_X\, n\cdot P\, S^\dagger_X. \tag{6.40}$$

Then Eq. (6.39) takes the form:

$$\begin{aligned}&\Big[W_X\bar{n}\cdot PW^\dagger_X,\, \Big[S_X n\cdot P\, S^\dagger_X,\, W_X\bar{n}\cdot PW^\dagger_X\Big]\Big] = 0,\\ &\Big[S_X\, n\cdot PS^\dagger_X,\, \Big[W_X\bar{n}\cdot PW^\dagger_X,\, S_X n\cdot P\, S^\dagger_X\Big]\Big] = 0.\end{aligned} \tag{6.41}$$

Because of the symmetry (6.38) the second equation is equivalent to the first. Expanding the first equation and using unitarity of the operators W_X and S_X

gives:

$$
\begin{aligned}
&2W_X\,\bar{n}\cdot P\,W_X^\dagger\,S_X\,n\cdot P_p\,S_X^\dagger W_X\,\bar{n}\cdot P\,W_X^\dagger \\
&-W_X(\bar{n}\cdot P)^2 W_X^\dagger S_X n\cdot P_p S_X^\dagger - S_X n\cdot P S_X^\dagger W_X(\bar{n}\cdot P)^2 W_X^\dagger = 0. \quad (6.42)
\end{aligned}
$$

By means of simple algebra one can check that this equation is identically zero if the following ansatz is used:

$$S_X^\dagger\,W_X = W\,S^\dagger, \tag{6.43}$$

where W and S are the path ordered exponents that depend only on A_c and A_p:

$$
\begin{aligned}
W &= P\exp\left(ig_s\int_{-\infty}^{y} ds\,\bar{n}\cdot A_c(s\,\bar{n})\right), \\
S &= P\exp\left(ig_s\int_{-\infty}^{y} ds\,n\cdot A_p(s\,n)\right). \qquad (6.44)
\end{aligned}
$$

The vanishing of the gauge fixing term in (6.35) gives an additional equation that must be satisfied in order to remove the gauge ambiguity from A_X.

Equation (6.43) is our master equation. It can be verified at lowest orders in g_s by direct tree-level matching calculation (see [12]). Eq. (6.43) simply extends the pattern to all orders in g_s. The method of auxiliary Lagrangian allows one to prove that the extension is valid.

An important observation is that the auxiliary Lagrangian (6.35) vanishes on classical solutions given by Eq. (6.43). This can be verified by writing the covariant derivative in the Lagrangian as (6.40) and using the master equation (6.43). Therefore no interaction between potential and collinear gluons at tree-level is induced by integrating out the off-shell field A_X (see [12] for an explicit example at order g_s^2).

6.3.4 ET Lagrangian at All Orders in g_s

The master equation (6.43) makes it possible to eliminate the auxiliary field A_X from the operator (6.34). The first step is to use the color identity

$$\mathbf{C}\otimes U^\dagger\mathbf{C}U = U\mathbf{C}U^\dagger\otimes\mathbf{C}, \tag{6.45}$$

where $\mathbf{C}$ is either $\mathbb{1}$ or T^a and U is a unitary operator in the fundamental representation of $SU(3)$, so the path ordered exponentials (6.33) and (6.44) qualify. Then (6.34) can be written as

$$\mathcal{L}^{(tree)} = [\bar{\xi}_{n,p}\, S_X^\dagger\, W_X \Gamma_j \mathbf{C}\ S\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\, \Gamma_j \mathbf{C}\, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]. \tag{6.46}$$

Using the master equation (6.43) and the identity (6.45) one more time gives finally:

$$\mathcal{L}^{(tree)} = [\bar{\xi}_{n,p} W \Gamma_j \mathbf{C}\ h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j\, S\, \mathbf{C}\, S^\dagger\, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]. \tag{6.47}$$

When expanded in powers of strong coupling constant g_s Eq. (4.9) reproduces the infinite set of tree-level Feynman diagrams of the ET generated by the set of local operators arising when matching to the FT.

6.3.5 Eliminating Ultrasoft Gluons

Here we give a derivation of Eqs. (4.11). At the leading order in v and λ the operator (6.47) can be written in terms of the fields which don't couple to ultrasoft gluons via the kinetic terms (see e.g. [12]):

$$\begin{aligned}
&\xi_{n,p} = Y_n\, \xi^{(0)}_{n,p}, \qquad A_{n,p} = Y_n\, A^{(0)}_{n,p} Y_n^\dagger \qquad \text{and} \qquad W = Y_n\, W^{(0)}\, Y_n^\dagger,\\
&h_{\beta_b} = Y_{\beta_b} h^{(0)}_{\beta_b},\\
&\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = Y_{\beta_{c\bar{c}}} \xi^{(0)}_{\beta_{c\bar{c}},\tilde{p}_\perp}, \qquad \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp} = Y_{\beta_{c\bar{c}}} \eta^{C\,(0)}_{\beta_{c\bar{c}},-\tilde{p}_\perp},\\
&A_{\beta_{c\bar{c}},\tilde{q}} = Y_{\beta_{c\bar{c}}}\, A^{(0)}_{\beta_{c\bar{c}},\tilde{q}} Y^\dagger_{\beta_{c\bar{c}}} \qquad \text{and} \qquad S = Y_{\beta_{c\bar{c}}}\, S^{(0)} Y^\dagger_{\beta_{c\bar{c}}}.
\end{aligned} \tag{6.48}$$

Here Y_l, with $l = n, \beta_b$, or $\beta_{c\bar{c}}$, stands for the path ordered exponent:

$$Y_l(y) = P \exp\left(i g_s \int_{-\infty}^{y} ds\, l \cdot A_{us}(l\, s) \right). \tag{6.49}$$

Using the field redefinitions above one can rewrite Eq. (4.9) in terms of the ultrasoft free fields (6.48):

$$
\begin{aligned}
&O^{(tree)} = [\bar{\xi}_{n,p} W \Gamma_j \mathbf{C}\ h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j\, S\, \mathbf{C}\, S^\dagger\, \eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}] \\
&= [\bar{\xi}^{(0)}_{n,p} Y^\dagger_n Y_n W^{(0)} Y^\dagger_n \Gamma_j \mathbf{C} Y_{\beta_b}\ h^{(0)}_{\beta_b}] \times \\
&\times [\bar{\xi}^{(0)}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} Y^\dagger_{\beta_{c\bar{c}}} Y_{\beta_{c\bar{c}}}\, S^{(0)}\, Y^\dagger_{\beta_{c\bar{c}}} \Gamma_j \mathbf{C} Y_{\beta_{c\bar{c}}}\, S^{(0)\,\dagger}\, Y^\dagger_{\beta_{c\bar{c}}} Y_{\beta_{c\bar{c}}} \eta^{C\,(0)}_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}] \\
&= [\bar{\xi}^{(0)}_{n,p} W^{(0)} Y^\dagger_n \Gamma_j \mathbf{C} Y_{\beta_b}\ h^{(0)}_{\beta_b}][\bar{\xi}^{(0)}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\, S^{(0)}\, Y^\dagger_{\beta_{c\bar{c}}} \Gamma_j \mathbf{C} Y_{\beta_{c\bar{c}}}\, S^{(0)\,\dagger}\, \eta^{C\,(0)}_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}] \quad (6.50)
\end{aligned}
$$

In the last line we used the fact that operators Y_n and $Y_{\beta_{c\bar{c}}}$ are unitary. For the singlet operator, $\mathbf{C} = \mathbb{1}$, and then from unitarity of $Y_{\beta_{c\bar{c}}}$ and S the first of Eqs. (4.11) follows:

$$
O^{(tree)}_{\mathbf{0}} = \sum_j [\bar{\xi}^{(0)}_{n,p} W^{(0)} \Gamma_j\, Y^\dagger_n Y_{\beta_b}\ h^{(0)}_{\beta_b}][\bar{\xi}^{(0)}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j \eta^{C\,(0)}_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}],
$$

However, for the octet operator, $\mathbf{C} = T^a$, further simplification seems impossible, so the second of Eqs. (4.11) follows:

$$
O^{(tree)}_{\mathbf{8}} = \sum_j [\bar{\xi}^{(0)}_{n,p} W^{(0)} \Gamma_j Y^\dagger_n T^a Y_{\beta_b}\ h^{(0)}_{\beta_b}][\bar{\xi}^{(0)}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \Gamma_j\, S\, Y^\dagger_{\beta_{c\bar{c}}} T^a Y_{\beta_{c\bar{c}}}\, S^\dagger\, \eta^{C\,(0)}_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}].
$$

In Chapter 4 we used Eqs. (4.11) to analyse factorization properties of the ET operator. After the field redefinition (6.50) ultrasoft fields do not couple anymore to the ET quark and gluons and that simplifies the analysis greatly. The derivation of Eqs. (4.9) and (4.11) given in this Chapter illustrates the power of functional methods presented in [12]. These methods make straightforward the resummation of the infinite subset of Feynman diagrams.

7

ET Lagrangians

In this Chapter we give a brief survey of the ET Lagrangians used to derive the results of the three previous Chapters. There are three of them: HQET, NRQCD, and SCET Lagrangians, and each one has given rise to a prolific field of research in B-physics and elsewhere.

The HQET section of this Chapter is based on [6]. The NRQCD section is based on [13], [14], and [6]. The SCET section is based on [11], [12], and [15].

7.1 HQET Lagrangian

There are three heavy quarks participating in the decay: the b and c quarks and the $\bar{c}$ anti-quark. In this section tree-level Lagrangians for heavy quarks and anti-quarks are derived.

7.1.1 Effective Lagrangian for a Heavy Quark

To derive the effective Lagrangian for a heavy quark we extract the large quark momentum mv and split the quark field into two spinor subspaces using the projectors $P_v = \frac{1+\not{v}}{2}$ and $P_{-v} = \frac{1-\not{v}}{2}$:

$$Q(x) = e^{-imv\cdot x}(h(x)+\tilde{h}(x)), \quad h(x) = e^{imv\cdot x}P_v Q(x), \quad \text{and} \quad \tilde{h}(x) = e^{imv\cdot x}P_{-v}Q(x). \tag{7.1}$$

One can easily check the following identities:

$$P_v + P_{-v} = 1, \quad P_v P_{-v} = 0, \qquad \not{v} h = h, \quad \not{v}\tilde{h} = -\tilde{h},$$
$$P_v \not{A} P_v = v\cdot A P_v, \quad P_{-v}\not{A} P_{-v} = -v\cdot A P_{-v},$$
$$P_{-v}\not{A} P_v = P_{-v}\not{A}_\perp P_v, \quad \text{and} \quad P_v\not{A} P_{-v} = P_v \not{A}_\perp P_{-v}. \tag{7.2}$$

In the last identities $A_\perp = A - (v\cdot A)v$.

The quark Lagrangian of the full theory is then written in terms of the heavy fields h and $\tilde{h}$:

$$\mathcal{L} = \bar{Q}(i\not{D} - m)Q = (\bar{h} + \bar{\tilde{h}})[i\not{D} + m\not{v} - m](h + \tilde{h}). \tag{7.3}$$

It is straightforward to verify that using identities (7.2) allows one to reduce the Lagrangian to a form that makes obvious the separation between the two subsets of quark fields in the limit $m \to \infty$:

$$\mathcal{L} = \bar{h} iv\cdot D h + \bar{h} i\not{D}_\perp \tilde{h} + \bar{\tilde{h}} i\not{D}_\perp h - \bar{\tilde{h}}(iv\cdot D + 2m)\tilde{h}. \tag{7.4}$$

For large m the derivative $iv\cdot D$ is small compared to m and can be neglected, so that $\tilde{h}$ is not a dynamical field and can be integrated out using equations of motion (EOM). When m is large but not infinite integrating it out results in the asymptotic expansion near the mass-shell of the heavy quark. The higher order terms in this expansion describe effects of quark-anti-quark coupling and are suppressed by inverse powers of m. The EOM for the field $\tilde{h}$ reads:

$$\frac{\delta\mathcal{L}}{\delta\bar{\tilde{h}}} = i\not{D}_\perp h - (iv\cdot D + 2m)\tilde{h} = 0 \quad \Longrightarrow \quad \tilde{h} = \frac{1}{iv\cdot D + 2m} i\not{D}_\perp h. \tag{7.5}$$

Substituting the solution back in Eq. (7.4) gives finally the tree-level Lagrangian for the heavy quark where the second term can be expanded now in $1/m$:

$$\mathcal{L}_h = \bar{h}\left[iv\cdot D + i\not{D}_\perp \frac{1}{iv\cdot D + 2m} i\not{D}_\perp\right] h. \tag{7.6}$$

The expansion in powers of $1/m$ produces an infinite number of terms whose coefficients can be improved systematically to account for radiative corrections by matching with the full theory at the scale $\mu = m$. The matching procedure gives the ET Lagrangian at tree-level, at one-loop, etc.

