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The Fukaya category of the elliptic curve as an algebra over the Feynman transform

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The Fukaya Category of the Elliptic Curve as an Algebra over the Feynman Transform

A dissertation submitted in partial satisfaction of the
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in

Mathematics

by

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The dissertation of Michael A. Slawinski is approved, and it is acceptable in quality and form for publication on microfilm:

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University of California, San Diego

2011

DEDICATION

To Kristen, my brother Thomas, and my parents.

EPIGRAPH

The reward for work well done is the opportunity to do more.

Jonas Salk

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I would first like to thank Professor Mark Gross for suggesting this topic, and for his continued support throughout the past four years. This project would not have been possible without his vision or the many useful conversations that continue to this day. Thank you to my father for fueling the fire of my curiosity for math and science from the beginning, and to my mother, who taught me to brave all challenges with a smile. I would also like to thank my brother Thomas for both his friendship and humor, which brighten my life each and every day. Thank you finally to my fiancee Kristen for her constant love and support. Because of her, I have not only been able to share my life as a whole, but the unique experience of graduate school with the person I love most, and for this I am forever in her debt.

VITA

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ABSTRACT OF THE DISSERTATION

The Fukaya Category of the Elliptic Curve as an Algebra over the Feynman Transform

by

Michael A. Slawinski
Doctor of Philosophy in Mathematics

University of California San Diego, 2011

Professor Mark Gross, Chair

In [4] Barannikov proves the equivalence between the existence of a morphism of twisted modular operads $\mathscr{F}_D\mathbb{S}[t] \longrightarrow \mathcal{E}_V$ ($\mathscr{F}$ and $\mathcal{E}_V$ as defined in [17]), and certain tensors of $\mathcal{E}_V$ satisfying the quantum master equation of Batalin-Vilkovisky geometry of an affine $\mathbb{S}[t]$-manifold. He then suggests the possibility of generalizing this morphism to the categorical case by replacing $\mathcal{E}_V$ with a twisted modular operad, referred to here has $\mathcal{E}_L$, constructed from the Lagrangian submanifolds of a fixed symplectic manifold.

The main result of this thesis is the construction of an explicit example of such a morphism in the case that the symplectic manifold is an elliptic curve.

Given a symplectic manifold X, one constructs a precategory closely related to the Fukaya Category, denoted $\mathrm{Fuk}(X)$, whose objects are the Lagrangian submanifolds $L \subset X$, and whose morphism spaces $\mathrm{Hom}(L_i, L_j)$ are finite dimensional modules over the Novikov ring, generated by the points of $L_i \cap L_j$. There is a nondegenerate bilinear pairing

$$B : \mathrm{Hom}(L_i, L_j) \otimes \mathrm{Hom}(L_j, L_i) \longrightarrow \mathbb{C},$$

which is degree -1 in the elliptic curve case.

Given a finite sequence of cyclic chains of Lagrangian submanifolds

$$\{L_{i0}, \dots, L_{id_i}\}_{i=1}^{b-2g+1}$$

in the elliptic curve, we construct elements

$$\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1}) \in \bigotimes_{i=1}^{b-2g+1} \bigotimes_{j=1}^{d_i} \mathrm{Hom}(L_{ij}, L_{i(j-1)}) \otimes \mathrm{Hom}(L_{i0}, L_{id_i}) \qquad (0.1)$$

of degree $(d+1)-(2-2b)$, defined by summing over zero-dimensional tropical Morse graphs $\mathcal{G}$ with $\dim \mathrm{H}_1(\mathcal{G}) = b$, where $d+1 = \sum_{i=1}^{b+1}(d_i+1)$ and $b-2g+1$ is the number of cycles.

The tensors (0.1) define the twisted modular operad

$$\mathcal{E}_L((d+1,b)) \coloneqq \bigotimes_{i=1}^{b-2g+1} \bigotimes_{j=1}^{d_i} \mathrm{Hom}(L_{ij}, L_{i(j-1)}) \otimes \mathrm{Hom}(L_{i0}, L_{id_i}),$$

whose contraction maps $\mu_G^{\mathcal{E}_L}$ are given by contraction via the bilinear form B.

We construct a morphism of twisted modular operads from the Feynman transform of a twist of $\tilde{\mathbb{S}}[t]$ to $\mathcal{E}_L$, where $\tilde{\mathbb{S}}[t]$ is an untwisted version of $\mathbb{S}[t]$, by mapping the generators $\{\sigma_1 \cdots \sigma_{b-2g+1}\}$ of $\tilde{\mathbb{S}}[t]$ to the elements $\{\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1})\}$. This algebra structure is equivalent to the set $\{\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1})\}$ being a solution to the quantum master equation of [4], or, equivalently, to $\{\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1})\}$ satisfying what will be referred to here as the *quantum A_∞-relations*. The usual A_∞-relations on $\mathrm{Fuk}(X)$ are recovered by setting $b=0$.

Chapter 1

Introduction

1.1 Mirror Symmetry

1.1.1 String Theory

String theory is a proposal which seeks to unify the standard model of particle physics with Einstein's theory of general relativity. By replacing point particles with 1-dimensional strings, string theory tempers the quantum fluctuations of space time, which on extremely small scales, render general relativity meaningless. The large families of particles, and the virtual particles which control their interaction are replaced by vibrating strings, where the vibrational pattern determines the species of particle or virtual particle. Just as point particles are replaced by 1-dimensional strings, the curves in spacetime traced out by the motion of the point particles are replaced by 2-dimensional worldsheets as the strings travel through spacetime. The motion of a string in terms of the vibrational pattern is restricted by the necessity of satisfying the Einstein field equation, and the supersymmetry (holonomy) requirement of string theory. More precisely, the manifold to which the string vibrations are restricted must have a Ricci-flat metric with $SU(3)$-holonomy. These requirements lead the team of Strominger, Candelas, Horowitz and Witten to Calabi-Yau manifolds in 1984 in [11].

A Calabi-Yau n-fold X is a compact n-dimensional Kähler manifold such that any one of the following equivalent conditions holds.

1. The canonical bundle of X is trivial, that is, $K_X \simeq \mathcal{O}_X$.

2. X has a holomorphic volume form that is nowhere vanishing.

3. The structure group of X can be reduced from $U(n)$ to $SU(n)$.

4. The first integral Chern class $c_1(X)$ vanishes.

5. X has a Kähler metric with global holonomy contained in $SU(n)$.

Examples include elliptic curves, $K3$ surfaces, and the non-singular quintic 3-fold X_ψ given by the solution set in $\mathbb{P}^4$ of the equation

$$x_0^5 + \cdots + x_4^5 + \psi x_0 x_1 x_2 x_3 x_4 = 0 \qquad \psi \in \mathbb{C} \tag{1.1}$$

It is easy to see that the first and third examples are Calabi-Yau. Indeed, any compact hypersurface of $\mathbb{P}^n$ is Kähler, as the metric is inherited from $\mathbb{P}^n$, and the canonical bundle of a hypersurface in $\mathbb{P}^n$ is given by $\mathcal{O}(d-n-1)$, where d is the degree of the hypersurface. Elliptic curves are cubic by definition, and the quintic is obviously degree 5, so in either case one has $K \simeq \mathcal{O}(n+1-n-1) = \mathcal{O}$. This in fact gives an easy recipe for producing Calabi-Yau's; one need only take a projective hypersurface of degree $n+1$, where n is the dimension of the ambient space.

One can now describe string theory in a more mathematically precise fashion. The worldsheet obtained by the vibration of a string as time passes is replaced by a holomorphic map from a Riemann surface to the target Calabi-Yau.

At present the five distinct types of string theory are open type I, closed type I, closed type IIA, closed type IIB, and the $SO(32)$ and $E_8 \times E_8$ flavors of heterotic string theory.

1.1.2 The Mathematics of Mirror Symmetry

Mirror Symmetry was discovered as a duality between pairs of Calabi-Yau 3-folds in 1989 by Greene and Plesser in [18], with a large number of examples calculated in four-dimensional weighted projective space by Candelas, Lynker, and Schimmrigk in [10]. In string theoretic terms, type IIA string theory on a Calabi-Yau X is mirror dual to type IIB on $X^\vee$, the mirror dual of X. This duality was expressed as the reversal of the Hodge numbers $h^{1,1}$ and $h^{2,1}$, that is, $h^{1,1}(X) = h^{2,1}(X^\vee)$ and $h^{2,1}(X) = h^{1,1}(X^\vee)$, for X and $X^\vee$ a mirror pair. A more tangible consequence of this is that $\chi(X) = -\chi(X^\vee)$, a result that dramatically increased the list of such 3-folds with positive Euler characteristic.

There have been several programs, with various points of view, to construct and study mirror pairs, and families of mirror pairs of Calabi-Yau manifolds. Strominger, Yau, and Zaslow study mirror symmetry geometrically in [37] by viewing a pair of mirror Calabi-Yau's as torus fibrations with dual fibers. However, the notable shortcoming of this approach is the lack of understanding of how the mirror correspondence behaves with respect to the singular fibers. Papers by Gross [19, 20, 21] show the program works at a topological level, and the analytic aspects are being tackled to this day by Kontsevich and Soibelman in [33, 34], using rigid analytic geometry, and by Gross and Siebert in [22, 27, 24, 25], using tropical and log geometry.

In [5], [6], and [7], Batyrev and Borisov construct families of mirror Calabi-Yau hypersurfaces by examining dual lattice polyhedra in $\mathbb{R}^n$. The construction is also studied by Gross in [23] using a toric degeneration, in which a family of toric varieties is studied as a fibration over an affine base, degenerating to the singular central fiber.

Finally, Kontsevich introduced a program in [32] called Homological Mirror Symmetry, which interprets mirror symmetry as the equivalence of two "categories", the bounded derived category of coherent sheaves on a Calabi-Yau X, denoted $D^b_\infty(\check{X})$ (B-side), and the Fukaya category, denoted $\mathrm{Fuk}(X)$(A-side). The objects of the former are bounded complexes $\mathcal{E}^\bullet$ of coherent sheaves on the variety $\check{X}$, and the Hom-spaces are given by

$$\mathrm{Hom}^n_{D^b_\infty(\check{X})}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \bigoplus_{p+q=n} \check{C}^p(\mathcal{U}, \mathcal{H}om^q_{\mathcal{O}_{\check{X}}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)),$$

where

$$\mathcal{H}om^q_{\mathcal{O}_{\check{X}}}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \bigoplus_m \mathcal{H}om_{\mathcal{O}_{\check{X}}}(\mathcal{E}^m, \mathcal{F}^{m+q})$$

Using a projector, this category can be transformed from a dg-category to an actual A_∞-category. By this I mean there are multiplication maps on tensor products of the Hom-spaces that satisfy a generalization of associativity. The objects of the Fukaya category are the Lagrangian submanifolds $L \subset X$, and the Hom-spaces $\mathrm{Hom}(L_i, L_j)$ are given by modules generated by the points of $L_i \cap L_j$ over the Novikov ring. Homological mirror symmetry is then expressed by comparing the A_∞ structures of both.

The version of the Fukaya category considered in this project is, loosely speaking, mirror dual as an A_∞-category to the derived category of a degeneration of elliptic curves, which has as its central fiber, a cyclic chain of copies of the projective line over $\mathbb{C}$.

1.2 Counting Curves

One of the most striking aspects of mirror symmetry is its application to enumerative geometry. In [9] the authors construct a one-dimensional family of mirror quintic Calabi-Yau 3-folds, and a map between the moduli space of these Calabi-Yau's, and the Kahler moduli space of a quintic threefold. By examining the relationship between the two coordinates on these respective spaces, one can write down a geometric series $5 + \sum n_k e^{2\pi i t k}$, where n_k is the number of degree k rational (genus 0) curves on a generic quintic 3-fold in $\mathbb{P}^4$.

In 1993 a genus 1 version of the above result was obtained by Bershadsky, Cecotti, Ooguri, and Vafa in [8]. Specifically, they used the mirror correspondence to derive a formula for the number of degree k elliptic curves on a quintic Calabi-Yau 3-fold. As of 2006, similar formulas have been found for curves up to and including genus 51. See [28], for example.

Counting the number of curves of arbitrary genus on a Calabi-Yau n-fold is an area of current research and connects such topics as higher-genus Gromov-Witten theory, operad theory, topological conformal field theory (TCFT), and homological mirror symmetry, the latter providing the motivation for the constructions given in chapters 4, 5, and 6 of the proceeding work. In the most ambitious homological approach to higher-genus curve counting, described in [12] and [13], Costello constructs the B-model mirror to a TCFT constructed from the Gromov-Witten invariants of a compact symplectic manifold. Taking the very different approach described in [4], Barannikov uses the Feynman transform of Getzler and Kapranov [17] to claim the existence of a combinatorial description of Gromov-Witten invariants, in terms of the periodic cyclic homology of the given twisted modular operads via the characteristic class map. He further suggests the possibility of constructing a Feynman transform-algebra structure on the Lagrangians of a symplectic manifold.

The main result of this project is the construction of an explicit example of such a structure, that is, the construction of a morphism Ω from the Feynman transform of a twisted modular operad, generated by the elements of the symmetric group $\mathbb{S}_n$, to a modular operad generated by the Lagrangian submanifolds of an elliptic curve. The morphism of modular operads constructed here can be used to calculate a partition function along the lines of [31], which defines the characteristic class with value in

$$\bigoplus_i \mathrm{H}_i(\overline{\mathfrak{M}}'_{\gamma,\nu}),$$

where $\overline{\mathfrak{M}}'_{\gamma,\nu}$ is a quotient of $\overline{\mathfrak{M}}_{\gamma,\nu}$, the compactified Deligne-Mumford moduli space of genus γ curves with ν marked points.

1.3 The Project

1.3.1 Stable Graphs

In order to define Ω, one must first understand the category of stable graphs, the objects of which are the inputs to the modular operads described in Chapter 3. The category is described as follows. The objects are connected, one-dimensional CW complexes formed by finite sets of vertices, edges, and legs. The morphisms are given by contractions of internal edges, that is, by continuous maps $G \longrightarrow G/S$, where $S \subseteq \mathrm{Edge}(G)$. Stability is a notion related to the labeling of the vertices $v \in \mathrm{Vert}(G)$ by integers $b(v)$, which controls the valencies of the vertices. The category of graphs G with n legs, and $b = \dim \mathrm{H}_1(G) + \sum_{\mathrm{Vert}(G)} b(v)$ is denoted by $\Gamma((n,b))$. Chapter 2 is devoted to describing this category in full detail, and noting various formulae relating the number of vertices, edges, and cycles of such graphs, which play a crucial role in the dimension of moduli spaces of tropical Morse graphs, defined in Chapter 6.

1.3.2 Modular Operads

Chain complexes called stable $\mathbb{S}$-modules form the foundation of operads and modular operads defined in Chapter 3. A modular operad is best viewed as a collection of functors from the categories of stable graphs to chain complexes, along with a contraction map on the chain complexes given by the contraction of edges of the input graph. Modular operads may also come with an $\mathbb{S}_n$-action. More precisely, given a modular operad P, and a morphism of graphs $G \longrightarrow G/S$, there is a degree 0 morphism of chain complexes $\mu^P_{G \longrightarrow G/S} : P((G)) \longrightarrow P((G/S))$. It is often necessary to shift the degree of the output chain complex, and change the $\mathbb{S}_n$ action of the modular operad itself, usually for the sake of the compatibility of a given morphism of modular operads. Cocycles and coboundaries are defined to serve this purpose. These are maps which take graphs as inputs, and output a chain complex isomorphic to $\mathbb{C}[d]$ for $d \in \mathbb{Z}$. Tensoring with such an object gives the desired twist of the output chain complex of the given modular operad.

Chapter 3 builds to the definition and characterization of a twisted free modular operad called the Feynman transform. It is defined by taking the dual of a given modular operad P and twisting by a coboundary that depends on the given twist of P. This is the algebraic structure which encodes the A_∞ and quantum A_∞-relations on a chain complex V, and later, on the Hom-spaces of the Fukaya category of the elliptic curve.

1.3.3 The Elliptic Curve

An elliptic curve E is a smooth, projective, algebraic curve of genus 1. One may use the double-periodicity of the Weierstrass $\wp$-function to give a homeomorphism between the given curve and a 2-torus, whose complex structure is defined by factoring $\mathbb{C}$ by a lattice, determined by the original equation of E in $\mathbb{P}^2$. Because of the low dimension, any 1-dimensional submanifold is Lagrangian. The elliptic curve used in this project will be viewed an an S^1-bundle over $B := \mathbb{R}/d\mathbb{Z}$. We call B the *base* of the fibration.

1.3.4 The Fukaya Category of the Elliptic Curve

Chapter 5 defines the categorical structures involved in defining the sought after morphism Ω. The first two sections are devoted to A_∞-categories and tropical Morse trees, which are the primary ingredients needed to define the Fukaya category of the elliptic curve. This is the subject of the third section. As stated above, this precategory carries the structure of an A_∞-precategory. The multiplication maps

$$m_{d,0} : \bigotimes_{i=1}^{d} \mathrm{Hom}(L_{i-1}, L_i) \longrightarrow \mathrm{Hom}(L_0, L_d)$$

are defined by summing over a zero-dimensional moduli space of holomorphic polygons, with boundary contained in the Lagrangians $L_0, L_1, \ldots, L_d$.

1.3.5 Tropical Morse Graphs

A *tropical Morse tree* T is a combinatorial tool used to replace the holomorphic polygons mentioned above. These are continuous maps from a given metric ribbon tree to the base of the fibration B, along with a given vector field which allows one to build an actual polygon from T. Chapter 6 can be thought of as the heart of the thesis, as the crucial generalization of the moduli spaces of tropical Morse trees to the moduli spaces of tropical Morse graphs is made. The idea is simply that we consider domains with nontrivial first homology, i.e., graphs with cycles, along with the contractible trees considered above. Just as tropical Morse trees correspond to holomorphic disks, tropical Morse graphs correspond to holomorphic maps of positive genus. For example, if G is a tropical Morse graph such that $\dim \mathrm{H}_1(G) = 1$, then G corresponds to a holomorphic annulus.

The A_∞-relations of the Fukaya category are obtained by examining compactified 1-dimensional moduli spaces of tropical Morse trees and carefully assigning signs to the pairs of degenerate trees bounding these spaces. Beyond the definition of tropical Morse graphs, a dimension formula for the moduli spaces is derived and proved.

1.3.6 The Fukaya Category as an Algebra over the Feynman Transform

In the final chapter we construct the morphism Ω in detail. The basis vectors of the Feynman transform are given by twisted cycles, and can be represented by graphs whose vertices are decorated with a fixed number of cycles. The morphism, when restricted to the genus zero ($b = 0$) case, is given by mapping single vertex graphs with no edges, each decorated with a single cycle, to the maps $m_{d,0}$ defined in 1.3.4, which are referred to as the genus zero *operations*. The relations among compositions of these genus zero operations are obtained by considering the restriction of Ω to genus zero graphs with a single non-intersecting edge, with each of the two vertices decorated with a single cycle. These relations give $\mathrm{Fuk}(E)$ the structure of an A_∞-precategory. Although $\mathrm{Fuk}(X)$ was already known to possess an A_∞-structure, its expression here as an algebra over the Feynman transform is a first.

The morphism Ω, when restricted to the genus one ($b = 1$) case, is given by mapping single vertex graphs with no edges, each decorated with two cycles, to the genus one operations, which are defined as follows. Rather than summing over a cyclic chain of Lagrangians that bound a holomorphic polygon, these new operations are defined by taking the sums over disjoint pairs of cyclic chains of Lagrangians that bound holomorphic annuli. The quantum A_∞ relations are defined by examining degenerations of genus one tropical Morse graphs, and are given over the Feynman transform by restricting Ω to stable graphs with $b = 1$, whose vertices are decorated by either one or two cycles each. This generalizes the Fukaya category from being an A_∞-precategory, to being a quantum A_∞-precategory.

Chapter 2

Graphs and S-modules

In this chapter we review basic concepts necessary for the discussion of modular operads, all of which can be found in [17].

2.1 Graphs

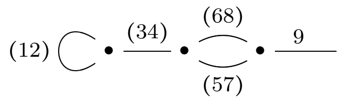
A graph G is a triple $(\mathrm{Flag}(G), \lambda, \sigma)$, where $\mathrm{Flag}(G)$ is a finite set, whose elements are called flags, λ is a partition of $\mathrm{Flag}(G)$, and σ is an involution acting on $\mathrm{Flag}(G)$. The vertices of G are the unordered blocks(subsets) of the partition and the set of all such is denoted by $\mathrm{Vert}(G)$. We write $\mathrm{Leg}(v)$ for the subset of $\mathrm{Flag}(G)$ corresponding to the vertex v. Let $\mathrm{Edge}(G)$ denote the set of two-cycles of σ and let $\mathrm{Leg}(G)$ denote the subset of $\mathrm{Flag}(G)$ fixed by σ. We say that two legs *meet* if they either belong to the same vertex, or comprise an edge.

A 1 dimensional CW-complex $|G|$ is associated to each graph G by taking a copy of $[0,1]$ for each flag, and imposing the following equivalence relation: the points $0 \in [0,1]$ are identified for all flags in a block of the partition λ, and the points $1 \in [0,1]$ are identified for pairs of flags exchanged by the involution σ. So two flags touch in $|G|$ if and only if they belong to the same vertex, or comprise an edge.

To each vertex we associate a point, out of which emanates line segments, one for each element of $\mathrm{Leg}(v)$. Two line segments are glued at their non-vertex ends if σ maps one to the other.

Example 2.1.1. $\mathrm{Flag}(G) = \{1, \ldots, 9\}$, $\lambda = \{1,2,3\} \cup \{4,5,6\} \cup \{7,8,9\}$,

$$\sigma = (12)(34)(57)(68)$$



If G is contractible, then G will be called a *tree*. We can break symmetry and call one of the legs the *output*, and the rest will be called the *inputs*. A tree of this type is a *rooted tree*.

Definition 2.1.2. A *stable graph* G is a connected labeled graph with a non-negative integer $b(v)$ assigned to each vertex $v \in \mathrm{Vert}(G)$, such that $2b(v) + n(v) - 2 > 0$. For a stable graph G, set

$$b(G) = \left(\sum_{v \in \mathrm{Vert}(G)} b(v) \right) + b_1(G),$$

where $b_1(G) = \dim H_1(G)$. Let $*^{l(v)}_{n(v),b(v)}$ be the unique graph with one vertex v, $n(v)$ legs, $l(v)$ loops, and integer $b(v)$.

Proposition 2.1.3. If G is a stable graph, then

$$|\mathrm{Edge}(G)| = b(G) - 1 + \sum_{\mathrm{Vert}(G)} (1 - b(v)),$$

and if $b(G) = 0$, then $|\mathrm{Edge}(G)| + 1 = |\mathrm{Vert}(G)|$.

Let S and S' be subsets of $\mathrm{Leg}(G)$ and $\mathrm{Leg}(G')$ respectively, and let $\varphi : S \to S'$ be a bijection. We can form a new graph $G \otimes_\varphi G'$ by gluing the elements of S to the elements of S' using φ. Then

$$\mathrm{Edge}(G \otimes_\varphi G') = \mathrm{Edge}(G) \cup \mathrm{Edge}(G') \cup \{(i\,\varphi(i))\}$$

for $i \in S$ and

$$\mathrm{Leg}(G \otimes_\varphi G') \subseteq \mathrm{Leg}(G) \sqcup \mathrm{Leg}(G').$$

We can then decompose any graph G as

$$G = \bigotimes_{v \in \mathrm{Vert}(G)} G_v,$$

where v is the unique vertex of G_v and φ is given implicitly by G. Note that $G_v = *^{l(v)}_{n(v),b(v)}$ for all $v \in \mathrm{Vert}(G)$.

Morphisms of Stable Graphs

Definition 2.1.4. Let G_0 and G_1 be two graphs. A morphism of *graphs* $f : G_0 \to G_1$ is an injection

$$f^* : \mathrm{Flag}(G_1) \to \mathrm{Flag}(G_0)$$

such that:

1. $\sigma_0 \circ f^* = f^* \circ \sigma_1$, where σ_i, $i = 0, 1$ are the involutions of $\mathrm{Flag}(G_i)$.

2. σ_0 acts freely on $\mathrm{Flag}(G_0) \smallsetminus f^*(\mathrm{Flag}(G_1))$.

3. Two flags a and b in G_1 meet if and only if there is a chain $(x_0, \ldots, x_k)$ of flags in G_0 such that $f^*a = x_0$, $\sigma_0 x_{i-1}$ and x_i meet for all $1 \leq i \leq k$, and $f^*b = \sigma_0 x_k$.

The motivation behind each of the axioms is not so clear. The first ensures edges are mapped either to edges or points, and legs are mapped to legs. Indeed, since f^* is an injection, σ_1 moves $l \in \mathrm{Flag}(G_1)$ if and only if σ_0 moves $f^*l \in \mathrm{Flag}(G_0)$. Two is another way of saying that G_1 is obtained from G_0 by doing nothing more than contracting a subset of $\mathrm{Edge}(G)$. The third ensures that no morphism is allowed to tear a leg from a vertex and re-attach it to another, i.e. that f is continuous. I illustrate this last point with an example.

Example 2.1.5.

Consider the graphs

$$G_0 = (\mathrm{Flag}(G_0) = \{1, 2\} \sqcup \{3\}, \sigma_0 = (23))$$

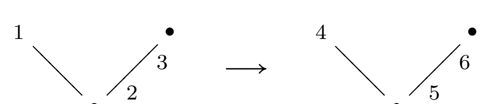
$$G_1 = (\mathrm{Flag}(G_1) = \{4, 5\} \sqcup \{6\}, \sigma_1 = (56))$$

and consider the function

$$f^* : \mathrm{Flag}(G_1) \to \mathrm{Flag}(G_0)$$

$$4, 5, 6 \mapsto 1, 3, 2$$

The induced map is



Intuitively this map removes the edge (23) and re-attaches it by its opposite end. The map f^* satisfies axioms 1 and 2, but not 3. Indeed, 4 and 6 do not meet, but the chain $(x_0 = 1, x_1 = 2, x_2 = 3)$ is such that $f^*a = f^*4 = 1 = x_0$, $\sigma_0 x_0 = 1$ meets $x_1 = 2, \sigma_0 x_1 = 3$ meets $x_2 = 3$, and $f^*b = f^*6 = 2 = \sigma_0 3 = \sigma_0 x_2$.

It is important to note that these three axioms force $\mathrm{Aut}(G)$ to act nontrivially only on the set of edges. In particular, if G is a tree, then $\mathrm{Aut}(G)$ is trivial.

Definition 2.1.6. Let $f : G_0 \to G_1$ be a morphism and let $v \in \mathrm{Vert}(G_1)$. Define the graph $f^{-1}(v)$ to be the subgraph of G_0 consisting of all flags of G_0 which either comprise edges collapsed to v, or are the flags attached to the vertices which bound the collapsed edges. This is uniquely defined by the arrangement of these flags in G_0.

Definition 2.1.7. A morphism of *stable graphs* $f : G_0 \longrightarrow G_1$ is a morphism of graphs such that $b(v) = b(f^{-1}(v))$ for every $v \in \mathrm{Vert}(G_1)$.

Denote by $\Gamma((n,b))$ the category of pairs (G, ρ_G), where G is a stable graph, $\rho_G : \mathrm{Leg}(G) \to \{1,\dots,n\}$ is a bijection, $b(G) = b$, and whose morphisms are morphisms of stable graphs, which preserve the labeling ρ_G of the legs. We write $\star^l_{n,b}$ for the graph with a unique vertex, n legs, integer b, and l loops attached. Write $\star_{n,b}$ for $\star^0_{n,b}$. Let $[\Gamma(n,b)]$ denote the set of isomorphism classes of this category.

2.1.1 Topological Definition for $\Gamma((n,b))$

Replace each object G of $\Gamma((n,b))$ with $|G|$, and each morphism $f : G_0 \longrightarrow G_1$ with a continuous map $\varphi_f : |G_0| \longrightarrow |G_1|$ which is constant with value a vertex of G_1 on edges which collapse, and is the identity everywhere else.

2.2 Stable S-modules

Definition 2.2.1. An $\mathbb{S}$-module is a sequence of chain complexes $P = \{P(n) | n \geq 0\}$ together with an action of S_n on $P(n)$ for each $n \geq 0$. The $\mathbb{S}$-module P is called cyclic if there is also an action of $\mathbb{Z}_{n+1} =< (0\,1 \cdots n) >$ on $P(n)$ for each n.

Denote by V^* the linear dual of the chain complex (graded vector space) V, where $(V^*)_i = (V_{-i})^*$. The differential is given by $\delta^* : V_i^* \to V_{i-1}^*$, the adjoint of $\delta : V_{-i+1} \to V_{-i}$. Tensor products of these chain complexes are permuted according to the rule

$$V \otimes W \longrightarrow W \otimes V$$

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v, \tag{2.1}$$

where $|v| = \deg v$ and is always viewed modulo 2. For a module U over a finite group G we denote via U_G the k-vector space of coinvariants, i.e. the quotient of U by the submodule generated by $\{gu - u | u \in U, g \in G\}$. The set of invariants $\{u \in U | gu = u, \forall g \in G\}$ will be denoted by U^G.

Let I be a finite set. Then

$$\bigotimes_{i \in I} V_i := \left(\bigoplus_{\substack{bijections,\\ \gamma:\{1,\dots,n\}\to I}} V_{\gamma(1)} \otimes \cdots \otimes V_{\gamma(n)} \right)_{S_n},$$

where

$$\sigma \cdot (v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(n)}) \quad = \quad (-1)^{\epsilon(\sigma)} v_{\gamma(\sigma(1))} \otimes \cdots \otimes v_{\gamma(\sigma(n))}$$

for $v_{\gamma(i)} \in V_{\gamma(i)}$, and $\epsilon(\sigma)$ is calculated by writing it as a sequence of transpositions of adjacent factors, and then using (2.1). This space will also be written as $V^{\otimes I}$ or V^I.

Let $\sigma = (ij)$ be a transposition with $i < j - 1$. Up to sign, the effect of action by σ is certainly the interchanging of $v_{\gamma(i)}$ and $v_{\gamma(j)}$, but the signs produced by commuting $v_{\gamma(i)}$ and $v_{\gamma(j)}$ through the factors within the string bounded by these two factors must not be ignored. The procedure is illustrated as follows: Commuting $v_{\gamma(j)}$ to the left through $v_{\gamma(i+1)} \otimes \cdots \otimes v_{\gamma(j-1)}$ yields the sign $(-1)^{|v_{\gamma(j)}|(|v_{\gamma(i+1)}|+\cdots+|v_{\gamma(j-1)}|)}$, and then commuting $v_{\gamma(i)}$ and $v_{\gamma(j)}$ yields $(-1)^{|v_{\gamma(i)}||v_{\gamma(j)}|}$. Finally, $v_{\gamma(i)}$ is commuted through $v_{\gamma(i+1)} \otimes \cdots \otimes v_{\gamma(j-1)}$ to the place originally held by $v_{\gamma(j)}$, yielding $(-1)^{|v_{\gamma(i)}|(|v_{\gamma(i+1)}|+\cdots+|v_{\gamma(j-1)}|)}$. The end result is

$$\begin{aligned} & \sigma : v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(i)} \otimes \cdots \otimes v_{\gamma(j)} \otimes \cdots \otimes v_{\gamma(n)} \\ \mapsto \; & (-1)^{|v_{\gamma(i)}||v_{\gamma(j)}|+(|v_{\gamma(i)}|+|v_{\gamma(j)}|)(|v_{\gamma(i+1)}|+\cdots+|v_{\gamma(j-1)}|)} v_{\gamma(1)} \otimes \cdots \otimes v_{\gamma(j)} \otimes \cdots \\ & \cdots \otimes v_{\gamma(i)} \otimes \cdots \otimes v_{\gamma(n)} \end{aligned} \tag{2.2}$$

One can check that the sign produced here is independent of the order in which $v_{\gamma(i)}$ and $v_{\gamma(j)}$ were moved.

In other words, we choose an arbitrary linear ordering on I and set

$$\bigotimes_{i \in I} V_i = V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n},$$

so that each vector $v_{i_1} \otimes \cdots \otimes v_{i_n}$ represents a unique equivalence class in the module of co-invariants.

Example 2.2.2. Let A be a chain complex which is finitely generated over $\mathbb{C}$ by homogeneous elements $\{a, b\}$, and set $\mathrm{I} = \{f, g, h\}$. Then $A^{\otimes\{f,g,h\}}$ is the eight-dimensional vector space generated by the tensors $a_f \otimes a_g \otimes a_h,\ a_f \otimes a_g \otimes b_h,\ a_f \otimes b_g \otimes a_h$, and so on. The action of S_3 on $A^{\{f,g,h\}}$ is induced by the bijection $f \mapsto 1, g \mapsto 2, h \mapsto 3$. If, for example, $\sigma = (12) = (fg)$, then $\sigma \cdot a_f \otimes b_g \otimes a_h = (-1)^{|a_f||b_g|} b_g \otimes a_f \otimes a_h = (-1)^{|a_g||b_f|} b_f \otimes a_g \otimes a_h$. This last equality follows from the fact that $A^{\otimes \mathrm{I}}$ is defined by taking a quotient, and any two labelings of the same element lie in the same equivalence class.

Definition 2.2.3. If P is a cyclic $\mathbb{S}$-module, and I is an $(n+1)$–element set, define

$$P((I)) := \left(\bigoplus_{\substack{\text{bij.} \\ \{0,1,\dots,n\} \to I}} P(n) \right)_{S_{(n+1)}.}$$

Once we have the notion of an operad, and have developed a few examples, we will be able to shed more light on the above definitions, namely the nature of the S_{n+1} action.

Definition 2.2.4. If there is a decomposition

$$P((n)) = \bigoplus_{b \in \mathbb{Z}_{\geq 0}} P((n,b))$$

such that S_n acts on $P((n,b))$ for each n, and $P((n,b)) = 0$ if $2b + n - 2 \leq 0$, then P is called stable.

Chapter 3

The Feynmann Transform

We now introduce the notion of a modular operad and Feynman transform, as defined by Getzler and Kapranov in [17]. In addition, we introduce Barannikov's modular operads $\mathbb{S}[t]$ and $\tilde{\mathbb{S}}[t]$, and explain his calculations showing what the structure of an algebra over the Feynman transform of twists of these modular operads means. This material is from [4].

3.1 Operads

Definition 3.1.1. An operad $(P, \circ_i)$ consists of an $\mathbb{S}$-module P, and maps $\circ_i : P(m) \times P(n) \to P(m+n-1)$ for $m, n \geq 1$ and $1 \leq i \leq m$ satisfying the following two relations:

i) For $a \in P(k)$, $b \in P(l)$ and $c \in P(m)$, and $1 \leq i < j \leq k$,

$$(a \circ_i b) \circ_{j+l-1} c = (a \circ_j c) \circ_i b$$

ii) For $a \in P(k), b \in P(l)$ and $c \in P(m)$ and $1 \leq i \leq k$, $1 \leq j \leq l$,

$$(a \circ_i b) \circ_{i+j-1} c = a \circ_i (b \circ_j c)$$

We also have an S_n-action on each $P(n)$, compatible with the compositions. If $\sigma \in S_m$, $\gamma \in S_n$, $a \in P(m)$, and $b \in P(n)$, then

$$\sigma a \circ_{\sigma(i)} \gamma b = (\sigma \circ_i \gamma)(a \circ_i b),$$

where $\sigma \circ_i \gamma$ acts as follows: The permutation γ acts on $\{i, \ldots, i+n-1\}$ and σ acts on

$$\{\{1\}, \ldots, \{i-1\}, \{i, \ldots, i+n-1\}, \{i+n\}, \ldots, \{m+n-1\}\}.$$

Example 3.1.2.

Let X be a topological space. The endomorphism operad $End_X := \{P(n) = \mathrm{Map}(X^n, X)\}_{n\geq 1}$ is defined as follows: The operations

$$\circ_i : P(n) \times P(m) \to P(n+m-1)$$

are given by

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) = f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+m-1}), x_{i+m}, \dots, x_{m+n-1})$$

for $1 \leq i \leq n$. The S_n action is given by $(\sigma f)(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. The equivariance condition is best illustrated with a simple example. Let $f, g \in \mathrm{Map}(X^2, X)$, and $\sigma = \rho = (12) \in S_2$. Then

$$\begin{aligned}((\sigma \circ_2 \rho)(f \circ_2 g))(a,b,c) &= (f \circ_2 g)(c,b,a)\\ &= f(c, g(b,a))\\ &= \sigma f(\rho g(a,b), c)\\ &= (\sigma f \circ_1 \rho g)(a,b,c)\\ &= (\sigma f \circ_{\sigma(2)} \rho g)(a,b,c)\end{aligned}$$

Remark 3.1.3. It is important to note the difference between an entry of f and an entry place. σf is defined by pre-composing f with $\sigma : X^n \to X^n$, so that, for example, $\sigma = (12)$ moves $f(x,y) = x + y^2$ to $(\sigma f)(x,y) = f(y,x) = y + x^2$, not $g(x,y) = x + y^2$.

The functions $f \in \mathrm{End}_X(n)$ have n inputs and one output, so it makes sense to represent a function as a tree T with one vertex, n "incoming" legs and 1 "outgoing" leg. Let $\mathrm{Flag}T(= \mathrm{Leg}T) = \{0, 1, \dots, n\}$. Fix $\{0\}$ as the outgoing leg. This structure can be exploited to further understand the subtlety mentioned above in the following way. Label the i^{th} incoming leg with x_i for all i, $1 \leq i \leq n$.

Let $f \in \mathrm{End}_X(3)$. It is unclear as to whether S_3 acts on the external labels $\{x_i\}$, or the intrinsic labels $\{i\}$ i.e., on the legs themselves. This ambiguity is resolved by requiring, for each object $\mathrm{End}_X(n)$, that x_i stays attached to the i^{th} leg. See figure 2.1.

The entries are again inserted into f from left to right.

Example 3.1.4.

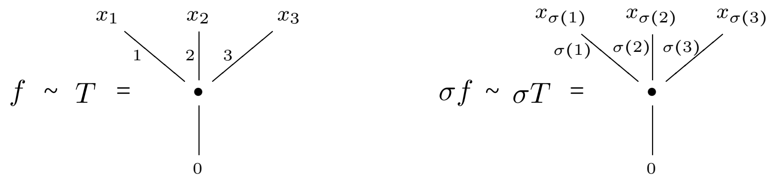


Figure 3.1: Both f and T are taken as above, with $\sigma \in S_3$.

The operad $\mathcal{A}ss$ is defined by

$$\mathcal{A}ss(n) = \{\sigma \cdot f_n | \sigma \in S_n,\ f_n(x_1, \ldots, x_n) = x_1 \cdots x_n\},$$

where f_n is the order-preserving operation which transforms a list of n elements into a string of the same n elements. Also impose the condition $f_2 \circ_1 f_2 = f_2 \circ_2 f_2$. This is the associativity condition, which only becomes important when we discuss algebras over operads. Let τ_i be the map which replaces j with $j - i + 1$ for $i \leq j \leq i + m - 1$, and let τ_m be the map which replaces j with $j - m + 1$ for $i + m \leq j \leq n + m - 1$. Then composition is done the same way as for End_X, i.e.

$$\circ_i : \mathcal{A}ss(n) \otimes \mathcal{A}ss(m) \longrightarrow \mathcal{A}ss(n + m - 1)$$

is given by

$$\begin{aligned}
& \sigma f_n \circ_i \gamma f_m(x_1, \ldots, x_{m+n-1}) \\
= \ & \sigma f_n(x_1, \ldots, x_{i-1}, \gamma f_m(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{n+m-1}) \\
= \ & f_n(x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}, f_m(x_{\tau_i^{-1}\gamma\tau_i(i)}, \ldots, x_{\tau_i^{-1}\gamma\tau_i(i+m-1)}), x_{\tau_m^{-1}\sigma\tau_m(i+m)}, \ldots \\
& \ldots, x_{\tau_m^{-1}\sigma\tau_m(n+m-1)}) \\
= \ & x_{\sigma(1)} \cdots x_{\sigma(i-1)} (x_{\tau_i^{-1}\gamma\tau_i(i)}, \ldots, x_{\tau_i^{-1}\gamma\tau_i(i+m-1)}) x_{\tau_m^{-1}\sigma\tau_m(i+m)} \cdots x_{\tau_m^{-1}\sigma\tau_m(n+m-1)}
\end{aligned}$$

where

$$x_{\tau_i^{-1}\gamma\tau_i(i)} \cdots x_{\tau_i^{-1}\gamma\tau_i(i+m-1)}$$

is in the $\sigma(i)^{\text{th}}$ spot of the outer product. I include the permutations τ_i and τ_m because I am thinking of σ and γ as acting strictly on the sets $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ respectively. This operad simply inserts a string of variables into another string of variables.

