String method of nonimaging optics from a radiation theory perspective

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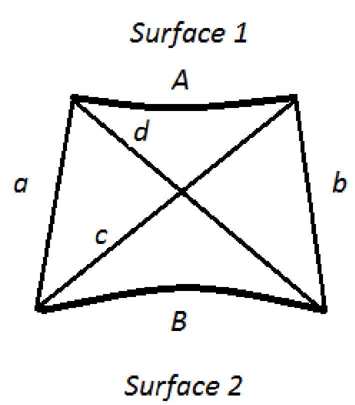
ABSTRACT

In this paper we will discuss the one-dimensional Hottel string method as it applies to symmetric, infinite sources (as in the case of constructing ideal solar concentrators) and extend the theory to asymmetric, finite sources and demonstrate that an ideal concentrator can be created in this case. Furthermore, we will discuss the concept of flowlines and explore the yet unknown relationship between strings and flowlines.

1. Hottel String Method

The well-known Hottel string method describes the probability of radiative transfer from one surface to another using crossed strings. In the symmetric case, we consider two surfaces of length A and B and draw strings labelled a, b, c and d:



Then the probability of radiative transfer from Surface 1 to 2 is given by the equation[3]:

$$P_{12} = \frac{(c+d)-(a+b)}{2A} \tag{1}$$

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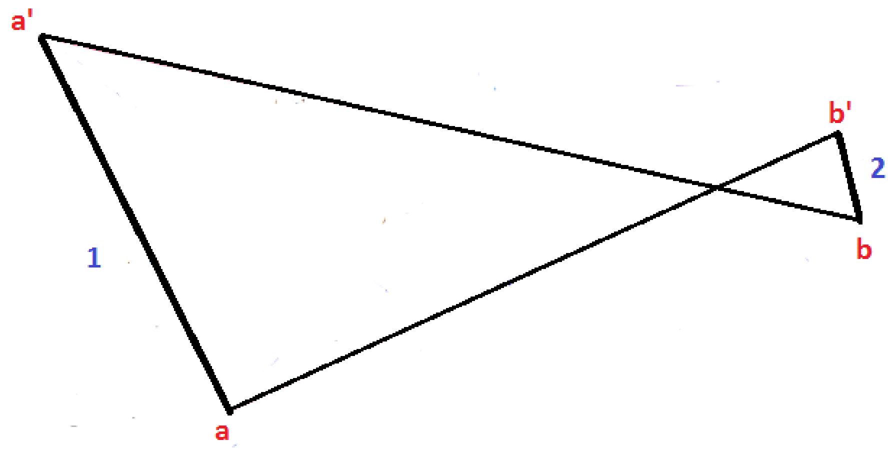
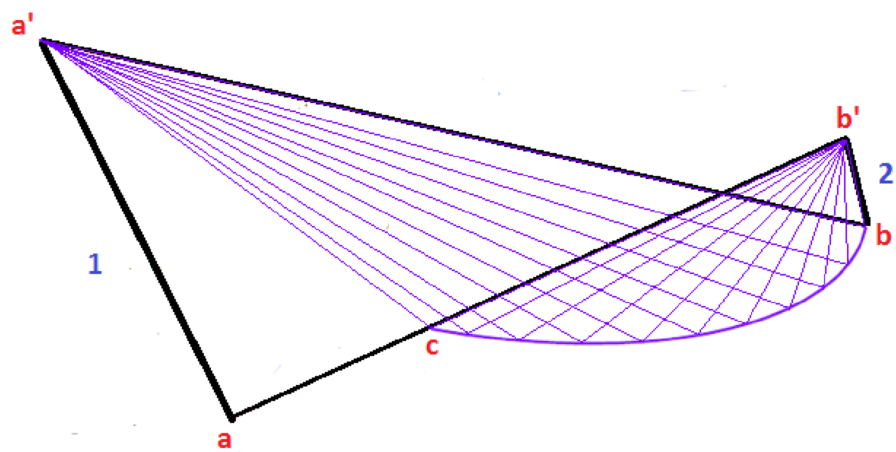
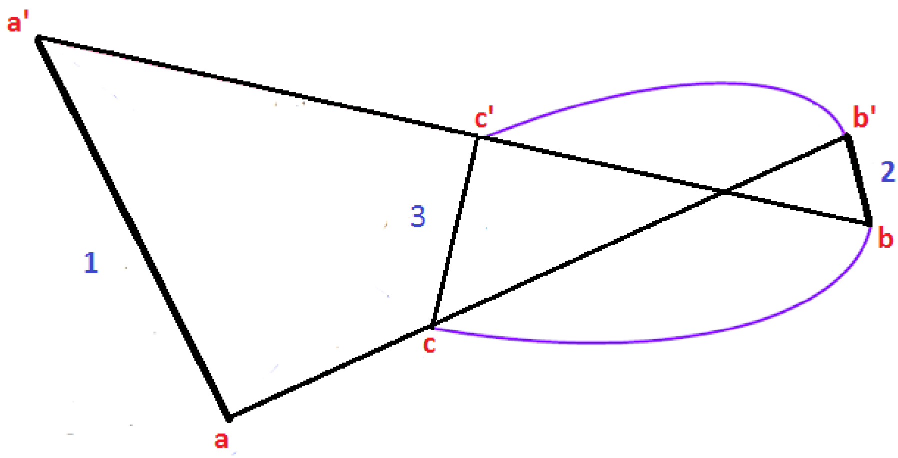
We can use this principle in constructing a thermodynamically efficient concentrator. Let us consider an asymmetric case with a finite source. First, we'll draw our strings:



We now take a string length $a'b + bb'$ and, using *a'* and *b'* as the foci, draw the elliptical profile until it meets point *c* on *ab'*.



Now, we will repeat the same thing with a string length $ab' + b'b$, using *a* and *b* as the foci, draw the elliptical profile until it meets point *c* on *ab'*.



Now, we can call *cc'* Surface 3. If we assume that no energy is lost when radiation travels from surface 1 to surface 2, then we have:

$$A_1 P_{13} = A_1 P_{12} \tag{2}$$

That is, energy is conserved when travelling from surface 1 to surface 2 *through* surface 3. Then, by the same conservation principle, we have the identities

$$A_1 P_{13} = A_3 P_{31} \tag{3}$$

$$A_1 P_{12} = A_2 P_{21} \tag{4}$$

Now, if we substitute Equation (3) into Equation (2), we have

$$A_3 P_{31} = A_1 P_{12} \tag{5}$$

and we then substitute Equation (5) into Equation (4) to arrive at

$$A_3 P_{31} = A_2 P_{21}. \tag{6}$$

Now, from Equation (6) the geometric concentration is given by

$$C = \frac{A_3}{A_2} = \frac{P_{21}}{P_{31}} \leq \frac{1}{P_{31}} \tag{7}$$

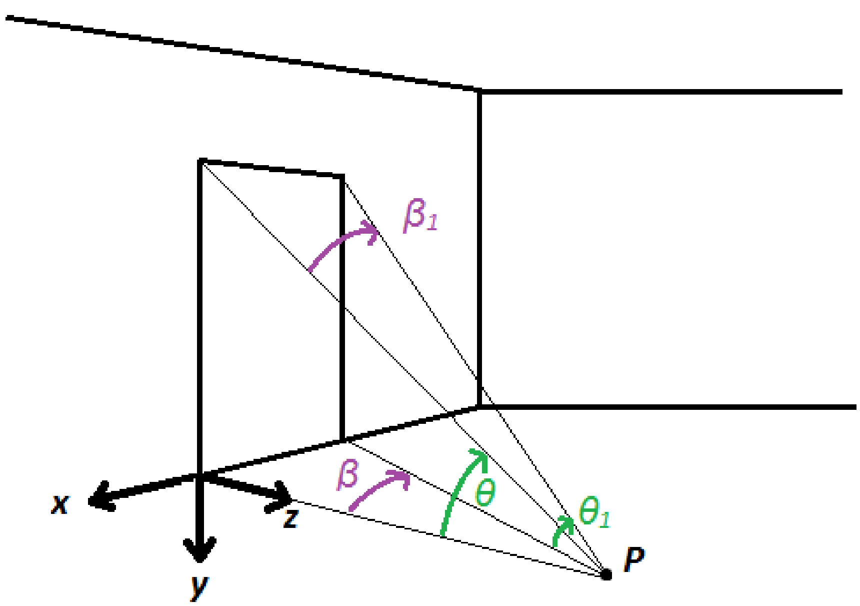
since is a probability and must be less than 1. Thus, we produce a thermodynamically efficient ideal concentrator.