7.1.2 Effective Lagrangian for Heavy Anti-Quark

The derivation of the effective Lagrangian for the anti-quark is almost identical. This time we split the field as

$$Q(x) = e^{imv\cdot x}(y(x) + \tilde{y}(x)), \quad y(x) = e^{-imv\cdot x}P_{-v}Q(x) \text{ and } \tilde{y}(x) = e^{-imv\cdot x}P_vQ(x). \tag{7.7}$$

Obviously $\not{v}y = -y$ and $\not{v}\tilde{y} = \tilde{y}$. The quark Lagrangian of the FT is then written in terms of the heavy fields y and $\tilde{y}$:

$$\mathcal{L} = \bar{Q}(i\not{D} - m)Q = (\bar{y} + \bar{\tilde{y}})[i\not{D} - m\not{v} - m](y + \tilde{y}). \tag{7.8}$$

Using the identities (7.2) allows one to reduce the Lagrangian to the form where the fields y and $\bar{y}$ manifestly decouple in the limit $m \to \infty$:

$$\mathcal{L} = -\bar{y}iv\cdot Dy + \bar{y}i\not{D}_\perp\tilde{y} + \bar{\tilde{y}}i\not{D}_\perp y + \bar{\tilde{y}}(iv\cdot D - 2m)\tilde{y}. \tag{7.9}$$

The EOM for the field $\tilde{y}$ reads:

$$\frac{\delta\mathcal{L}}{\delta\bar{\tilde{y}}} = i\not{D}_\perp y + (iv\cdot D - 2m)\tilde{y} = 0 \quad \Longrightarrow \quad \tilde{y} = -\frac{1}{iv\cdot D - 2m}i\not{D}_\perp y. \tag{7.10}$$

Substituting the solution back in Eq. (7.9) gives the tree-level Lagrangian of the heavy anti-quark:

$$\mathcal{L}_y = -\bar{y}\left[iv\cdot D + i\not{D}_\perp\frac{1}{iv\cdot D - 2m}i\not{D}_\perp\right]y. \tag{7.11}$$

Note that the Lagarangian (7.11) is not normal ordered. That is, applying the quantization procedure to it will give a Hamiltonian where creation operators are on the right. To get the normal ordered Lagrangian we should rewrite Eq. (7.11) in terms of the charge conjugated fields:

$$\mathcal{L}_y = \bar{y}^c\left[iv\cdot\bar{D} + i\bar{\not{D}}_\perp\frac{1}{iv\cdot\bar{D} + 2m}i\bar{\not{D}}_\perp\right]y^c \qquad \text{where} \qquad y^c = Cy^* = e^{imv\cdot x}P_vQ^c, \tag{7.12}$$

and C is the charge conjugation matrix which in the Dirac basis is $C = i\gamma^2$. The symbol $\bar{D}$ stands for the covariant derivative in which the generators T^a are replaced with the generators for the complex conjugate representation: $-(T^a)^*$. Other than that Eq. (7.12) is identical to Eq. (7.6), so its quantization yields the normally ordered Hamiltonian.

7.1.3 HQET Gluons

In HQET there are only ultrasoft gluons $A_{us}(x)$. In the ET $A_{us}(x)$ play the role of a background field. Each component of the $A_{us}(x)$ field scales homogeneously like Λ_c. The gauge group of the ultrasoft gluons is the gauge group of the full theory. For the ultrasoft field we can take Feynman gauge with the gauge fixing Lagrangian

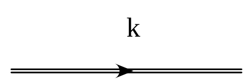
$$\mathcal{L}_{us} = -\frac{1}{2}(\partial \cdot A_{us})^2. \tag{7.13}$$

Then the propagator of the ultrasoft gluons becomes the usual Feynman propagator

$$D^{us}_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2 + i0}. \tag{7.14}$$

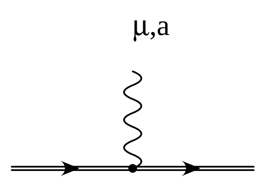
7.1.4 Feynman Rules for HQET

Here Feynman rules for HQET Lagrangian that are sufficient for one-loop calculation in the ET are given. The Feynman rules for the ultrasoft gluons are the same as in the full theory. The symbol β_b now replaces the symbol v which have been used above during the derivation of the HQET Lagrangian.



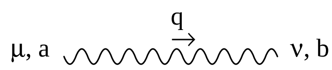
$$= \quad \frac{i\,P_{\beta_b}}{\beta_b \cdot k + i0}$$

Figure 7.1: Propagator of the heavy b-quark; β_b is the 4-velocity of b-quark.



$$= \quad -ig_s\beta_b^\mu T^a$$

Figure 7.2: Interaction vertex of the ultrasoft gluon and the heavy b-quark.



$$= \frac{-i g_{\mu\nu}\delta_{ab}}{q^2 + i0}$$

Figure 7.3: Propagator of the ultrasoft gluon.

7.2 NRQCD Lagrangian in Covariant Form

To describe the bound state of two heavy quarks we need an effective theory which is different from HQET (see, e.g. section 4.8 in [6]). In the center-of-mass (COM) of the heavy $c\bar{c}$-pair the effective Lagrangian for the pair is given by the NRQCD Lagrangian. The COM has 4-velocity $\beta_{c\bar{c}}$ in the b-quark rest frame. Boosting the NRQCD Lagrangian to the frame where the COM velocity is $\beta_{c\bar{c}}$ gives the desired covariant Lagrangian. Our derivation proceeds through the same steps as the derivation of the NRQCD Lagrangian [14], only we don't specify the frame of reference.

7.2.1 Covariant NRQCD Lagrangian

To get started we combine the Lagrangians (7.6) and (7.12) and keep only the first term in $1/m$ expansion:

$$\mathcal{L}_{c\bar{c}} = \bar{h}_c \left[i\beta_{c\bar{c}} \cdot D + \frac{(i\not{D}_{c\bar{c},\perp})^2}{2m} \right] h_c + \bar{h}_{\bar{c}} \left[i\beta_{c\bar{c}} \cdot \bar{D} + \frac{(i\not{\bar{D}}_{c\bar{c},\perp})^2}{2m} \right] h_{\bar{c}}. \tag{7.15}$$

In this equation

$$h_c = e^{im\beta_{c\bar{c}}\cdot x} P_{\beta_{c\bar{c}}} Q \qquad \text{and} \qquad h_{\bar{c}} = e^{im\beta_{c\bar{c}}\cdot x} P_{\beta_{c\bar{c}}} Q^c \tag{7.16}$$

where Q is the c-quark field operator in the full theory.

The Lagrangian (7.15) is the HQET Lagrangian for the heavy quark and anti-quark. To obtain the covariant NRQCD Lagrangian we need to split the residual momentum of heavy quarks further as $\tilde{p}+k$. The component of $\tilde{p}$ along the Wilson line $\beta\cdot\tilde{p}$ scales like mv^2 and the perpendicular component $\tilde{p}_\perp = \tilde{p} - \beta(\beta\cdot\tilde{p})$ scales like mv as it can be seen from the on-shell condition $(m\beta + \tilde{p})^2 = m^2$, or $2m\beta\cdot\tilde{p} + \tilde{p}^2 = 0$. The residual momentum k is of the order Λ_c.

So, we separate the large momentum component by taking the Fourier transform and assigning the label $\tilde{p}_\perp$ to the quark fields:

$$h(x) = \sum_{\tilde{p}_\perp} \exp(-i\tilde{p}_\perp x) h_{\beta_{c\bar{c}},\tilde{p}_\perp}(x) \tag{7.17}$$

Then the Lagrangian (7.15) becomes:

$$\begin{aligned} \mathcal{L}_{c\bar{c}} &= \sum_{\tilde{p}_\perp} \bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \left[i\beta_{c\bar{c}} \cdot D + \frac{(\not{\tilde{p}}_\perp + i\not{D}_{c\bar{c},\perp})^2}{2m} \right] \xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp} + (\xi \to \eta, T^a \to \overline{T}^a) \\ \text{with} \quad & P_{-\beta_{c\bar{c}}} \xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = 0 \quad \text{and} \quad P_{-\beta_{c\bar{c}}} \eta_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = 0. \end{aligned} \tag{7.18}$$

Here $D^\mu = (\partial^0 + ig_s A^0_{us}, -\nabla + ig_s \mathbf{A}_{us})$ and we use the conventional NRQCD symbols ξ and η for the quark and anti-quark spinors, respectively.

However unlike in the standard derivation we keep all four components of the spinors to ensure relativistic covariance and treat the constraints (7.18) as equations which the 4-spinors must satisfy. The solutions of Eqs. (7.18) are such that in the COM frame their upper components become the conventional NRQCD two-component spinors and the lower components are zero: $\xi_{\mathbf{p}}$ and $\eta_{\mathbf{p}}$:

$$\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = \sqrt{\frac{1+\beta^0_{c\bar{c}}}{2}} \begin{pmatrix} \xi_{\mathbf{p}} \\ \frac{\vec{\beta}_{c\bar{c}}\vec{\sigma}}{1+\beta^0_{c\bar{c}}} \xi_{\mathbf{p}} \end{pmatrix} \quad \text{and} \quad \eta_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = \sqrt{\frac{1+\beta^0_{c\bar{c}}}{2}} \begin{pmatrix} \eta_{\mathbf{p}} \\ \frac{\vec{\beta}_{c\bar{c}}\vec{\sigma}}{1+\beta^0_{c\bar{c}}} \eta_{\mathbf{p}} \end{pmatrix}. \tag{7.19}$$

In other words $\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ and $\eta_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ are the NRQCD spinors $\xi_{\mathbf{p}}$ and $\eta_{\mathbf{p}}$ boosted into the frame moving at the velocity $-\vec{\beta_c c}(\beta^0)^{-1}$ with respect to the COM frame. In the frame $\beta_{c\bar{c}} = (1, \mathbf{0})$ we reproduce the standard form of the NRQCD Lagrangian. This formalism is essentially the same as that one discussed in [25].

When calculating one-loop corrections to the ET Lagrangian it is more convenient to work with the charge conjugated field $\eta^C_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ because it creates the anti-quark $\bar{c}$ in the final state and this is exactly what we need in the ET for the $B \to J/\psi K$ decay. To write down a covariant NRQCD Lagrangian for the antiquark in terms of $\eta^C_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ remember that the covariant NRQCD Lagrangian for the heavy anti-quark (the second term in Eq. (7.15)) follows from the Lagrangian (7.11) after charge conjugation. Therefore Eq. (7.15) is the Lagrangian for the heavy anti-quark written in terms of the charge conjugated field $\eta^C_{\beta_{c\bar{c}},\tilde{p}_\perp}$:

$$\mathcal{L}_{\bar{c}} = -\sum_{\tilde{p}_\perp} \bar{\eta}^C_{\beta_{c\bar{c}},\tilde{p}_\perp} \left[i\beta_{c\bar{c}} \cdot D + i\not{D}_{c\bar{c}\perp} \frac{1}{i\beta_{c\bar{c}} \cdot D - 2m_c} i\not{D}_{c\bar{c}\perp} \right] \eta^C_{\beta_{c\bar{c}},\tilde{p}_\perp},$$
$$\text{with} \qquad P_{-\beta}\eta^C_{\beta_{c\bar{c}},\tilde{p}_\perp} = \eta^C_{\beta_{c\bar{c}},\tilde{p}_\perp}. \tag{7.20}$$

Note that the fields $\eta^C_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ and $\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ transform under the same representation of $SU(3)$, therefore the symbol D instead of $\bar{D}$.

