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Barut, Erol

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BARCODE GROWTH FOR TORIC DOMAINS

A dissertation submitted in partial satisfaction
of the requirements for the degree of

DOCTOR OF PHILOSOPHY
in
MATHEMATICS

by
Erol Barut

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The Dissertation of Erol Barut
is approved:

Professor Viktor L. Ginzburg, Chair

Professor Başak Gürel

Professor Longzhi Lin

Donald Smith
Interim Vice Provost and Dean of Graduate Studies

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2026

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Abstract

Barcode Growth for Toric Domains

by

Erol Barut

In this thesis, we study barcode growth functions of the symplectic homology of Liouville domains. We show that the barcode growth function of a smooth convex/concave or real analytic toric domain grows at most like a polynomial of degree half the dimension of the domain. In particular, the barcode entropy of such a domain is zero and its slow barcode entropy is bounded by half the dimension, reflecting the complete integrability of the Reeb flow on the boundary. We extend the definition of barcode growth functions to star-shaped domains with non-smooth boundary via smooth approximations in the symplectic Banach–Mazur distance, and show that the polynomial bound persists for non-smooth convex/concave toric domains. We also show that barcode entropy and slow barcode entropy are invariants of the exact symplectomorphism type of the interior. As applications, we obtain obstructions to exact symplectomorphisms between interiors of Liouville domains.

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1 Introduction

In this thesis, based on [BG], we study barcode growth functions of Floer/symplectic homology associated to Hamiltonian systems and their relation to the dynamics of the underlying Hamiltonian system.

The action filtration on Floer homology has been a vital tool in symplectic dynamics and has been used extensively (see, for example, [BPS, G, G²09, G²16]). The filtered Floer homology has also been used to define invariants related to symplectic manifolds and Hamiltonian dynamics [Us1].

Persistence modules are algebraic objects used to encode the change of homology with respect to a filtration, accompanied by an object of combinatorial nature called a barcode that encodes the same data. For an aspherical symplectic manifold, Hamiltonian Floer homology can be viewed as a pointwise finite-dimensional persistence module. This perspective was pioneered in [PS]. In general, Floer homology cannot be viewed as a persistence module according to the definition adopted in this dissertation; despite this, a way to define a barcode associated to a Hamiltonian diffeomorphism (up to a shift) was given in [UZ]. Since then, several invariants built from the barcode of a Hamiltonian diffeomorphism or a Liouville domain have been introduced [ÇG²24b, KS].

The invariants we are interested in are the barcode growth function and barcode entropy. The former is described as the number of “not-so-short” bars in the barcode of filtered symplectic homology, and the latter is the exponential growth rate of the former. Barcode entropy was first introduced in [ÇG²24a] in the context of filtered

Hamiltonian Floer theory and Lagrangian Floer theory. Since then, it has appeared in other flavors of Floer homology [Fe24, ÇG²M25, FeLS]. In each of these settings, it has been shown that barcode entropy bounds the topological entropy of the time-one map of the related flow, a result known as Theorem A in the context of barcode entropy. Another important result proved in each flavor is Theorem B, which states that in the presence of a hyperbolic invariant set the barcode entropy is nonzero. These two results show that barcode entropy detects features of the related dynamics, and the Theorem B type results in [Fe25, ÇG²M25, FeLS, ÇG²24a] give examples of Hamiltonian/Reeb dynamics with nonzero barcode entropy.

In contrast to those results, we show that the barcode (as defined in [ÇG²M25]) of convex/concave or real analytic toric domains grows at most like a polynomial of degree half the dimension of the domain, hence showing that the barcode entropy is zero and the slow barcode entropy is bounded by half the dimension. This result reflects the ‘simplicity’ of the dynamics of the Reeb flow on the boundary: the Reeb flow (when defined) on the boundary of such a domain is completely integrable with well-behaved (toric) singularities [Ar].

Besides detecting dynamical features, barcode entropy and barcode growth functions can be viewed as invariants of a Liouville domain and can be used to distinguish Liouville domains. We also visit the question of whether the symplectic homology of a Liouville domain is determined by its interior (see Proposition 5.11). This has already been proven in different versions of the statement in different settings; the proofs translate to our setting, nonetheless we give a proof in Section 5.3. We also define barcode growth functions for star-shaped domains that do not necessarily have smooth boundary. The invariance of barcode growth functions does not extend to star-shaped domains when both domains are non-smooth; however, we show that barcode entropy and slow barcode entropy agree for two star-shaped domains with exact symplectomorphic interiors. We also show that the polynomial bound for the barcode growth function still holds for convex/concave domains that are not smooth.

1.1 Main Results

For a Liouville domain W , the filtered symplectic homology $SH^s(W)$ over a fixed field $\mathbb{F}$ can be viewed as a persistence module (see Corollary 5.4). The decomposition of persistence modules gives an invariant of persistence modules called a barcode (see Theorem 2.5). A barcode is simply a collection of intervals. Let $\mathfrak{b}_\epsilon(W, s)$ denote the number of bars of length greater than ϵ starting before s in the barcode of the symplectic homology of W . We will give a polynomial upper bound for the barcode growth when W is a smooth convex/concave toric domain or a real analytic toric domain. The result is summarized in the following theorem.

Theorem 1.1. *Let W_Ω be a smooth convex/concave toric domain or a real analytic toric domain. Then there exist constants $C_n(\Omega)$ and $C_0(\Omega)$, independent of s and ϵ , such that*

$$\mathfrak{b}_\epsilon(W_\Omega, s) \leq C_n(\Omega)s^n + C_0(\Omega) \quad (1.1)$$

The proof relies on counting the number of Reeb orbits with contact action less than s . The constants $C_n(\Omega)$ and $C_0(\Omega)$ will be determined explicitly in the proof of the theorem.

$\mathfrak{b}_\epsilon(s)$ is monotone increasing in s and decreasing in ϵ (see Section 2.2). The exponential and polynomial growth rates of $\mathfrak{b}_\epsilon$ are called the ϵ -barcode entropy and the ϵ -slow barcode entropy, respectively, and their limits as $\epsilon \rightarrow 0^+$ are called the barcode entropy and the slow barcode entropy, denoted by $\hbar(W)$ and $\hbar^{Slow}(W)$ (see Section 2.2 and [CG²M25]).

As an immediate consequence of the theorem, the barcode entropy of a smooth convex/concave toric domain or a real analytic toric domain is zero.

$$\hbar(W) = 0 \quad (1.2)$$

The theorem also gives an upper bound on the slow barcode entropy of the Liouville

domain:

$$\hbar^{Slow}(W) \leq n \quad (1.3)$$

This reflects the trivial nature of the dynamics due to complete integrability with toric singularities.

Let φ denote the time-one map of the Reeb flow of ∂W and denote its topological entropy by $h_{\text{top}}(\varphi)$. Theorem A of [ÇG²M25] states the following (see also [FeLS]):

$$\hbar(W) \leq h_{\text{top}}(\varphi) \quad (1.4)$$

Moreover, if K is a hyperbolic invariant set of φ , then

$$h_{\text{top}}(\varphi|_K) \leq \hbar(W) \quad (1.5)$$

A corollary of the inequalities above in dimension three is the following (see [ÇG²M25, Cor. C]):

$$\hbar(W) = h_{\text{top}}(\varphi_\alpha) \quad (1.6)$$

The filtered symplectic homology can be extended to star-shaped domains with non-smooth boundary via continuity. However, it is not expected to form a persistence module, even in a more general sense than Definition 2.1, and we do not know if it has a well-defined barcode growth function. At the end of Section 5.2 we instead define the barcode growth function directly, using smooth approximations in the symplectic Banach–Mazur distance, and hence also the barcode entropy and the slow barcode entropy for star-shaped domains.

Let W be a convex/concave toric domain. Recall that W is non-smooth if the extension of f to a neighborhood U of $S_{\geq 0}^{n-1}$ cannot be taken smooth. In this case W is not a Liouville domain. Using the definition of the barcode growth function for star-shaped domains, we extend our result to convex/concave toric domains.

Theorem 1.2. *Let W_Ω be a convex/concave toric domain. Then there exist constants $C_n(\Omega)$ and $C_0(\Omega)$, independent of s and ϵ , such that*

$$\mathfrak{b}_\epsilon(W_\Omega, s) \leq C_n(\Omega)s^n + C_0(\Omega) \quad (1.7)$$

Symplectic homology is an invariant of the exact symplectomorphism type of the interior of a Liouville domain. This is a rather well-known property of the symplectic homology of Liouville domains; see [CFHW, Gu, Hu24] and the discussion in Section 5.3. In particular, if W_1 and W_2 are star-shaped domains with smooth boundary such that there is an exact symplectomorphism between $\mathring{W}_1$ and $\mathring{W}_2$, then $\mathfrak{b}_\epsilon(W_1, s) = \mathfrak{b}_\epsilon(W_2, s)$. We generalize this result to the case where at least one of the star-shaped domains has smooth boundary. We do not know if the equality holds for two non-smooth star-shaped domains, but we show that $\hbar(W_1) = \hbar(W_2)$ and $\hbar^{Slow}(W_1) = \hbar^{Slow}(W_2)$ for any star-shaped domains in $\mathbb{R}^{2n}$.

Combining the results from [ÇG²M25], (1.2) and the invariance of $\hbar$ for star-shaped domains, we conclude the following corollary.

Corollary 1.3. *Let W be a Liouville domain and W_Ω be a convex/concave toric domain or a real analytic toric domain. If $\mathfrak{h}_{\text{top}}(\varphi) > 0$, where φ denotes the time-one map of the Reeb flow of ∂W , then there is no exact symplectomorphism between $\mathring{W}$ and $\mathring{W}_\Omega$.*

Corollary 1.3 is a generalization of [Hu24, Prop. 1.3].

Lastly, the invariance of $\hbar$ for Liouville domains together with (1.6) implies the invariance of topological entropy in dimension four. Thus we have the following corollary.

Corollary 1.4. *Let W_1 and W_2 be four-dimensional Liouville domains and let φ_1 and φ_2 denote the time-one maps of the Reeb flows on their boundaries. If $\mathring{W}_1$ and*

$\dot{W}_2$ are exact symplectomorphic, then

$$h_{\text{top}}(\varphi_1) = h_{\text{top}}(\varphi_2) \tag{1.8}$$

The dissertation is organized as follows. In this chapter, we have stated our main results and discussed their further implications. In Chapter 2, we review basics of persistence modules and barcodes. We will define persistence modules and the interleaving distance between them. We discuss barcodes associated to a persistence module and give the definition of the bottleneck distance on barcodes. Finally, we give the definition of barcode growth functions and barcode entropy of a persistence module and discuss their properties. Chapter 3 consists of a quick recap of Liouville domains and the definition of convex/concave toric domains that we adopt in this dissertation. We give the definition of a Liouville domain and its symplectic completion in the first part. In the second part, we give the definition of toric domains. We spell out our definition of convex/concave toric domains and elaborate on it. We start Chapter 4 by giving the construction of the filtered Floer chain complex and continuation maps. Then, in Chapter 5, we define symplectic homology persistence modules; we also recall the symplectic Banach–Mazur distance and prove the invariance results there. Finally, our main results, which give a polynomial upper bound on the barcode growth functions (Theorems 1.1 and 1.2), are proven in Chapter 6. We first prove the result for smooth convex/concave toric domains and real analytic toric domains by basically counting the number of generators of the associated filtered Floer chain complex. Then we give the proof for convex/concave toric domains that are not necessarily smooth.