3.1.1 Cyclic Operads

The idea of cyclic operads is to include the idea of cyclic symmetry into the definition of an operad. Instead of the input of an operad being simply a natural number, we can give a new definition which replaces the natural number with a tree. Let T_n be a rooted tree with n incoming legs and one outgoing leg. One should think of the composition $P(n) \circ_i P(m) \to P(m+n-1)$ as being induced by inserting the root of T_m into the ith branch of T_n, much in the same way $f \circ_i g$ was defined in End_X. For a general operad P, $P(n)$ can contain more than one element, but thinking of each element as a rooted tree helps to illuminate the associativity relations and the S_n equivariance. The notion of the cyclic operad is such that it allows us to view these trees as being symmetric, not labeling certain branches as being *inputs* and the last as being the *output*. One then views composition as being induced by inserting *any* leg of the first tree into *any* leg of the second. Alternatively, one can think of these two trees as being glued along legs l_1 and l_2, and then composition is induced by collapsing the edge $(l_1 l_2)$.

Let $\tau_n = (0\,1\,2 \cdots n)$.

Definition 3.1.5. An operad is cyclic if the underlying $\mathbb{S}$-module P is cyclic, and $\tau_{m+n-1}(a \circ_m b) = \tau_n b \circ_1 \tau_m a$.

The Operad $\mathcal{E}_V$

Let V be a chain complex such that its homogeneous subspaces V_i are finite-dimensional for all i, and let $|x| = \deg x$. An inner product on V is a non-degenerate bilinear form B such that $B(dx,y) + (-1)^{|x|}B(x,dy) = 0$, where d is the differential of V. Such a bilinear form is symmetric (resp. antisymmetric) if $B(y,x) = (-1)^{|x||y|}B(x,y)$ (resp. $B(y,x) = -(-1)^{|x||y|}B(x,y)$), and has degree k if $B(x,y) = 0$ unless $|x| + |y| = k$.

Let V be a chain complex with symmetric inner product $B(x,y)$ of degree 0. Kapranov defines a cyclic $\mathbb{S}$–module $\mathcal{E}_V$ by putting $\mathcal{E}_V((n+1)) = \mathcal{E}_V(n) = V^{\otimes(n+1)}$ with the action of $\mathbb{S}_{n+1}$ being

$$\sigma \cdot (v_1 \otimes v_2 \otimes \cdots v_{n+1}) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n+1)}$$

This cyclic $\mathbb{S}$–module can be given the structure of a cyclic operad in the following way: if $a \in V^{\otimes(m+1)}$ and $b \in V^{\otimes(n+1)}$, the product $a \circ_i b \in V^{\otimes(m+n)}$ is defined by contracting $a \otimes b$ with the bilinear form B, applied to the ith factor of a and the 1st factor of b.

3.2 Modular Operads

Definition 3.2.1. Let P be a stable $\mathbb{S}$-module and let G be a stable graph and set $P((G)) = \bigotimes_{\mathrm{Vert}(G)} P((\mathrm{Leg}(v), b(v)))$. Note that this implies

$$P((G \otimes_{\varphi} G')) = P((G)) \otimes P((G')) \tag{3.1}$$

for any G, G', φ.

Definition 3.2.2. A *modular pre-operad* is a stable $\mathbb{S}$-module P together with a chain map

$$\mu : \mathbb{M}P((n,b)) := \bigoplus_{G \in [\Gamma((n,b))]} P((G))_{\mathrm{Aut}(G)} \longrightarrow P((n,b)) \tag{3.2}$$

called the *structure map.*

If P is a modular pre-operad and $G \in \mathrm{Ob}\Gamma((n,b))$, denote by

$$\mu_G : P((G)) \longrightarrow P((n,b)) \tag{3.3}$$

the $\mathbb{S}_n$-equivariant map obtained by composing the universal map

$$P((G)) \longrightarrow \mathbb{M}P((n,b)) = \bigoplus_{G \in [\Gamma((n,b))]} P((G))_{\mathrm{Aut}(G)}$$

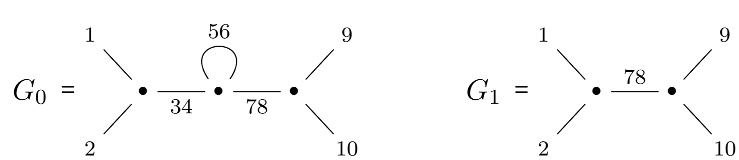
with the structure map $\mu : \mathbb{M}P((n,b)) \longrightarrow P((n,b))$. This map is called composition along the graph G.

Given a morphism $f : G_0 \longrightarrow G_1$ of stable graphs, define a morphism $P((f)) : P((G_0)) \longrightarrow P((G_1))$ to be the composition

$$P((G_0)) = \bigotimes_{u \in \mathrm{Vert}(G_0)} P((\mathrm{Leg}(u), b(u))) \simeq \bigotimes_{v \in \mathrm{Vert}(G_1)} P((f^{-1}(v))) \tag{3.4}$$

$$\xrightarrow{\otimes_v \mu_{f^{-1}(v)}} \bigotimes_{v \in \mathrm{Vert}(G_1)} P((\mathrm{Leg}(v), b(v))) = P((G_1))$$

Example 3.2.3. Consider the following graphs:



$$G_0 = (u_1 := u_{\{1,2,3\}}, u_2 := u_{\{4,5,6,7\}}, u_3 := u_{\{8,9,10\}}, \sigma_0 = (34)(56)(78))$$

$$G_1 = (v_1 := v_{\{1,2,7\}}, v_2 := v_{\{8,9,10\}}, \sigma_1 = (78))$$

and let $f : G_0 \to G_1$ be given by

$$f^* : \{1,2,7,8,9,10\} \mapsto \{1,2,7,8,9,10\}$$

respectively. Then $f^{-1}(v_1) = (G_{u_1} \otimes_{3\to 4} G_{u_2})$, $f^{-1}(v_2) = G_{u_3}$, and $P((f))$ is given by

$$\begin{aligned}
P((G_0)) &= P((G_{u_1} \otimes_{3\to4} G_{u_2} \otimes_{7\to8} G_{u_3})) \\
&= P(((G_{u_1} \otimes_{3\to4} G_{u_2}) \otimes_{7\to8} G_{u_3})) \\
&= P((f^{-1}(v_1) \otimes_{7\to8} f^{-1}(v_2))) \\
&= P((f^{-1}(v_1))) \otimes P((f^{-1}(v_2))) \\
&\xrightarrow{\mu^{\otimes 2}} P((G_{v_1})) \otimes P((G_{v_2})) \\
&= P((G_{v_1} \otimes_{7\to8} G_{v_2})) \\
&= P((G_1))
\end{aligned}$$

Definition 3.2.4. A *modular operad* is a modular pre-operad P such that

i) for any $G \in \Gamma((n,b))$ and $f \in \mathrm{Mor}(\Gamma((n,b)))$, the associations

$$\begin{aligned} G &\mapsto P((G)) \\ f &\mapsto P((f)) \end{aligned}$$

define a functor from the category of stable graphs to the category of chain complexes over some field k. In other words, $P((f \circ g)) = P((f)) \circ P((g))$ for any composition

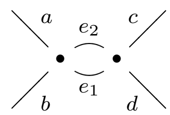
$$G_0 \xrightarrow{g} G_1 \xrightarrow{f} G_2$$

of morphisms of stable graphs.

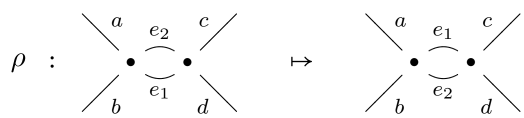
ii) $\otimes_v \mu_{f^{-1}(v)} \circ \mathrm{d}_{P((G))} = \mathrm{d}_{P((G/J))} \circ \otimes_v \mu_{f^{-1}(v)}$, where $\mathrm{d}_{P((G))}$ is the differential on $P((G))$, $J \subseteq \mathrm{Edge}(G)$, and v runs through $\mathrm{Vert}(G/J)$.

The equality $\mu_G \circ P((\rho)) = \mu_G$ holds for any $\rho \in \mathrm{Aut}(G)$, as $\mathbb{M}P((n,b))$ is defined by taking coinvariants with respect to action by $\mathrm{Aut}(H)$ for any $H \in \Gamma((n,b))$. The morphism of chain complexes $P((G)) \longrightarrow P((n,b))$ therefore depends only on the labeling of the legs, and not on any automorphic relabeling of the edges.

Example 3.2.5. Let $G \in \Gamma((4,1))$ be given by



and let $\rho \in \mathrm{Aut}(G)$ be the automorphism

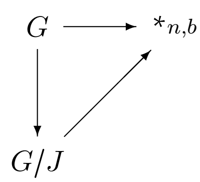


Then $\mu_G = \mu_{\rho(G)} \circ P((\rho))$.

Let $G \in \Gamma((n,b))$ be some graph. For any subset $J \subseteq \mathrm{Edge}(G)$, there is a map $\pi : G \longrightarrow G/J$ defined by the inclusion $\mathrm{Flag}(G/J) \hookrightarrow \mathrm{Flag}(G)$. Writing the map $\otimes_v \mu_{\pi^{-1}(v)}$ for $v \in \mathrm{Vert}(G/J)$ from (3.4) as $\mu_{G \longrightarrow G/J}$, the functoriality of P yields

$$\mu_{G/J} \circ \mu_{G \longrightarrow G/J} = \mu_G \tag{3.5}$$

Indeed, simply apply P to the diagram



where $G/(\mathrm{Edge}(G)) = *_{n,b} = (G/J)/(\mathrm{Edge}(G/J))$. Equation 3.5 will be referred to as the *associativity condition* for composition. Condition ii) of 3.2.4 can now be written as $\mu_{G \to G/H} \circ \mathrm{d}_{P((G))} = \mathrm{d}_{P((G/H))} \circ \mu_{G \to G/H}$.

The group $\mathrm{Aut}(G)$ acts only on the edges of G, and since some of those edges are formed by gluing the free legs of the $f^{-1}(v)$, we have the inclusion

$$\prod_{v \in \mathrm{Vert}(G/J)} \mathrm{Aut}(f^{-1}(v)) \hookrightarrow \mathrm{Aut}(G)$$

In particular, μ_G factors through not only $P((G))_{\mathrm{Aut}(G)} = P\left(\left(\bigotimes_{v\in\mathrm{Vert}(G/J)} f^{-1}(v)\right)\right)_{\mathrm{Aut}(G)}$, but through

$$P\left(\left(\bigotimes_{v\in\mathrm{Vert}(G/J)} f^{-1}(v)\right)\right)_{\prod_{v\in\mathrm{Vert}(G/J)}\mathrm{Aut}(f^{-1}(v))}$$

as well. In other words, any automorphism $\rho \in \mathrm{Aut}(f^{-1}(v))$ can be extended to G, and by both axiom 3 of the definition of a stable graph, and the definition of $f^{-1}(v)$, any automorphism of G restricts to a subgraph of the form $\otimes_S f^{-1}(v)$, where S is some subset of $\mathrm{Vert}(f(G))$.

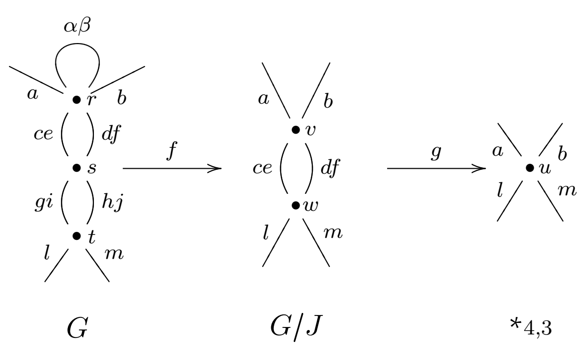
Any morphism $G \longrightarrow G/J$ can be factored into a sequence of morphisms

$$G \longrightarrow G_1/J_1 \longrightarrow G_2/J_2 \longrightarrow \cdots \longrightarrow G_n/J_n = G/J,$$

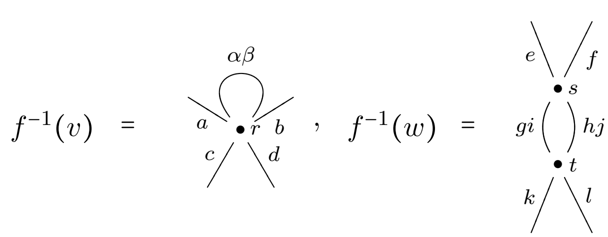
where $J_i \subseteq \mathrm{Edge}(G_i)$ and $G_i \coloneqq G_{i-1}/J_{i-1}$, and the composition μ_G can therefore be factored via associativity as $\mu_G = \mu_{G_{n-1}/J_{n-1}\longrightarrow G_{n-1}/J_{n-1}} \circ \cdots \circ \mu_{G\longrightarrow G_1/J_1}$.

Remark 3.2.6. This property means that for any operad P, any composition map μ_G^P can be built from the compositions along graphs with either two vertices and one edge, or one vertex and one loop.

Example 3.2.7. Consider the morphisms



Pulling back the vertices v and w via f yields the graphs



The map $\mu_G : P((G)) \longrightarrow P((*_{4,3}))$ coincides with

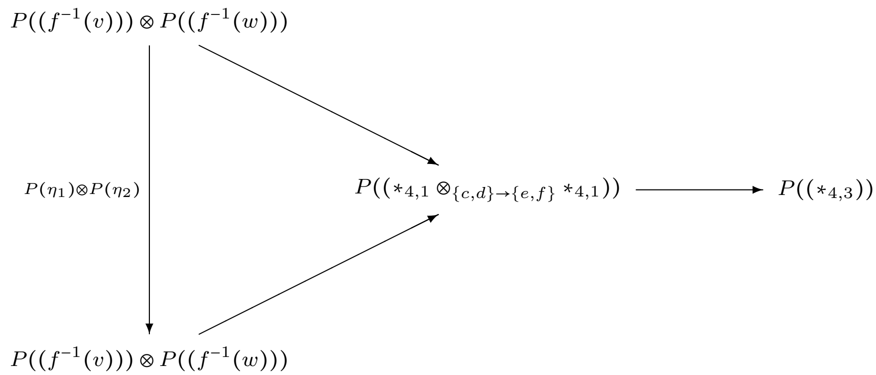
$$\mu_{G/J} \circ \mu_{G \to G/J} : P((f^{-1}(v))) \otimes P((f^{-1}(w))) \xrightarrow{\mu^{\otimes 2}} P((*_{4,1})) \otimes P((*_{4,1})) =$$

$$= P((*_{4,1} \otimes_{\{c,d\} \to \{e,f\}} *_{4,1})) \xrightarrow{\mu_{G/J}} P((*_{4,3}))$$

Furthermore, if we take

$$(\eta_1, \eta_2) \in \mathrm{Aut}(f^{-1}(v)) \oplus \mathrm{Aut}(f^{-1}(w)) \text{ to be } \eta_1 = (\alpha \leftrightarrow \beta)\ \eta_2 = ((gi) \leftrightarrow (hj))$$

then the diagram



commutes.

3.2.1 Composition for The Free Modular Operad

Let P be any stable $\mathbb{S}$-module. We are now going to define the free modular operad whose underlying stable $\mathbb{S}$-module is

$$\mathbb{M}P((n,b)) = \bigoplus_{G \in [\Gamma((n,b))]} P((G))_{\mathrm{Aut}(G)}$$

Let us try to understand the structure of the composition map

$$\mu_G : \mathbb{M}P((G)) \to \mathbb{M}P((n,b))$$

First, take $G \in [\Gamma((n,b))]$ and let $S_G = \{\oplus\eta_j \,|\, \eta_j : \{v_j\} \to [\Gamma((\mathrm{Leg}(v_j), b(v_j)))]\}$, where η_j is a choice function, and v_j runs over the set of vertices of G. Then

$$\begin{aligned}
\mathbb{M}P((G)) &= \bigotimes_{v \in \mathrm{Vert}(G)} \mathbb{M}P((\mathrm{Leg}(v), b(v))) \\
&= \bigotimes_{v \in \mathrm{Vert}(G)} \left(\bigoplus_{H \in [\Gamma((\mathrm{Leg}(v), b(v)))]} P((H))_{\mathrm{Aut}(H)} \right) \\
&= \bigoplus_{S_G} \left(\bigotimes_{\mathrm{Vert}(G)} P\left((\eta_j(v_j))\right)_{\mathrm{Aut}(\eta_j(v_j))} \right) \\
&= \bigoplus_{S_G} \left(\bigotimes_{\mathrm{Vert}(G)} P((G_{v_j}))_{\mathrm{Aut}(G_{v_j})} \right) \\
&= \bigoplus_{S_G} P((\bigotimes_{\mathrm{Vert}(G)} G_{v_j}))_{\prod \mathrm{Aut}(G_{v_j})},
\end{aligned} \tag{3.6}$$

where for fixed $\eta = \oplus\eta_j$, each G_{v_j} replaces v_j in G, and a new graph, G_η, is defined as follows:

Let $G \in \Gamma((n,b))$ be such that $\mathrm{Vert}(G) = \{1, \ldots, n\}$ with involution σ and let $G_{v_i} \in [\Gamma((\mathrm{Leg}(v_i), b(v_i)))]$ for $i = 1, \ldots, n$. For any category $\Gamma((n,b))$ there is a fixed labeling of the legs of any graph $G \in \Gamma((n,b))$, so every graph $G_{v_i} \in [\Gamma((\mathrm{Leg}(v_i), b(v_i)))]$ comes with a bijection $f_i : \mathrm{Leg}(G_{v_i}) \longrightarrow \mathrm{Leg}(v_i)$. It is important to recall that by a *vertex* we mean not only the associated point in the geometric realization, but the set of flags emanating from that point as well. Now, replace v_i with G_{v_i} and glue $l_\alpha, l_\beta \in \bigsqcup_i \mathrm{Leg}(G_{v_i})$ together if and only if $\sigma f_i(l_\alpha) = f_j(l_\beta)$ with $l_\alpha \in \mathrm{Leg}(G_{v_i})$ and $l_\beta \in \mathrm{Leg}(G_{v_j})$. Since $\mathrm{Leg}(G_\eta) = \mathrm{Leg}(G)$ and $b(v_i) = b(G_{v_i})$, it must be the case that G_η belongs to $\Gamma((n,b))$.

Remark 3.2.8. It should be noted that P is in not in general "injective" in the sense that $P((G)) = P((H))$ does not imply that G and H are even in the same category. In particular, it need not be the case that $\otimes G_{v_j} \in \Gamma((n,b))$ above. See Example 2.2.5 for more details.

The composition μ_G is therefore obtained by canonically projecting the summands of

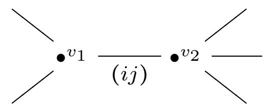
$$\mathbb{M}P((G)) = \bigoplus_{S_G} P((\bigotimes_{\mathrm{Vert}(G)} G_{v_j}))_{\prod \mathrm{Aut}(G_{v_j})}$$

onto the summands of $\bigoplus_{S_G} P((G_\eta))_{\mathrm{Aut}(G_\eta)} \subseteq \mathbb{M}P((n,b))$.

For any graph G we have $G = \otimes G_{v_j}$, and if $\oplus\eta_j$ is such that $\eta_j(v_j) = G_{v_j}$, then $G_\eta = G$. Furthermore, whenever $G_{v_j} = \star_{n(v_j), b(v_j)}$ for all j, then $\mathrm{Aut}(G_{v_j}) = \mathrm{id}$

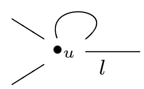
and $\mu_G^{\mathbb{M}P}|_{P((G_\eta))\otimes \mathrm{Aut}(G_{v_j})}$ is the projection $P((G_\eta)) = P((G)) \longrightarrow P((G))_{\mathrm{Aut}(G)}$. For example, if G is the graph $v_1 = \{1,2,3\}, v_2 = \{4,5,6\}, \sigma = (24)(35)$, then $\mathrm{Aut}(G_{v_i}) = \mathrm{id}$, but $\mathrm{Aut}(G) = \mathbb{Z}/2\mathbb{Z}$.

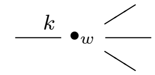
Example 3.2.9. Let

G = 

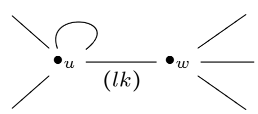
$\in \quad [\Gamma((5,2))]$

with $b(v_1) = b(v_2) = 1$, and take $G_{v_1} \in [\Gamma((3,1))]$, $G_{v_2} \in [\Gamma((4,1))]$ to be

G_{v_1} = 

, G_{v_2} = 

where $b(u) = 0$, $b(w) = 1$, and the f_i's are the obvious bijections. Then $(ij)f_1(l) = (ij)(i) = j = f_2(k)$, and so

G_η = 

Furthermore,

$$b(G_{v_1}) = b(u) + \dim H_1(G_{v_1}) = 0 + 1 = 1$$

$$b(G_{v_2}) = b(w) + \dim H_1(G_{v_2}) = 1 + 0 = 1$$

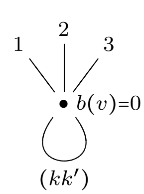
and $b(G_\eta) = b(G_{v_1}) + b(G_{v_2}) = 2$.

As in this example, if $\mathrm{H}_1(G) = 0$, then the identification

$$\mathbb{M}P(G) = \bigoplus P((G_\eta))_{\mathrm{Aut}(G_\eta)},$$

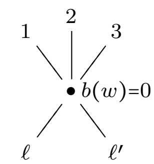
and therefore the map $\mu_G^{\mathbb{M}P}$, is quite intuitive. If $\mathrm{H}_1(G) \neq 0$, or more specifically, if G contains a self-intersecting loop, then the map $\mu_G^{\mathbb{M}P}$ is more subtle:

Example 3.2.10. For any modular operad P, $P((G)) = \bigotimes_{\mathrm{Vert}(G)} P((\mathrm{Leg}(v), b(v)))$. Let

G = 

and let

$H =$



As G has only one vertex v, each $\oplus\eta_j$ from above is replaced by a single map $\eta : \{v\} \longrightarrow [\Gamma((\mathrm{Leg}(v), b(v)))]$, and S_G can be replaced by $[\Gamma((\mathrm{Leg}(v), b(v)))]$. Let G_{η_0} correspond to the η such that $\eta(v) = G_v = H$. The bijection f in this case is

$$f : \mathrm{Leg}(G_v) \longrightarrow \mathrm{Leg}(v)$$

$$1, 2, 3, \ell, \ell' \mapsto 1, 2, 3, k, k',$$

and $f(\ell) = (kk')f(\ell') = \sigma f(\ell')$, so $G_{\eta_0} = G$. Examine the composition map $\mu_G^{\mathbb{M}P}$:

$$\mu_G^{\mathbb{M}P} : \mathbb{M}P((G)) = \bigotimes_{\mathrm{Vert}G} \mathbb{M}P((\mathrm{Leg}(v), b(v))) = \mathbb{M}P((\mathrm{Leg}(v), 0)) =$$

$$= \bigoplus_{K \in [\Gamma((\mathrm{Leg}(v),0))]} P((K))_{\mathrm{Aut}(K)} \longrightarrow \bigoplus_{L \in [\Gamma((3,1))]} P((L))_{\mathrm{Aut}(L)} =$$

$$= \mathbb{M}P((3,1))$$

The map $\mu_G^{\mathbb{M}P}|_{P((H))_{\mathrm{Aut}(H)}} : P((H))_{\mathrm{Aut}(H)} \longrightarrow P((G))_{\mathrm{Aut}(G)}$ is the canonical projection $P((G)) \longrightarrow P((G))_{\mathrm{Aut}(G)}$, since $P((G)) = P((H))$, and $\mathrm{Aut}(H) = \mathrm{id}$. The critical step here is replacing the vertex of $H \in \Gamma((\mathrm{Leg}(v), 0))$ with the graph $G_{\eta_0} = G$, and gluing the legs ℓ and ℓ' via f to obtain G itself.

3.2.2 Morphisms of Modular Operads

Definition 3.2.11. A *morphism* $P \longrightarrow Q$ of modular operads consists, for each graph G, of a morphism of chain complexes $P((G)) \longrightarrow Q((G))$ subject to the commutativity of

$$\begin{array}{ccc} P((G)) & \longrightarrow & Q((G)) \\ \downarrow{\scriptstyle \mu^P_{G\to H}} & & \downarrow{\scriptstyle \mu^Q_{G\to H}} \\ P((H)) & \longrightarrow & Q((H)) \end{array}$$

where $H = G/I$ for some set $I \subseteq \mathrm{Edge}(G)$.

Proposition 3.2.12. Let P be a cyclic $\mathbb{S}$–module, Q a modular operad, and let

$$F : \text{ModularOperads} \longrightarrow \mathbb{S}\text{–modules}$$

be the forgetful functor. There is a bijection $\text{Hom}_{\text{ModOp}}(\mathbb{M}P, Q) \simeq \text{Hom}_{\mathbb{S}-\text{mod}}(P, FQ)$, that is, $\mathbb{M}$ is the left adjoint to the forgetful functor F.

Proof. For a fixed category $\Gamma((n,b))$, a morphism $\varphi : \mathbb{M}P \longrightarrow Q$ is a collection of chain complex morphisms $\varphi(G) : \mathbb{M}P((G)) \longrightarrow Q((G))$, indexed by the objects G of $\Gamma((n,b))$. Since $\mathbb{M}P$ and Q are modular operads, $\varphi(G)$ can be decomposed as

$$\bigotimes_{v \in \text{Vert}(G)} \mathbb{M}P((n(v), b(v))) \longrightarrow \bigotimes_{v \in \text{Vert}(G)} Q((n(v), b(v))),$$

and is therefore determined by the collection of morphisms

$$\mathbb{M}P((n(v), b(v))) \longrightarrow Q((n(v), b(v)))$$

By the definition of $\mathbb{M}P$, any morphism $\mathbb{M}P((n,b)) \longrightarrow Q((n,b))$ is given by a collection of morphisms $P((G))_{\text{Aut}(G)} \longrightarrow Q((n,b))$, each of which factors as

$$\begin{array}{ccc} P((G))_{\text{Aut}(G)} & \xrightarrow{\mu_G^P} & P((n,b)) \\ & \searrow & \downarrow \\ & & Q((n,b)) \end{array}$$

Any morphism $\varphi : \mathbb{M}P \longrightarrow Q$ of modular operads is therefore uniquely defined by a collection of chain complex morphisms $P((n,b)) \longrightarrow Q((n,b))$, indexed by pairs n, b such that $n - 2 + 2b > 0$; that is, a morphism $P \longrightarrow Q$ of stable $\mathbb{S}$-modules.

Conversely, given a morphism of modular operads $\mathbb{M}P \longrightarrow Q$, restricting the morphism $\mathbb{M}P((n,b)) \longrightarrow Q((n,b))$ to $P((*_{n,b}))$ gives a morphism $P((n,b)) = P((*_{n,b})) \longrightarrow Q((n,b))$. □

3.2.3 Cocycles and Coboundaries

Definition 3.2.13. A cocycle D is a functor (not necessarily monoidal) $\Gamma((n,b)) \longrightarrow$ Graded Vector Spaces satisfying the following conditions:

i) $\dim D(G) = 1$ for all $G \in \Gamma((n,b))$

ii) $D(*_{n,b}) = \mathbb{C}$

iii) For any morphism of stable graphs $f : G \longrightarrow G/I$, there is an isomorphism

$$\nu_f : D(G/I) \otimes \bigotimes_{v \in \mathrm{Vert}(G/I)} D(f^{-1}(v)) \longrightarrow D(G)$$

iv) Given a composition of morphisms $G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2$, the following diagram commutes:

$$\begin{array}{ccc}
D(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} D(f_2^{-1}(v)) \otimes \bigotimes_{v \in \mathrm{Vert}(G_1)} D(f_1^{-1}(v)) & \xrightarrow{\nu_{f_2}} & D(G_1) \otimes \bigotimes_{v \in \mathrm{Vert}(G_1)} D(f_1^{-1}(v)) \\
\downarrow \simeq & & \\
D(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} \left(D(f_2^{-1}(v)) \otimes \bigotimes_{w \in f_2^{-1}(v)} D(f_1^{-1}(w)) \right) & & \downarrow \nu_{f_1} \\
\downarrow D(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} \nu_{f_1}|_{f_2^{-1}(v)} & & \\
D(G_2) \otimes \bigotimes_{v \in \mathrm{Vert}(G_2)} D((f_2 \circ f_1)^{-1}(v)) & \xrightarrow{\nu_{f_2 \circ f_1}} & D(G_0)
\end{array}$$

v) If $f : G_0 \longrightarrow G_1$ is an isomorphism, the following diagram commutes:

$$\begin{array}{ccc}
D(G_1) \otimes \bigotimes_{v \in \mathrm{Vert}(G_1)} D(f^{-1}(v)) & \xrightarrow{\nu_f} & D(G_0) \\
& \searrow^{\mathrm{id} \otimes g} & \downarrow D(f) \\
& & D(G_1)
\end{array}$$

where the map g is given as follows. Let π_v be the projection $\pi_v : f^{-1}(v) \longrightarrow *_{n(v),b(v)}$. Then $D(\pi_v)$ is the map $D(\pi_v) : D(f^{-1}(v)) \longrightarrow D(*_{n(v),b(v)}) = \mathbb{C}$, and we set $g = \bigotimes_{\mathrm{Vert}(G_1)} D(\pi_v)$.

Let D be a cocycle. Define $D^n(G) := D(G)^{\otimes n}$ and $D^{-1}(G) := D(G)^*$. By definition, for any stable graph G, we have $D(G) = V[m]$ where $V \simeq \mathbb{C}$ and $m \in \mathbb{Z}$. Therefore $D^n(G) = V^{\otimes n}[nm]$, and $D^{-1}(G) = V^*[-m]$ are again one-dimensional and concentrated in a single degree.

Recall the determinant of a vector space V is the one-dimensional vector space $\bigwedge^{\mathrm{top}}(V)[\dim V]$, which is concentrated in degree $(-\dim V)$. If S is a finite set, write $\mathbb{C}^S$ for $\bigoplus_S \mathbb{C}$ and set $Det(S) := Det(\mathbb{C}^S) = \bigwedge^{|S|}(\mathbb{C}^S)[|S|]$. The familiar isomorphism $\bigwedge^{\mathrm{top}}(V \oplus W) \simeq \bigwedge^{\mathrm{top}}(V) \otimes \bigwedge^{\mathrm{top}}(W)$ tells us that $Det(\coprod_I S_i) \simeq \bigotimes_I Det(S_i)$ for any finite sequence of disjoint sets $\{S_i\}$. If $V_* = \bigoplus_I V_i$ is a graded finite dimensional $\mathbb{C}$-vector space, then set

$$Det(V_*) = \bigotimes_{\mathbb{Z}} Det^{(-1)^i}(V_i)$$

The two cocycles which will be important for the purposes of this project are

$$\begin{aligned} \mathcal{K}(G) &= \mathrm{Det}(\mathrm{Edge}(G)) \\ \mathcal{L}(G) &= \mathrm{Det}(\mathrm{Flag}(G))\mathrm{Det}^{-1}(\mathrm{Leg}(G)) \end{aligned}$$

For example, if G is the graph

then $\mathcal{K}(G) = \bigwedge^2(\mathbb{C}^2)[2]$, and $\mathcal{L}(G) = \bigwedge^2 \mathbb{C}^{\{f,f'\}}[2] \otimes \bigwedge^2 \mathbb{C}^{\{g,g'\}}[2]$, where (ff') and (gg') are the edges of G.

Lemma 3.2.14. Let l be an integer. Then $\mathcal{K}^{l-2}\mathcal{L}(G) = \mathcal{K}^l(G) \otimes \bigotimes_{\mathrm{Edge}(G)} \bigwedge^2 \mathbb{C}^{\{f,f'\}}$.

Proof. Let G be a stable graph. Calculating yields

$$\begin{aligned}
& \mathcal{K}^{l-2}\mathcal{L}(G) \\
= \; & \mathcal{K}^l(G) \otimes \mathcal{K}^{-2}(G) \otimes \mathcal{L}(G) \\
= \; & \mathcal{K}^l(G) \otimes \mathbb{C}[-2|\mathrm{Edge}(G)|] \otimes \mathrm{Det}(\mathrm{Flag}(G)) \otimes \mathrm{Det}^{-1}(\mathrm{Leg}(G)) \\
= \; & \mathcal{K}^l(G) \otimes \mathbb{C}[-2|\mathrm{Edge}(G)|] \otimes \bigwedge^{\mathrm{top}} \mathbb{C}^{\mathrm{Flag}(G)}[|\mathrm{Flag}(G)|] \\
& \otimes (\bigwedge^{\mathrm{top}} \mathbb{C}^{\mathrm{Leg}(G)}[|\mathrm{Leg}(G)|])^* \\
= \; & \mathcal{K}^l(G) \otimes \mathbb{C}[-2|\mathrm{Edge}(G)|] \otimes \bigwedge^{\mathrm{top}} \mathbb{C}^{\mathrm{Leg}(G) \sqcup \bigsqcup_{\mathrm{Edge}(G)} \{s_e,t_e\}}[|\mathrm{Leg}(G)| + 2|\mathrm{Edge}(G)|] \\
& \otimes (\bigwedge^{\mathrm{top}} \mathbb{C}^{\mathrm{Leg}(G)}[|\mathrm{Leg}(G)|])^* \\
= \; & \mathcal{K}^l(G) \otimes \mathbb{C}[-2|\mathrm{Edge}(G)|] \otimes \bigwedge^{\mathrm{top}} \mathbb{C}^{\mathrm{Leg}(G)}[|\mathrm{Leg}(G)|] \otimes \\
& \bigwedge^{\mathrm{top}} \mathbb{C}^{\bigsqcup_{\mathrm{Edge}(G)} \{s_e,t_e\}}[2|\mathrm{Edge}(G)|] \otimes (\bigwedge^{\mathrm{top}} \mathbb{C}^{\mathrm{Leg}(G)}[|\mathrm{Leg}(G)|])^* \\
= \; & \mathcal{K}^l(G) \otimes \bigwedge^{\mathrm{top}} \mathbb{C}^{\bigsqcup_{\mathrm{Edge}(G)} \{s_e,t_e\}} \\
= \; & \mathcal{K}^l(G) \otimes \bigotimes_{\mathrm{Edge}(G)} \bigwedge^2 \mathbb{C}^{\{s_e,t_e\}} \qquad (3.7)
\end{aligned}$$

□

Definition 3.2.15. Let $D^{\vee}$ denote the *dualizing cocycle* $\mathcal{K} \otimes D^{-1}$.

Let s be a stable $\mathbb{S}$-module such that $\dim_{\mathbb{C}} s((n,b)) = 1$ for all b, n.

Definition 3.2.16. The coboundary of s is the cocycle

$$D_s(G) := s((n,b)) \otimes \bigotimes_{v \in \mathrm{Vert}(G)} s^{-1}((n(v), b(v)))$$

Later on when we examine algebras over certain operads, it will be important to shift degrees and alter the $\mathbb{S}_n$-action for the sake of compatibility. This will be done using the coboundaries induced by the stable $\mathbb{S}$-modules

$$\begin{aligned} \Sigma((n,b)) &= \mathbb{C}[-1] \\ \alpha((n,b)) &= \mathbb{C}[n] \\ \beta((n,b)) &= \mathbb{C}[b-1] \\ \tilde{\mathfrak{s}} &= sgn_n[n] \\ \chi_l &= \tilde{\mathfrak{s}}\Sigma\beta^{2(1-l)} \ l \in \mathbb{Z} \end{aligned}$$

Let $\sigma \in S_n$. If s is among the first three, then $S_n \times D_s(G) \longrightarrow D_s(G)$ is given by $(\sigma, v) \mapsto v$, and if $s = \tilde{\mathfrak{s}}$, then $(\sigma, v) \mapsto \mathrm{sgn}_n(\sigma) \cdot v$. The same holds for χ_l.

The upshot here is that if s is a stable $\mathbb{S}$-module of the type mentioned above, and if P is a modular D-operad, then sP is a modular $D_s \otimes D$-operad.

Remark 3.2.17. It is easy to see that $D_{\beta^2} = \mathcal{K}^2$ and $D_{\tilde{\mathfrak{s}}} = \mathcal{L}^{-1}$.

Definition 3.2.18. Twisted modular operads are defined by simply replacing $P((G))$ in Definitions (3.2.2) and (3.2.4) with $D(G) \otimes P((G))$. More specifically, the map (3.2) becomes

$$\mu : \mathbb{M}_D P((n,b)) := \bigoplus_{G \in [\Gamma((n,b))]} (D(G) \otimes P((G)))_{\mathrm{Aut}(G)} \longrightarrow P((n,b)) \tag{3.8}$$

and (3.3) becomes

$$\mu_G : D(G) \otimes P((G)) \longrightarrow D(*_{n,b}) \otimes P((*_{n,b})) = P((n,b)) \tag{3.9}$$

A twisted modular operad should be thought of as a modular operad whose composition maps μ_G are twisted by a factor of $D(G)$.

