Title

A Multi-Interface Transmission Problem and Crack Approximation

Author

Wiegmann, Andreas

Publication Date

1999-02-01

LBNL-42875
Preprint

ERNEST ORLANDO LAWRENCE
BERKELEY NATIONAL LABORATORY

A Multi-Interface Transmission Problem and Crack Approximation

Andreas Wiegmann

Computing Sciences Directorate
Mathematics Department

February 1999

To be submitted
for publication.

DISCLAIMER

LBNL-42875

A MULTI-INTERFACE TRANSMISSION PROBLEM AND CRACK APPROXIMATION*

Andreas Wiegmann
Mathematics Department
Computing Sciences Directorate
Lawrence Berkeley National Laboratory
Berkeley, CA 94720

February 1999

*This work was supported in part by the Office of Science, Office of Computational and Technology Research, Mathematical, Information, and Computational Sciences Division, Applied Mathematical Sciences Subprogram, of the U.S. Department of Energy, under Contract No. DE-AC03-76SF00098.

Abstract

Analytic solutions to the Laplace equation in annulus and disk are combined with transmission conditions to find analytic solutions for multiple concentric circular interface transmission problems. Coefficients are found rapidly by solving 2×2 linear systems for each interface. The special case of two very close interfaces is used to quantitatively test crack jump conditions which result from approximately combining transmission conditions at the two interfaces into conditions at a single interface.

1 Introduction

Transmission problems arise in settings with composite materials. One equation models the behavior of a physical quantity (such as displacement, flow, etc.) but at least on of the coefficients is discontinuous across the interface between different materials.

In the inverse transmission problem, the location of the interface(s) is unknown and should be determined. To study the inverse problem ([1, 2, 3]), good knowledge and good solvers for the forward problem are needed. The efficiently computable solutions which are found here are useful as test cases for forward solvers (especially those that can deal with discontinuities) and can also be used to quantitatively evaluate the crack jump conditions found in [4, 5].

2 Analytic solutions for layered conductivities in a disk

We are interested in analytic solutions to

$$\nabla \cdot (\beta \nabla u) = 0 \quad \text{in } \mathbf{D}, \tag{1}$$

$$\frac{\partial u}{\partial \xi} = g \quad \text{on } \partial\mathbf{D} \text{ with } \int_{\partial \mathbf{D}} g = 0. \tag{2}$$

Here β is positive piecewise constant with k concentric circular interfaces inside the unit disk $\mathbf{D}$ and ξ denotes the outward normal directions on the boundary and on interfaces. The existence of a unique solution to this problem follows from the regularity of β and since β is bounded from below by a positive constant. Such solutions are useful to test our *approximations for crack-like inclusions* from [4, 5] and also our implementation of the *Explicit Jump Immersed Interface Method* from [4]. The procedure extends the idea in [3] in matching coefficients for the nth Fourier coefficient g_n and writing the solution for Neumann boundary conditions with Fourier coefficients[1], to multiple interfaces.

2.1 General solution

Consider k interfaces at $0 < s_1 < s_2 < \ldots < s_k < s_{k+1} = 1$ between constant conductivities β_0 in $\Omega_0 = \{r < s_1\}$, β_1 in $\Omega_1 = \{s_1 < r < s_2\}$, ..., β_{k-1} in $\Omega_{k-1} = \{s_{k-1} < r < s_k\}$ and[2] $\beta_k = 1$ in $\Omega_k = \{s_k < r < 1\}$. Define the contrasts $\rho_k = \beta_{k-1}/1$, $\rho_{k-1} = \beta_{k-2}/\beta_{k-1}$, ..., $\rho_1 = \beta_0/\beta_1$.

[1] For computations with real numbers, we use cosine expansions; due to $\int g = 0$ we have $g_0 = 0$.

[2] For normalization purposes we define the conductivity closest to the boundary as 1.

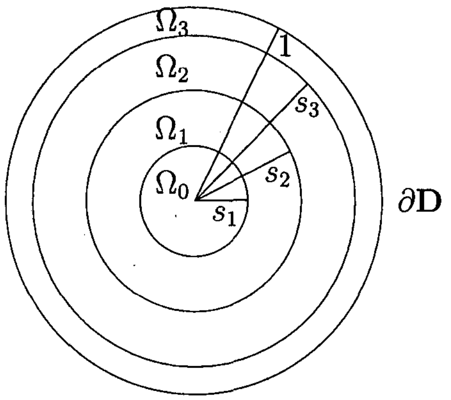


Figure 1: The unit disk, divided into four concentric layers by three interfaces of radii s_1, s_2 and s_3.

