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Transversality of CR mappings between CR submanifolds of complex spaces

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Transversality of CR mappings between CR submanifolds of complex spaces

A dissertation submitted in partial satisfaction of the
requirements for the degree
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in

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by

Son Ngoc Duong

Committee in charge:

Professor Peter Ebenfelt, Chair
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Professor Kenneth Intriligator
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2012

The dissertation of Son Ngoc Duong is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2012

DEDICATION

To my parents, my only heroes.

Kính tặng ba mẹ.

EPIGRAPH

The art of doing mathematics
consists in finding that special case
which contains all the germs of generality.
—David Hilbert

TABLE OF CONTENTS

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Chapter 3, in full, is a revision of the material as it appear in "Ebenfelt, P., Duong, S., CR transversality of holomorphic mapping between generic submanifolds in complex spaces, *Proc. Amer. Math. Soc.* 140 (2012) 1729–1738".

Chapter 4, in full, is a revision of the material as it is going to appear in *Illinois Journal of Mathematics* under the title "Ebenfelt, P.; Duong, S.: Transversality of holomorphic mappings between real hypersurfaces in complex spaces of different dimensions".

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ABSTRACT OF THE DISSERTATION

Transversality of CR mappings between CR submanifolds of complex spaces

by

Son Ngoc Duong
Doctor of Philosophy in Mathematics

University of California San Diego, 2012

Professor Peter Ebenfelt, Chair

We investigate the geometric property of transversality of holomorphic, formal or CR mappings between real-analytic, formal or smooth generic submanifolds of complex spaces of equidimension as well as of different dimensions.

In Chapter 3, we shall consider the CR transversality in equidimension case. The main purpose of this chapter is to show that a holomorphic, formal or smooth CR mapping sending a real-analytic, smooth or formal generic submanifold M into such another M' is CR transversal to the target, provided that the source manifold is of finite bracket type and the mapping is of generic full rank. This result and its corollary completely resolve two questions posed by Peter Ebenfelt and Linda Preiss Rothschild in a paper from 2006. We also show that under a very mild assumption on the source manifold, the generic full rank condition imposed on the mapping is also necessary for the CR transversality to hold. This result confirms a conjecture in a paper by Bernhard Lamel and Nordine Mir.

In Chapter 4, we consider the transversality of mappings when the target manifold is of higher dimension. We will restrict ourself to the situation in which both manifolds M and M' are hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$ respectively, where $1 < n < N$. A main result of this chapter implies that under certain restrictions on the dimensions

n, N and the rank of the Levi-form of the target hypersurface, if the set of points at which the mapping H fails to be a local embedding has codimension at least 2, then the mapping must be transversal to the target at all points. Another result of this chapter implies that under some more restrictive assumptions, any finite holomorphic mapping sending M into M' is transversal at all points, unless the source hypersurface is of infinite type. This result may be considered as a different dimension analogue of a theorem by M. Salah Baouendi and Linda Preiss Rothschild from 1990.

Chapter 1

Summary of results

1.1 Introduction

The study of CR mappings between real hypersurfaces, or more generally, real submanifolds in complex spaces dates back to 1907 when Poincaré posed the local biholomorphic equivalence problem for real hypersurfaces in $\mathbb{C}^2$ [53]. Later, B. Segre [55], É. Cartan [19, 20], N. Tanaka [57], S.-S. Chern—J. Moser [21] and J. Moser—S. Webster [50] solved the equivalence problem in various settings. On the other hand, the work of Charles Fefferman [33] in 1974 relates the study of biholomorphic equivalence of strictly pseudoconvex domains to that of the equivalence of their boundaries. Since then, real hypersurfaces or more generally, real submanifolds and their mappings now have been studied extensively by both mathematicians and physicists (see, e.g., [6, 38, 47, 51]). We refer the reader to a beautiful survey paper by Baouendi, Ebenfelt and Rothschild [8] for the development which is closest to the topics discussed in this dissertation.

In this work, we shall consider a particular geometric property of CR mappings between CR manifolds. Namely, we are interested in the transversality of such mappings between generic submanifolds of complex spaces. The transversality in general is an important notion in many area of mathematics; for example, the Hopf boundary lemma has been frequently used in the study of elliptic PDE and potential theory (cf. [36]). In CR geometry and several complex variables, this notion either plays an important role or the main object of study in many works such as [52, 34, 35, 16, 17, 12, 23, 24, 30, 41] just to mention a few. For more detail examples, it plays a role in understanding the extension of CR mappings [15], regularity of CR mappings [13], convergence of formal mappings [7, 49] and rigidity phenomena of holomorphic mappings between real hypersurfaces as well as higher codimension submanifolds [28, 29, 11, 3, 32].

To be more clear, let us recall the basic notion of our work.

Definition 1.1 (CR transversality [30])**.** Let U is an open subset of $\mathbb{C}^N$, H a holomorphic mapping $U \to \mathbb{C}^N$, and M' a generic submanifold through a point $p' := H(p)$ for some $p \in U$, then H is said to be *CR transversal* to M' at p if

$$T^{1,0}_{p'}M' + dH(T^{1,0}_p\mathbb{C}^N) = T^{1,0}_{p'}\mathbb{C}^N, \tag{1.1}$$

where $T^{0,1}M' = \mathbb{C}TM' \cap T^{0,1}\mathbb{C}^N$ denotes the CR bundle on M' and $T^{1,0}M' = \overline{T^{0,1}M'}$ its complex conjugate.

It is worth mentioning that in this definition, we use the holomorphic tangent space $T^{1,0}_pM'$ instead of the real tangent space T_pM' as in the usual definition of transversality (cf. [37]). This turns out to be natural for our purposes. In fact, the CR transversality of a holomorphic mapping is, in general, strictly stronger than that of transversality when the map is considered as a real smooth mapping (cf. [30]).

Let $M \subset \mathbb{C}^{n+d}$ and $M' \subset \mathbb{C}^{n'+d'}$ be smooth generic submanifolds of codimension d and d', respectively (so that the CR dimensions are n and n', respectively), and H a holomorphic mapping from an open neighborhood U of M in $\mathbb{C}^{n+d}$ into $\mathbb{C}^{n'+d'}$ such that $H(M) \subset M'$. Consider the following question.

"Under what conditions is the mapping H CR transversal to M'?" (1.2)

This question has been of interest for a long time (cf. [16, 17, 39]). In the present work, we will consider the following conditions for the mapping H.

- Generic full rank property: H is said to be of generic full rank if the Jacobian matrix of H has full rank at some point, and hence at generic points.

- Finite multiplicity: H is said to be finite at a point p if the ideal generated by the germs of components of H near p has finite codimension in the ring of germs of holomorphic functions at p.

- Conditions on the zero varieties of minors of Jacobian matrix of H.

In equidimension case, the finite multiplicity condition was considered in [16] for hypersurfaces by Baouendi and Rothschild. It has been proved that if the target hypersurface is of finite type, then finite multiplicity mappings are CR transversal [16, Theorem 1]. This result was generalized to higher codimension case in [30] by Ebenfelt and Rothschild; in the same paper, the authors posed two questions regarding the possibility of extending the result for mappings of generic full rank [30, Question 1-2] (see also [46, Conjecture 2.7]). In Chapter 3, we shall provide affirmative answers to both questions (see Theorem 1.2). We also confirm the aforementioned conjecture for all codimension. These results will be stated in detail in Section 1.2.

The different dimension case has been studied in several papers, mostly in the situation that the target is a hyperquadric of higher dimension (cf. [11, 3, 41]). For example, in [11], Baouendi and Huang proved that "not totally degenerate" mappings between hyperquadrics of the same signature are CR transversal to the target. For more general hypersurfaces, Baouendi, Ebenfelt and Rothschild [10] showed that under certain assumptions on the Levi forms of the hypersurfaces M and M' and the codimension $\dim M' - \dim M$, "not totally degenerate" holomorphic mappings sending M into M' are CR transversal to the target along M except on a proper subvariety.

In Chapter 4, we shall consider the question of CR transversality at *all* points. Using a different method, we shall provide several sufficient conditions guaranteeing the CR transversality at all points of the mapping. These results, which have been written up a joint paper with Peter Ebenfelt in [27], will be stated in detail in Section 1.3.

Through out the present dissertation, we will also give various examples to illustrate that certain conditions imposed in our theorems cannot be relaxed.

1.2 Equidimension case

In this section, we will state our results on CR transversality of holomorphic mappings between complex spaces of the same dimension. Elaborate discussions of the results will be given in Chapter 3. Let M and M' be two generic submanifolds of the same dimension in a complex space $\mathbb{C}^N$ (i.e., $n = n'$, $d = d'$ and $N = n+d$). Assume that H is a holomorphic mapping from a neighborhood of some distinguished point $p \in M$ such that $H(M) \subset M'$. Our first result in Chapter 3 is the following theorem.

Theorem 1.2 ([26])**.** *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$. Assume that M is of finite type at p and* $\operatorname{Jac} H \not\equiv 0$. *Then H is CR transversal to M' at p'.*

Here, we denote by $\operatorname{Jac} H$ the determinant of the Jacobian matrix of H and the notation $\operatorname{Jac} H \not\equiv 0$ means $\operatorname{Jac} H$ does not vanish identically. In this case, we also say that H is of generic full rank.

Our next result is the following necessary condition for CR transversality.

Theorem 1.3 ([26])**.** *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$. Assume that M is holomorphically nondegenerate at p. If H is CR transversal to M' at p' then* $\operatorname{Jac} H \not\equiv 0$. *Furthermore, M' is also holomorphically nondegenerate.*

By combining two theorems above, we get the following necessary and sufficient condition for CR transversality.

Theorem 1.4 ([26])**.** *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$. Assume that M is holomorphically nondegenerate and of finite type at p. Then H is CR transversal to M' at p' if and only if* $\operatorname{Jac} H \not\equiv 0$.

As suggested in [30], results on CR transversality can be applied to the problem of local biholomorphic equivalence of generic submanifolds as in the following theorem.

Theorem 1.5 ([26]). *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$ and $\operatorname{Jac} H \not\equiv 0$. If M is of finite type and essentially finite at p, then H is a finite mapping. If, in addition, M is finitely nondegenerate at p, then H is a local biholomorphism near p.*

Theorem 1.2 provides an affirmative answers to two questions posed in [30] by Ebenfelt and Rothschild mentioned in the previous section. Theorem 1.4 in special case of hypersurfaces settles the conjecture by Lamel and Mir in [46]. The last part of Theorem 1.5, in the special setting of Levi-nondegenerate hypersurfaces, used to be a conjecture by Vitushkin [59]. It was confirmed by Isaev for Levi-nondegenerate hypersurfaces in [42] and for generic Levi-nondegenerate submanifolds of any codimension in [43]. Our method in this chapeter is different and provides affirmative answer to the conjecture for much broader class of submanifolds, namely, the class of finitely nondegenerate submanifolds. We refer the reader to [43] for related results and examples.

1.3 Different dimension case

To formulate our results in different dimension case, we shall need to introduce a little more notations. Given a holomorphic mapping $H\colon U \subset \mathbb{C}^{n+1} \to \mathbb{C}^{N+1}$, the subvariety W_H defined by

$$W_H := \{Z \in U\colon\ \operatorname{rk} H_Z(Z) < n+1\}, \tag{1.3}$$

where H_Z denotes the $(N+1) \times (n+1)$ matrix of partial derivatives of the components of H, is precisely the variety of points where H has degenerate rank. Our first result in this situation is the following theorem.

Theorem 1.6 ([27]). *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{N+1}$ be smooth real hypersurfaces through p and p' respectively, and $H : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{N+1}, p')$ a germ at p of a holomor-*

phic mapping such that $H(M) \subset M'$. Denote by r the rank of the Levi form of M' at p' and assume that

$$2N - r \leq 2n - 2. \tag{1.4}$$

If the germ at p of the analytic variety W_H, given by (1.3)*, has codimension at least* 2*, then H is transversal to M' at p'.*

We remark that finite holomorphic mappings need not satisfy the condition on the codimension of W_H in Theorem 1.6. However, we shall also prove the following transversality theorem for finite mappings.

Theorem 1.7 ([27])**.** *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{N+1}$ be smooth real hypersurfaces through p and p' respectively, and $H : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{N+1}, p')$ a germ at p of a holomorphic mapping such that $H(M) \subset M'$. Denote by r the rank of the Levi form of M' at p' and assume that*

$$2N - r \leq 2n - 3. \tag{1.5}$$

Assume also that M is of finite type at p and H is a finite mapping at p. Then H is transversal to M' at p'.

This theorem is a consequence of a more general theorem which involves conditions on all minors of Jacobian matrix of H of a smaller dimension. The theorem will be stated and proved in Chapter 4. We shall also give an example showing that generic full rank mappings may fail to be transversal to the target. Furthermore, we shall show that if the mapping fails to be transversal, then the non-transversal locus of the complexified mapping must be of a special form. We leave open several situations which will be summarized at the end of Chapter 4 (cf. [41]).

In our proofs, we shall work with formal power series and mappings, and thus the results are also valid for smooth CR mappings and formal mappings between smooth or formal submanifolds, modulo a necessary (and fairly obvious) modification in their statements.

Chapter 2

Preliminaries

In this chapter we recall some basic facts about CR geometry which will be needed in subsequent chapters. The reader should consult, for example, [6, 18, 22, 25, 44, 48] for more background on CR geometry.

2.1 Complex Euclidean spaces

Let $\mathbb{C}^N$ be the complex Euclidean space of complex dimension N. For $Z \in \mathbb{C}^N$ we write $Z = (Z_1, \ldots, Z_N)$ where $Z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{R}$ and $i = \sqrt{-1}$. We identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$ by $Z \to (x_1, y_1, \ldots, x_N, y_N)$ and denote the complex conjugation of Z by $\bar{Z}$. A function f on a subset of $\mathbb{C}^N$ will be denote by $f(Z, \bar{Z})$ instead of $f(Z)$ to emphasis that f may not be holomorphic.