7.2.2 Full Theory Spinors in Terms of ξ and η

In order to match a full QCD ME onto the ME of the covariant NRQCD we have to write down the spinors u and v in terms of ξ and η. This can be done in the same way as in NRQCD only the frame of reference is not specified. For example:

$$(m\not{\beta} + \tilde{\not{p}}_c - m)u = 0,$$
$$(-2mP_{-\beta} + \tilde{\not{p}}_c)u = 0,$$
$$(-2mP_{-\beta} + P_{-\beta}\tilde{\not{p}}_c)(P_\beta + P_{-\beta})u = 0,$$
$$(-2mP_{-\beta} + P_{-\beta}\tilde{\not{p}}_{c\perp}P_\beta - \beta \cdot \tilde{p}_c P_{-\beta})u = 0,$$
$$P_\beta u = \xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp}, \qquad P_{-\beta}u = u_-, \qquad u = \xi_{\beta_{c\bar{c}}p_\perp} + u_-,$$
$$-(2m + \beta \cdot \tilde{p}_c)u_- + P_{-\beta}\tilde{\not{p}}_{c\perp}\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = 0,$$
$$u_- = \frac{1}{2m + \beta \cdot \tilde{p}_c} P_{-\beta}\tilde{\not{p}}_{c\perp}\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp},$$
$$u(m\beta + \tilde{p}_c) = \left[1 + \frac{1}{2m + \beta \cdot \tilde{p}_c} P_{-\beta}\tilde{\not{p}}_{c\perp} \right] \xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp} \overset{\beta=(1,\vec{0})}{\longrightarrow} \begin{pmatrix} \xi_{\mathbf{p}} \\ \frac{\vec{p}\vec{\sigma}}{E+m}\xi_{\mathbf{p}} \end{pmatrix}. \tag{7.21}$$

In the fifth line the ET spinor $\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp}$ is introduced as a projection of the full QCD spinor u onto the subspace restricted by the constraint (7.18). The second half of the bispinor subspace u_- accounts for the difference of the order v between the full and effective theories. One can see that in the COM frame the full QCD spinor is reproduced. Unlike the ξ-spinor which has only two upper components in the COM frame, the full QCD spinor has also two non-zero lower components suppressed by the first power of velocity in the COM frame.

To parameterize the v-spinor of the FT by the η^C-spinor notice that the charge conjugated spinor $v^c = Cv^*$ satisfies the same equation as the u-spinor. The covariant NRQCD Lagrangian for the anti-quark is derived from the FT using charge conjugation. Therefore the v^c-spinor must be written in terms of the ET η-spinor in the same way as the u-spinor is written in terms of the ξ-spinor and then the v-spinor follows after charge conjugation:

$$\begin{aligned} v = i\gamma^2 v^{c*} \quad &= i\gamma^2 \left[1 + \frac{1}{2m + \beta \cdot \tilde{p}_c} P_{-\beta} \not{\tilde{p}}_{c\perp} \right]^* \eta^*_{\beta_{c\bar{c}},\, \tilde{p}_\perp}, \\ &= \left[1 - \frac{1}{2m + \beta \cdot \tilde{p}_c} P_{\beta} \not{\tilde{p}}_{c\perp} \right] (i\gamma^2 \eta^*_{\beta_{c\bar{c}},\, \tilde{p}_\perp}), \\ &= \left[1 - \frac{1}{2m + \beta \cdot \tilde{p}_c} P_{\beta} \not{\tilde{p}}_{c\perp} \right] \eta^C_{\beta_{c\bar{c}},\, \tilde{p}_\perp}. \end{aligned} \tag{7.22}$$

Here, componentwise:

$$\eta^C_{\beta_{c\bar{c}},\, \tilde{p}_\perp} = (i\gamma^2 \eta^*_{\beta_{c\bar{c}},\, \tilde{p}_\perp}) = \begin{pmatrix} \frac{\vec{\beta}\vec{\sigma}}{1+\beta^0} (-i\sigma^2 \eta^*_{\mathbf{p}}) \\ (-i\sigma^2 \eta^*_{\mathbf{p}}) \end{pmatrix} \qquad \text{where} \qquad (-i\sigma^2 \eta^*) = (-\eta^\dagger_\downarrow, \eta^\dagger_\uparrow)^T. \tag{7.23}$$

Notice that the spinor $\eta^C_{\beta_{c\bar{c}},\, \tilde{p}_\perp}$ satisfies $P_{-\beta} \eta^C_{\beta_{c\bar{c}},\, \tilde{p}_\perp} = \eta^C_{\beta_{c\bar{c}},\, \tilde{p}_\perp}$, i.e. for the charge conjugated spinor the 4-velocity is reversed.

7.2.3 Potential Gluon Field

The Lagrangians (7.15) and (7.20) contain only the ultrasoft field A_{us} and are label-diagonal. However we can add a term that changes label $\tilde{p}_\perp$ on the quark field without knocking the quark off-shell:

$$\mathcal{L}_p = -g_s \sum_{\tilde{p}_\perp,\, \tilde{q} \neq 0} \bar{\xi}_{\beta_{c\bar{c}}\, \tilde{p}_\perp + \tilde{q}} (\beta_{c\bar{c}} \cdot A_{\beta_{c\bar{c}},\, \tilde{q}}) \xi_{\beta_{c\bar{c}}\, \tilde{p}_\perp}. \tag{7.24}$$

Notice that $\tilde{q} = 0$ corresponds to the ultrasoft field $A_{us}(x)$ and should be excluded to avoid double counting. The field $A_{\beta_{c\bar{c}},\, \tilde{q}}$ will not remove the quark off-shell provided that its longitudinal component $A_\parallel = \beta_{c\bar{c}} \cdot A_{\beta_{c\bar{c}},\, \tilde{q}}$ transfers the on-shell momentum $\sim (mv^2,\, mv)$. This condition implies that the propagator of $A_\parallel$ should be:

$$D^{\parallel}_{\mu\nu}(q) = \frac{-i\delta_{ab}}{\tilde{q}_\perp^2 + i0}\beta_{c\bar{c},\,\mu}\beta_{c\bar{c},\,\nu}. \tag{7.25}$$

Hereafter we refer to the field $A_{\beta_{c\bar{c}},\,\tilde{q}}$ as a potential gluon. It is straightforward to check that potential gluon exchange at one loop gives the same expression as the tree-level amplitude generated by the local (in ET) four-fermion Coulomb interaction in the standard approach to NRQCD (see [13]). In the covariant version of NRQCD it is necessary to develop a covariant description of the force binding together quarks of the $c\bar{c}$-pair.

To make sure that the only interaction between potential gluons and heavy quarks at leading order in v is due to Eq. (7.24) we have to choose a gauge fixing condition for the potential field $A_{\beta_{c\bar{c}},\,\tilde{q}}$ such that

$$\beta^{\mu}_{c\bar{c}}\, D^{(0)}_{\mu\nu} = -\frac{i\,\beta_{\beta_{c\bar{c}},\,\nu}}{q_\perp^2 + i0}, \tag{7.26}$$

where $D^{(0)}_{\mu\nu}$ is the tree-level propagator in this gauge. In an arbitrary gauge the Compton scattering of potential gluons off a heavy quark gives rise to an infinite set of effective theory interactions at higher orders in α_s which have the same power counting as the four-fermion Coulumb interaction that scales like $\alpha_s(mv)^4$ (see [13]). Figure 7.4 shows explicitly how the first of these terms arises upon matching with the FT.

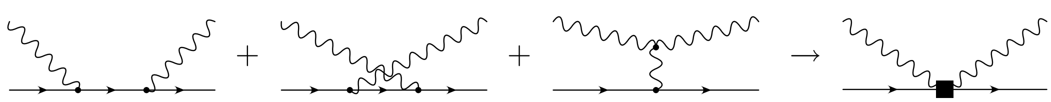


Figure 7.4: Effective interaction in NRQCD due to Compton scattering of two gluons carrying momenta $\sim mv$ off the quark in the FT. Note that in QED the first two diagrams cancel each other and there is no third, so the effective interaction due to Compton scattering appear only in a non-abelian gauge theory.

However if Eq. (7.26) holds then after taking the effective theory limit for quark propagators in the first three FT diagrams in Fig. (7.4) the momentum transferred does not take the quark off-shell. Then all three diagrams become covariant NRQCD diagrams and do not produce a new vertex in the ET. It should be obvious that the same conclusion will hold for Compton scattering involving arbitrary number of gluons.

Covariant Coulomb Gauge

Let's now check that choosing the covariant Coulomb gauge (CCG) for the potential gluon $A_{\beta_{c\bar{c}\tilde{q}}}(x)$ gives a propagator that satisfies Eq. (7.26). The CCG is defined by the gauge fixing Lagrangian:

$$\mathcal{L}_{CG} = \lambda(x)\,(\partial_\perp \cdot A_\perp) = \lambda(x)\,[\partial \cdot A_{\beta_{c\bar{c}\tilde{q}}} - (\beta_{c\bar{c}} \cdot \partial)(\beta_{c\bar{c}} \cdot A_{\beta_{c\bar{c}\tilde{q}}})], \tag{7.27}$$

where $\lambda(x)$ is the Lagrange multiplier. In the CCG the propagator of the potenial gluon takes the form:

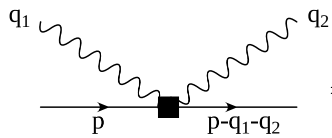
$$D^{CG}_{\mu\nu}(q) = \frac{-i\delta_{ab}}{q^2 + i0}\left[g_{\mu\nu} + \frac{(q_\mu\beta_\nu + \beta_\mu q_\nu)(\beta \cdot q) - q_\mu q_\nu}{q^2 - (\beta \cdot q)^2 + i0}\right]. \tag{7.28}$$

In the COM frame $\beta_{c\bar{c}} = (1, \vec{0})$ and the propagator splits into the propagator of the longitudinal off-shell gluons and the propagator of transverse gluons:

$$D^{CG}_{00} = \frac{i\delta_{ab}}{\mathbf{k}^2 - i0}, \qquad D^{CG}_{0,i} = 0, \qquad D^{CG}_{i\,j} = \frac{i\delta_{ab}}{q^2 + i0}\left[\delta_{i\,j} - \frac{k_i\,k_j}{\mathbf{k}^2}\right], \tag{7.29}$$

and Eq. (7.26) is manifestly satisfied. The CCG imposes a gauge fixing condition only on the transversal component $A_\perp = A - A_\|$ of the potential gluon $A_{\beta_{c\bar{c}\tilde{q}}}$. As a result the longitudinal component $A_\|$ does not propagate in time (along $\beta_{c\bar{c}}$) and becomes an auxiliary field that must be integrated out.

Decoupling of the longitudinal $A_\|$ and the transverse components $A_\perp$ of the potential gluon implies that they can be assigned different power counting. The longitudinal component $A_\|$ must scale like mv^2 if we want the Lagrangian (7.24) to have the same scaling as Eq. (7.18). This is consistent with treating the longitudinal component of the *ultrasoft* field in Eq. (7.18) as a zero bin of Eq. (7.24). The realistic scaling for the ultrasoft field $A_{us}(x)$ in the bound state of the $c\bar{c}$-pair



$$= \quad -ig_s^2[\varepsilon^{\mu}_{2\perp},\, \varepsilon_{1\mu\perp}]\frac{\beta_{c\bar{c}}\cdot(q_2-q_1)}{(q_1+q_2)^2_\perp + i0}$$

Figure 7.5: An ET vertex generated by the three-gluon coupling of the FT in the covariant NRQCD. Both gluons correspond to transversal fields $A_\perp$.

in the J/ψ is mv^2 (see [13]). The scaling of the transverse fields is determined in [13] and here we repeat their argument.

In the CCG the first three diagrams of the FT in Fig. 7.4 become the covariant NRQCD diagrams when the ET limit is taken. Nevertheless we can think of the third diagram in Fig. (7.4) as giving rise to a new interaction (the fourth diagram) in the ET at order g_s^2, where the longitudinal field $A_\parallel$ has been integrated out. Taking the ET limit in the third diagram in Fig. (7.4) and assuming that both gluons correspond to the transverse fields $A_\perp$ gives (see Fig. (7.5)):

$$\mathcal{L}_2 = -\frac{1}{2}g_s^2 \sum_{\tilde{p}_\perp\,\tilde{q}_1\,\tilde{q}_2\neq 0} \bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp+\tilde{q}_1+\tilde{q}_2}[A^{\mu}_{\tilde{q}_2\perp},\, A_{\tilde{q}_1\mu\perp}]\xi_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\frac{\beta_{c\bar{c}}\cdot(q_2-q_1)}{(q_1+q_2)^2_\perp + i0}. \tag{7.30}$$

According to [13] the new interaction should be thought of as giving rise to a correction to the Coulomb interaction, so it must scale like $\alpha_s v^4$. Recall that the fermion fields scale like $(mv)^{3/2}$. Therefore the transverse field $A_\perp$ must scale like mv and we were justified in keeping only the transverse component in the vertex because terms involving the longitudinal component $A_\parallel$ are subleading. So we conclude that *the potential gluon field $A_{\beta_{c\bar{c}},\tilde{q}}$ scales like the momentum that leaves the heavy-quark on-shell*:

$$A_\parallel \sim mv^2 \qquad \text{and} \qquad A_\perp \sim mv. \tag{7.31}$$

At one loop we don't need the self-interaction terms for the potential gluons. If we want to go beyond one loop we will need the effective Lagrangian for the potential field at the leading order in v to all orders in g_s. On the grounds of gauge invariance it must be the full QCD Lagrangian for the non-abelian gauge field in which the scaling (7.31) has been employed plus the action for ghosts following from the gauge fixing condition (7.27). The Lagrangian must be a usual trace of the square of the commutator of two covariant derivatives. Each derivative is written as a sum of the longitudinal and transversal components and only the leading terms in each components are retained:

$$i\mathcal{D}^{\mu} = i\mathcal{D}^{\mu}_{\parallel} + i\mathcal{D}^{\mu}_{\perp} = i\mathcal{D}^{\mu}_{us\,\parallel} - g_s A^{\mu}_{p\,\parallel} + \mathcal{P}^{\mu}_{\perp} - g_s A^{\mu}_{p\,\perp} \tag{7.32}$$

The spatial derivative in the transversal component is replaced by the label operator $\mathcal{P}_{\perp}$ and its ultrasoft component has been discarded. Also we don't keep the perpendicular component of the ultrasoft field here. From this expression one could see that at leading order in v only the longitudinal component of the ultrasoft field couples to the potential field and therefore the field redefinition (6.48) removes the ultrasoft field dependence from the potential gluons.