We make the following ansatz for the solution: $u(r,\theta) = u^{(i)}(r,\theta)$ in $\Omega_i (i = 0, 1, \ldots k)$ and

$$\frac{\partial u^{(k)}}{\partial \xi}(1,\theta) = \sum_{n\in\mathbf{N}} \cos(n\theta) g_n = \frac{\partial u}{\partial \xi} \qquad \text{on } \partial\mathbf{D}, \tag{3}$$

$$u^{(i)}(r,\theta) = \sum_{n\in\mathbf{N}} \cos(n\theta)\left(a_n^{(i)} r^n + b_n^{(i)} r^{-n}\right) \quad \text{in } \Omega_i,\ i = 1, 2, \ldots, k, \tag{4}$$

$$u^{(0)}(r,\theta) = \sum_{n\in\mathbf{N}} \cos(n\theta) f_n r^n \qquad \text{in } \Omega_0. \tag{5}$$

Continuity of u and βu_ξ across the interfaces and the boundary condition yield the following $2k+1$ equations ((8) and (9) for $i = 2, 3, \ldots, k$) in g_n, f_n and $a_n^{(i)}, b_n^{(i)}$ for $i = 1, 2, \ldots, k$

$$s_1^n a_n^{(1)} + s_1^{-n} b_n^{(1)} = f_n s_1^n, \tag{6}$$

$$n s_1^{n-1} a_n^{(1)} - n s_1^{-n-1} b_n^{(1)} = n\rho_1 f_n s_1^{n-1}, \tag{7}$$

$$s_i^n a_n^{(i)} + s_i^{-n} b_n^{(i)} = s_i^n a_n^{(i-1)} + s_i^{-n} b_n^{(i-1)}, \tag{8}$$

$$n s_i^{n-1} a_n^{(i)} - n s_i^{-n-1} b_n^{(i)} = n\rho_i \left(s_i^{n-1} a_n^{(i-1)} - s_i^{-n-1} b_n^{(i-1)}\right), \tag{9}$$

$$n a_n^{(k)} - n b_n^{(k)} = g_n. \tag{10}$$

We observe that g_n and $a_n^{(i)}, b_n^{(i)}$ for $i = 1, 2, \ldots, k$ are linear in f_n and that for given $\tilde{f}_n$ (for simplicity we use $\tilde{f}_n = 1$), the system decouples into k (2×2) systems with solutions

$$\tilde{b}_n^{(1)} = \frac{1-\rho_1}{2} s_1^{2n}, \tag{11}$$

$$\tilde{a}_n^{(1)} = \frac{1+\rho_1}{2}, \tag{12}$$

for $i = 2, 3, \dots, k$

$$\tilde{b}_n^{(i)} = \frac{1-\rho_i}{2} s_i^{2n} a_n^{(i-1)} + \frac{1+\rho_i}{2} b_n^{(i-1)}, \tag{13}$$

$$\tilde{a}_n^{(i)} = \frac{1+\rho_i}{2} a_n^{(i-1)} + \frac{1-\rho_i}{2} s_i^{-2n} b_n^{(i-1)}, \tag{14}$$

and finally

$$\tilde{g}_n = n\tilde{a}_n^{(k)} - n\tilde{b}_n^{(k)}. \tag{15}$$

For the case of a single interface (see [3]) and two interfaces (see below, §2.2) we have proved directly that $\tilde{g}_n \neq 0$ as long as $\rho > 0$, which by the next Lemma guarantees the unique solution to the original problem. The direct proof for the general case has evaded us so far, but we know that always $\tilde{g}_n \neq 0$ because a unique solution to for (1) and (2) is known to exist and has to satisfy our ansatz, (3)—(5).

By linearity in f_n, we have the following result.