For any point $p \in \mathbb{C}^N$ the real tangent space $T_p\mathbb{C}^N$ of $\mathbb{C}^N$ at p is spanned by the following vectors

$$\left.\frac{\partial}{\partial x_j}\right|_p, \ \left.\frac{\partial}{\partial y_j}\right|_p, \ j = 1, 2, \ldots, N. \tag{2.1}$$

We will also work with the complexified tangent space $\mathbb{C}T_p\mathbb{C}^N := T_p\mathbb{C}^N \otimes_{\mathbb{R}} \mathbb{C}$ which is a complex vector space of complex dimension $2N$ spanned by the vectors in (2.1) over $\mathbb{C}$. The following complex vector fields are of great importance.

$$\frac{\partial}{\partial Z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{Z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Then one can see easily that these vector fields span $\mathbb{C}T_p\mathbb{C}^N$ at every point. Recall that the *holomorphic tangent space* $T_p^{1,0}\mathbb{C}^N$ is defined to be the subspace of $\mathbb{C}T_p\mathbb{C}^N$ spanned by the vectors $\frac{\partial}{\partial Z_j}\big|_p$, $j = 1, \dots N$. By the chain rule, this is well defined, i.e., this definition is independent of the choice of holomorphic coordinates. The *anti-holomorphic tangent space* $T_p^{0,1}\mathbb{C}^N$ is defined to be $\overline{T_p^{1,0}\mathbb{C}^N}$. It is easy to verify that

$$\mathbb{C}T_p\mathbb{C}^N = T_p^{1,0}\mathbb{C}^N \oplus T_p^{0,1}\mathbb{C}^N.$$

2.2 Cauchy–Riemann submanifolds of complex spaces

Let $M \subset \mathbb{C}^N$ be a $\mathcal{C}^k$-smooth embedded real submanifold of real codimension d. Thus, for every point $p \in M$, there is a neighborhood U of p and a C^k-smooth real vector-valued function $\rho = (\rho_1, \dots \rho_d)$ defined in U such that

$$M \cap U = \{Z \in U : \rho(Z, \bar{Z}) = 0\}, \tag{2.2}$$

and such that $d\rho_1 \wedge d\rho_2 \cdots \wedge d\rho_d$ does not vanish in U. We shall refer to ρ as a *local defining function* for M near p. We shall mainly concern with $\mathcal{C}^\infty$-smooth submanifolds and thus ρ_j can be chosen to be $\mathcal{C}^\infty$. Furthermore, if ρ can be chosen to be real-analytic, then M is said to be real-analytic.

If $p \in M$ then the tangent space T_pM of M at p can be identified with the subspace of the tangent space $T_p\mathbb{C}^N$ consists of vectors which annihilate all local defining function at p. Namely,

$$T_pM = \{X \in T_p\mathbb{C}^N : X\rho_k(p, \bar{p}) = 0,\ k = 1, 2, \dots d\}.$$

Clearly, this identification does not depend on the choice of local defining functions. We also define the complexified tangent space $\mathbb{C}T_pM = T_pM \otimes_{\mathbb{R}} \mathbb{C}$. Furthermore, we write

$$T_p^{0,1}M = T_p^{0,1}\mathbb{C}^N \cap \mathbb{C}T_pM. \tag{2.3}$$

It is well-known that (cf.[6, page 7])

$$\dim_{\mathbb{C}} T_p^{0,1}M = N - \operatorname{rk}\left(\frac{\partial \rho_k}{\partial \bar{Z}_j}(p, \bar{p})\right)_{1 \le j \le N, 1 \le k \le d}. \tag{2.4}$$

Note that $\dim_{\mathbb{C}} T_p^{0,1}M$ may vary as p varies on M. An important class of real submanifolds of $\mathbb{C}^N$ is the class of those for which $\dim_{\mathbb{C}} T_p^{0,1}M$ is constant as p varies on M. More precisely, we have the following definition (cf. [6, page 9]).

Definition 2.1 (Cauchy-Riemann submanifolds)**.** A real submanifold $M \in \mathbb{C}^N$ is called a CR manifold if $\dim_{\mathbb{C}} T_p^{0,1}M$ is constant for $p \in M$. If M is a CR submanifold (or Cauchy–Riemann submanifold), then $\dim_{\mathbb{C}} T_p^{0,1}M$ is called the *CR dimension* of M.

Let M be a CR submanifold of $\mathbb{C}^N$. The importance of the constancy of $\dim_{\mathbb{C}} T_p^{0,1}M$ is that the mapping $p \mapsto T_p^{0,1}M$ determines a subbundle $T^{0,1}M$ of the complex tangent bundle $\mathbb{C}TM$. We shall refer to $T^{0,1}M$ as the CR tangent bundle of the CR manifold M.

Remark 2.2. If M is a CR submanifold of $\mathbb{C}^N$ and $L = T^{0,1}M$ the CR tangent bundle. Then

(i) L is formally integrable, i.e., $[L, L] \subset L$.

(ii) L is almost Lagrangian, i.e., $L \cap \bar{L}$ is trivial.

A differentiable manifold M together with a distinguished subbundle L of $\mathbb{C}TM$ satisfying (i) and (ii) is called an *abstract* CR manifold. The problem whether a smooth abstract CR manifold arises (even locally) from an embedded CR submanifold of complex spaces, first posed by Kohn, is difficult and has a long history. We will not touch this problem in this work. The reader is referred to, for example, [18, pp. 169-172] for a discussion on this problem. However, by a theorem of Andreotti and Fredricks [1], *real-analytic* CR manifolds can always be realized as generic submanifolds of complex manifolds.

Definition 2.3 (CR vector fields)**.** Let M be a CR submanifold and $T^{0,1}M$ its CR tangent bundle. A vector field L on M is a *CR vector field* if for every $p \in M$, $L_p \in T_p^{0,1}M$.

In other words, a CR vector field on M is a smooth section over M of the bundle $T^{0,1}M$.

Definition 2.4. Let M be a CR submanifold and $k \geq 1$. A $\mathcal{C}^k$-smooth function f is said to be CR if $Lf \equiv 0$ for every CR vector field L on M.

It follows immediately that if h is a holomorphic function in a neighborhood of M and $f = h|_M$ then f is a CR function on M. The converse does not hold in general. However, in real analytic category, CR functions is nothing but the restrictions of holomorphic functions (cf. [6, Corollary 1.7.13]).

Theorem 2.5 (cf. [6])**.** *Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold and f a CR function in a neighborhood of $p \in M$. If f is real-analytic in a neighborhood of p in M then f extends as a holomorphic function in a neighborhood of p in $\mathbb{C}^N$.*

Definition 2.6. Let M and N be CR submanifolds of complex spaces and $F : M \to N$ a $\mathcal{C}^1$-smooth mapping. We say that F is a CR mapping if $F_*(T_p^{0,1}M) \subset T_p^{0,1}N$ for every $p \in M$.

It turns out that if $N \subset \mathbb{C}^N$ and $F = (F_1, \ldots, F_N)$ are components of F, i.e., $F_j = Z_j \circ F$ for some choice of holomorphic coordinates in $\mathbb{C}^N$, then F is CR mapping if and only if F_j are CR functions for $j = 1, 2, \ldots, N$ (cf. [6, Proposition 2.3.3]).

We will concern only a class of CR submanifolds whose CR dimension is as small as possible. In view of (2.4), those are real submanifolds which possess a local defining function ρ such that $\partial\rho_1, \ldots \partial\rho_d$ are linearly independent over $\mathbb{C}$.

Definition 2.7 (Generic submanifolds)**.** A real submanifold $M \subset \mathbb{C}^N$ is generic if, for every $p \in M$, there is a defining function ρ such that the complex differentials $\partial\rho_1, \ldots, \partial\rho_d$ are linearly independent over $\mathbb{C}$ near p.

We note that this definition depends neither on the choice of holomorphic coordinates in $\mathbb{C}^N$, nor the choice of local defining function. Furthermore, from (2.4), we see that a generic submanifold M of codimension d in $\mathbb{C}^N$ is a CR manifold of CR dimension $n := N - d$.

2.3 Normal coordinates for generic submanifolds

An important tool for our work is normal coordinates for generic submanifolds. If M is real-analytic and $p \in M$, then it is well-known (cf. [6, Theorem 4.2.6]) that there exist normal coordinates $Z = (z, w)$ for M vanishing at p, where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots w_d)$, such that M is defined by a complex equation of the following form.

$$w = Q(z, \bar{z}, \bar{w}), \tag{2.5}$$

where $Q(z, \chi, \tau)$ is a $\mathbb{C}^d$-valued holomorphic function defined in a neighborhood of $p = (0, 0)$ satisfying

$$Q(z, 0, \tau) \equiv Q(0, \chi, \tau) \equiv \tau. \tag{2.6}$$

The fact that the d complex equations in (2.5) define a submanifold of real codimension d is equivalent to one of the following equivalent identities (cf. [6])

$$Q(z, \chi, \bar{Q}(\chi, z, w)) \equiv w, \quad \bar{Q}(\chi, z, Q(z, \chi, \tau)) \equiv \tau; \tag{2.7}$$

here and in the rest of this dissertation, we use the following notation: if $u(x)$ is an analytic function or formal power series in some set of variables x, then $\bar{u}(x)$ is the function or power series given by $\bar{u}(x) := \overline{u(\bar{x})}$.

It should be noted that we can consider the variables $\bar{z}, \bar{w}$ in (2.5) as an independent set of complex variables $\xi = (\chi, \tau)$ and thus the complexified equation $w = Q(z, \chi, \tau)$ defines a complex submanifold $\mathcal{M}$ of codimension d in $\mathbb{C}^N_Z \times \mathbb{C}^N_\xi$. We shall refer to $\mathcal{M}$ as the (extrinsic) *complexification* (or Segre family—the term coined by James Faran) of M , which has been used to study the mapping problems between real submanifolds for a long time (cf. [60]).

2.4 Segre mappings and a criteria for finite type of Baouendi, Ebenfelt and Rothschild

Segre varieties was introduced by Beniamino Segre in his 1931 paper [55]. The iterated Segre mappings, introduced by Baouendi, Ebenfelt and Rothschild in [4], have played an important role in the study of the mapping problem between real submanifolds in complex spaces (see, e.g., [54] and reference there in).

Definition 2.8 (cf. [4]). Let M be a generic analytic (or formal) submanifold defined by $w = Q(z, \bar{z}, \bar{w})$. For a positive integer k, the k^{th} *Segre mapping* of M at 0 is the mapping $v^k : \mathbb{C}^{kn} \to \mathbb{C}^N$ defined by

$$\mathbb{C}^{kn} \ni t = (t^1, \ldots, t^k) \mapsto v^k(t) := (t^k, u^k(t^1, \ldots, t^k)), \tag{2.8}$$

where $u^k : \mathbb{C}^{kn} \to \mathbb{C}^d$ is given inductively by

$$u^1(t^1) = 0,\ u^k(t^1, \ldots, t^k) = Q(t^k, t^{k-1}, \overline{u^{k-1}}(t^1, \ldots, t^{k-1})),\ k \geq 2. \tag{2.9}$$

The importance of iterated Segre mappings lies under the following theorem.

Theorem 2.9 (cf. [4]). *Let $M \subset \mathbb{C}^N$ be a formal generic submanifold of codimension d through 0 and $v^k(t) = (t^k, u^k(t)) : \mathbb{C}^{k(N-d)} \to \mathbb{C}^N$ the k^{th} Segre mapping of M. Then the following are equivalent.*

1. *M is of finite type at 0.*

2. *$\operatorname{rk} v^k = N$ for $k \geq d + 1$.*

3. *$\operatorname{rk} u^k = d$ for $k \geq d + 1$.*

Here we use the notation $\operatorname{rk} u$ to denote the generic rank of u.

Here, we recall that M is said to be of *finite type* at p if the complex Lie algebra $\mathfrak{g}_M$ generated by all smooth CR vector fields on M and their conjugates satisfies $\mathfrak{g}_M(p) = \mathbb{C}T_pM$. This notion is sometimes referred to as finite type in the sense of Kohn [45] (and Bloom-Graham) or finite bracket type. This should not be confused with the finite type condition in the sense of D'Anglelo (cf. [25]) which is related but different.

A CR submanifold M is said to be *minimal* (in the sense of Tumanov [58]) at a point $p \in M$ if there is no real submanifold $N \subset M$ passing through p with $\dim N < \dim M$ and such that $T_q^{\mathbb{C}} M \subset T_q S$, for all $q \in S$. It is well-known that if M is real-analytic, then the minimality condition of M is equivalent to the finite type condition. However, this is not true for smooth generic submanifolds of $\mathbb{C}^N$ (cf. [6]).

2.5 Nondegeneracy conditions for CR manifolds

Beside the well-known Levi-nodegeneracy condition for CR submanifolds, there are several other nondegeneracy conditions which play important roles in CR geometry. We recall these conditions which will be needed in our later discussions. The reader is referred to [6, Chapter XI] for a more elaborate discussion of these conditions.

By a germ at p of holomorphic vector fields we shall mean a germ of vector fields of the form $L = \sum_{j=1}^{N} \varphi_j(Z)\frac{\partial}{\partial Z_j}$ where the φ_j are germs at p of holomorphic functions.

Definition 2.10 (Holomorphic nondegeneracy)**.** We say that a smooth CR submanifold M is *holomorphically nondegenerate* at p if there is no non-trivial germ at p of a holomorphic vector fields tangent to M near p.

This notion was introduced by Stanton in [56] for hypersurfaces. We remark that for connected real-analytic CR submanifolds, holomorphic nondegeneracy at a point is equivalent to holomorphic nondegeneracy at all point (cf. [4]).

We also need the notion of essential finiteness which was first introduced by Baouendi, Jacobowitz and Treves in [14].

Definition 2.11 (Essential finiteness)**.** Let M be a generic submanifold given in a normal coordinates by $w = Q(z, \bar{z}, \bar{w})$. We write

$$Q^j(z, \chi, 0) = \sum_{\alpha} q^j_{\alpha}(z)\chi^{\alpha}, \quad j = 1, 2, \ldots d.$$

Then M is said to be *essentially finite* at 0 if the ideal I_M generated by q^j_{α} is of finite codimension in $\mathbb{C}[[\chi]]$. We also call the codimension of I_M the essential type of M at 0 and denote by $\operatorname{Ess}_0(M)$.

Finally, we recall the notion of finite nondegenerate.

Definition 2.12 (Finite nondegeneracy)**.** Let $M \subset \mathbb{C}^N$ be a generic submanifold through p defined by $\rho = 0$ where $\rho = (\rho_1, \ldots \rho_d)$. We say that M is *finitely nondegenerate* at p if

$$\operatorname{span}_{\mathbb{C}} \left\{ L^{\alpha} \left(\frac{\partial \rho^j}{\partial Z} \right)(p) : \; j = 1, \ldots d, \; \alpha \in \mathbb{N}^n_+ \right\} = \mathbb{C}^N, \tag{2.10}$$

where $\mathrm{span}_{\mathbb{C}}$ denotes the vector space spanned over $\mathbb{C}$ and $L^\alpha := L_1^{\alpha_1} \dots L_n^{\alpha_n}$. Here, $L_1, \dots L_n$ is a basis for the smooth CR vector fields tangent to M near p.

The following theorem gives a comparison between various nondegeneracy conditions (see [6] for a proof).

Theorem 2.13 (cf. [6]). *Let M be a generic submanifold of $\mathbb{C}^N$ through 0. Consider the following statements*

1. *M is finitely nondegenerate at 0;*
2. *M is essentially finite at 0;*
3. *M is holomorphic nondegenerate at 0.*

Then (i) implies (ii) and (ii) implies (iii).

The importance of these nondegeneracy conditions will be clear in Chapter 3.

2.6 Transversality

We begin by recalling the following theorem which will be very useful determine whether a mapping is CR transversal.