2 Persistence Modules and Barcodes

Persistence modules arose in the field of topological data analysis in the early 2000s [CZCG, ZC], and have since found applications far beyond it. In this chapter we recall basic definitions and notions in persistence modules and barcodes. In the first part we give the definition of persistence modules and their barcodes. We adopt our setting from [ÇG²M25] and borrow some notation from [PRSZ]. We continue the discussion with the interleaving distance and the bottleneck distance. In the second part we define barcode growth and barcode entropy, and discuss their properties and their relation to the interleaving/bottleneck distance. The reader is referred to [PRSZ] for a general discussion and applications of persistence modules. Unlike in [PRSZ], we do not define persistence modules to be pointwise finite; however, most if not all persistence modules in this dissertation will be pointwise finite. For further reference see [BV, CdSGO, UZ].

2.1 Basic Definitions and the Structure Theorem

Definition 2.1. *A persistence module (V, π) is a pair consisting of a family of finite dimensional $\mathbb{F}$ -vector spaces $\{V_t\}_{t \in \mathbb{R}}$ equipped with a family of maps $\pi_{s,t} : V_s \rightarrow V_t$ for all $s \leq t$ in $\mathbb{R}$ such that:*

1. *For all $s \leq t \leq r$, $\pi_{t,r} \circ \pi_{s,t} = \pi_{s,r}$.*

2. There exists a countable discrete set denoted by $\mathcal{S}(V)$ such that $\pi_{s,t}$ are isomorphisms whenever $s \leq t$ in $\mathbb{R}$ belong to the same connected component of $\mathbb{R} \setminus \mathcal{S}(V)$; in particular, for all $t \in \mathbb{R}$, there exists ϵ such that for all $s \in (t - \epsilon, t]$, $\pi_{s,t}$ is an isomorphism.

3. For all $s < t$, the map $\pi_{s,t}$ has finite rank (q-tameness).

4. There exists $s_0 \in \mathbb{R}$ such that $V_s = 0$ for all $s < s_0$.

A morphism Φ from persistence modules (V, π) to (W, θ) is a family of maps $\Phi_t : V_t \rightarrow W_t$ satisfying $\Phi_t \circ \pi_{s,t} = \theta_{s,t} \circ \Phi_s$.

Example 2.2. The most basic example of a persistence module is given by

$$(\mathbb{F}_{(a,b)})_t = \begin{cases} \mathbb{F} & \text{if } t \in (a, b] \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

with the structure maps

$$\pi_{s,t} = \begin{cases} \mathbb{I} & \text{if } a < s \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

Such persistence modules are called interval modules, and they are simple persistence modules. Let $\mathbb{F}_{(a,b]}$ and $\mathbb{F}_{(c,d]}$ be two interval modules such that $c \leq a$ and $a < d \leq b$, and define $\iota : \mathbb{F}_{(a,b]} \rightarrow \mathbb{F}_{(c,d]}$ by

$$\iota_t := \begin{cases} \mathbb{I} & \text{if } t \in (a, b] \cap (c, d] \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

Then ι is a morphism of persistence modules.

For a persistence module (V, π) , its δ -shift module is given by $(V[\delta])_t = V_{t+\delta}$ and $\pi[\delta]_{s,t} = \pi_{s+\delta, t+\delta}$. It is easy to verify that the map $(\Phi_V^\delta)_t := \pi_{t, t+\delta}$ gives

a morphism between V and $V[\delta]$, called the δ -shift morphism. Similarly, for a morphism $\Psi : (V, \pi) \rightarrow (W, \theta)$ of persistence modules, the δ -shift of Ψ is the morphism $\Psi[\delta] : V[\delta] \rightarrow W[\delta]$ defined by $(\Psi[\delta])_t = \Psi_{t+\delta}$.

Definition 2.3. Let (V, π) and (W, θ) be persistence modules and let $\delta \in \mathbb{R}_{>0}$. We say that V and W are δ -interleaved if there exist morphisms $F : V \rightarrow W[\delta]$ and $G : W \rightarrow V[\delta]$ such that

$$G[\delta] \circ F = \Phi_V^{2\delta} \quad (2.4)$$

and

$$F[\delta] \circ G = \Phi_W^{2\delta} \quad (2.5)$$

The pair (F, G) is called a δ -interleaving between V and W .

A module is δ -interleaved with itself for all $\delta \in \mathbb{R}_+$. If V and W are δ_1 -interleaved and W and Z are δ_2 -interleaved, then by shifting and composing the interleaving maps one can show that V and Z are $(\delta_1 + \delta_2)$ -interleaved. As a result, if V and W are δ -interleaved, they are δ' -interleaved for all $\delta' > \delta$. These properties of δ -interleavings motivate the following definition.

Definition 2.4. Let (V, π) and (W, θ) be two persistence modules. The interleaving distance between them is defined as

$$d_{int}(V, W) := \inf\{\delta \in \mathbb{R}_{\geq 0} \mid V \text{ and } W \text{ are } \delta\text{-interleaved}\} \quad (2.6)$$

d_{int} satisfies the triangle inequality by the additivity property of the interleaving relation. Furthermore, d_{int} defines a so-called extended metric on the set of isomorphism classes of persistence modules: the fact that $d_{int}(V, W) = 0$ if and only if V and W are isomorphic persistence modules is a consequence of Theorems 2.5 and 2.10. It is only an extended metric because d_{int} takes values in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. This occurs even for interval modules: one can easily deduce that $d_{int}(\mathbb{F}_{(a,b]}, \mathbb{F}_{(c,\infty)}) = \infty$.

Theorem 2.5. *For every persistence module (V, π) , there exists a family of intervals $\{I_\alpha\}_{\alpha \in A}$, indexed by a countable set A , such that*

$$V \cong \bigoplus_{\alpha \in A} \mathbb{F}_{I_\alpha} \quad (2.7)$$

The decomposition is unique up to reordering.

This theorem is called the normal form theorem or the structure theorem; its proof can be found in [CdSGO] and [BV]; see also [CB, CZCG, ZC] for related results.

Intervals may appear with multiplicities, so it is best to view the collection $\{I_\alpha\}$ as a multiset. A multiset is a pair (X, m) , where X is a set and $m : X \rightarrow \mathbb{N}$ counts multiplicities; equivalently, we view a multiset as a subset of $X \times \mathbb{N}$ of the form $\{(a, m(a)) \mid a \in X\}$.

Definition 2.6. *A barcode $\mathcal{B}$ is a multiset consisting of countably many intervals of the form $(a, b]$ or (a, ∞) .*

Indeed, by the normal form theorem, for every persistence module V there exists a unique barcode $\mathcal{B}(V)$ whose elements are the intervals I_α from Theorem 2.5, counted with their multiplicities. There is a very similar notion of distance to interleaving defined on the set of barcodes, as follows.

The Cartesian product of two multisets (X, m_X) and (Y, m_Y) is defined by $X \times Y$ together with the multiplicity function $m_{X \times Y}(x, y) = m_X(x) m_Y(y)$. A bijection between two multisets (X, m_X) and (Y, m_Y) is a subset $f \subseteq X \times Y$ together with a multiplicity function $m_f : X \times Y \rightarrow \mathbb{N}$ (with finite support) such that for every $a \in X$ and $b \in Y$,

$$m_X(a) = \sum_{b \in Y} m_f(a, b), \quad m_Y(b) = \sum_{a \in X} m_f(a, b). \quad (2.8)$$

For a given barcode $\mathcal{B}$, we define $\mathcal{B}_\delta := \{((a, b], m) \in \mathcal{B} \mid b - a > \delta\}$, the submultiset of $\mathcal{B}$ consisting of the bars of length greater than δ . We will adopt

the same notation for the barcode of a persistence module: $\mathcal{B}_\delta(V)$ will denote the multiset of bars having length greater than δ in the barcode of V .

Definition 2.7. Two barcodes $\mathcal{B}$ and $\mathcal{B}'$ are called δ -matched if there exists a bijection $f : C \rightarrow C'$, where $C \subset \mathcal{B}$ and $C' \subset \mathcal{B}'$ are submultisets, such that

- $\mathcal{B}_{2\delta} \subset C$ and $\mathcal{B}'_{2\delta} \subset C'$
- $((a, b], (c, d]) \in f \implies (a, b] \subset (c - \delta, d + \delta]$ and $(c, d] \subset (a - \delta, b + \delta]$
where $a, b \in \mathbb{R}$ and $c, d \in \mathbb{R} \cup \{\infty\}$

Remark 2.8. In the definition above, c or d can be infinite, and in that case the interval should be understood as (a, ∞) ; we adopt the conventions $\infty + \delta = \infty$ and $\infty - \delta = \infty$. Hence finite intervals are matched with finite intervals and infinite intervals are matched with infinite intervals. For two infinite intervals (a, ∞) and (b, ∞) , the condition should be read as $|a - b| < \delta$.

Definition 2.9. The bottleneck distance between two barcodes $\mathcal{B}$ and $\mathcal{B}'$ is defined by

$$d_{bit}(\mathcal{B}, \mathcal{B}') = \inf\{\delta \in \mathbb{R}_{\geq 0} \mid \mathcal{B} \text{ and } \mathcal{B}' \text{ are } \delta\text{-matched}\} \quad (2.9)$$

Theorem 2.10. Let V and W be persistence modules and let $\mathcal{B}(V)$ and $\mathcal{B}(W)$ be their barcodes. Then

$$d_{int}(V, W) = d_{bit}(\mathcal{B}(V), \mathcal{B}(W)) \quad (2.10)$$

In other words, the map $\mathcal{B}$ sending a persistence module to its barcode is an isometry onto its image (see [CdSGO, Le]).

Remark 2.11. An interleaving may be regarded as an approximate isomorphism. Indeed, by the composition conditions (2.4) and (2.5), a δ -interleaving between V and W is an isomorphism except on a disjoint union of intervals of length less than 2δ . One can similarly define an injective version of this notion by requiring only

one of the two composition conditions (2.4)–(2.5) (see [Us, Def. 3.2]). By arguing with δ -matchings of barcodes (Definition 2.7), one can show that if two persistence modules admit such one-sided interleavings in both directions, of sizes δ_1 and δ_2 , then they are $\max(\delta_1, \delta_2)$ -interleaved. This observation will be useful in Section 5.2.

As mentioned before, the interleaving distance is not necessarily finite, and neither is the bottleneck distance. However, if there is a t_0 such that $V_\infty \cong V_t$ and $W_t \cong W_\infty$ for $t > t_0$, then $d_{\text{int}}(V, W) < \infty$. The truncation of (V, π) at s is defined by

$$(\text{Tr}_s V)_t = \begin{cases} V_t & t \leq s \\ 0 & s < t \end{cases} \quad (2.11)$$

$$(\text{Tr}_s \pi)_{t,r} = \begin{cases} \pi_{t,r} & t, r \leq s \\ 0 & \text{otherwise} \end{cases} \quad (2.12)$$

The truncations of any two persistence modules always have finite interleaving distance. This gives rise to a family of pseudo-metrics defined by

$$d_s = d_{\text{int}}(\text{Tr}_s(V), \text{Tr}_s(W)) \quad (2.13)$$

A choice of a sequence $s_k \rightarrow \infty$ as $k \rightarrow \infty$ gives a countable family of pseudo-metrics. This family induces a metrizable topology on the space of equivalence classes of persistence modules, and likewise on the space of barcodes. We will refer to convergence in this topology as truncated convergence.