3.2.4 The Twisted Modular D-operad $\mathbb{M}_D P$

The free modular D-operad $\mathbb{M}_D P$ is defined by

$$\mathbb{M}_D P((n,b)) = \bigoplus_{G \in [\Gamma((n,b))]} (D(G) \otimes P((G)))_{\mathrm{Aut}(G)},$$

with composition map

$$D(G) \otimes \mathbb{M}_D P((G)) \longrightarrow \mathbb{M}_D P((n,b)),$$

and is said to be generated by the stable $\mathbb{S}$-module P. As is the case with any operad, $\mathbb{M}_D P$ is generated by the simple tensors in $\mathbb{M}_D P((G)) = \bigotimes_{\mathrm{Vert}(G)} \mathbb{M}_D P((G_{v_i}))$. For each graph G, there exists a choice function $\eta = \oplus \eta_j$ as in section 2.2.1, such that $G_\eta = G$. Since $G_{v_j} = *_{n(v_j),b(v_j)}$ for all $v_j \in \mathrm{Vert}(G)$, the groups $\mathrm{Aut}(G_{v_j})$ are trivial, and because $D(*_{n,b}) = \mathbb{C}$ for any n, b, the restriction

$$\mu_G^{\mathbb{M}_D P}|_{(\otimes_{\mathrm{Vert}(G)}(D(G_{v_j}) \otimes P((G_{v_j}))))_{\prod \mathrm{Aut}(G_{v_j})}}$$

is just the projection

$$D(G) \otimes \bigotimes_{\mathrm{Vert}(G)} P((G_{v_j})) = D(G) \otimes P((G)) \longrightarrow (D(G) \otimes P((G)))_{\mathrm{Aut}(G)}$$

3.2.5 The Twisted Modular Operad $\mathcal{E}_V$

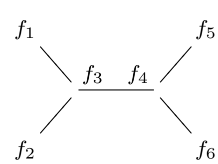
Set $\mathcal{E}_V((G)) \coloneqq V^{\otimes \mathrm{Flag} G}$, and define

$$\mu_G^{\mathcal{E}_V} : \mathcal{K}^{-1}\mathcal{L}(G) \otimes \mathcal{E}_V((G)) \longrightarrow \mathcal{E}_V((*_{n,b}))$$

by

$$(\bigwedge^{\epsilon_G} e_i)[\epsilon_G] \otimes \bigotimes_{\mathrm{Edge}(G)} (f \wedge f') \otimes \bigotimes_{f \in \mathrm{Flag}(G)} v_f \mapsto (-1)^\epsilon \prod_{\substack{\mathrm{Edges} \\ (f,f')}} B(v_f, v_{f'}) \bigotimes_{f \in \mathrm{Leg}(G)} v_f,$$

where B is an anti-symmetric pairing $V \otimes V \longrightarrow \mathbb{C}$ of degree 1, $\epsilon_G = |\mathrm{Edge}(G)|$, and the sign $(-1)^\epsilon$ is defined below. The first factor $(\bigwedge^{\epsilon_G} e_i)[\epsilon_G]$ accounts for the degree of B, and the second factor $\bigotimes_{\mathrm{Edge}(G)}(f \wedge f')$ accounts for the anti-symmetry of B. For example, if G is the graph



then μ_G is the map

$$\mathcal{K}^{-1}\mathcal{L}(G)\otimes\mathcal{E}_V((G)) = V^{\otimes\{f_1,f_2,f_3,f_4,f_5,f_6\}} \longrightarrow V^{\otimes\{f_1,f_2,f_5,f_6\}} = \mathcal{E}_V((*_{4,0}))$$

$$e[1]\otimes(f_3\wedge f_4)\otimes v_{f_1}\otimes\cdots\otimes v_{f_6} \mapsto (-1)^{\epsilon}B(v_{f_3},v_{f_4})v_{f_1}\otimes v_{f_2}\otimes v_{f_5}\otimes v_{f_6}$$

The Sign $(-1)^{\epsilon}$

Let G be a graph with two vertices labeled by b_1 and b_2, and one edge given by (ff'). One then has $\mathcal{E}_V((G)) = \mathcal{E}_V((*_{n_1,b_1}))\otimes\mathcal{E}_V((*_{n_2,b_2}))$. The simple tensors have the form

$$(v_1\otimes\cdots\otimes v_f\otimes\cdots\otimes v_{n_1})\otimes(w_1\otimes\cdots\otimes w_{f'}\otimes\cdots\otimes w_{n_2})$$

Composition is done in several steps, each involving a permutation of the factors with the goal being to insert, in the operadic sense, v into w. Assume v_f sits in the α^{th} spot of $v_1\otimes\cdots\otimes v_{n_1}$ and $w_{f'}$ sits in the β^{th} spot of $w_1\otimes\cdots\otimes w_{n_2}$.

First, move v_f through $v_{f+1}\otimes\cdots\otimes v_{n_1}$. This produces the sign $(-1)^{|v_f|(\sum_{i=\alpha+1}^{n_1}|v_i|)}$. Then move w_i $1\le i\le\beta-1$ successively all the way to the right. Moving w_i gives the sign $(-1)^{|w_i|(|w|-|w_i|)}$. The product now has the form

$$(-1)^{|v_f|(\sum_{i=\alpha+1}^{n_1}|v_i|)}(-1)^{\sum_{j=1}^{\beta-1}|w_j|(|w|-|w_j|)} v_1\otimes\cdots\otimes v_{n_1}\otimes v_f\otimes w_{f'}\otimes w_{f'+1}\otimes\cdots$$
$$\cdots\otimes w_{n_2}\otimes w_1\otimes\cdots\otimes w_{f'-1}$$

Contracting this product via $\mathrm{id}\otimes\cdots\otimes B\otimes\cdots\otimes\mathrm{id}$ gives the sign $(-1)^{|B|(|v|-|v_f|)}$. Finally, move $w_1,\ldots,w_{\beta-1}$ all the way to the left. Moving each w_l for $1\le l\le\beta-1$ gives the sign $(-1)^{|w_\ell|(|v|+|w|-|v_f|-|w_{f'}|-|w_\ell|)}$. The end result is

$$(-1)^{|v_f|(\sum_{i=\alpha+1}^{n_1}|v_i|)+\sum_{j=1}^{\beta-1}|w_j|(|w|-|w_j|)+|B|(|v|-|v_f|)+\sum_{\ell=1}^{\beta-1}|w_\ell|(|v|+|w|-|v_f|-|w_{f'}|-|w_\ell|)}$$
$$B(v_f,w_{f'})w_1\otimes\cdots\otimes w_{\beta-1}\otimes v_1\otimes\cdots\otimes v_{n_1}\otimes w_{\beta+1}\otimes\cdots\otimes w_{n_2} \tag{3.10}$$

Now let $G = *^1_{n,b}$. Then $\mu_G^{\mathcal{E}_V}:\mathcal{E}_V((G))\longrightarrow\mathcal{E}_V((n,b+1))$ is given by

$$v_1\otimes\cdots\otimes v_f\otimes\cdots\otimes v_{f'}\otimes\cdots\otimes v_n$$
$$\mapsto (-1)^{|v_f|(\sum_{i=\alpha+1}^{\beta-1}|v_i|)+|B|((\sum_{j=1}^{\beta-1}|v_j|)-|v_f|)}B(v_f,v_{f'})v_1\otimes\cdots\otimes v_{f-1}\otimes v_{f+1}\otimes\cdots$$
$$\cdots\otimes v_{f'-1}\otimes v_{f'+1}\otimes\cdots\otimes v_n \tag{3.11}$$

3.2.6 The Twisted Modular Operad $\mathbb{S}[t]$

Let $\mathbb{C}[S_n]'$ denote the $\mathbb{N}$-graded $\mathbb{C}$-vector space with basis indexed by elements (σ, a_σ) with $\deg(\sigma, a_\sigma) \coloneqq \deg a_\sigma = -i_\sigma$, where $\sigma \in S_n$ is a permutation with i_σ cycles and $a_\sigma = \sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma} \in Det(cycle\, \sigma)$. We also impose the relation $(\sigma, -a_\sigma) = -(\sigma, a_\sigma)$. Let $\mathbb{C}[t]$ denote the space of polynomials in the variable t and let $\deg t = -2$. Note that deg $t^g = -2g$.

The underlying $\mathbb{S}$-module of the modular operad $\mathbb{S}[t]$ is

$$\mathbb{S}((n)) \coloneqq \mathbb{C}[S_n]'[-1] \otimes_{\mathbb{C}} \mathbb{C}[t],$$

and $\mathbb{S}((n)) = \bigoplus_{b \geq 0} \mathbb{S}[t]((n,b))$ with

$$\mathbb{S}[t]((n,b)) = \bigoplus_{\substack{\sigma, g \\ b = 2g + i_\sigma - 1}} \mathbb{C} \cdot (\sigma, a_\sigma) t^g [-1],$$

where $\deg(\sigma, a_\sigma) t^g = \deg t^g + \deg a_\sigma = -2g - i_\sigma$. This gives a shift of $2g + i_\sigma$, and as a vector space, $\mathbb{C} \cdot (\sigma, a_\sigma) t^g [-1] \simeq \mathbb{C}[2g + i_\sigma - 1]$. In particular, $\mathbb{S}[t]((n,b))$ is a one-dimensional $\mathbb{C}$-vector space sitting in degree $(-b)$. Note that if $b \leq 1$, then $g = 0$. If $n \leq 2$, set $\mathbb{S}[t]((n,0)) = 0$.

Definition 3.2.19. A *stable ribbon graph* is a connected graph G together with

i) partitions of the set of flags adjacent to every vertex into $i(v)$ subsets $\mathrm{Leg}(v)_j \subseteq \mathrm{Leg}(v)$ with $\mathrm{Leg}(v)_j \cap \mathrm{Leg}(v)_k = \varnothing$ if $j \neq k$, $1 \leq j, k \leq i(v)$.

ii) a fixed cyclic order on every subset $\mathrm{Leg}(v)_j$.

iii) a map $g : \mathrm{Vert}(G) \longrightarrow \mathbb{Z}_{\geq 0}$ such that for any vertex v, $2(2g(v) + i(v) - 2) + n(v) > 0$, so that putting $b(v) = 2g(v) + i(v) - 1$ defines a stable graph.

The basis vectors $(\sigma, a_\sigma) t^g$ can be represented by single-vertex stable ribbon graphs with associated orientations in the following way: Let $\mathbb{C}[S_n]$ be the vector space $\mathrm{Span}_{\mathbb{C}}\{\sigma \in S_n\}$ concentrated in degree 0, where $\mathbb{C}[S_n]^*$ is identified with $\mathbb{C}[S_n]$ via $\sigma^*(\tau) = \delta_{\sigma\tau}$. Let $\sigma = \sigma_1 \cdots \sigma_{i_\sigma}$ act on the flags of $\star_{n,b}$ and identify σ_j, $1 \leq j \leq i_\sigma$, with the corresponding orbit in $\mathrm{Flag}(\star_{n,b}) \coloneqq \{1, \ldots, n\}$. Then $(\sigma, a_\sigma) t^g$ can be represented by $\star_{n,b}$, thought of as the stable ribbon graph, and is best written as

$$\begin{aligned} \star_{n,b}[-1] \otimes (\sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma})[i_\sigma] \otimes t^g \quad &\in \quad \mathbb{S}[t]((n, 2g + i_\sigma - 1)) \\ &= \quad \mathbb{S}[t]((n,b)) \\ &\subset \quad \mathbb{C}[S_n]'[-1] \otimes \mathbb{C}[t] \end{aligned}$$

where the shift by -1 coincides with $[-1]$ in $\mathbb{C}[S_n]'[-1]\otimes\mathbb{C}[t]$, and $*_{n,b}$ contributes no shift. The idea is to write the vectors $(\sigma,a_\sigma)t^g$ in such way that the composition maps $\mu_G^{\mathbb{S}[t]}$ can be defined as explicitly as possible.

Tensoring with t^g is nothing but a shift, so we can rewrite the latter expression as

$$*_{n,b}\otimes(\sigma_1\wedge\cdots\wedge\sigma_{i_\sigma})[2g+i_\sigma-1]=*_{n,b}\otimes(\sigma_1\wedge\cdots\wedge\sigma_{i_\sigma})[b] \tag{3.12}$$

By definition, $\mathbb{S}[t]((n,b))$ is finite dimensional, so we can put a finite order on the basis elements. If we write $\mathbb{S}[t]((n,b))=\mathrm{Span}_{\mathbb{C}}\{e_i\}$, then the canonical basis, $\{f_i\}$, for $\mathbb{S}[t]((n,b))^*$ is defined by $f_i(e_j)=\delta_{ij}$. If

$$*_{n,b}\otimes(\sigma_1\wedge\cdots\wedge\sigma_{i_\sigma})[b]$$

corresponds to e_i, then f_i will be represented by

$$*_{n,b}\otimes(\sigma_1\wedge\cdots\wedge\sigma_{i_\sigma})[-b].$$

Composition for $\mathbb{S}[t]$

$\mathbb{S}[t]$ is a modular Det operad, so the compositions all have the form

$$\mu^{\mathbb{S}[t]}_{G\to G/e}:\mathrm{Det}(H_1(G))\otimes\mathbb{S}[t]((G))\longrightarrow\mathrm{Det}(H_1(G/e))\otimes\mathbb{S}[t]((G/e)),$$

where this map has degree 0.

As was mentioned in Remark 3.2.6, all compositions in $\mathbb{S}[t]$ can be built from the compositions along graphs with either two vertices and one edge, or one vertex and one loop. The former is defined as follows.

Composition along $*_{n_1,b}\otimes *_{n_2,b'}$

Let G be the graph with vertices $v_1=\{I\sqcup\{f\}\}$, $v_2=\{J\sqcup\{f'\}\}$, involution (ff'), and $b(v_1)=b$, $b(v_2)=b'$. Choose an arbitrary ordering on v_1 and identify $\mathbb{S}[t]((I\sqcup\{f\},b))$ with $\mathbb{S}[t]((|I\sqcup\{f\}|,b))$. Since $\mathrm{Det}(H_1(G))=\mathrm{Det}(H_1(G/e))\simeq\mathbb{C}$, $\mu^{\mathbb{S}[t]}_{G\to *_{n,b}}=\mu^{\mathbb{S}[t]}_G$ is a map

$$\mathbb{S}[t]((I\sqcup\{f\},b))\otimes\mathbb{S}[t]((J\sqcup\{f'\},b'))\longrightarrow\mathbb{S}[t]((|I\sqcup J|,b+b'))$$

Let $(\sigma,a_\sigma)t^g$, $(\rho,a_\rho)t^{g'}$ be basis vectors of $\mathbb{S}[t]((I\sqcup\{f\},b))$ and $\mathbb{S}[t]((J\sqcup\{f'\},b'))$ respectively, with $a_\sigma=\sigma_1\wedge\cdots\wedge\sigma_{i_\sigma}[i_\sigma]$, $a_\rho=\rho_1\wedge\cdots\wedge\rho_{i_\rho}[i_\rho]$ and $f\in\sigma_k$, $f'\in\rho_l$.

Define

$$\pi_{ff'} : \mathrm{Aut}(\{1,\dots,n\} \sqcup \{f,f'\}) \longrightarrow S_n$$

$$\sigma \mapsto \begin{cases} (f'\,\sigma(f'))(f\,\sigma(f))\sigma & \text{if } \sigma(f) \neq f' \text{ and } \sigma(f') \neq f \\ (f'\,\sigma(f'))(f\,\sigma(f))\sigma & \text{if } \sigma(f) = f' \\ (f\,\sigma(f))(f'\,\sigma(f'))\sigma & \text{if } \sigma(f') = f \end{cases}$$

and let $\mu = \pi_{ff'}(\sigma\rho(ff'))$. Intuitively, $\pi_{ff'}$ is the operation which erases f and f' from the permutation σ. For example, if $n = 5$ and $\sigma\rho = (12f3)(f'45)$, then $\sigma\rho(ff') = (12f3)(f'45)(ff') = (12f45f'3)$ and $\pi_{ff'}((12f45f'3)) = (12453)$, so $\pi_{ff'}(\sigma\rho(ff')) = (12453)$. This can be seen graphically as follows:

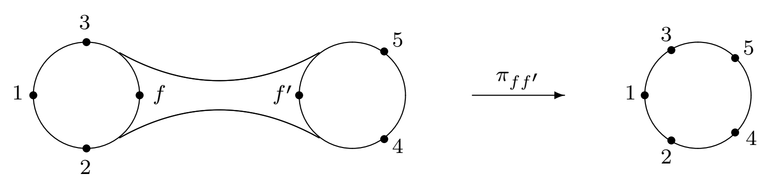


Figure 3.2: Right multiplication by the transposition (ff'), followed by application of the map $\pi_{ff'}$ has the effect of gluing the cycles at the points f and f'. One could also think of this as inserting (45) into (123).

We then have $\mu_G^{\mathbb{S}[t]}((\sigma, a_\sigma)t^g \otimes (\rho, a_\rho)t^{g'}) = (\mu, a_\mu)t^{g+g'}$, where

$$a_\mu = (-1)^{k+l}\pi_{ff'}\sigma_k\rho_l(ff') \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma} \wedge \rho_1 \wedge \cdots \wedge \hat{\rho_l} \wedge \cdots \wedge \rho_{i_\rho}[i_\mu]$$

and $f \in \sigma_k$, $f' \in \rho_l$.

I claim this is the natural choice for a_μ. As before, write $(\sigma, a_\sigma)t^g$ as $\star_{|I|,b} \otimes \sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma}[b]$ using the identification in the previous section. The element $(\sigma, a_\sigma)t^g \otimes (\rho, a_\rho)t^{g'}$ then becomes

$$\star_{|I|+1,b} \otimes (\sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma})[b] \otimes \star_{|J|+1,b'} \otimes (\rho_1 \wedge \cdots \wedge \rho_{i_\rho})[b']$$

and this is rewritten as

$$(\star_{|I|+1,b} \otimes_{f,f'} \star_{|J|+1,b'})(\sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma} \wedge \rho_1 \wedge \cdots \wedge \rho_{i_\rho})[b + b'].$$

There should be a natural contraction

$$\mathrm{Det}(cycle\,\sigma) \otimes \mathrm{Det}(cycle\,\rho) \to \mathrm{Det}(cycle\,\mu)$$

mapping $a_\sigma \otimes a_\rho$ to a_μ, compatible with the contraction

$$*_{|I|+1,b} \otimes_{f,f'} *_{|J|+1,b'} \mapsto *_{|I|+|J|,b+b'} = *_{n,b+b'},$$

which records the sewing of σ_k with ρ_l along (ff'), and the deletion of σ_k and ρ_l. The isomorphism $\mathrm{Det}(S \sqcup T) \simeq \mathrm{Det}(S) \otimes \mathrm{Det}(T)$, when applied to the sets $\{\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k\}\}$ and $\{\sigma_k\}$, gives

$$\mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k\}) \xrightarrow{\sim} \mathrm{Det}(\{\sigma_k\}) \otimes \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k\}) \xrightarrow{\sim} \mathrm{Det}(\mathit{cycle}\,\sigma)$$

$$\sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 1] \mapsto \sigma_k[1] \otimes \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 1] \mapsto (-1)^{k-1} \sigma_1 \wedge \cdots \wedge \sigma_k \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma]$$

The factor $(-1)^{k-1}$ comes from moving σ_k from the first spot to the kth spot by applying the permutation $(k\ k\text{-}1)(k\text{-}1\ k\text{-}2)\cdots(k\,1)$ to $\sigma_k \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma}$. Inverting this isomorphism gives rise to the sequence

$$\mathrm{Det}(\mathit{cycle}\,\sigma) \otimes \mathrm{Det}(\mathit{cycle}\,\rho) \longrightarrow \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k\}) \otimes \mathrm{Det}(\mathit{cycle}\,\rho \smallsetminus \{\rho_l\}) \longrightarrow$$

$$\longrightarrow \mathrm{Det}(\pi_{ff'}\sigma_k\rho_l(ff')) \otimes \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k\}) \otimes \mathrm{Det}(\mathit{cycle}\,\rho \smallsetminus \{\rho_l\}) \longrightarrow$$

$$\longrightarrow \mathrm{Det}(\pi_{ff'}(\sigma\rho(ff'))) = \mathrm{Det}(\mathit{cycle}\,\mu)$$

On elements,

$$a_\sigma \otimes a_\rho = \sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma] \otimes \rho_1 \wedge \cdots \wedge \rho_{i_\rho}[i_\rho] \mapsto$$

$$(-1)^{k+l-2} \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 1] \otimes \rho_1 \wedge \cdots \wedge \hat{\rho_l} \wedge \cdots \wedge \rho_{i_\rho}[i_\rho - 1] \mapsto$$

$$\pi_{ff'}\sigma_k\rho_l(ff')[1] \otimes (-1)^{k+l} \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 1] \otimes \rho_1 \wedge \cdots \wedge \hat{\rho_l} \wedge \cdots \wedge \rho_{i_\rho}[i_\rho - 1] \mapsto$$

$$(-1)^{k+l} \pi_{ff'}\sigma_k\rho_l(ff') \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma} \wedge \rho_1 \wedge \cdots \wedge \hat{\rho_l} \wedge \cdots \wedge \rho_{i_\rho}[(i_\sigma + i_\rho - 1) = i_\mu],$$

which is a_μ.

It is important to note that if we want to think of the elements $(\sigma, a_\sigma)t^g$ as $*_{n,b}[-1] \otimes \sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma] \otimes t^g$, when the map $\mu_G^{\mathbb{S}[t]}$ is applied, the contraction $*_{|I|,b} \otimes_{\{f,f'\}} *_{|J|,b'} \to *_{n,b+b'}$ is done in degree zero, and the result is then shifted by -1.

Composition along $*^1_{n,b-1}$

Let G be the graph $*^1_{n,b-1}$ with a single vertex $v = \{1,\dots,n\} \sqcup \{f,f'\}$ and single loop defined by the edge $e = (ff')$. Then $G/e = *^0_{n,b}$ and $\mu^{\mathbb{S}[t]}_G$ has the form

$$\mathrm{Det}(H_1(G)) \otimes \mathbb{S}[t]((G)) \longrightarrow \mathbb{C} \otimes \mathbb{S}[t]((n,b))$$

Assume first that $f, f' \in \sigma_k, \sigma_l$ respectively, with $k < l$. Let $\mu = \pi_{ff'}\sigma(ff')$. Then $\mu^{\mathbb{S}[t]}_G(e_{ff'}[1] \otimes (\sigma, a_\sigma)t^g) = 1 \otimes (\mu, a_\mu)t^{g+1}$, where $e_{ff'}[1] \mapsto 1$ is a map of degree -1, and

$$a_\mu = (-1)^{k+l-1}\pi_{ff'}\sigma_k\sigma_l(ff') \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \hat{\sigma_l} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\mu].$$

As before, there should be a natural contraction $\mathrm{Det}(\mathit{cycle}\,\sigma) \to \mathrm{Det}(\mathit{cycle}\,\mu)$ compatible with sewing the cycles σ_k, σ_l, and the contraction $*^1_{n,b-1} \to *^0_{n,b}$. Also as before, there is a sequence of natural isomorphisms which yield a_μ. The first of which is as follows:

$$\mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k,\sigma_l\}) \longrightarrow \mathrm{Det}(\sigma_k) \otimes \mathrm{Det}(\sigma_l) \otimes \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k,\sigma_l\}) \longrightarrow$$

$$\longrightarrow \mathrm{Det}(\sigma_k) \otimes \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \sigma_k) \longrightarrow \mathrm{Det}(\mathit{cycle}\,\sigma)$$

On elements,

$$\sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \hat{\sigma_l} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 2] \mapsto$$

$$\sigma_k[1] \otimes \sigma_l[1] \otimes \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \hat{\sigma_l} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 2] \mapsto$$

$$\sigma_k[1] \otimes (-1)^{l-1-1}\sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_l \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 1] \mapsto$$

$$(-1)^{k-1+l-1-1}\sigma_1 \wedge \cdots \wedge \sigma_k \wedge \cdots \wedge \sigma_l \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma]$$

Inverting this map gives rise to the sequence

$$\mathrm{Det}(\mathit{cycle}\,\sigma) \to \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k,\sigma_l\}) \to \mathrm{Det}(\pi_{ff'}\sigma_k\sigma_l(ff')) \otimes \mathrm{Det}(\mathit{cycle}\,\sigma \smallsetminus \{\sigma_k,\sigma_l\}) \to$$

$$\to \mathrm{Det}(\mathit{cycle}\,\mu)$$

On the element $\sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma]$, this takes the form

$$\sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma] \mapsto$$

$$(-1)^{k+l-1}\sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \hat{\sigma_l} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 2] \mapsto$$

$$(-1)^{k+l-1}\pi_{ff'}\sigma_k\sigma_l(ff')[1] \otimes \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \hat{\sigma_l} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 2] \mapsto$$

$$(-1)^{k+l-1}\pi_{ff'}\sigma_k\sigma_l(ff') \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \hat{\sigma_l} \wedge \cdots \wedge \sigma_{i_\sigma}[i_\sigma - 1]$$

Remark 3.2.20. The number $b(G)$ is defined as

$$\left(\sum_{v\in \mathrm{Vert}(G)} b(v)\right) + \dim \mathrm{H}_1(G),$$

so in this case, $b-1 = b(v) = 2g + i_\sigma - 1$, where v is the vertex of $*^1_{n,b-1}$. Therefore $b = 2g + i_\sigma = 2(g+1) + (i_\sigma - 1) - 1$ should be the total shift of $\mu_G^{\mathbb{S}[t]}(e_{ff'}[1] \otimes (\sigma, a_\sigma)t^g) = *^0_{n,b}[-1] \otimes a_\mu[i_\mu = i_\sigma - 1] \otimes t^{g+1}$. By considering this shift, one sees that the element $e_{ff'}[1]$ does not act as an orientation correction, or as a shift of $(\mu, a_\mu)t^{g+1}$. It simply maps to $1 \in \mathrm{Det}(*^0_{n,b}) \simeq \mathbb{C}$, so as to ensure $\deg \mu_G^{\mathbb{S}[t]} = 0$.

Now assume $f, f' \in \sigma_k$. Then $\mu_G^{\mathbb{S}[t]}$ dissects the cycle σ_k into two cycles, whose relative order in the basis vector of $\mathrm{Det}(cycle\, \pi_{ff'}\sigma(ff'))$ is determined by the order in which f and f' appear in σ_k. In this case, $\sigma(ff')$ is the product of two cycles, σ_k^f and $\sigma_k^{f'}$, where σ_k^f contains f and $\sigma_k^{f'}$ containsf'. We have $\mu_G^{\mathbb{S}[t]}(e_{ff'}[1] \otimes (\sigma, a_\sigma)t^g) = 1 \otimes (\mu, a_\mu)t^g$, and

$$a_\mu = \begin{cases} (-1)^{k-1}(\pi_f \sigma_k^f) \wedge (\pi_{f'}\sigma_k^{f'}) \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma} & \text{if } f \text{ appears before } f' \text{ in } \sigma \\ (-1)^{k-1}(\pi_{f'} \sigma_k^{f'}) \wedge (\pi_{f}\sigma_k^{f}) \wedge \sigma_1 \wedge \cdots \wedge \hat{\sigma_k} \wedge \cdots \wedge \sigma_{i_\sigma} & \text{if } f' \text{ appears before } f \text{ in } \sigma \end{cases}$$

This is well-defined as long as all of the cycles of a given length begin with the same letter. The factor of $(-1)^{k-1}$ comes from the following two sequences of natural isomorphisms. The first is

$$\mathrm{Det}(cycle\, \sigma \smallsetminus \sigma_k) \longrightarrow \mathrm{Det}(cycle\, \sigma),$$

which was defined above. The second is

$$\mathrm{Det}(cycle\, \sigma) \longrightarrow \mathrm{Det}(cycle\, \sigma \smallsetminus \sigma_k) \longrightarrow \mathrm{Det}(cycle\, \pi_{ff'}\sigma_k(ff')) \otimes \mathrm{Det}(cycle\, \sigma \smallsetminus \sigma_k) \longrightarrow$$

$$\longrightarrow \mathrm{Det}(\pi_f \sigma_k^f) \otimes \mathrm{Det}(\pi_{f'}\sigma_k^{f'}) \otimes \mathrm{Det}(cycle\, \sigma \smallsetminus \sigma_k) \longrightarrow \mathrm{Det}(cycle\, \pi_{ff'}\sigma(ff'))$$

If f and f' are neighbors in σ_k, then either $\sigma_k^f = (f)$, or $\sigma_k^{f'} = (f')$. In either case, $\mu_G^{\mathbb{S}[t]}(e_{ff'}[1] \otimes (\sigma, a_\sigma)t^g) = 0$.

3.2.7 The Modular Operad $\tilde{\mathbb{S}}[t]$

The underlying $\mathbb{S}$-module is $\tilde{\mathbb{S}}[t]((n)) = k[S_n] \otimes k[t]$ and

$$\tilde{\mathbb{S}}[t]((n,b)) = \bigoplus_{\substack{\sigma \\ b = 2g + i_\sigma - 1}} k \cdot \sigma t^g$$

where $\deg \sigma t^g = -2g$. The compositions $\mu_G^{\tilde{\mathbb{S}}[t]} : \tilde{\mathbb{S}}[t]((G)) \longrightarrow \tilde{\mathbb{S}}[t]((n,b))$ are given by $\sigma t^g \otimes \tau t^{g'} \mapsto \pi_{ff'}(\sigma\tau(ff'))t^{g+g'}$, where σ and τ act nontrivially on f and f', respectively. In the elliptic curve case, we always take $g = 0$. This is indeed an untwisted modular operad, with a twisted version playing a key role in the main result of this project.

3.3 The Feynman Transform

As a stable $\mathbb{S}$-module, but ignoring differentials, $\mathscr{F}_D P$ equals $\mathbb{M}_{D^\vee} P^*$, the underlying stable $\mathbb{S}$-module of the free modular $D^\vee$-operad generated by the linear dual P^* of P.

The Feynman Differential

The differential of the Feynman Transform, $d_{\mathscr{F}_D P}$, is the sum $d_{\mathscr{F}_D P} := \partial_{P^*} + \delta$, where ∂_{P^*} and δ are defined as follows: For any G, $D^\vee(G)$ is a graded vector space concentrated in degree $-(|\mathrm{Edge}(G)| + \deg D(G))$. So thinking of $D^\vee(G) \otimes P((G))^*$ as the total complex of a double complex, only one column of $D^\vee(G) \otimes P((G))^*$ is nonzero. Define ∂_{P^*} by $\partial_{P^*}(x \otimes y) = x \otimes \mathrm{d}_{P^*}(y)$, where d_{P^*} is the differential on P^*.

The map δ is induced by the composition maps $\mu^P_{G\to G/e}$ of the modular D-operad P. Let G be a stable graph in $[\Gamma((n,b))]$, and let e be an edge of G. The morphism $G \to G/e$ gives rise to a map

$$D(G) \otimes P((G)) \longrightarrow D(G/e) \otimes P((G/e)).$$

Dualizing, and taking the adjoint map, gives

$$D^{-1}(G/e) \otimes P((G/e))^* = D(G/e)^* \otimes P((G))^* \longrightarrow D(G)^* \otimes P((G))^* = D^{-1}(G) \otimes P((G))^*.$$

The subtle point here is that the map $(\mu^P_{G\to G/e})^*$ descends to a map

$$(D^{-1}(G/e) \otimes P((G/e))^*)_{\mathrm{Aut}(G/e)} \to (D^{-1}(G) \otimes P((G))^*)_{\mathrm{Aut}(G)}.$$

Both maps will be written $(\mu^P_{G\to G/e})^*$ as this will cause no confusion.

The cocycle $\mathcal{K}$ takes G to $\mathcal{K}(G) := \Lambda^{|\mathrm{Edge}(G)|}\mathbb{C}^{\mathrm{Edge}(G)}[\mathrm{Edge}(G)]$, and setting $e_k = e$, the natural isomorphism $\mathrm{Det}(\{e\}) \otimes \mathrm{Det}(\mathrm{Edge}(G) \smallsetminus \{e\}) \longrightarrow \mathrm{Det}(\mathrm{Edge}(G))$ gives the map

$$\eta_e^G : \mathcal{K}(G/e) \longrightarrow \mathcal{K}(G)$$

$$(\bigwedge_{i\neq k} e_i)[\epsilon_G - 1] \mapsto (-1)^{k-1} \bigwedge_i e_i[\epsilon_G]$$

Our grading is homological, so $\deg(\eta_e^G) = -1$. Then $\deg(\eta_e^G \otimes (\mu^P_{G\to G/e})^*) = \deg(\eta_e^G) + \deg(\mu^P_{G\to G/e})^* = -1 + 0 = -1$.

Let $G, H \in [\Gamma((n,b))]$ be such that $H/e \simeq G$ for a fixed edge e. Choose an isomorphism $\varphi : H/e \longrightarrow G$ and define

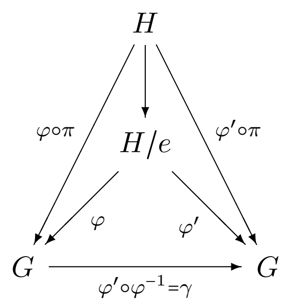
$$\delta|_{(D^\vee(G)\otimes P((G))^*)_{\mathrm{Aut}(G)}} = \bigoplus_{\substack{H\in[\Gamma((n,b))] \\ H/e\simeq G}} \eta_e^H \otimes (\mu^P_{H\to G})^*,$$

where the map $H \to G$ is the composition $H \overset{\pi}{\to} H/e \overset{\varphi}{\to} G$. I claim $\eta_e^H \otimes (\mu^P_{H\to G})^*$, and therefore δ, is well-defined.

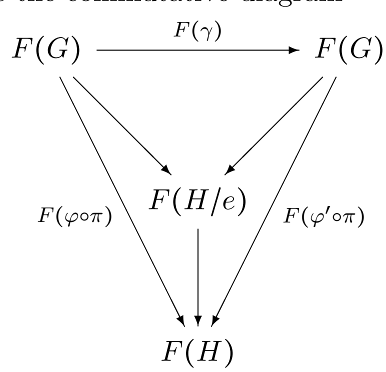
Let φ, φ' be isomorphisms $H/e \longrightarrow G$. These induce maps

$$\delta, \delta' : (D^\vee(G) \otimes P((G))^*)_{\mathrm{Aut}(G)} \longrightarrow (D^\vee(H) \otimes P((H))^*)_{\mathrm{Aut}(G)}.$$

We have a commutative diagram



Applying $F := D^\vee \otimes P^*$ gives the commutative diagram



Let $x \in F(G)$, and let $\overline{x} \in F(G)_{\mathrm{Aut}(G)}$. Recall the action of $\mathrm{Aut}(G)$ on $F(G)$ is given by $\gamma \cdot x = F(\gamma)(x)$. Commutativity, and the fact that $F(\varphi \circ \pi)$ and $F(\varphi' \circ \pi)$ descend to

$$\delta, \delta' : F(G)_{\mathrm{Aut}(G)} \longrightarrow F(H)_{\mathrm{Aut}(H)},$$

respectively, gives $\delta(\overline{x}) = \delta'(\overline{F(\gamma)(x)}) = \delta'(\overline{\gamma \cdot x}) = \delta'(\overline{x})$, so δ is well-defined.

Theorem 3.3.1. $d^2_{\mathscr{F}_D P} = 0$.

Proof. I first claim $\delta^2 = 0$. Let G be a stable graph. Let K be such that $G = K/\{e_1, e_2\}$ and set $H_i = K/\{e_i\}$ for $i = 1, 2$. As above, set $F = D^\vee \otimes P^*$. Then δ^2 is the map

$$\delta^2 : F(G)_{\mathrm{Aut}(G)} \longrightarrow F(H_1)_{\mathrm{Aut}(H_1)} \oplus F(H_2)_{\mathrm{Aut}(H_2)} \longrightarrow F(K)_{\mathrm{Aut}(K)},$$

and

$$\delta^2 = ((\eta^K_{e_1} \otimes \mu^*_{K\to H_1}) \oplus (\eta^K_{e_2} \otimes \mu^*_{K\to H_2})) \circ ((\eta^{H_1}_{e_2} \otimes \mu^*_{H_1\to G}) \oplus (\eta^{H_2}_{e_1} \otimes \mu^*_{H_2\to G}))$$

Recall $D^\vee(G) = \mathcal{K}(G) \otimes D(G)^*$ and let $x \in D(G)^* \otimes P(G)^*$. On elements one has

$$\begin{aligned}
& \delta^2(\bigwedge_{i\neq 1,2} e_i[\epsilon_K - 2] \otimes x) \\
= \; & e_1 \wedge e_2 \wedge (\bigwedge_{i\neq 1,2} e_i[\epsilon_K]) \otimes (\mu^*_{K\to H_1} \circ \mu^*_{H_1\to G})(x) \\
& + e_2 \wedge e_1 \wedge (\bigwedge_{i\neq 1,2} e_i[\epsilon_K]) \otimes (\mu^*_{K\to H_2} \circ \mu^*_{H_2\to G})(x) \\
= \; & e_1 \wedge e_2 \wedge (\bigwedge_{i\neq 1,2} e_i[\epsilon_K]) \otimes (\mu^*_{K\to G})(x) - e_1 \wedge e_2 \wedge (\bigwedge_{i\neq 1,2} e_i[\epsilon_K]) \otimes (\mu^*_{K\to G})(x) \\
= \; & 0
\end{aligned}$$

Since the map $\delta^2|_{F(G)_{\mathrm{Aut}(G)}}$ is a sum over all pairs of edges e_1, e_2, and graphs K such that $K/\{e_1, e_2\} = G$, one sees $\delta^2|_{F(G)_{\mathrm{Aut}(G)}} = 0$.

By definition of P, $\mu \circ \mathrm{d}_P + \mathrm{d}_P \circ \mu = 0$. Noting the definition of d^*_P, taking the adjoint yields $\mathrm{d}_{P^*} \circ \mu^* + \mu^* \circ \mathrm{d}_{P^*} = 0$. Then

$$\begin{aligned}
\delta \circ \mathrm{d}_{P^*} + \mathrm{d}_{P^*} \circ \delta &= (\eta_e \otimes \mu^*) \circ (1 \otimes \mathrm{d}_{P^*}) + (1 \otimes d_{P^*}) \circ (\eta_e \otimes \mu^*) \\
&= \eta_e \otimes (\mu^* \circ \mathrm{d}_{P^*}) + \eta_e \otimes (d_{P^*} \circ \mu^*) \\
&= \eta_e \otimes (\mu^* \circ \mathrm{d}_{P^*}) + \eta_e \otimes (-\mu^* \circ \mathrm{d}_{P^*}) \\
&= 0
\end{aligned}$$

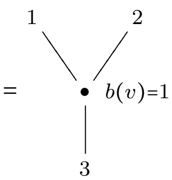
and $\mathrm{d}^2_{\mathscr{F}_D P} = (\delta + \mathrm{d}_{P^*})^2 = \delta^2 + \delta \circ \mathrm{d}_{P^*} + \mathrm{d}_{P^*} \circ \delta + \mathrm{d}^2_{P^*} = 0$. □

Example 3.3.2.

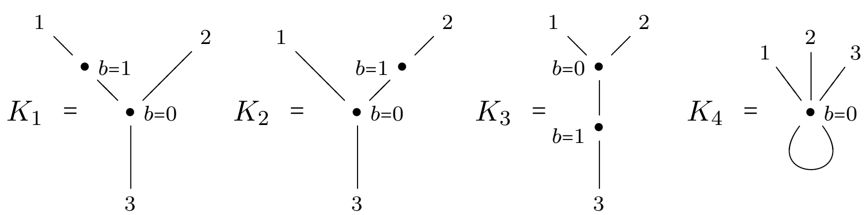
Consider the category $\Gamma((3,1))$ of stable graphs G with $b(G) = 1$ and the elements of $\mathrm{Leg}(G)$ labeled by $\{1, 2, 3\}$. Recall that a stable graph G is one such that $2b(v) + n(v) - 2 > 0$ for all vertices v of G, where $n(v)$ is the valence of v and $b(v)$ is a nonnegative integer assigned to v.

Let $P = \mathbb{S}[t]$, $D = \text{Det}$, and let

G =

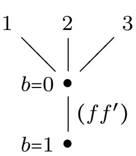


Let $K \in \Gamma((3,1))$ be such that $K/e \simeq G$ for some $e \in \text{Edge}(K)$. Then K has one internal edge and at most two vertices. If K has only one vertex v, then K contains a loop, and $b(v) = 1 - b_1(K) = 1 - 1 = 0$. If K has two vertices v_1 and v_2, then K is a tree, and $b(v_1) + b(v_2) = 1 - b_1(K) = 1 - 0 = 1$. Assume $b(v) = 0$ for some vertex in either case. Then $i_{\sigma_v} = 1 - 2g(v)$, which forces $n(v) \geq 3$, $g(v) = 0$, and $i_{\sigma_v} = 1$. If $b(v) = 1$, then $i_{\sigma_v} = 2 - 2g(v)$ and in this case $i_{\sigma_v} = 2$ with $n(v) \geq 1$. The graph K must therefore be one of



or

K =



Note that $\mathbb{S}[t]((K)) = \mathbb{S}[t]((4,0)) \otimes \mathbb{S}[t]((1,1))$, where $\mathbb{S}[t]((1,1)) = 0$ by stability, so the graph K need not be considered.