Theorem 2.14 (Ebenfelt-Rothschild [30]). *Let $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a formal holomorphic mapping and $M' \subset \mathbb{C}^N$ a formal generic submanifold of codimension d. Then the following are equivalent.*

1. *H is CR transversal to M' at 0.*
2. *H is transversal to M' at 0 and the formal real submanifold $H^{-1}(M') \subset \mathbb{C}^N$ is generic.*
3. *If (z', w') are normal coordinates for M' at 0 and $H = (F, G)$, then*

   $$dG : T_0^{1,0}\mathbb{C}^N \to T_0^{1,0}\mathbb{C}^d$$

   has rank d.

4. *There exists a formal generic submanifold* $M \subset \mathbb{C}^N$ *of codimension* d *through* 0 *such that* $H(M) \subset M'$ *and*

$$\det \frac{\partial G}{\partial w}(0) \neq 0, \tag{2.11}$$

where (z, w) *are normal coordinates for* M *at* 0*,* (z', w') *are normal coordinates for* M' *at* 0*, and* $H(z, w) = (F(z, w), G(z, w))$*.*

In view of second condition, CR transversality implies transversality in real sense. The reverse implication does not hold in general (see [30, Example 5.1]). On the other hand, when considering the situation in which the mapping sends one generic submanifold into such another, the two notions coincide. Finally, the last condition in this theorem will be very useful in determining whether a CR mapping given in normal coordinates is CR transversal.

2.7 Formal submanifolds and mappings

A standard technique to extend results from real-analytic case to smooth case is using formal manifolds and mappings. This has been done in many earlier works (e.g., [15, 16, 7, 30]). In this dissertation, we shall use the same technique. Namely, we shall work with formal manifolds and formal mappings and the results will be automatically valid for smooth and real-analytic case.

Chapter 3

Transversality of mappings: Equidimension case

Let M and M' be generic submanifolds of codimension d in $\mathbb{C}^N$ and H a holomorphic mapping from a neighborhood U of $p \in M$ sending $M \cap U$ into M'. In this chapter, We shall investigate the CR transversality of H to M' at p. A main result of this chapter implies that if the source manifold M is of finite type and H is of generic full rank, then H is CR transversal to M' along M (Theorem 3.1). We also show that the condition on H is necessary, if the source manifold is assumed to be holomorphically nondegenerate. Furthermore, using formal power series techniques, these results extend to the case when M and M' are smooth or formal submanifolds and H is a smooth CR mapping or a formal mapping.

3.1 A sufficient condition for CR transversality

In this section, we state and prove the following theorem which gives a sufficient condition for a holomorphic mapping to be CR transversal to the target manifold.

Theorem 3.1 (Ebenfelt - D. [26]). *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$. Assume that M is of finite type at p and $\operatorname{Jac} H \not\equiv 0$. Then H is CR transversal to M' at p'.*

Since the problem is purely local, we will assume that p and p' are the origin, M and M' are generic formal submanifolds through 0, given in normal coordinates by $\rho(Z,\bar{Z}) = 0$ and $\rho'(Z',\bar{Z}') = 0$ where $\rho(Z,\xi) = w - Q(z,\chi,\tau)$ and $\rho'(Z',\xi') = w' - Q'(z',\chi',\tau')$. Here we recall that $Z = (z,w)$ and $\xi = (\chi,\tau)$. Let $H(Z)$ be a formal mapping sending M into M'. We begin with the following lemma.

Lemma 3.2. *Let M and M' be generic formal submanifolds defined as above and $H : (\mathbb{C}^N,0) \to (\mathbb{C}^N,0)$ a formal mapping sending M into M', that is*

$$\rho'(H(Z),\bar{H}(\xi)) = a(Z,\xi)\cdot\rho(Z,\xi) \tag{3.1}$$

where $a(Z,\xi)$ is a $d\times d$ matrix of formal power series in $\mathbb{C}[[Z,\xi]]$. Then, there are $d\times d$ matrices $C(Z,\xi)$ and $E(Z,\xi)$ of formal power series in $\mathbb{C}[[Z,\xi]]$ such that the following holds

$$\det H_Z(Z))\cdot I_d = a(Z,(\chi,\bar{Q}(\chi,Z)))\cdot C(Z,\xi); \tag{3.2}$$

$$\det \bar{H}_\xi(\xi)\cdot I_d = a((z,Q(z,\xi)),\xi)\cdot E(z,\xi) \tag{3.3}$$

where I_d denotes the $d\times d$ identity matrix.

Proof. By differentiating (3.1) with respect to Z we get.

$$\rho'_{Z'}(H(Z),\bar{H}(\xi))\cdot H_Z(Z) = a_Z(Z,\xi)\cdot\rho(Z,\xi) + a(Z,\xi)\cdot\rho_Z(Z,\xi). \tag{3.4}$$

Now we substitute $\xi = (\chi,\bar{Q}(\chi,Z))$ into (3.4) and note that $\rho(Z,(\chi,\bar{Q}(\chi,Z)))) = 0$. Hence

$$\rho'_{Z'}(H(Z),\bar{H}(\chi,\bar{Q}(\chi,Z)))\cdot H_Z(Z) = a(Z,(\chi,\bar{Q}(\chi,Z)))\cdot\rho_Z(Z,(\chi,\bar{Q}(\chi,Z))). \tag{3.5}$$

By Cramer's rule, there is an $N\times N$ matrix of formal power series $B(Z)$ such that $H_Z B = BH_Z = (\det H_Z)I_N$. Thus, it follows from equation (3.5) that

$$\rho'_{Z'}(H(Z),\bar{H}(\chi,\bar{Q}(\chi,Z)))\cdot\det H_Z(Z) = a(Z,(\chi,\bar{Q}(\chi,Z)))\cdot\rho_Z(Z,(\chi,\bar{Q}(\chi,Z)))\cdot B(Z). \tag{3.6}$$

Recall that $\rho'(Z',\xi') = w' - Q(z',\xi')$ where $Z' = (z',w')$. It then follows that the last d columns of $\rho'_{Z'}$ is the identity matrix I_d. Thus, by considering the last d columns of equation (3.6), we find that there is a $d \times d$ matrix $C = C(Z,\xi)$ whose entries are power series such that

$$\det H_Z(Z) \cdot I_d = a(Z, (\chi, \bar{Q}(\chi, Z))) \cdot C(Z,\xi).$$

Thus, equation (3.2) is proved.

Now, to prove (3.3), we differentiate (3.1) with respect to ξ and substitute $Z = (z, Q(z,\xi))$. Since $\rho((z,Q(z,\xi)),\xi) = 0$ we have

$$\rho'_{\xi'}(H(z,Q(z,\xi)), \bar{H}(\xi)) \cdot \bar{H}_\xi(\xi) = a((z,Q(z,\xi)),\xi) \cdot \rho_\xi((z,Q(z,\xi)),\xi). \tag{3.7}$$

Note that $\bar{H}_\xi(\xi) \cdot \bar{B}(\xi) = \det \bar{H}_\xi(\xi) I_N$, where $B(Z)$ is the matrix introduced above. Multiplying both sides of (3.7) with $\bar{B}(\xi)$, we obtain

$$\rho'_{\xi'}(H(z,Q(z,\xi)), \bar{H}(\xi)) \cdot \det \bar{H}_\xi(\xi) = a((z,Q(z,\xi)),\xi) \cdot \rho_\xi((z,Q(z,\xi)),\xi) \cdot \bar{B}(\xi). \tag{3.8}$$

Taking the last d columns of (3.8), we obtain

$$\det \bar{H}_\xi(\xi) \cdot \rho'_{\tau'}(H(z,Q(z,\xi)), \bar{H}(\xi))) = a((z,Q(z,\xi)),\xi) \cdot D(Z,\xi), \tag{3.9}$$

where $D(Z,\xi)$ is the matrix formed by the last d columns of $\rho_\xi((z,Q(z,\xi)),\xi) \cdot \bar{B}(\xi)$. Now, it follows from (2.6) that $\rho'_{\tau'}(0,0) = Q'_{\tau'}(0,0,0) = I_d$ and hence $\rho'_{\tau'}(H(z,Q(z,\xi)), \bar{H}(\xi)))$ is invertible over the ring $\mathbb{C}[[Z,\xi]]$. Consequently, it follows from (3.9) that there is a matrix $E(Z,\xi)$ such that

$$\det \bar{H}_\xi(\xi) \cdot I_d = a((z,Q(z,\xi)),\xi) \cdot E(Z,\xi),$$

which is (3.3). The proof is complete. □

Lemma 3.3. *Assume that* $\det H_Z(0) = 0$*, but* $\det H_Z(Z) \not\equiv 0$*. Then there exist units* $u(Z,\xi)$*,* $v(Z,\xi)$ *in* $\mathbb{C}[[Z,\xi]]$ *and formal power series* $b(Z) \in \mathbb{C}[[Z]]$*,* $c(\xi) \in \mathbb{C}[[\xi]]$ *such that*

$$\det a(Z, (\chi, \bar{Q}(\chi, Z))) = u(Z,\xi) \cdot b(Z); \tag{3.10}$$

$$\det a((z,Q(z,\xi)),\xi)) = v(Z,\xi) \cdot c(\xi). \tag{3.11}$$

Furthermore, $b(Z)$ *is a divisor of* $(\det H_Z(Z))^d$ *in* $\mathbb{C}[[Z]]$ *and* $c(\xi)$ *is a divisor of* $(\det \bar{H}_\xi(\xi))^d$ *in the ring* $\mathbb{C}[[\xi]]$.

Proof. It follows from (3.2) that

$$(\det H_Z(Z))^d = \det a(Z, (\chi, \bar{Q}(\chi, Z)) \det C(Z, (\chi, \bar{Q}(\chi, Z))). \tag{3.12}$$

We now factor both sides of (3.12) into products of irreducible elements in the unique factorization domain $\mathbb{C}[[Z, \xi]]$. Since the left hand side of (3.12) is a non-trivial formal power series in the ring $\mathbb{C}[[Z]] \subset \mathbb{C}[[Z, \xi]]$, its factorization involves factors that are power series in Z only. Thus, by the uniqueness of the factorization, we obtain

$$\det a(Z, (\chi, \bar{Q}(\chi, Z))) = u(Z, \xi) \cdot b(Z).$$

where $b(Z) \in \mathbb{C}[[Z]]$ is a divisor of $(\det H_Z(Z))^d$ and $u(Z, \xi)$ is a unit in $\mathbb{C}[[Z, \xi]]$.

Similarly, it follows from (3.3) that

$$(\det \bar{H}_\xi(\xi))^d = \det a((z, Q(z, \xi)), \xi)) \cdot \det E(Z, \xi).$$

A similar argument to the one above shows that

$$\det a((z, Q(z, \xi)), \xi) = v(Z, \xi) \cdot c(\xi),$$

where $v(Z, \xi) \in \mathbb{C}[[Z, \xi]]$ is a unit and $c(\xi) \in \mathbb{C}[[\xi]]$ is a divisor of $(\det \bar{H}_\xi(\xi))^d$. The proof is complete. □

Lemma 3.4. *Let* u, v, b, c *be as in Lemma* 3.3*. Then, there is a unit* $s(Z, \xi)$ *in* $\mathbb{C}[[Z, \xi]]$ *such that*

$$b(z, Q(z, \xi)) = s(Z, \xi) \cdot c(\xi). \tag{3.13}$$

Proof. We substitute $Z = (z, Q(z, \xi))$ into (3.10), and use (2.7) and (3.11) to obtain

$$\begin{aligned} u((z, Q(z, \xi)), \xi) \cdot b(z, Q(z, \xi)) &= \det a((z, Q(z, \xi)), (\chi, \bar{Q}(\chi, z, Q(z, \xi))) \\ &= \det a((z, Q(z, \xi)), \xi) \\ &= v(Z, \xi) \cdot c(\xi). \end{aligned}$$

Since $u(Z, \xi)$ and $v(Z, \xi)$ are units, we can take $s(Z, \xi) = (u((z, Q(z, \xi)), \xi))^{-1} v(Z, \xi)$ to obtain (3.13). It is obvious that $s(Z, \xi)$ is also a unit. The proof is complete. □

We will need the following lemma whose proof may be found, for instance, in [6, Proposition 5.3.5] for the case $m = N$. The proof bellow for the case $m \geq N$ which we reproduce for reader's convenience, is almost the same.

Lemma 3.5 (cf. [6])**.** *Let $K(x) : (\mathbb{C}^N, 0) \to (\mathbb{C}^k, 0)$ and $\phi(y) : (\mathbb{C}^m, 0) \to (\mathbb{C}^N, 0)$ be formal mappings. Assume that ϕ has generic rank N. If $K \not\equiv 0$ then $K \circ \phi \not\equiv 0$.*

Proof. Without lost of generality, we can assume that $k = 1$. We will show that for $K \in \mathbb{C}[[x]]$, $K \not\equiv 0$, it holds that $K \circ \phi \not\equiv 0$. By Weierstrass Preparation Theorem for formal power series (cf. [6, Theorem 5.3.1]), after a holomorphic change of coordinates, $K(x) = U(x)P(x)$ where U is an unit in $\mathbb{C}[[x]]$ and $P(x)$ is a Weierstrass polynomial in some x_j. We may also assume $j = N$. Thus, we can write

$$P(x) = x_N^q + \sum_{j=0}^{q-1} a_j(x')x_N^j \tag{3.14}$$

where $a_j \in \mathbb{C}[[x']]$ with no constant term and $x' = (x_1, \ldots x_{N-1})$. Observe that $K \circ \phi \equiv 0$ if and only if $P \circ \phi \equiv 0$. Thus, it's enough to prove the lemma for K is a Weierstrass polynomial in x_N of degree q for some $q \geq 1$. Now we will use induction on q. First, assume $q = 1$. Then K is of the following form,

$$K(x) = x_N + a_0(x'). \tag{3.15}$$

Now assume for contradiction that $K \circ \phi \equiv 0$. By differentiating this identity we get

$$K_x(\phi(y)) \cdot \phi_y(y) \equiv 0. \tag{3.16}$$

More precisely, $K_x(\phi(y))$ is the gradient vector of K and $\phi_y(y)$ is the Jacobian matrix of $\phi(y)$. By assumption, $\phi_y(y)$ has maximal rank in the quotient field of $\mathbb{C}[[y]]$, it follow that $K_x(\phi(y)) \equiv 0$. In particular, $K_{x_N} \circ \phi \equiv 0$. This is absurd since $K_{x_N}(x) \equiv 1$. Thus, assertion holds for $q = 1$.