2.2 Barcode growth functions and barcode entropy

Given a persistence module (V, π) , one can define its slow barcode entropy as the polynomial growth rate of the number of not-so-short bars starting before the

parameter s . More precisely, define

$$\mathcal{B}_\epsilon(V, s) := \{((a, b], n) \in \mathcal{B}_\epsilon(V) \mid a < s\}, \quad (2.14)$$

namely, the bars of V starting before s and having length greater than ϵ . Define

$$\mathbf{b}_\epsilon(V, s) := |\mathcal{B}_\epsilon(V, s)| \quad (2.15)$$

$\mathbf{b}_\epsilon(V, s)$ is called the barcode growth function of V . We will often write $\mathbf{b}_\epsilon(s)$ instead of $\mathbf{b}_\epsilon(V, s)$ when the persistence module is clear from the context. Barcode growth functions are monotone increasing (non-decreasing) and left semicontinuous in s , and monotone decreasing and right semicontinuous in ϵ .

Lemma 2.12. *For any persistence module V , $\mathbf{b}_\epsilon(V, s)$ is finite for all $\epsilon > 0$ and $s \in \mathbb{R}$.*

Proof. Assume on the contrary that, for some ϵ and s , $\mathbf{b}_\epsilon(V, s)$ is infinite. Then there exist infinitely many intervals I_n of length greater than ϵ starting before s . Let a_n be the left end point of I_n . Since $0 \leq a_n < s$, the sequence a_n has a convergent subsequence, which we still denote by a_n . Let a be the limit of a_n . Then there is $N \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{2}$ for all $n > N$.

When $a_k < a$ for some $k > N$, we have $a < a_k + \frac{\epsilon}{2}$, and hence $(a, a + \frac{\epsilon}{2}) \subset (a_k, a_k + \epsilon) \subset I_k$.

When $a < a_k$ for some $k > N$, we have $a_k < a + \frac{\epsilon}{2}$, and hence $(a + \frac{\epsilon}{2}, a + \epsilon) \subset (a_k, a_k + \epsilon) \subset I_k$. There are infinitely many k satisfying $a_k < a$ or infinitely many k satisfying $a < a_k$. In each case, there is an interval contained in infinitely many of the I_k , which implies that the image of $\pi_{s,t}$ has infinite dimension for some $s < t$, contradicting q -tameness. $\square$

Let $d_{btl}(V, W) < \frac{\delta}{2}$ and let f be a $\frac{\delta}{2}$ -matching between $\mathcal{B}(V)$ and $\mathcal{B}(W)$. Then

$f(\mathcal{B}_{\epsilon+\delta}(V, s)) \subset \mathcal{B}_\epsilon(W, s + \delta)$, and for $\epsilon > \delta$ we get

$$\mathbf{b}_{\epsilon+\delta}(V, s - \delta) \leq \mathbf{b}_\epsilon(W, s) \leq \mathbf{b}_{\epsilon-\delta}(V, s + \delta) \quad (2.16)$$

Proposition 2.13. *Let V_n be a sequence of persistence modules converging to V in the truncated convergence topology. Then*

$$\mathbf{b}_\epsilon(V, s) \leq \liminf_{n \rightarrow \infty} \mathbf{b}_\epsilon(V_n, s) \quad (2.17)$$

The proposition can be interpreted as saying that barcode growth functions are lower semicontinuous with respect to the truncated convergence topology on the space of (isomorphism classes of) persistence modules. The proof of the proposition uses (2.16) together with

$$\mathbf{b}_\epsilon(\text{Tr}_s(V)) \leq \mathbf{b}_\epsilon(V, s) \leq \mathbf{b}_\epsilon(\text{Tr}_{s+\epsilon}(V)) \leq \mathbf{b}_\epsilon(V, s + \epsilon). \quad (2.18)$$

Proof. We will prove it for convergence in d_{int} first. For any given $\delta > 0$, by (2.16),

$$\mathbf{b}_{\epsilon+\delta}(V, s - \delta) \leq \mathbf{b}_\epsilon(V_n, s) \quad (2.19)$$

for all n large enough. Taking the limit inferior of the right-hand side, we get

$$\mathbf{b}_{\epsilon+\delta}(V, s - \delta) \leq \liminf_{n \rightarrow \infty} \mathbf{b}_\epsilon(V_n, s) \quad (2.20)$$

for any $\delta > 0$. Notice that the left-hand side, as a function of δ , is right-semicontinuous. Thus

$$\lim_{\delta \rightarrow 0^+} \mathbf{b}_{\epsilon+\delta}(V, s - \delta) = \mathbf{b}_\epsilon(V, s) \quad (2.21)$$

and hence

$$\mathbf{b}_\epsilon(V, s) \leq \liminf_{n \rightarrow \infty} \mathbf{b}_\epsilon(V_n, s) \quad (2.22)$$

Now applying (2.18) together with the above result, we get

$$\mathbf{b}_\epsilon(V, s) = \mathbf{b}_\epsilon(\text{Tr}_{s+\epsilon}(V), s) \leq \liminf_{n \rightarrow \infty} \mathbf{b}_\epsilon(\text{Tr}_{s+\epsilon}(V_n), s) = \liminf_{n \rightarrow \infty} \mathbf{b}_\epsilon(V_n, s) \quad (2.23)$$

□

The barcode entropy and the slow barcode entropy of a persistence module are defined as the exponential and the polynomial growth rate of the barcode growth functions, respectively. First, set $\log^+(x) := \max\{\log(x), 0\}$. The ϵ -barcode entropy of V is defined by

$$\hbar_\epsilon(V) := \limsup_{s \rightarrow \infty} \frac{\log^+(\mathbf{b}_\epsilon(V, s))}{s} \quad (2.24)$$

and the barcode entropy is defined by

$$\hbar(V) := \lim_{\epsilon \rightarrow 0^+} \hbar_\epsilon(V) \quad (2.25)$$

Similarly, the ϵ -slow barcode entropy is defined by

$$\hbar_\epsilon^{\text{Slow}}(V) := \limsup_{s \rightarrow \infty} \frac{\log^+(\mathbf{b}_\epsilon(V, s))}{\log(s)} \quad (2.26)$$

and the slow barcode entropy is defined by

$$\hbar^{\text{Slow}}(V) := \lim_{\epsilon \rightarrow 0^+} \hbar_\epsilon^{\text{Slow}}(V) \quad (2.27)$$

Both barcode entropy and slow barcode entropy fail to be continuous in d_{int} , and hence also in the truncated convergence topology.

Example 2.14. *Let*

$$\mathcal{B}(V_n) = \left\{ \left(\ln k - \frac{1}{n}, \ln k + \frac{1}{n} \right] \mid k \in \mathbb{N} \right\}. \quad (2.28)$$

Then $\mathcal{B}(V_n)_{2\delta}$ is empty for $\delta > \frac{1}{n}$, and thus V_n converges to the zero module. However, a simple calculation shows that $\hbar(V_n) = 1$ for all n . This is simply because the limit in ϵ does not have to be uniform.

Example 2.15. *Let V be a persistence module with $\hbar(V) \neq 0$. Then $\text{Tr}_n(V)$ gives a sequence of persistence modules that converges to V in the truncated convergence topology, but*

$$\hbar(\text{Tr}_n(V)) = 0, \quad (2.29)$$

since the barcode growth functions of $\text{Tr}_n(V)$ are constant for $s > n$.

However, similarly to barcode growth functions, they do have semicontinuity properties.

3 Liouville Domains and Toric Domains

In this chapter we recall basic definitions and facts about Liouville domains, and we introduce toric domains, the main class of examples in this dissertation.

3.1 Liouville Domains and Completions

Definition 3.1. A Liouville domain (W, ω, λ) consists of a compact symplectic manifold with boundary (W, ω) together with a 1-form λ , called the Liouville form, satisfying:

1. $\omega = d\lambda$.
2. The unique vector field X_λ defined by the equation $\iota_{X_\lambda}\omega = \lambda$ is transversally outward-pointing along the boundary ∂W .

The vector field X_λ is called the Liouville vector field.

Definition 3.2. A symplectomorphism $\phi : (W_1, d\lambda_1) \rightarrow (W_2, d\lambda_2)$ is a strict Liouville morphism if

$$\lambda_1 = \phi^* \lambda_2 \tag{3.1}$$

Condition (2) guarantees that $\alpha := \lambda|_{\partial W}$ is a contact form on the boundary, and that the orientation on ∂W defined by $\alpha \wedge (d\alpha)^{n-1}$ agrees with the orientation induced by the volume form ω^n on W .

The first condition implies that $\mathcal{L}_{X_\lambda}\omega = \omega$ by Cartan's formula. Hence the flow ϕ_λ^t of X_λ expands the symplectic form (i.e., $(\phi_\lambda^t)^*\omega = e^t\omega$). In view of this property, an open neighborhood of ∂W can be identified with $((1 - \delta, 1] \times \partial W, d(\rho\alpha))$ for some $\delta > 0$, as follows: the flow defined by

$$\begin{cases} \frac{d\phi_\lambda^t}{dt} = X_\lambda \circ \phi_\lambda^t \\ \phi_\lambda^0(p) = p, \quad \forall p \in \partial W \end{cases} \quad (3.2)$$

exists on $(-\epsilon, 0]$ for sufficiently small $\epsilon > 0$. Set $\delta = e^{-\epsilon}$. Then the map

$$\Psi : (1 - \delta, 1] \times \partial W \rightarrow W, \quad (\rho, p) \mapsto \phi_\lambda^{\ln \rho}(p) \quad (3.3)$$

is a proper local diffeomorphism, and hence a diffeomorphism onto its image. Moreover, the map Ψ is a strict Liouville morphism.

To verify that $\Psi^*\lambda = \rho\alpha$, note that $T((1 - \delta, 1] \times \partial W) = \mathbb{R}\frac{\partial}{\partial\rho} \oplus T(\partial W)$. So,

$$\Psi^*\lambda = (\Psi^*\lambda)|_{\mathbb{R}\frac{\partial}{\partial\rho}} + (\Psi^*\lambda)|_{T(\partial W)} \quad (3.4)$$

One can easily show that $(\Psi^*\lambda)|_{\mathbb{R}\frac{\partial}{\partial\rho}} = \lambda(X_\lambda) d\rho = 0$, and hence

$$\Psi^*\lambda = (\Psi^*\lambda)|_{T(\partial W)} = \rho\lambda|_{T(\partial W)} = \rho\alpha \quad (3.5)$$

We can now give the definition of the completion of W to an open symplectic manifold:

$$\hat{W} := W \cup_\Psi [1, \infty) \times \partial W, \quad (3.6)$$

extending ω by

$$\omega := d(\rho\alpha) \quad \text{on } [1, \infty) \times \partial W. \quad (3.7)$$

The form ω is smooth, since $d(\rho\alpha)$ and ω agree for $\rho \in (1 - \delta, 1]$.