The map δ will be constructed explicitly for H_1 and H_4, the cases H_2 and H_3 being similar to that of H_1. The vector spaces corresponding to the three relevant graphs are

$$\text{Det}(\text{H}_1(G)) \otimes \mathbb{S}[t]((G)) = \mathbb{C} \otimes \text{Span}_{\mathbb{C}} \left\{ \begin{array}{l} u_1 = ((1)(23), (1) \wedge (23))t^0 \\ u_2 = ((2)(13), (2) \wedge (13))t^0 \\ u_3 = ((3)(12), (3) \wedge (12))t^0 \end{array} \right\}$$

$$
\begin{aligned}
&\mathrm{Det}(\mathrm{H}_1(K_1)) \otimes \mathbb{S}[t]((K_1)) \\
= \quad &\mathbb{C} \otimes \mathrm{Span}_{\mathbb{C}} \left\{ \begin{array}{l} v_{11} = ((1)(f),(1)\wedge(f))t^0 \otimes ((f'23),(f'23))t^0 \\ v_{12} = ((1)(f),(1)\wedge(f))t^0 \otimes ((f'32),(f'32))t^0 \end{array} \right\}
\end{aligned}
$$

$$\mathrm{Det}(\mathrm{H}_1(K_4)) \otimes \mathbb{S}[t]((K_4)) = \mathbb{C}[1] \otimes \mathrm{Span}_{\mathbb{C}}\{w_i = (\sigma, a_\sigma)t^0 | \sigma \in \mathrm{Cycle}\{1,2,3,f,f'\}\}$$

Write σ for $(\sigma, a_\sigma)t^0$ and define the order of the basis elements of $\mathbb{S}[t]((K_4))$ as follows: $\{(12f3f'),(1f2f'3),(12f'3f),(13f2f'),(13f'2f),(1f23f'),(1f3f'2),(1f32f'),$ $(1f'3f2),(1f'32f),(1f'23f),(1f'2f3),\{\sigma | f,f' \text{ neighbors in } \sigma\}\}$.

The first of the three vector spaces above sits in degree -1, the second in degree $0+(-1)=-1$, and the third in degree $-1+0=-1$.

Let $\mu_i := \mu_{K_i}^{\mathbb{S}[t]}|_{1\otimes\mathbb{S}[t]((K_i))}$. For $i=1$, $\mathrm{Det}(\mathrm{H}_1(K_i))$ is just $\mathbb{C}$, so the composition is

$$\mu_{K_i}^{\mathbb{S}[t]} : \mathbb{S}[t]((K_1)) \longrightarrow \mathbb{S}[t]((G))$$
$$v_{11}, v_{12} \mapsto ((1)(23),(-1)^{2+1}(23)\wedge(1)) = u_1$$

We have $[\mu_i]_{v_1}^{u} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. This implies $[\mu_i^*]_{u^*}^{v_1^*} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and

$$\eta_{K_1}^e \otimes (\mu_{K_1}^{\mathbb{S}[t]})^* : 1 \otimes (1 \otimes u_i^*) \mapsto \begin{cases} e[1] \otimes 1 \otimes (v_{11}^* + v_{12}^*) & \text{if} \quad i=1 \\ 0 & \text{if} \quad i=2 \\ 0 & \text{if} \quad i=3 \end{cases}$$

The element $1 \otimes (1 \otimes u_i^*)$ sits in degree 1, and $e[1] \otimes 1 \otimes (v_{11}^* + v_{12}^*)$ sits in degree 0. For $i=4$, $\mathrm{Det}(K_i) = \mathbb{C}[1]$, and so the composition is

$$\mu_{K_4}^{\mathbb{S}[t]} : \mathbb{C}[1] \otimes \mathbb{S}[t]((K_4)) \longrightarrow \mathbb{C} \otimes \mathbb{S}[t]((G))$$

$$l[1] \otimes w_i \mapsto \begin{cases} 1 \otimes u_1 & \text{if} \quad i = 6,8,10,11 \\ 1 \otimes -u_2 & \text{if} \quad i = 2,4,5,12 \\ 1 \otimes -u_3 & \text{if} \quad i = 1,3,7,9 \\ 0 & \text{if} \quad 13 \le i \le 24 \end{cases}$$

Applying the transpose $[\mu_i^*]_{u^*}^{w^*}$ gives us

$$\eta^e_{K_4} \otimes (\mu^{\mathbb{S}[t]}_{K_4})^* : 1 \otimes (1 \otimes u_j^*) \mapsto \begin{cases} e[1] \otimes l[-1] & \otimes \quad (w_6^* + w_8^* + w_{10}^* + w_{11}^*) \quad \text{if} \quad j = 1 \\ e[1] \otimes l[-1] & \otimes \quad (-w_2^* - w_4^* - w_5^* - w_{12}^*) \quad \text{if} \quad j = 2 \\ e[1] \otimes l[-1] & \otimes \quad (-w_1^* - w_3^* - w_7^* - w_9^*) \quad \text{if} \quad j = 3 \end{cases}$$

The elements $1 \otimes (1 \otimes u_j^*)$, $e[1] \otimes l[-1] \otimes (w_a^* + w_b^* + w_c^* + w_d^*)$ sit in degrees 1 and 0 respectively.

The operad $\mathbb{S}[t]$ takes, as an input, a stable graph and outputs a stable ribbon graph decorated with cycles. Using this point of view, the Feynman differential on $\mathscr{F}_{\text{Det}}\mathbb{S}[t]$ can be written purely in terms of stable ribbon graphs. The degree in which the vector, represented by a stable ribbon graph, sits can be read directly from the graph using the equation $b(v) = 2g(v) + i_\sigma - 1$.

The Basis of $\mathscr{F}_{\mathbf{Det}}\mathbb{S}[t]$

Given n, b satisfying the stability condition, the space $\mathscr{F}_{\text{Det}}\mathbb{S}[t]((n,b))$ is given by

$$\begin{aligned} \mathscr{F}_{\text{Det}}\mathbb{S}[t]((n,b)) &= \bigoplus_{G \in [\Gamma((n,b))]} \text{Det}^\vee(G) \otimes \mathbb{S}[t]((G))^* \\ &= \bigoplus_{G \in [\Gamma((n,b))]} \mathcal{K}\text{Det}^{-1}(G) \otimes \mathbb{S}[t]((G))^* \\ &= \bigoplus_{G \in [\Gamma((n,b))]} \text{Det}(\text{Edge}(G)) \otimes \text{Det}^{-1}(\text{H}_1(G)) \otimes \mathbb{S}[t]((G))^* \end{aligned} \tag{3.13}$$

Given $\sigma \in \mathbb{S}_n$, one can, as in (3.12), write the basis vector $[(\sigma, a_\sigma)t^g]^*$ of $\mathbb{S}[t]((n,b))^*$ as $*_{n,b} \otimes \sigma_1 \wedge \cdots \wedge \sigma_{i_\sigma}[-b]$, the canonical basis element of the space $*_{n,b} \otimes \bigwedge^{\text{top}}(\mathbb{C}^{cycle\,\sigma})^*[-b]$, where $\sigma = \sigma_1 \cdots \sigma_{i_\sigma}$. Since $\mathbb{S}[t]((G))^* = \otimes_{\text{Vert}(G)}\mathbb{S}[t]((n(v), b(v)))^*$ for any $G \in \Gamma((n,b))$, the basis vectors of $\mathbb{S}[t]((G))^*$ are best represented as elements of the spaces

$$\begin{aligned} G(\sigma_{v_1}, \ldots, \sigma_{v_{|\text{Vert}(G)|}}) &:= \bigotimes_{\text{Vert}(G)} (*_{n(v),b(v)} \otimes \bigwedge\nolimits^{\text{top}}(\mathbb{C}^{cycle\,\sigma_v})^*[-b(v)]) \\ &= (\bigotimes_{\text{Vert}(G)} *_{n(v),b(v)}) \otimes \bigotimes_{\text{Vert}(G)} (\bigwedge\nolimits^{\text{top}}(\mathbb{C}^{cycle\,\sigma_v})^*[-b(v)]) \\ &= (\bigotimes_{\text{Vert}(G)} *_{n(v),b(v)}) \otimes (\bigotimes_{\text{Vert}(G)} \bigwedge\nolimits^{\text{top}}(\mathbb{C}^{cycle\,\sigma_v})^*)[-\textstyle\sum b(v)], \end{aligned}$$

indexed by the set $\prod_{v \in \text{Vert}(G)} \mathbb{S}_{n(v)}$.

Letting $\epsilon_G = |\mathrm{Edge}(G)|$, and let $\lambda_G = \dim \mathrm{H}_1(G)$, the basis elements of $\mathscr{F}_{\mathrm{Det}}\mathbb{S}[t]((n,b))$ can then be realized as elements of the spaces

$$\begin{aligned}
&\mathcal{K}\mathrm{Det}^{-1}(G) \otimes G(\sigma_{v_1}, \ldots, \sigma_{v_{|\mathrm{Vert}(G)|}}) \\
= \;& \mathrm{Det}(\mathrm{Edge}(G)) \otimes \mathrm{Det}^{-1}(\mathrm{H}_1(G)) \otimes G(\sigma_{v_1}, \ldots, \sigma_{v_{|\mathrm{Vert}(G)|}}) \\
= \;& \bigwedge^{\mathrm{top}}(\mathbb{C}^{\epsilon_G}) \otimes \bigwedge^{\mathrm{top}}(\mathrm{H}_1(G,\mathbb{C}))^*[\epsilon_G - \lambda_G] \otimes G(\sigma_{v_1}, \ldots, \sigma_{v_{|\mathrm{Vert}(G)|}}) \\
= \;& (\bigotimes_{\mathrm{Vert}(G)} *_{n(v),b(v)}) \otimes \bigwedge^{\mathrm{top}}(\mathbb{C}^{\epsilon_G}) \otimes \bigwedge^{\mathrm{top}}(\mathrm{H}_1(G,\mathbb{C}))^* \\
& \otimes (\bigotimes_{\mathrm{Vert}(G)} \bigwedge^{\mathrm{top}}(\mathbb{C}^{cycle\,\sigma_v})^*)[\epsilon_G - \lambda_G - \sum b(v)]
\end{aligned} \tag{3.14}$$

and take the form

$$(\bigotimes_{\mathrm{Vert}(G)} *_{n(v),b(v)}) \otimes \alpha_G,$$

where

$$\alpha_G := e_1 \wedge \cdots \wedge e_{\epsilon_G} \otimes \ell_1 \wedge \cdots \wedge \ell_{\lambda_G} \otimes \bigotimes_{\mathrm{Vert}(G)} \sigma_{v_1} \wedge \cdots \wedge \sigma_{v i_{\sigma_v}}[\epsilon_G - b] \tag{3.15}$$

We have proved the following:

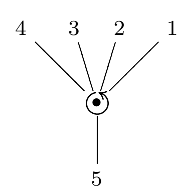
Proposition 3.3.3. The basis vectors of $\mathscr{F}_{\mathrm{Det}}\mathbb{S}[t]((n,b))$ can be labeled by pairs (G, α_G) for G a stable ribbon graph, and α_G as defined by (3.15).

As the basis of $\mathscr{F}_{\mathrm{Det}}\mathbb{S}[t]$ can be represented by stable ribbon graphs, the action of $d_{\mathscr{F}}$ can also be represented by such graphs. Let $\mathscr{G}$ be a stable ribbon graph with one vertex and $d+1$ legs, and for the sake of intuition, mark the leg labeled by $d+1$ as the *output* and the rest as *inputs*. Modulo degrees, $d_{\mathscr{F}}$ can then be represented by

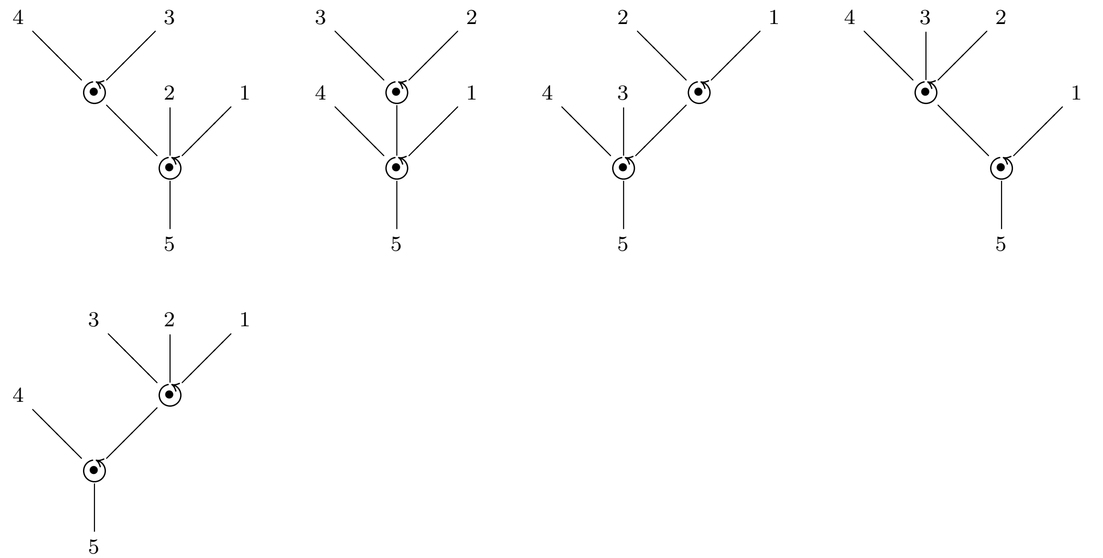
$$d_{\mathscr{F}}(\mathscr{G}) = \sum_{i=2}^{d-1} \sum_{j=1}^{d-i+1} \mathscr{G}_{ij}, \tag{3.16}$$

where $\mathscr{G}_{ij}$ is a stable ribbon graph with a single edge corresponding to inserting the output of a single-vertex stable ribbon tree with i inputs into the j^{th} input of a second single-vertex stable ribbon tree with $d-i+1$ inputs.

Example 3.3.4. Consider the stable ribbon graph $\mathscr{G}$



Applying $d_{\mathscr{F}}$ to $\mathscr{G}$ gives a sum of five vectors indexed by the following stable ribbon graphs $\mathscr{G}_{ij}$.



Visualizing the Feynman differential in this way, the connection between the Feynman transform and the A_∞ relations is realized as the correspondence between the graphs $\mathscr{G}_{ij}$ and the compositions $m_{d-i+1} \circ_j m_i$ for $2 \le i \le d-1$.

Chapter 4

Algebras over The Feynman Transform

The first part of this chapter is devoted to giving a detailed discussion of what it means for a chain complex to be an algebra over the Feynman transform. This material can be found in [4]. In the second half of the chapter, we construct a morphism from the Feynman transform of a twisted version of $\tilde{\mathbb{S}}[t]$, to $\mathcal{E}_V$, where V is specifically chosen to mimic the chain complexes encountered in the categorical version of such a construction. In the last part we define what is meant by a quantum A_∞-algebra and give an explicit example of these relations in the genus 1 case. The notion of a quantum A_∞-algebra can be found in [4], with a slightly different formulation. Specifically, a set of operations on a chain complex satisfies the quantum A_∞-relations if and only if they are a solution to the quantum master equation, as described in [4].

4.1 The Basic Structure

A $\mathscr{F}_D P$-algebra structure on the chain complex V is a morphism of $D^\vee$-modular operads

$$\hat{m} : \mathscr{F}_D P \longrightarrow \mathcal{E}_V,$$

given, for each stable graph G, by a morphism of chain complexes

$$\mathscr{F}_D P((G)) \longrightarrow \mathcal{E}_V((G)) = V^{\otimes \mathrm{Flag}(G)}$$

In other words, there are commutative diagrams

$$\begin{array}{ccc} \mathscr{F}_D P((G)) & \xrightarrow{\hat{m}} & \mathcal{E}_V((G)) \\ \Big\downarrow d_{\mathscr{F}} & & \Big\downarrow d_{\mathcal{E}_V} \\ \mathscr{F}_D P((G)) & \xrightarrow[\hat{m}]{} & \mathcal{E}_V((G)) \end{array} \tag{4.1}$$

and

$$\begin{array}{ccc} D^\vee(H) \otimes \mathscr{F}_D P((H)) & \xrightarrow{\hat{m}} & D^\vee(H) \otimes \mathcal{E}_V((H)) \\ \Big\downarrow \mu^{\mathscr{F}}_{H\to G} & & \Big\downarrow \mu^{\mathcal{E}_V}_{H\to G} \\ D^\vee(G) \otimes \mathscr{F}_D P((G)) & \xrightarrow[\hat{m}]{} & D^\vee(G) \otimes \mathcal{E}_V((G)) \end{array} \tag{4.2}$$

for every $G, H \in \Gamma((n,b))$ such that $G = H/J$ for some $J \subseteq \mathrm{Edge}(H)$.

When $G = \star_{n,b}$, the former takes the form

$$\begin{array}{ccc} \mathscr{F}_D P((n,b)) & \xrightarrow{\hat{m}} & V^{\otimes n} \\ \Big\downarrow d_{\mathscr{F}} & & \Big\downarrow d_{\mathcal{E}_V} \\ \mathscr{F}_D P((n,b)) & \xrightarrow[\hat{m}]{} & V^{\otimes n} \end{array}$$

By definition,

$$\mathscr{F}_D P((n,b)) = \bigoplus_{G\in[\Gamma((n,b))]} (D^\vee(G) \otimes P((G))^*)_{\mathrm{Aut}(G)},$$

so maps $(D^\vee(G) \otimes P((G))^*)_{\mathrm{Aut}(G)} \longrightarrow V^{\otimes n}$ must be defined for all $G \in [\Gamma((n,b))]$.

The difficulty in defining these maps is that if $|\mathrm{Edge}(G)| > 0$, then $|\mathrm{Flag}(G)| > |\mathrm{Leg}(G)| = n$ and the map

$$P((G))^* = \bigotimes_{\mathrm{Vert}(G)} P((\mathrm{Leg}(v), b(v)))^* \longrightarrow \mathcal{E}_V((n,b))$$

must involve the contraction μ^P_G, because $\sum_{v\in\mathrm{Vert}(G)} |\mathrm{Leg}(v)| = |\mathrm{Flag}(G)| > n$.

The maps $(D^\vee(G) \otimes P((G))^*)_{\mathrm{Aut}(G)} \longrightarrow V^{\otimes n}$ are defined by replacing

$$\hat{m}|_{(D^\vee(G)\otimes P((G))^*)_{\mathrm{Aut}(G)}}$$

in the first diagram with $(\mu^{\mathcal{E}_V}_{G\to\star_{n,b}} \circ \hat{m})|_{(D^\vee(G)\otimes P((G))^*)_{\mathrm{Aut}(G)}}$ from the second. Recalling the definition of the composition map for a free modular twisted D-operad, any element of $(D^\vee(G)\otimes P((G))^*)_{\mathrm{Aut}(G)}$ is the result of applying $\mu^{\mathscr{F}_D P}_G$ to an element of the summand

$$D^\vee(G) \otimes P((G))^* \;\; = \;\; D^\vee(G) \otimes \bigotimes_{\mathrm{Vert}(G)} P((\mathrm{Leg}(v_i), b(v_i)))^* \subseteq D^\vee(G) \otimes \mathscr{F}_D P((G))$$

By the commutativity of the diagram

$$\begin{array}{ccc}
D^{\vee}(G) \otimes \bigotimes_{\mathrm{Vert}(G)} P((\mathrm{Leg}(v_i), b(v_i)))^* & \xrightarrow{\hat{m}} & D^{\vee}(G) \otimes V^{\otimes \mathrm{Flag}(G)} \\
\Big\downarrow {\scriptstyle \mu_G^{\mathscr{F}_D P}} & & \Big\downarrow {\scriptstyle \mu_G^{\mathcal{E}}} \\
\mathscr{F}_D P((n,b)) \supseteq (D^{\vee}(G) \otimes P((G))^*)_{\mathrm{Aut}(G)} & \xrightarrow{\hat{m}} & \mathcal{E}_V((n,b)) = V^{\otimes n}
\end{array}$$

the map $\hat{m}$, when restricted to the subspace $(D^{\vee}(G) \otimes P((G))^*)_{\mathrm{Aut}(G)}$, is given by $\mu_G^{\mathcal{E}_V} \circ (\mathrm{id} \otimes \bigotimes_{\mathrm{Vert}(G)} \hat{m}_{\mathrm{Leg}(v),b(v)})$, where

$$\hat{m}_{\mathrm{Leg(v),b(v)}} \in \mathrm{Hom}(P((\mathrm{Leg}(v), b(v)))^*, V^{\otimes \mathrm{Leg(v)}}) \simeq P((\mathrm{Leg}(v), b(v))) \otimes V^{\otimes \mathrm{Leg}(v)}$$

Any map between these two twisted modular $D^{\vee}$-operads therefore corresponds to a set $\{\hat{m}_{\mathrm{Leg(v),b(v)}} | v \in G\}$, where G runs through all elements of $[\Gamma((n,b))]$, and must satisfy

$$d_{\mathcal{E}} \circ \hat{m} = \hat{m} \circ d_{\mathscr{F}} \tag{4.3}$$

It is important to note the maps $\hat{m}_{\mathrm{Leg}(v),b(v)}$ do not depend on G, but are indexed by the vertices of G.

The Feynman differential is given by

$$d_{\mathscr{F}}|_{(D^{\vee}(G) \otimes P((G))^*)_{\mathrm{Aut}(G)}} = d_{P^*} + \sum_{\substack{H \in [\Gamma((n,b))] \\ H/e \simeq G}} \eta_e^H \otimes (\mu^P_{H \to G})^*$$

where η_e^H is multiplication by $e[1]$, the canonical element of degree (-1) from $\mathrm{Det}(\{e\})$. By taking $G = \star_{n,b}$, $(D^{\vee}(G) \otimes P((G))^*)_{\mathrm{Aut}(G)}$ becomes $P((\star_{n,b}))$ and the equation $(d_{\mathcal{E}} \circ \hat{m} = \hat{m} \circ d_{\mathscr{F}})|_{(D^{\vee}(G) \otimes P((G))^*)_{\mathrm{Aut}(G)}}$ can be expressed in full detail. If $H/e \simeq G = \star_{n,b}$, then H is one of

i) $G_{n,b} := \star^1_{n,b-1}$, the unique graph with one vertex, one self-intersecting edge $e = (ff')$, and n external legs.

ii) $G_{(I_1,I_2,b_1,b_2)} := \star_{I_1 \sqcup \{f\}, b_1} \otimes_{\{f,f'\}} \star_{I_2 \sqcup \{f'\}, b_2}$ with $e = (ff')$, $b_1 + b_2 = b$, and $I_1 \sqcup I_2 = \mathrm{Leg}(G)$. If $b = 0$ there are $\frac{1}{2}\sum_{2 \le i \le n-2} \binom{n}{i}$ of these, and if $b > 0$, there are $\frac{1}{2}\sum_{1 \le i \le n-1} \binom{n}{i}$.

The differential can then be written more specifically as

$$d_{\mathscr{F}} = d_{P^*} + e[1] \otimes (\mu^P_{G_{n,b}})^* + \frac{1}{2}\sum_S e[1] \otimes (\mu^P_{G_{(I_1,I_2,b_1,b_2)}})^*,$$

where $S = \{(I_1, I_2) | I_1 \sqcup I_2 = \{1, 2, \dots, n\}, |I_i| + 2b(v_i) \geq 2\}$.

Recall that $\mathcal{E}_V$ is a modular $D^\vee = \mathcal{K}^{-1}\mathcal{L}$ - operad, so the contraction map $\mu^{\mathcal{E}}_{G_{(I_1,I_2,b_1,b_2)}}$ is given by

$$\mathcal{K}^{-1}\mathcal{L}(G_{(I_1,I_2,b_1,b_2)}) \otimes V^{\otimes I_1 \sqcup \{f\}} \otimes V^{\otimes I_2 \sqcup \{f'\}} \longrightarrow V^{\otimes n}.$$

Note 4.1.1. Even powers of $\mathcal{K}$ contribute nothing but a degree shift, because any permutation of edges $e_i \leftrightarrow e_j$ produces the sign -1 an even number of times.

Because $\mathcal{E}_V$ is a modular $D^\vee = \mathcal{K}D^{-1} = \mathcal{K}^{-1}\mathcal{L}$-operad, P must be a modular $D = \mathcal{K}(D^\vee)^{-1} = \mathcal{K}^2\mathcal{L}^{-1}$-operad, so the contraction map $\mu^{P}_{G_{(I_1,I_2,b_1,b_2)}}$ is given by

$$\mathcal{K}^2\mathcal{L}^{-1}(G_{(I_1,I_2,b_1,b_2)}) \otimes P((I_1 \sqcup \{f\}, b_1)) \otimes P((I_2 \sqcup \{f'\}, b_2)) \longrightarrow P((n, b_1 + b_2)),$$

where $\mathcal{K}^2\mathcal{L}^{-1}(G) = \bigotimes_{\mathrm{Edge}(G)} \bigwedge^2 \mathbb{C}^{\{f,f'\}}$. Indeed, we have the relation

$$\begin{aligned}
\mathcal{K}^{3-l}\mathcal{L}^{-1}(G) &= D(G) \\
&= \mathcal{K}(D^\vee)^{-1}(G) \\
&= \mathcal{K}(G) \otimes D^\vee(G)^* \\
&= \mathcal{K}(G) \otimes (\mathcal{K}^l(G) \otimes \bigotimes_{\mathrm{Edge}(G)} \bigwedge^2 \mathbb{C}^{\{f,f'\}})^* \\
&= \mathcal{K}(G) \otimes \mathcal{K}^{-l}(G) \otimes \bigotimes_{\mathrm{Edge}(G)} \bigwedge^2 \mathbb{C}^{\{f,f'\}} \\
&= \mathcal{K}^{1-l}(G) \otimes \bigotimes_{\mathrm{Edge}(G)} \bigwedge^2 \mathbb{C}^{\{f,f'\}},
\end{aligned} \tag{4.4}$$

and setting $l = 1$ yields the result. Diagram (4.1) reduces to

$$\begin{array}{ccc}
P((n,b))^* & \xrightarrow{\hat{m}} & \mathcal{E}_V((n,b)) = V^{\otimes n} \\
\downarrow d_{\mathscr{F}} & & \downarrow d_{\mathcal{E}} \\
\bigoplus_{H/e \simeq *_{n,b}} (\mathcal{K}^l(H) \otimes \bigwedge^2 \mathbb{C}^{\{f,f'\}} \otimes P((H))^*)_{\mathrm{Aut}(H)} & \xrightarrow{\hat{m}} & V^{\otimes n}
\end{array} \tag{4.5}$$

For each such graph H, diagram (4.2) reduces to

$$\begin{array}{ccc} \mathcal{K}^l(H)\otimes\bigwedge^2\mathbb{C}^{\{f,f'\}}\otimes P((H))^* & \xrightarrow{\hat{m}} & \mathcal{K}^l(G)\otimes\bigwedge^2\mathbb{C}^{\{f,f'\}}\otimes V^{\otimes\{f,f'\}}\otimes V^{\otimes n} \\ \downarrow{\scriptstyle proj} & & \downarrow{\scriptstyle \mu^{\mathcal{E}}_H} \\ (\mathcal{K}^l(H)\otimes\bigwedge^2\mathbb{C}^{\{f,f'\}}\otimes P((H))^*)_{\mathrm{Aut}(H)} & \xrightarrow{\hat{m}} & V^{\otimes n} \end{array} \tag{4.6}$$

If $H = G_{n,b}$, then write $\hat{m}|_{P((H))^*}$ as $\hat{m}_{\{1,\ldots,n\}\sqcup\{f,f'\},b-1}$, and if $H = G_{(I_1,I_2,b_1,b_2)}$, then write $\hat{m}|_{P((H))^*}$ as $\hat{m}_{I_1\sqcup\{f\},b_1}\otimes\hat{m}_{I_2\sqcup\{f'\},b_2}$.

Certainly $\mathrm{im}\, d_{P^*}\subseteq P((\mathrm{Leg}(v),b(v)))^*$, so

$$\hat{m}|_{\mathrm{im}d_{P^*}} = \mu^{\mathcal{E}}_{*_{n,b}}\circ\hat{m}_{\mathrm{Leg}(v),b(v)} = \mathrm{id}\circ\hat{m}_{\mathrm{Leg}(v),b(v)} = \hat{m}_{\mathrm{Leg}(v),b(v)}$$

We also have

$$\hat{m}|_{\mathrm{im}\left(e[1]\otimes(\mu^P_{G_{n,b}})^*\right)} = \mu^{\mathcal{E}}_{G_{n,b}}\circ\hat{m}_{\{1,\ldots,n\}\sqcup\{f,f'\},b-1}$$

and

$$\hat{m}|_{\mathrm{im}\left(e[1]\otimes(\mu^P_{G_{(I_1,I_2,b_1,b_2)}})^*\right)} = \mu^{\mathcal{E}}_{G_{(I_1,I_2,b_1,b_2)}}\circ\hat{m}_{I_1\sqcup\{f\},b_1}\otimes\hat{m}_{I_2\sqcup\{f'\},b_2}$$

The map $\hat{m}$ is degree 0, so commutes with tensoring by $e[1]$ and

$$\begin{aligned} \hat{m}\circ d_{\mathscr{F}} &= \hat{m}\circ d_{P^*} + \hat{m}\circ(e[1]\otimes(\mu^P_{G_{n,b}})^*) + \frac{1}{2}\hat{m}\circ\sum_S e[1]\otimes(\mu^P_{G_{(I_1,I_2,b_1,b_2)}})^* \\ &= \hat{m}_{\mathrm{Leg}(v),b(v)}\circ d_{P^*} + (\mu^{\mathcal{E}}_{G_{n,b}}\circ\hat{m}_{\{1,\ldots,n\}\sqcup\{f,f'\},b-1})\circ(e[1]\otimes(\mu^P_{G_{n,b}})^*) \\ &\quad + \frac{1}{2}\sum_S(\mu^{\mathcal{E}}_{G_{(I_1,I_2,b_1,b_2)}}\circ(\hat{m}_{I_1\sqcup\{f\},b_1}\otimes\hat{m}_{I_2\sqcup\{f'\},b_2}))\circ(e[1]\otimes(\mu^P_{G_{(I_1,I_2,b_1,b_2)}})^*) \\ &= \hat{m}_{\mathrm{Leg}(v),b(v)}\circ d_{P^*} + \mu^{\mathcal{E}}_{G_{n,b}}\circ(e[1]\otimes(\hat{m}_{\{1,\ldots,n\}\sqcup\{f,f'\},b-1}\circ(\mu^P_{G_{n,b}})^*)) \\ &\quad + \frac{1}{2}\sum_S\mu^{\mathcal{E}}_{G_{(I_1,I_2,b_1,b_2)}}\circ(e[1]\otimes(\hat{m}_{I_1\sqcup\{f\},b_1}\otimes\hat{m}_{I_2\sqcup\{f'\},b_2})\circ(\mu^P_{G_{(I_1,I_2,b_1,b_2)}})^*) \end{aligned}$$

Equation 4.3 then becomes

$$\begin{aligned} & d_{V^{\otimes n}}\circ\hat{m}_{\mathrm{Leg}(v),b(v)} \\ = \; & \hat{m}_{\mathrm{Leg}(v),b(v)}\circ d_{P^*} + \mu^{\mathcal{E}}_{G_{n,b}}\circ(e[1]\otimes(\hat{m}_{\{1,\ldots,n\}\sqcup\{f,f'\},b-1}\circ(\mu^P_{G_{n,b}})^*)) \\ + \; & \frac{1}{2}\sum_S\mu^{\mathcal{E}}_{G_{(I_1,I_2,b_1,b_2)}}\circ(e[1]\otimes(\hat{m}_{I_1\sqcup\{f\},b_1}\otimes\hat{m}_{I_2\sqcup\{f'\},b_2})\circ(\mu^P_{G_{(I_1,I_2,b_1,b_2)}})^*) \end{aligned} \tag{4.7}$$

Equation (4.7) gives V the structure of an A_∞-algebra for $b = 0$, and the structure of what will be referred to as a *quantum A_∞-algebra* for general b.

4.2 A Genus Zero Example

Definition 4.2.1. An (non-unital) A_∞-algebra A is a cohomologically $\mathbb{Z}$-graded k-vector space

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

with graded k-linear maps, for $d \geq 1$,

$$m_d : A^{\otimes d} \longrightarrow A$$

of degree $2 - d$ satisfying for each $d \geq 1$ the relation

$$\sum_{\substack{1\leq p\leq d \\ 0\leq q\leq d-p}} (-1)^{\deg a_1+\cdots+\deg a_q - q} m_{d-p+1}(a_d, \ldots, a_{p+q+1}, m_p(a_{p+q}, \ldots, a_{q+1}), a_q, \ldots, a_1) = 0. \tag{4.8}$$

The sign $(-1)^{\deg a_1+\cdots+\deg a_q - q}$ is given by Seidel in [36].

The map m_1 is degree 1 and(4.8) takes the form $m_1(m_1(a_1)) = 0$, making A into a complex with differential m_1.

The map m_2 is degree 0, that is, $\deg m_2(a_2, a_1) = \deg a_2 + \deg a_1$ and should be viewed as a multiplication map. Equation (4.8) is

$$m_2(a_2, m_1(a_1)) + (-1)^{\deg a_1 - 1} m_2(m_1(a_2), a_1) + m_1(m_2(a_2, a_1)) = 0$$

Up to sign, this says m_1 is a graded derivation with respect to the multiplication operation. More precisely, setting

$$\partial(a) = (-1)^{\deg a} m_1(a), a_2 \cdot a_1 = (-1)^{\deg a_1} m_2(a_2, a_1), \tag{4.9}$$

(4.8) can be rewritten as

$$\partial(a_2 \cdot a_1) = (\partial a_2) \cdot a_1 + (-1)^{\deg a_2} a_2 \cdot (\partial a_1).$$

If multiplication were associative, this would yield what is known as a *differential graded algebra*.

Setting $d = 3$, (4.8) becomes

$$\begin{aligned} m_3(a_3, a_2, m_1(a_1)) \;&+\; (-1)^{\deg a_1 - 1} m_3(a_3, m_1(a_2), a_1) \\ &+\; (-1)^{\deg a_1 + \deg a_2 - 2} m_3(m_1(a_3), a_2, a_1) \\ &+\; m_2(m_2(a_3, a_2), a_1) + (-1)^{\deg a_1 - 1} m_2(m_2(a_3, a_2), a_1)) \\ &+\; m_1(m_3(a_3, a_2, a_1)) = 0 \end{aligned}$$

Using (4.9) $m_2(m_2(a_3,a_2),a_1) + (-1)^{\deg a_1 - 1} m_2(m_2(a_3,a_2),a_1))$ can be rewritten as

$$(-1)^{\deg a_2} a_3 \cdot (a_2 \cdot a_1) - (-1)^{\deg a_2} (a_3 \cdot a_2) \cdot a_1,$$

so this expression tells us by how much multiplication fails to be associative.

If $d = 4$ (4.8) is

$$\begin{aligned} m_4(a_4,a_3,a_2,m_1(a_1)) &+ (-1)^{\deg a_1 - 1} m_4(a_4,a_3,m_1(a_2),a_1) \\ &+ (-1)^{\deg a_1 + \deg a_2 - 2} m_4(a_4,m_1(a_3),a_2,a_1) \\ &+ (-1)^{\deg a_1 + \deg a_2 + \deg a_3 - 3} m_4(m_1(a_4),a_3,a_2,a_1) \\ &+ m_3(a_4,a_3,m_2(a_2,a_1)) + (-1)^{\deg a_1 - 1} m_3(a_4,m_2(a_3,a_2),a_1) \\ &+ (-1)^{\deg a_1 + \deg a_2 - 2} m_3(m_2(a_4,a_3),a_2,a_1) \\ &+ m_2(a_4,m_3(a_3,a_2,a_1)) + (-1)^{\deg a_1 - 1} m_2(m_3(a_4,a_3,a_2),a_1) \\ &+ m_1(m_4(a_4,a_3,a_2,a_1)) = 0 \end{aligned}$$

In Chapter 5 we introduce chain complexes, built from certain Lagrangians in the elliptic curve, that form the building blocks of the categorical generalization, $\mathcal{E}_L$, of the twisted modular operad $\mathcal{E}_V$, as defined in Section 3.2.5. These complexes are related via a degree 1 bilinear form B to their dual complexes by a 1-shift. In order to gain intuition into the structures described in the final three chapters, we now construct a morphism from the Feynman transform of a twist of $\tilde{\mathbb{S}}[t]$ to $\mathcal{E}_V$, where V is defined to mimic the relationship the complexes employed in later chapters have to their dual complexes.

Let V be a homologically graded complex given by

$$\cdots \longrightarrow 0 \longrightarrow V_1 \xrightarrow{d_V} V_0 \longrightarrow 0 \longrightarrow \cdots \tag{4.10}$$

and let $\{v_i\}_{i=1}^n$, $\{w_i\}_{i=1}^n$ be bases of V_1 and V_0 respectively. There is a degree 1 antisymmetric pairing B defined by

$$B : V \otimes V \longrightarrow \mathbb{C}$$

$$B(w_i, v_j) = \delta_{ij}, B(v_i, w_j) = -\delta_{ij}$$

Degree 1 means $B(v,w) \neq 0$ only if $|v| + |w| = 1$ for homogeneous elements v and w. The pairing B can also be written as a morphism $V \otimes V \longrightarrow \mathbb{C}[-1]$ of degree 0.

This pairing gives an isomorphism $\varphi : V \longrightarrow V^*$ of degree one defined by $\varphi(v)(w) = B(v,w)$, seen together as

$$\begin{array}{ccccccccc} 2 & & 1 & & 0 & & -1 & & -2 \\ & \longrightarrow & V_1 & \xrightarrow{d_V} & V_0 & \longrightarrow & & & \\ & & & \searrow^{\varphi} & & \searrow^{\varphi} & & & \\ & & & \longrightarrow & V_0^* & \xrightarrow[d_{V^*}]{} & V_1^* & \longrightarrow & \end{array} \tag{4.11}$$

The vectors spaces V and V^* can therefore be identified via φ, which gives rise to an isomorphism

$$\xi_n : V^{\otimes(n+1)} \longrightarrow (V^*)^{\otimes n} \otimes V \xrightarrow{\delta} \mathrm{Hom}(V^{\otimes n}, V) \tag{4.12}$$

$$(\bigotimes_{i=1}^{n} v_i) \otimes v_{n+1} \mapsto \bigotimes_{i=1}^{n}(\varphi(v_i)) \otimes v_{n+1} \mapsto (\bigotimes_{i=1}^{n} u_i \mapsto (-1)^{\sum_{i=1}^{n}(n-i)|v_i|} \prod B(v_i, u_i) v_{n+1})$$

Note that up to sign

$$B(v,w) =< \varphi(v), w >, \tag{4.13}$$

where $< -,- >: V \otimes V^* \longrightarrow \mathbb{C}$ is the standard pairing $v \otimes f \mapsto < v, f >= f(v)$.

Let W be the cohomological complex such that $W^i = V_{1-i}$ for V defined in (4.10). The complexes W and W^* and the map φ between them then take the form

$$\begin{array}{ccccccccc} -2 & & -1 & & 0 & & 1 & & 2 \\ & & & \longrightarrow & W^1 & \xrightarrow{d_W} & W^0 & \longrightarrow & \\ & & & \swarrow^{\varphi} & & \swarrow^{\varphi} & & & \\ & \longrightarrow & (W^0)^* & \xrightarrow[d_{W^*}]{} & (W^1)^* & \longrightarrow & & & \end{array} \tag{4.14}$$

The isomorphism $W^{\otimes(n+1)} \longrightarrow \mathrm{Hom}(W^{\otimes n}, W)$ is defined in precisely the same way as ξ_n, so this isomorphism will also be called ξ_n.