Lemma 1 *The solution to the original system (6)-(10) with given g_n can be obtained as $(i = 1, 2, \dots, k)$*

$$f_n = g_n/\tilde{g}_n, \tag{16}$$

$$a_n^{(i)} = \tilde{a}_n^{(i)}/\tilde{g}_n, \tag{17}$$

$$b_n^{(i)} = \tilde{b}_n^{(i)}/\tilde{g}_n. \tag{18}$$

2.2 Exact solution for a problem with a crack-like double interface

The above solution for the case of two very close interfaces can be viewed as an analytic solution for a crack problem, that is a problem posed in a region including a thin layer of very different material properties. We will use it to probe both analytically and numerically the jump conditions for cracks derived in [5]. Denote the two interface locations by $s_- = s - \epsilon$ and $s_+ = s + \epsilon$, and let $\beta_0 = 1$, $\beta_1 = \rho$ and by convention, $\beta_2 = 1$. Then, in the previous notation, $\rho_1 = \rho^{-1}$ and $\rho_2 = \rho$. As a special case of Lemma 1, we find the following Corollary.

Corollary 2 *The solution for a crack-like double interface has coefficients*

$$f_n = \frac{4n^{-1}g_n}{((1+\rho^{-1})((1-\rho)s_+^{2n}+1+\rho)-(1-\rho^{-1})s_-^{2n}(1+\rho+(1-\rho)s_+^{-2n}))}, \tag{19}$$

$$a_n^{(1)} = \frac{1+\rho^{-1}}{2}f_n, \tag{20}$$

$$b_n^{(1)} = \frac{1-\rho^{-1}}{2}s_-^{2n}f_n, \tag{21}$$

$$a_n^{(2)} = \frac{1+\rho}{2}a_n^{(1)} + \frac{1-\rho}{2}s_+^{-2n}b_n^{(1)}, \tag{22}$$

$$b_n^{(2)} = \frac{1-\rho}{2}s_+^{2n}a_n^{(1)} + \frac{1+\rho}{2}b_n^{(1)}, \tag{23}$$

where $s_- = s-\epsilon$ *and* $s_+ = s+\epsilon$.

Proof. As long as we didn't show $\tilde{g}_n \neq 0$ in general, it remains to be shown that f_n exists. We check that the denominator in (19) is zero if and only if ρ is one of

$$\frac{-1+s_-^{2n}s_+^{-2n}+\sqrt{-4\,s_-^{2n}s_+^{-2n}+(s_+^{2n})^2+2\,s_+^{2n}s_-^{2n}+(s_-^{2n})^2}}{-s_+^{2n}+1-s_-^{2n}+s_-^{2n}s_+^{-2n}},$$

$$\frac{-1+s_-^{2n}s_+^{-2n}-\sqrt{-4\,s_-^{2n}s_+^{-2n}+(s_+^{2n})^2+2\,s_+^{2n}s_-^{2n}+(s_-^{2n})^2}}{-s_+^{2n}+1-s_-^{2n}+2\,s_-^{2n}s_+^{-n}}.$$

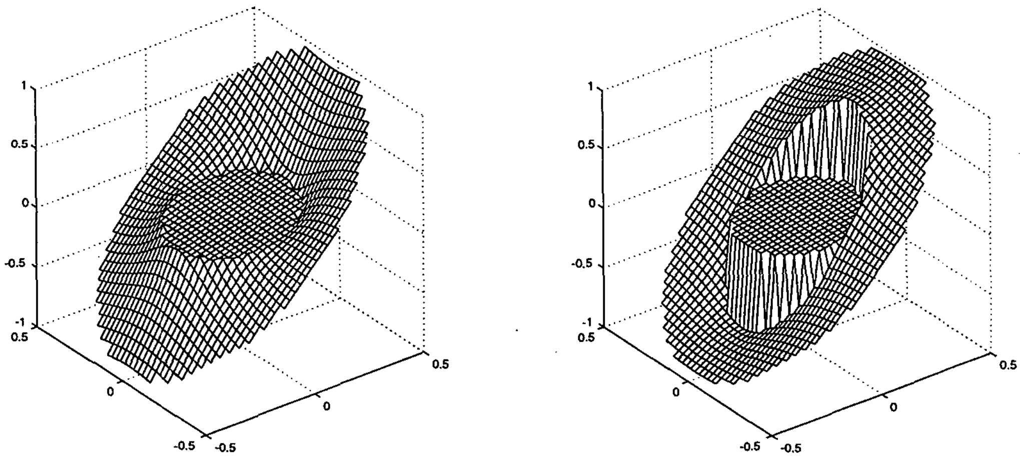
Observing that $1 > s_+ > s_- > 0$ it is easily seen that the denominator in these expressions is always positive, and that the argument of the root is always negative. This shows that f_n can be computed by (19) for any real ρ, and in particular positive ρ that we are considering. □