The Liouville vector field X_λ can be extended to $\hat{W}$, and the extension is a complete vector field. Thus the flow of $\hat{X}_\lambda$ gives an action of $\mathbb{R}$. We define an action of $c \in \mathbb{R}_+$ on $\hat{W}$ by

$$c \cdot_\lambda x := \phi_\lambda^{\ln c}(x) \quad (3.8)$$

We will drop λ , and also the dot, from the notation when there is no risk of confusion. If $(\rho, p) \in [1, \infty) \times \partial W$, then $c(\rho, p) = (c\rho, p)$, which explains the notation.

Similarly, for $\xi \in \mathbb{R}_+$ we will denote by $\xi W := \{\xi x \in \hat{W} \mid x \in W\}$ the Liouville domain with Liouville form $\hat{\lambda}|_{\xi W}$. We will also define a left and a right action of $\mathbb{R}_+$ on symplectomorphisms between Liouville domains in the obvious manner: if $\Phi : W_1 \rightarrow W_2$, then $\xi\Phi : W_1 \rightarrow \xi W_2$ and $\Phi\xi : \xi^{-1}W_1 \rightarrow W_2$.

A symplectomorphism between two Liouville domains does not necessarily preserve the Liouville form, unlike the map Ψ in (3.3).

Definition 3.3. A symplectomorphism $\phi : (W_1, d\lambda_1) \rightarrow (W_2, d\lambda_2)$ is an exact symplectomorphism if

$$\lambda_1 - \phi^*\lambda_2 = df \quad (3.9)$$

for some function f . An exact symplectomorphism ϕ is called compactly supported if $df = 0$ outside a compact subset of the interior of W_1 .

A compactly supported exact symplectomorphism $\phi : W_1 \rightarrow W_2$ can be extended to a symplectomorphism of the completions satisfying

$$\rho_2 \circ \hat{\phi} = \rho_1 \quad (3.10)$$

on the cylindrical end, where ρ_1, ρ_2 are the Liouville coordinates of $\hat{W}_1$ and $\hat{W}_2$, respectively.

3.2 Toric Domains

$\mathbb{C}^n = \mathbb{R}^{2n}$ carries a canonical $\mathbb{T}^n$ -action by coordinate-wise complex multiplication. This action preserves the standard symplectic form ω_0 , and it is Hamiltonian with moment map given by

$$\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto \pi(|z_1|^2, \dots, |z_n|^2) \quad (3.11)$$

A toric domain is a closed star-shaped domain in $\mathbb{R}^{2n}$ that is invariant under the $\mathbb{T}^n$ -action. If Ω is a closed domain given by the intersection of a star-shaped domain with the positive quadrant in $\mathbb{R}^n$, then

$$W_\Omega := \mu^{-1}(\Omega), \quad \partial W_\Omega = \mu^{-1}(\partial\Omega) \quad (3.12)$$

is a toric domain, and every toric domain is determined by such a domain in $\mathbb{R}^n$. Any such domain can be described by a map $f : S_{\geq 0}^{n-1} \rightarrow \mathbb{R}_{>0}$ as follows:

$$\Omega_f := \{rx \in \mathbb{R}^n \mid r \leq f(x), x \in S_{\geq 0}^{n-1}, r \in \mathbb{R}_{\geq 0}\} \quad (3.13)$$

and

$$\partial\Omega_f := \{rx \in \mathbb{R}^n \mid r = f(x), x \in S_{\geq 0}^{n-1}, r \in \mathbb{R}_{\geq 0}\} \quad (3.14)$$

The domain defined by

$$W_f = \mu^{-1}(\Omega_f) \subset \mathbb{R}^{2n}. \quad (3.15)$$

is the toric domain defined by f . We will write W_Ω or W_f interchangeably, depending on whether the domain Ω or the parametrizing function f is more convenient; the two notations are related by (3.15). The domain W_f will be called smooth whenever f has a smooth extension to an open neighborhood of $S_{\geq 0}^{n-1}$ in S^{n-1} . When W_f is

smooth, its boundary

$$\partial W_f = \mu^{-1}(\partial \Omega_f) \quad (3.16)$$

is a smooth hypersurface, and lines through the origin intersect ∂W_f transversely; thus W_f is a Liouville domain with the standard Liouville form λ_0 . Likewise, W_f will be called real analytic if f has a real analytic extension to some open neighborhood of $S_{\geq 0}^{n-1}$ in S^{n-1} .

The Reeb flow on ∂W_f , whenever well defined, is completely integrable in the sense that the functions μ_i , which generate the Hamiltonian $\mathbb{T}^n$ -action, are invariant under the Reeb flow, are in involution (i.e., $\{\mu_i, \mu_j\} = 0$), and are linearly independent almost everywhere. The points where linear independence fails have a nice structure as well: more precisely, the inverse image of a singular value of the moment map μ is a torus as well.

Definition 3.4. A toric domain W_f is called *concave* if, for an extension of f , still denoted by f , to a small open neighborhood U of $S_{\geq 0}^{n-1}$ in S^{n-1} , the set

$$\{rx \in \mathbb{R}^n \mid r \geq f(x), x \in U \subset S^{n-1}\} \subset \mathbb{R}^n \quad (3.17)$$

is convex.

Definition 3.5. A toric domain W_f is called *convex* if, for an extension of f to a small open neighborhood U of $S_{\geq 0}^{n-1}$ in S^{n-1} , the set

$$\{rx \in \mathbb{R}^n \mid r \leq f(x), x \in U \subset S^{n-1}\} \subset \mathbb{R}^n \quad (3.18)$$

is convex.

The domain W_f is called a *smooth convex/concave toric domain* if, in the setting of the definitions above, the extension of f can be taken smooth. Some words of caution are due at this point. For any $f : S_{\geq 0}^{n-1} \rightarrow (0, \infty)$, W_f defines a star-shaped

domain in $\mathbb{R}^{2n}$; however, according to the definition we have adopted, W_f being a convex or concave toric domain does not mean that it is a convex subset of $\mathbb{R}^{2n}$ (see [BG, Example 2.2]). Another possible point of confusion is that a toric domain that is smooth and convex/concave does not have to be a smooth convex/concave toric domain.

The completion of a smooth toric domain $\hat{W}_f$ is symplectomorphic to $\mathbb{R}^{2n}$ with the standard symplectic form ω_0 , and the Liouville vector field is given by

$$X_{\lambda_0} = \sum_i x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \quad (3.19)$$

Its flow is $\phi^t(x) = e^t x$; thus the Liouville coordinate for W_f is given by $\rho_f := \frac{r}{f} \circ \mu$, where r is the radial component of the polar coordinates on $\mathbb{R}^n$.

One can easily deduce that the action defined in (3.8) is simply scalar multiplication, which further explains the notation. Abusing notation, we will sometimes write f both for the function $f : S_{\geq 0}^{n-1} \rightarrow (0, \infty)$ and for $f \circ \mu$, and write W_f for the toric domain defined by either, as in (3.15).

4 Hamiltonian Floer Homology

Throughout this dissertation, the grading of Floer and symplectic homology is inessential for our purposes, and we treat all Floer and symplectic homology groups as ungraded vector spaces over the fixed field $\mathbb{F}$.

In this chapter we briefly review Hamiltonian Floer homology, mainly in order to fix the conventions and notation. Throughout this chapter, (M, ω) will be either an exact symplectic manifold (i.e., $\omega = d\lambda$ for a global 1-form λ) or a closed symplectically aspherical manifold (i.e., $\int_{S^2} u^* \omega = 0$ for every smooth map $u : S^2 \rightarrow M$).

4.1 The Action Functional and Periodic Orbits

A time-dependent Hamiltonian is a function $H : S^1 \times M \rightarrow \mathbb{R}$. Its Hamiltonian vector field X_H is given by the equation $\iota_{X_H} \omega = -dH_t$, and the flow of X_{H_t} is a symplectomorphism. A 1-periodic orbit γ of X_H is called non-degenerate whenever $d\Phi_H^1(\gamma(0)) : T_{\gamma(0)}M \rightarrow T_{\gamma(0)}M$ does not have 1 as an eigenvalue; a Hamiltonian is called non-degenerate if every 1-periodic orbit of X_H is non-degenerate. The condition of a hamiltonian being on-degenerate is generic and can be achieved by small perturbation.

A contractible 1-periodic orbit γ of X_{H_t} can be viewed as a critical point of the action functional $\mathcal{A}_{H_t} : \mathcal{L}_0 M \rightarrow \mathbb{R}$, where $\mathcal{L}_0 M = C_0^\infty(S^1, M)$ is the space of

contractible loops. For a capping $u : D^2 \rightarrow M$ with $u(e^{2\pi it}) = \gamma(t)$, the action is

$$\mathcal{A}_H(\gamma) := \int_{D^2} u^* \omega - \int_0^1 H_t(\gamma(t)) dt \quad (4.1)$$

The action functional does not depend on the choice of u . Let γ_s be a family of loops with $\gamma_0 = \gamma$ and $\frac{\partial}{\partial s}|_{s=0} \gamma_s = Y$. The critical points of the action functional are given by

$$d\mathcal{A}_H(\gamma)(Y) := \frac{\partial}{\partial s} \Big|_{s=0} \mathcal{A}_H(\gamma_s) = \int_0^1 \omega(Y, \frac{d\gamma}{dt} - X_H) dt = 0 \quad \text{for all } Y \quad (4.2)$$

Thus $\frac{d\gamma}{dt} = X_H(\gamma)$, and the set of critical points of the action functional is $C(H) := \{\gamma \in \mathcal{L}_0 M \mid \dot{\gamma} = X_H \circ \gamma\}$.

4.2 Floer Trajectories and the Chain Complex

Let J be an almost complex structure compatible with ω (i.e., $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric). Then $\langle Y, X \rangle_J := \int_0^1 g_J(Y(t), X(t)) dt$ defines an L^2 metric on the loop space. The negative gradient flow of the action functional with respect to this metric is given by the Floer equation

$$\frac{\partial u}{\partial s} - J \left(\frac{\partial u}{\partial t} - X_H \right) = 0 \quad (4.3)$$

where $u : \mathbb{R} \times S^1 \rightarrow M$. A solution u of the Floer equation is called a Floer trajectory; solutions carry a natural $\mathbb{R}$ -action by translation. The energy of a Floer trajectory is defined by

$$E(u) = \int_{-\infty}^{\infty} \int_0^1 \left\| \frac{\partial u}{\partial s} \right\|_J^2 dt ds \quad (4.4)$$

If $\lim_{s \rightarrow \pm\infty} u(s, \cdot) = \gamma_{\pm}$, then u is said to connect γ_- to γ_+ , and

$$E(u) = \mathcal{A}_H(\gamma_-) - \mathcal{A}_H(\gamma_+) \quad (4.5)$$

Since $E(u) \geq 0$, it follows that $\mathcal{A}_H(\gamma_+) \leq \mathcal{A}_H(\gamma_-)$, i.e., the action decreases along a Floer trajectory. Conversely, a finite-energy solution whose image is contained in a compact subset of M is asymptotic to 1-periodic orbits γ_-, γ_+ as $s \rightarrow \mp\infty$; see [Sa]. The space of unparametrized Floer trajectories connecting two 1-periodic orbits γ_1 and γ_2 , endowed with the C_{loc}^∞ topology, is denoted by $\mathcal{M}_{H,J}(\gamma_1, \gamma_2)$. For a C^∞ -generic perturbation $(\tilde{H}, \tilde{J})$ of (H, J) [HS], it is a manifold of dimension

$$\dim \mathcal{M}_{\tilde{H}, \tilde{J}}(\gamma_1, \gamma_2) = \mu_{CZ}(\gamma_2) - \mu_{CZ}(\gamma_1) - 1 \quad (4.6)$$

and it is compact, and hence a finite set, whenever it is 0-dimensional. See [HS].