2. Flowline Derivation

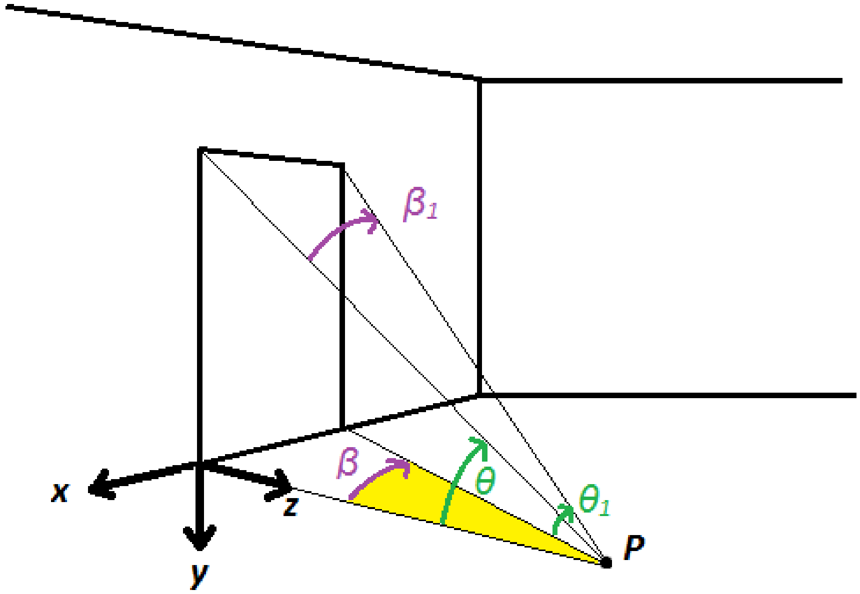
A useful approach in concentrator design is using the lines of flow from the geometrical vector flux to model light rays as fluid flow in phase space. The advantage of this form is that mirrors can be placed along the flow lines without disturbing the vector field. The direction of these flow lines will be the the direction of energy flow from one body to another. The geometrical vector flux **J** is defined by its components as

$$(J_x, J_y, J_z) = \left(\iint dp_y dp_z, \iint dp_z dp_x, \iint dp_x dp_y\right) \tag{8}$$

where p_x, p_y and p_z are the optical direction cosines of a ray from a Lambertian source. [2]Let us first consider a finite source with point *P* on the *z*-axis:



We define angles β_1 and θ_1 and projection angles β and θ. Now, $\iint dp_z dp_x$ will be the solid angle projection of the shaded area $\frac{\beta_1}{2} cos\theta$ onto the "floor" (the *x*-*z* plane), which will simply be $\frac{1}{2}(\beta - \beta_1 \cos\theta)$. [1] This is the *y*-component of the flowline vector.
In the two dimensional case, we model the flowline vector by considering a very long aperature extending to infinity, in this case in the *x*-direction. Then, we can see that in the infinite case, $\beta = \beta_1 = \frac{\pi}{2}$ and $\theta_1 = 0$. Then, the *y*-component of our flowline vector becomes



$$J_y = \frac{1}{2}(1 - \cos\theta) \tag{9}$$

Similarly, if we take the solid angle projections onto the *y*-*z* plane and *x*-*y* planes, we can find the *x*-component and *z*-components respectively. In the infinite case, the *x*-component is 0 by symmetry. [1]The *z*-component is then, for the finite case

$$J_z = \frac{1}{2}(\theta_1 \sin\beta + \beta_1 \sin\theta) \tag{10}$$

and in the infinite case

$$J_z = \frac{1}{2}\sin\theta. \tag{11}$$

Then, to find the magnitude of the flowline vector in two dimensions, we first consider an arbitrary point (not on the axis) so we have

$$J_{y_1} = (1 - \cos\theta_1) \tag{12}$$

$$J_{y_2} = (1 - \cos\theta_2) \tag{13}$$

and

$$J_{z_1} = \sin\theta_1 \tag{14}$$
$$J_{z_2} = \sin\theta_2 \tag{15}$$

so our components become

$$J_y = J_{y_1} - J_{y_2} = -(\cos\theta_1 - \cos\theta_2) \tag{16}$$
$$J_z = J_{z_1} - J_{z_2} = \sin\theta_1 - \sin\theta_2 \tag{17}$$

or, if we apply trigonometric identites,

$$J_y = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right) \tag{18}$$
$$J_z = 2\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right). \tag{19}$$