Consider a map $m_{n,b} : W^{\otimes n} \longrightarrow W$ of degree $2-2b-n$. If A and B are any two finite-dimensional vector spaces, the degree of the isomorphism

$$\mathrm{Hom}(A,B) \longrightarrow A^* \otimes B$$

$$f \mapsto \sum a^* \otimes f(a)$$

is zero, because $\deg(a^* \otimes f(a)) = \deg a^* + \deg f(a) = -\deg a + (\deg a + \deg f) = \deg f$. Noting the shift in (4.11), one therefore has

$$\deg \xi_n^{-1}(m_{n,b}) = n + \deg \delta^{-1}(m_{n,b}) = n + \deg m_{n,b} = n + (2-2b-n) = 2-2b, \tag{4.15}$$

where the first equality is a result of the isomorphisms

$$W^{\otimes n} \otimes W \simeq (W^*[-1])^{\otimes n} \otimes W \simeq (W^*)^{\otimes n} \otimes W[-n]$$

Because the Feynman transform is a homological complex, the stable $\mathbb{S}$-module underlying the twisted modular operad $\mathcal{E}_V$ should be given by a homological complex as well. The elements $m_{n,b}$ should therefore be considered as elements of $V^{\otimes(n+1)}$. As an element of $V^{\otimes(n+1)}$, $\overline{m}_{n,b} := \xi^{-1}(m_{n,b})$ sits in degree $(n+1) - (2-2b)$.

Consider the stable $\mathbb{S}$-modules $\beta((n,b)) = \mathbb{C}[b-1]$ and $\tilde{\mathfrak{s}}((n,b)) = sgn_n[n]$. As $\deg \tilde{\mathbb{S}}[t] = 0$, the twisted modular $D_{\beta^2} D_{\tilde{\mathfrak{s}}} = \mathcal{K}^2\mathcal{L}^{-1}$-operad $\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((n+1,b))$ sits in degree $(2-2b)-(n+1)$. The vector space $(\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((n+1,b)))^*$ therefore sits in degree $(n+1)-(2-2b)$.

Theorem 4.2.2. Let $P = \beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]$ and $\mathcal{E}_V((n,b)) = V^{\otimes n}$, with V and B as above. The element $(-1)^{\sum_{i=1}^{n}(n-i)|v_i|} v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} = \overline{m}_{n,0}(\sigma) \in V^{\otimes(n+1)}$ corresponds via ξ_n to the degree $2 - 2\cdot 0 - n$ map

$$m_{n,0}(\sigma) : V^{\otimes n} \longrightarrow V$$

$$u_1 \otimes \cdots \otimes u_n \mapsto \prod B(v_i, u_i) v_{n+1},$$

where $\sigma = (12\cdots n\,(n+1))$ and $\deg \overline{m}_{n,0}(\sigma) = n-1$ by (4.15). The Feynman transform-algebra structure on $\mathcal{E}_V$ is given, for $b=0$, by the map

$$\hat{m}_{n+1,0} : (\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((n+1,0)))^* \longrightarrow \mathcal{E}_V((n+1,0))$$

$$((12\cdots n\,(n+1))[n-1])^* \mapsto \overline{m}_{n,0}(\sigma), \tag{4.16}$$

and gives V the structure of an A_∞-algebra.

Remark 4.2.3. By using (4.13), the map $m_{n,0}$ is seen to be the same as the one obtained from $\varphi(v_1)\otimes\cdots\otimes\varphi(v_n)\otimes v_{n+1}$ via the usual isomorphism $(V^*)^{\otimes n}\otimes V \simeq \mathrm{Hom}(V^{\otimes n}, V)$.

Proof. The vector space $\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((n,0))^*$ is concentrated in a single degree, so $d^*_{\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]} = 0$ and since the algebra structure $\mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t] \longrightarrow \mathcal{E}_V$ is being considered only for $b=0$, the map $e[1]\otimes(\mu^{\tilde{\mathbb{S}}[t]}_{G_{n,b}})^*$ does not contribute to the Feynman differential, which takes the following simplified form:

$$d_{\mathscr{F}} = \sum_{\substack{H\in\Gamma((n,0)) \\ H/e\simeq *_{n,0}}} e[1]\otimes(\mu_H^{\tilde{\mathbb{S}}[t]})^*$$

The sum $d_{\mathscr{F}}((12\cdots n)^*)$ may be visualized using (3.16) by noting that the graphs $\{\mathscr{G}_{ij}\}$ from above are obtained by decorating the vertices of the graphs $\{H \in \Gamma((n,0)) | H/e \simeq \star_{n,0}\}$ with cycles, which compose via $\mu_H^{\tilde{\mathbb{S}}[t]}$ to $(12\cdots n)$.

The equation (4.7) takes the form

$$\begin{aligned} & d_{V^{\otimes n}} \circ \hat{m}_{n,0} \\ = & \sum_{\substack{H \in \Gamma((n,0)) \\ H/e \simeq \star_{n,0}}} \mu_H^{\mathcal{E}_V} \circ (e[1] \otimes (\bigotimes_{v \in \mathrm{Vert}(H)} \hat{m}_{\mathrm{Leg}(v),0}) \circ (\mu_H^{\tilde{\mathbb{S}}[t]})^*) \end{aligned} \tag{4.17}$$

The coefficient $\frac{1}{2}$ has been dropped here since the sum is more carefully indexed by the specific graphs obtained by inserting an edge into $\star_{n,b}$.

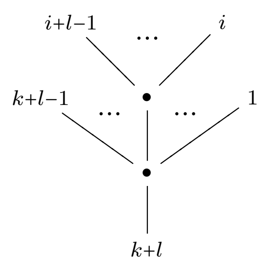
By Remark 3.2.17, the relevant $\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]$-composition maps $\mu_H^{\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]}$ ($|\mathrm{Edge}(H)| = 1$) are

$$\mu_H^{\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]} : \mathcal{K}^2\mathcal{L}^{-1} \otimes \beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((H)) \longrightarrow \beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((n,0))$$

$$f \wedge f' \otimes (\sigma \otimes \tau)[\sum_{i=1}^{2}(n(v_i) - 2)] \mapsto \pi_{ff'}(ff')\sigma\tau[n-2]$$

These maps are indeed degree zero because $\sum_{i=1}^{2}(n(v_i) - 2) = n - 2$ for any stable graph $H \in \Gamma((n,b))$.

If H_i has the form



then

$$\begin{aligned} & (e[1] \otimes (\mu_{H_i}^{\tilde{\mathbb{S}}[t]})^*)(12\cdots n)^* \\ = & (f \wedge f')[1] \otimes (i \cdots (i+l-1)\ f)^* \otimes (1\,2 \cdots (i-1)\ f'\ (i+l) \cdots n)^* \end{aligned} \tag{4.18}$$

Applying $\mu_H^{\mathcal{E}_V} \circ \hat{m}$ to this tensor will give, along with the appropriate sign, the tensor that corresponds to the composition $m_{k,0} \circ_i m_{l,0}$ under the isomorphism (4.12), with $k = n - l$.

Write $\overline{m}_{l,0}$ for $\overline{m}_{l,0}((i\cdots(i+l-1)\,f))$ and similarly for $\overline{m}_{k,0}$. The element $\overline{m}_{l,0}$ is given by $(-1)^{\sum_{j=i}^{i+l-1}(l-(j-i+1))|v_j|} v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_f$, and $\overline{m}_{k,0}$ is given by
$(-1)^{\sum_{j=1}^{i-1}(k-j)|v_j| + (k-i)|v_{f'}| + \sum_{j=i+l}^{l+k-1}(k-(j-l+1))|v_j|} v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{f'} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l}$.

For this same graph, $\mu_{H_i}^{\mathcal{E}_V}$ is given by

$$\begin{aligned} & (-1)^{\epsilon_1} v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_f \otimes v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{f'} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l} \\ \mapsto \quad & (-1)^{\epsilon_1+\epsilon_2} v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l}, \end{aligned} \tag{4.19}$$

where

$$\epsilon_1 = \sum_{j=i}^{i+l-1} (l-(j-i+1))|v_j| + \sum_{j=1}^{i-1}(k-j)|v_j| + (k-i)|v_{f'}| + \sum_{j=i+l}^{l+k-1}(k-(j-l+1))|v_j|$$

and

$$\epsilon_2 = \sum_{j=1}^{i-1} |v_j|(|\overline{m}_{k,0}| - |v_j|) + |B|(|\overline{m}_{l,0}| - |v_f|) + |v_f| + \sum_{j=1}^{i-1} |v_j|(|\overline{m}_{l,0}| + |\overline{m}_{k,0}| - |v_f| - |v_{f'}| - |v_j|)$$

The calculation of the sign $(-1)^{\epsilon_2}$ is given in detail in Section 3.2.5, and the extra term $|v_f|$ in ϵ_2 is due to $B(v_f, v_{f'})$ having the sign $(-1)^{|v_f|}$. Noting that $|B| = 1$, $|v_f| + |v_{f'}| = 1$, and $|\overline{m}_{n,0}| = n-1$ for any n, ϵ_2 can be simplified to

$$\epsilon_2 = \sum_{j=1}^{i-1} k|v_j| + l - 1 + \sum_{j=1}^{i-1}(k+l)|v_j|$$

Write d_V for $d_{V^{\otimes n}}$. Applying $d_V \hat{m}_{n,0}$ to $(12\cdots n)^* \in \tilde{\mathbb{S}}[t]((n,0))^*$ gives

$$\begin{aligned} d_V \hat{m}_{n,0}((12\cdots n)^*) &= d_V(u_1 \otimes \cdots \otimes u_{n-1} \otimes u_n) \\ &= \sum_{j=1}^{n} (-1)^{\sum_{k=1}^{j-1}|u_{j-k}|} u_1 \otimes \cdots \otimes d_V(u_j) \otimes \cdots \otimes u_n \\ &= \sum_{j=1}^{n} (-1)^{\sum_{k=1}^{j-1}|u_{j-k}|} u_1 \otimes \cdots \otimes d_V(u_j) \otimes \cdots \otimes u_n \end{aligned} \tag{4.20}$$

Applying the right-hand side of (4.17) to $(12\cdots n)^*$ gives

$$\begin{aligned} & \sum_{\substack{H\in\Gamma((n,0)) \\ H/e \simeq *_{n,0}}} \mu_H^{\mathcal{E}_V} \circ (e[1] \otimes (\bigotimes_{v\in \mathrm{Vert}(H)} \hat{m}_{\mathrm{Leg}(v),0}) \circ (\mu_H^{\tilde{\mathbb{S}}[t]})^*)((12\cdots n)^*) \\ = & \sum_{\substack{H\in\Gamma((n,0)) \\ H/e \simeq *_{n,0}}} \mu_H^{\mathcal{E}_V} \circ (e[1] \otimes \hat{m}_{\mathrm{Leg}(u_1),0}((i \cdots (i+l-1)\ f)^*) \\ & \otimes \hat{m}_{\mathrm{Leg}(u_2),0}((12\cdots (i-1)\ f\ (i+l) \cdots n)^*)) \\ = & \sum_{\substack{H\in\Gamma((n,0)) \\ H/e \simeq *_{n,0}}} \mu_H^{\mathcal{E}_V}((f \wedge f')[1] \otimes v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_f \otimes v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{f'} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l}) \\ = & \sum_{\substack{H\in\Gamma((n,0)) \\ H/e \simeq *_{n,0}}} (-1)^{\epsilon_1+\epsilon_2} v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l} \\ = & \sum_i (-1)^{\epsilon_1+\epsilon_2} v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l} \end{aligned} \tag{4.21}$$

One must also include the sign

$$(-1)^{\epsilon_3} = (-1)^{\sum_{j=1}^{k+l-1}(k+l-1-j)|v_j|}$$

coming from applying ξ_{k+l-1} to the contracted tensor $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l}$.

Summing ϵ_1, ϵ_2, and ϵ_3 and noting $|v_f| = 1 - |v_{f'}|$ for v_f and $v_{f'}$ such that $B(v_f, v_{f'}) \neq 0$, gives

$$l - 1 + \sum_{j=1}^{i-1} |v_j| + (k+i)(l-1) + (k-i) = |\overline{m}_{l,0}| + \sum_{j=1}^{i-1} |v_j| + (|\overline{m}_{k,0}| + 1 + i)|\overline{m}_{l,0}| + |\overline{m}_{k,0}| + 1 - i$$

Converting back to cohomological degrees gives

$$2 + (i-1) + \sum_{j=1}^{i-1} |w_j| + (2+1+i)2 + 2 + 1 - i = \sum_{j=1}^{i-1} |w_j| \mod 2,$$

where w_j corresponds to v_j under the identification $W^i = V_{1-i}$.

When it is being applied to a tensor $t_1 \otimes \cdots \otimes t_{k+l-1}$, (4.21) may be rewritten as

$$\sum_i (-1)^{\sum_{j=1}^{i-1} |t_j| + i - 1} w_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes w_{i+l-1} \otimes w_{i+l} \otimes \cdots \otimes w_{k+l}, \tag{4.22}$$

because $B(w_i, t_i) \neq 0$ only if $|w_i| + |t_i| = 1$. Similarly for (4.20). Putting (4.20) and (4.22) together and noting that the element $u_1 \otimes \cdots \otimes d_V(u_j) \otimes \cdots \otimes u_{n-1} \otimes v_n$ corresponds to the composition $m_{n,0} \circ_j m_{1,0}$, yields the result. □

4.3 The Quantum A_∞-relations

Definition 4.3.1. A *Quantum A_∞-algebra* is a $\mathbb{Z}$-graded k-vector space

$$V = \bigoplus_{p \in \mathbb{Z}} V_p$$

with k-linear maps

$$\begin{aligned} \hat{m}_{d+1,b} : (\beta^2 \tilde{\mathfrak{s}} \tilde{\mathbb{S}}[t]((d+1,b)))^* &\longrightarrow \mathcal{E}_V((d+1,b)) \\ ((\sigma_1 \cdots \sigma_{b-2g+1})[(d+1) - (2-2b)])^* &\mapsto \overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1}), \end{aligned} \tag{4.23}$$

of degree 0, satisfying for each product of cycles $\sigma_1 \cdots \sigma_{b-2g+1}$, the relation

$$\begin{aligned} & d_V(\hat{m}_{d+1,b}(\sigma[n])^*) \qquad (4.24) \\ = \; & \mu^{\mathcal{E}_V}_{G_{d+1,b}}((f \wedge f')[1] \otimes (\hat{m}_{d+3,b-1} \circ (\mu^{\tilde{\mathbb{S}}[t]}_{*^1_{d+1,b-1}})^*)((\sigma[n])^*) \\ + \; & \sum_{\substack{H \in \Gamma((d+1,b)) \\ H/e \simeq *^1_{d+1,b}}} \mu^{\mathcal{E}_V}_H((f \wedge f')[1] \otimes (\hat{m}_{k,b_1} \otimes \hat{m}_{(d+1)-k+2,b_2}) \circ (\mu^{\tilde{\mathbb{S}}[t]}_H)^*)((\sigma[n])^*), \end{aligned}$$

where $\sigma[n] = \sigma_1 \cdots \sigma_{b-2g+1}[(d+1)-(2-2b)]$, $b_1 + b_2 = b$, and the sequence of disjoint cycles $\sigma_1 \cdots \sigma_{b-2g+1}$ is such that $d+1 = \sum_{i=1}^{b-2g+1} |\sigma_i|$. The elements $\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1})$ are tensors in $V^{\otimes(d+1)}$ sitting in degree $(d+1)-(2-2b)$. Note that we take $g = 0$ in the case of the elliptic curve. By setting $b = 0$, one recovers the structure of an A_∞-algebra by Lemma 4.2.2.

Notice the definition involves a homologically graded complex, while the usual definition of an A_∞-algebra involves cohomologically graded complexes. The degree of the map $m_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1}) : V^{\otimes d} \longrightarrow V$ corresponding to the tensor $\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1})$, when viewed using cohomological degrees, is $2-2b-d$. Indeed, consider a degree $2-2b-d$ map $m_{d,b} : W^{\otimes d} \longrightarrow W$ of cohomologically graded complexes. When viewed as an element of $W^{\otimes(d+1)}$, it has degree $2-2b$. Switching to the homological complex V given by $V_i = W^{1-i}$, the degree becomes $(d+1)-(2-2b)$. Finally, viewing this same element as a map $V^{\otimes d} \longrightarrow V$, the degree is seen to be $(d+1)-(2-2b)-d = 2b-1$.

4.3.1 The $b = 1$ Case of The Quantum A_∞-relations

As above, let $\overline{m}_{n,b}(\sigma) = \xi_n^{-1}(m_{n,b}(\sigma))$, where $m_{n,b}(\sigma) : V^{\otimes n} \longrightarrow V$ is a degree $2b-1$ map of homological complexes. This element $\overline{m}_{n,b}(\sigma)$ sits in degree $(n+1)-(2-2b)$ and depends on the permutation σ as in (4.23). The map (4.23) takes the form

$$\hat{m}_{|\sigma|+|\tau|,1} : (\beta^2 \tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((|\sigma|+|\tau|,1)))^* \longrightarrow \mathcal{E}_V((|\sigma|+|\tau|,1)) = V^{\otimes(|\sigma|+|\tau|)}$$

$$\begin{aligned} ((\sigma\tau)[|\sigma|+|\tau|])^* &\mapsto \xi^{-1}_{|\sigma|+|\tau|-1}(m_{|\sigma|+|\tau|-1,1}(\sigma\tau)) \\ &= \overline{m}_{|\sigma|+|\tau|-1,1}(\sigma\tau) \end{aligned} \tag{4.25}$$

Proposition 4.3.2. Let $\gamma(\sigma\tau)$ be the number of stable ribbon graphs which contract to $(\sigma\tau)^*$ under $\mu^{\tilde{\mathbb{S}}[t]}$, where σ and τ are disjoint cycles, and let $n = |\sigma|$ and $m = |\tau|$. Then

$$\gamma(\sigma\tau) = \begin{cases} 3 + \frac{1}{2}(n^2-n-2) + \frac{1}{2}(m^2-m-2) & \text{if } n, m \geq 3 \\ 1 + n + \frac{1}{2}(m^2-m-2) & \text{if } 1 \leq n \leq 2,\ m \geq 3 \\ m & \text{if } n = 1, 1 \leq m \leq 2 \end{cases}$$

Proof. Let $G = \star_{n+m,1}$ and let $\mathcal{G}$ be the corresponding stable ribbon graph whose single vertex is decorated with two cycles σ, τ such that $|\sigma| = n$ and $|\tau| = m$. Let $L_\sigma \subset \mathrm{Leg}(\mathcal{G})$ be the subset corresponding to σ. Similarly for τ. The basis elements spanning $d_{\mathscr{F}}(\mathcal{G})$

are indexed by stable ribbon graphs $\mathcal{H}$ such that $\mu^{\tilde{\mathbb{S}}[t]}(\mathcal{H}) = \mathcal{G}$, $|\text{Edge}(\mathcal{H})| = 1$, and $b_1(\mathcal{H}) + \sum_{\text{Vert}(\mathcal{H})} b(v) = 1$. If $b(v_i) = 0$ then $n(v_i) \geq 3$ and $n(v_i) = 2$ only if $b(v_i) = 1$. Because $b(v) = i_\sigma(v) - 1$ for any vertex v, if $|\text{Vert}(\mathcal{H})| = 2$ then one vertex will be decorated with two cycles and the other with a single cycle.

Assume first that $n, m \geq 3$ and let k be an integer such that $2 \leq k \leq n-1$. There are $n - k + 1$ ways of isolating k adjacent legs from L_σ. Each gives rise to a graph $\mathcal{H}$ in the following way. Let $\sigma = (1 \cdots n)$ and let $\sigma_k = (i \cdots j)$ be such that $1 \leq i < j \leq n$ and $j - i + 1 = k$. Then the first vertex of $\mathcal{H}$ is decorated by $(i \cdots j\, f)$ and the second by $(1 \cdots (i-1)\, f'\, (j+1) \cdots n)\tau$. If $j = n$, then the latter permutation is $(1 \cdots (i-1)\, f')$. Summing over all k between 2 and $n-1$ gives

$$\begin{aligned} \sum_{k=2}^{n-1} (n - k + 1) &= n(n-2) - \left(\frac{(n-1)n}{2} - 1 \right) + (n-2) \\ &= \frac{n^2 - n - 2}{2} \end{aligned}$$

The same argument holds for τ, so the subtotal is $\frac{1}{2}(n^2 - n - 2) + \frac{1}{2}(m^2 - m - 2)$.

The remaining 3 graphs come from the two distinct ways of partitioning $\text{Leg}(\mathcal{G})$ while keeping both σ and τ intact, and from drawing a loop around either L_σ, or equivalently, around L_τ. Specifically, if we let $\sigma = (1 \cdots n)$ and $\tau = ((n+1) \cdots p)$, where $p - (n+1) + 1 = m$, then $\mathcal{H}$ can be defined either by decorating the first vertex with $(1 \cdots n\, f)$ and the second with $(f') \cdot \tau$, or the first with $\sigma \cdot (f)$ and the second with $(f'\, (n+1) \cdots p)$. If $\mathcal{H}$ is obtained by inserting a loop, then the single cycle decorating the unique vertex is $(f\, 1 \cdots n\, f'\, (n+1) \cdots p)$.

By precisely the same reasoning, there are $n + 1 + \frac{1}{2}(m^2 - m - 2)$ possibilities if $1 \leq n \leq 2$ and $m \geq 3$. Indeed, if $n = 2$ then there is no way to split σ into two cycles because only one vertex of $\mathcal{H}$ can carry two cycles at a time, and stability requires $n(v) \geq 3$ if $b(v) = 0$. We are left with the $\frac{1}{2}(m^2 - m - 2)$ ways of splitting τ along with the 3 mentioned in the previous paragraph. If $n = 1$ and $m \geq 3$ there is only one way, again by stability, to split σ and τ between two vertices while keeping them both intact, and one way of inserting a loop.

If $n = 1$ and $m = 2$, then there are two possibilities. There is one way of splitting σ and τ between two vertices and one way of inserting a loop.

If $n = m = 1$, then the only possibility is that given by drawing a loop around either leg.

□

Given a stable ribbon graph $\mathcal{G}$ with its unique vertex decorated by two cycles σ and τ, I divide the possible graphs $\mathcal{H}$ such that $\mu_H^{\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]}(\mathcal{H}) = \mathcal{G}$ into five groups, denoted by $H(i)$ for $1 \leq i \leq 5$. The first group is given by inserting a single edge in such a way as to split σ into two cycles, each of which having a strictly smaller order than σ. The second is given by doing precisely the same thing, but with τ. The third and fourth each consist of a single graph, given by the two ways of decorating a single-edge, two-vertex graph with cycles without splitting σ or τ into smaller cycles. The fifth is given by the unique way of attaching a loop to $\mathcal{G}$. The five types are illustrated in Figure 3.3.

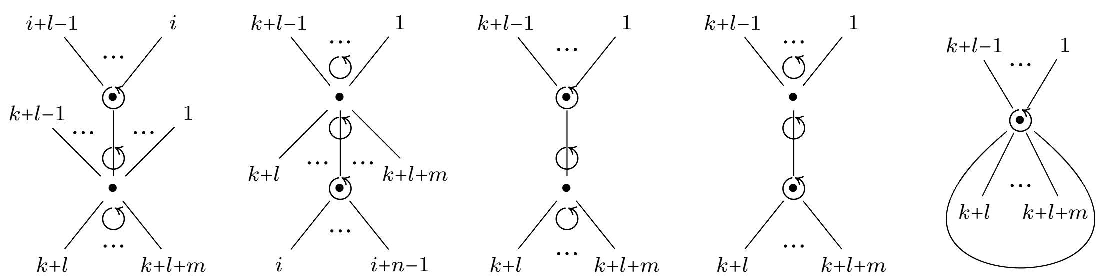


Figure 4.1: There are $\frac{1}{2}(n^2 - n - 2)$ of the first graph, $\frac{1}{2}(m^2 - m - 2)$ of the second, and one each of the remaining three. These five graphs correspond to $\overline{m}_{l,0}(\sigma_1) \otimes \overline{m}_{k+m,1}(\sigma_2\sigma_3)$, $\overline{m}_{n,0}(\sigma_4) \otimes \overline{m}_{k+l+m-n,1}(\sigma_5\sigma_6)$, $\overline{m}_{k+l-1,0}(\sigma_7) \otimes \overline{m}_{m+1,1}(\sigma_8\sigma_9)$, $\overline{m}_{m+1,0}(\sigma_{10}) \otimes \overline{m}_{k+l-1,1}(\sigma_{11}\sigma_{12})$, and $\overline{m}_{k+l+m+1,0}(\sigma_{13})$, respectively, where $\{\sigma_i\}_{i=1}^{13}$ can be read directly from the above five figures.

The $b = 1$ case of the Quantum A_∞ relation on $V^{\otimes(k+l+m)}$ is given by applying

$$\begin{aligned} & d_V \circ \hat{m}_{\mathrm{Leg}(v),1} \\ = \quad & \hat{m}_{\mathrm{Leg}(v),1} \circ d_{(\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t])^*} + \mu^{\mathcal{E}_V}_{G_{k+l+m,1}} \circ (e[1] \otimes (\hat{m}_{\{1,\cdots,k+l+m\}\sqcup\{f,f'\},1} \circ (\mu^{\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]}_{G_{k+l+m,1}})^*)) \\ + \quad & \frac{1}{2}\sum \mu^{\mathcal{E}_V}_{G_{(I_1,I_2,b_1,b_2)}} \circ (e[1] \otimes (\hat{m}_{I_1\sqcup\{f\},b_1} \otimes \hat{m}_{I_2\sqcup\{f'\},b_2}) \circ (\mu^{\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]}_{G_{(I_1,I_2,b_1,b_2)}})^*), \end{aligned}$$

to $(\sigma\tau)^*$, where $b_1 + b_2 = 1$ and $I_1 \sqcup I_2 = \mathrm{Leg}(\mathcal{G})$.

For the remainder of the section, I will write $\overline{m}_{d,b}$ for $\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1})$ ($g = 0$ for the elliptic curve). Keep in mind that every element $\overline{m}_{d,b}$ depends on a specific product of cycles $\sigma_1 \cdots \sigma_{b-2g+1}$.

The Composition $(-1)^{\epsilon(1)}\overline{m}_{k+m,1} \circ_i \overline{m}_{l,0}$

The tensor $\overline{m}_{l,0} \otimes \overline{m}_{k+m,1} \in \mathcal{E}_V((l+1,0)) \otimes \mathcal{E}_V((k+m+1,1))$ is given by

$$(v_i \otimes \cdots \otimes v_{i+l-1} \otimes v_f) \otimes (v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{f'} \otimes v_{i+l} \otimes \cdots \otimes v_{k+l-1} \otimes v_{k+l} \otimes \cdots \otimes v_{k+l+m}),$$

where $1 \le i \le k-1$, and $\mu^{\mathcal{E}_V}_{H(1)}(\overline{m}_{l,0} \otimes \overline{m}_{k+m,1}) = (-1)^{\epsilon_1+\epsilon_2+l-1+l\sum_{j=1}^{i-1}|v_j|}\overline{m}_{k+m,1} \circ_i \overline{m}_{l,0}$, where

$$\epsilon_1 = \sum_{j=i}^{i+l-1}(l-(j-i+1))|v_j|$$

and

$$\epsilon_2 = \sum_{j=1}^{i-1}(k+m-j)|v_j| + (k+m-i)|v_{f'}| + \sum_{j=i+l}^{k+l+m-1}(k+m-(j-l+1))|v_j|$$

give the signs associated to $\overline{m}_{l,0}$ and $\overline{m}_{k+m,1}$, respectively, by (4.12), and $\deg \overline{m}_{l,0} = l-1$ and $\deg \overline{m}_{k+m,1} = k+m+1$. The sign $(-1)^{l-1+l\sum_{j=1}^{i-1}|v_j|}$ is calculated as follows. For each j such that $1 \le j \le i-1$, one has the three terms

$$\begin{aligned} |v_j|(|\overline{m}_{k+m,1}|-|v_j|) &= |v_j|(k+m+1)-|v_j|^2 \\ &= (k+m)|v_j|, \end{aligned} \tag{4.26}$$

$$|B|(|\overline{m}_{l,0}|-|v_f|)+|v_f| = l-1, \tag{4.27}$$

and

$$\begin{aligned} |v_j|(|\overline{m}_{l,0}|+|\overline{m}_{k+m,1}|-|v_f|-|v_{f'}|-|v_j|) &= |v_j|(l-1+k+m+1-1-|v_j|) \\ &= (k+l+m)|v_j| \end{aligned} \tag{4.28}$$

Summing over the j such that $1 \le j \le i-1$ yields the result. In order to calculate the final sign, one must include the sign $(-1)^{\sum_{j=1}^{k+l+m-1}(k+l+m-1-j)|v_j|}$ corresponding to the contracted tensor $v_1 \otimes \cdots \otimes v_{k+l+m-1} \otimes v_{k+l+m}$. Summing this sign, ϵ_1, ϵ_2, and $l-1+l\sum_{j=1}^{i-1}|v_j|$ gives $(-1)^{\epsilon(1)}$, where

$$\epsilon(1) = |\overline{m}_{l,0}| + \sum_{j=1}^{i-1}|v_j| + (|\overline{m}_{k+m,1}|-1+i)(1+|\overline{m}_{l,0}|) \tag{4.29}$$

Converting to cohomological signs gives

$$(-1)^{\sum_{j=1}^{i-1}|w_j|},$$

the usual A_∞-sign.

The Composition $(-1)^{\epsilon(2)}\overline{m}_{k+l+m-n,1} \circ_i \overline{m}_{n,0}$

The tensor $\overline{m}_{n,0} \otimes \overline{m}_{k+l+m-n,1} \in \mathcal{E}_V((n+1,0)) \otimes \mathcal{E}_V((k+l+m-n+1,1))$ is given by

$$(v_i \otimes \cdots \otimes v_{i+n-1} \otimes v_f) \otimes (v_1 \otimes \cdots \otimes v_{k+l} \otimes \cdots \otimes v_{i-1} \otimes v_{f'} \otimes v_{i+n} \otimes \cdots \otimes v_{k+l+m-1} \otimes v_{k+l+m}),$$

and $\mu^{\mathcal{E}_V}_{H(2)}(\overline{m}_{n,0} \otimes \overline{m}_{k+l+m-n,1}) = (-1)^{\epsilon_3+\epsilon_4+n-1+n\sum_{j=1}^{i-1}|v_j|}\overline{m}_{k+l+m-n,1} \circ_i \overline{m}_{n,0}$, where $k+l \leq i \leq k+l+m-n+1$ and

$$\epsilon_3 = \sum_{j=i}^{i+n-1}(n-(j-i+1))|v_j|,$$

$$\epsilon_4 = \sum_{j=1}^{i-1}(k+l+m-n-j)|v_j| + (k+l+m-n-i)|v_{f'}| + \sum_{i+n}^{k+l+m-1}(k+l+m-n-(j-n+1))|v_j|$$

As with the previous case, including the sign $(-1)^{\sum_{j=1}^{k+l+m-1}(k+l+m-1-j)|v_j|}$ gives the sign $(-1)^{\epsilon(2)}$, where

$$\epsilon(2) = |\overline{m}_{n,0}| + \sum_{j=1}^{i-1}|v_j| + (|\overline{m}_{k+l+m-n,1}| - 1 + i)(1 + |\overline{m}_{n,0}|) \tag{4.30}$$

The Composition $(-1)^{\epsilon(3)}\overline{m}_{m+1,1} \circ \overline{m}_{k+l-1,0}$

The tensor $\overline{m}_{k+l-1,0} \otimes \overline{m}_{m+1,1}$ is given by

$$(v_1 \otimes \cdots \otimes v_{k+l-1} \otimes v_f) \otimes (v_{f'} \otimes v_{k+l} \otimes \cdots \otimes v_{k+l+m-1} \otimes v_{k+l+m}),$$

and $\mu^{\mathcal{E}_V}_{H(3)}(\overline{m}_{k+l-1,0} \otimes \overline{m}_{m+1,1}) = (-1)^{\epsilon_5+\epsilon_6+k+l}\overline{m}_{m+1,1} \circ \overline{m}_{k+l-1,0}$ and the associated signs are

$$\epsilon_5 = \sum_{j=1}^{k+l-1}(k+l-1-j)|v_j|$$

and

$$\epsilon_6 = ((m+1)-1)|v_{f'}| + \sum_{j=k+l}^{k+l+m-1}(m+1-(j-(k+l-2)))|v_j|$$

Including the sign $(-1)^{\sum_{j=1}^{k+l+m-1}(k+l+m-1-j)|v_j|}$ gives $(-1)^{\epsilon(3)}$, where

$$\epsilon(3) = |\overline{m}_{k+l-1,0}| + |\overline{m}_{m+1,1}|(1 + |\overline{m}_{k+l-1,0}|) \tag{4.31}$$

The Composition $(-1)^{\epsilon(4)}\overline{m}_{k+l-1,1} \circ \overline{m}_{m+1,0}$

The tensor $\overline{m}_{m+1,0} \otimes \overline{m}_{k+l-1,1}$ is given by

$$(v_{k+l} \otimes \cdots \otimes v_{k+l+m} \otimes v_f) \otimes (v_1 \otimes \cdots \otimes v_{k+l-1} \otimes v_{f'}),$$

and $\mu^{\mathcal{E}_V}_{H(4)}(\overline{m}_{m+1,0} \otimes \overline{m}_{k+l-1,1}) = (-1)^{\epsilon_7+\epsilon_8+m+(m+1)\sum_{j=1}^{k+l-1}|v_j|}\overline{m}_{k+l-1,1} \circ \overline{m}_{m+1,0}$ the associated signs are

$$\epsilon_7 = \sum_{j=k+l}^{k+l+m}(m+1-(j-(k+l-1)))|v_j|$$

and

$$\epsilon_8 = \sum_{j=1}^{k+l-1}(k+l-1-j)|v_j|$$

Including $(-1)^{\sum_{j=1}^{k+l+m-1}(k+l+m-1-j)|v_j|}$ gives $(-1)^{\epsilon(4)}$, where

$$\epsilon(4) = |\overline{m}_{k+l-1,1}| + 1 - |v_{k+l+m}| \tag{4.32}$$

The Element $(-1)^{\epsilon(5)}\overline{m}_{k+l+m-1,1}$

The tensor $\overline{m}_{k+l+m+1,0}$ is given by

$$v_1 \otimes \cdots \otimes v_{k+l-1} \otimes v_{f'} \otimes v_{k+l} \otimes \cdots \otimes v_{k+l+m-1} \otimes v_{k+l+m} \otimes v_f,$$

and $\mu^{\mathcal{E}_V}_{H(5)}(\overline{m}_{k+l+m+1,0}) = (-1)^{\epsilon_9+|v_{f'}|\sum_{j=k+l}^{k+l+m}|v_j|+|\overline{m}_{k+l+m+1,0}|-1}\overline{m}_{k+l+m-1,1}$, where

$$\epsilon_9 = \sum_{j=1}^{k+l-1}(k+l+m+1-j)|v_j| + (k+l+m+1-(k+l))|v_{f'}| + \sum_{j=k+l}^{k+l+m}(k+l+m+1-(j+1))|v_j|$$

Including $(-1)^{\sum_{j=1}^{k+l+m-1}(k+l+m-1-j)|v_j|}$ gives $(-1)^{\epsilon(5)}$, where

$$\epsilon(5) = |v_{f'}|(m+1+|v_{k+l+m}|) + (|v_{f'}|+1)\sum_{j=k+l}^{k+l+m-1}|v_j| + |\overline{m}_{k+l+m+1,0}| - 1 \tag{4.33}$$

We have proved

Lemma 4.3.3. The $b = 1$ case of the Quantum A_∞-relations is given by

$$\begin{aligned}
& \sum_{i=1}^{k-1} (-1)^{|\overline{m}_{l,0}|+\sum_{j=1}^{i-1}|v_j|+(|\overline{m}_{k+m,1}|-1+i)(1+|\overline{m}_{l,0}|)} \overline{m}_{k+m,1} \circ_i \overline{m}_{l,0} \\
+ & \sum_{i=k+l}^{k+l+m-n+1} (-1)^{|\overline{m}_{n,0}|+\sum_{j=1}^{i-1}|v_j|+(|\overline{m}_{k+l+m-n,1}|-1+i)(1+|\overline{m}_{n,0}|)} \overline{m}_{k+l+m-n,1} \circ_i \overline{m}_{n,0} \\
+ & (-1)^{|\overline{m}_{k+l-1,0}|+|\overline{m}_{m+1,1}|(1+|\overline{m}_{k+l-1,0}|)} \overline{m}_{m+1,1} \circ \overline{m}_{k+l-1,0} \\
+ & (-1)^{|\overline{m}_{k+l-1,1}|+1-|v_{k+l+m}|} \overline{m}_{k+l-1,1} \circ \overline{m}_{m+1,0} \\
+ & (-1)^{|v_{f'}|(m+1+|v_{k+l+m}|)+(|v_{f'}|+1)\sum_{j=k+l}^{k+l+m-1}|v_j|+|\overline{m}_{k+l+m+1,0}|-1} \overline{m}_{k+l+m-1,1} \\
= & \; 0
\end{aligned} \tag{4.34}$$

Converting to cohomological signs gives

$$\begin{aligned}
& \sum_{i=1}^{k-1} (-1)^{\sum_{j=1}^{i-1}|w_j|} \overline{m}_{k+m,1} \circ_i \overline{m}_{l,0} \\
+ & \sum_{i=k+l}^{k+l+m-n+1} (-1)^{\sum_{j=1}^{i-1}|w_j|} \overline{m}_{k+l+m-n,1} \circ_i \overline{m}_{n,0} \\
+ & \overline{m}_{m+1,1} \circ \overline{m}_{k+l-1,0} \\
+ & (-1)^{|w_{k+l+m}|} \overline{m}_{k+l-1,1} \circ \overline{m}_{m+1,0} \\
+ & (-1)^{m-1+|w_f||w_{k+l+m}|+|w_{f'}|\sum_{j=k+l}^{k+l+m-1}|w_j|} \overline{m}_{k+l+m-1,1} \\
= & \; 0
\end{aligned} \tag{4.35}$$

Chapter 5

The Fukaya Category of the Elliptic Curve

The goal of the previous two chapters was to establish the definition and an initial use of the Feynman transform, that is, the relationship between morphisms of modular operads with the Feynman transform as their source, and quantum A_∞-algebras. In this chapter the relationship between the former and the latter is taken a step further, i.e., by replacing the quantum A_∞ algebra with a categorical analogue. Specifically, this chapter will be devoted to the construction of the Fukaya category of the elliptic curve given by $\mathbb{C}/(d\mathbb{Z} \oplus \mathbb{Z})$ for a natural number d. With the exception of the final section, this material is entirely expository and can be found in [2] and [3].

5.1 A_∞-precategories

The notion of an A_∞-algebra V generalizes easily to the case of categories. One simply substitutes the copies of V in $m_{d,0} : V^{\otimes d} \longrightarrow V$ with Hom-spaces between objects of the given category. More precisely, we have

Definition 5.1.1. An (non-unital) A_∞-category $\mathcal{A}$ consists of a collection of objects Ob$\mathcal{A}$, a $\mathbb{Z}$-graded k-vector space $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$ for any $X_0, X_1 \in \mathrm{Ob}\mathcal{A}$, and for every $d \geq 1$, k-linear composition maps

$$m_{d,0} : \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_0, X_d)$$

of degree $2 - d$, satisfying (4.8)

It turns out that the category of interest, the Fukaya category of the elliptic curve, is not quite an A_∞-category, because Hom-spaces do not necessarily exist for any choice of objects X_0, X_1 in the category. It is actually something called a *precategory*, defined as follows.

