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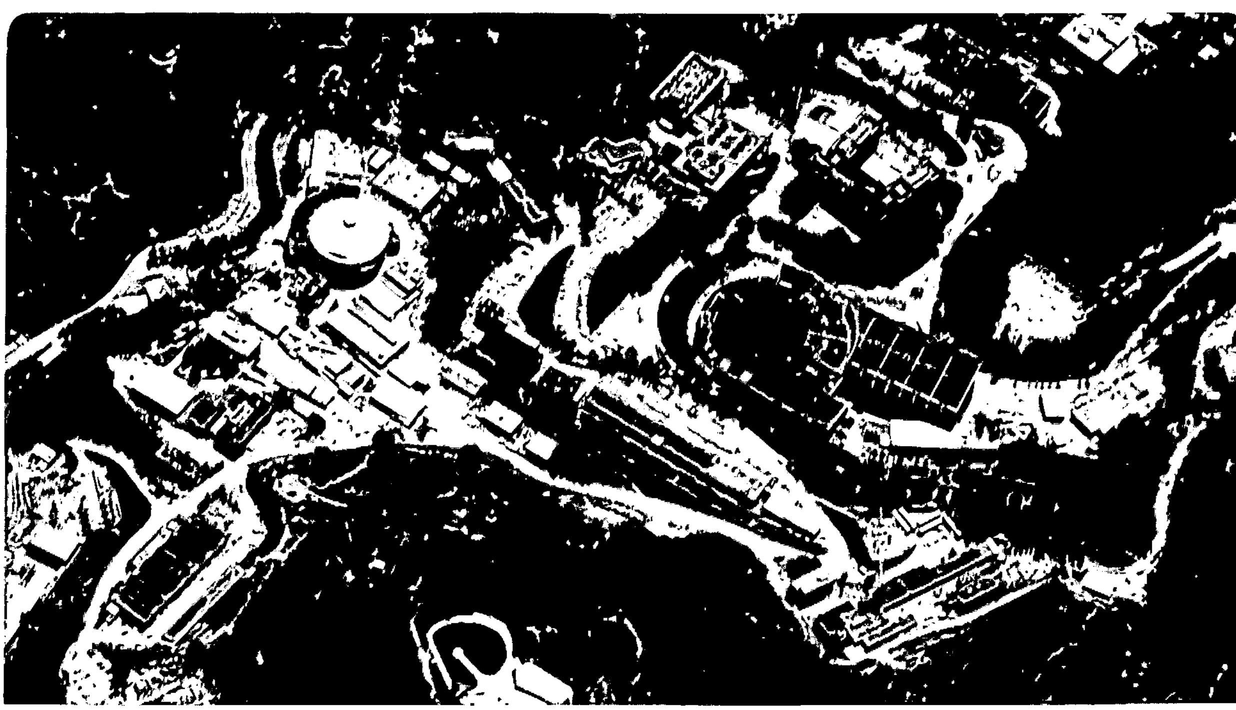

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$N = 2$ String Amplitudes *

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In physics, solvable models have played very important roles. Understanding a simple model in detail teaches us a lot about more complicated models in generic situations. Five years ago, C. Vafa and I found [1] that the closed $N = 2$ string theory, that is a string theory with the $N = 2$ local supersymmetry on the worldsheet, is classically equivalent to the self–dual Einstein gravity in four spacetime dimensions. Thus this string theory is solvable at the classical level.

More recently, we have examined the $N = 2$ string partition function for spacial compactifications, and computed it to all order in the string perturbation expansion [2]. The fact that such computation is possible at all suggests that the $N = 2$ string theory is solvable even quantum mechanically.

Another important aspect of the $N = 2$ string theory is that various perturbative computations in the Type II superstring theory compactified down to six or four dimensions can be transformed into problems in the $N = 2$ string theory and they become easier to handle. This is because the Type II superstring can be reformulated as $N = 2$ string theory in which the worldsheet $N = 2$ supersymmetry is spontaneously broken down to $N = 1$ by super–Higgs mechanism. This was discovered by Berkovits and Vafa [3].

Because of this $N = 2$ string $\Leftrightarrow$ Type II string correspondence, our $N = 2$ string computation can be described in the context of the Type II superstring theory. In view of the nature of this workshop, I will present our results from this point of view.