Denote $C^t(H) := \{\gamma \in C(H) \mid \mathcal{A}_H(\gamma) < t\}$. For $t \notin \mathcal{S}(H)$, the underlying vector space of the filtered Floer chain complex is defined by

$$CF^t(H) := \mathbb{F} C^t(H) \quad (4.7)$$

The differential is defined by counting Floer trajectories with signs. To define the signs, one needs to give a coherent orientation to each moduli space; then the sign, denoted by $\varepsilon(u)$, is 1 when the natural orientation of u matches the coherent orientation and -1 otherwise. For details, see [FH].

$$n(\gamma, \beta) = \begin{cases} \sum_{u \in \mathcal{M}_{\tilde{H}, \tilde{J}}(\gamma, \beta)} \varepsilon(u) & \text{if } \dim \mathcal{M}_{\tilde{H}, \tilde{J}}(\gamma, \beta) = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.8)$$

$$\partial_J \gamma = \sum_{\beta} n(\gamma, \beta) \beta \quad (4.9)$$

Due to (4.5), the differential ∂_J is action decreasing: if $n(\gamma, \beta) \neq 0$, then $\mathcal{A}_H(\beta) \leq \mathcal{A}_H(\gamma)$. Hence $CF^t(H)$ is a subcomplex of the full Floer chain complex, and the inclusions $CF^s(H) \hookrightarrow CF^t(H)$ for $s \leq t$ induce maps on homology. One has $\partial_J^2 = 0$ for a generic perturbation, and the filtered Floer homology of H is defined

by

$$HF^t(H, J) := H((CF^t(H), \partial_J)) \quad (4.10)$$

The chain complex is defined using a sufficiently small perturbation; however, the resulting homology is independent of the perturbation as long as $t \notin \mathcal{S}(H)$.

4.3 Continuation Maps

Let (H_0, J_0) and (H_1, J_1) be two pairs of a Hamiltonian and an almost complex structure. For an isotopy (H_s, J_s) between the two pairs (Assumed to be constant on $\|s\| > s_0$ for some s), one can write the s -dependent Floer equation

$$\frac{\partial u}{\partial s} - J_s \left(\frac{\partial u}{\partial t} - X_{H_s} \right) = 0 \quad (4.11)$$

When a solution of (4.11) connects a 1-periodic orbit γ_0 of H_0 to a 1-periodic orbit γ_1 of H_1 , its energy is given by

$$E(u) = \mathcal{A}_{H_0}(\gamma_0) - \mathcal{A}_{H_1}(\gamma_1) - \int_{-\infty}^{\infty} \int_0^1 \frac{\partial H_s}{\partial s}(u(s, t), t) dt ds \quad (4.12)$$

Thus, the energy of a connecting trajectory is finite whenever $\int \frac{\partial H_s}{\partial s}(u(s, t), t) ds$ is bounded from above for all u . Under the assumption that the images of all the Floer trajectories connecting two orbits are confined to a compact set, the space of connecting trajectories is compact whenever it is 0-dimensional, and it defines a map

$$\theta_{H_s, J_s} : CF^t(H_0, J_0) \rightarrow CF^{t+C}(H_1, J_1) \quad (4.13)$$

by a definition similar to (4.9), for a generic perturbation of (H_s, J_s) . One can show that $\theta_{H_s, J_s} \circ \partial_{H_0, J_0} = \partial_{H_1, J_1} \circ \theta_{H_s, J_s}$. Thus we obtain a map

$$\theta_{H, K} : HF^t(H) \rightarrow HF^{t+C}(K) \quad (4.14)$$

for some $C \in (0, \infty)$.

Furthermore, the maps θ commute with the inclusion maps, and hence preserve the persistence structure on filtered Floer homology.

One case where a uniform bound applies to all connecting trajectories is when $H_0 = H_1 = H$, for then H_s can be taken to be constant in s . In this case the continuation map is an isomorphism and automatically action preserving, which means that the filtered Floer homology is independent of the almost complex structure, and it will be denoted by $HF^t(H)$.

5 Symplectic Homology

Symplectic homology is an invariant of Liouville domains. It is defined as the direct limit, over a certain family of Hamiltonians, of the Floer homology of the symplectic completion. In this chapter we recall the definition and basic properties of symplectic homology. Throughout this chapter, (W, ω, λ) will be a Liouville domain, its completion will be denoted by $(\hat{W}, \omega, \lambda)$, and ρ will denote its Liouville coordinate. In this setting, since ω is exact, the action functional can be defined without cappings, and it is given by

$$\mathcal{A}_H(\gamma) := \int_{S^1} \gamma^* \lambda - \int_0^1 H_t(\gamma(t)) dt \quad (5.1)$$

Hence it is also no longer necessary to restrict to contractible orbits.

5.1 Symplectic Homology as a Persistence Module

Definition 5.1. *A Hamiltonian H defined on $\hat{W}$ is called a linear at infinity Hamiltonian of W if*

1. $H(\rho, p) = h(\rho)$ on $(1, \infty) \times \partial W$, where $h'(\rho) > 0$ and $h''(\rho) > 0$;
2. $h(\rho) = m\rho + b$ on $[\rho_{\max}, \infty) \times \partial W$ for some $\rho_{\max} > 1$ and $m \in \mathbb{R} \setminus \mathcal{S}(\alpha)$; m is called the slope of H and is denoted by $\text{slope}(H)$;
3. $H < 0$ on W ;

3*. H is C^2 -small on W .

Here ρ is the Liouville coordinate of $\hat{W}$, and $\mathcal{S}(\alpha)$ denotes the action (period) spectrum of the contact form α , i.e., the set of periods of closed Reeb orbits; similarly, $\mathcal{S}(H)$ will denote the action spectrum of H . The set of such Hamiltonians is denoted by $\mathcal{H}_W$.

In addition, if $H|_W = c$ is constant, H is called admissible when $c < 0$ and semi-admissible when $c = 0$. The terminology used here is not standard; we emphasize that admissible Hamiltonians are not semi-admissible.

Let $H \in \mathcal{H}_W$. Then

$$\iota_{X_H} \omega = d\rho(X_H) \alpha - \alpha(X_H) d\rho + \rho \iota_{X_H} d\alpha = -dH = -h'(\rho) d\rho \quad (5.2)$$

on $(1, \infty) \times \partial W$. Thus

$$\alpha(X_H) = h'(\rho) \quad (5.3)$$

and

$$\iota_{X_H} d\alpha = 0 = d\rho(X_H) \quad (5.4)$$

Hence $X_H = h'(\rho)R_\alpha$, where R_α is the Reeb vector field of $(\partial W, \alpha)$. So every 1-periodic orbit of the Hamiltonian flow is of the form $(\rho, \gamma(h'(\rho)t))$, where γ is an $h'(\rho)$ -periodic Reeb orbit of $(\partial W, \alpha)$. In other words, the 1-periodic orbits of the Hamiltonian flow of H are in one-to-one correspondence with the Reeb orbits of $(\partial W, \alpha)$ of period less than $\text{slope}(H)$. For $\rho > \rho_{\max}$, the condition $h'(\rho) = m \notin \mathcal{S}(\alpha)$ ensures that X_H has no 1-periodic orbits beyond $\rho_{\max}$, and the periodic orbits of X_H are confined to a compact region, which in turn ensures that a small non-degenerate perturbation will have finitely many periodic orbits.

$$\mathcal{A}_H(\gamma) = \int_{S^1} \rho(\gamma(t)) \alpha(\gamma'(t)) dt - \int_{S^1} h(\rho(\gamma(t))) dt = \rho h'(\rho) - h(\rho) \quad (5.5)$$

A compatible almost complex structure J on $\hat{W}$ will be called admissible if, on $(1, \infty) \times \partial W$, it is invariant under the $\mathbb{R}$ -action, satisfies $J \frac{\partial}{\partial \rho} = R_\alpha$ (equivalently, $d\rho \circ J = -\alpha$), and preserves the contact distribution $\ker \alpha$, where it is compatible with $d\alpha$. Let u be a solution of the Floer equation that is asymptotic to γ as $s \rightarrow \pm\infty$, for a linear at infinity Hamiltonian and an admissible almost complex structure. The maximum principle ensures that $f := \rho \circ u$ cannot have a local maximum in the interior. Therefore, any solution of the Floer equation stays in a compact subset. So, for $t \in \mathbb{R} \setminus \mathcal{S}(H)$, $HF^t(H)$ is defined and finite dimensional. Let $\iota_{s,t}$ be the map induced by the inclusion at the chain level. Then

$$\iota_{s,t} = \iota_{r,t} \circ \iota_{s,r} \quad (5.6)$$

for $s \leq r \leq t$ in $\mathbb{R} \setminus \mathcal{S}(H)$. We will call the $\iota_{s,t}$ inclusion maps even though, at the homology level, they are not necessarily injective. They will sometimes be referred to as structure maps as well. The Floer homology of H for $t \in \mathcal{S}(H)$ will be defined by

$$HF^t(H) := \varinjlim_{s < t} HF^s(H) \quad (5.7)$$

Similarly, $\iota_{s,t}$ can be defined by

$$\iota_{s,t} := \varinjlim_{s < s' < t} \iota_{s,s'} \quad (5.8)$$

Defining it this way ensures that left-semicontinuity is satisfied. Hence we define the Floer persistence module of H , denoted by $(\mathbb{HF}(H), \iota^H)$, by

$$(\mathbb{HF}(H))_t := HF^t(H) \quad (5.9)$$

Let $H_- < H_+$ be two linear at infinity Hamiltonians and let H_s be a monotone homotopy such that $\lim_{s \rightarrow \pm\infty} H_s = H_\pm$. Let u be a solution of the s -dependent Floer

equation. A similar maximum principle applies to $\rho \circ u$, which provides a C^0 -bound for the space of connecting trajectories. Thus, the maps defined as in (4.14),

$$(\theta_{H_-, H_+})_t : HF^t(H_-) \rightarrow HF^t(H_+) \quad (5.10)$$

are well defined, and they are action preserving because $\frac{\partial H_s}{\partial s} \geq 0$. The maps θ_{H_-, H_+} are called continuation maps.