Definition 5.1.2. A (non-unital) A_∞-precategory $\mathcal{A}$ consists of:

1. A collection of objects $\mathrm{Ob}(\mathcal{A})$.
2. For each $n \geq 2$, a collection of transversal sequences $\mathrm{Ob}^n_{\mathrm{tr}}(\mathcal{A}) \subseteq \mathrm{Ob}(\mathcal{A})^n$, i.e., a set of n-tuples of sequences.
3. For $(X_0, X_1) \in \mathrm{Ob}^2_{\mathrm{tr}}(\mathcal{A})$, a $\mathbb{Z}$-graded chain complex $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$.
4. For $(X_0, \ldots, X_d) \in \mathrm{Ob}^{d+1}_{\mathrm{tr}}(\mathcal{A})$, a map

$$m_d : \mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{A}}(X_{d-1}, X_d) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X_0, X_d)[2-d]$$

   We require in addition:

5. Every subsequence of a transversal sequence is transversal.
6. The A_∞-relations are satisfied by the m_d's.

The objects of the Fukaya category of the elliptic curve $\mathbb{C}/(d\mathbb{Z} \oplus \mathbb{Z})$ belong to a certain class of Lagrangian submanifolds, and the maps m_d are defined by considering polygons bounded by these Lagrangians, where the polygons are parameterized by what are called *tropical Morse trees.*

5.2 Tropical Morse Trees

In this section, a class of decorated graphs and their moduli spaces will be described. When these moduli spaces are zero dimensional, their graphs are used to define the A_∞-structure of the Fukaya category of the elliptic curve. The majority of the material in section 5.2 can be found in [3].

Definition 5.2.1. Let B be the affine manifold $\mathbb{R}/d\mathbb{Z}$ with coordinate y. Define the local system Λ on B by $\Lambda_p = \mathbb{Z} \cdot \partial/\partial y|_p$. Set $X(B) := TB/\Lambda$ and define a section $\sigma_n : B \longrightarrow X(B)$ locally by $\sigma_n(y) = (y, -ny\partial/\partial y)$. Set $L_n = \sigma_n(B)$ and define its orientation as being left to right.

The quotient $X(B)$ is precisely our elliptic curve $\mathbb{C}/(d\mathbb{Z}\oplus\mathbb{Z})$, and for each $k \geq 2$, the set of all tuples $(L_{n_1}, \ldots, L_{n_k})$ is exactly the set of transversal sequences mentioned in part 2 of Definition 5.1.2.

Definition 5.2.2. Denote by $B(\frac{1}{n}\mathbb{Z})$ the set of points of $B = \mathbb{R}/d\mathbb{Z}$ with coordinates in $\frac{1}{n}\mathbb{Z}$.

In general, this space is always finite whenever the affine manifold B is compact. However, compactness is not needed to see this here. For example, if $d = 2$ and $n = 3$, then $B(\frac{1}{n}\mathbb{Z}) = \{0, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}\}$ The spaces $B(\frac{1}{n}\mathbb{Z})$ and $B(\frac{1}{-n}\mathbb{Z})$ are dual by the pairing

$$< p, q >= \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

Definition 5.2.3. Let B be as in Definition 5.2.2. Denote by $X(B)$ the elliptic curve TB/Λ, where Λ is the integral lattice defined by $\Lambda_p = \mathbb{Z}$ for $p \in B$ and TB is the tangent bundle of B.

The significance of the space $B(\frac{1}{n_j - n_i}\mathbb{Z})$ is that it parameterizes the equivalence classes mod Λ of the set of intersection points of $L_{n_i} \cap L_{n_j}$, where L_{n_i} and L_{n_j} are lagrangian submanifolds of $X(\mathbb{R}/d\mathbb{Z})$. Indeed, if $(x, y_1) \in L_{n_i}$ and $(x, y_2) \in L_{n_j}$ with $y_1 = y_2 \mod \Lambda$, then $y_1 = -n_i x + kd$, $y_2 = -n_j x + k'd$ for integers k, k' and $(n_j - n_i)x + (k - k')d = y_1 - y_2 \in \mathbb{Z}$, which gives $x \in B(\frac{1}{n_j - n_i}\mathbb{Z})$.

Definition 5.2.4. The Novikov ring, Λ_{nov}, is the ring of formal power series of the form $\sum_{i\in\mathbb{Z}} a_i q^{\lambda_i}$, where the coefficients $a_i \in \mathbb{Z}$ vanish for all sufficiently negative i, and λ_i are a sequence of real numbers satisfying $\lim_{i\to\infty} \lambda_i = \infty$.

Definition 5.2.5. A *(metric) ribbon tree* is a connected tree with a finite number of vertices and edges, with no bivalent vertices, with the additional data of a cyclic ordering of edges at each vertex and a length assigned to each edge in $(0, \infty]$.

Definition 5.2.6. (Tropical Morse Trees)
Let B be the integral affine manifold $\mathbb{R}/d\mathbb{Z}$. Given a sequence of distinct integers $n_0, \ldots, n_d \in \mathbb{Z}$ and any metric ribbon tree S we can label the edges e of S with integers n_e as follows. If e is an external incoming edge, attached to the ith external vertex, then $n_e = n_i - n_{i-1}$; otherwise, if e comes out of a vertex v, then n_e is the sum of all

numbers labeling the edges coming into v. Then given in addition points

$$p_{i,i+1} \in B\left(\frac{1}{n_{i+1} - n_i}\mathbb{Z}\right)$$

and

$$p_{0,d} \in B\left(\frac{1}{n_d - n_0}\mathbb{Z}\right)$$

we define $S_d^{\text{trop}}(p_{0,d}; p_{0,1}, \ldots, p_{d-1,d})$ to be the moduli space of *tropical Morse trees* on B, i.e., continuous maps $\phi : S \longrightarrow B$ from a ribbon tree with a collection of *affine displacement vectors*, i.e., for each edge e of S, a section $\mathbf{v}_e \in \Gamma(e, (\phi|_e)^* TB)$, satisfying the following properties:

(1) If v is the ith external incoming vertex, then $\phi(v) = p_{i-1,i}$; if v is the external outgoing vertex, then $\phi(v) = p_{0,d}$.

(2) If e is an edge S, then $\phi(e)$ is locally an affine line segment on B. (This line segment can have irrational slope). If e is an external edge, we also allow $\phi(e)$ to be a point.

(3) If v is an external vertex and e the unique edge of S containing v, then $\mathbf{v}_e(v) = 0$.

(4) For an edge e of S identified with $[0,1]$ with coordinate s, with the edge e oriented from 0 to 1, then $\mathbf{v}_e(s)$ is tangent to $\phi(e)$ at $\phi(s)$, pointing in the same direction as the orientation on $\phi(e)$ induced by that on e. Furthermore, using the affine structure to identify $(\phi|_e)^* TB$ with the trivial bundle on e, we have

$$\frac{d}{ds}\mathbf{v}_e(s) = n_e \phi_*(\partial/\partial s).$$

If $B = \mathbb{R}^n/M$ for some lattice $M \subseteq \mathbb{Z}^n$, then this equation takes the form $\mathbf{v}_e(s) = \mathbf{v}_e(0) + n_e(\phi(s) - \phi(0))$.

(5) If v is an internal vertex of S with incoming edges $e_1, \ldots, e_p$ and outgoing edge e_{out}, then

$$\mathbf{v}_{e_{\text{out}}}(v) = \sum_{i=1}^{p} \mathbf{v}_{e_i}(v).$$

(6) The length of an edge e in S (remember that each edge of a ribbon tree comes along with a length, the external edges having infinite length) coincides with

$$\frac{1}{n_e}\log\left(\frac{\mathbf{v}_e(1)}{\mathbf{v}_e(0)}\right).$$

Since $\mathbf{v}_e(0)$ and $\mathbf{v}_e(1)$ are proportional vectors pointing in the same direction, their quotient makes sense as a positive number. There is one special case: if e is an external edge that is contracted by ϕ, then $\mathbf{v}_e(0) = \mathbf{v}_e(1) = 0$, but we still take the length to be infinite.

Remark 5.2.7. Write S_d^{trop} for $S_d^{\text{trop}}(p_{0,d}; p_{0,1}, \ldots, p_{d-1,d})$. Because the metric ribbon trees are simply connected, a tropical Morse tree ϕ factors through $\mathbb{R}$, the universal cover of $B = \mathbb{R}/d\mathbb{Z}$. Given points $p_{0,1}, \ldots, p_{d-1,d}, p_{0,d}$, the element $\phi \in S_d^{\text{trop}}$ can be thought of as a collection of lifts $\tilde{p}_{0,1}, \ldots, \tilde{p}_{d-1,d}, \tilde{p}_{0,d}$ up to the action by $d\mathbb{Z}$. If $\dim S_d^{\text{trop}} = 0$ the fixing of a lift of one of $p_{0,1}, \ldots, p_{d-1,d}, p_{0,d}$ gives a bijection between S_d^{trop} and the set of lifts of the remaining points.

Example 5.2.8.

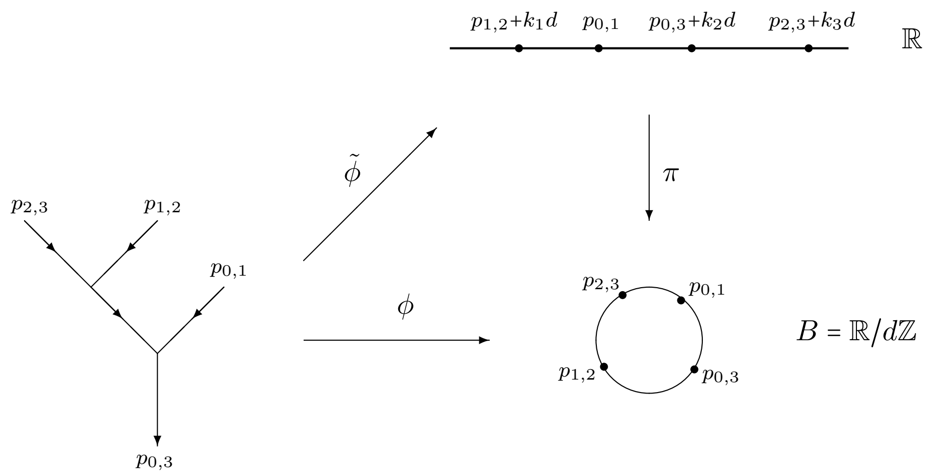


Figure 5.1: Assume $\dim S_3^{\mathrm{trop}}(p_{0,3}; p_{0,1}, p_{1,2}, p_{2,3}) = 0$. By fixing a lift of $p_{0,1}$, the elements of $S_3^{\mathrm{trop}}(p_{0,3}; p_{0,1}, p_{1,2}, p_{2,3})$ are indexed by the set of triples $(k_1, k_2, k_3) \in \mathbb{Z}^3$. The given tree is just one of the possible domains of the elements ϕ.

5.2.1 Holomorphic Polygons

A tropical Morse tree $\phi : S \longrightarrow B$ defines a piecewise linear disk in the following way. Any edge e of S is labeled by $n_e = n_j - n_i$ for some $j > i$.

Consider the map

$$R_e : e \times [0,1] \longrightarrow X(B) = T(B)/\Lambda$$

$$(s,t) \mapsto \sigma_{n_i}(\phi(s)) - t \cdot \mathbf{v}_e(s)$$

$$\begin{aligned} \sigma_{n_i}(\phi(s)) - t \cdot \mathbf{v}_e(s) &= (\phi(s), -n_i\phi(s)) - t \cdot \mathbf{v}_e(s) \\ &= (\phi(s), -n_i\phi(s)\partial/\partial y) - (\phi(s), t \cdot \mathbf{v}_e(s)) \\ &= (\phi(s), (-n_i\phi(s) - t \cdot \mathbf{v}_e(s))) \end{aligned}$$

Write the vertices of e as v_{in} and v_{out}. We have $R_e(s,0) = \sigma_{n_i}(\phi(s)) \in L_{n_i}$ and this implies $R_e(e \times \{0\}) \subseteq L_{n_i}$. Assuming $R_e(v_{\mathrm{in}} \times \{1\}) \subseteq L_{n_j}$, we have

$$\sigma_{n_i}(\phi(v_{\mathrm{in}})) - \mathbf{v}_e(v_{\mathrm{in}}) = (\phi(v_{\mathrm{in}}), -n_i\phi(v_{\mathrm{in}})\partial/\partial y - \mathbf{v}_e(v_{\mathrm{in}})) \in L_{n_j} \qquad (5.1)$$

Since TB is a bundle, the only way we can have $(y, \alpha\partial/\partial y) \in \sigma(B)$ for some σ, is if $\sigma(y) = \alpha\partial/\partial y$. The section L_{n_j} is in TB/Λ, so the inclusion (5.1) above implies

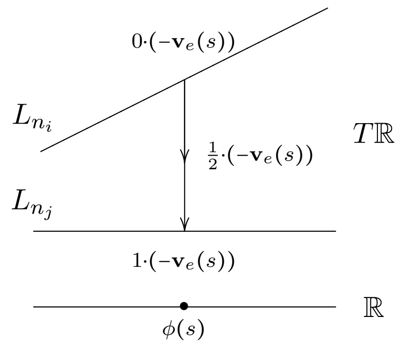


Figure 5.2: If $t = 0$, then $\sigma_{n_i}(\phi(s)) - t \cdot \mathbf{v}_e(s) = \sigma_{n_i}(\phi(s)) \in L_{n_i}$, and if $t = 1$, then $\sigma_{n_i}(\phi(s)) - t \cdot \mathbf{v}_e(s) = (\phi(s), -n_j\phi(s) + (n_j - n_i)\phi(0) - \mathbf{v}_e(0)) = (\phi(s), -n_j\phi(s)) = \sigma_{n_j}(\phi(s)) \in L_{n_j}$.

$\sigma_{n_i}(\phi(v_{\text{in}})) - \mathbf{v}_e(v_{\text{in}}) = \sigma_{n_j}(\phi(v_{\text{in}})) \mod \Lambda$. Now, condition 4 of Definition 5.2.6 tells us that $\frac{d}{ds}\mathbf{v}_e(s) = n_e\phi_*(\partial/\partial s)$, so

$$\begin{aligned} -\frac{d}{ds}\mathbf{v}_e(s) &= -n_e\phi_*(\partial/\partial s) \\ &= -(n_j - n_i)\phi_*(\partial/\partial s) \\ &= -(n_j - n_i)\frac{d}{ds}\phi(s)\partial/\partial y \\ &= \frac{d}{ds}(n_i\phi(s)\partial/\partial y - n_j\phi(s)\partial/\partial y) \\ &= \frac{d}{ds}((\phi(s), n_i\partial/\partial y) - (\phi(s), n_j\phi(s)\partial/\partial y)) \\ &= \frac{d}{ds}(\sigma_{n_j}(\phi(s)) - \sigma_{n_i}(\phi(s))) \end{aligned}$$

Solving this differential equation with the initial condition

$$-\mathbf{v}_e(v_{\text{in}}) = \sigma_{n_j}(\phi(v_{\text{in}})) - \sigma_{n_i}(\phi(v_{\text{in}})) \mod \Lambda$$

gives

$$-\mathbf{v}_e(s) = \sigma_{n_j}(\phi(s)) - \sigma_{n_i}(\phi(s)) \mod \Lambda$$

Then $R_e(e \times \{1\}) = \{\sigma_{n_i}(\phi(s)) - \mathbf{v}_e(s) | s \in e\} = \{\sigma_{n_j}(\phi(s)) | s \in e\} \subseteq L_{n_j}$. Thus $R_e(v_{\text{in}} \times \{1\}) \subseteq L_{n_j} \Rightarrow R_e(e \times \{1\}) \subset L_{n_j}$.

Let v be a vertex of the tree S. If v is an external incoming vertex v_i, then $\phi(v_i) \in B(\frac{1}{n_i - n_{i-1}}\mathbb{Z})$ by definition of ϕ, and so $\phi(v_i) = \frac{n}{n_i - n_{i-1}} + kd$ for some $n, k \in \mathbb{Z}$. We then have

$$\begin{aligned}
\sigma_{n_i}(\phi(v_i)) - \sigma_{n_{i-1}}(\phi(v_i)) &= -n_i\phi(v_i)\partial/\partial y + n_{i-1}\phi(v_i)\partial/\partial y \\
&= (-n_i + n_{i-1})\phi(v_i)\partial/\partial y \\
&= (-n_i + n_{i-1})\left(\frac{n}{n_i - n_{i-1}} + kd\right)\partial/\partial y \\
&\in \mathbb{Z}\cdot\partial/\partial y = \Lambda|_{\phi(v_i)} \subseteq TB|_{\phi(v_i)}
\end{aligned}$$

and this implies $\sigma_{n_i}(\phi(v_i)) = \sigma_{n_{i-1}}(\phi(v_i)) \in L_{n_i} \cap L_{n_i}$, since the Lagrangians are being viewed modulo Λ. But then

$$\begin{aligned}
R_e(v_i \times \{1\}) &= \sigma_{n_{i-1}}(\phi(v_i)) - \mathbf{v}_e(v_i) \\
&= \sigma_{n_{i-1}}(\phi(v_i)) - 0 \\
&= \sigma_{n_i}(\phi(v_i)) \in L_{n_i}
\end{aligned}$$

The conclusion here is that if the edge e of $e \times [0,1]$ is mapped to the Lagrangian $L_{n_{i-1}}$, then the opposite edge is mapped to the next Lagrangian L_{n_i}.

We want to use induction to prove that if v is any interior vertex with incoming edges $e_1, \dots, e_p$, outgoing edge e_{out}, with e_j weighted by $n_{i_j} - n_{i_{j-1}}$ for $i_0 < \cdots < i_p$, then the inclusions

$$R_{e_{\text{out}}}(e_{\text{out}} \times \{0\}) \subseteq L_{n_{i_0}}, R_{e_{\text{out}}}(v \times \{1\}) \subseteq L_{n_{i_p}}$$

imply $R_{e_{\text{out}}}(e_{\text{out}} \times \{1\}) \subseteq L_{n_{i_p}}$. Inductively assume

$$R_{e_j}(e_j \times \{0\}) \subseteq L_{n_{i_{j-1}}}, R_{e_j}(e_j \times \{1\}) \subseteq L_{n_{i_j}}$$

The second inclusion implies $\sigma_{n_{i_j}}(\phi(s)) = \sigma_{n_{i_{j-1}}}(\phi(s)) - \mathbf{v}_e(s) \mod \Lambda$. Keeping in mind the edge e_{out} is labeled by $n_{i_p} - n_{i_0}$, part 5 of Definition 5.2.6 tells us that

$$\begin{aligned}
R_{e_{\text{out}}}(v \times \{1\}) &= \sigma_{n_{i_0}}(\phi(v)) - \mathbf{v}_{e_{\text{out}}}(v) \\
&= \sigma_{n_{i_0}}(\phi(v)) - \sum_j \mathbf{v}_{e_j}(v) \\
&= \sigma_{n_{i_0}}(\phi(v)) - \sum_j (\sigma_{n_{i_{j-1}}}(\phi(v)) - \sigma_{n_{i_j}}(\phi(v))) \\
&= \sigma_{n_{i_0}}(\phi(v)) - \sum_j ((\phi(v), -n_{i_{j-1}}\phi(v)\partial/\partial y) - (\phi(v), -n_{i_j}\phi(v)\partial/\partial y)) \\
&= \sigma_{n_{i_0}}(\phi(v)) - \sum_j (\phi(v), (n_{i_j} - n_{i_{j-1}})\phi(v)\partial/\partial y) \\
&= (\phi(v), -n_{i_0}\phi(v)\partial/\partial y) - (\phi(v), \sum_j (n_{i_j} - n_{i_{j-1}})\phi(v)\partial/\partial y) \\
&= (\phi(v), -n_{i_0}\phi(v)\partial/\partial y - \sum_j (n_{i_j} - n_{i_{j-1}})\phi(v)\partial/\partial y) \\
&= (\phi(v), -n_{i_0}\phi(v)\partial/\partial y - (n_{i_p} - n_{i_0})\phi(v)\partial/\partial y) \\
&= (\phi(v), -n_{i_p}\phi(v)\partial/\partial y) \\
&= \sigma_{n_{i_p}}(\phi(v)) \in L_{n_{i_p}}
\end{aligned}$$

We now have $R_{e_{\text{out}}}(e_{\text{out}} \times \{0\}) \subseteq L_{n_{i_0}}$ by definition, and $R_{e_{\text{out}}}(v \times \{1\}) \subseteq L_{n_{i_p}}$, so $R_{e_{\text{out}}}(e_{\text{out}} \times \{1\}) \subseteq L_{n_{i_p}}$. For examples of such polygons see [3] page 608.

Definition 5.2.9. Let (L_{n_i}, L_{n_j}) be an ordered pair of Lagrangians and let $p_{i,j} \in L_{n_i} \cap L_{n_j}$. Then set

$$\deg p_{i,j} := \begin{cases} 1 & \text{if } n_j < n_i \\ 0 & \text{if } n_j > n_i \end{cases} \tag{5.2}$$

Example 5.2.10. Graphically, the degree of an intersection is given by Figure 5.2.10.

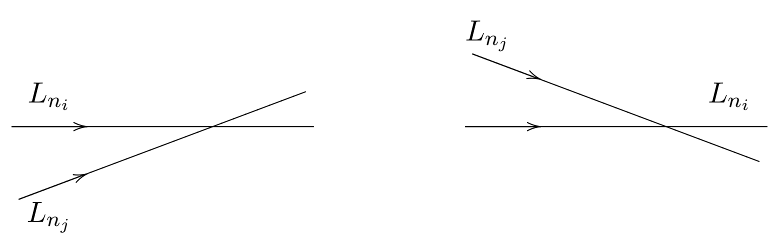


Figure 5.3: The first intersection is degree 1 and the second is degree 0, where the arrows indicate orientation. It is important to keep in mind that the slope of L_{n_k} is $-n_k$ for any $n_k \in \mathbb{Z}$.

Attached to each tropical Morse tree T is a sign $(-1)^{s(u)}$ defined by Abouzaid in [2]. Each degree one point $p_{i,j}$ with $i < j$ contributes a sign $(-1)^{s(p_{i,j})}$, which is positive if the natural orientation on $\mathbb{C}$ induces an orientation on the corresponding polygon, and the sign is positive if the induced boundary orientation on L_{n_j} agrees with the fixed orientation on L_{n_j}, and is negative otherwise. Then

$$(-1)^{s(u)} = \prod_{\{p_{i,j} \mid \deg p_{i,j} = 1\}} (-1)^{s(p_{i,j})} \tag{5.3}$$

See Figure 4.4 for the defining examples of how to calculate $(-1)^{s(u)}$.

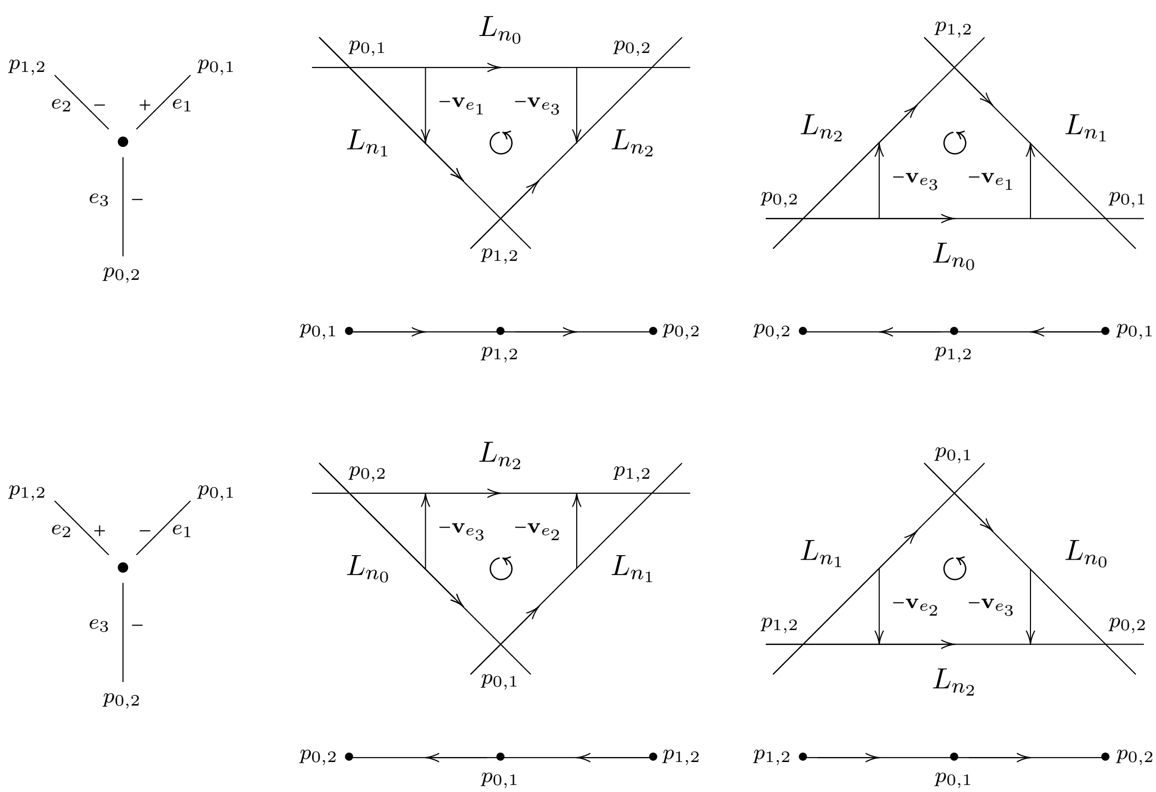


Figure 5.4: The contributions, counterclockwise from upper left, are: +1, +1, −1, and −1. Note that even if the outgoing edge e of a vertex v is such that $n_e > 0$, the signs $(-1)^{s(u)}$ remain the same. The only difference is the polygon may not close once e terminates. The arrow in the segment below each triangle indicates the direction of motion of $\phi(s) \in \mathbb{R}$.

A tropical Morse tree $\phi \in S_d^{\text{trop}}(p_{0,d}; p_{0,1}, \ldots, p_{d-1,d})$ can then be identified with a holomorphic map from the abstract disk, obtained by gluing the sets $R_e(e \times [0,1]$ along the edges $1 \times [0,1]$, where e terminates when $s = 1$, to $X(B)$.

5.3 Fuk($X(\mathbb{R}/d\mathbb{Z})$)

Definition 5.3.1. The objects of the precategory Fuk($X(\mathbb{R}/d\mathbb{Z})$) are the Lagrangian submanifolds of $X(\mathbb{R}/d\mathbb{Z})$ defined by $\sigma_n(\mathbb{R}/d\mathbb{Z})$. Note that since $X(\mathbb{R}/d\mathbb{Z})$ is an elliptic curve, any 1-dimensional submanifold is Lagrangian. The Hom-spaces of this category, $\text{Hom}(L_{n_i}, L_{n_j})$, are given by

$$\text{Hom}(L_{n_i}, L_{n_j}) = \begin{cases} \bigoplus_{p \in B(\frac{1}{n_j - n_i}\mathbb{Z})}[p]\Lambda_{\text{nov}} \longrightarrow 0 & \text{if } n_i < n_j \\ 0 \longrightarrow \bigoplus_{p \in B(\frac{1}{n_j - n_i}\mathbb{Z})}[p]\Lambda_{\text{nov}} & \text{if } n_i > n_j \end{cases}$$

The first column is degree 0 and the second is degree 1. A typical element of $\text{Hom}(L_{n_i}, L_{n_j})$ for $B(\frac{1}{n_j - n_i}\mathbb{Z}) = \{p_1, \ldots, p_n\}$ has the form

$$\sum_{i \in \mathbb{Z}}(a_{p_1})_i q^{(\lambda_{p_1})_i}[p_1] + \cdots + \sum_{i \in \mathbb{Z}}(a_{p_n})_i q^{(\lambda_{p_n})_i}[p_n],$$

but notice it makes perfect sense to write $p \in \text{Hom}(L_i, L_j)$ for $p \in L_i \cap L_j$.

5.3.1 Fuk($X(\mathbb{R}/d\mathbb{Z})$) is an A_∞-precategory

Define morphisms

$$m_{d,0} : \bigotimes_{i=1}^{d} \text{Hom}(L_{i-1}, L_i) \longrightarrow \text{Hom}(L_0, L_d)$$

by

$$m_{d,0}(p_{0,1}, \ldots, p_{d-1,d}) = \sum_{B(\frac{1}{n_d - n_0}\mathbb{Z})} \sum_{S_d^{\text{trop}}(p_{0,d}; p_{0,1}, \ldots, p_{d-1,d})} (-1)^{s(\phi)} q^{\deg \phi} p_{0,d} \tag{5.4}$$

where $\deg \phi$ is the area of the polygon defined by ϕ, $s(\phi)$ is the sign defined in (5.3), and $\dim S_d^{\text{trop}}(p_{0,d}; p_{0,1}, \ldots, p_{d-1,d}) = 0$ for all sequences $\{p_{0,1}, \ldots, p_{d-1,d}, p_{0,d}\}$. Recall from 5.2.7 that if a single lift $\widetilde{p}_{i,j}$ is fixed, one can think of $S_d^{\text{trop}}(p_{0,d}; p_{0,1}, \ldots, p_{d-1,d})$ as the set of all lifts to $T(\mathbb{R}/d\mathbb{Z})$ of the remaining points $p_{0,1}, \ldots, \widehat{p}_{i,j}, \ldots, p_{0,d} \in \mathbb{R}/d\mathbb{Z}$, making the inner sum of (5.4) countably infinite with the outer sum finite.

Example 5.3.2. Let $B = \mathbb{R}/d\mathbb{Z}$ and consider the sum

$$m_{2,0}(p_{0,1} \otimes p_{1,2}) = \sum_{B(\frac{1}{n_2 - n_0}\mathbb{Z})} \sum_{S_2^{\text{trop}}(p_{0,2}; p_{0,1}, p_{1,2})} (-1)^{s(\phi)} q^{\deg \phi} \in \text{Hom}(L_0, L_2)$$

In this example $\pi(\widetilde{p_{i,j}}) = \pi(\widetilde{p_{i,j}} + dk) \mod d\mathbb{Z}$, but the triangles give distinct contributions to the sum.

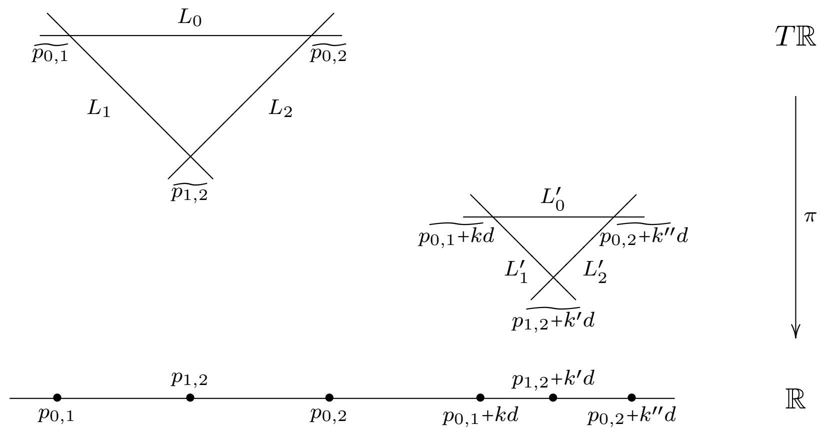


Figure 5.5: Two of the countably infinite number of triangles contributing to the coefficient $\sum_{S_2^{\text{trop}}(p_{0,2};p_{0,1},p_{1,2})}(-1)^{s(\phi)}q^{\deg\phi}$ of $p_{0,2}$. The lines L and L' are distinct in $T\mathbb{R}$, but are equivalent modulo Λ.

The maps m_d can be composed as follows. Let $l+k-1=d$ and write $S_k^{\text{trop}}, S_l^{\text{trop}}$ for

$$S_k^{\text{trop}}(p_{i-1,k+i-1};p_{i-1,i},\dots,p_{k+i-1,k+i-1})$$

and

$$S_l^{\text{trop}}(p_{0,d};p_{0,1},\dots,p_{i-2,i-1},p_{i-1,k+i-1},p_{k+i-1,k+i},\dots,p_{d-1,d}),$$

respectively, where each space is defined by fixing the same lift of $p_{i-1,k+i-1}$.

The map $m_{l,0}\circ_i m_{k,0}:\text{Hom}(L_0,L_1)\otimes\cdots\otimes\text{Hom}(L_{d-1},L_d)\longrightarrow\text{Hom}(L_0,L_d)$ takes the form

$$\begin{aligned}
&m_{l,0}\circ_i m_{k,0}(p_{0,1},\dots,p_{d-1,d})\\
=\;& m_{l,0}(p_{0,1},\dots,p_{i-2,i-1},m_{k,0}(p_{i-1,i},\dots,p_{k+i-2,k+i-1}),p_{k+i-1,k+i},\dots,p_{d-1,d})\\
=\;& \sum_{B(\frac{1}{n_{k+i-1}-n_{i-1}}\mathbb{Z})}\sum_{S_k^{\text{trop}}}(-1)^{s(\phi)}q^{\deg\phi}m_{l,0}(p_{0,1},\dots,p_{i-2,i-1},p_{i-1,k+i-1},p_{k+i-1,k+i},\dots,p_{d-1,d})\\
=\;& \sum_{B(\frac{1}{n_{k+i-1}-n_{i-1}}\mathbb{Z})}\sum_{S_k^{\text{trop}}}(-1)^{s(\phi)}q^{\deg\phi}\sum_{B(\frac{1}{n_d-n_0}\mathbb{Z})}\sum_{S_l^{\text{trop}}}(-1)^{s(\varphi)}q^{\deg\varphi}p_{0,d}\\
=\;& \sum_{B(\frac{1}{n_d-n_0}\mathbb{Z})}\sum_{B(\frac{1}{n_{k+i-1}-n_{i-1}}\mathbb{Z})}\sum_{(\phi,\varphi)\in S_k^{\text{trop}}\times S_l^{\text{trop}}}(-1)^{s(\phi)+s(\varphi)}q^{\deg\phi+\deg\varphi}p_{0,d}
\end{aligned}$$

Compositions are defined by summing over pairs of zero-dimensional tropical Morse trees, which are fitted together by matching the output of the first with the correct input of the second. This is why the same lift is chosen to define both moduli spaces.

Each of these pairs corresponds to a boundary point of a 1-dimensional moduli space of tropical Morse trees with, in this case, $k+l-1$ inputs. Since the moduli spaces in question are 1-dimensional, their boundary points come in pairs. The A_∞-relations among the $m_{d,0}$'s are then obtained by assigning signs to trees in such a way that the two boundary points of a fixed 1-dimensional moduli space have opposite signs. Pairwise cancelation yields the result.

The proof that the $m_{n,0}$'s satisfy the A_∞-relations is done by considering the identification made at the conclusion of section 5.2.1, and then using lemma 3.6 of [2].

Example 5.3.3. Consider the compositions

$$m_{3,0} \circ_3 m_{2,0}, m_{2,0} \circ_1 m_{3,0} : \bigotimes_{i=1}^{4} \mathrm{Hom}(L_{i-1}, L_i) \longrightarrow \mathrm{Hom}(L_0, L_4),$$

and the elements $m_{3,0}(p_{0,1}, p_{1,2}, m_{2,0}(p_{2,3}, p_{3,4})), m_{2,0}(m_{3,0}(p_{0,1}, p_{1,2}, p_{2,3}), p_{3,4})$, illustrated by the first and last graphs of Figure 4.6, and by the two graphs of Figure 4.7.

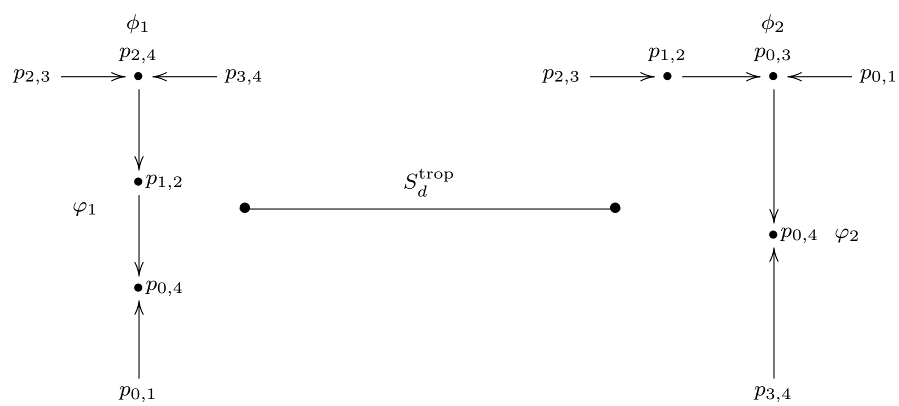


Figure 5.7: These degenerate trees contribute to $m_{3,0}(p_{0,1}, p_{1,2}, m_{2,0}(p_{2,3}, p_{3,4}))$ and $m_{2,0}(m_{3,0}(p_{0,1}, p_{1,2}, p_{2,3}), p_{3,4})$, respectively. If (ϕ_1, φ_1) corresponds to the first tree and (ϕ_2, φ_2) corresponds to the second, then $s(\phi_1) + s(\varphi_1) + \deg p_{0,1} + \deg p_{1,2} - 2 + s(\phi_2) + s(\varphi_2) = 1 \mod 2$, and $\deg \phi_1 + \deg \varphi_1 = \deg \phi_2 + \deg \varphi_2$.

5.4 Quantum A_∞ Categories

Definition 5.4.1. A *Quantum A_∞-Category* $\mathcal{A}$ is a collection of objects $\mathrm{Ob}\mathcal{A}$, a $\mathbb{Z}$-graded k-vector space $\mathrm{Hom}_{\mathcal{A}}(X_0, X_1)$ for any $X_0, X_1 \in \mathrm{Ob}\mathcal{A}$, and for every pair of integers d, b such that $d \geq 1$ and $b \geq 0$, a finite sequence of cyclic chains of objects

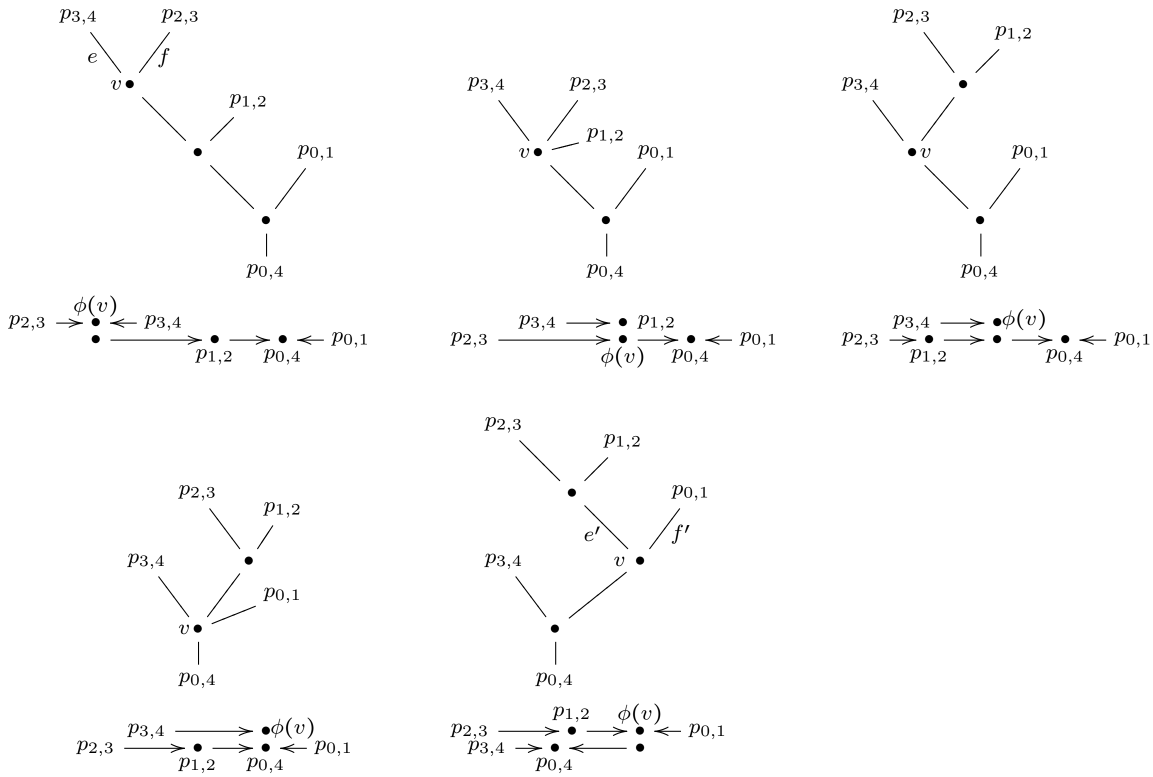


Figure 5.6: The first and last graphs represent the endpoints of the one-dimensional space $S_4^{\text{trop}}(p_{0,4}; p_{0,1}, p_{1,2}, p_{2,3}, p_{3,4})$, as $\mathbf{v}_e(1) + \mathbf{v}_f(1) = 0 = \mathbf{v}_{e'}(1) + \mathbf{v}_{f'}(1)$. The legs labeled by $p_{1,2}$ and $p_{0,4}$ are contracted, fixing their respective internal vertices. Notice the change in topology of the domain of ϕ as $\phi(v)$ moves left to right in B.