Now assume the assertion holds for all Weierstrass polynomial of degree $q - 1$ with $q \geq 2$. Let K be a Weierstrass polynomial of degree q in x_N. We write

$$K(x) = x_N^q + \sum_{j=0}^{q-1} a_j(x')x_N^j \tag{3.17}$$

We want to show that $K \circ \phi \not\equiv 0$. Assume for contradiction that $K \circ \phi \equiv 0$. Now as before, we can conclude that $K_{x_N} \circ \phi \equiv 0$. But notice that

$$K_{x_N}(x) = q x_N^{q-1} + \sum_{j=1}^{q-1} j a_j(x') x_N^{j-1}$$

is a Weierstrass polynomial in x_N of degree $q-1$. By induction hypothesis, $K_{x_N} \circ \phi \not\equiv 0$. This is a contradiction. The proof is complete. □

We will briefly recall some basic constructions of formal manifolds and mappings which has been used in many earlier works (e.g., [15, 16, 7, 30, 26]). A formal, generic submanifold M of codimension d through 0 in $\mathbb{C}^N$ is defined by a formal equation of the form (2.5), where $Q(z, \chi, \tau)$ is a $\mathbb{C}^d$-valued power series in $(z, \chi, \tau) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$ satisfying the normality condition (2.6) and the reality condition (2.7).

A formal holomorphic mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ (i.e. a $\mathbb{C}^N$-valued power series in $Z = (z, w)$ with no constant term) is said to send the formal submanifold M into a formal submanifold M' if there exists a $d \times d$ matrix $a(Z, \xi)$ of formal power series such that the following holds.

$$\rho'(H(Z), \bar{H}(\xi)) = a(Z, \xi)\, \rho(Z, \xi) \tag{3.18}$$

If M and M' are smooth, generic submanifolds through p and p' in $\mathbb{C}^N$ and H is a holomorphic mapping $(\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ (or a smooth CR mapping defined on M) sending M into M', then one can associate to M and M' formal manifolds, still denoted by M and M', through 0 and a formal holomorphic mapping, also denoted by $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$, sending M into M'; the reader is e.g. referred to [5] for this natural construction. It is also straightforward to verify that the holomorphic mapping H sending the smooth manifold M into the smooth manifold M' is CR transversal to M' at p if and only if the formal manifolds and mapping satisfy (3.18) with $\det a(0) \neq 0$.

We shall now prove our main result.

Proof of Theorem 3.1. As explained above, we may assume that M and M' are formal manifolds through $0 \in \mathbb{C}^N$, $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ a formal mapping satisfying (3.18), and then to prove Theorem 3.1, it suffices to prove that the matrix $a(Z,\xi)$ in (3.18) satisfies $\det a(0) \neq 0$. Thus, we assume, in order to reach a contradiction, that $\det a(0) = 0$. We deduce from (3.11) that

$$v(0,0) \cdot c(0) = \det a(0) = 0.$$

Thus, $c(0) = 0$ since $v(Z,\xi)$ is a unit. Setting $\xi = 0$ in equation (3.13) yields

$$b(z,0) = b(z,Q(z,0)) = s((z,0),0) \cdot c(0) = 0. \tag{3.19}$$

Thus, it follows from (3.10) that

$$\det a(z,0,\chi,\bar{Q}(\chi,z,0)) = u(z,0,\chi,\bar{Q}(\chi,z,0)) \cdot b(z,0) = 0. \tag{3.20}$$

By taking determinants on both sides of equation (3.3), substituting $\xi = (\chi,\bar{Q}(\chi,z,0))$, and using (2.7), we conclude that

$$\begin{aligned} &(\det \bar{H}_\xi(\chi,\bar{Q}(\chi,z,0)))^d \\ &\qquad = \det a(z,0,\chi,\bar{Q}(\chi,z,0)) \cdot \det E(Z,(\chi,\bar{Q}(\chi,z,0))) \equiv 0. \end{aligned} \tag{3.21}$$

In the hypersurfaces case, the proof is complete. Indeed, if M is finite type at 0, then the map $(z,\chi) \mapsto (\chi,\bar{Q}(\chi,z,0))$ has rank $n+1$ (see e.g [4]) and thus, by Lemma 3.5, equation (3.21) implies that $\det \bar{H}_\xi \equiv 0$. This is a contradiction.

For general case, we will need the iterated Segre mappings introduced in [4] (see Definition 2.8, and also [9, 30]). For a positive integer k, the *kth Segre mapping* of M at 0 is the mapping $v^k : \mathbb{C}^{kn} \to \mathbb{C}^N$ defined by

$$\mathbb{C}^{kn} \ni t = (t^1,\ldots,t^k) \mapsto v^k(t) := (t^k, u^k(t^1,\ldots,t^k)), \tag{3.22}$$

where $u^k : \mathbb{C}^{kn} \to \mathbb{C}^d$ is given inductively by

$$u^1(t^1) = 0,\ u^k(t^1,\ldots,t^k) = Q(t^k, t^{k-1}, \overline{u^{k-1}}(t^1,\ldots,t^{k-1})),\ k \geq 2. \tag{3.23}$$

The crucial property of the Segre mappings needed here is the result (see [4, 6, 5, 9]) that M is of finite type at 0 if and only if the maps v^k has generic rank N for k large enough (Theorem 2.9). Thus, by Lemma 3.5, the following lemma implies that $\det H_Z \equiv 0$, which is a contradiction and completes the proof of Theorem 3.1. □

Lemma 3.6. *For every $j \geq 0$, the following holds.*

$$\det H_Z \circ v^{2j+1} \equiv 0. \tag{3.24}$$

Proof. We may consider $b(Z)$ and $c(\xi)$ in Lemmas 3.3 and 3.4 as power series in (Z, ξ) by $b(Z, \xi) = b(Z)$ and $c(Z, \xi) = c(\xi)$. Since the complexification $\mathcal{M}$ of M is parametrized by

$$(z, \chi, \tau) \mapsto (z, Q(z, \chi, \tau), \chi, \tau),$$

it follows from (3.13) that $b \cong c$ on $\mathcal{M}$, where we use the notation $\alpha \cong \beta$ to mean $\alpha = \gamma\beta$ for some unit γ. Now, another crucial property of the Segre mappings v^k (see e.g. [5]) is that $(v^{k+1}, \overline{v^k}) \in \mathcal{M}$ and $(v^{k-1}, \overline{v^k}) \in \mathcal{M}$ for every k. Consequently, equation (3.13) implies that

$$b \circ v^{k+1} \cong c \circ \overline{v^k}, \quad c \circ \overline{v^k} \cong b \circ v^{k-1}. \tag{3.25}$$

We deduce that $b \circ v^{k+1} \cong b \circ v^{k-1}$ for all $k \geq 2$. By induction, we obtain, for every positive integer j,

$$b \circ v^{2j+1} \cong b \circ v^1.$$

Hence, since $b \circ v^1 \equiv 0$ by (3.19), we conclude that $b \circ v^{2j+1} \equiv 0$. Since, by Lemma 3.3, $b(Z)$ is a divisor of $(\det H_Z(Z))^d$, it follows that

$$\det H_Z \circ v^{2j+1} \equiv 0.$$

This completes the proof of Lemma 3.6. □

We have the following corollary which strengthens [30, Theorem 1.1] and also extends Theorem 4.1 in the same paper to higher codimension case.

Corollary 3.7. *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$. Assume that M' is of finite type at $p' = H(p)$ and H is not totally degenerate at p. Then H is CR transversal to M' at p'.*

Proof. It follows from Proposition 2.3 in [30] that M is of finite type at p and $\operatorname{Jac} H \not\equiv 0$. Thus, we can apply Theorem 3.1 to deduce that H is CR transversal to M' at p'. The proof is complete. □

3.2 Necessary conditions for CR transversality

3.2.1 Mappings of generic full rank

We first prove a necessary condition for CR transversality under the assumption that M is holomorphically nondegenerate.

Theorem 3.8 (Ebenfelt - D. [26]). *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifold of the same dimension through p and p' and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of holomorphic mapping such that $H(M) \subset M'$. Assume that M is holomorphically nondegenerate at 0. If H is CR transversal to M' at 0 then $\operatorname{Jac} H \not\equiv 0$. Furthermore, M' is also holomorphically nondegenerate.*

Before going to proof, we recall the following example showing that the holomorphic nondegeneracy condition on M cannot be relaxed.

Example 3.9. Let $M = M' \subset \mathbb{C}^3$ be the hypersurface given by

$$\Im w = |z_1|^2$$

where (z_1, z_2, w) are coordinates in $\mathbb{C}^3$. Observe that the holomorphic vector field $L = \partial/\partial z_2$ tangent to M. Hence, M is holomorphically degenerate. Consider the mapping $H(z_1, z_2, w) = (z_1, 0, w)$. Then clearly H is CR transversal to M' at 0. However, $\operatorname{Jac} H \equiv 0$ as easy to be seen.

Proof of Theorem 3.8. The idea of the proof is taken from R. Angle [2]. Assume, in order to reach a contradiction, that $\operatorname{Jac} H \equiv 0$. Then, there is a non trivial N-vector $U(Z)$ with components in the field of fractions of $\mathbb{C}[[Z]]$ such that

$$H_Z(Z) \cdot U(Z) \equiv 0. \tag{3.26}$$

By multiplying (3.26) with a suitable power series if necessary, we may assume that $U(Z)$ has components in $\mathbb{C}[[Z]]$. Thus, we can consider the following nontrivial formal holomorphic vector field

$$L = \sum_{j=1}^{n} U_j(Z) \frac{\partial}{\partial Z_j}$$

It follows from (3.26) that $LH_j = 0$ for all $j = 1, \dots N$. Now, since H sends M into M' we have

$$\rho'(H(Z), \bar{H}(\xi)) = a(Z,\xi) \cdot \rho(Z,\xi). \tag{3.27}$$

Applying L to the left hand side of (3.27), we obtain

$$\sum_{j=1}^{N} \rho'_{Z'_j}(H(Z), \bar{H}(\xi)) LH_j(Z) \equiv 0.$$

Consequently, we must also have $L\big(a(Z,\xi)\cdot\rho(Z,\xi)\big) \equiv 0$. In other words,

$$(La)\cdot\rho + a\cdot(L\rho) \equiv 0 \tag{3.28}$$

Since H is CR transversal to M' at 0, we have $\det a(0) \neq 0$ and hence $a(Z,\xi)$ is invertible in $\mathbb{C}[[Z,\xi]]$. We deduce from (3.28) that

$$L\rho = -(a)^{-1}(La)\rho.$$

It follows that L is tangent to M. This is a contradiction since M is holomorphically nondegenerate. The proof is complete. □

We get the following necessary and sufficient condition for CR transversality. This result in the particular case of hypersurfaces confirms a conjecture by Lamel and Mir [46, Conjecture 2.7].

Theorem 3.10 (Ebenfelt - D. [26]). *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds of the same dimension through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$. Assume that M is holomorphically nondegenerate and of finite type at p. Then H is CR transversal to M' at p if and only if* $\operatorname{Jac} H \not\equiv 0$.

Proof. The theorem follows directly from Theorem 3.1 and 3.8. □

3.2.2 Not totally nondegenerate, finite mappings and local biholomorphisms

A nondegeneracy condition for holomorphic mappings sending a hypersurface M into another hypersurface M' is the following. A holomorphic mapping H sending M into M' is said to be *not totally degenerate* if H does not send a neighborhood in $\mathbb{C}^N$ of a point in $p \in M$ into M'. It is known that when M is holomorphically nondegenerate, then this condition is equivalent to $\operatorname{Jac} H \not\equiv 0$ (see [10, Theorem 5.1, Corollary 5.2]). Thus, we have the following theorem about the equivalence of several conditions related to the CR transversality. For the sake of completeness, we also provide a proof.

Theorem 3.11. *Let M and M' be real analytic hypersurfaces in $\mathbb{C}^N$ such that M is holomorphically nondegenerate and of finite type at 0. If H is a holomorphic mapping sending M into M'. Then the following are equivalent.*

(i) H does not send a neighborhood of 0 in $\mathbb{C}^N$ into the Segre variety of M' at 0.

(ii) H does not send a neighborhood of 0 in $\mathbb{C}^N$ into M'.

(iii) $\operatorname{Jac} H \not\equiv 0$.

(iv) H is CR transversal to M' at 0.

Furthermore, if (i) holds then M' is also holomorphically nondegenerate.

Proof. (i)⇒(ii): Let H be a holomorphic mapping sending M into M'. Then we have

$$\rho' \circ H = a \cdot \rho$$

In our normal coordinates, this reads

$$G(Z) - Q(F(Z), \bar{H}(\xi)) = a(Z, \xi) \cdot (w - Q(z, \xi)).$$

Setting $\xi = 0$ we have

$$G(Z) = a(Z, 0)\, w.$$

Since H does not send a neighborhood of 0 in $\mathbb{C}^N$ into the Segre varieties of M' at 0, we deduce that $G(Z) \not\equiv 0$ and so $a(Z,\xi) \not\equiv 0$. In other words, H does not send a neighborhood of 0 in $\mathbb{C}^N$ into M'.

(ii) $\Rightarrow$ (iii): We will use an idea similar to one in [2]. Now assume that H does not send a neighborhood of 0 in $\mathbb{C}^N$ into M', then $a \not\equiv 0$. By factorization, we can write $a(Z,\xi) = \tilde{a}(Z,\xi)\rho^k(Z,\xi)$ for $k \geq 0$ and $\tilde{a}$ is not divisible by ρ. In other words, $\tilde{a}|_{\mathcal{M}} \not\equiv 0$. Thus we can write

$$\rho' \circ H = \tilde{a} \cdot \rho^{k+1}, \quad k \geq 0. \tag{3.29}$$

Now assume for contradiction that $\operatorname{Jac} H \equiv 0$, then as in the proof of Theorem 3.8, we can find a non trivial holomorphic vector field L such that $L(\rho' \circ H) \equiv 0$. Apply L to both sides of (3.29) we get

$$L(\tilde{a} \cdot \rho^{k+1}) \equiv 0.$$

This implies

$$(L\tilde{a})\rho^{k+1} + (k+1)\tilde{a} \cdot \rho^k L\rho \equiv 0.$$

Consequently

$$(L\tilde{a})\rho + (k+1)\tilde{a} \cdot L\rho \equiv 0.$$

Substitute $w = Q(z,\xi)$ into previous equation we get

$$\tilde{a}(z, Q(z,\xi), \xi) \cdot L\rho(z, Q(z,\xi), \xi) \equiv 0.$$

Since $\tilde{a}|_{\mathcal{M}} \not\equiv 0$, i.e., $\tilde{a}(z, Q(z,\xi), \xi) \not\equiv 0$ we can deduce that

$$L\rho|_{\mathcal{M}} \equiv 0.$$

This implies L tangent to M which is a contradiction since M is assumed to be holomorphically nondegenerate. Consequently we have (iii).

(iii)$\Rightarrow$(iv): This follows from Theorem 3.1.

(iv)$\Rightarrow$(i): By Theorem 1.4, we have $\partial G/\partial Z_N(0) \neq 0$. Thus $G(Z) \not\equiv 0$. Recall that the Segre varieties of M' at 0 in normal coordinates is given by the equation $w' = 0$. Thus H does not send any neighborhood of 0 in $\mathbb{C}^N$ into the Segre varieties at 0 which is given by $\{w' = 0\}$. The proof is complete. $\square$

We conclude this section by the following application of CR transversality. First, we recall the following standard definition.