2.3 Approximation via a single interface

When representing the crack as a single interface, we will compare the approximation on the outside with $u^{(2)}$ and on the inside with $u^{(0)}$. So it is natural to consider the function $\bar{u}$ obtained by using the coefficients of $u^{(0)}$ on $\{r < s\}$, and those of $u^{(2)}$ on $\{s < r < 1\}$. For this function, seen for $s = 0.25$ and $\epsilon = 0.01$ in a conductive case (a) and a resistive case (b) in Figure 2, expansions in ϵ yield the following jumps:

$$[\bar{u}] = \left(\frac{s}{n}\frac{a_n^{(2)}}{f_n} + \frac{s}{n}\frac{b_n^{(2)}}{f_n}s^{-2n} - \frac{s}{n}\right)\underbrace{ns^{n-1}f_n\cos(n\theta)}_{=\bar{u}_\xi^-} = \left(2\frac{1-\rho}{\rho}\epsilon + O(\epsilon^2)\right)\bar{u}_\xi^-, \tag{24}$$

$$[\bar{u}_\xi] = \left(\frac{a_n^{(2)}}{f_n} - \frac{b_n^{(2)}}{f_n}s^{-2n} - 1\right)\underbrace{ns^{n-1}f_n\cos(n\theta)}_{=\bar{u}_\xi^-} = \left(2n\frac{\rho-1}{s}\epsilon + O(\epsilon^2)\right)\bar{u}_\xi^-. \tag{25}$$



a) b)

Figure 2: *Analytic solutions for* $g = \lambda\cos\theta$*, with* λ *chosen so that* $u(x,y) = x$ *on* $\partial\mathbf{D}$*, according to §2.3 in the a) conductive (*$\rho = 100$*) and b) resistive (*$\rho = 0.01$*) case. In both cases* $s = 0.25$ *and* $\epsilon = 0.01$.

This analytically known comparison function $\bar{u}$ and its jumps led to the modifications of the jump conditions from [4] to the jump conditions in [5]. The jump conditions in [4] agreed to $O(\epsilon)$ with (24) (error $-\epsilon u_\xi^-$) and (25) (error $-n\epsilon s^{-1}$), while those in [5] agree to $O(\epsilon^2)$. This improvement is most noticable for small s, or more generally, for large curvature. Note that (25) implies that superposition for $[\bar{u}_\xi]$ fails, so from this point on the analysis is only valid in each Fourier mode.

3 Analytic solutions for crack-approximations

In [5], we derived jump conditions describing the behavior of the solution to an elliptic equation in the presence of cracks. For the simple geometry considered in the previous sections and for boundary data consisting of a single Fourier mode, we can find analytic solutions satisfying these jump conditions, and compare them with the "exact jump conditions" for $\bar{u}$ derived above.

3.1 Conductive Crack

In Theorem 1 b) in [5] (for $\rho \gg 1$), the jumps are approximated as

$$[u] = 0, \tag{26}$$

$$[u_\xi] = -2\epsilon(\rho - 1)(u_\eta^-)'. \tag{27}$$

The approximation (26) for (24) is $O(\epsilon)$ since for large ρ we have $2(\rho^{-1}-1)\epsilon \approx -2\epsilon$. Recall that the arclength derivative along the interface $(u_\eta^-)'$ can also be expressed as $(u_\eta^-)' = u_{\eta\eta}^- - \theta' u_\xi^- = -u_{\xi\xi}^- - \theta' u_\xi^-$ where θ' is the curvature of the crack, and equal to $1/s$ for a circular crack of mean radius s. So (27) becomes

$$[u_\xi] = 2\epsilon(\rho-1)(u_{\xi\xi}^- + \theta' u_\xi^-).$$

Making the ansatz that for $r < s$ the solution u has the form $f_n \cos(n\theta) r^n$, we find that $u_{\xi\xi}^- = (n-1)/s\, u_\xi^-$, and using this and that for the circle $\theta' = 1/s$ we arrive at

$$[u_\xi] = 2\epsilon(\rho-1)\left(\frac{1}{s} + \frac{n-1}{s}\right) u_\xi^-. \tag{28}$$