Let me first talk about the Type II string compactified down to four dimensions.

$$\mathbf{R}^4 \times M \ , \qquad M : \mathrm{Calabi-Yau \ \ three-fold}.$$

The supersymmetric sigma–model on M has the $N = 2$ superconformal symmetry with the central charge $c = 9$. The $\mathbf{R}^4$ part of the Type II string can be described by using so–called New Green–Schwarz variables [4]. It was found by Berkovits that this part also have the $N = 2$ superconformal symmetry with $c = -3$ [5]. Therefore the total system has $c = 6$, which is the critical central charge for the $N = 2$ string.

Although Berkovits and Vafa showed that any Type II string amplitude can be transformed into $N = 2$ string amplitude [3], the computation becomes particularly easy for the class of amplitudes I now describe.

The low energy effective theory of the Type IIA string has the $N = 2$ supersymmetry on $\mathbf{R}^4$. There are $h^{1,1} = \mathrm{dim}H^{1,1}(M)$ vector multiplets corresponding to the Kähler moduli of M and one vector multiples containing the gravi–photon which is in the Ramond–Ramond sector of the Type II string. On the other hand, the dilaton field, whose expectation value determines the string coupling constant, belongs to the hyper–multiplet. The gravi–photon decouples at zero momentum, and the F–term for the Kähler moduli is determined at the string tree level since there is no neutral coupling between the vector and the hyper–multiplets.

To describe coupling of the gravi–photon, it is useful to introduce the Weyl superfield $W_{\mu\nu}^{ij}$ $(i,j=1,2;\ \mu,\nu=0,...,3)$ whose lowest component is the gravi–photon field strength $T_{\mu\nu}$ as

$$W^2 = T_{\mu\nu}T^{\mu\nu} + \cdots.$$

Let us consider an F–term of the following form

$$F_g(t)(W^2)^g,$$

where t^a $(a = 1,...,h^{1,1})$ are vector superfields corresponding to the Kähler moduli of M. Since W^2 has the weight 2, assuming that there is no neutral coupling between the vector and the hyper–multiplets, we conclude that F_g gets contribution only from g–loop string amplitude. This computation can be done exactly by transforming it into $N = 2$ string problem [6], [7].

In order for this to make sense as an F–term, F_g must be a chiral superfield, which means F_g is holomorphic in the Kähler moduli t^a.

$$\bar{\partial}_{\bar{t}^a} F_g = 0$$

However the exact computation by Bershadsky, Cecotti, Vafa and myself showed that there is a holomorphic anomaly to this equation.

$$\begin{aligned}\bar{\partial}_{\bar{t}^a} F_g &= \frac{1}{2}\bar{C}_{\bar{a}\bar{b}\bar{c}} e^{2K} G^{b\bar{b}} G^{c\bar{c}} \times \\ &\times \Big(D_b D_c F_{g-1} + \sum_{r=1}^{g-1} D_b F_r D_c F_{g-r} \Big),\end{aligned}$$

where C_{abc} and K are the Yukawa coupling and the Kähler potential for t^a's and $G_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K$ is the Zamolodchikov metric. From the target space point of view this holomorphic anomaly may be interpreted as due to massless particle propagations in internal loops, thus the holomorphy should be restored if we consider an Wilsonian effective action. In [6], F_g is computed exactly by integrating the holomorphic anomaly equation.

Now let me turn to six-dimensional compactification of Type II string

$$\mathbf{R}^6 \times M, \quad \mathrm{dim} M = 4.$$

The low energy effective theory on $\mathbf{R}^6$ has $N=2$ supersymmetry. Each matter multiplet comes with four scalar fields, each corresponding to the moduli of M. I denote them by t^{ij} $(i,j=1,2)$, with the reality condition

$$(t^{ij})^* = \epsilon^{ik}\epsilon^{jl}t^{kl}.$$

The analytic field strength W for the gravity multiplet contains the gravi–photon as its lowest component.