Then, we find the magnitude of the two-dimensional flowline vector using

$$|J| = \sqrt{{J_y}^2 + {J_z}^2} \tag{20}$$

and with a bit of simple algebra we arrive at the magnitude of the flowline being

$$|J| = 2\sin\left(\frac{\theta_1-\theta_2}{2}\right) \tag{21}$$

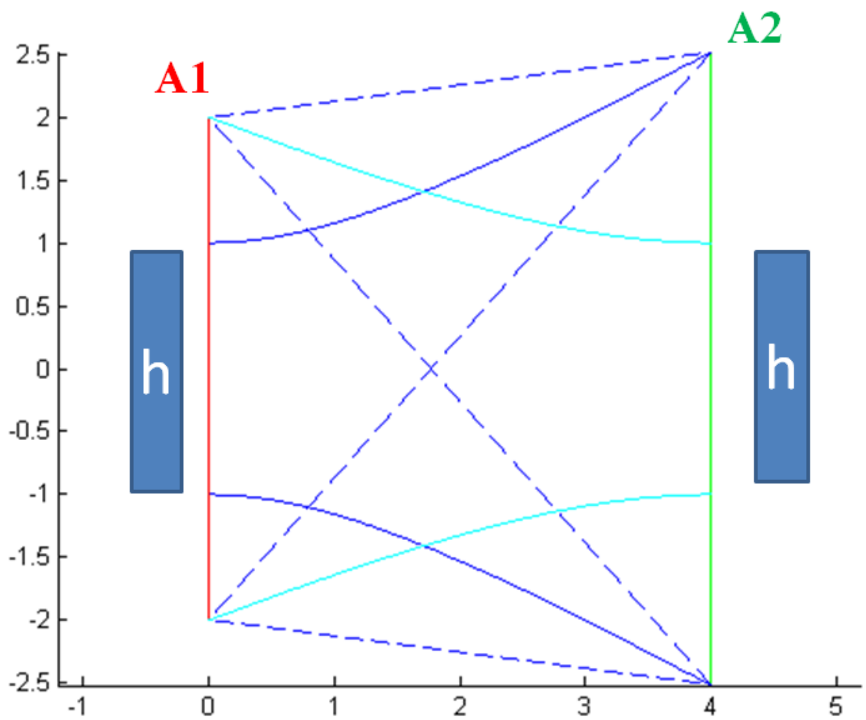
with a direction of

$$\frac{\theta_1-\theta_2}{2}. \tag{22}$$

This means that in two dimensions, the flowline will always bisect the extreme angles.

3. Relationship between Strings and Flowlines

If we draw the flowlines between two surfaces along with the strings, we see some remarkable results:



Here the curved lines extending from the center of A2 to the edges of A1 are the flowlines which form hyperbolas using the ends of A1 and A2 as foci. As an inherent property of a hyperbola, however, this also means that at each point on these hyperbolas, a pair of strings can be drawn to the extrema and the difference between these strings (long string - short string) will be constant. In this example, that constant length is h. This means that the amount of radiation received by A2 from A1 is equal to the amount of radiation h emits everywhere. And this is the same as the amount A1 receives from A2–satisfying the second law of thermodynamics! However, the flowline is a purely geometrical concept, while strings are based only on the laws of thermodynamics. The fact that they agree in concentrator design is neither trivial nor coincidental.

4. Conclusions

Although concentrator design is typically viewed from a strictly optical perspective, ultimately we are concerned with the transfer of light which should also be studied from a thermodynamics perspective. Using radiation theory to study light flow alllows us a different outlook on optical design problems. Both the string method and flowline method of concentrator design agree despite originating from completely different physical theories, suggesting that radiative heat transfer theory and geometrical optics are more closely linked than one might expect, despite being separated by centuries of study. Uncovering the relationship between strings and flowlines could be key to further nonimaging optic design.

References

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