To conclude: in the covariant NRQCD the force binding the heavy $c\bar{c}$-pair is described by the Lagrangian (7.24) with the potential field $A_{\beta_{c\bar{c}},\tilde{q}}$ in a gauge that satisfies Eq. (7.26). One example is the covariant Coulomb gauge (7.27). We have constructed a covariant version of NRQCD formalism discussed in many papers (see e.g. [13] and [14]).

7.2.4 Feynman Rules in Covariant NRQCD

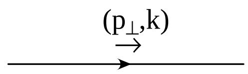
To do perturbative calculations with the Lagrangian (7.18) we need the quark/anti-quark propagator. Most straightforwardly it can be obtained from the full theory propagator by making the substitution $p \to m\beta + \tilde{p} + k$ and keeping only the leading terms in $v = p/m$ both in the numerator and denominator:

$$\frac{i\,P_{\beta_{c\bar{c}}}}{\beta\cdot k + \frac{\tilde{p}_{\perp}^{2}}{2m_c} + i0}. \tag{7.33}$$

Since we work with the four-component spinors we should keep the projector P_{β} as the Dirac structure of the propagator. Recall that in the canonical formulation of the NRQCD the propagator has no Dirac structure. The propagator for the charge conjugated spinor can be obtained from the Lagrangian (7.20) using canonical methods (see Chapter 8 for the derivation):

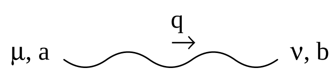
$$\frac{i\,P_{-\beta_{c\bar{c}}}}{-\beta\cdot k+\frac{\tilde{p}_\perp^2}{2m_c}+i0}. \tag{7.34}$$

The Feynman rules necessary for calculating one-loop diagrams in covariant NRQCD are shown in Figs. 7.6-7.9.



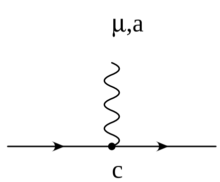
$$= \frac{i\,P_{\beta_{c\bar{c}}}}{\beta_{c\bar{c}}\cdot k+\frac{p_\perp^2}{2m_c}+i0} \quad \text{or} \quad = \frac{i\,P_{-\beta_{c\bar{c}}}}{-\beta_{c\bar{c}}\cdot k+\frac{\tilde{p}_\perp^2}{2m_c}+i0}$$

Figure 7.6: Propagator of c-quark and/or $\bar{c}$-antiquark in covariant NRQCD; $\beta_{c\bar{c}}$ is 4-velocity of COM of $c\bar{c}$ pair. The second expression is the propagator of the charge conjugated field η^C.

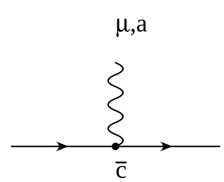


$$= \frac{-i\delta_{ab}}{\tilde{q}_\perp^2+i0}\beta_{c\bar{c},\,\mu}\beta_{c\bar{c},\,\nu}.$$

Figure 7.7: Propagator of potential gluon.



$$= \quad -ig_s\beta^\mu_{c\bar{c}}T^a$$



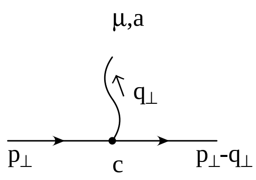
$$= \quad -ig_s\beta^\mu_{c\bar{c}}\overline{T}^a \quad \text{or} \quad ig_s\beta^\mu_{c\bar{c}}T^a$$

Figure 7.8: Interaction vertex of ultrasoft gluon and c-quark and/or $\bar{c}$-antiquark in covariant NRQCD. The last expression is the vertex for η^C field.

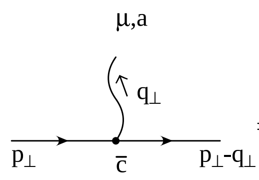
7.3 SCET Lagrangian

7.3.1 Lagrangian of Collinear Quarks

Now we have to work out the effective Lagrangian for the light s-quark. This is done by introducing the light-cone coordinates parameterized by vectors n



$= \quad -ig_s\beta^{\mu}_{c\bar{c}}T^a$



$= \quad -ig_s\beta^{\mu}_{c\bar{c}}\overline{T}^a \quad \text{or} \quad ig_s\beta^{\mu}_{c\bar{c}}T^a$

Figure 7.9: Interaction vertex of potential gluon and c-quark and/or $\bar{c}$-antiquark in covariant NRQCD. The last expression is for the interaction with η^C.

and $\bar{n}$, which satisfy $n^2 = \bar{n}^2 = 0$, and $n \cdot \bar{n} = 2$. For motion in the z-direction $n^\mu = (1, 0, 0, 1)$ and $\bar{n}^\mu = (1, 0, 0, -1)$. The momentum vector in light-cone coordinates becomes

$$p^\mu = p^- \frac{n^\mu}{2} + (p_\perp)^\mu + p^+ \frac{\bar{n}^\mu}{2}, \tag{7.35}$$

where $p^+ = n \cdot p$ and $p^- = \bar{n} \cdot p$. The on-shell condition then reads $p^+p^- + p_\perp^2 = 0$. At large energies the light-cone components are widely separated and can be assigned power counting according to $p^- \sim \lambda^0$, $p_\perp \sim \lambda$, and $p^+ \sim \lambda^2$. The effective theory that describes this kinematic domain is called the soft-collinear effective theory (SCET).

The tree-level Lagrangian for the soft-collinear quark can be obtained by expanding the full theory Lagrangian in powers of λ. The derivation is similar to that of covariant NRQCD. We start with the QCD Lagrangian for a massless quark:

$$\mathcal{L}_q = \bar{\psi} i \not{D} \psi. \tag{7.36}$$

Here $D^\mu = (\partial^0 + ig_s A^0, -\nabla + ig_s \mathbf{A})$. The first step is to split the quark momentum as

$$p = \frac{1}{2}(\bar{n} \cdot p)n + p_\perp + k, \tag{7.37}$$

where the components scale like λ^0, λ, and λ^2, respectively. The momentum $k \sim \lambda^2$ stands for the residual momentum. To derive the ET Lagrangian we subtract the largest component $p = \frac{1}{2}(\bar{n} \cdot p)n + p_\perp$ and put the label n, p on the fields:

$$\psi(x) = \psi_{us}(x) + \sum_{p \neq 0} e^{-i(\frac{1}{2}(\bar{n} \cdot p) + p_\perp)x} \psi_{n,p}(x) \tag{7.38}$$

Note that zero bin $p = 0$ corresponds to the ultrasoft degrees of freedom and is excluded from the collinear part of the field. The residual component k is of the order of Λ_c and gives the position dependence to the field $\psi_{n,p}(x)$. Substituting Eq. (7.38) into Eq. (7.36) we obtain:

$$\begin{aligned}
&\mathcal{L}_q = \mathcal{L}_c + \mathcal{L}_{cus} + \mathcal{L}_{us}, \qquad \text{where} \\
&\mathcal{L}_c = \sum_{p,p'\neq 0} e^{i(p'-p)x}\bar{\psi}_{n,p'}\left[(\bar{n}\cdot p + i\bar{n}\cdot D)\frac{\not{n}}{2} + (\not{p}_\perp + i\not{D}_\perp) + (in\cdot D)\frac{\not{\bar{n}}}{2}\right]\psi_{n,p}, \\
&\mathcal{L}_{cus} = \sum_{p\neq 0} e^{ipx}\bar{\psi}_{n,p}\left[(i\bar{n}\cdot D)\frac{\not{n}}{2} + (i\not{D}_\perp)\right]\psi_{us} + h.\,c., \\
&\mathcal{L}_{us} = \bar{\psi}_{us} i\not{D}\psi_{us}.
\end{aligned} \tag{7.39}$$

In $\mathcal{L}_{cus}$ we keep only the terms which contain collinear gluon fields, so that momentum conservation is preserved.

The collinear quark field $\psi_{n,p}$ can be split into the large and small components $\xi_{n,\,p}$ and $\xi_{\bar{n},\,p}$, respectively, by means of the projection operators

$$P_n = \frac{\not{n}\not{\bar{n}}}{4}, \quad \text{and} \quad P_{\bar{n}} = \frac{\not{\bar{n}}\not{n}}{4}, \quad \text{so that} \quad \xi_{n,\,p} = P_n\psi_{n,\,p}, \quad \text{and} \quad \xi_{\bar{n},\,p} = P_{\bar{n}}\psi_{\bar{n},\,p}. \tag{7.40}$$

One can see that the spinors $\xi_{n,\,p}$ and $\xi_{\bar{n},\,p}$ satisfy

$$\not{n}\,\xi_{n,\,p} = 0 \qquad \text{and} \qquad \not{\bar{n}}\,\xi_{\bar{n},\,p} = 0. \tag{7.41}$$

Now we write down the Lagrangian (7.39) in terms of these spinors. The following identities involving the projection operators will be useful:

$$P_n\not{n}P_{\bar{n}} = \not{n}, \qquad P_{\bar{n}}\not{\bar{n}}P_n = \not{\bar{n}}, \qquad P_n\not{X}_\perp P_{\bar{n}} = 0, \quad \text{and} \quad P_{\bar{n}}\not{X}_\perp P_n = 0. \tag{7.42}$$

The collinear $\mathcal{L}_c$ and collinear-ultrasoft Lagrangians written in terms of ET spinors are:

$$\begin{aligned}
\mathcal{L}_c = \sum_{p,\,p'\neq 0}&\Big[\bar{\xi}_{\bar{n},\,p'}(\bar{n}\cdot p + i\bar{n}\cdot D)\frac{\not{n}}{2}\xi_{\bar{n},\,p} + \bar{\xi}_{n,\,p'}(\not{p}_\perp + i\not{D}_\perp)\xi_{\bar{n},\,p} \\
&+\bar{\xi}_{\bar{n},\,p'}(\not{p}_\perp + i\not{D}_\perp)\xi_{n,\,p} + \bar{\xi}_{n,\,p'}(in\cdot D)\frac{\not{\bar{n}}}{2}\xi_{n,\,p}\Big]. \\
\mathcal{L}_{cus} = \sum_{p\neq 0}&\left[\bar{\xi}_{\bar{n},\,p}(i\bar{n}\cdot D)\frac{\not{n}}{2} + \bar{\xi}_{n,\,p}(i\not{D}_\perp) + \bar{\xi}_{\bar{n},\,p}(i\not{D}_\perp)\right]\psi_{us} + h.\,c.
\end{aligned} \tag{7.43}$$

At large energies the term $\bar{n} \cdot p$ dominates over the derivative in the collinear Lagrangian. The collinear component of the gluon field could in principle cancel it but it is multiplied by g_s which is small in the perturbative regime. Therefore in this kinematic domain the field $\xi_{\bar{n},p}$ ceases to be dynamical and can be integrated out by using the EOM:

$$\begin{aligned} \frac{\partial \mathcal{L}_q}{\partial \bar{\xi}_{\bar{n},p'}} &= (\bar{n} \cdot p + i\bar{n} \cdot D)\frac{\not{n}}{2}\xi_{\bar{n},p} + (\not{p}_\perp + i\not{D}_\perp)\xi_{n,p} = 0, \\ \text{or} \qquad (\bar{n} \cdot p + i\bar{n} \cdot D)\xi_{\bar{n},p} &= (\not{p}_\perp + i\not{D}_\perp)\frac{\not{\bar{n}}}{2}\xi_{n,p} \end{aligned} \tag{7.44}$$