$\mathcal{H}_W$ forms a directed set with the partial order given by $<$; thus $HF^t(H)$, together with the continuation maps, becomes a directed system of vector spaces. The filtered symplectic homology of W is defined by

$$SH^t(W, \alpha) := \varinjlim_{H \in \mathcal{H}_W} HF^t(H) \quad (5.11)$$

The direct limit is taken over all Hamiltonians that are linear at infinity. The direct limit can be restricted to admissible Hamiltonians, as they are cofinal in $\mathcal{H}_W$. One could even take the limit over semi-admissible Hamiltonians as well. Note that a semi-admissible Hamiltonian $H \notin \mathcal{H}_W$, since the inequality in Definition 5.1 is strict. Nonetheless, if H is semi-admissible, the family of Hamiltonians $a_i H - \epsilon_i$, where $a_i, \epsilon_i > 0$, $a_i \rightarrow \infty$ and $\epsilon_i \rightarrow 0$, forms a cofinal family in the class of admissible Hamiltonians:

$$HF^t(a_i H) = HF^{t-\epsilon_i}(a_i H - \epsilon_i) \quad (5.12)$$

for $t \notin S(a_i H)$. Thus, taking the direct limits of both sides:

$$SH^t(W) = \varinjlim_{i \in \mathbb{N}} HF^t(a_i H)$$

Theorem 5.2. *Let $H_0 < H_1 \in \mathcal{H}_W$ and let $\theta_{H_0, H_1}^t : HF^t(H_0) \rightarrow HF^t(H_1)$ be the*

continuation maps. Then the following diagram commutes:

$$\begin{array}{ccc}
HF^s(H_1) & \xrightarrow{\iota^{H_1}} & HF^t(H_1) \\
\theta_{H_0, H_1} \uparrow & & \uparrow \theta_{H_0, H_1} \\
HF^s(H_0) & \xrightarrow{\iota^{H_0}} & HF^t(H_0)
\end{array} \tag{5.13}$$

In other words, the continuation maps are not only linear maps for each $t \in \mathbb{R}$, but together they form a morphism of persistence modules.

Therefore,

$$\mathbb{SH}(W, \alpha) := \varinjlim_{H \in \mathcal{H}_W} \mathbb{HF}(H) \tag{5.14}$$

defines what will be called the symplectic homology persistence module of W . We will sometimes suppress the field $\mathbb{F}$, and sometimes also the contact form α or the domain W , in the notation and simply write $\mathbb{SH}(W)$ or $\mathbb{SH}(\alpha)$. A direct limit of persistence modules is not necessarily a persistence module in the sense of Definition 2.1.

Proposition 5.3 (Cor. 3.7, [CG²M25]). *Let semi-admissible Hamiltonian H for W with $\text{slope}(H) = s$,*

$$SH^s(W) \cong HF(H). \tag{5.15}$$

Moreover, if $H_1 \leq H_2$ are semi-admissible for W with $s_1 = \text{slope}(H_1)$ and $s = \text{slope}(H_2)$, following diagram commutes

$$\begin{array}{ccc}
SH^{s_1}(\alpha) & \xrightarrow{\cong} & HF(H_1) \\
\downarrow \iota_{s_1, s_2} & & \downarrow \theta_{H_1, H_2} \\
SH^{s_2}(\alpha) & \xrightarrow{\cong} & HF(H_2)
\end{array} \tag{5.16}$$

map.

Corollary 5.4. $\mathbb{SH}(W)$ is a persistence module.

The proposition is not only useful for guaranteeing that $\mathbb{SH}^t(W)$ is finite dimen-

sional. Since symplectic homology as a direct limit is quite inconvenient to work with, the proposition will be essential for working around the inconveniences.

Note that isomorphism of in Proposition 5.3 are isomorphism of vector space. So claim is not $SH^s(W) \cong HF^t(H)$. However, there exist for a monotone increasing function $\mathfrak{a}$ which can be made C^0 -close to the identity by a suitable choice of H such that $\mathbb{SH}^t(\alpha) = \mathbb{HF}^{\mathfrak{a}(t)}(H)$ see [ÇG²M25, Thm. 3.5]. Two of these are reparametrization of each other as persistence modules. Advantage of working with floer homology is that it stems from a filtered chain complex and its generator for before $\mathfrak{a}(t)$ is a bound for its epsilon barcode. (See. [ÇG²24a])

Fix a semi-admissible Hamiltonian H with $\text{slope}(H) = 1$; then $\text{slope}(sH) = s$. Set

$$V_s := HF(sH) \tag{5.17}$$

for $s \notin \mathcal{S}(\alpha)$, and define V_s for $s \in \mathcal{S}(\alpha)$ by taking the direct limit as above. With $\theta_{s,s'} := \theta_{s'H,sH}$, (V, θ) is a persistence module. Moreover, as a corollary of Proposition 5.3, we have

Corollary 5.5. *(V, θ) and $\mathbb{SH}(\alpha)$ are isomorphic as persistence modules.*

It is now possible to define the barcode growth function of a Liouville domain W by

$$\mathfrak{b}_\epsilon(W, s) := \mathfrak{b}_\epsilon(\mathbb{SH}(W), s) \tag{5.18}$$

As pointed out in Section 2.2, $\mathfrak{b}_\epsilon(W, s)$ is increasing in s and decreasing in ϵ . Thus the barcode entropy $\hbar(W) := \hbar(\mathbb{SH}(W))$ is well defined as an element of $\mathbb{R}_+ \cup \{\infty\}$.

Lastly, we will introduce some persistence module morphisms induced by maps between Liouville domains that will be needed in Section 5.2 and in the proof of Proposition 5.11. The first one will be the Viterbo transfer map.

Theorem 5.6. *Let (W_1, λ_1) and (W_2, λ_2) be two Liouville domains and let $\Phi : W_1 \rightarrow \mathring{W}_2$ be a codimension zero exact symplectic embedding. Then there exists a*

persistence module morphism

$$\Phi^* : \mathcal{SH}(W_2) \rightarrow \mathcal{SH}(W_1) \quad (5.19)$$

Furthermore, if $\Phi : W_1 \rightarrow \mathring{W}_2$ and $\Psi : W_2 \rightarrow \mathring{W}_3$ are codimension zero exact symplectic embeddings, then

$$(\Psi \circ \Phi)^* = \Phi^* \circ \Psi^* \quad (5.20)$$

Finally, if $\Phi, \Psi : W_1 \rightarrow \mathring{W}_2$ are two exact symplectic embeddings that are isotopic, then

$$\Phi^* = \Psi^* \quad (5.21)$$

The Viterbo transfer map was introduced in [Vi] for codimension zero symplectic embeddings on total symplectic homology. The construction uses continuation map for monotone homotopy therefore maps are action preserving. More precisely, if W is a Liouville domain and V is a subdomain, then there exists a map

$$SH^t(W) \rightarrow SH^t(V) \quad (5.22)$$

One can modify the Liouville form on the target manifold so that an exact symplectic embedding becomes a strict Liouville embedding, while leaving the Liouville form unchanged in a neighborhood of the boundary (see [GH, Lemma 7.5]). The following lemma guarantees that the symplectic homology of the target manifold stays unchanged under this modification.

Lemma 5.7. *Let (W_1, λ_1) and (W_2, λ_2) be two Liouville domains and let $\Phi : W_1 \rightarrow W_2$ be a compactly supported exact symplectomorphism. Then*

$$\mathcal{SH}(W_1) \cong \mathcal{SH}(W_2) \quad (5.23)$$

as persistence modules.

The lemma can be easily proved by showing that Φ extends to a symplectomorphism between the completions of the two Liouville domains such that whenever H is an admissible Hamiltonian for W_2 , $H \circ \Phi$ is an admissible Hamiltonian for W_1 .

Another map we will need is the so-called rescaling map. Let U be a Liouville subdomain of an open Liouville manifold W , and let CU be its rescaling defined in (3.8). If ρ_U is the Liouville coordinate for U , then the Liouville coordinate of CU is $\rho_{CU} = \frac{1}{C}\rho_U$. Thus, if $H : W \rightarrow \mathbb{R}$ is an admissible Hamiltonian of slope s for U , then it is also an admissible Hamiltonian of slope Cs for CU , which gives an isomorphism between $SH^{C^{-1}s}(U)$ and $SH^s(CU)$, called the rescaling map. We summarize this result and the properties of the map in the theorem below.

Theorem 5.8. *There exists an isomorphism $r_\lambda : SH^{\lambda^{-1}s}(U) \rightarrow SH^s(\lambda U)$ induced by the scaling action of $\lambda > 1$, making the following diagram commute:*

$$\begin{array}{ccc} SH^{\lambda^{-1}s}(U) & \xrightarrow{\sim} & SH^s(\lambda U) \\ & \searrow & \swarrow \\ & SH^s(U) & \end{array} \quad (5.24)$$

where the left diagonal arrow is the structure map of the symplectic homology persistence module and the right diagonal arrow is the Viterbo transfer map.

The proof for commutative diagram can be find in [GU, PRSZ].

5.2 Symplectic Banach–Mazur Distance

The symplectic Banach–Mazur distance is a pseudo-metric defined on certain collections of Liouville domains, inspired by a similar notion of distance on convex bodies. It can be considered a symplectic homology version of Hofer’s distance. The symplectic Banach–Mazur distance has been introduced on several occasions,

in slightly different settings and with small differences in the definition. Treatments of symplectic Banach–Mazur distances can be found in [PRSZ, SZ, Us] and the references therein.

We will only define the symplectic Banach–Mazur distance for compact star-shaped domains (with respect to the origin) of $\mathbb{R}^{2n}$, since this will be sufficient for our purposes; from now on, by a star-shaped domain we will mean a compact star-shaped domain with respect to the origin in $\mathbb{R}^{2n}$. The construction applies to any Liouville manifold of finite type using the rescaling defined in (3.8).

For a continuous map $f : S^{2n-1} \rightarrow (0, \infty)$, $W_f = \{(r, \theta) \in \mathbb{R}^{2n} \mid r \leq f(\theta)\} \subset \mathbb{R}^{2n}$ is a compact star-shaped domain. This is consistent with the notation in (3.15) upon replacing f with $f \circ \mu$. We will call a star-shaped domain smooth whenever its boundary is a smooth hypersurface. A smooth exact symplectomorphism $\Phi : K \rightarrow U$, where K and U are a compact and an open subset of $\mathbb{R}^{2n}$ respectively, is a smooth exact symplectomorphism from an open neighborhood of K into U . An exact symplectic embedding $\phi : U \rightarrow \mathring{V}$ is called unknotted if there exists an isotopy of exact symplectic embeddings $\Psi_t : U \rightarrow \mathring{V}$ such that Ψ_0 is the natural inclusion and $\Psi_1 = \phi$.

Definition 5.9. *Let U and V be two star-shaped domains and let λ be a positive real number. We call λ (U, V) -admissible if there exist exact symplectic embeddings*

$$\varphi : \lambda^{-1}U \rightarrow \mathring{V} \tag{5.25}$$

and

$$\psi : V \rightarrow \lambda \mathring{U} \tag{5.26}$$

such that the map

$$\psi \circ \varphi : \lambda^{-1}U \rightarrow \lambda \mathring{U} \tag{5.27}$$

is unknotted.