Hence (27) matches (25) exactly up to $O(\epsilon^2)$. This is important because the $O(\epsilon^2)$ term contains the large factor ρ. Defining

$$\tilde{\rho} = 2n\frac{(\rho-1)}{s}\epsilon + 1,$$

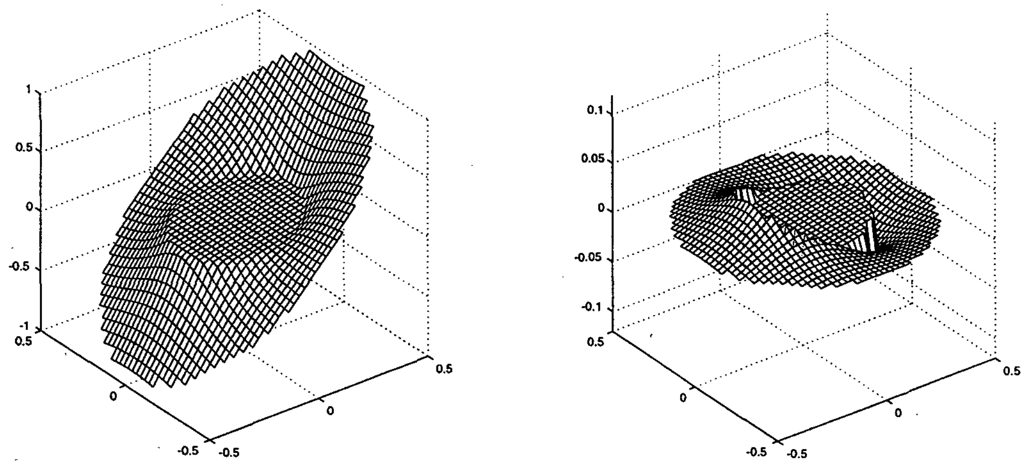
(28) is just the jump condition for a single interface case of the general solution considered in §2.1, with $\rho_0 = \tilde{\rho}$ and $s_1 = s$. We write the analytic solution for the conductive crack approximation as the solution of the single interface problem with "effective conductivity" $\rho_0 = \tilde{\rho}$ and using Lemma 1.

Corollary 3 *The conductive crack approximation has the following solution*

$$\begin{aligned} f_n &= \frac{2g_n}{n(1+\tilde{\rho}-(1-\tilde{\rho})s^{2n})}, \\ a_n &= (1+\tilde{\rho})f_n/2, \\ b_n &= (1-\tilde{\rho})s^{2n} f_n/2. \end{aligned}$$

Proof. Algebra shows that f_n exists except when $\tilde{\rho} = (s^{2n}-1)/(s^{2n}+1) < 0$. But the assumption that the crack is conductive, i.e. $\rho \geq 1$, which together with $0 < s < 1$ yields $\tilde{\rho} > 0$. □

Figure 3 a) shows the solution accoring to Corollary 3 with the same Dirichlet data, $s = 0.25$ and $\epsilon = 0.01$ as in Figure 2 a). Figure 3 b) shows the difference between the two. The magnitude of the error corresponds quite well to the fact that we do not model the discontinuity of the solution.



a) b)

Figure 3: *a) Analytic solution according to Corollary 3 and b) error when comparing with the analytic solution in Figure 2 a).*

3.2 Resistive Crack

In Theorem 1 a) in [5] (for $\rho \ll 1$), the jumps are approximated as

$$[u] = 2\epsilon \left(\rho^{-1} - 1\right) u_\xi^-, \tag{29}$$

$$[u_\xi] = 0. \tag{30}$$

The approximation (29) matches $[\bar{u}]$ in (24) exactly up to $O(\epsilon^2)$; the approximation (30) for (24) is $O(\epsilon)$. This is true because for small ρ we have $2n(\rho - 1)s^{-1}\epsilon \approx -2ns^{-1}\epsilon$. This example implies that s cannot be too small (i.e. the curvature of the crack should not be too small) and n cannot be too large (i.e. the boundary values have to be sufficiently regular) for the derivations in [5] to be valid.