The first step in deriving the SCET Lagrangian is to use Eq. (7.44) and write down Eq. (7.43) in terms of the spinor $\xi_{n,p}$ only:

$$\mathcal{L}_q = \sum_{p,p'} \bar{\xi}_{n,p'} \left[in \cdot D + (\not{p}_\perp + i\not{D}_\perp)\frac{1}{\bar{n} \cdot p + i\bar{n} \cdot D}(\not{p}_\perp + i\not{D}_\perp)\right]\frac{\not{\bar{n}}}{2}\xi_{n,p}. \tag{7.45}$$

The kinematics of energetic light quarks is more peculiar than that of the heavy quarks. An energetic quark with momentum p_1 can emit a collinear energetic gluon with momentum q and remain on-shell, as it follows from the simple relation, where each term is of the order $\sim \lambda^2$:

$$p_1^2 = (p_2 + q)^2 = p_2^2 + q^2 + (p_2 \cdot \bar{n})(q \cdot n) + 2p_\perp q^\perp + (p_2 \cdot n)(q \cdot \bar{n}). \tag{7.46}$$

To account for the possibility of collinear gluon emission we have to split the gluon field into collinear and ultrasoft parts: $A^\mu = A^\mu_c + A^\mu_{us}$. The light-cone components of A^μ_c scale like the components of the collinear momentum (7.37) $A^\mu_c \sim (\lambda^0, \lambda, \lambda^2)$ and the components of A^μ_{us} scale like λ^2. Collinear gluons can change quark labels, so we have to single them out. It is done by putting labels on gluon fileds, so that $A^\mu_c \to e^{-i(\frac{1}{2}(\bar{n}\cdot q)+q_\perp)}A_{n,q}(x)$. Under the ultrasoft gauge transformations collinear gluons transform homogeneously like matter fields without a derivative term. Physically this means that we can treat collinear gluons as matter fields propagating on the ultrasoft background. The Lagrangian (7.45) then becomes:

$$\begin{aligned} \mathcal{L}_q = \sum_{p,p',q} \bar{\xi}_{n,p'} \Big[& in \cdot D - g_s n \cdot A_{n,q} + (\not{p}_\perp + i\not{D}_\perp - g_s \not{A}^\perp_{n,q}) \times \\ & \times \frac{1}{\bar{n} \cdot p + i\bar{n} \cdot D - g_s \bar{n} \cdot A_{n,q}}(\not{p}_\perp + i\not{D}_\perp - g_s \not{A}^\perp_{n,q})\Big]\frac{\not{\bar{n}}}{2}\xi_{n,p}. \end{aligned} \tag{7.47}$$

where conservation of label momenta is assumed.

Expanding the denominator in Eq. (7.47) gives the SCET Lagrangian as a series expansion in powers of λ similarly to that of the covariant NRQCD Lagrangian that is expanded in powers of v. The elegant way to ensure momentum conservation is to treat $\bar{n}\cdot p$ as the momentum operator $\bar{\mathcal{P}}$ applied to the labels of field operators on the right. Then up to $O(\lambda)$ terms:

$$\mathcal{L}_{sc} = \bar{\xi}_{n,p}\left[in\cdot D + \frac{p_\perp^2}{\bar{n}\cdot p}\right]\frac{\bar{\not{n}}}{2}\xi_{n,p} - g_s\,\bar{\xi}_{\bar{n},p+q}\Big[n\cdot A_{n,q} + \\ + \not{A}^\perp_{n,q}\frac{\not{p}_\perp}{\bar{n}\cdot p} + \frac{\not{p}_\perp + \not{q}_\perp}{\bar{n}\cdot(p+q)}\not{A}^\perp_{n,q} - \frac{\not{p}_\perp + \not{q}_\perp}{\bar{n}\cdot(p+q)}\bar{n}\cdot A_{n,q}\frac{\not{p}_\perp}{\bar{n}\cdot p}\Big]\frac{\bar{\not{n}}}{2}\xi_{n,p}. \quad (7.48)$$

7.3.2 Lagrangian of Collinear Gluons

The components of the collinear gluon field A_c scale like the components of collinear momentum: $\bar{n}\cdot A_c \sim 1$, $A_\perp \sim \lambda$, and $n\cdot A_c \sim \lambda^2$. The Lagrangian for collinear gluons must remain invariant under the soft-collinear gauge transformations which preserve power counting. This can be done by introducing the covariant derivative

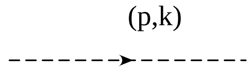
$$i\mathcal{D}^\mu = \frac{n^\mu}{2}\bar{\mathcal{P}}^\mu + \mathcal{P}_\perp^\mu + \frac{\bar{n}^\mu}{2}i\,n\cdot D \qquad \text{where} \qquad iD^\mu = i\partial^\mu - g_s A^\mu_{us} \quad (7.49)$$

and writing the QCD Lagrangian in the background gauge with collinear gluons as a matter field and ultrasoft gluons as a background field. For completeness we write here the Lagrangian in λ^0 order, although for our purposes it will be sufficient to retain only the terms of the order g_s^0:

$$\mathcal{L}_c = \frac{1}{2g_s^2}\mathrm{Tr}\left\{\left[i\mathcal{D}^\mu - g_s A^\mu_{n,q}, i\mathcal{D}^\nu - g_s A^\nu_{n,q}\right]\right\}^2 \\ +2\mathrm{Tr}\left\{\bar{c}_{n,p'}\left[i\mathcal{D}_\mu\left[i\mathcal{D}^\mu - g_s A^\mu_{n,q}, c_{n,p}\right]\right]\right\} + \mathrm{Tr}\left\{\left[i\mathcal{D}_\mu, A^\mu_{n,q}\right]\right\}^2 \quad (7.50)$$

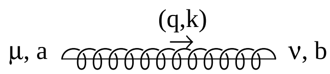
7.3.3 Feynman Rules of SCET

In this section we give Feynman rules for the SCET that are sufficient for one-loop calculation.



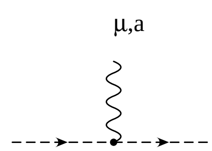
$$= \quad i\frac{\not{n}}{2}\frac{\bar{n}\cdot p}{n\cdot k\bar{n}\cdot p + p_\perp^2 + i0}$$

Figure 7.10: Propagator of collinear quark; n is the light vector of the s-quark.



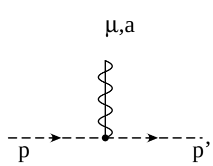
$$= \frac{-ig_{\mu\nu}\delta_{ab}}{\bar{n}\cdot q\, n\cdot k + q_\perp^2 + i0}.$$

Figure 7.11: Propagator of collinear gluon in Feynman gauge.



$$= \quad -ig_s T^a n^\mu \frac{\not{\bar{n}}}{2}$$

Figure 7.12: Interaction vertex of the ultrasoft gluon and collinear quark.



$$= -ig_s T^a \left[n^\mu + \frac{\gamma_\perp^\mu \not{p}_\perp}{\bar{n}\cdot p} + \frac{\not{p}'_\perp \gamma_\perp^\mu}{\bar{n}\cdot p'} - \frac{\not{p}'_\perp \not{p}_\perp}{\bar{n}\cdot p\bar{n}\cdot p'}\bar{n}^\mu \right] \frac{\not{\bar{n}}}{2}$$

Figure 7.13: Interaction vertex of the collinear gluon and collinear quark.

8

Technical Notes

In this Chapter we give some details of the derivations omitted in the body of the dissertation.

8.1 Transformation of Operator Basis

In this section we present without derivation (which is technically complicated and will be given in the upcoming paper) the relation used to calculate the Wilson coefficients $C_0^{NLO}(m_b)$ and $C_8^{NLO}(m_b)$ given in Eq. (4.8).

The Wilson coefficients for the set of operators

$$P_1 = [\bar{s}\gamma_\mu P_L T^a c][\bar{c}\gamma^\mu P_L T^a b], \qquad P_2 = [\bar{s}\gamma_\mu P_L c][\bar{c}\gamma^\mu P_L b]. \tag{8.1}$$

have been calculated in [9] at the NNLO in α_s. We need however the Wilson coefficients in the different operator basis

$$\mathcal{O}_1 \equiv [\bar{s}\gamma_\mu P_L T^a b][\bar{c}\gamma^\mu P_L T^a c], \qquad \mathcal{O}_2 \equiv [\bar{s}\gamma_\mu P_L b][\bar{c}\gamma^\mu P_L c]. \tag{8.2}$$

At the LO the two bases are related by a linear transformation which follows when one applies the Fierz transformations in the color and spinor spaces:

$$\begin{pmatrix} \mathcal{O}_1 \\ \mathcal{O}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{4}{9} \\ 2 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}. \tag{8.3}$$

At the NLO the one-loop corrections have to be taken into account and that brings complications due to the so-called evanescent operators. The dimensional regularization used when calculating Feynman integrals doesn't have an unambiguous definition for the matrix γ_5, which is essentially a four-dimensional object. There is a systematic way of coping with this difficulty (see, e.g. [23]) by introducing the infinite set of the operators vanishing in the limit $D = 4$. Working out the relation between the bases (8.1) and (8.2) at the NLO requires calculating six more diagrams involving the evanescent operators. The relation is given by:

$$
\begin{aligned}
\vec{C}'(\mu) &= \begin{pmatrix} -\frac{1}{3} & 2 \\ \frac{4}{9} & \frac{1}{3} \end{pmatrix} \left[\begin{pmatrix} C_1^{(0)}(\mu) \\ C_2^{(0)}(\mu) \end{pmatrix} + \frac{\alpha_s(\mu)}{4\pi} \begin{pmatrix} C_1^{(1)}(\mu) \\ C_2^{(1)}(\mu) \end{pmatrix} \right] \\
&+ \frac{\alpha_s(\mu)}{4\pi} \begin{pmatrix} 0 & 6 \\ -\frac{4}{3} & 0 \end{pmatrix} \begin{pmatrix} C_1^{(0)}(\mu) \\ C_2^{(0)}(\mu) \end{pmatrix}.
\end{aligned} \tag{8.4}
$$

Here $\vec{C}'(\mu)$ are the Wilson coefficients for the basis (8.2) and $\vec{C}(\mu)$ for the basis (8.1). The first matrix in Eq. (8.4) is the transposed matrix in Eq. (8.3) corresponding to the tree-level transformation of the operator basis. The second matrix in Eq. (8.4) is the contribution of the evanescent operators.

The explicit expressions for the Wilson coefficients $\vec{C}(\mu)$ can be found in Eqs. (39)-(44) in [9]. Substituting them into Eq. (8.4) gives the expressions for $C_0^{NLO}(m_b)$ and $C_8^{NLO}(m_b)$ whose numerical values are presented in Eq. (4.8).

8.2 Effective Theory Amplitude

In this section we present the results of one-loop calculation of the ET amplitude for the decay $b \to (c\bar{c})s$. Feynman diagrams have been calculated in the MS-scheme. The momenta of the four external particles have been set on-shell and the corresponding IR divergences have been regulated with dimensional regularization. Technically all ET diagrams calculated according to this prescription are zero because the corresponding Feynman integrals are scaleless. However, if the off-shellness of the external particles is kept the amplitude is non-zero and after the counterterm removes the UV-divergence, the resulting IR-diverging terms reproduce exactly the IR-diverging terms of the FT amplitude. The latter is expected because the ET is the low-energy limit of the FT.

The above observation suggests that *the UV-divergences of an ET amplitude are equal to the IR-divergences of the FT amplitude with negative sign when momenta of external particles are set on-shell and dimensional regularization is used*. By itself this is an elegant technical trick but there is also a clear physical interpretation behind it. The IR-singularities of the FT amplitudes are actually special kinematic domains where the number of degrees of freedom is limited compared to the FT and the dynamics is simplified. For example, non-realitivistic quantum electrodynamics (NRQED) is a special limit of QED when the speed of light goes to infinity [14]. In a sense this is the limit of Newtonian physics.

8.2.1 ET Diagrams

Overall there are eight amputated diagrams in the ET which contribute to the decay amplitude at one loop. Seven of them are due to loop corrections to the zero order term in the g_s expansion of the tree-level Lagrangian (4.9):

$$\mathcal{L}^{(0)} = [\bar{\xi}_{n,p}\Gamma_j \mathbf{C}\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j \mathbf{C}\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}]. \tag{8.5}$$

Six diagrams where the ultrasoft gluon is being exchanged between the ET quarks are shown in Fig. 8.1. The seventh diagram corresponds to the potential gluon exchange between the quarks of the ($c\bar{c}$)-pair and is shown in Fig. 8.2. The last diagram is a one-loop correction to the first order term in the g_s expansion of (4.9),

$$\mathcal{L}^{(1),col} = -g_s[\bar{\xi}_{n,p}\frac{(\bar{n}\cdot A_{n,q})}{(\bar{n}\cdot q)}\Gamma_j \mathbf{C}\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j \mathbf{C}\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{8.6}$$

due to collinear gluon exchange. It is shown in Fig. 8.3. The first order correction in g_s due to the potential gluon,

$$\mathcal{L}^{(1),p} = -g_s[\bar{\xi}_{n,p}\Gamma_j \mathbf{C} h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j[\frac{n\cdot A_{\tilde{q}}}{n\cdot q_\perp}\,, \mathbf{C}]\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{8.7}$$

doesn't give rise to a one-loop correction because the corresponding Feynman integral vanishes identically.