Set

$$\delta(U, V) := \inf \{ \ln \lambda \mid \lambda \text{ is } (V, U)\text{-admissible} \}. \quad (5.28)$$

The maps induce

$$(\psi^*)_s : \mathbb{SH}^{\lambda^{-1}s}(U) \cong \mathbb{SH}^s(\lambda U) \rightarrow \mathbb{SH}^s(V) \quad (5.29)$$

and

$$(\varphi^*)_s : \mathbb{SH}^s(V) \rightarrow \mathbb{SH}^{\lambda s}(U) \quad (5.30)$$

The composition gives a map

$$(\varphi^*)_s \circ (\psi^*)_s : \mathbb{SH}^{\lambda^{-1}s}(U) \rightarrow \mathbb{SH}^{\lambda s}(U) \quad (5.31)$$

Due to the unknottedness condition, $(\varphi^*) \circ (\psi^*) = \iota_{\lambda^{-1}U, \lambda U}^*$, which together with the rescaling isomorphism of Theorem 5.8 gives one side of an interleaving equation between $\mathbb{SH}(U)$ and $\mathbb{SH}(V)$. (See Remark 2.11). The unknottedness condition is necessary to make the connection to filtered symplectic homology, but it only guarantees one side of the interleaving: for maps as in the definition, we can define

$$\lambda \psi \lambda^{-1} : \lambda^{-1}V \rightarrow \mathring{U} \quad (5.32)$$

and

$$\lambda^{-1} \varphi \lambda : U \rightarrow \lambda \mathring{V} \quad (5.33)$$

however, it is not guaranteed that the composition $\lambda^{-1} \varphi \lambda \circ \lambda \psi \lambda^{-1}$ is unknotted. Hence $\delta(U, V)$ is not symmetric, due to the unknottedness condition (see [GU, Us] for more details). Here we do not require the exact symplectic embeddings to be defined on the whole of $\mathbb{R}^{2n}$ or to be the identity outside of a compact subset, as opposed to [PRSZ].

Definition 5.10. *The symplectic Banach–Mazur distance between two star-shaped domains U and V is defined by*

$$d_{\text{SBM}}(U, V) := \max\{\delta(U, V), \delta(V, U)\}. \quad (5.34)$$

The symplectic Banach–Mazur distance is an extended pseudo-metric on the set of star-shaped domains (see Lemma 5.12). It is symmetric by definition, and it satisfies the triangle inequality (see [Us]).

Discussion following (5.31) and Remark 2.11 leads to the property of the symplectic Banach–Mazur distance essential to our purpose, which is that it gives a bound on the truncated distance between the symplectic homology persistence modules of two domains with smooth boundary (see [SZ, Us]):

$$d_s(\mathbb{SH}(W), \mathbb{SH}(U)) := d(\text{Tr}_s(\mathbb{SH}(W)), \text{Tr}_s(\mathbb{SH}(U))) \leq s(e^{d_{\text{SBM}}(W, U)} - 1), \quad (5.35)$$

Let U_n be a sequence of smooth star-shaped domains and let V be a smooth star-shaped domain such that $d_{\text{SBM}}(U_n, V) \rightarrow 0$. As a consequence of Proposition 2.13, we have

$$\mathfrak{b}_\epsilon(V, s) \leq \liminf_{n \rightarrow \infty} \mathfrak{b}_\epsilon(U_n, s) \quad (5.36)$$

since the corresponding symplectic homology persistence modules converge in the truncated convergence topology. In particular, if $g, f : S^{2n-1} \rightarrow (0, \infty)$, then

$$d_s(\mathbb{SH}(W_f), \mathbb{SH}(W_g)) := d(\text{Tr}_s(\mathbb{SH}(W_f)), \text{Tr}_s(\mathbb{SH}(W_g))) \leq s(e^\eta - 1), \quad (5.37)$$

where $\eta = \max |\ln \frac{g}{f}|$. Similarly to (5.36), whenever $f_n, f : S^{2n-1} \rightarrow (0, \infty)$ with f_n smooth and $f_n \rightarrow f$ uniformly, we have $d_{\text{SBM}}(W_f, W_{f_n}) \leq \max |\ln(f/f_n)|$, and thus

$$d_{\text{SBM}}(W_f, W_{f_n}) \rightarrow 0.$$

$$b_\epsilon(W_f, s) \leq \liminf_{n \rightarrow \infty} b_\epsilon(W_{f_n}, s) \quad (5.38)$$

We would like to extend the definition of the barcode growth function of a smooth star-shaped domain to non-smooth star-shaped domains. This is necessary for proving Theorem 1.2, and for (5.38) to remain correct when f is allowed to be merely continuous. There are a few directions one could take. The filtered symplectic homology of a non-smooth star-shaped domain could be defined using an inverse limit construction as in the proof of Proposition 5.11, and it would still have structure maps defined; it would be an invariant of the exact symplectomorphism type of the interior of the domain. However, it is not guaranteed to be a persistence module according to the definition we have adopted, and it may or may not have a well-defined barcode. Another possible method is taking the limit of $\text{Tr}_s(\text{SH}(W_k))$ with respect to the interleaving distance, for a sequence of smooth star-shaped domains satisfying $d_{\text{SBM}}(W_k, W) \rightarrow 0$, in the completion of the space of persistence modules. Both would be rather laborious constructions and would require more background on persistence modules. Instead, we will directly define the barcode growth function as follows. First, define the set

$$\mathfrak{S}(U) = \{\{U_k\} \mid U_k \text{ is smooth and } d_{\text{SBM}}(U, U_k) \rightarrow 0\} \quad (5.39)$$

and set

$$\mathfrak{b}_\epsilon(U, s) := \inf_{\{U_k\} \in \mathfrak{S}(U)} \liminf_{k \rightarrow \infty} \mathfrak{b}_\epsilon(U_k, s) \quad (5.40)$$

When U is smooth, one can take the constant sequence $U_k = U$ and apply (5.36) to see that the definition above agrees with the smooth case. Moreover, $\mathfrak{b}_\epsilon(W, s)$ is finite for all ϵ and s , whether W is smooth or not, due to (2.16) applied to a smooth

approximation. The definition can also be expressed as

$$\mathbf{b}_\epsilon(U, s) = \lim_{\delta \rightarrow 0^+} \inf \{ \mathbf{b}_\epsilon(W, s) \mid d_{\text{SBM}}(U, W) < \delta \text{ and } W \text{ is smooth} \} \quad (5.41)$$

The infimum can be replaced by a minimum, since the infimum is taken over a set of integers. Hence, for fixed ϵ and s , there exists a sequence $\{U_k\} \in \mathfrak{S}(U)$ such that $\mathbf{b}_\epsilon(U, s) = \lim_{k \rightarrow \infty} \mathbf{b}_\epsilon(U_k, s)$.

The barcode growth functions of non-smooth domains inherit monotonicity properties: $\mathbf{b}_\epsilon(W, s)$ is monotone increasing in s for fixed ϵ , and monotone decreasing in ϵ for fixed s , since a pointwise limit of monotone functions is monotone and a pointwise infimum of a set of monotone functions is monotone. We do not see any apparent reason for left semicontinuity of the barcode growth function to hold for non-smooth domains, nor do we have a counterexample.

5.3 Invariance of Symplectic Homology and Barcode Growth

The filtered symplectic homology is an invariant under exact symplectomorphisms of the interior. A version of the statement first appeared in [CFHW], where the Floer homologies are defined on contractible orbits. Symplectic homology as defined in [CFH] inherently only depends on the interior. The fact has also been proven in [Gu, Hu24] in settings similar to ours. The result is simply a corollary of the existence of the Viterbo transfer map for exact symplectic embeddings and the universal property of the inverse limit.

Proposition 5.11. *Let (W_1, λ_1) and (W_2, λ_2) be two Liouville domains. If there exists an exact symplectomorphism $\phi : (\mathring{W}_1, \lambda_1) \rightarrow (\mathring{W}_2, \lambda_2)$, then*

$$\mathbb{SH}(\lambda_1) \cong \mathbb{SH}(\lambda_2) \quad (5.42)$$

as persistence modules.

Proof. For a Liouville domain W , the set of Liouville subdomains together with the Viterbo transfer maps forms an inverse system. Choose a sequence $\eta_n > 0$ with $\eta_n \rightarrow 1$ such that $\eta_n W$ is well defined; then $\eta_n W$ is cofinal in the inverse system. By Theorem 5.8,

$$SH^s(\eta_n W) \cong SH^{\eta_n^{-1}s}(W) \quad (5.43)$$

If $s \notin \mathcal{S}(\lambda)$ and η_n is sufficiently close to 1, the structure map $\iota_{\eta_n^{-1}s, s}$ is an isomorphism. The Viterbo transfer map of the inclusion $\eta_n W \hookrightarrow W$ is an isomorphism, since it can be written as a composition of the rescaling map and structure maps. So we can conclude that

$$SH^s(W) = \varprojlim_U SH^s(U), \quad (5.44)$$

for $s \notin \mathcal{S}(\lambda)$. Since the Viterbo transfer maps are persistence module morphisms, we can actually deduce that

$$\mathbb{SH}(W) = \varprojlim_U \mathbb{SH}(U) \quad (5.45)$$

Now let $\phi : \mathring{W}_1 \rightarrow \mathring{W}_2$ be an exact symplectomorphism and let $\{V_n\}$ be a cofinal sequence of subdomains of W_1 . The idea is that we can find a cofinal sequence of subdomains $\{U_n\}$ of W_2 such that

$$\phi(V_n) \subset \mathring{U}_n \quad (5.46)$$

and

$$\phi^{-1}(U_{n-1}) \subset \mathring{V}_n \quad (5.47)$$

Compositions of ϕ and ϕ^{-1} with the inclusion maps of the subdomains give the following maps:

$$f_n : \mathbb{SH}(U_n) \rightarrow \mathbb{SH}(V_n) \quad (5.48)$$

and

$$g_n : \mathbb{SH}(V_n) \rightarrow \mathbb{SH}(U_{n-1}) \quad (5.49)$$

The following sequence forms an inverse system of persistence modules

$$\cdots \xrightarrow{f_{n+1}} \mathbb{SH}(V_{n+1}) \xrightarrow{g_{n+1}} \mathbb{SH}(U_n) \xrightarrow{f_n} \mathbb{SH}(V_n) \xrightarrow{g_n} \mathbb{SH}(U_{n-1}) \xrightarrow{f_{n-1}} \cdots \quad (5.50)$$

because $f_n \circ g_n = \iota_{U_n, V_n}^*$.

Both $\{\mathbb{SH}(V_n)\}$ and $\{\mathbb{SH}(U_n)\}$ are cofinal in it. Thus

$$\varprojlim_{n \in \mathbb{N}} \mathbb{SH}(V_n) = \varprojlim_{n \in \mathbb{N}} \mathbb{SH}(U_n), \quad (5.51)$$

which completes the proof. $\square$

As the notation suggests, the isomorphism between the two symplectic homologies is an isomorphism of persistence modules. Thus $\mathbf{b}_\epsilon(W_1, s) = \mathbf{b}_\epsilon(W_2, s)$ when W_1 and W_2 are Liouville domains with exact symplectomorphic interiors. We will see that the barcode growth functions are the same when at least one of the domains has smooth boundary. We do not know whether the statement holds for two non-smooth domains, but barcode entropy and slow barcode entropy are invariants of the exact symplectomorphism type of the interior. The following lemma will be helpful for proving the invariance results.

Lemma 5.12. *Let W_1 and W_2 be closed star-shaped domains (not necessarily with smooth boundary). If W_1 and W_2 have exact symplectomorphic interiors, then $d_{\text{SBM}}(W_1, W_2) = 0$.*

Proof. Let $F : \mathring{W}_1 \rightarrow \mathring{W}_2$ be an exact symplectomorphism. For a given $\lambda > 1$, define $\phi := F|_{\lambda^{-1}W_1}$. Then

$$\phi : \lambda^{-1}W_1 \rightarrow \mathring{W}_2 \quad (5.52)$$

is an exact symplectic embedding.