In order to describe a low energy effective action with the $N=2$ supersymmetry in six dimensions, it is convenient to use the harmonic superspace. As in the case of the four–dimensional compactification, we may consider an effective action term of the form

$$\int F_g(t) W^{4g}$$

where the integral is over the harmonic superspace. For this integral to make sense, F_g must satisfy

$$\begin{aligned} &\epsilon^{ij} \frac{\partial}{\partial u_L^i} \frac{D}{Dt^{jk}} F_g = 0 \\ &\epsilon^{ij} \frac{\partial}{\partial u_R^i} \frac{D}{Dt^{kj}} F_g = 0, \end{aligned} \tag{1}$$

where (u_L, u_R) are the harmonic superspace coordinates [8].

According to the $N=2$ string $\Leftrightarrow$ Type II string correspondence, we should be able to interpret F_g as an $N=2$ string amplitude. It turned out that a g–loop contribution to F_g is equal to the g–loop partition function of the $N=2$ string on M [8]. Since F_g depends on the harmonic superspace coordinates (u_L, u_R), the precise statement is the following. The $N=2$ string has the local $N=2$ supersymmetry on the worldsheet, and the $N=2$ supergravity multiplet contains the $U(1)$ gauge field. Therefore the $N = 2$ supermoduli integral involves a summing over the instanton number of the $U(1)$ gauge field. In fact, due to the holomorphic splitting of the worldsheet theory, it is possible to define separate instanton numbers for the left and the right–moving sectors. Let us denote by $\mathcal{A}^g_{nm}$ the g–loop partition function function of the $N = 2$ string with the left and right instanton numbers n and m. We should restrict n and m to be $|n|, |m| \leq 2g - 2$ since otherwise there will be a bosonic ghost zero mode which cannot be absorbed. Now the statement is that the $N = 2$ string partition function $\mathcal{A}^g_{nm}$ appears in the expansion of the Type II string effective action F_g in powers of u_L and u_R as

$$\begin{aligned} F_g &= (u^1_L u^2_L u^1_R u^2_R)^{2g-2} \sum_{n,m=-(2g-2)}^{2g-2} \mathcal{A}^g_{nm} \times \\ & \times \begin{pmatrix} 4g-4 \\ 2g-2+n \end{pmatrix} \begin{pmatrix} 4g-4 \\ 2g-2+m \end{pmatrix} \times \\ & \times (u^1_l/u^2_L)^n (u^1_R/u^2_R)^m . \end{aligned} \tag{2}$$

As in the case of the four–dimensional compactification, the equations (1) are potentially anomalous due to propagation of massless particles in internal loops. Vafa and I examined these equations carefully and found that in fact there is an anomaly. This anomaly, however, can be canceled if we consider the following combination.

$$\begin{aligned} \epsilon^{ij} u^k_R \frac{\partial}{\partial u^i_L} \frac{D}{Dt^{jk}} F_g = 0 \\ \epsilon^{ij} u^k_L \frac{\partial}{\partial u^i_R} \frac{D}{Dt^{kj}} F_g = 0, \end{aligned} \tag{3}$$

These are exact equations for the $N = 2$ string partition function. It should be very interesting to study it on the 80–dimensional moduli space of $K3$ surface.

We were able to solve the equations (3) to all order in g for a target space $M = T^2 \times \mathbf{R}^2$, where T^2 is a two-dimensional torus. Let us denote the Kähler moduli of T^2 by $\rho = \theta + ir$ where r is proportional to the size of T^2. Our solution is

$$F^g = c^{(g)} \sum_{(r,s)\in Z^2\backslash(0,0)} |r + s\rho|^{2g-4} \times$$

$$\times \left(\frac{u_L^1 u_R^1}{r+s\rho} + \frac{u_L^2 u_R^2}{r+s\bar{\rho}} \right)^{4g-4},$$

where $c^{(g)}$ is a constant independent of ρ and is related to some topological invariant on the moduli space of genus–g Riemann surface. By comparing this with (2), we find that the g–loop partition function of the $N = 2$ string is given by

$$\mathcal{A}^g_{nn} \propto \sum_{(r,s)\in Z^2\setminus(0,0)} \frac{1}{(r+s\rho)^{g+n}(r+s\bar{\rho})^{g-n}}$$

and $\mathcal{A}^g_{nm} = 0$ if $n \neq m$.

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