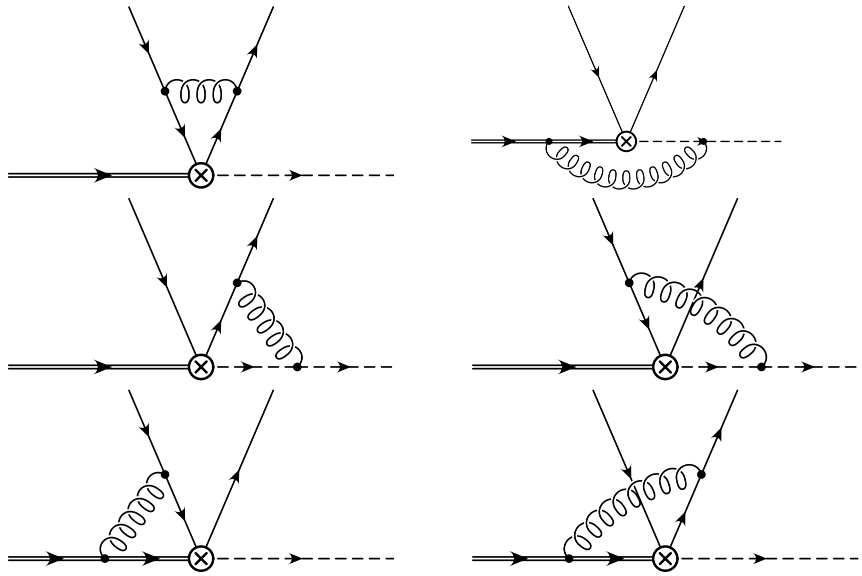


Figure 8.1: Diagrams with the ultrasoft gluon exchange in ET.

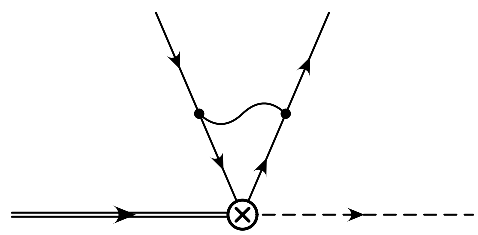


Figure 8.2: The potential gluon exchange between the quarks of $c\bar{c}$-pair.

8.2.2 ET Amplitude

The non-renormalized amplitude is given by the sum of the diagrams in Figs. 8.1, 8.2, and 8.3. The UV-divergences in the non-renormalized amplitude are then removed by the field renormalization factors for three heavy and one light quarks, Z_H and Z_l, and by the renormalization matrix Z_{ij}, where $i, j = 0, 8$ stands

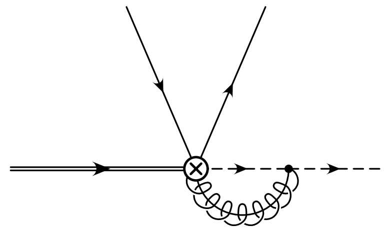


Figure 8.3: Correction due to collinear gluon.

for the singlet and octet operators:

$$\begin{aligned}
Z_H &= 1+\frac{\alpha_s}{4\pi}2C_F\frac{1}{\varepsilon_{UV}}, \qquad Z_l = 1-\frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon_{UV}}, \qquad C_F=\frac{4}{3},\\
Z_{00} &= 1+\frac{\alpha_s}{4\pi}C_F\left\{\frac{1}{\varepsilon_{UV}^2}+\frac{2}{\varepsilon_{UV}}\ln\frac{\mu}{2E_s}+\frac{5}{2}\frac{1}{\varepsilon_{UV}}\right\}, \qquad Z_{08}=1, \qquad Z_{80}=1,\\
Z_{88} &= 1+\frac{\alpha_s}{4\pi}\left\{C_F\left[\frac{1}{\varepsilon_{UV}^2}+\frac{2}{\varepsilon_{UV}}\ln\frac{\mu}{2E_s}\right]+\frac{6}{\varepsilon_{UV}}\frac{m_b}{2E_s}\ln\frac{2m_c}{m_b}+\frac{3\pi i}{\varepsilon_{UV}}+\frac{19}{3}\frac{1}{\varepsilon_{UV}}\right\},
\end{aligned} \tag{8.8}$$

Here $E_s = \frac{m_b^2-4m_c^2}{2m_b}$ is the energy of s-quark in the b-quark frame. Finally the MS-renormalized amplitude is multiplied by the LSZ-factor to get the expression corresponding to the on-shell renormalized ET amplitude:

$$\begin{aligned}
&R_H = 1-\frac{\alpha_s}{4\pi}2C_F\frac{1}{\varepsilon_{IR}}, \qquad R_l = 1+\frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon_{IR}},\\
&\Longrightarrow \sqrt{R_H^3\cdot R_l} = 1-\frac{\alpha_s}{4\pi}\frac{5}{2}C_F\frac{1}{\varepsilon_{IR}}.
\end{aligned} \tag{8.9}$$

The resulting expression is given by:

$$\sum_{i=0,8} c_i[\bar{\xi}_{n,p}\Gamma_j\mathbf{C}_i\, h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma_j\mathbf{C}_i\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}] \qquad \text{where} \tag{8.10}$$

$$\begin{aligned}
c_0 &= 1+\frac{\alpha_s}{4\pi}C_F\left\{-\frac{1}{\varepsilon_{IR}^2}-\frac{2}{\varepsilon_{IR}}\ln\frac{\mu}{2E_s}-\frac{5}{2}\frac{1}{\varepsilon_{IR}}+\frac{2\pi i}{v}\left[-\frac{1}{\varepsilon_{IR}}-\ln\frac{\mu^2}{m_c^2v^2}-i\pi\right]\right\},\\
c_8 &= 1+\frac{\alpha_s}{4\pi}\left\{C_F\left[-\frac{1}{\varepsilon_{IR}^2}-\frac{2}{\varepsilon_{IR}}\ln\frac{\mu}{2E_s}\right]-\frac{6}{\varepsilon_{IR}}\frac{m_b}{2E_s}\ln\frac{2m_c}{m_b}-\frac{3\pi i}{\varepsilon_{IR}}-\frac{19}{3}\frac{1}{\varepsilon_{IR}}\right.\\
&\left.-\frac{\pi i}{3v}\left[-\frac{1}{\varepsilon_{IR}}-\ln\frac{\mu^2}{m_c^2v^2}-i\pi\right]\right\}.
\end{aligned} \tag{8.11}$$

At one loop there is no mixing between the singlet and the octet operators: $Z_{08} = Z_{80} = 0$. One can see why by writing down the color structure of the diagrams that can potentially mix the singlet and the octet. These are the four last diagrams in Fig. 8.1, the rest of the ET diagrams obviously contribute independently to the renormalization of $(\bar{s}b)$ and $(c\bar{c})$ currents. The interaction vertices for the c and $\bar{c}$ quarks have different signs as can be seen in Fig. 7.8 and therefore the color structure of the sum of the third and the fourth and/or the fifth and the sixth diagrams in Fig. 8.1 will be:

$$T^a\mathbf{C} \otimes [T^a, \mathbf{C}] \qquad \text{and} \qquad \mathbf{C}T^a \otimes [T^a, \mathbf{C}]. \tag{8.12}$$

Using color identities it is not difficult to check that both structures are proportional to $\mathbf{C} \otimes \mathbf{C}$ both for the singlet and the octet operators and therefore do not mix them.

There is no reason to believe that this pattern will hold at the higher orders in α_s. For example, the finite parts in the FT amplitude at one loop (which are precisely the coefficients $C_{ij;k}^{ET,(0)}$) do mix singlet and octet operators suggesting that already at the NLO singlet and octet contributions become entangled. As we have already shown in Chapter 5 the entanglement at $\mu = m_b$ is of the same order of magnitude as the LO coefficient.

8.2.3 Anomalous Dimension at the LO

In this section we derive the expression for the anomalous dimension (AD) at the LO and solve the RG equation to get the RG improved Wilson coefficient at the LO (see Eq. (5.1)).

The renormalization matrix Z_{ij} in Eq. (8.8) is diagonal and contains terms $1/\varepsilon^2$, $(1/\varepsilon)\log\mu$, and $1/\varepsilon$. The terms proportional to $1/\varepsilon$ are subleading and will contribute to the AD at NLO (see [11]). One of them is imaginary indicating that it should be canceled by a term coming from the two-loop diagrams. Therefore at LO the renormalization factors $Z_{00} = Z_{88}$ are the same and equal to the renormalization factor for the $(\bar{s}b)$-current (see [11]):

$$Z_{00} = Z_{88} = 1 + \frac{\alpha_s(\mu)}{4\pi} C_F \left[\frac{1}{\varepsilon_{UV}^2} + \frac{2}{\varepsilon_{UV}} \ln \frac{\mu}{2E_s} \right]. \tag{8.13}$$

This observation can be understood as an independent check of the factorization at one loop for the singlet ET Lagrangian discussed in 4.3.3. At one loop the $c\bar{c}$-current doesn't get renormalizied and therefore the renormalization factor is completely due to the renormalization of the $(\bar{s}b)$-current only.

The anomalous dimension is given by

$$\gamma(\mu) = Z^{-1} \mu \frac{d}{d\mu} Z = -\frac{\alpha_s(\mu)}{\pi} C_F \ln \frac{\mu}{m_b} + o(1), \tag{8.14}$$

where in addition to Eq. (8.13) we have used the explicit expression for the β-function, $\beta(g_s) = -\varepsilon g_s + o(g_s^3)$ and replaced $2E_s \to m_b$. Now we use Eq. (8.14) to solve the RG equation for the Wilson coeffcients $C_j^{ET,(0)}(\mu/m_b, r)$. The RG equation is the same for all Dirac structures $j = 1, \ldots, 4$ and has the form

$$\mu \frac{d}{d\mu} C(\mu) = \gamma(\mu) C(\mu), \tag{8.15}$$

with the $C_j^{ET,(0)}(\mu = m_b, r)$ given by Eq. (5.2) as initial conditions. When solving this equation we use the explicit expression for the running coupling constant α_s at one loop (see Eq. (4.6)). The result is:

$$C(\mu) = C(m_b) \exp \left(\frac{4\pi C_F}{\beta_0^2 \alpha_s(m_b)} \left[\ln \frac{\alpha_s(m_b)}{\alpha_s(\mu)} + 1 - \frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right] \right). \tag{8.16}$$

The last two terms in the exponent probably exceed the logarithmic accuracy of the calculation and at the LO should be discarded. Then keeping only the large logarithm we can write down the RG improved values for the Wilson coefficients in the ET at the LO as

$$C_j^{ET}(\mu) = C_j^{ET}(m_b) \left[\frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right]^{\frac{4\pi C_F}{\beta_0 \alpha_s(m_b)}}. \tag{8.17}$$

8.3 Full Theory Amplitude

The FT amplitude is given by the sum of six diagrams which look exactly like those in Fig. (8.1) except that all four quark propagators are represented by the same solid lines. Calculating the FT diagrams is much more difficult because now we have to deal with the full quark propagators which depend on the mass of the particle and have a non-trivial Dirac structure. Unlike in the ET the interaction vertices in the FT change the Dirac structure of a diagram. The diagrams are calculated with the external particle momenta set on-shell in the MS-renormalization scheme. The expression for the FT amplitude follows when one multiplies the sum of the six diagrams by the field renormalization factors, the operator renormalization matrix, and the LSZ factors to get the on-shell value of the amplitude. These factors are given below:

$$\begin{aligned}
&Z_q = 1 - \frac{1}{\varepsilon_{UV}} C_F, \\
&Z_{00} = 0, \quad Z_{08} = -\frac{4}{3\varepsilon_{UV}}, \quad Z_{08} = -\frac{6}{\varepsilon_{UV}}, \quad Z_{88} = \frac{2}{\varepsilon_{UV}}, \\
&\sqrt{R_b R_c^2 R_s} = -\frac{10}{3\varepsilon_{UV}} - 2\ln\frac{\mu^2}{m_b^2} - 4\ln\frac{\mu^2}{m_c^2} - 8,
\end{aligned} \tag{8.18}$$

The explicit expression for the FT amplitude is given by the sum of the IR-divergent pieces (see 8.11) which are the same both for the ET and FT amplitudes and the finite piece which is given by Wilson coefficients $C_{ij:k}^{ET,(1)}$ in Chapter 5. The next section explains how the Wilson coefficients are obtained and would help to restore the expression for the FT amplitude if one needs it.