Definition 3.12. Let $H : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$ be a formal holomorphic mapping (or a germ of holomorphic mapping such that $H(0) = 0$. We say that H is *finite* if

$$\dim_{\mathbb{C}} \mathbb{C}[[Z]]/\mathcal{I}(H(Z)) < \infty,$$

where $\mathcal{I}(H(Z))$ denotes the ideal generated by the component of H in $\mathbb{C}[[Z]]$. We note in passing that in the case H is holomorphic, then one can replace $\mathbb{C}[[Z]]$ by $\mathbb{C}\{Z\}$ in the definition above.

Theorem 3.13 (Ebenfelt - D. [26]). *Let $M, M' \subset \mathbb{C}^N$ be smooth generic submanifolds through p and p' respectively, and $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ a germ of a holomorphic mapping such that $H(M) \subset M'$ and $\operatorname{Jac} H \not\equiv 0$. If M is of finite type and essentially finite at p, then H is a finite mapping. If, in addition, M is finitely nondegenerate at p, then H is a local biholomorphism near p.*

Proof. The proof follows from Theorem 3.1 and two theorems by Ebenfelt and Rothschild in [30]. We omit the detail. □

3.3 Examples

In this section, we collect several examples to illustrate that certain conditions imposed in our theorems cannot be relaxed. The first is the following example showing that the finite type condition of M in Theorem 3.1 is optimal.

Example 3.14 ([26]). Let $H : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be the mapping $H(z, w) = (z, w^2)$, and $M \subset \mathbb{C}^2$ the hypersurface given by $\operatorname{Im} w = (\operatorname{Re} w)\varphi(z, \bar{z})$ where φ is a smooth real function such that $\varphi(z, 0) \equiv \varphi(0, \bar{z}) \equiv 0$. Observe that M is parametrized by $(z, \bar{z}, s) \mapsto (z, s + is\varphi(z, \bar{z}))$. Furthermore

$$H(z, s + is\varphi(z, \bar{z})) = (z, s^2(1 - \varphi^2(z, \bar{z})) + 2is^2\varphi(z, \bar{z})).$$

Then it is easy to see that H sends M into the smooth real hypersurface M' given by

$$\operatorname{Im} w' = \frac{2(\operatorname{Re} w')\varphi(z', \bar{z}')}{1 - \varphi^2(z', \bar{z}')}.$$

Observe that M is of infinite type at 0 since it contains the complex hypersurface $\{w = 0\}$, however, φ can be chosen such that M is of finite type at most point except a proper subvariety containing 0. The reader can check that the map H satisfies $\operatorname{Jac} H \not\equiv 0$ but is not transversal to M' at 0.

The following example shows that for higher codimension generic submanifolds, we cannot replace the finite type condition by essential finiteness in Theorem 3.1. We remark that however, for hypersurfaces, essential finiteness implies finite type condition (cf. [6, Proposition 9.4.16]).

Example 3.15. Let $M = M' \in \mathbb{C}^3$ be the generic submanifold given by

$$\Im w_1 = |z|^2, \quad \Im w_2 = 0.$$

One can check easily that M is essentially finite at 0. In fact, M is 1-nondegenerate at every points. On the other hand, along M, the CR vector field

$$X = \frac{1}{2i}\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial \bar{w}_1}$$

tangent to M and thus spans the 1-dimensional $T_p^{0,1}M$ space at every $p \in M$. Furthermore, it is easy to see that at every point $p \in E := \{w_2 = 0\} \cap M$, X and $\overline{X}$ tangent to the submanifold E. In fact, one can see that E is the CR orbit of M at 0. Consequently, M is not of finite type at 0. Now, consider the following mapping

$$H(z, w_1, w_2) = (z, w_1, 1 + w_2 - e^{w_2}).$$

It is easy to verify that $H(M) \subset M$. Furthermore, $\operatorname{Jac} H = 1 - e^{w_2} \not\equiv 0$. Because

$$G_w(z, w) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - e^{w_2} \end{bmatrix} \tag{3.30}$$

we have $\det G_w(0) = 0$ and hence H is not CR transversal to M at 0. Observe also that the Jacobian $\operatorname{Jac} H$ vanishes identically on the complex hypersurface $\{w_2 = 0\}$, as expected. In fact, any holomorphic self-map of M fixing the origin, having nonvanishing Jacobian at some point $p = (z, w_1, w_2)$ with $w_2 \neq 0$ must be local biholomorphic at every point.

We conclude this section by the following example which is borrowed from Ebenfelt and Rothschild [31]. It shows that we cannot drop the condition on genericity of the target manifold in Theorem 3.1.

Example 3.16 (cf. [31])**.** Let $M \subset \mathbb{C}^3$ be given in a neighborhood of the origin by

$$M := \{(z, w_1, w_2) \in \mathbb{C}^3 \mid \operatorname{Im} w_1 = |z|^2, \operatorname{Im} w_2 = |z|^4\} \tag{3.31}$$

Consider the holomorphic mapping $H : \mathbb{C}^3_{z,w_1,w_2} \to \mathbb{C}^3_{z'_1,z'_2,w'}$

$$H(z, w_1, w_2) = (z, w_1 + iw_2, (w_1 - iw_2)^2). \tag{3.32}$$

Then H sends M into the submanifold $M' \subset \mathbb{C}^3_{z'_1,z'_2,w'}$ defined by

$$M' = \{(z'_1, z'_2, w') \in \mathbb{C}^3 \mid w' = (\bar{z}'_2 + i|z'_1|^2 + |z'_1|^4)^2\} \tag{3.33}$$

Observe that M is of finite type at 0 and H is a finite holomorphic mapping. One can check that

$$T_0^{1,0} M' = \operatorname{span}\{\partial/\partial z'_1|_0, \partial/\partial z'_2|_0\} \tag{3.34}$$

Thus $T_0^{1,0} M'$ is of two dimension and hence M is not generic at 0. One can also check that $H_*(\partial/\partial_z|_0)$, $H_*(\partial/\partial_{w_1}|_0)$ and $H_*(\partial/\partial_{w_2}|_0)$ annihilate ρ' and thus H is not CR transversal to M' at 0.

3.4 Acknowledgements

Chapter 3, in full, is a revision of the material as it appear in "Ebenfelt, P., Duong, S., CR transversality of holomorphic mapping between generic submanifolds in complex spaces, *Proc. Amer. Math. Soc.* 140 (2012) 1729–1738". Example 3.14, taken from the same paper, was suggested to the authors of [26] by a referee at Proc. Amer. Math. Soc. The author also thanks Ravi Shroff for sending a copy of [43] which was mentioned in Chapter 1.

Chapter 4

Transversality of mappings: Different dimension case

In the last chapter, we consider the transversality problem for mappings between real submanifolds of a complex space of equal dimensions. It is natural to consider the same problem for the different dimensions case. More precisely, let M and M' be two real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$ respectively (with $1 \leq n < N$) and H a holomorphic mappings sending M into M'. Consider the following question:

"*Under what conditions is H CR transversal to M' at all points?*".

When the target is strictly pseudoconvex, the tranversality holds for all non trivial mappings in any codimensions by mean of Hopf's Lemma. For non-pseudoconvex case, the situation is more delicate. The first result in non-pseudoconvex case may be a result by Baouendi and Huang in [11] where the author proved that "not totally degenerate" mappings between hyperquadrics of the same signature must be transversal to the target hyperquadric. In [3], Baouendi, Ebenfelt and Huang generalized the result to the case in which the hyperquadrics has small signature difference. For more general Levi-nondegenerate hypersurfaces, Huang and Yuan Zhang proved in [41] a transversality result for codimension one case in which the target is a nondegenerate hyperquadric and the source is a non-umbilical, Levi-nondegenerate hypersurface of the same signature. On the other hand, Baouendi, Ebenfelt and Rothschild gave various sufficient geometric conditions for M and M' so that "not totally degenerate" mappings is transversal to the target at *most* points (see [10]).

We first give a sufficient condition for the CR transversality to hold (Theorem 4.3). The result implies that if $N \leq 2n-2$, M' is Levi-nondegenerate, and H is local embedding outside a complex subvariety of codimension 2, then H is transversal to M' at all points of M. We show by examples that this conclusion fails in general if $N \geq 2n$, or if the set W_H of points where H fails to be local embedding has codimension one. Furthermore, we shall refine the result to show that under some additional assumptions on the minors of the Jacobian matrix of H, the mapping is CR transversal to the target. Finally, as an application of this refinement, we shall show that every finite mapping is transversal at all points, provided that the stronger inequality $N \leq 2n-3$ holds and that M is of finite type (see Theorem 4.18).

4.1 Preliminaries

4.1.1 Levi forms

Let M be a (real analytic, smooth or formal) real hypersurface. Associated to M at p, there is a Hermitian form

$$\mathcal{L}_p \colon T_p^{(1,0)}M \times T_p^{(1,0)}M \to \mathbb{C}T_pM/(T_p^{(1,0)}M + T_p^{(0,1)}M) \cong \mathbb{C}$$

called the E. E. Levi form of M at p. In terms of a local defining equation $\rho = 0$, the space $T_p^{(1,0)}M$ can be identified with the subspace of $c \in \mathbb{C}^{n+1}$ such that

$$\sum_{j=1}^{n+1} \frac{\partial \rho}{\partial Z_j}(p,\bar{p})c_j = 0,$$

and then the Levi form $\mathcal{L}_p$ is represented by the restriction to this space of the Hermitian $(n+1)\times(n+1)$-matrix

$$\rho_{Z\bar{Z}}(p,\bar{p}) := \left(\frac{\partial^2 \rho}{\partial Z_i \partial \bar{Z}_j}(p,\bar{p})\right), \quad 1 \leq i,j \leq n+1.$$

Remark 4.1. We remark that in normal coordinates, the $T_0^{(1,0)}M$ space can be identified with the space of $c \in \mathbb{C}^n \times \mathbb{C}$ such that $c_{n+1} = 0$ and the Levi form of M at 0 can be represented by the $n \times n$ matrix $Q_{z\chi}(0,0,0)$. Moreover, M is of finite type at 0 if and only if $Q(\chi, z, 0) \not\equiv 0$.

4.1.2 Complexifications of real hypersurfaces and mappings

If U is an open neighborhood of M in $\mathbb{C}^{n+1}$ and $H\colon U \to \mathbb{C}^{N+1}$ a holomorphic mapping, then H sends M into a smooth real hypersurface $M' \subset \mathbb{C}^{N+1}$ if and only if there is a smooth function a in $U \subset \mathbb{C}^{n+1}$ such that $\rho' \circ H = a\rho$, where ρ' denotes a defining function for M'. Moreover, H is transversal to M' precisely at those points $p \in M$ where $a \neq 0$ (see e.g. [30]). In what follows, we shall always assume, without loss of generality of course, that the given points $p \in M$ and $p' = H(p) \in M'$ are both the origin $p = 0 \in \mathbb{C}^{n+1}$, and $p' = 0 \in \mathbb{C}^{N+1}$.

When M and M' are real-analytic, then ρ and ρ' are given by convergent power series in $(Z, \bar{Z}) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ and $(Z', \bar{Z}') \in \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$, respectively. By replacing $\bar{Z}$ and $\bar{Z}'$ by independent variables ξ and $\bar{\xi}$, we obtain a holomorphic mapping $\mathcal{H} := (H, \bar{H})\colon U \times U^* \to \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$, where

$$\bar{H}(\xi) := \overline{H(\bar{\xi})}, \quad U^* := \{\xi \in \mathbb{C}^{n+1} \colon \bar{\xi} \in U\},$$

sending 0 to 0 and $\mathcal{M}$ into $\mathcal{M}'$, where $\mathcal{M} := \{\rho(Z,\xi) = 0\} \subset U \times U^*$ and $\mathcal{M}' = \{\rho'(Z',\xi') = 0\} \subset \mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$ denote the complexifications of M and M', respectively. Thus, we have

$$\rho'(H(Z), \bar{H}(\xi)) = a(Z,\xi)\rho(Z,\xi), \tag{4.1}$$

and $\mathcal{H}$ fails to be transversal to $\mathcal{M}'$ precisely along the common zero set of $a(Z,\xi)$ and $\rho(Z,\xi)$. If M and M' are merely C^∞-smooth, then we can replace ρ, a, and ρ' by their formal Taylor series at 0 in $(Z, \bar{Z})$ and $(Z', \bar{Z}')$ and H by its convergent (or formal if H is a C^∞-smooth CR mapping) Taylor series at 0 and obtain (4.1) as an identity of formal power series. This is standard procedure in the field, and is referred to as identifying M and M' with their formal manifolds and considering H as a formal mapping sending M into M'; the reader is referred to e.g. [16, 6, 5] for further discussion of this procedure.

Remark 4.2. In what follows, we shall work over the rings of formal power series with formal manifolds and mappings, unless explicitly specified otherwise. For convenience, we shall also drop the $'$ on the target space coordinates (Z', ξ'), as it will be clear from the context to which space the variables belong.

4.2 A sufficient condition for transversality

4.2.1 Main result

Given a holomorphic mapping $H \colon U \subset \mathbb{C}^{n+1} \to \mathbb{C}^{N+1}$, we shall consider the complex analytic subvariety

$$W_H := \{Z \in U \colon \operatorname{rk} H_Z(Z) < n+1\}, \tag{4.2}$$

where H_Z denotes the $(N+1) \times (n+1)$ matrix of partial derivatives of the components of H,

$$H_Z := \left(\frac{\partial H_i}{\partial Z_j}\right), \quad 1 \leq i \leq N+1,\ 1 \leq j \leq n+1.$$

We shall only consider situations where W_H is a proper subvariety (i.e. the rank of H is of generic maximal rank); just as in the equidimension case mentioned above, this is essentially necessary for transversality to hold under some mild conditions on M. Observe that if $\delta_l(Z)$, for $l = 1, \dots, m \leq \binom{N+1}{N-n}$, denote the collection of all non-trivial $(n+1) \times (n+1)$-minors of the matrix $H_Z(Z)$, then W_H coincides with the set defined by

$$\delta_1(Z) = \ldots = \delta_m(Z) = 0.$$

Thus, when $N > n$ the codimension of this set is in general large, and the codimension is one only when all the minors have a common divisor. Our first result is the following:

Theorem 4.3 (Ebenfelt - D. [27]). *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{N+1}$ be smooth real hypersurfaces through p and p' respectively, and $H : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{N+1}, p')$ a germ at p of holomorphic mapping such that $H(M) \subset M'$. Denote by r the rank of the Levi form of M' at p' and assume that*

$$2N - r \leq 2n - 2. \tag{4.3}$$

If the germ at p of the analytic variety W_H, given by (4.2)*, has codimension at least* 2*, then H is transversal to M' at p.*

The proof of this theorem will be given in Section 4.2.3.

4.2.2 Examples

The following example shows that condition (4.3) in Theorem 4.3 is at least "almost" sharp.