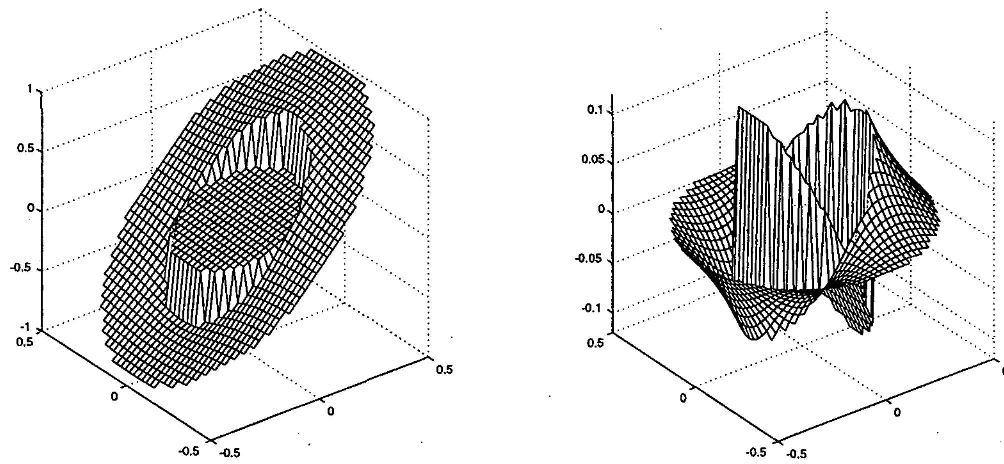
As in §2.1 we find that now the following equations hold

$$a_n s^n + b_n s^{-n} = \left(2\epsilon\left(\rho^{-1} - 1\right) + 1\right) s^n f_n,$$
$$s^{n-1} a_n - s^{-n-1} b_n = s^{n-1} f_n.$$

Now we combine this with the boundary condition (10).

Lemma 4 *The resistive crack approximation has the following solution*

$$f_n = \frac{g_n}{n((1 - s^{2n})\epsilon\left(\rho^{-1} - 1\right) + 1)}, \tag{31}$$
$$a_n = (\epsilon\left(\rho^{-1} - 1\right) + 1) f_n,$$
$$b_n = \epsilon\left(\rho^{-1} - 1\right) s^{2n} f_n.$$



a) b)

Figure 4: *a) Analytic solution according to Lemma 4 and b) error when comparing with the analytic solution in Figure 2 b).*

Proof. Again, we have to prove that f_n exists. But the assumption that the crack is resistivetive, $\rho \leq 1$, shows that the denominator in (31) is positive for any $\epsilon > 0$ and $0 < s < 1$. □

Figure 4 a) shows the solution accoring to Lemma 4 with the same Dirichlet data, $s = 0.25$ and $\epsilon = 0.01$ as in Figure 2 b). Figure 3 b) shows the difference between the two. The larger error of this approximation compared with the conductive case is due to the larger discontinuity in the solution.

4 Conclusion

We have found analytic solutions to certain multi-interface transmission problems. The special case of solutions for two very close interfaces was used to quantitatively test the crack jump conditions from [5]. The analysis here illustrates that the use of these conditions for conductive cracks shouls pose no problem, while for resistive cracks the quality of the approximation was much poorer than for the resistive cracks. Also, restrictions on the curvature of the crack and regularity of the solution in order to apply the crack conditions need to be carefully specified.

References

[1] K. M. Bryan. Numerical recovery of certain discontinuous electrical conductivities. *Inverse Problems*, 7:827—840, 1991.

[2] F. Hettlich and W. Rundell. The determination of a discontinuity in a conductivity from a single boundary measurement. *Inverse Problems*, 14:67—82, 1998.

[3] C. F. Tolmasky and A. Wiegmann. Recovery of small Perturbations of an Interface for an Elliptic Inverse Problem via Linearization. To appear in *Inverse Problems*, 1998.

[4] A. Wiegmann. *The Explicit–Jump Immersed Interface Method and Interface Problems for Differential Equations*. PhD thesis, University of Washington, 1998.

[5] A. Wiegmann, Z. Li, and R. J. LeVeque. Jump Conditions for Cracks in Elliptic Problems. To appear in Applied Mathematics Letters and appeared as LBNL Technical Report Nr. 41863.

Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

Prepared for the U.S. Department of Energy under Contract No. DE-AC03-76SF00098