$\{X_{i0}, \ldots, X_{id_i}\}_{i=1}^{b-2g+1}$ giving rise to a tensor

$$\overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b-2g+1}) \in \bigotimes_{i=1}^{b-2g+1} \bigotimes_{j=1}^{d_i} \operatorname{Hom}(X_{ij}, X_{i(j-1)}) \otimes \operatorname{Hom}(X_{i0}, X_{id_i})$$

of degree $(d+1) - (2-2b)$, satisfying the relation (4.24), where, as in Definition 4.3.1, $b - 2g + 1 = i_\sigma$ is the number of disjoint cycles comprising $\sigma = \sigma_1 \cdots \sigma_{b-2g+1}$, $b_1 + b_2 = b$, $d + 1 = \sum_{i=1}^{b-2g+1} |\sigma_i|$, and $\hat{m}_{d+1,b}((\sigma_1 \cdots \sigma_{b+1})[d+1-(2-2b)]^*) = \overline{m}_{d,b}(\sigma_1 \cdots \sigma_{b+1})$.

In section 4 the notion of a quantum A_∞ algebra was generalized from that of an A_∞ algebra by replacing stable trees with stable graphs. Analogously, in order to generalize A_∞-precategories to quantum A_∞-precategories, one must replace tropical Morse trees with *tropical Morse graphs*.

Chapter 6

Tropical Morse Graphs

In this chapter we define a generalization of tropical Morse trees, called tropical Morse graphs, and formulate and prove a formula for the dimension of moduli spaces of such graphs.

6.1 Definition and Basic Properties

Definition 6.1.1. Let $G^{\text{trop}}_{d,b}(p_1, p_2, \ldots, p_{d+1})$ be the set of continuous maps $\phi : G \longrightarrow B$ from metric ribbon graphs G of genus b, each with a collection of affine displacement vectors satisfying properties 1,2,3,4, and 6 in the definition of tropical Morse trees. Replace property 5 with

(5*) Fix an orientation on G. If v is an internal vertex of G with incoming edges $e_1, \ldots, e_p$, and outgoing edges $f_1, \ldots, f_q$, then

$$\sum_{i=1}^{p} \mathbf{v}_{e_i}(v) = \sum_{j=1}^{q} \mathbf{v}_{f_j}(v)$$

The notation $G^{\text{trop}}_{d,b}(p_1, p_2, \ldots, p_{d+1})$ differs from that of $S^{\text{trop}}_{d}(p_{0,1}, p_{1,2}, \ldots; p_{0,d})$ to reflect the fact that the points labeling the external legs of a given tropical Morse graph are organized into $b+1$ disjoint cycles, as opposed to a single cycle as in the tropical Morse tree case. We have $G^{\text{trop}}_{d,0} = S^{\text{trop}}_{d}$.

Remark 6.1.2. Internal edges are not allowed to collapse. More specifically, if $\phi : G \longrightarrow B$ is such that the balancing condition at a certain vertex forces $\phi|_e$ to be constant, then the domain G is redefined by contracting e.

Definition 6.1.3. The fatgraph of a graph G is a thickening of the edges and legs of G. One replaces the edges with rectangles and glues them according to the given cyclic order at each vertex.

Let G be a stable graph. Orient the boundary of the fatgraph so that as the boundary is traversed, the interior lies on the left. The vertices bounding the external legs correspond to points of intersection between the chosen set of Lagrangians. The legs therefore partition the boundary into segments, each labeled with L_{n_j} for some j. If an external vertex v, labeled with $p_{i,j}$, is a transition point from L_{n_i} to L_{n_j}, and the relevant edge is oriented away from v, then label that leg with $n_j - n_i$. If the edge is oriented toward v, then label the edge with $n_i - n_j$.

Similarly, label the internal edges according to which boundary segments bound the edge. Let e be such an edge bounded by segments labeled L_{n_i} and L_{n_j}. If the orientation of the segment of the boundary of the fatgraph labeled L_{n_j} agrees with the orientation of the internal edge, then label the internal edge with $n_j - n_i$. Otherwise, label it $n_i - n_j$. The corresponding holomorphic maps (holomorphic annuli in the $b = 1$ case) are constructed in essentially the same way as holomorphic disks are constructed from tropical Morse trees. The signs are defined in precisely the same way as well.

6.1.1 The Lagrangian Condition

For any external edge labeled by e, $-\mathbf{v}_e(s) = \sigma_{n_j}(\phi(s)) - \sigma_{n_i}(\phi(s)) = -n_j\phi(s) - (-n_i\phi(s)) = (-n_j + n_i)\phi(s) \mod \Lambda$, so $\mathbf{v}_e(s) = (n_j - n_i)\phi(s) = n_e\phi(s) \mod \Lambda$. Call this last equality the *Lagrangian condition* for the edge e. In the tree case, the balancing conditions force $\mathbf{v}_e(s) = n_e\phi(s) \mod \Lambda$ for the internal edges.

Example 6.1.4. Consider the general tree contributing to $m_2 \circ_1 m_2$. Label the incoming edges e_i for $1 \le i \le 3$, and label the single internal edge e. Let $p_{i-1,i}$ be the input on e_i. The balancing condition at the first internal vertex v is

$$\begin{aligned}
\mathbf{v}_e(v) &= \mathbf{v}_{e_1}(1) + \mathbf{v}_{e_2}(1) \\
&= n_{e_1}(\phi(v) - p_{0,1}) + n_{e_2}(\phi(v) - p_{1,2}) \\
&= (n_{e_1} + n_{e_2})\phi(v) - n_{e_1}p_{0,1} - n_{e_2}p_{1,2} \\
&= n_e\phi(v) \mod \Lambda,
\end{aligned} \tag{6.1}$$

where $n_{e_i}p_{i-1,i} \in \mathbb{Z}$, since $p_{i-1,i} \in B(\frac{1}{n_i - n_{i-1}}\mathbb{Z})$. Furthermore, the balancing condition at the second internal vertex w yields

$$\begin{aligned}
\mathbf{v}_{e_{\text{out}}}(w) &= \mathbf{v}_e(w) + \mathbf{v}_{e_3}(w) \\
&= \mathbf{v}_e(v) + n_e(\phi(w) - \phi(v)) + n_{e_3}(\phi(w) - p_{2,3}) \qquad (6.2) \\
&= n_e\phi(v) - n_{e_1}p_{0,1} - n_{e_2}p_{1,2} + n_e\phi(w) - n_e\phi(v) + n_{e_3}(\phi(w) - p_{2,3}) \\
&= (n_e + n_{e_3})\phi(w) - n_{e_1}p_{0,1} - n_{e_2}p_{1,2} - n_{e_3}p_{2,3} \\
&= n_{e_{\text{out}}}\phi(w) \mod \Lambda
\end{aligned}$$

Not only is this the Lagrangian condition on the outgoing edge, but line 4 reveals it is in fact independent of the location of $\phi(v)$.

Let e be an edge labeled by $n_j - n_i$. The reasoning as to why the Lagrangian condition holds for an arbitrary edge in any TMT is given in detail on pages 631 and 632 of [3]. In short, the statement that the vector $\mathbf{v}_e(s)$ is bound at its tail by L_{n_i} and at its head by L_{n_j} for all $s \in e$, is equivalent to the statement that $R_e(e \times \{0\}) \subseteq L_{n_i}$ and $R_e(e \times \{1\}) \in L_{n_j}$.

In the graph case, one must impose the Lagrangian condition on an arbitrary edge within each generator of $\mathrm{H}_1(G)$. The balancing conditions then force the condition on each edge comprising each generator.

Proposition 6.1.5. Let G be trivalent. Imposing the Lagrangian condition on a single edge of each generator of $H_1(G)$ is enough to guarantee the condition holds throughout G, as long the condition is imposed once and only once per generator.

Lemma 6.1.6. Given an arbitrary vertex v in G, if all but one of the attached edges have the Lagrangian condition, then so does the remaining edge.

Proof. Orient the attached edges with the Lagrangian condition toward v, and the remaining edge, labeled e, away from v. Let $\{l_i\}$ be the subset of the attached edges which are external legs, and let $\{e_j\} = \mathrm{Leg}(v) - \{l_i\} - \{e\}$ be the remaining edges. Note that by the definition of a tropical Morse graph, $\sum n_{l_i} + \sum n_{e_j} = n_e$. The balancing condition at v yields

$$\begin{aligned}\mathbf{v}_e(0) &= \sum \mathbf{v}_{l_i}(1) + \sum \mathbf{v}_{e_j}(1) \\ &= \sum(\mathbf{v}_{l_i}(0) + n_{l_i}(\phi(v) - p_{i-1,i})) + \sum(\mathbf{v}_{e_j}(0) + n_{e_j}(\phi(v) - \phi(w_j))) \\ &= (\sum n_{l_i})\phi(v) - \sum n_{l_i}p_{i-1,i} + \sum \mathbf{v}_{e_j}(0) + (\sum n_{e_j})\phi(v) - \sum n_{e_j}\phi(w_j) \\ &= (\sum n_{l_i} + \sum n_{e_j})\phi(v) + \sum(\mathbf{v}_{e_j}(0) - n_{e_j}\phi(w_j)) - \sum n_{l_i}p_{i-1,i} \\ &= n_e\phi(v) \mod \Lambda,\end{aligned} \tag{6.3}$$

where e_j is bound by v and w_j. □

Proof. (of Proposition 6.1.5)

Let $S = \{c_i\}$ be the set of minimal generators of $H_1(G)$, and give each the counterclockwise orientation. Say that an edge or external leg of G is *marked* if it has the Lagrangian condition. Say that a generator is marked if at least one of its edges is marked. I claim there exists a vertex with no less than $n(v) - 1$ marked edges.

Suppose the claim does not hold for a graph G. Let $n'(v)$ be the number of marked edges emanating from the vertex v. Let n be the number of external legs of G. Each marked internal edge is bounded by two vertices, and all external edges are marked, so

$$\sum_{\mathrm{Vert}(G)} n'(v) = n + 2b_1(G)$$

For the sake of contradiction, suppose $n'(v) \leq n(v) - 2$ for all $v \in \mathrm{Vert}(G)$. Letting $E = |\mathrm{Edges}(G)|$ and $V = |\mathrm{Vert}(G)|$, one has

$$\begin{aligned} E - V + 1 &= b_1(G) \\ &= \frac{1}{2}(\sum n'(v) - n) \\ &\leq \frac{1}{2}(\sum(n(v) - 2) - n) \qquad (6.4) \\ &= \frac{1}{2}(\sum n(v) - 2V - n) \\ &= \frac{1}{2}(2E + n - 2V - n) \\ &= E - V, \end{aligned}$$

a contradiction.

So there exists a vertex v_1 such that $n'(v_1) \geq n(v_1) - 1$. If v_1 comprises part of a cycle, then it comprises part of a minimal generator. If $n'(v_1) = n(v_1)$, then this minimal generator is marked more than once, which is impossible. If v_1 does not comprise part of a cycle, then G is the unique graph with one vertex and n legs, as internal edges not comprising cycles are initially unmarked. It therefore must be the case that $n'(v_1) = n(v_1) - 1 = 2$.

I claim that upon finding and subsequently marking the remaining unmarked edges of k vertices with $n'(v) = 2$, there always exists another vertex v with $n'(v) = 2$. Suppose not. Then there are k vertices such that $n'(v) = n(v) = 3$ and $V - k$ vertices such that $n'(v)$ is either 0 or 1. Then

$$\begin{aligned} 2k + 2b_1 + n &= \sum_{\text{Vert}(G)} n'(v) \\ &\leq 3k + V - k \\ &= V + 2k, \end{aligned}$$

giving $2b_1 + n \leq V$. But $2b_1 - 2 + n = V$ by equation (2.10) of [17], so this is a contradiction. The entire graph can therefore be inductively marked by finding vertices v with $n'(v) = 2$ and using Lemma 6.1.6.

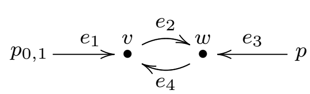
□

6.1.2 The Output is Determined

Let G be a tropical morse graph with n external legs. The image of each external vertex is determined by the images of the remaining external vertices.

Example 6.1.7.

Let G be



The balancing conditions are

i) $n_{e_1}(\phi(v) - p_{0,1}) + \mathbf{v}_{e_4}(0) + n_{e_4}(\phi(v) - \phi(w)) = \mathbf{v}_{e_2}(0)$

ii) $\mathbf{v}_{e_2}(0) + n_{e_2}(\phi(w) - \phi(v)) + n_{e_3}(\phi(w) - p) = \mathbf{v}_{e_4}(0)$

When $b_1(G) > 0$, $\phi|_G$ cannot be lifted to the universal cover, so ϕ is lifted edge by edge. In particular, the difference $\phi(p) - \phi(q)$, for p, q points on G, is viewed on the universal cover. So if e is a segment bounded by p and q, and $\phi|_e$ wraps around $\mathbb{R}/d\mathbb{Z}$ k times, then $\phi(p) - \phi(q)$ will have the form $\alpha + kd$, where α is the length of the directed segment between $\phi(p)$ and $\phi(q)$ in $\mathbb{R}/d\mathbb{Z}$.

Certainly these equations alone do not imply the Lagrangian condition on either edge, since $\mathbf{v}_{e_2}(0)$ and $\mathbf{v}_{e_4}(0)$ could be simultaneously replaced by $\mathbf{v}_{e_2}(0)+\epsilon$ and $\mathbf{v}_{e_4}(0)+\epsilon$. Solving the Lagrangian condition on e_2, $\mathbf{v}_{e_2}(s) = n_{e_2}\phi(s) \mod \Lambda$, with the first balancing condition, gives

$$(n_{e_1} + n_{e_4} - n_{e_2})\phi(v) - n_{e_1}p_{0,1} + \mathbf{v}_{e_4}(0) = -n_{e_1}p_{0,1} + \mathbf{v}_{e_4}(0) = n_{e_4}\phi(w) \mod \Lambda$$

As $n_{e_1} + n_{e_4} = n_{e_2}$, this is $\mathbf{v}_{e_4}(0) = n_{e_4}\phi(w) \mod \Lambda$.

Substituting the left hand side of the first equation for $\mathbf{v}_{e_2}(0)$ in the second gives

$$n_{e_1}\phi(v) - n_{e_1}p_{0,1} + \mathbf{v}_{e_4}(0) + n_{e_4}\phi(v) - n_{e_4}\phi(w) + n_{e_2}\phi(w) - n_{e_2}\phi(v) + n_{e_3}\phi(w) - n_{e_3}\phi(p) = \mathbf{v}_{e_4}(0)$$

Because $n_{e_1} + n_{e_4} = n_{e_2}$ and $n_{e_2} + n_{e_3} = n_{e_4}$ one has

$$\begin{aligned} n_{e_3}\phi(p) &= -n_{e_1}p_{0,1} + (n_{e_1} + n_{e_4} - n_{e_2})\phi(v) + (-n_{e_4} + n_{e_2} + n_{e_3})\phi(w) \\ &= -n_{e_1}p_{0,1} \end{aligned} \tag{6.5}$$

so $\phi(p) = \frac{-n_{e_1}p_{0,1}}{n_{e_3}} \mod d\mathbb{Z}$.

Theorem 6.1.8. Let e be the leg attached to the exceptional vertex p, and let e be oriented outward if $b_1(G) = 0$, and inward if $b_1(G) > 0$. If $p_{i-1,i}$ bounds e_i, then $\phi(p) = \frac{\sum n_{e_i}p_{i-1,i}}{n_e}$ if $b_1(G) = 0$, and $\phi(p) = \frac{\sum -n_{e_i}p_{i-1,i}}{n_e}$ if $b_1(G) > 0$.

Proof. Let e be an arbitrary edge of a contractible tropical morse graph T, bounded at its incoming end by v, and by w at its outgoing end. Then $\mathbf{v}_e(1) = \mathbf{v}_e(0) + n_e(\phi(w) - \phi(v))$ and $\mathbf{v}_e(0) = \sum \mathbf{v}_{e_i}(0) + n_{e_i}(\phi(v) - \phi(u_i))$ where each e_i feeds directly into v from u_i. So

$$\begin{aligned} \mathbf{v}_e(1) &= n_e\phi(w) + \sum \mathbf{v}_{e_i}(0) + (\sum n_{e_i} - n_e)\phi(v) - \sum n_{e_i}\phi(u_i) \qquad (6.6) \\ &= n_e\phi(w) + \sum \mathbf{v}_{e_i}(0) - \sum n_{e_i}\phi(u_i) \end{aligned}$$

If u_i is external, then $\mathbf{v}_{e_i}(0) = 0$ and $n_{e_i}\phi(u_i) = n_{e_k}p_{k-1,k}$ where $p_{k-1,k}$ is the k^{th} input. If u_i is internal, then there is a set of edges $\{e_j\}$ that feed into u_i, and

$$\begin{aligned} \mathbf{v}_{e_i}(0) &= \sum \mathbf{v}_{e_j}(1) \qquad (6.7) \\ &= \sum \mathbf{v}_{e_j}(0) + n_{e_j}(\phi(u_i) - \phi(u_j)) \end{aligned}$$

Substituting gives

$$\begin{aligned}-\mathbf{v}_{e_i}(0) + n_{e_i}\phi(u_i) &= -\sum \mathbf{v}_{e_j}(0) + (-\sum n_{e_j} + n_{e_i})\phi(u_i) + \sum n_{e_j}\phi(u_j) \qquad (6.8)\\ &= -\sum \mathbf{v}_{e_j}(0) + \sum n_{e_j}\phi(u_j)\end{aligned}$$

This process can be repeated inductively until each u_j is external. Let e_{out} be the outgoing edge bounded by v and p. Then

$$\begin{aligned}0 &= \mathbf{v}_{e_{\text{out}}}(1) \qquad (6.9)\\ &= \mathbf{v}_{e_{\text{out}}}(0) + n_{e_{\text{out}}}(\phi(p) - \phi(v))\end{aligned}$$

and therefore

$$\begin{aligned}n_{e_{\text{out}}}\phi(p) &= n_{e_{\text{out}}}\phi(v) - \mathbf{v}_{e_{\text{out}}}(0)\\ &= n_{e_{\text{out}}}\phi(v) - \sum \mathbf{v}_{e_i}(1)\\ &= n_{e_{\text{out}}}\phi(v) - \sum(\mathbf{v}_{e_i}(0) + n_{e_i}(\phi(v) - \phi(u_i))) \qquad (6.10)\\ &= (n_{e_{\text{out}}} - \sum n_{e_i})\phi(v) - \sum \mathbf{v}_{e_i}(0) + \sum n_{e_i}\phi(u_i)\\ &= -\sum \mathbf{v}_{e_i}(0) + \sum n_{e_i}\phi(u_i)\\ &= \sum n_{e_k}\phi(u_k)\\ &= \sum n_{e_k}p_{k-1,k}\end{aligned}$$

In the $b_1(G) > 0$ case, all of the external legs are oriented inward. Let e be the edge connected to the exceptional vertex p, bounded on the other end by v. Let $\{u_k\}$ be the set of internal vertices. Let $\mathbf{v}_{e_i}^k$ denote the vector $\mathbf{v}_{e_i}$ at u_k. The balancing condition at k can be written as

$$\sum_i \mathbf{v}_{e_i}^k = \sum_j \mathbf{v}_{e_j}^k \mod d\mathbb{Z} \qquad (6.11)$$

Although redundant, I will write 6.11 as $\sum_i \mathbf{v}_{e_i}^k(0) = \sum_j \mathbf{v}_{e_j}^k(1)$ for clarity. Summing the balancing conditions yields

$$n_e(\phi(v) - \phi(p)) + \sum_k\sum_i \mathbf{v}_{e_i}^k(1) = \sum_k\sum_j \mathbf{v}_{e_j}^k(0) \mod d\mathbb{Z}, \qquad (6.12)$$

where $\mathbf{v}_e(0) = 0$, so is not included. If $\mathbf{v}_{e_j}^k(0) \neq 0$, then u_k is internal, and there exists a summand $\mathbf{v}_{e_i}^l(1)$ on the left such that $e_i = e_j$ is bounded by u_k and u_l. As $\mathbf{v}_{e_i}^l(1) = \mathbf{v}_{e_i}^l(0)+n_{e_i}(\phi(u_l)-\phi(u_k))$, the terms $\mathbf{v}_{e_i}^k(0)$ can be canceled from (6.12) and this equation becomes $-n_e\phi(p) + n_e\phi(v) + \sum_k\sum_i n_{e_i}^k(\phi(v_k) - \phi(u_{k_i})) = 0$, where for each k, e_i begins at u_{k_i} and terminates at v_k. If u_{k_i} is external, then $\phi(u_{k_i}) = p_{j-1,j}$ for some j. Since $\sum n_{e_{\text{in}}} = \sum n_{e_{\text{out}}}$ for all internal vertices u_k,

$$\begin{aligned} n_e\phi(v) + \sum_k \sum_i n^k_{e_i}(\phi(v_k) - \phi(u_{k_i})) &= \sum_{\text{internal } u_k} (\sum_i n^k_{e_i})\phi(u_k) - \sum_{\text{internal } u_k} (\sum_j n^k_{e_j})\phi(u_k) \\ &\quad - \sum_{\text{external } u_k} n_{e_k} p_{k-1,k} \mod d\mathbb{Z} \\ &= - \sum_{\text{external } u_k} n_{e_k} p_{k-1,k} \mod d\mathbb{Z} \end{aligned} \tag{6.13}$$

Equation (6.12) becomes $-n_e\phi(p) = \sum_{\text{external } u_k} n_{e_k} p_{k-1,k}$. Notice the minus sign is a simply a result of writing $\phi(v) - \phi(p)$ instead of $\phi(p) - \phi(v)$ in (6.12). □

6.2 The Dimension of $G^{\text{trop}}_{d,b}(p_1, \ldots, p_{d+1})$

The dimension of $G^{\text{trop}}_{d,b}$ is equal to $\dim\{\phi : G \longrightarrow B\}$ for G a sufficiently general tropical morse graph with $d+1$ legs and $b = b_1$. This means $n(v) = 3$ and $b(v) = 0$ for all $v \in \text{Vert}(G)$. Let G be such a graph. Heuristically, the dimension should be the number of free variables minus the number of constraints.

Letting Vert(G) denote the set of internal vertices, the free variables consist of the elements $\{\phi(v)\}$ for $v \in \text{Vert}(G)$, the element $\phi(p)$ for p the exceptional vertex, and the vectors $\{\mathbf{v}_e(0)\}$ for $e \in \text{Edge}(G)$, as each of these vectors is free to vary in a 1-dimensional subspace of the corresponding fiber of TB. The constraints consist of the set of balancing conditions, the requirement that $\mathbf{v}_e(0) = n_e\phi(0) \mod \mathbb{Z}$ for each edge e, and the set of degree one external legs. Assuming all external legs are oriented inward, this translates here to

$$\begin{aligned} &|\{\phi(v)|v \in \text{Vert}(G)\}| + |\{\phi(p)\}| + |\{\mathbf{v}_e(0)\}| - |\{\text{balancing conditions}\}| - \\ &\qquad |\{\text{Lagrangian conditions}\}| - |\{\text{degree 1 external vertices}\}| = \\ &|\text{Vert}(G)| + 1 + |\text{Edge}(G)| - |\text{Vert}(G)| - b_1(G) - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}). \end{aligned}$$

Definition 6.2.1. Two legs of a graph G are said to be *adjacent* if they emanate from the same internal vertex.

Theorem 6.2.2. If $n_e \neq 0$ for all legs e, then $\dim G^{\text{trop}}_{d,b} = d-1+2b_1-(\deg p_{d,0} - \sum \deg p_{i-1,i})$

Proof. Let S be the set of marked edges of G. The balancing conditions and the lagrangian conditions give rise to a

$$(b_1 + |\text{Vert}(G)|) \times (|\text{Vert}(G)| + 1 + |\text{Edge}(G)|)$$

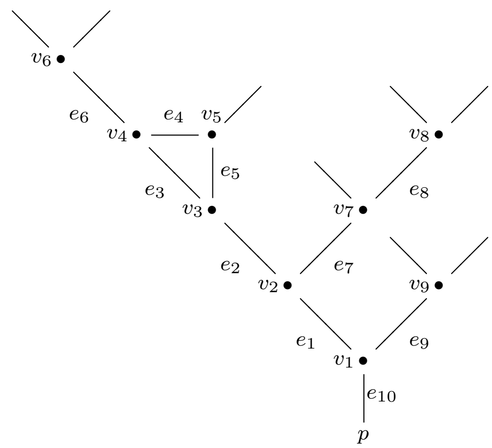
matrix A_G, whose columns are indexed by the elements

$$\{\phi(v)|v \in \mathrm{Vert}(G)\} \cup \{\phi(p)\} \cup \{\mathbf{v}_e(0)|e \in \mathrm{Edge}(G)\},$$

and whose rows are indexed by $\mathrm{Vert}(G) \cup \{l_e|e \in S\}$. Since $b_1 + |\mathrm{Vert}(G)| = |\mathrm{Edge}(G)| + 1$, A_G will have maximal rank if the square submatrix given by the rightmost $|\mathrm{Edge}(G)| + 1$ columns is invertible. Denote this square matrix by B_G.

The first row of B_G corresponds to the balancing condition at the unique internal vertex u which is connected to p. Order the remaining rows and all of the columns as follows. From u, choose one of the edges e connected to u, and traverse that edge until reaching the vertex which bounds it on its opposite end. The balancing condition at this second vertex gives the second row, and the second column is indexed by $\mathbf{v}_e(0)$. If e comprises part of a loop in $\mathrm{H}_1(G)$, then continue in this way around the minimal loop containing e. If not, then continue along an arbitrary path until it terminates with a vertex which is connected to an external leg. For the next vertex, backtrack from the current vertex until reaching the most recent junction, and move along the untraced edge to the next vertex. This new edge marks the next column, and the new vertex marks the new row. Continue in this way until all internal edges of G have been traversed. The marked edges will be those traversed last within their respective minimal loops.

Example 6.2.3.



	$\phi(p)$	$\mathbf{v}_{e_1}(0)$	$\mathbf{v}_{e_2}(0)$	$\mathbf{v}_{e_3}(0)$	$\mathbf{v}_{e_4}(0)$	$\mathbf{v}_{e_5}(0)$	$\mathbf{v}_{e_6}(0)$	$\mathbf{v}_{e_7}(0)$	$\mathbf{v}_{e_8}(0)$	$\mathbf{v}_{e_9}(0)$
v_1	$-n_{e_{10}}$	1	0	0	0	0	0	0	0	1
v_2	0	-1	1	0	0	0	0	1	0	0
v_3	0	0	-1	1	0	1	0	0	0	0
v_4	0	0	0	-1	-1	0	1	0	0	0
v_5	0	0	0	0	1	-1	0	0	0	0
v_6	0	0	0	0	0	0	-1	0	0	0
v_7	0	0	0	0	0	0	0	-1	1	0
v_8	0	0	0	0	0	0	0	0	-1	0
v_9	0	0	0	0	0	0	0	0	0	-1
l	0	0	0	0	0	-1	0	0	0	0

Figure 6.1: B_G for the given G. The vertices and edges are listed in the order in which they are traversed, starting from v_1.

If A_G has maximal rank, i.e. if $\text{rank}(A_G) = |\text{Vert}(G)| + b_1$, then the solution space of this system will have dimension

$$\begin{aligned} |\text{Vert}(G)| + 1 + |\text{Edge}(G)| - (|\text{Vert}(G)| + b_1) &= |\text{Edge}(G)| + 1 - b_1 \qquad (6.14) \\ &= 3(b_1 - 1) + d + 1 + 1 - b_1 \\ &= d - 1 + 2b_1 \end{aligned}$$

Let A_{ij} be the entry of A that lies in the ith row and the jth column, and let

$\tilde{A}_{ij}$ be the submatrix of A given by eliminating the ith row and the jth column. Then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

Let G be a trivalent graph with $b_1 > 0$, and let e be the marked edge of an arbitrary cycle. As the lagrangian condition l_e is $n_e \phi(v) = \mathbf{v}_e(0)$, the corresponding row of A_G is $l_e = (0 \cdots n_e \cdots 0 \cdots - 1_e \cdots 0)$, and considered as a row of B_G, takes the form $l_e = (0 \cdots - 1_e \cdots 0)$, where 1_e sits in the column corresponding to $\mathbf{v}_e(0)$. Then $\det(B_G) = (-1)^{i_e + j_e}(-1_e)(\det(\tilde{B}_G)_{i_e j_e})$, where -1_e sits in the i_eth row and j_eth column of B_G. The final b_1 rows of the matrix B_G all have this form, so

$$\det(B_G) = \pm \det C_G,$$

where C_G is the matrix resulting from eliminating the i_eth row and j_eth column of each of the last b_1 rows of B_G. The sign depends on the location of each 1_e.

Claim 6.2.4. The matrix C_G is upper triangular and is such that

$$(C_G)_{ii} = \begin{cases} -n_l & \text{if } i = 1 \\ \pm 1 & \text{if } i \neq 1 \end{cases}$$

where l is the external leg to which the exceptional vertex p is attached.

Proof. (of claim 6.2.4) Each edge e is bounded by two vertices, so a column vector marked by $\mathbf{v}_e(0)$ will consist of at most 3 nonzero entries, the third coming from the edge e possibly being marked.

The structure of these column vectors depends on when the edge is traversed in the sequence. There are three possibilities:

i) e is not the first edge traversed after having backtracked and is not the final edge traversed in a loop

ii) e is the first edge traversed after having backtracked

iii) e is the final edge traversed in a minimal loop

These vectors take the forms

$$\begin{pmatrix} * \\ \pm 1 \\ \mp 1 \\ * \\ * \\ * \\ * \end{pmatrix}, \quad \begin{pmatrix} * \\ \pm 1 \\ * \\ \mp 1 \\ * \\ * \\ * \end{pmatrix}, \quad \begin{pmatrix} * \\ \pm 1 \\ * \\ \mp 1 \\ * \\ -1 \\ * \end{pmatrix}$$

respectively, where $*$ represents some number of 0 entries. Indeed, if e is of the first type, then the vertices bounding this edge give rise to successive rows in B_G. If e is the first traversed after having backtracked, then there will be several rows, depending on how many edges were retraced, in between the rows indexed by the vertices bounding e. If e is the final edge traversed in a loop consisting of n edges, then there will be a gap of $n-1$ rows between the nonzero entries. The -1 entry near the end comes from the fact that edges of the final type are exactly the marked edges.

Let v_i be a vertex of G. The nonzero entries of the row indexed by v_i correspond to the edges which are connected to v_i. The first nonzero entry corresponds either to the edge traversed just before v_i, or just after. Because the first column is indexed by $\phi(p)$, the first nonzero entry of the row v_i will lie at or after the i^{th} spot. The matrix C_G is therefore upper triangular. Since $(B_G)_{11} = (C_G)_{11} = -n_l \neq 0$, it remains only to show $(C_G)_{ii} \neq 0$ for $2 \leq i \leq |\text{Vert}(G)|$.

Let e_n be an edge of the first type listed above, bounded by v_n and v_{n+1}. Depending on the orientations of the edges connected to v_{n+1}, either the vector $\mathbf{v}_{e_n}(0)$ or $\mathbf{v}_{e_n}(1)$ contributes to the balancing condition at v_{n+1}, so $(B_G)_{(n+1)(n+1)} = \pm 1$.

Now let e_n be the first edge traversed after having backtracked, bounded by v_i and v_j. Assume v_j is the one vertex of the two that has not yet been traversed. The sequence of vertices and edges has the form

$$\cdots \longrightarrow e_{n-1} \longrightarrow v_{j-1} \longrightarrow v_i \longrightarrow e_n \longrightarrow v_j \longrightarrow \cdots,$$

where v_i was traversed before v_{j-1}, so does not index a new row. Therefore, $(B_G)_{(j-1)(n)}$ and $(B_G)_{(j)(n+1)}$ are both nonzero. In other words, if i and j are such that $(B_G)_{ij}$ is the second nonzero entry of the j^{th} column, and the $(j+1)$st column is indexed by an edge which is the first to be traversed after having backtracked, then $(B_G)_{(i+1)(j+1)}$ is nonzero as well.

Let e_n be of the third type, and let e_{n-1} be the second to last edge traversed in the given minimal loop. The final entry of $\mathbf{v}_{e_{n-1}}(0)$ and the second entry of $\mathbf{v}_{e_n}(0)$ will lie in the same row, as e_{n-1} and e_n flow, in terms of the order in which they are traversed, through the same vertex. This means that the second entry of $\mathbf{v}_{e_n}(0)$ has coordinates $(i, i+k)$, where k is the number of minimal loops traversed, up to and including the one containing e_n. Now, since $(B_G)_{ij} = (C_G)_{(i)(j-k)}$ for $1 \le i \le |\mathrm{Vert}(G)|$, where k is the number of minimal loops traversed before column j, the second entries of all of the column vectors of B_G will slide into the diagonal upon taking cofactors at the nonzero entries of B_G corresponding to the marked edges.

□

Then $\det(B_G) = \pm n_l$ and since $n_l \neq 0$ for all external legs l, the matrix B_G is invertible, and $\mathrm{rank}(A_G) = d - 1 + 2b_1$.

Let l be a leg of G and recall that $\mathbf{v}_l(s) = \mathbf{v}_l(0) + n_l(\phi(s) - p_{i,j})$ must point in the same direction as $\phi'(s)$, where $p_{i,j}$ labels l in G. If $n_l < 0$, then $n_l(\phi(s) - p_{i,j})$ and $\mathbf{v}_l(s)$ point in opposite directions, so $|\mathbf{v}_l(0)| \geq |n_l(\phi(s) - p_{i,j})|$ for all $s \in [0,1]$. Since l is a leg of G, $\mathbf{v}_l(0) = 0$ by definition, so ϕ necessarily contracts l to $p_{i,j}$. The image under ϕ of the vertex bounding l on its opposite end is therefore determined. If $n_l > 0$ the image under ϕ of the vertex bounding l on its opposite end is unrestricted. Because $\deg p_{i,j} = 1$ if and only if $n_l < 0$, and $\deg p_{i,j} = 0$ if and only if $n_l > 0$, the dimension of $\dim G^{\mathrm{trop}}_{d,b}$ is given by

$$\begin{aligned}
\dim G^{\mathrm{trop}}_{d,b} &= |\mathrm{Vert}(G)| + 1 + |\mathrm{Edge}(G)| - |\mathrm{Vert}(G)| - b_1 - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}) \\
&= |\mathrm{Edge}(G)| + 1 - b_1 - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}) \\
&= 3(b_1 - 1) + d + 1 + 1 - b_1 - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}) \\
&= d - 1 + 2b_1 - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i})
\end{aligned}$$

□

Notice that when $b(v) = 0$ for all $v \in \mathrm{Vert}(G)$, we have $b = b_1 = -|\mathrm{Vert}(G)| + |\mathrm{Edge}(G)| + 1$, so the dimension formula can be written

$$\dim G^{\mathrm{trop}}_{d,b} = |\mathrm{Vert}(G)| - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}),$$

and the dimension is seen to be a function of the number of elements $\phi(v)$ for $v \in \mathrm{Vert}(G)$, which can vary in 1-dimensional subspaces of $B = S^1$.

There are two other versions of this same formula. Letting n be the total number of external legs, the dimension becomes

$$n - 2 + 2b_1 - (\deg p_{d,0} + \sum \deg p_{i-1,i})$$

If, in the original formula, the $n_d - n_0$ leg is oriented outward, then the formula changes as

$$\begin{aligned} n - 2 + 2b_1 - (\deg p_{d,0} + \sum \deg p_{i-1,i}) &= (d+1) - 2 + 2b_1 - (\deg p_{d,0} + \sum \deg p_{i-1,i}) \\ &= d - 1 + 2b_1 - (\deg p_{d,0} + \sum \deg p_{i-1,i}) \\ &= d - 1 + 2b_1 - (1 - \deg p_{0,d} + \sum \deg p_{i-1,i}) \\ &= d - 2 + 2b_1 + \deg p_{0,d} - \sum \deg p_{i-1,i} \end{aligned}$$

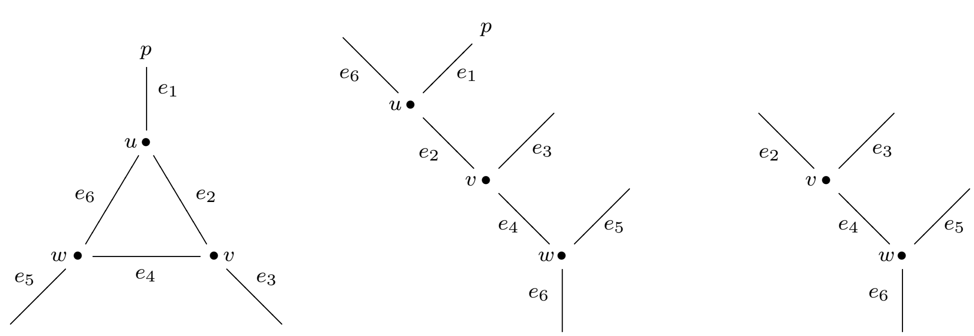
where

$$\deg p_{0,d} = \begin{cases} 0 & \text{if } n_d - n_0 > 0 \\ 1 & \text{if } n_d - n_0 < 0 \end{cases}$$

Remark 6.2.5. One actually has $p_{d,0} = p_{0,d}$. The labeling simply reflects the orientation of the relevant edge of S, where $\phi : S \longrightarrow B$ is the TMT in question.

Intuitively, taking the cofactor with respect to the single nonzero element of the row marked by l_e, can be visualized as breaking the edge e and considering the matrix of the resulting graph G', where $b_1(G') = b_1(G) - 1$. Note that since one is only comparing the matrices $B_{G'}$ and $\tilde{B}_G$, the images of the endpoints of the newly formed external legs are irrelevant.

Example 6.2.6. Let G, H, and K be the graphs



respectively.