Example 4.4. Consider the strictly pseudoconvex hyperquadric $M \subset \mathbb{C}^{n+1}$ (biholomorphically equivalent to the sphere) given by

$$\mathrm{Im}\ w - \sum_{j=1}^{n} |z_j|^2 = 0$$

and the nondegenerate hyperquadric $M' \subset \mathbb{C}^{2n+1}$ given by

$$\mathrm{Im}\ w' + \sum_{j=1}^{n} |z'_{2j-1}|^2 - \sum_{j=1}^{n} |z'_{2j}|^2 = 0,$$

where we use coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $(z', w,) \in \mathbb{C}^{2n} \times \mathbb{C}$. Now, consider the polynomial mapping $H = (F_1, F_2, \ldots, F_{2n-1}, F_{2n}, G)\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{2n+1}, 0)$ given by

$$H(z,w) := \Big(z_1 + [z]\, z_1 + \frac{i}{2} w, z_1 - [z]\, z_1 - \frac{i}{2} w, \ldots, \\ z_n + [z]\, z_n + \frac{i}{2} w, z_n - [z]\, z_n - \frac{i}{2} w, -2[z]w \Big), \quad (4.4)$$

where we have used the notation $[z] := \sum_{j=1}^{n} z_j$.

Lemma 4.5. *We claim that H sends M into M', H is a local embedding at* 0 *(and hence as germs at* 0*, we have $W_H = \emptyset$), but H is not transversal to M' along the intersection of M and the real hypersurface* $\mathrm{Re}\,[z] = 0$*, and hence, in particular, is not transversal at* 0.

The proof of this Lemma will be given in Section 4.6. In this example, $N = 2n$ and $r = N = 2n$ (since M' is Levi nondegenerate). Thus, we have $2N - r = N = 2n$, which is equal to $(2n-2)+2$ and hence condition (4.3) is violated. However, the authors do not know of an example where $2N-r=(2n-2)+1=2n-1$, which leaves open the possibility that condition (4.3) could be sharpened to $2N-r\leq 2n-1$ in Theorem 4.3.

We would like to point out that when the target M' is Levi-nondegenerate at p' (i.e. $r=N$, as in Example 4.4 above), then the condition (4.3) can be rewritten $N-n\leq n-2$. (The number $N-n$, the difference between dimension of the target space and the source space, is often referred to as the *codimension* of the mapping.) That is, transversality holds at p for mappings H up to a codimensional gap $N-n$ that increases with the CR dimension n of the source manifold, *provided* that the codimension of W_H is at least 2. The following example shows that this phenomenon fails if we allow W_H to have codimension one.

Example 4.6. Consider the sphere $M\subset\mathbb{C}^{n+1}$ given by

$$\sum_{j=1}^{n+1}|Z_j|^2-1=0$$

and the nondegenerate hyperquadric $M'\subset\mathbb{C}^{n+3}$ given by

$$\operatorname{Im} w'-\left(\sum_{j=1}^{n+1}|z_j'|^2-|z_{n+2}'|^2\right)=0.$$

It is straightforward to verify that the polynomial mapping $H\colon\mathbb{C}^{n+1}\to\mathbb{C}^{n+3}$ given by

$$H(Z):=(Z_1^2,Z_1Z_2,\ldots Z_1Z_{n+1},Z_1,0)$$

sends M into M'. The set W_H is given by $Z_1=0$ (and hence has codimension one), and the mapping H is not transversal to M' along the intersection of the sphere M with W_H (cf. Example 2.3 in [10]). Thus, this is a family of examples where W_H has codimension one, the mapping has codimension 2 (i.e. $N-n=2$), and transversality fails at certain points regardless of the CR dimension n of the source.

Example 4.6 shows that even for Levi-nondegenerate hypersurfaces and mappings of generic full rank, transversality may fail at specific points unless further conditions are imposed. One direction is to assume conditions relating the signatures of the Levi forms as in [11, 40, 41] in which transversality is proved for mappings between hyperquadrics of the same signature. We shall not pursue this direction here.

4.2.3 Proof of Theorem 4.3

In this section, we will assume M and M' are (analytic, smooth, or formal) real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, and as mentioned in the previous section we shall assume that $p = 0 \in M$ and $p' = 0 \in M'$. We shall identify M and M' with formal hypersurfaces as explained in the previous section. We shall also assume in this section that

$$2N - r \leq 2n - 2, \tag{4.5}$$

where r is the rank of Levi form of M' at $p' = 0$. We shall use the notation $\rho(Z, \xi)$ and $\rho'(Z', \xi')$ for (complexified) formal defining functions for M and M', respectively. In normal coordinates, ρ has the following form

$$\rho(Z, \xi) = w - Q(Z, \xi); \quad \bar{\rho} = \tau - \bar{Q}(\chi, Z), \tag{4.6}$$

where Q and $\bar{Q}$ are described in Section 2.3.

Let $H\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ be a formal holomorphic mapping sending M into M', i.e.

$$\rho'(H(Z), \bar{H}(\xi)) = a(Z, \xi)\rho(Z, \xi) \tag{4.7}$$

where $a(Z, \xi)$ is a formal power series in $\mathbb{C}[[Z, \xi]]$. We shall assume that H has generic rank $n+1$, i.e., there is at least one $(n+1) \times (n+1)$-minors of H_Z which does not vanish identically. We shall denote by $\{\delta_l(Z), l = 1, 2, \ldots, m\}$ the collection of $(n+1) \times (n+1)$-minors which do not vanish identically. (Thus, we have $1 \leq m \leq \binom{N+1}{N-n}$.)

We first observe that $a \neq 0$ when (4.5) holds. For the readers convenience, we sketch the simple proof. Assume that $a \equiv 0$. Then, by differentiating (4.7) once with respect to Z and once with respect to ξ, we obtain

$$H_Z^t(Z)\, \rho'_{Z\xi}(H(Z), \bar{H}(\xi))\, \bar{H}_\xi(\xi) = 0, \tag{4.8}$$

where as before H_Z is the $(N+1)\times(n+1)$ Jacobian matrix of H; superscript t denotes transpose of a matrix, and $\rho'_{Z\xi}$ is an $(N+1)\times(N+1)$-matrix. If we let S denote the field of fractions of $\mathbb{C}[[Z,\xi]]$, then we can regard (4.8) as a matrix identity over S. Note that the ranks of H_Z^t and $\bar{H}_\xi$ over K are both $n+1$, and the rank of $\rho'_{Z\xi}$ is at least r (since the rank of $\rho'_{Z\xi}(0,0)$ is at least r). Elementary linear algebra implies that $n+1-(N+1-r)\geq(N+1)-(n+1)$ or, equivalently, $2N-t\geq 2n$, proving our claim that $a\not\equiv 0$ under condition (4.5). It now follows from Theorem 1.1 in [10] that a is not a multiple of ρ. In other words, in normal coordinates $Z=(z,w)$ and $\xi=(\chi,\tau)$ as in the previous section, we have

$$a(Z,(\chi,\bar{Q}(\chi,Z)))\not\equiv 0,\quad a((z,Q(z,\chi,\tau)),\xi)\not\equiv 0$$

where Q and $\bar{Q}$ are as in (4.6).

As mentioned in the first section, the mapping H is transversal to M' at 0 if and only if $a(0,0)\neq 0$. We shall consider the ideal $I:=I(a,\rho)$ in the ring $\mathbb{C}[[Z,\xi]]$ of formal power series in (Z,ξ). We that note the power series a depends on the choices of defining power series ρ and ρ', but the ideal I clearly does not. In the case $a(0,0)=0$, the ideal I is proper. When M, M' and H are analytic, I defines a complex analytic variety $\mathcal{X}$ in $\mathbb{C}^{n+1}_Z\times\mathbb{C}^{n+1}_\xi$ consisting of the points at which the complexified mapping $\mathcal{H}(Z,\xi)=(H(Z),\bar{H}(\xi))$ fails to be transversal to the complexified hypersurface $\mathcal{M}'$. In this section, we shall also give a description (Corollary 4.13) of the non-transversality locus $\mathcal{X}$ (in the analytic case) of a (complexified) holomorphic mapping of generic full rank when the condition on the codimension of W_H in Theorem 4.3 fails; when the codimensional condition on W_H holds, we shall show that $\mathcal{X}$ is empty. But first let us observe some simple properties of I.

The following definition is standard (see, e.g., [61]).

Definition 4.7. Let $I\subset R$ be a ideal. The radical of I, denotes by $\sqrt{I}$ is the following.

$$\sqrt{I}=\{f\in R: f^q\in I \text{ for some positive integer } q\}. \tag{4.9}$$

Lemma 4.8. *The ideal I and its radical $\sqrt{I}$ are Hermitian, i.e. if $\alpha(Z,\xi) \in \mathbb{C}[[Z,\xi]]$, then $\alpha(Z,\xi) \in I$ if and only if $\bar{\alpha}(\xi,Z) \in I$, and similarly for $\sqrt{I}$.*

Proof. Recall that we can choose real-valued defining functions ρ and ρ' for M and M' respectively and, hence, the corresponding function a is real-valued as well. At the level of formal power series, this is equivalent to ρ, ρ', a being Hermitian; i.e. if u equals ρ, ρ', or a, then $u(Z,\xi) = \bar{u}(\xi,Z)$. The conclusion of Lemma 4.8 follows immediately. □

In the following lemma, we use normal coordinates $Z = (z,w)$, $\xi = (\chi,\tau)$ as above, and $\bar{Q}(\chi,Z) = \bar{Q}(\chi,z,w)$ is the power series appearing in (4.6). Recall that $a(Z,(\chi,\bar{Q}(\chi,Z))) \not\equiv 0$.

Lemma 4.9. *Assume that $a(0,0) = 0$ and let*

$$a(Z,(\chi,\bar{Q}(\chi,Z))) = a_1^{t_1}(Z,\chi)\dots a_k^{t_k}(Z,\chi)$$

be the unique (modulo units) factorization into irreducible (or prime) elements in $\mathbb{C}[[Z,\chi]] \subset \mathbb{C}[[Z,\xi]]$. Let $I_j = I(a_j,\rho)$. Then,

$$\sqrt{I} = \cap_{j=1}^{k} I_j$$

is a Lasker–Noether decomposition of $\sqrt{I}$.

Proof. Recall that we may choose $\rho(Z,\xi) = \tau - \bar{Q}(\chi,Z)$. It then follows that for some $\tilde{a}(Z,\xi) \in \mathbb{C}[[Z,\xi]]$

$$a(Z,\xi) = a(Z,(\chi,\bar{Q}(\chi,Z))) + \tilde{a}(Z,\xi)\rho(Z,\xi).$$

Hence $a \in I_j$ and so $I = I(a,\rho) \subset I_j$ for all $j = 1,\dots k$.

Next, we claim that, for each j, the ideal I_j is prime. Indeed, fix j and let $f,g \in \mathbb{C}[[Z,\xi]]$ such that $fg \in I_j$. Then

$$f(Z,\xi)g(Z,\xi) = r(Z,\xi)a_j(Z,\chi) + s(Z,\xi)\rho(Z,\xi),$$

for some $r,s \in \mathbb{C}[[Z,\xi]]$. If we substitute $\xi = (\chi,\bar{Q}(\chi,Z))$ in this identity, then we obtain

$$f(Z,(\chi,\bar{Q}(\chi,Z)))g(Z,(\chi,\bar{Q}(\chi,Z))) = r(Z,(\chi,\bar{Q}(\chi,Z)))a_j(Z,\chi)$$

Since $a_j(Z,\chi)$ is irreducible, we deduce that it divides, say, $f(Z,(\chi,\bar{Q}(\chi,Z)))$. It follows that

$$\begin{aligned} f(Z,\xi) &= f(Z,(\chi,\bar{Q}(\chi,Z))) + \tilde{f}(Z,\xi)\rho(Z,\xi) \\ &= r(Z,(\chi,\bar{Q}(\chi,Z)))a_j(Z,\chi) + \tilde{f}(Z,\xi)\rho(Z,\xi) \end{aligned} \tag{4.10}$$

for some $\tilde{f}(Z,\xi)$ and so f belongs to I_j. We conclude that I_j is prime, as desired. Since $I \subset I_j$, for all j, and I_j is prime, we conclude that $\sqrt{I} \subset I_j$, for all j, proving $\sqrt{I} \subset \cap_{j=1}^k I_j$.

Now assume $f(Z,\xi) \in I_j$ for all j. Then we can write, for any fixed j,

$$f(Z,\xi) = r(Z,\xi)a_j(Z,\chi) + s(Z,\xi)\rho(Z,\xi),$$

for some power series r and s. If we substitute $\tau = \bar{Q}(\chi,Z)$, then we get

$$f(Z,(\chi,\bar{Q}(\chi,Z))) = r(Z,(\chi,\bar{Q}(\chi,Z)))a_j(Z,\chi).$$

Thus $f(Z,(\chi,\bar{Q}(\chi,Z)))$ is divisible by $a_j(Z,\chi)$ for all $j = 1,2,\ldots k$. It then follows that, for some integer l, $f(Z,(\chi,\bar{Q}(\chi,Z)))^l$ is divisible by $a(Z,(\chi,\bar{Q}(\chi,Z)))$. We conclude that

$$f(Z,\xi)^l = f(Z,(\chi,\bar{Q}(\chi,Z)))^l + \tilde{f}(Z,\xi)\rho(Z,\xi),$$

for some $\tilde{f}(Z,\chi)$, belongs to I and hence $f(Z,\xi) \in \sqrt{I}$. Consequently, $\cap_{j=1}^k I_j \subset \sqrt{I}$. The proof is complete. □

A key point in the proof of our main results is the following lemma.

Lemma 4.10. *Assume that $a(0,0) = 0$. Then, for each j, either $\delta_l(Z) \in I_j$ for every $l = 1,2,\ldots m$, or $\bar{\delta}_l(\xi) \in I_j$ for every $l = 1,2,\ldots m$. As above, $\delta_l(Z), l = 1,2,\ldots m$ denotes the collection of all $(n+1)\times(n+1)$-minors of $H_Z(Z)$ that do not vanish identically.*

Proof. If we differentiate (4.7) with respect to Z we obtain

$$H_Z(Z)^t\,\rho_Z'(H(Z),\bar{H}(\xi)) = a_Z(Z,\xi)\rho(Z,\xi) + a(Z,\xi)\rho_Z(Z,\xi). \tag{4.11}$$

Here, as above, the Jacobian matrix H_Z is regarded as an $(N+1)\times(n+1)$-matrix, the superscript t denotes transposition of a matrix, and the gradient vectors are regarded as column vectors. Let $K = \mathbb{C}[[Z]]$. Then $\mathbb{C}[[Z,\xi]]$ can be identified with the ring $K[[\xi]]$. We can regard equation (4.11) as an identity in $(K[[\xi]])^{N+1}$. Thus, we may rewrite this identity as follows

$$H_Z^t\,\rho_Z'(\bar{H}(\xi)) = a_Z(\xi)\rho(\xi) + a(\xi)\rho_Z(\xi), \tag{4.12}$$

where we have used the notation $\rho_Z'(\xi') := \rho_Z'(H(Z),\xi')$; H_Z^t is a matrix with components in the field K and e.g. $a(\xi)$ is a formal power series in ξ whose coefficients are elements in K.