8.4 Matching Procedure

In this section we give a schematic outline of the matching procedure at one loop which allows one to determine the Wilson coefficients in the ET at the first order in α_s.

8.4.1 Tree-Level Matching

On the FT side we have the ME which at the leading order in the ET power expansion can be schematically written as

$$A_i^{(0)}\langle O_i\rangle. \tag{8.19}$$

This is basically Eq. (5.1), where $A_i^{(0)}$ stands for the product $C_i(m_b)\, C_j^{ET,(0)}(1,\, r)$ with i being a cumulative index for (i, j) and $\langle O_i\rangle$ stands for the Dirac-color structure. At tree-level the Wilson coefficients of the ET coincide with the Wilson coefficients of the FT. This is how we obtained Eq.(5.1) in the first place. So, the tree-level matching is simply:

$$A_i^{(0)}\langle O_i\rangle = C_i^{(0)}\langle O_i\rangle \qquad \Longrightarrow \qquad A_i^{(0)} = C_i^{(0)}, \tag{8.20}$$

where $C_i^{(0)}$ are the Wilson coefficients of the ET at the leading order in α_s. We used them as the initial values for the RG equations when deriving the RG-improved ET Lagrangian at the LO.

8.4.2 One-Loop Matching

At one loop the ME on the FT side upon expanding the Dirac structures in terms of the ET structures becomes:

$$[A_i^{(0)} + \frac{\alpha_s}{4\pi} A_i^{(1)}]\langle O_i\rangle. \tag{8.21}$$

On the ET side the ME now has the form:

$$\left[C_i^{(0)} + \frac{\alpha_s}{4\pi} C_i^{(1)}\right]\left[\langle O_i\rangle + \frac{\alpha_s}{4\pi} X_{ij}^{(1)}\langle O_j\rangle\right], \tag{8.22}$$

where $C_i^{(1)}$ is the first order correction to the Wilson coefficient and the second bracket is the ME in the ET at first order in α_s as it comes from the ET diagrams at one-loop. Equating (8.21) and (8.22) gives the matching condition at one-loop:

$$C_i^{(1)} = A_i^{(1)} - C_j^{(0)}\, X_{j\,i}^{(1)} = A_i^{(1)} - A_j^{(0)}\, X_{j\,i}^{(1)}. \tag{8.23}$$

The coefficients $X^{(1)}_{j\,i}$ are IR divergent and don't contain any finite parts, they are given by Eq. (8.11). The coefficients $A^{(1)}$ contain both the finite part and the IR divergences originated by the same prescription. Cancellation of the IR divergences between the full and the effective theory amplitudes serves as an independent check of the calculation.

8.5 Effective Theory Dirac Structures

In this section we derive the reduction formulae necessary to project the ME of the FT into the ME of the ET. The generic Dirac structure of the ME on the FT side has the form

$$[\bar{s}\,\Gamma_1\,b]\,[\bar{c}\,\Gamma_2\,c]\,, \tag{8.24}$$

where $\Gamma_{1,2}$ is one of the following products of Dirac matrices

$$\Gamma = \{1;\ \gamma_5;\ \gamma^\mu;\ \gamma^\mu\gamma_5;\ \gamma^\mu\gamma^\nu;\ \gamma^\mu\gamma^\nu\gamma_5;\ \gamma^\mu\gamma^\nu\gamma^\eta;\ \gamma^\mu\gamma^\nu\gamma^\eta\gamma_5\}\,. \tag{8.25}$$

Here the last four entries come from loop corrections to the tree-level FT operator. On the ET side the generic ME is

$$[\bar{\xi}_{n,p}\Gamma^{ET}_{sb}\,h_{\beta_b}][\bar{\xi}_{\beta_{c\bar{c}}\,\tilde{p}_\perp}\Gamma^{ET}_{c\bar{c}}\,\eta^C_{\beta_{c\bar{c}}\,-\tilde{p}_\perp}], \tag{8.26}$$

where Γ^{ET}_{sb} and $\Gamma^{ET}_{c\bar{c}}$ are the set of operators which form the complete set on the corresponding subspace of Dirac spinors.

In this section we derive the relations between the ME (8.24) of the FT side and the ME (8.26) on the ET side. Firstly, for $[\bar{c}\ldots c]$, then for $[\bar{s}\ldots b]$, and finally for their dot-products $[\bar{s}\ldots b][\bar{c}\ldots c]$.

8.5.1 Reduction Formulae for $[\bar{c}\ldots c]$

The Dirac structures $\Gamma^{ET}_{c\bar{c}}$ follow by sandwiching the FT structures in Eq. (8.25) between the projectors $P_{\beta_{c\bar{c}}}$ and $P_{-\beta_{c\bar{c}}}$

$$P_{\beta_{c\bar{c}}}\Gamma P_{-\beta_{c\bar{c}}} \to \Gamma^{ET}_{c\bar{c}}\,. \tag{8.27}$$

The calculations are straightforward and the results are summarized below. The structure on the left comes from the FT side and the structure on the right is it's ET counterpart. Below β stands for $\beta_{c\bar{c}}$.

$$\begin{aligned}
1 &\to 0, \\
\gamma_5 &\to \gamma_5, \\
\gamma^\mu &\to \gamma_\perp^\mu \qquad \text{where} \qquad \gamma_\perp^\mu = \gamma^\mu - \beta^\mu \not{\beta}, \\
\gamma^\mu \gamma_5 &\to \beta^\mu \gamma_5, \\
\gamma^\mu \gamma^\nu &\to \beta^\mu \gamma_\perp^\nu - \beta^\nu \gamma_\perp^\mu.
\end{aligned} \tag{8.28}$$

This is a complete set of Dirac structures in $D = 4$. It follows that any of the sixteen basic Dirac structures of the FT in $D = 4$ is a linear combination of the four Dirac structures in the ET, so

$$\Gamma_{c\bar{c}}^{ET} = \{\gamma_5; \gamma_\perp^\mu\}, \qquad \text{where} \qquad \gamma_\perp^\mu = \gamma^\mu - \beta^\mu \not{\beta}. \tag{8.29}$$

The remaining structure

$$\gamma^\mu \gamma^\nu \gamma_5 \to g^{\mu\nu} \gamma_5 + i\varepsilon^{\mu\nu\rho\eta} \beta_\rho \gamma_{\eta,\perp}, \tag{8.30}$$

where $\varepsilon^{0123} = 1$ and $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, follows from the identity $\sigma^{\mu\nu} = \frac{i}{2}\varepsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\gamma_5$ valid in $D = 4$ and the last line of (8.28). Eqs. (8.28) and (8.30) complete the necessary list of the relations between the MEs of the FT and the ET for the $[\bar{c} \ldots c]$ current.

8.5.2 Reduction Formulae for $[\bar{s} \ldots b]$

Working out the Dirac structures Γ_{sb}^{ET} requires more work because now the original Dirac structure from the list (8.25) is sandwiched between the two projectors for different subspaces: $P_{\bar{n}}$ and P_{β_b}. The most straightforward way to work out the necessary relations is to use the "orthogonality" relation Eq. (17) from [22] that reads

$$\Gamma_{sb}^{ET} = \frac{\not{\bar{n}}}{2} \operatorname{Tr}\left[\frac{\not{n}}{2} P_{\bar{n}} \Gamma P_{\beta_b}\right] - \frac{\not{\bar{n}}}{2}\gamma_5 \operatorname{Tr}\left[\frac{\not{n}}{2}\gamma_5 P_{\bar{n}} \Gamma P_{\beta_b}\right] + \gamma_\perp^\mu \operatorname{Tr}\left[\gamma_\mu^\perp P_{\bar{n}} \Gamma P_{\beta_b}\right]. \tag{8.31}$$

The formulae below summarize the results. The symbol β stands for β_b.

$$
\begin{aligned}
1 \quad &\to (\beta\cdot n)\frac{\not{\bar{n}}}{2} + \not{\beta}_\perp, \\
\gamma_5 \quad &\to -(\beta\cdot n)\frac{\not{\bar{n}}}{2}\gamma_5 + \frac{i}{2}\gamma_\perp^\mu \varepsilon_{\mu\nu\eta\rho}\bar{n}^\nu n^\eta \beta^\rho, \\
\gamma^\mu \quad &\to n^\mu \frac{\not{\bar{n}}}{2} + \gamma_\perp^\mu \qquad \text{where} \qquad \gamma_\perp^\mu = \gamma^\mu - n^\mu\frac{\not{\bar{n}}}{2} - \bar{n}^\mu\frac{\not{n}}{2}, \\
\gamma^\mu\gamma_5 \quad &\to n^\mu\frac{\not{\bar{n}}}{2}\gamma_5 + \frac{i}{2}\gamma_{\nu\perp}\varepsilon^{\mu\nu\alpha\beta}\bar{n}_\alpha n_\beta, \\
\gamma^{[\mu}\gamma^{\nu]} \quad &\to (n^\mu\beta^\nu - n^\nu\beta^\mu)\frac{\not{\bar{n}}}{2} - i\varepsilon^{\mu\nu\alpha\gamma}n_\alpha\beta_\gamma\frac{\not{\bar{n}}}{2}\gamma_5 + \gamma_\perp^\mu\beta_\perp^\nu - \gamma_\perp^\nu\beta_\perp^\mu \\
&+(\beta\cdot\bar{n})(\gamma_\perp^\mu n^\nu - \gamma_\perp^\nu n^\mu) - \frac{1}{2}\not{\beta}_\perp(\bar{n}^\mu n^\nu - \bar{n}^\nu n^\mu), \\
\gamma^{[\mu}\gamma^{\nu]}\gamma_5 \quad &\to i\varepsilon^{\mu\nu\alpha\gamma}n_\alpha\beta_\gamma\frac{\not{\bar{n}}}{2} - (n^\mu\beta^\nu - n^\nu\beta^\mu)\frac{\not{\bar{n}}}{2}\gamma_5 + \frac{i}{2}(\varepsilon^{\beta\gamma\alpha\mu}\beta^\nu - \varepsilon^{\beta\gamma\alpha\nu}\beta^\mu)\bar{n}_\beta n_\gamma \gamma_{\alpha\perp} \\
&+\frac{i}{2}\varepsilon^{\mu\nu\alpha\eta}(\beta_\eta\gamma_{\alpha\perp} - \beta_\alpha\gamma_{\eta\perp}).
\end{aligned}
\tag{8.32}
$$

where $\gamma^{[\mu}\gamma^{\nu]} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

8.5.3 Specifying the Rest Frame of the b-Quark

The relations (8.32) can be considerably simplified by means of a standard trick. Using this trick one can reduce the relations (8.32) to a more simple set of relations derived in [11] which is equivalent to working in the basis

$$\Gamma_{sb}^{ET} = \{1,\, \gamma_5,\, \gamma_\perp^\mu\}\,. \tag{8.33}$$

The sum of the two light-like vectors n and $\bar{n}$ is always a time-like vector. This time-like vector can be chosen to be any unit time-like vector in the problem. In particular, one can choose β_b, the unit vector along the world-line of b-quark. So,

$$\frac{n}{2} + \frac{\bar{n}}{2} = \beta_b. \tag{8.34}$$

This property follows from the observation that for any light-like vector n one can always find a frame in which the vector has the components $n^\mu = (1,0,0,1)$ and therefore $\bar{n}^\mu = (1,0,0,-1)$. Time direction in this frame is specified by the unit time-like vector $\beta^\mu = (1,0,0,0)$. In our case, the choice (8.34) corresponds to the rest frame of the b-quark. If we choose $\beta_{c\bar{c}}$ instead we would end up with the equivalent set of relations corresponding to the rest frame of the J/ψ. It follows from Eq. (8.34) that:

$$(\beta_b \cdot n) = (\beta_b \cdot \bar{n}) = 1, \qquad (\beta_b \cdot \gamma_\perp) = \not{\beta}_\perp = 0, \qquad \text{and} \qquad \beta_\perp^\mu = 0. \tag{8.35}$$

One more relation follows trivially from (8.34) but it is important when casting the structures of Eq. (8.32) into a more simple form:

$$\frac{\not{\bar{n}}}{2} = \not{\beta}_b - \frac{\not{n}}{2}. \tag{8.36}$$

The operator $\not{\beta}_b$ applied to the heavy b-quark spinor h_{β_b} gives the spinor back, and the operator $\frac{\not{n}}{2}$ annihilates the SCET spinor $\xi_{n,p}$.