The balancing conditions on G are

i) $n_{e_1}(\phi(u) - \phi(p)) + \mathbf{v}_{e_6}(0) + n_{e_6}(\phi(u) - \phi(w)) = \mathbf{v}_{e_2}(0)$

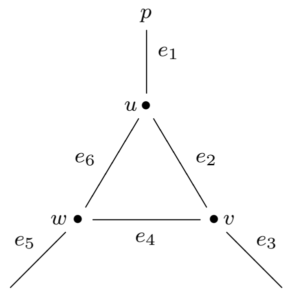
ii) $n_{e_3}(\phi(v) - p_{0,1}) + \mathbf{v}_{e_2}(0) + n_{e_2}(\phi(v) - \phi(u)) = \mathbf{v}_{e_4}(0)$

iii) $n_{e_5}(\phi(w) - p_{1,2}) + \mathbf{v}_{e_4}(0) + n_{e_4}(\phi(w) - \phi(v)) = \mathbf{v}_{e_6}(0)$

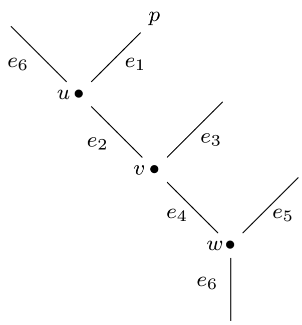
and the lagrangian condition is $n_{e_6}\phi(w) = \mathbf{v}_{e_6}(0)$. The corresponding matrix is

$$A = \begin{pmatrix} (n_{e_1} + n_{e_6}) & 0 & -n_{e_6} & -n_{e_1} & -1 & 0 & 1 \\ -n_{e_2} & (n_{e_2} + n_{e_3}) & 0 & 0 & 1 & -1 & 0 \\ 0 & -n_{e_4} & (n_{e_4} + n_{e_5}) & 0 & 0 & 1 & -1 \\ 0 & 0 & n_{e_6} & 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.15}$$

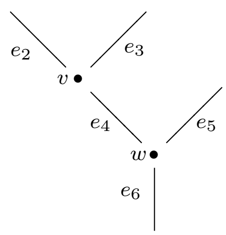
where the rows are labeled u, v, w, and l, and the columns are labeled $\phi(u)$, $\phi(v)$, $\phi(w)$, $\phi(p)$, $\mathbf{v}_{e_2}(0)$, $\mathbf{v}_{e_4}(0)$, and $\mathbf{v}_{e_6}(0)$.



	$\phi(p)$	$\mathbf{v}_{e_2}(0)$	$\mathbf{v}_{e_4}(0)$	$\mathbf{v}_{e_6}(0)$
u	$-n_{e_1}$	-1	0	1
v	0	1	-1	0
w	0	0	1	-1
l	0	0	0	-1



	$\phi(p)$	$\mathbf{v}_{e_2}(0)$	$\mathbf{v}_{e_4}(0)$
u	$-n_{e_1}$	-1	0
v	0	1	-1
w	0	0	1



	$\phi(p)$	$\mathbf{v}_{e_4}(0)$
v	$-n_{e_3}$	-1
w	0	1

Figure 6.2: Taking the cofactor with respect to the last row and the last column yields the matrix corresponding to the graph obtained from G by breaking the edge e_6. Similarly, taking the cofactor with respect to the first row and the second column yields the matrix corresponding to the graph obtained from H by breaking the edge e_2.

Chapter 7

Fuk($X(\mathbb{R}/d\mathbb{Z})$) as an Algebra over The Feynman Transform

This chapter is devoted to the main result of the thesis, i.e., the construction of a morphism from the Feynman transform of a twist of $\tilde{\mathbb{S}}[t]$ to $\mathcal{E}_L$, an operad built from the Lagrangian submanifolds of an elliptic curve. The final section describes a perturbation of the relevant moduli spaces, which could be used to extend the non-triviality of the above morphism to genus one.

7.1 The Operad $\mathcal{E}_L$

There is an anti-symmetric pairing $B : \mathrm{Hom}(L_i, L_j) \otimes \mathrm{Hom}(L_j, L_i) \longrightarrow \mathbb{C}[-1]$ defined by composing

$$\mathrm{Hom}(L_i, L_j) \otimes \mathrm{Hom}(L_j, L_i) \xrightarrow{m_2} \mathrm{Hom}(L_i, L_i) \simeq \mathrm{H}^*(L_i, \mathbb{C})$$

with

$$\mathrm{H}^*(L_i, \mathbb{C}) \simeq \mathrm{H}^*(S^1, \mathbb{C}) \xrightarrow{\text{proj}} \mathrm{H}^{top}(S^1, \mathbb{C}) = \mathbb{C}[-1]$$

The pairing B gives the degree 1 isomorphism

$$\gamma : \mathrm{Hom}(L_i, L_j) \longrightarrow \mathrm{Hom}(L_j, L_i)^* \tag{7.1}$$

$$p_{i,j} \mapsto (q_{j,i} \mapsto B(p_{i,j}, q_{j,i}))$$

This can be visualized as follows.

$$\begin{array}{ccccc} 1 & & 0 & & -1 \\ \mathrm{Hom}(L_i, L_j) & & \mathrm{Hom}(L_j, L_i) & & \\ & \overset{\gamma}{\searrow} & & \overset{\gamma}{\searrow} & \\ & & \mathrm{Hom}(L_j, L_i)^* & & \mathrm{Hom}(L_i, L_j)^* \end{array} \tag{7.2}$$

As V was constructed to mimic the behavior of the chain complexes $\mathrm{Hom}(L_i, L_j)$ and their dual spaces, the diagram bares resemblance to (4.11), but note there is no map between $\mathrm{Hom}(L_i, L_j)$ and $\mathrm{Hom}(L_j, L_i)$.

Because B is anti-symmetric, a convention must be chosen for the sign associated to B. The convention here will be $B(\deg 0, \deg 1) = +1$ and $B(\deg 1, \deg 0) = -1$, and this gives $(-1)^{|p_{i,j}|} B(p_{i,j}, p_{j,i}) = +1$ for $p_{i,j} \in \mathrm{Hom}(L_i, L_j)$.

As in Chapter 4, define the morphism

$$m_{d,0} : \bigotimes_{i=1}^{d} \mathrm{Hom}(L_{i-1}, L_i) \longrightarrow \mathrm{Hom}(L_0, L_d)$$

by

$$p_{0,1} \otimes \cdots \otimes p_{d-1,d} \mapsto \sum_{B(\frac{1}{n_d - n_0}\mathbb{Z})} \sum_{G_{d,0}^{\mathrm{trop}}} (-1)^{s(\phi)} q^{\deg \phi} p_{0,d},$$

where $s(\phi)$ and $\deg \phi$ are the sign and area of the holomorphic disk ϕ, respectively.

Consider the isomorphism

$$\begin{aligned} \varphi : & \bigotimes_{i=1}^{d} \mathrm{Hom}(L_i, L_{i-1}) \otimes \mathrm{Hom}(L_0, L_d) \longrightarrow \\ & \bigotimes_{i=1}^{d} \mathrm{Hom}(L_{i-1}, L_i)^* \otimes \mathrm{Hom}(L_0, L_d) \longrightarrow \\ & \mathrm{Hom}\Big(\bigotimes_{i=1}^{d} \mathrm{Hom}(L_{i-1}, L_i), \mathrm{Hom}(L_0, L_d)\Big) \end{aligned}$$

$$p_{1,0} \otimes \cdots \otimes p_{d,d-1} \otimes p_{0,d} \mapsto \Big(q_{0,1} \otimes \cdots \otimes q_{d-1,d} \mapsto (-1)^{\sum_{i=1}^{d} (d-i)|p_{i,i-1}|} \prod_{i=1}^{d} B(p_{i,i-1}, q_{i-1,i}) p_{0,d}\Big), \tag{7.3}$$

where the intermediate map is given by $(\bigotimes_{i=1}^{d} \gamma) \otimes \mathrm{id}$. Let $\overline{m}_{d,0} = \varphi^{-1}(\tilde{m}_{d,0})$, where $\tilde{m}_{d,0} \in \mathrm{Hom}(\bigotimes_{i=1}^{d} \mathrm{Hom}(L_{i-1}, L_i), \mathrm{Hom}(L_0, L_d))$ is defined by $\tilde{m}_{d,0} = (-1)^{|p_{0,d}|} m_{d,0}$ for $p_{0,d}$ an arbitrary element in $\mathrm{Hom}(L_0, L_d)$. This makes sense as $\mathrm{Hom}(L_0, L_d)$ is concentrated in a single degree.

The isomorphism (7.3) mimics the isomorphism

$$V^{\otimes(n+1)} \simeq (V^*)^{\otimes n} \otimes V \simeq \mathrm{Hom}(V^{\otimes n}, V)$$

$$(\bigotimes_{i=1}^{n} v_i) \otimes v_{n+1} \mapsto \bigotimes_{i=1}^{n}(\varphi(v_i)) \otimes v_{n+1} \mapsto (\bigotimes_{i=1}^{n} u_i \mapsto (-1)^{\sum_{i=1}^{n}(n-i)|v_i|} \prod B(v_i, u_i) v_{n+1})$$

from section 2.2.9, so φ^{-1} is given by

$$\begin{aligned} f \quad \mapsto & \sum_{\substack{\text{sequences} \\ p_{0,1},\dots,p_{d-1,d}}} (-1)^{\sum_{i=1}^{d}(d-i)|p_{i,i-1}|} (-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \otimes (-1)^{|p_{d,d-1}|} p_{d,d-1} \otimes f(p_{0,1} \otimes \cdots \otimes p_{d-1,d}) \\ = & \sum_{\substack{\text{sequences} \\ p_{0,1},\dots,p_{d-1,d}}} (-1)^{\sum_{i=1}^{d}(d-i)|p_{i,i-1}|} (-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \otimes (-1)^{|p_{d,d-1}|} p_{d,d-1} \otimes \left(\sum_{B(\frac{1}{n_d - n_0}\mathbb{Z})} \alpha_{0,d}\, p_{0,d} \right) \\ = & \sum_{\substack{\text{sequences} \\ p_{0,1},\dots,p_{d-1,d},p_{0,d}}} \alpha_{0,d} (-1)^{\sum_{i=1}^{d}(d-i)|p_{i,i-1}|} (-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \otimes (-1)^{|p_{d,d-1}|} p_{d,d-1} \otimes p_{0,d} \end{aligned}$$

and

$$\begin{aligned} \overline{m}_{d,0} \quad = & \quad \varphi^{-1}(\tilde{m}_{d,0}) \qquad (7.4) \\ = & \sum_{\substack{\text{sequences} \\ p_{0,1},\dots,p_{d-1,d},p_{0,d}}} \sum_{S_d^{\text{trop}}(p_{0,d};p_{0,1},\dots,p_{d-1,d})} (-1)^{s(\phi)} q^{\deg \phi} (-1)^{\sum_{i=1}^{d}(d-i)|p_{i,i-1}|} (-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \\ & \cdots \otimes (-1)^{|p_{d,d-1}|} p_{d,d-1} \otimes (-1)^{|p_{0,d}|} p_{0,d} \end{aligned}$$

The extra signs $(-1)^{|p_{1,0}|}, \dots, (-1)^{|p_{d,d-1}|}$ account for the anti-symmetry of B, and $(-1)^{|p_{0,d}|}$ is included by definition of $\tilde{m}_{d,0}$. Indeed,

$$\begin{aligned} \varphi(\overline{m}_{d,0})(q_{0,1} \otimes \cdots \otimes q_{d-1,d}) \quad = & \sum_{\substack{\text{sequences} \\ p_{0,1},\dots,p_{0,d}}} \sum_{S_d^{\text{trop}}} (-1)^{s(\phi)} q^{\deg \phi} (-1)^{|p_{0,1}|} p_{1,0} \otimes \cdots \\ & \cdots \otimes (-1)^{|p_{d,d-1}|} p_{d,d-1} \otimes (-1)^{|p_{0,d}|} p_{0,d}(q_{0,1} \otimes \cdots \otimes q_{d-1,d}) \\ = & \sum_{B(\frac{1}{n_d - n_0}\mathbb{Z})} \sum_{S_d^{\text{trop}}} (-1)^{s(\phi)} q^{\deg \phi} (-1)^{|q_{1,0}|} B(q_{1,0}, q_{0,1}) \cdots \\ & \cdots (-1)^{|q_{d,d-1}|} B(q_{d,d-1}, q_{d-1,d}) (-1)^{|p_{0,d}|} p_{0,d} \\ = & \sum_{B(\frac{1}{n_d - n_0}\mathbb{Z})} \sum_{S_d^{\text{trop}}(p_{0,d};q_{0,1},\dots,q_{d-1,d})} (-1)^{s(\phi)} q^{\deg \phi} (-1)^{|p_{0,d}|} p_{0,d} \\ = & \quad (-1)^{|p_{0,d}|} m_{d,0}(q_{0,1} \otimes \cdots \otimes q_{d-1,d}) \\ = & \quad \tilde{m}_{d,0}(q_{0,1} \otimes \cdots \otimes q_{d-1,d}), \qquad (7.5) \end{aligned}$$

where

$$(-1)^{|p_{j,j-1}|}B(p_{j,j-1},q_{j-1,j}) = \begin{cases} 0 & \text{if } q_{j-1,j} \neq p_{j-1,j} \\ 1 & \text{otherwise} \end{cases}$$

for $1 \leq j \leq d$.

Let $b \in \mathbb{Z}_{\geq 0}$ and let $\{L_{i0},\dots,L_{id}\}_{i=1}^{b+1}$ be a finite sequence of cyclic chains of Lagrangians. Define the element

$$\overline{m}_{d,b}(\sigma) \in \mathcal{E}_L((d+1,b)) = \bigotimes_{i=1}^{b+1}\bigotimes_{j=1}^{d_i} \mathrm{Hom}(L_{ij},L_{i(j-1)}) \otimes \mathrm{Hom}(L_{i0},L_{id_i}) \tag{7.6}$$

by

$$\overline{m}_{d,b}(\sigma) = \sum_{\substack{\text{sequences}\\ p_1,\dots,p_{d+1}}} \sum_{G^{\text{trop}}_{d,b}} (-1)^{s(\phi)} q^{\deg\phi} (-1)^{\sum_{i=1}^{d}(d-i)|p_i|} (-1)^{|p_1|} p_1 \otimes \cdots \otimes (-1)^{|p_{d+1}|} p_{d+1},$$

where $p_1,\dots,p_{d+1}$ can be partitioned into $b+1$ cycles $\sigma_1\cdots\sigma_{b+1} = \sigma$ according to the cyclic chains $\{L_{i0},\dots,L_{id_i}\}_{i=1}^{b+1}$.

The complexes $\mathrm{Hom}(L_i,L_j)$ are usually considered as cohomological complexes, and are related to the given homological complexes by $W^i = V_{1-i}$, where W represents $\mathrm{Hom}(L_i,L_j)$ as a cohomological complex and V as a homological complex.

One may break symmetry in $\mathcal{E}_L$ and use φ to write the element $\overline{m}_{d,b}(\sigma)$ as a map

$$\begin{aligned} \bigotimes_{i=1}^{b}\bigotimes_{j=1}^{d_i} \mathrm{Hom}(L_{i(j-1)},L_{ij}) \otimes \mathrm{Hom}(L_{i0},L_{id_i}) \otimes \bigotimes_{j=1}^{d_{b+1}} \mathrm{Hom}(L_{(b+1)(j-1)},L_{(b+1)j}) \\ \longrightarrow \mathrm{Hom}(L_{(b+1)0},L_{(b+1)d_{b+1}}) \end{aligned} \tag{7.7}$$

The appropriate form of the dimension formula to take when viewing this map cohomologically is

$$\dim G^{\text{trop}}_{d,b} = d - 2 + 2b + \deg p_{d+1} - \sum_{i=1}^{d} \deg p_i \tag{7.8}$$

When $\dim G^{\text{trop}}_{d,b} = 0$, Equation (7.6), and therefore the element $\overline{m}_{d,b}(\sigma)$, sits in degree

$$\begin{aligned} \deg p_{d+1} + \sum_{i=1}^{d} \deg p_i^{\vee} &= d + \deg p_{d+1} - \sum_{i=1}^{d} \deg p_i \\ &= d + (2 - d - 2b) \\ &= 2 - 2b, \end{aligned} \tag{7.9}$$

where $p^\vee$ is the element in $\mathrm{Hom}(L_j, L_i)$ dual to the element p in $\mathrm{Hom}(L_i, L_j)$. Considered as a homological tensor, (7.6) sits in degree $(d+1)-(2-2b)$.

For example, if $b=0$, then (7.7) takes the form

$$\mathrm{Hom}(L_0, L_1) \otimes \cdots \otimes \mathrm{Hom}(L_{d-1}, L_d) \longrightarrow \mathrm{Hom}(L_0, L_d),$$

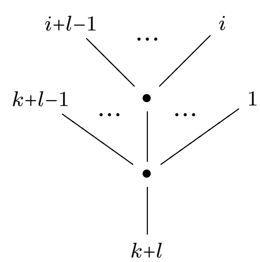
with

$$\dim G^{\mathrm{trop}}_{d,0} = d - 2 + \deg p_{0,d} - \sum_{i=1}^{d} \deg p_{i-1,i},$$

and

$$\begin{aligned} \deg p_{0,d} + \sum_{i=1}^{d} \deg p_{i,i-1} &= d + \deg p_{0,d} - \sum_{i=1}^{d} \deg p_{i-1,i} \\ &= d + (2-d) \\ &= 2, \end{aligned}$$

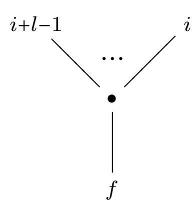
Letting H be the graph



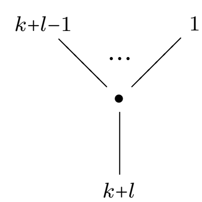
one has

$$\begin{aligned} \mathcal{E}_L((H)) = & \bigotimes_{j=i}^{i+l-1} \mathrm{Hom}(L_j, L_{j-1}) \otimes \mathrm{Hom}(L_{i-1}, L_{i+l-1}) \otimes \bigotimes_{m=1}^{i-1} \mathrm{Hom}(L_m, L_{m-1}) \otimes \\ & \bigotimes_{m=i+l}^{k+l} \mathrm{Hom}(L_m, L_{m-1}) \otimes \mathrm{Hom}(L_0, L_{k+l}), \end{aligned}$$

where $L_{i-1}, \ldots, L_{i+l-1}$ and $L_0, \ldots, L_{i-1}, L_{i+l}, \ldots, L_{k+l}$ bound the fatgraphs of



and



respectively, in TB.

Because $\mu_G^{\mathcal{E}_V}$ is defined by contraction via the anti-symmetric degree 1 pairing B, the appropriate twist for $\mathcal{E}_V$ is

$$\mathcal{K}^{-1}\mathcal{L} = \mathcal{K} \otimes \Big(\bigotimes_{e \in \mathrm{Edge}(G)} \bigwedge^{2} (\mathbb{C}^{\{f,f'\}}) \Big),$$

and setting $D^\vee = \mathcal{K}^{-1}\mathcal{L}$, gives $D = \mathcal{K}^2\mathcal{L}^{-1}$. This is precisely the correct twist for the modular operad $\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]$, and allows for the construction of a morphism of modular operads $\mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t] \longrightarrow \mathcal{E}_L$.

7.2 The Algebra Structure $\mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t] \longrightarrow \mathcal{E}_L$

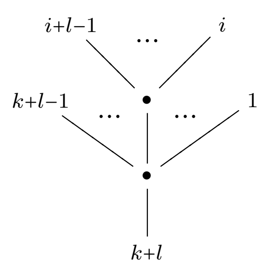
Theorem 7.2.1. There is a nontrivial morphism of modular operads

$$\Omega : \mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t] \longrightarrow \mathcal{E}_L$$
$$(\sigma_1\cdots\sigma_{b+1})[(d+1)-(2-2b)])^* \mapsto \overline{m}_{d,b}(\sigma_1\cdots\sigma_{b+1}), \tag{7.10}$$

where $\sum_{i=1}^{b+1} |\sigma_i| = d+1$.

Proof. The $b=0$ part of the proof follows the proof of Theorem 4.2.2, that is, it must be shown that for $b=0$, the morphism Ω gives an A_∞-structure to $\mathcal{E}_L$. Because $\mathrm{Hom}(L_i, L_j)$ is concentrated in a single degree for any two Lagrangians L_i and L_j, one has $m_{1,0} := d_{\mathcal{E}_L} = 0$, and the relation (4.8) reduces to one involving compositions arising strictly through the application of $\mu^{\mathcal{E}_L}$ to stable graphs with one non-intersecting edge. As in the proof of Theorem 4.2.2, write $\overline{m}_{n,0}$ for $\overline{m}_{n,0}(\sigma)$ for $n = k, l$, where the cycles on which $\overline{m}_{k,0}$ and $\overline{m}_{l,0}$ depend can be read directly from the graph H_i below.

Let H_i be the graph



Consider the following tensors.

$$\overline{m}_{l,0} = \sum_{\substack{\text{sequences}\\ r_{i-1,i},\dots,r_{i-1,i+l-1}}} \sum_{S_l^{\mathrm{trop}}(r_{i-1,i+l-1};r_{i-1,i},\dots,r_{i-1,i+l-1})} (-1)^{s(\varphi)} q^{\deg\varphi} (-1)^{\epsilon_1} (-1)^{|r_{i,i-1}|} r_{i,i-1} \otimes \cdots$$
$$\cdots \otimes (-1)^{|r_{i+l-1,i+l-2}|} r_{i+l-1,i+l-2} \otimes (-1)^{|r_{i-1,i+l-1}|} r_{i-1,i+l-1}$$

$$\overline{m}_{k,0} = \sum_{\substack{\text{sequences}\\ p_{0,1},\dots,p_{k+l-1,k+l},p_{0,k+l}}} \sum_{S_k^{\mathrm{trop}}(p_{0,k+l};p_{0,1},\dots,p_{k+l-1,k+l})} (-1)^{s(\phi)} q^{\deg\phi} (-1)^{\epsilon_2} (-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots$$
$$\cdots \otimes (-1)^{|p_{i-1,i-2}|} p_{i-1,i-2} \otimes (-1)^{|p_{i+l,i+l-1}|} p_{i+l,i+l-1} \otimes \cdots \otimes (-1)^{|p_{0,k+l}|} p_{0,k+l}$$

The signs $(-1)^{\epsilon_1}$ and $(-1)^{\epsilon_2}$ are given by

$$\epsilon_1 = \sum_{j=i}^{i+l-1} (l - (j - i + 1))|p_{j,j-1}|$$

and

$$\epsilon_2 = \sum_{j=1}^{i-1}(k - j)|p_{j,j-1}| + (k - i)|p_{i+l-1,i-1}| + \sum_{j=i+l}^{l+k-1}(k - (j - l + 1))|p_{j,j-1}|$$

The tensor $\overline{m}_l \otimes \overline{m}_k$ is an element of $\mathcal{E}_L((H_i))$. Recalling that $\mathcal{K}^{-1}\mathcal{L}(H) = \mathcal{K}(H) \otimes \bigwedge^2 \mathbb{C}^{\{f,f'\}} = \mathbb{C}[1] \otimes \bigwedge^2 \mathbb{C}^{\{f,f'\}} = \bigwedge^2 \mathbb{C}^{\{f,f'\}}[1]$, the diagrams (7.11) and (7.12) become

$$\begin{array}{ccc}
(\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((\mathrm{Leg}(v), b(v))))^* & \xrightarrow{\hat{m}} & \mathcal{E}_L((*_{d+1,0})) \\
\Big\downarrow{\scriptstyle d_{\mathscr{F}}} & & \Big\downarrow{\scriptstyle d_{\mathcal{E}_L}=0} \\
\bigoplus_{H/e\simeq *_{d+1,0}} (\mathcal{K}^{-1}\mathcal{L}(H) \otimes \beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((H))^*)_{\mathrm{Aut}(H)} & \xrightarrow{\hat{m}} & \mathcal{E}_L((*_{d+1,0}))
\end{array} \tag{7.11}$$

and

$$\begin{array}{ccc}
\mathcal{K}^{-1}\mathcal{L}(H) \otimes \beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((H))^* & \xrightarrow{\hat{m}} & \mathcal{K}^{-1}\mathcal{L}(H) \otimes \mathcal{E}_L((H)) \\
\Big\downarrow{\scriptstyle proj} & & \Big\downarrow{\scriptstyle \mu_H^{\mathcal{E}_L}} \\
(\mathcal{K}^{-1}\mathcal{L}(H) \otimes \beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((H))^*)_{\mathrm{Aut}(H)} & \xrightarrow{\hat{m}} & \mathcal{E}_L((*_{d+1,0}))
\end{array} \tag{7.12}$$

respectively.

Composing $\overline{m}_{l,0} \otimes \overline{m}_{k,0}$ via $\mu_{H_i}^{\mathcal{E}_L}$ yields

$$\begin{aligned}
& \mu_{H_i}^{\mathcal{E}_L}((f \wedge f')[1] \otimes \overline{m}_{l,0} \otimes \overline{m}_{k,0}) \\
= & \sum_{\substack{\text{sequences} \\ p_{0,1},\ldots,p_{i-1,i+l-1},\ldots,p_{0,k+l}}} \sum_{\substack{\text{sequences} \\ r_{i-1,i},\ldots,r_{i-1,i+l-1}}} \sum_{S_l \times S_k} (-1)^{s(\phi)+s(\varphi)} q^{\deg\phi+\deg\varphi} (-1)^{\epsilon_1+\epsilon_2} \\
& (-1)^{|p_{i+l-1,i-1}|+|r_{i-1,i+l-1}|}(-1)^{\epsilon} B(p_{i+l-1,i-1}, r_{i-1,i+l-1})(-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \\
& \cdots \otimes (-1)^{|p_{i-1,i-2}|} p_{i-1,i-2} \otimes (-1)^{|r_{i,i-1}|} r_{i,i-1} \otimes \cdots \otimes (-1)^{|r_{i+l-1,i+l-2}|} r_{i+l-1,i+l-2} \\
& \otimes \cdots \otimes (-1)^{|p_{0,k+l}|} p_{0,k+l} \\
= & (-1)^{|p_{i+l-1,i-1}|+|p_{i-1,i+l-1}|}(-1)^{\epsilon} \sum_{\substack{p_{0,1},\ldots,p_{i-1,i+l-1},\ldots,p_{0,k+l} \\ r_{i-1,i},\ldots,r_{i-1,i+l-1} \\ \text{s.t. } p_{i-1,i+l-1}=r_{i-1,i+l-1}}} \sum_{S_k \times S_l} (-1)^{s(\phi)+s(\varphi)} q^{\deg\phi+\deg\varphi} \\
& (-1)^{\epsilon_1+\epsilon_2}(-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \otimes (-1)^{|p_{i-1,i-2}|} p_{i-1,i-2} \otimes (-1)^{|r_{i,i-1}|} r_{i,i-1} \otimes \cdots \\
& \cdots \otimes (-1)^{|r_{i+l-1,i+l-2}|} r_{i+l-1,i+l-2} \otimes \cdots \otimes (-1)^{|p_{0,k+l}|} p_{0,k+l} \\
= & (-1)(-1)^{\epsilon}(-1)^{\epsilon_1+\epsilon_2} \sum_{\substack{p_{0,1},\ldots,\hat{p}_{i-1,i+l-1},\ldots,p_{0,k+l} \\ r_{i-1,i},\ldots,r_{i+l-2,i+l-1},r_{i-1,i+l-1}}} \sum_{S_k \times S_l} (-1)^{s(\phi)+s(\varphi)} q^{\deg\phi+\deg\varphi} \\
& (-1)^{|p_{1,0}|} p_{1,0} \otimes \cdots \otimes (-1)^{|p_{i-1,i-2}|} p_{i-1,i-2} \otimes (-1)^{|r_{i,i-1}|} r_{i,i-1} \otimes \cdots \\
& \cdots \otimes (-1)^{|r_{i+l-1,i+l-2}|} r_{i+l-1,i+l-2} \otimes \cdots \otimes (-1)^{|p_{0,k+l}|} p_{0,k+l},
\end{aligned}$$

where

$$\epsilon = |\overline{m}_{l,0}| + \sum_{j=1}^{i-1} |p_{1,0}| + (|\overline{m}_{k,0}| + 1 + i)|\overline{m}_{l,0}| + |\overline{m}_{k,0}| + 1 - i \tag{7.13}$$

This element corresponds to $(-1)(-1)^{\epsilon}(-1)^{\epsilon_1+\epsilon_2} \widetilde{m_{k,0} \circ_i m_{l,0}}$ under φ. Following (7.5), one has

$$\begin{aligned}
& \varphi(\mu_{H_i}^{\mathcal{E}_L}((f \wedge f')[1] \otimes (\overline{m}_{l,0} \otimes \overline{m}_{k,0})))(q_{0,1} \otimes \cdots \otimes q_{i-2,i-1} \otimes \cdots \\
& \cdots \otimes q_{i+l-2,i+l-1} \otimes \cdots \otimes q_{k+l-1,k+l}) \\
= & (-1)^{1+\epsilon+\epsilon_1+\epsilon_2}(-1)^{|p_{0,k+l}|} \sum_{B(\frac{1}{n_d-n_0}\mathbb{Z})} \sum_{B(\frac{1}{n_{i+l-1}-n_{i-1}}\mathbb{Z})} \sum_{S_l \times S_k} (-1)^{s(\phi)+s(\varphi)} q^{\deg\phi+\deg\varphi} p_{0,k+l} \\
= & (-1)^{1+\epsilon+\epsilon_1+\epsilon_2}(-1)^{|p_{0,k+l}|} m_{k,0} \circ_i m_{l,0}(q_{0,1} \otimes \cdots \otimes q_{k+l-1,k+l}) \\
= & (-1)^{1+\epsilon+\epsilon_1+\epsilon_2} \widetilde{m_{k,0} \circ_i m_{l,0}}(q_{0,1} \otimes \cdots \otimes q_{k+l-1,k+l})
\end{aligned}$$

Including the sign $(-1)^{\epsilon_3}$ corresponding to the contracted tensor

$$p_{1,0} \otimes \cdots \otimes p_{i-1,i-2} \otimes r_{i,i-1} \otimes \cdots \otimes r_{i+l-1,i+l-2} \otimes \cdots \otimes p_{0,k+l},$$

where $\epsilon_3 = \sum_{j=1}^{k+l-1} (k+l-1-j)|p_{j,j-1}|$, and converting back to cohomological signs gives

$$(-1)^{1+|p_{0,k+l}|}(-1)^{i-1+\sum_{j=1}^{i-1}|q_{j-1,j}|} m_{k,0} \circ_i m_{l,0}(q_{0,1} \otimes \cdots \otimes q_{k+l-1,k+l})$$

The contracted tensors $(-1)^{1+\epsilon+\epsilon_1+\epsilon_2+\epsilon_3}\overline{m}_{k,0}\circ_i\overline{m}_{l,0}$ cancel pairwise with the same signs as for the usual compositions $m_{k,0}\circ_i m_{l,0}$. Indeed, the extra signs $(-1)^{|p_{i+l-1,i-1}|+|p_{i-1,i+l-1}|}$ and $(-1)^{|p_{0,k+l}|}$ always cancel as $|p|+|q|=1$ for any pair p,q on which B is nonzero, and the compositions all have the same target, so $(-1)^{|p_{0,k+l}|}$ shows up in every term. The tensors $\overline{m}_{n,0}$ therefore satisfy the A_∞-relations using the signs of Seidel as in [36].

Without perturbing the moduli spaces $G^{\text{trop}}_{d,b}$, the map (7.10) is necessarily trivial for $b>0$. Consider the diagram

$$\begin{array}{ccc} \mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t](n,1) & \xrightarrow{\ \Omega\ } & \mathcal{E}_L(n,1) \\ \Big\downarrow{\scriptstyle d_{\mathscr{F}}} & & \Big\downarrow{\scriptstyle d_{\mathcal{E}_L}=0} \\ \mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t](n,1) & \xrightarrow[\ \Omega\ ]{} & \mathcal{E}_L(n,1) \end{array}$$

A basis element $\mathscr{G}=\sigma^*\in(\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t](n,1))^*\subseteq\mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t](n,1)$ will be such that $i_\sigma=b(v)+1=b+1=2$, and the map Ω is defined as

$$\Omega:\mathscr{F}_{\mathcal{K}^2\mathcal{L}^{-1}}\beta^2\tilde{\mathfrak{s}}\tilde{\mathbb{S}}[t]((n,1))\longrightarrow\mathcal{E}_L((n,1))$$

$$\begin{aligned}(\sigma\tau)[n]^* \quad\mapsto\quad & \sum_{\substack{\text{sequences}\\ p_1,\dots,p_n}}\ \sum_{G^{\text{trop}}_{n-1,1}}(-1)^{s(\phi)}q^{\deg\phi}(-1)^{|p_1|}p_1\otimes\cdots \\ & \cdots\otimes(-1)^{|p_{|\sigma|}|}p_{|\sigma|}\otimes(-1)^{|p_{|\sigma|+1}|}p_{|\sigma|+1}\otimes\cdots\otimes(-1)^{|p_{|\sigma|+|\tau|}|}p_{|\sigma|+|\tau|}\end{aligned}\tag{7.14}$$

Among the stable ribbon graphs in $\mathscr{F}_{\mathcal{L}}\tilde{\mathbb{S}}[t](n,1)$ which map to $\mathscr{G}$ are the ones with $b(v)=0$ and one self-intersecting edge with some legs in the loop and some outside of it. More specifically, since $i_\sigma=2$, σ may be written as a product of two disjoint cycles, one of which fixes the inner legs and acts transitively on the outer ones, or vice-versa.

The TMG's defining the summands of $\Omega\circ d_{\mathscr{F}}(\mathscr{G})$ are defined by labeling the fatgraphs of the stable ribbon graphs corresponding to these summands with a set of chosen Lagrangians in $X(B)$. For this reason, the stable ribbon graphs in question must have at least two legs in the loop and two out of it. This is because a single leg either in or out would have to be labeled by $n_e=0$, as the relevant lagrangian would intersect itself at the corner defined by the end of e. But because the sums of the inner labels and outer labels are both zero, two of the legs must be labeled with positive integers. This implies $\deg p_{d,0} + \sum \deg p_{i-1,i} \le n-2$, and we have

$$\begin{aligned}\dim G_{d,1}^{\text{trop}} &= n - 2 + 2b_1 - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}) \\ &= n - (\deg p_{d,0} + \textstyle\sum \deg p_{i-1,i}) \\ &\ge n - (d-2) \\ &= 2,\end{aligned}$$

So there is no possibility of having a one-dimensional space of such TMG's, which would give rise to relations between pairs of TMG's from a zero-dimensional space. The only possibility of having a new operation is if we allow all external legs to be labeled by 0. The new operation would be obtained by summing over TMG's with $b = 1$ and all inward and outward pointing external legs labeled by 0. If the inner and outer Lagrangians were equal, the element $\overline{m}_{n-1,1}$ would be given by covering the torus with annuli. □

7.3 A Perturbation

As the previous paragraph indicates, the moduli spaces $G_{d,b}^{\text{trop}}(p_1, \ldots, p_{d+1})$ should be not only generalized to allow for graphs whose legs can be labeled by zero, but perturbed to force the inputs into sufficiently general positions.

The objects of the new category are pairs $(L, \mathbf{v})$, where L is a Lagrangian given as before, $\mathbf{v}$ is a constant section of TB, and $(L, \mathbf{v})$ is taken to mean the Lagrangian $L + \epsilon\mathbf{v}$ in TB. The Hom-space $\text{Hom}((L_{n_i}, \mathbf{v}), (L_{n_j}, \mathbf{w}))$ is generated over Λ_{nov} by the points of $B(\frac{1}{n_j - n_i}\mathbb{Z})$, and sits in degree 0 if $n_j > n_i$ and degree 1 if $n_i > n_j$. If $n_i = n_j$ then $\text{Hom}((L_{n_i}, \mathbf{v}), (L_{n_j}, \mathbf{w}))$ is given by the chain complex $\mathbb{C}[B] \longrightarrow \mathbb{C}[p]$, where p is an arbitrarily chosen point of B and the differential m_1 is zero.

A sequence $\{(L_{n_1}, \mathbf{v}_1), \ldots, (L_{n_k}, \mathbf{v}_k)\}$ is transversal if $\mathbf{v}_1 < \cdots < \mathbf{v}_k$, and

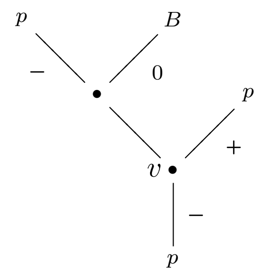
$$\begin{aligned}&\max\{|\mathbf{v}_j - \mathbf{v}_i| : 1 \le i < j \le k\} \\ < \;&\frac{1}{2}\min\{|p-q| : p \ne q \text{ and } p, q \in \bigcup_{\substack{i,j \\ i \ne j}} \pi((L_{n_i} + \mathbf{v}_i) \cap (L_{n_j} + \mathbf{v}_j))\}\end{aligned}$$

Every point of $\pi((L_{n_i} + \mathbf{v}) \cap (L_{n_j} + \mathbf{w}))$ can be written $p_{i,j} + (\mathbf{w} - \mathbf{v})$ for $p_{i,j} \in B(\frac{1}{n_j - n_i}\mathbb{Z}) = \pi(L_{n_i} \cap L_{n_j})$. The space $G_{k-1,b}^{\text{trop}}(p_1 + (\mathbf{v}_2 - \mathbf{v}_1), \ldots, p_k + (\mathbf{v}_{k+1} - \mathbf{v}_k))$ is defined as usual if $n_i \ne n_j$ for all adjacent Lagrangians L_{n_i} and L_{n_j} in a given cycle. If v is an external vertex of a graph bounding a leg labeled by $n_{i+1} - n_i = 0$, then $\phi(v)$ is unrestricted if

$\deg p + (\mathbf{v}_{i+1} - \mathbf{v}_i) = 0$, and $\phi(v) = p + (\mathbf{v}_{i+1} - \mathbf{v}_i)$ if $\deg p + (\mathbf{v}_{i+1} - \mathbf{v}_i) = 1$. Any external leg labeled by $n_e = 0$ is contracted.

The transversality condition ensures that the points $p_i + (\mathbf{v}_{i+1} - \mathbf{v}_i)$ defining the moduli spaces $G^{\text{trop}}_{k-1,b}(p_1 + (\mathbf{v}_2 - \mathbf{v}_1), \dots, p_k + (\mathbf{v}_{k+1} - \mathbf{v}_k))$ are always distinct, which ensures the actual and expected dimensions of $G^{\text{trop}}_{k-1,b}$ are equal. With these redefined moduli spaces, the category $\text{Fuk}(X(B))$ is now a genuine A_∞-precategory.

For example, if G is the graph



then ϕ contracts G to the point p, but the dimension formula gives $\dim\{\phi : G \longrightarrow B\} = 3 - 1 - 1 = 1$. By shifting slightly the inputs and output, the image of v becomes unrestricted, and $\phi(G)$ becomes a line segment.

The map Ω can now be defined nontrivially for $b = 1$ by sending $(\sigma\tau[|\sigma| + |\tau|])^*$ to $\overline{m}_{|\sigma|+|\tau|-1,1}$, defined by summing over the elements of the zero-dimensional space $G^{\text{trop}}_{|\sigma|+|\tau|-1,1}(p + (\mathbf{v}_2 - \mathbf{v}_1), \dots p + (\mathbf{v}_{k+1} - \mathbf{v}_k))$, where $|\sigma| + |\tau| = k$ and the points $p + (\mathbf{v}_2 - \mathbf{v}_1), \dots p + (\mathbf{v}_{k+1} - \mathbf{v}_k)$ are partitioned into two distinct cycles.

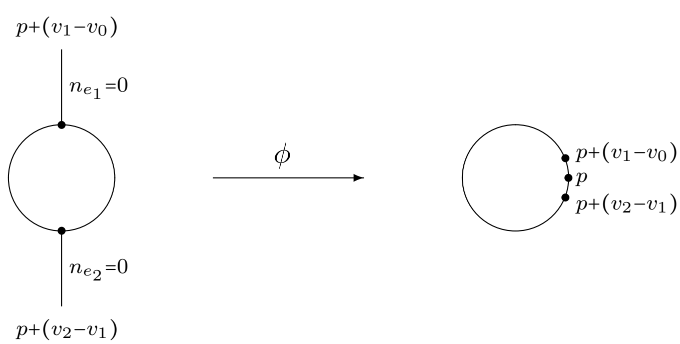


Figure 7.1: An example of a zero-dimensional tropical Morse graph G with $b = 1$, whose corresponding fatgraph is an annulus with inner and outer boundaries each lying on a single Lagrangian.

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