Since I_j is proper prime ideal of $K[[\xi]]$, it follows that $K[[\xi]]/I_j$ is an integral domain. Let us fix a j, and define S to be the field of fractions of $K[[\xi]]/I_j$. Denote by π the canonical projection: $\pi\colon K[[\xi]] \to K[[\xi]]/I_j,\ x \mapsto x + I_j$.

Now, let L be a formal vector field (or a derivation) in $K[[\xi]]$, i.e.

$$L = \sum_{l=1}^{n+1} \beta_l(\xi)\frac{\partial}{\partial \xi_l}$$

where $\beta_l(\xi) \in K[[\xi]]$. We say that L is Zariski tangent to I_j if $L(f) = 0 \mod I_j$ for all $f \in I_j$, or equivalently,

$$\sum_{l=1}^{n+1} \beta_l(\xi)\frac{\partial a_j}{\partial \xi_l}(\xi) = \sum_{l=1}^{n+1} \beta_l(\xi)\frac{\partial \rho}{\partial \xi_l}(\xi) = 0 \mod I_j. \tag{4.13}$$

It is straightforward to see that there are at least $n-1$ formal vector fields $L_1, \ldots L_{n-1}$ tangent to I_j,

$$L_k = \sum_l \beta_l^k(\xi)\partial/\partial\xi_l, \tag{4.14}$$

such that the collection of corresponding vectors in S^{n+1}:

$$\hat{V}_k = (\pi(\beta_1^k(\xi)), \ldots, \pi(\beta_{n+1}^k(\xi))), \quad k = 1, 2, \ldots n-1,$$

is linearly independent over the quotient field S of $K[[\xi]]/I_j$. Indeed, let us consider the following system of two linear equations over S with unknowns X_l, $l = 1, 2, \ldots n+1$,

$$\sum_{l=1}^{n+1} X_l\,\pi(a_{j,\xi_l}(\xi)) = 0, \quad \sum_{l=1}^{n+1} X_l\,\pi(\rho_{\xi_l}(\xi)) = 0.$$

This system has at least $n-1$ linearly independent solutions in S^{n+1}, denoted by

$$\tilde{V}_k = (\tilde{\beta}_1^k, \ldots, \tilde{\beta}_{n+1}^k), \quad k = 1, \ldots, n-1.$$

where $\tilde{\beta}_l^k \in S$. Since each component $\tilde{\beta}_l^k$ is a fraction $\tilde{\beta}_l^k = \mu_l^k / \nu_l^k$ with $\mu_l^k, \nu_l^k \in K[[\xi]]/I_j$, we can clear the denominators and obtain $n-1$ linearly independent vectors

$$\hat{V}_k = (\hat{\beta}_1^k, \ldots, \hat{\beta}_{n+1}^k),$$

whose components belong to $K[[\xi]]/I_j$. Since $\pi\colon K[[\xi]] \to K[[\xi]]/I_j$ is surjective, we can find $\beta_l^k(\xi) \in K[[\xi]]$ such that $\pi(\beta_l^k(\xi)) = \hat{\beta}_l^k$. The corresponding formal vector fields L_k, given by (4.14), satisfy the desired properties.

We now apply the vector fields L_k to (4.12). It follows from (4.13) and the fact that $I \subset I_j$ (Lemma 4.9) that

$$L_k\big(a_Z(\xi)\rho(\xi) + a(\xi)\rho_Z(\xi)\big) = 0 \mod I_j \text{ for } k = 1, 2, \ldots n-1.$$

Consequently, we have the following identity

$$L_k\big(H_Z^t\, \rho'_Z(\bar{H}(\xi))\big) = 0 \mod I_j \text{ for } k = 1, 2, \ldots n-1. \tag{4.15}$$

Using chain rule, we can rewrite (4.15) in matrix notations as follows

$$H_Z^t\, \Phi(\xi)\, \bar{H}_\xi(\xi)\, V_k(\xi) = 0 \mod I_j \text{ for } k = 1, 2, \ldots n-1, \tag{4.16}$$

where we have used the notation $V_k(\xi) = (\beta_1^k(\xi), \ldots, \beta_{n+1}^k(\xi))^t$ and $\Phi(\xi)$ for the $(N+1) \times (N+1)$ matrix $(\rho'_{Z\xi}(\bar{H}(\xi)))$. Note that since the Levi form of M' at 0 (which is represented by the restriction of $\rho'_{Z\xi}(0,0)$ to the holomorphic tangent space of M' at 0) has rank r by assumption, there is an $r \times r$-minor of Φ which is a unit in $K[[\xi]]$.

We now go to the quotient field S of $K[[\xi]]/I_j$. We will put a hat over elements of $K[[\xi]]$ (including vectors and matrices with elements in $K[[\xi]]$) to indicate their images in $K[[\xi]]/I_j$ under the canonical projection π. Thus, (4.16) implies

$$\hat{H}_Z^t\, \hat{\Phi}\, \hat{\bar{H}}_\xi\, \hat{V}_k = 0 \text{ for } k = 1, 2, \ldots n-1. \tag{4.17}$$

Let us assume, in order to reach a contradiction, that there is at least one $\delta_l(Z) \notin I_j$ and at least one $\bar{\delta}_{l'}(\xi) \notin I_j$. Consequently, the corresponding minors $\hat{\delta}_l$ and $\hat{\bar{\delta}}_{l'}$ do not vanish in S and it follows that the matrices $\hat{H}_Z^t$ and $\hat{\bar{H}}_\xi$ have rank $n+1$ over the field S. Furthermore, since there is an $r \times r$-minor of the matrix Φ which is a unit in $K[[\xi]]$, we deduce that $\hat{\Phi}$ has rank at least r over S. Now, consider the following collection of vectors in S^{N+1}:

$$Y_k = \hat{\Phi}\ \hat{\bar{H}}_\xi\ \hat{V}_k, \quad k = 1, \dots n-1 \quad \text{and} \quad Y_n = (\hat{\rho}'_{z_1}, \dots \hat{\rho}'_{z_{N+1}}). \tag{4.18}$$

Here we recall that $\hat{\rho}'_{z_l} = \pi(\rho'_{z_l}(\bar{H}(\xi)))$. We claim that the rank of the collection of vectors $Y_1, \dots Y_n \in S^{N+1}$ is at least $n+r-N-1$. Indeed, since $\hat{\Phi}$ has rank at least r over S and the $(N+1) \times (n+1)$-matrix $\hat{\bar{H}}_\xi$ has full rank $(= n+1)$, we deduce that the collection $Y_1, \dots Y_{n+1}$ has rank at least $n+r-N-2$. On the other hand, observe that in normal coordinates in the target space we may choose $\rho'(Z,\xi) = w - Q'(z,\chi,\tau)$ and, hence, the last row of $\hat{\Phi}$ contains only zeros resulting in the last component of each Y_k, for $k = 1, \dots n-1$, being 0. On the other hand, the last component of Y_n is 1, so that Y_n cannot be a linear combination of Y_k for $k = 1, \dots, n-1$. The claim that the rank of the collection $\{Y_k\}_{k=1}^n$ is $n+r-N-1$ follows. To complete the proof of the lemma, we observe from (4.12) and (4.17) that $\hat{H}_Z^t\, Y_k = 0$ for $k = 1, \dots n$. Since the $(n+1) \times (N+1)$-matrix $\hat{H}_Z^t$ has rank $n+1$, we deduce that $(N+1)-(n+1) \geq n+r-N-1$. This implies $2N - r \geq 2n - 1$ which contradicts (4.5). The proof of Lemma 4.10 is complete. □

Proposition 4.11. *Suppose that $a(0,0) = 0$. Then there are $B(Z), C(Z) \in \mathbb{C}[[Z]]$ such that*

$$a(Z,\xi) = B(Z)\bar{C}(\xi)t(Z,\xi) + \tilde{a}(Z,\xi)\rho(Z,\xi), \tag{4.19}$$

for some $\tilde{a}(Z,\xi), t(Z,\xi) \in \mathbb{C}[[Z,\xi]]$ such that $t(Z,\xi)$ is a unit. Moreover, each irreducible divisor of $B(Z)$ and each irreducible divisor of $C(Z)$ divides $\delta_l(Z)$ for every $l = 1, \dots, m$.

Proof. By Lemma 4.10, for each j, we have either $\delta_l(Z) \in I_j$ for all $l = 1, \dots, m$ or $\bar{\delta}_j(\xi) \in I_j$ for all l. Let us first assume that $\delta_l(Z) \in I_j$ for all l. Consequently,

$$\delta_l(Z) = r_l(Z,\xi)\ a_j(Z,\chi) + s_l(Z,\xi)\ \rho(Z,\xi)$$

for some r_l and s_l. By substituting $\tau = \bar{Q}(\chi, Z)$ we obtain

$$\delta_l(Z) = r_l(Z, (\chi, \bar{Q}(\chi, Z)))\ a_j(Z, \chi)$$

Now, recalling that $a_j(Z, \chi)$ is irreducible, we conclude that

$$a_j(Z, \chi) = b_j(Z)\ u_l(Z, \chi) \tag{4.20}$$

where $b_j(Z)$ is an irreducible (or prime) divisor of $\delta_l(Z)$ and $u_l(Z, \chi)$ is an unit. This holds for all $l = 1, 2, \ldots m$ (with the same $b_j(Z)$, modulo units) and hence $b_j(Z)$ is an irreducible divisor of $\delta_l(Z)$ for every $l = 1, \ldots, m$.

If $\bar{\delta}_l(\xi) \in I_j$ for all l, then we can write $\bar{\delta}_l(\xi) = r_l(Z, \xi)\ a_j(Z, \chi) + s_l(Z, \xi)\ \rho(Z, \xi)$. Thus, by substituting $w = Q(Z, \xi)$ this time, we obtain

$$\bar{\delta}_l(\xi) = r_l(Z, (\chi, \bar{Q}(\chi, Z)))\ a_j((z, Q(z, \xi)), \chi)$$

Observe that $a_j((z, Q(z, \xi)), \chi)$ is also irreducible, a fact that follows easily from the identity

$$a_j(Z, \chi) = a_j((z, w), \xi) = a_j((z, Q(z, \chi, \bar{Q}(\chi, z, w))), \chi),$$

where we recall that $\xi = (\chi, \tau)$. It then follows as above that $a_j((z, Q(z, \xi)), \chi)$ is an irreducible divisor of $\bar{\delta}_l(\xi)$ and thus $a_j((z, Q(z, \xi)), \chi) = \bar{c}_j(\xi)\ v_l(Z, \xi)$ for some irreducible divisor $\bar{c}_j(\xi)$ of $\bar{\delta}_l(\xi)$ and unit $v_l(Z, \xi)$. Again, since $\bar{\delta}_l(\xi) \in I_j$ for every l, we conclude that $\bar{c}_j(\xi)$ divides $\bar{\delta}_l(\xi)$ for every l. If we substitute $\tau = \bar{Q}(\chi, Z)$, then we obtain

$$a_j(Z, \chi) = \bar{c}_j(\chi, \bar{Q}(\chi, Z))\ \tilde{v}_l(Z, \chi),$$

where $\tilde{v}_l(Z, \chi) = v(Z, (\chi, \bar{Q}(\chi, Z))$. Putting all this together, we conclude (via Lemma 4.9) that

$$a(Z, (\chi, \bar{Q}(\chi, Z))) = a_1^{t_1}(Z, \chi) \cdots a_k^{t_k}(Z, \chi) = B(Z)\bar{C}(\chi, \bar{Q}(\chi, Z))\ t(Z, \xi) \tag{4.21}$$

where $t(Z, \xi)$ is a unit. By construction, every irreducible divisor of $B(Z)$ divides $\delta_l(Z)$ for all l, and every irreducible divisor of $\bar{C}(\xi)$ divides $\bar{\delta}_l(\xi)$ for all l. We conclude that

$$\begin{aligned} a(Z, \xi) &= a(Z, (\chi, \bar{Q}(\chi, Z))) + \tilde{a}(Z, \xi)\rho(Z, \xi) \qquad (4.22) \\ &= B(Z)\ C(\chi, \bar{Q}(\chi, Z))\ t(Z, \xi) + \tilde{a}(Z, \xi)\rho(Z, \xi). \end{aligned}$$

Similarly, we can also write

$$\bar{C}(\chi, \bar{Q}(\chi, Z)) = \bar{C}(\xi) + \tilde{C}(Z, \xi)\rho(Z, \xi),$$

which by substituting into (4.22) yields the desired form of $a(Z, \xi)$. The proof is complete.

□

We may now prove the following result, which as explained above is a reformulation of Theorem 4.3 in the formal setting (and hence has Theorem 4.3 as a direct consequence).

Theorem 4.12 (Ebenfelt - D. [27]). *Let M and M' be formal real hypersurfaces through 0 in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, and $H\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{N+1}, 0)$ a formal holomorphic mapping sending M into M'. Assume that*

$$2N - r \leq 2n - 2,$$

where r is the rank of Levi form of M' at 0. Assume further that the Jacobian matrix H_Z is of generic rank $n+1$ (i.e. at least one $(n+1) \times (n+1)$-minor is not identically zero) and that the collection of its not-identically-zero $(n+1) \times (n+1)$-minors $\delta_1, \ldots, \delta_m$ have no nontrivial common divisor. Then, H is transversal to M' at 0.

Proof. Assume, in order to reach a contradiction, that H is not transversal to M' at 0, i.e. $a(0,0) = 0$ where $a(Z, \xi)$ is given by (4.7). By Proposition 4.11, there are nontrivial power series $B(Z)$ and $C(Z)$ such that (4.19) holds, and such that every irreducible divisor of $B(Z)$ and every irreducible divisor of $C(Z)$ divides $\delta_l(Z)$ for all l. Also, note that at least one of $B(Z)$ or $C(Z)$ has to be 0 at $Z = 0$, since $a(0,0) = \rho(0,0) = 0$. This contradicts the assumption that $\delta_1(Z), \ldots, \delta_m(Z)$ have no common divisor. The proof is complete. □

4.3 The non-transversality locus

We conclude this section by giving a description in the analytic case (i.e. M, M', and H are analytic) of the non-transversality locus

$$\mathcal{X} := \{(Z,\xi) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \colon a(Z,\xi) = \rho(Z,\xi) = 0\} = \{(Z,\xi) \in \mathcal{M} \colon a(Z,\xi) = 0\}$$

of the complexified mapping $\mathcal{H}(Z,\xi) = (H(Z), \bar{H}(\xi))$ when (4.5) holds but the codimension of W_H is one. (Of course, when the codimension of W_H is at least two, we just proved that $\mathcal{X}$ is empty.) Recall (see e.g. Example 4.6) that $\mathcal{X}$ may be non-empty in this situation, but it turns out that the variety must have a special form. In context of Segre preserving maps, a similar description was given in [62].