Using Eqs. (8.34), (8.35), and (8.32) we reproduce Eq. (27) from [11]:

$$\begin{aligned}
1 &\to 1, \\
\gamma_5 &\to \gamma_5, \\
\gamma^\mu &\to n^\mu + \gamma_\perp^\mu, \\
\gamma^\mu\gamma_5 &\to -n^\mu\gamma_5 + i\varepsilon_\perp^{\mu\nu}\gamma_{\nu\,\perp} \qquad \text{where} \qquad \varepsilon_\perp^{\mu\nu} = \varepsilon^{\mu\nu\eta\rho}\beta_{b\,\eta}n_\rho, \\
\gamma^{[\mu}\gamma^{\nu]} &\to (n^\mu\beta_b^\nu - n^\nu\beta_b^\mu) - i\varepsilon_\perp^{\mu\nu}\gamma_5 + (\gamma_\perp^\mu n^\nu - \gamma_\perp^\nu n^\mu), \\
\gamma^{[\mu}\gamma^{\nu]}\gamma_5 &\to -i\varepsilon_\perp^{\mu\nu} + (n^\mu\beta_b^\nu - n^\nu\beta_b^\mu)\gamma_5 + i(n^\mu\varepsilon_\perp^{\nu\alpha} - n^\nu\varepsilon_\perp^{\mu\alpha})\gamma_{\alpha\,\perp}.
\end{aligned} \tag{8.37}$$

8.5.4 Reduction Formulae for $[\bar{s}\ldots b][\bar{c}\ldots c]$

Here we present the reduction formulae which are necessary when reducing the Dirac structures of the FT to the structures of ET *at the leading order* in v and λ. To simplify the results we use the relations

$$\gamma_\perp^\mu - i\varepsilon_\perp^{\mu\nu}\gamma_{\nu\,\perp} = -2\varepsilon_+^\mu \not{\varepsilon}_-, \qquad \gamma_\perp^\mu + i\varepsilon_\perp^{\mu\nu}\gamma_{\nu\,\perp} = -2\varepsilon_-^\mu \not{\varepsilon}_+, \tag{8.38}$$

where $\varepsilon_\pm$ are the polarization vectors of s-quark with $\pm$ corresponding to the spin direction along/opposite to the quark momentum. These relations can be verified in the frame where $n^\mu = (1,0,0,1)$, $\beta_b^\mu = (1,0,0,0)$, and $\varepsilon_\pm^\mu = \frac{1}{\sqrt{2}}(0,1,\pm i,0)$. Using the polarization vectors we can write down the "cumulative" reduction formulae. Eqs. (8.28), (8.30), and (8.37) together with Eq. (8.38) are used below to write down the reduction relations for Dirac structures of the FT in terms of polarization vectors.

On the $c\bar{c}$-side the following relations hold:

$$\begin{aligned}
\otimes\gamma^\mu P_L &\to \otimes\frac{1}{2}\gamma_\perp^\mu - \otimes\frac{1}{2}\beta_{c\bar{c}}^\mu\gamma_5, \\
\otimes\gamma^\mu P_R &\to \otimes\frac{1}{2}\gamma_\perp^\mu + \otimes\frac{1}{2}\beta_{c\bar{c}}^\mu\gamma_5, \\
\otimes\gamma^{[\mu}\gamma^{\nu]} P_L &\to \otimes\frac{1}{2}(\beta_{c\bar{c}}^\mu\gamma_\perp^\nu - \beta_{c\bar{c}}^\nu\gamma_\perp^\mu) - \otimes\frac{i}{2}\varepsilon^{\mu\nu\rho\eta}\beta_{c\bar{c}\,\rho}\gamma_{\eta_\perp}, \\
\otimes\gamma^{[\mu}\gamma^{\nu]} P_R &\to \otimes\frac{1}{2}(\beta_{c\bar{c}}^\mu\gamma_\perp^\nu - \beta_{c\bar{c}}^\nu\gamma_\perp^\mu) + \otimes\frac{i}{2}\varepsilon^{\mu\nu\rho\eta}\beta_{c\bar{c}\,\rho}\gamma_{\eta_\perp}.
\end{aligned} \tag{8.39}$$

On the $\bar{s}b$-side we can use relations

$$\begin{aligned}
\gamma^\mu P_L\otimes &\to n^\mu P_R \otimes -\varepsilon_+^\mu \not{\varepsilon}_- \otimes, \\
\gamma^\mu P_R\otimes &\to n^\mu P_L \otimes -\varepsilon_-^\mu \not{\varepsilon}_+ \otimes, \\
\gamma^{[\mu}\gamma^{\nu]} P_L\otimes &\to (n^\mu\beta_b^\nu - n^\nu\beta_b^\mu)P_L \otimes +i\varepsilon_\perp^{\mu\nu} P_L \otimes +(n^\mu\varepsilon_-^\nu - n^\nu\varepsilon_-^\mu)\not{\varepsilon}_+ \otimes, \\
\gamma^{[\mu}\gamma^{\nu]} P_R\otimes &\to (n^\mu\beta_b^\nu - n^\nu\beta_b^\mu)P_R \otimes -i\varepsilon_\perp^{\mu\nu} P_R \otimes +(n^\mu\varepsilon_+^\nu - n^\nu\varepsilon_+^\mu)\not{\varepsilon}_- \otimes .
\end{aligned} \tag{8.40}$$

We also use the leading order relations between the quark momenta:

$$p_b = m_b\beta_b, \qquad p_c = p_{\bar{c}} = m_c\beta_{c\bar{c}}, \qquad \text{and} \qquad p_s = E_s n. \tag{8.41}$$

At the leading order the conservation of momentum requires that the 4-vectors satisfy

$$m_b\beta_b = 2m_c\beta_{c\bar{c}} + E_s n, \qquad \text{where} \qquad n^2 = 1 \qquad \text{and} \qquad \beta_b \cdot n = 1. \tag{8.42}$$

The last condition is due to the convention (8.34). Then the vector $\beta_{c\bar{c}}$ becomes a linear combination of vectors β_b and n:

$$\beta_{c\bar{c}} = \frac{m_b}{2m_c}\beta_b - \frac{E_s}{2m_c}n. \tag{8.43}$$

In particular the latter relation says that the dot-product $(\beta_{c\bar{c}} \cdot \varepsilon_{\pm})$, where $\varepsilon_{\pm}$ is the s-quark polarization vector, vanishes. Also, using Eq. (8.42) allows one to write the dot-products of four velocities in terms of the quark masses, which gives

$$(\beta_{c\bar{c}} \cdot n) = \frac{m_b}{2m_c} \qquad \text{and} \qquad (\beta_b \cdot \beta_{c\bar{c}}) = \frac{m_b - E_s}{2m_c} \qquad \text{with} \qquad E_s = \frac{m_b^2 - (2m_c)^2}{2m_b}. \tag{8.44}$$

The last step is to make sure that the ET structures are linearly independent. Remember that the structures have the form:

$$\Gamma_{sb}^{ET} \otimes \Gamma_{c\bar{c}}^{ET} \qquad \text{where} \qquad \Gamma_{sb}^{ET} \in \{P_L, P_R, \gamma_\perp^\mu\}, \quad \text{and} \quad \Gamma_{c\bar{c}}^{ET} \in \{\gamma_5, \gamma_\perp^\mu\}. \tag{8.45}$$

Note that $\gamma_\perp^\mu$ on sb and $c\bar{c}$ sides are different, and have different number of components. If the matrices $\gamma_\perp^\mu$ are contracted with the orthogonal vectors we can be sure that the ET structures are linearly independent. We use the following basis of orthogonal vectors:

$$\beta_b, \;\; \varepsilon_+, \;\; \varepsilon_- \;\; \text{and} \;\; n^* = n - \beta_b, \tag{8.46}$$

which in the b-quark restframe become

$$\beta_b^\mu = (1,0,0,0) \qquad \varepsilon_\pm^\mu = \frac{1}{\sqrt{2}}(0,1,\pm i,0) \quad \text{and} \quad n^{*\mu} = (0,0,0,1). \tag{8.47}$$

Finally, using the equations of this section we can write down the expression for the FT Dirac structure in terms of the ET Dirac structures introduced in Eq. (5.2):

$$\gamma_\mu P_L \otimes \gamma^\mu P_L \to \frac{1}{2} P_R \otimes \not{n}_\perp^* + \frac{1}{2} P_R \otimes \not{\beta}_{b\perp} - \frac{1}{2}(\beta_{c\bar{c}} \cdot n) P_R \otimes \gamma_5 - \frac{1}{2} \not{\varepsilon}_- \otimes \not{\varepsilon}_{+\perp}. \tag{8.48}$$

8.6 The Propagator of η^C Field

Let's outline the derivation of the propagator (7.34). We'll work in the COM frame and then write the outcome in the covariant form. The field η^C is a subject to the constraint $P_{-\beta_{c\bar{c}}} \eta_{\beta_{c\bar{c}}\,\tilde{p}_\perp} = 0$, so the Dirac structure of the propagator is given by the projector $P_{-\beta_{c\bar{c}}}$. In the COM frame the Lagrangian (7.20) has the form:

$$\mathcal{L} = -\int \frac{d^3\mathbf{x}}{(2\pi)^3} \left[\eta^{C\dagger}\, i\, \dot{\eta}^C - \eta^{C\dagger} \frac{\Delta}{2m} \eta^C \right] \tag{8.49}$$

where we've dropped all unnecessary indices. The canonical momentum is then

$$P = \frac{\delta \mathcal{L}}{\delta \dot{\eta}^C} = -i\eta^{C\dagger} \qquad \Rightarrow \qquad \eta^{C\dagger} = i\, P. \tag{8.50}$$

The canonical momentum and the field η^C satisfy the anticommutation relation for grassman fields (see Ch. 9 in [1]):

$$\left\{ P(\mathbf{x}), \eta^C(\mathbf{y}) \right\} = +i\delta^3(\mathbf{x} - \mathbf{y}). \tag{8.51}$$

The fields η^C and P are then expanded in terms of the creation and the annihilation operators $b^\dagger_{\mathbf{k}}$ and $b_{\mathbf{k}}$ for the anti-quarks:

$$\begin{aligned} &\eta^C(\mathbf{y}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\mathbf{y}} b^\dagger_{\mathbf{k}}, \qquad -i\, P(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{x}} b_{\mathbf{k}}, \\ &\text{where} \qquad \{b_{\mathbf{k}'}\,, b^\dagger_{\mathbf{k}}\} = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}'). \end{aligned} \tag{8.52}$$

Then the Hamiltonian where $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$ is

$$\mathcal{H} = P{\cdot}\dot{\eta}^C - \mathcal{L} = \int \frac{d^3\mathbf{x}}{(2\pi)^3} i\, P \frac{-\Delta}{2m} \eta^C = -\int \frac{d^3\mathbf{k}}{(2\pi)^3} \epsilon_{\mathbf{k}} b_{\mathbf{k}} b^\dagger_{\mathbf{k}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \epsilon_{\mathbf{k}} b^\dagger_{\mathbf{k}} b_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \epsilon_{\mathbf{k}}. \tag{8.53}$$

The last term here corresponds to the contribution due to the Dirac sea of negative energy states, the constant that must be dropped.

With the Hamiltonian (8.53) the time-dependent fields become

$$\eta^C(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{-i\mathbf{k}\mathbf{x} + i\epsilon_{\mathbf{k}} t_1} b^\dagger_{\mathbf{k}}, \qquad \bar{\eta}^C(y) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\mathbf{y} - i\epsilon_{\mathbf{k}} t_2} b_{\mathbf{k}}. \tag{8.54}$$

The vacuum is the state without anti-quarks:

$$b_{\mathbf{k}} |\text{vac}\rangle = 0. \tag{8.55}$$

Finally the propagator for field η^C follows. In the covariant form and with the Dirac structure explicit it becomes

$$\langle \mathrm{T} \eta^C(x) \bar{\eta}^C(y) \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{iP_{-\beta_{c\bar{c}}}}{-\beta_{c\bar{c}} \cdot k + \frac{\vec{p}_\perp^{\,2}}{2m_c} + i0}. \tag{8.56}$$

Note that this propagator corresponds to the reversed fermion flow, so the direction of the fermion flow for the $c\bar{c}$-part of a diagram in the effective theory is the same as in the FT. The line corresponding to the $\bar{c}$-quark goes into the diagram. Like in the FT the sign of the momentum of the ET propagator must be reversed if the direction of momentum of a fermion line is opposite to that of the fermion flow.

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