Corollary 4.13 (Ebenfelt - D. [27]). *Let M and M' be real-analytic hypersurfaces through 0 in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, and $H \colon (\mathbb{C}^{n+1},0) \to (\mathbb{C}^{N+1},0)$ a holomorphic mapping sending M into M'. Assume that*

$$2N - r \leq 2n - 2,$$

where r is the rank of Levi form of M' at 0, and that the Jacobian matrix H_Z is of generic rank $n+1$ (i.e.W_H is a proper subvariety). If H is not transversal to M' at 0 then the non-transversality locus $\mathcal{X} = \{(Z,\xi) \in \mathcal{M} : a(Z,\xi) = 0\}$ of the complexified mapping $\mathcal{H} = (H, \bar{H})$ has a decomposition into irreducible components of the following form

$$\mathcal{X} = \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_k,$$

where $\mathcal{X}_j$ is either of the form $\{(Z,\xi) : Z \in W_i, \xi \in S^*_Z\}$ or $\{(Z,\xi) : \xi \in W^*_i, Z \in S_{\bar{\xi}}\}$. Here, the W_i denote the irreducible, codimension one components of W_H, * denotes the complex conjugate of a set, and

$$S_p := \{Z \in \mathbb{C}^{n+1} \colon \rho(Z, \bar{p}) = 0\}$$

denotes the Segre varieties of M at p. Moreover, $\mathcal{X}$ is Hermitian symmetric, i.e. $\mathcal{X}^* = \mathcal{X}$.

Proof. Observe that by Proposition 4.11, $(Z,\xi)\in\mathcal{X}$ if and only if $B(Z)\bar{C}(\xi)=0$, where each irreducible factor of B and C divide every $(n+1)\times(n+1)$-minor δ_l of H. The decomposition of $\mathcal{X}$ in Corollary 4.13 follows readily from this fact. The Hermitian symmetry is immediate from Lemma 4.8. The proof is complete. □

4.4 A refinement

The aim of this section is to prove a refinement of Theorem 4.3 which will lead to a tranversality result for finite mappings presented in the next section. First, let us introduce some notations. For an integer s with $1\leq s\leq n+1$, we define

$$W_H^s := \{Z\in\mathbb{C}^{n+1}\colon\ \operatorname{rk} H_Z(Z)<s\}. \tag{4.23}$$

Thus, $W_H^{n+1}=W_H$. Clearly, each W_H^s is a complex analytic variety defined by the vanishing of all $k\times k$ minors of H_Z, for $k=s,\ldots,n+1$, and we have a nesting

$$W_H^1\subset W_H^1\subset\ldots\subset W_H^{n+1}=W_H.$$

Our first result in this section is the following theorem.

Theorem 4.14 (Ebenfelt - D. [27]). *Let $M\subset\mathbb{C}^{n+1}$ and $M'\subset\mathbb{C}^{N+1}$ be smooth real hypersurfaces through p and p' respectively, and $H:(\mathbb{C}^{n+1},p)\to(\mathbb{C}^{N+1},p')$ a germ at p of holomorphic mapping such that $H(M)\subset M'$. Denote by r the rank of the Levi form of M' at p'. Assume that M is of finite type at p and that, for some $1\leq s\leq n+1$,*

$$2N-r\leq n+s-3. \tag{4.24}$$

If the germ at p of the analytic variety W_H, given by (4.2)*, is proper (i.e. H has generic rank $n+1$) and the germ at p of W_H^s, given by* (4.23)*, has codimension at least* 2*, then H is transversal to M' at p.*

As explained in Chapter 2 that if H is a smooth CR mapping from M to M', then we can identify H with a formal holomorphic power series mapping in the variable $Z\in\mathbb{C}^{n+1}$ centered at $Z=p$ and sending M into M' (formally). Thus, Theorem 4.14 is a consequent of the following.

Theorem 4.15 (Ebenfelt - D. [27]). *Let $M \subset \mathbb{C}^{n+1}$ and $M \subset \mathbb{C}^{N+1}$ be formal real hypersurface and H formal mapping such that $H(M) \subset M'$ and the collection of not identically zero $(n+1) \times (n+1)$-minors of $H_Z(Z)$, denoted as before by $\delta_l(Z)$ for $l = 1, \ldots, m$, is non-empty. Let r be the rank of the Levi form of M' at 0 and assume that, for some $1 \leq s \leq n+1$,*

$$2N - r \leq n + s - 3. \tag{4.25}$$

Suppose further that M is of finite type at 0, and that for every common divisor $d(Z)$ of the collection $\delta_l(Z)$, $l = 1, \ldots, m$, there is at least one $k \times k$-minor $\delta'(Z)$ of the Jacobian matrix $H_Z(Z)$ such that $k \geq s$ and $\delta'(Z)$ is relatively prime to $d(Z)$. Then H is transversal to M' at 0.

For the proof, we will need the following lemma.

Lemma 4.16. *Assume that M is of finite type at 0 and H is not transversal to M', i.e., $a(0) = 0$. Let I_j be the ideal defined in Lemma 4.9. If there exists a non-trivial $\alpha(Z) \in I_j$ then there are no non-trivial $\beta(\xi) \in I_j$.*

Proof. We assume, in order to obtain a contradiction, that there are non-trivial power series $\alpha(Z), \beta(\xi) \in I_j = I(a_j, \rho)$. We can argue as in the proof of Proposition 4.11 to deduce that

$$a_j(Z, \chi) = b(Z)\ u(Z, \chi). \tag{4.26}$$

where $u(Z, \chi)$ is a unit and $b(Z)$ an irreducible divisor of $\alpha(Z)$ in $\mathbb{C}[[Z]]$. Similarly, we can deduce from the fact that $\beta(\xi) \in I$ that

$$a_j(Z, \chi) = c(\chi, \bar{Q}(\chi, Z))\ v(Z, \xi) \tag{4.27}$$

where $v \in \mathbb{C}[[z, \xi]] \subset \mathbb{C}[[Z, \xi]]$ is a unit and $c(\xi)$ is a divisor of $\beta(\xi)$ in $\mathbb{C}[[\xi]]$. Now, we deduce from (4.26) and (4.27) that

$$u(Z, \xi)\ b(Z) = a_j(Z, \chi) = v(Z, \xi)\ c(\chi, \bar{Q}(\chi, Z)).$$

Hence, for some unit $s(Z, \xi)$,

$$b(Z) = c(\chi, \bar{Q}(\chi, Z))\ s(Z, \xi). \tag{4.28}$$

If we substitute $Z = 0$ into (4.28), then we get

$$c(\chi, 0) = c(\chi, \bar{Q}(\chi, 0)) = s(0, \xi)^{-1} \, b(0) = 0.$$

We deduce that $c(\xi) = \tau \tilde{c}(\xi)$, where $\tilde{c}(\xi)$ is an unit since $c(\xi)$ is irreducible. Then, by setting $\chi = 0$ in (4.28) and recalling that $\bar{Q}(0, Z) = w$, we deduce that for some unit $\tilde{b}(Z, \xi)$:

$$b(Z) = c(0, \bar{Q}(0, Z)) \; s(Z, 0, \tau) = w \; \tilde{b}(Z, \xi). \tag{4.29}$$

Consequently:

$$w \; \tilde{b}(Z, \xi) = b(Z) = c(\chi, \bar{Q}(\chi, Z)) \; s(Z, \xi) = \bar{Q}(\chi, Z) \; \tilde{c}(\chi, \bar{Q}(\chi, Z) \; s(Z, \xi). \tag{4.30}$$

This and the fact that s and $\tilde{c}$ are units imply $\bar{Q}(\chi, z, 0) = 0$. This contradicts the fact that M is of finite type at 0. The proof is complete. □

Now we can prove the Theorem 4.15.

Proof of Theorem 4.15. We will argue by contradiction. Assume that H is not transversal to M' at 0, i.e. $a(0, 0) = 0$. Recall that I_j, $j = 1, \ldots, k$, denote the ideals defined in Lemma 4.9. By Lemma 4.10, for each j either $\delta_l(Z) \in I_j$ for all l or $\bar{\delta}_l(\xi) \in I_j$ for all l. We claim that for some j, $\delta_l(Z) \in I_j$ for all l. Indeed, even if $\bar{\delta}_l(\xi) \in I_j$ for all l and all j, then (by Lemma 4.9) $\bar{\delta}_l(\xi) \in \sqrt{I}$ and, hence, by Lemmas 4.8 and 4.9 we would also have $\delta_l(Z) \in I_j$ for all l and j. (Although this does not matter for the proof, we point out that this latter situation cannot occur by Lemma 4.16.)

Let us now fix a j be such that $\delta_l(Z) \in I_j$ for all l. We claim that there is an $k \times k$-minor $\delta'(Z)$ of $H_Z(Z)$ such that $k \geq s$ and $\delta'(Z) \notin I_j$. Indeed, if $\delta'_i(Z)$, for $i = 1, \ldots, p$, denote the collection of all $k \times k$-minors of $H_Z(Z)$ for $k \geq s$ and $\delta'_i(Z) \in I_j$ for all i, then we can argue as in the proof of Proposition 4.11 (considering the collection of $\delta'_i(Z)$ and $\delta_l(Z)$, for $i = 1, \ldots, p$ and $l = 1 \ldots, m$) and conclude that there is a common irreducible divisor $b(Z)$ of $\delta'_i(Z)$ and $\delta_l(Z)$ for all $i = 1, \ldots, p$ and $l = 1, \ldots m$ (which is also a divisor of $a_j(Z, \chi)$, although this does not matter here). This contradicts the fact that to every common divisor of the $\delta_l(Z)$, $l = 1, \ldots, m$, there is at least one $\delta_i'(Z)$ which is relatively prime to it. Thus, let $\delta'(Z)$ denote a $k \times k$-minor, with $k \geq s$, such that $\delta'(Z) \notin I_j$. We will now proceed along the lines of the proof of Lemma 4.10, using the same notation as in that proof. We first observe that, by Lemma 4.16, no $\bar{\delta}_l(\xi) \in I_j$ by our choice of j. We conclude that the rank of $\hat{\bar{H}}_\xi(\xi)$ over the quotient field S of $K[[\xi]]/I_j$ is $n+1$. On the other hand, since $\delta'(Z) \notin I_j$, it also follows that the rank of $\hat{H}_Z(Z)$ over S is at least $k \geq s$. We then argue in the same way as in the proof of Lemma 4.10 to deduce from (4.17) that $2N - r \geq n + s - 2$, which contradicts our assumption (4.25). The proof is complete. □

4.5 Finite mappings

The aim of this section is to prove a transversality theorem for finite mappings.

Definition 4.17. A germ at p of a mapping $H\colon (\mathbb{C}^{n-1}, p) \to (\mathbb{C}^{N+1}, p')$ is finite if $H^{-1}(p') = \{p\}$ as germs at p, or equivalently if the vector space $\mathbb{C}[[Z]]/I(H)$ is finite dimensional over $\mathbb{C}$; here, $\mathbb{C}[[Z]]$ denotes the ring of formal power series in Z and $I(H)$ denotes the ideal generated by the components of H.

Theorem 4.18 (Ebenfelt - D. [27])**.** *Let $M \subset \mathbb{C}^{n+1}$ and $M' \subset \mathbb{C}^{N+1}$ be smooth real hypersurfaces through p and p' respectively, and $H : (\mathbb{C}^{n+1}, p) \to (\mathbb{C}^{N+1}, p')$ a germ at p of holomorphic mapping such that $H(M) \subset M'$. Denote by r the rank of the Levi form of M' at p' and assume that*

$$2N - r \leq 2n - 3. \tag{4.31}$$

Assume also that M is of finite type at p and H is a finite mapping at p. Then, H is transversal to M' at p.

Proof. We note that if H is a finite mapping at p, then W_H is proper and W_H^n has codimension at least 2. Thus, Theorem 4.18 follows from Theorem 4.14 with $s = n$. The proof is complete. □

4.6 Proof of the Lemma 4.5

Proof of Lemma 4.5. We first note that

$$\begin{aligned}\left|z_j + [z]\, z_j + \frac{i}{2}w\right|^2 - \left|z_j - [z]\, z_j - \frac{i}{2}w\right|^2 \\ &= 2\left(z_j\left(\overline{[z]z_j} - \frac{i}{2}\bar{w}\right) + \bar{z}_j\left([z]z_j + \frac{i}{2}w\right)\right) \\ &= 2\left(\left([z] + \overline{[z]}\right)|z_j|^2 + \frac{i}{2}(\bar{z}_j w - z_j\bar{w})\right)\end{aligned} \tag{4.32}$$

Thus, it follows that the expression

$$2i\left(\mathrm{Im}\ G + \sum_{j=1}^{n}|F_{2j-1}|^2 - \sum_{j=1}^{n}|F_{2j}|^2\right) = G - \bar{G} + 2i\sum_{j=1}^{n}\left(|F_{2j-1}|^2 - |F_{2j}|^2\right) \tag{4.33}$$

is equal to

$$\begin{aligned}-2\left([z]w - \overline{[z]w} - 2i\left([z] + \overline{[z]}\right)\sum_{j=1}^{n}|z_j|^2 + \overline{[z]}w - [z]\bar{w}\right) \\ = -2\left([z] + \overline{[z]}\right)(w - \bar{w} - 2i\sum_{j=1}^{n}|z_j|^2),\end{aligned} \tag{4.34}$$

or, in other words,

$$\mathrm{Im}\ G + \sum_{j=1}^{n}|F_{2j-1}|^2 - \sum_{j=1}^{n}|F_{2j}|^2 = i\left([z] + \overline{[z]}\right)(w - \bar{w} - 2i\sum_{j=1}^{n}|z_j|^2), \tag{4.35}$$

proving that H sends M into M', and that H is not transversal to M' along the intersection of M with $\mathrm{Re}\,[z] = 0$ as claimed in Example 4.4. The fact that H is a local embedding at 0 is trivial. □

4.7 Conclusion

We have provided several sufficient conditions for transversality of holomorphic mappings between real hypersurfaces. Also, we have analyzed examples showing that certain conditions in our theorems cannot be avoid. However, there are several situations which haven't been settled including:

- The situation when $N - n = 1$, $n > 1$ and the mapping is of generic full rank. We have shown by examples for any n that in codimension two case, holomorphic mappings with generic full rank need not be transversal. However, these example cannot be adapted to the codimension one case. We also mention here the conjecture in [41] for the case when the hypersurfaces are Levi-nondegenerate of the same signature.
- The situation when $N = 2n - 1$, $n > 1$ and the variety W_H has codimension two.
- The situation when $N = 2n - 2$, $n > 2$ and H is of finite multiplicity.

They would be very interesting questions for future research.

4.8 Acknowledgements

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Index