Let $\iota : t\mathring{W}_1 \rightarrow \lambda\mathring{W}_1$ be the natural inclusion. Fix $\eta_0 \in (1, \lambda)$ satisfying $\lambda^{-1}W_1 \subset F^{-1}(\eta_0^{-1}\mathring{W}_2)$, and define the map $\psi := \iota \circ (\eta_0 F^{-1} \eta_0^{-1})|_{W_2}$. Then

$$\psi : W_2 \rightarrow \lambda\mathring{W}_1 \quad (5.53)$$

is an exact symplectic embedding. The composition

$$\psi \circ \phi = \iota \circ (\eta_0 F^{-1} \eta_0^{-1}) \circ F|_{\lambda^{-1}W_1} : \lambda^{-1}W_1 \rightarrow \lambda\mathring{W}_1 \quad (5.54)$$

is isotopic to the inclusion: the map $\iota \circ (t F^{-1} t^{-1}) \circ F|_{\lambda^{-1}W_1}$ is defined for all $t \in [1, \eta_0]$; for $t = 1$ it is the inclusion $\lambda^{-1}W_1 \hookrightarrow \lambda\mathring{W}_1$, and for $t = \eta_0$ it is $\psi \circ \phi$, giving the desired isotopy. Thus every $\lambda > 1$ is (W_1, W_2) -admissible, and hence

$$\delta(W_1, W_2) = 0 \quad (5.55)$$

Applying the exact same construction with F replaced by F^{-1} , we see that

$$\delta(W_2, W_1) = 0 \quad (5.56)$$

as well, and thus we get the desired result:

$$d_{\text{SBM}}(W_1, W_2) = 0 \quad (5.57)$$

□

As mentioned earlier, d_{SBM} is only a pseudo-metric on the set of star-shaped domains. We do not know if this is the only case where d_{SBM} is zero. Proposition 5.11, restricted to star-shaped domains, is a direct consequence of the lemma, as the lemma implies that $d_{\text{int}}(\mathbb{S}\mathbb{H}(W_1), \mathbb{S}\mathbb{H}(W_2)) = 0$. Using the lemma, we will show that the barcode growth is an invariant of the exact symplectomorphism type of the interior

of a star-shaped domain when at least one of the domains is smooth. Applying the triangle inequality to the lemma, we get the following corollary:

Corollary 5.13. *If W_1 and W_2 have exact symplectomorphic interiors, then*

$$d_{\text{SBM}}(U, W_1) = d_{\text{SBM}}(U, W_2) \quad (5.58)$$

for every star-shaped domain U .

Theorem 5.14. *Let W_1 be a smooth star-shaped domain and let W_2 be a star-shaped domain. If W_1 and W_2 have exact symplectomorphic interiors, then*

$$b_\epsilon(W_1, s) = b_\epsilon(W_2, s) \quad (5.59)$$

for all ϵ and s .

Proof. By Lemma 5.12, for the constant sequence $U_n = W_1$, we have $d_{\text{SBM}}(U_n, W_2) \rightarrow 0$. Thus, by the definition (5.40),

$$b_\epsilon(W_2, s) \leq \lim_{n \rightarrow \infty} b_\epsilon(U_n, s) = b_\epsilon(W_1, s). \quad (5.60)$$

For any sequence $\{V_n\}$, Corollary 5.13 implies that $d_{\text{SBM}}(V_n, W_2) \rightarrow 0$ if and only if $d_{\text{SBM}}(V_n, W_1) \rightarrow 0$. Therefore, by Proposition 2.13,

$$b_\epsilon(W_1, s) \leq \liminf_{n \rightarrow \infty} b_\epsilon(V_n, s), \quad (5.61)$$

and taking the infimum over all sequences $\{V_n\} \in \mathfrak{S}(W_2)$ gives $b_\epsilon(W_1, s) \leq b_\epsilon(W_2, s)$. This concludes the proof. $\square$

Theorem 5.15. *Let W_1 and W_2 be two star-shaped domains. If W_1 and W_2 have exact symplectomorphic interiors, then*

$$b_{\epsilon+\delta}(W_1, s - \delta) \leq b_\epsilon(W_2, s) \quad (5.62)$$

and

$$\mathfrak{b}_{\epsilon+\delta}(W_2, s - \delta) \leq \mathfrak{b}_{\epsilon}(W_1, s) \quad (5.63)$$

for all $\delta > 0$ and $\epsilon > \delta$.

Proof. By symmetry, it is enough to prove (5.62). Let $\{U_k\}$ and $\{V_k\}$ be a sequence such that $\mathfrak{b}_{\epsilon}(W_1, s) = \lim_{k \rightarrow \infty} \mathfrak{b}_{\epsilon}(U_k, s)$ and $\mathfrak{b}_{\epsilon}(W_2, s) = \lim_{k \rightarrow \infty} \mathfrak{b}_{\epsilon}(V_k, s)$ then $d_{\text{SBM}}(U_k, V_k) \rightarrow 0$ by triangle inequality. By Proposition 2.13

$$\mathfrak{b}_{\epsilon+\delta}(W_1, s - \delta) \leq \liminf_{k \rightarrow \infty} \mathfrak{b}_{\epsilon}(U_k, s - \delta) \leq \lim_{k \rightarrow \infty} \mathfrak{b}_{\epsilon}(V_k, s) = \mathfrak{b}_{\epsilon}(W_2, s)$$

,

□

Corollary 5.16. *Let W_1 and W_2 be two star-shaped domains. If W_1 and W_2 have exact symplectomorphic interiors, then*

$$\hbar(W_1) = \hbar(W_2) \quad \text{and} \quad \hbar^{\text{Slow}}(W_1) = \hbar^{\text{Slow}}(W_2) \quad (5.64)$$

The corollary follows from Theorem 5.15 by the definitions (2.25) and (2.27).

With these preliminaries in place, we are now ready to prove the main theorems.

6 Proofs of the Main Theorems

6.1 Periodic Orbits on Toric Domains

In this chapter we prove Theorems 1.1 and 1.2. We will start by calculating the flow of a semi-admissible Hamiltonian on the cylindrical end of a toric domain and the Reeb flow on the boundary, and finding the periodic orbits. Similar calculations can be found in [GH].

Let W_f be a smooth or real analytic toric domain and define $F := \frac{r}{f}$. Then its completion is $\hat{W}_f = \mathbb{R}^{2n}$ with the standard symplectic structure, and the Liouville coordinate is given by $\rho = F \circ \mu$. Let $H := h \circ \rho$, where h is as in Definition 5.1. We will calculate its Hamiltonian vector field and the flow. The coordinates μ_i, θ_i are well defined unless $\mu_i = 0$. Thus we will use a mix of the coordinates x_i, y_i and μ_i, θ_i . Let us start the calculation with the left-hand side of $\iota_{X_H} \omega = -dH$:

$$\omega_0 = \sum_{i, \mu_i \neq 0} d\mu_i \wedge d\theta_i + \sum_{i, \mu_i = 0} dx_i \wedge dy_i \quad (6.1)$$

$$\iota_{X_H} \omega = \sum_{i, \mu_i \neq 0} d\mu_i(X_H) d\theta_i - d\theta_i(X_H) d\mu_i + \sum_{i, \mu_i = 0} dx_i(X_H) dy_i - dy_i(X_H) dx_i \quad (6.2)$$

Now, let us calculate the right-hand side:

$$dH = h'(F) \left(\sum_{i, \mu_i = 0} 2\pi x_i \partial_i F dx_i + 2\pi y_i \partial_i F dy_i + \sum_{i, \mu_i \neq 0} \partial_i F d\mu_i \right) \quad (6.3)$$

When $\mu_i = 0$, $x_i = y_i = 0$; thus

$$dH = h'(F) \sum_{i, \mu_i \neq 0} \partial_i F d\mu_i \quad (6.4)$$

Thus the Hamiltonian vector field of an admissible Hamiltonian is given by

$$X_H = h'(F) \sum_{i, \mu_i \neq 0} \partial_i F \frac{\partial}{\partial \theta_i} \quad (6.5)$$

The flow of X_H can be calculated explicitly and is given by

$$\phi_H^t(z_1, \dots, z_n) = (e^{2\pi i h'(F) \partial_1 F t} z_1, \dots, e^{2\pi i h'(F) \partial_n F t} z_n) \quad (6.6)$$

Note that $\mu_i = 0$ implies $|z_i| = 0$.

We can also deduce that $\sum_{i, \mu_i \neq 0} \partial_i F \frac{\partial}{\partial \theta_i}$, restricted to ∂W_f , gives the Reeb vector field, and its flow is given by

$$\varphi^t(z_1, \dots, z_n) = (e^{2\pi i \partial_1 F(\mu) t} z_1, \dots, e^{2\pi i \partial_n F(\mu) t} z_n) \quad (6.7)$$

For a subset I of $\{1, \dots, n\}$, denote by $\mathbb{R}^I \subset \mathbb{R}^n$ the subspace of points whose coordinates x_i vanish for all $i \notin I$. Let $\iota_I : \mathbb{R}^d \rightarrow \mathbb{R}^I \subset \mathbb{R}^n$ be the identification of $\mathbb{R}^I$ with $\mathbb{R}^d$, where $d = |I|$, denote by π_I the projection to $\mathbb{R}^I$, and define $\partial\Omega_I := \partial\Omega \cap \mathbb{R}^I$. Consider

$$d\pi_I \nabla F : \mathbb{R}_{>0}^I \rightarrow \mathbb{R}^d \quad (6.8)$$

For $p \in \partial\Omega_I$, the torus $\mu^{-1}(p)$ consists of T -periodic orbits if and only if

$$T d\pi_I \nabla F(p) = \vec{q} \in \mathbb{Z}^d \quad (6.9)$$

We will call such tori periodic. For a point satisfying the above equality, the period

is given by

$$T = \frac{\|\vec{q}\|}{\|d\pi_I \nabla F(p)\|} \quad (6.10)$$

Denoting by $\nabla_\theta f$ the gradient of f as a function on the sphere, we have

$$d\pi_I \nabla F = \frac{1}{f} \frac{\partial}{\partial r} - \frac{r}{f^2} d\pi_I \nabla_\theta f \quad (6.11)$$

and

$$\|d\pi_I \nabla F\| = \frac{1}{f} \sqrt{1 + \frac{\|d\pi_I \nabla_\theta f\|^2}{f^2}} \quad (6.12)$$

Hence we can define $\tau_I : S_{\geq 0}^{d-1} \rightarrow \mathbb{R}_+$ by

$$\tau_I(p) = \frac{f}{\sqrt{1 + \|d\pi_I \nabla_\theta f(\hat{p})\|^2 / f^2(\hat{p})}} \quad (6.13)$$

We will write S^{d-1} for the sphere in $\mathbb{R}^I$, as we did above, in order not to encumber the notation.

Note that $d\pi_I \nabla F$ is normal to the surface $\partial\Omega_I = \partial\Omega \cap \mathbb{R}^I$; thus we can rephrase the condition for $L_p := \mu^{-1}(p)$ to be periodic in terms of the Gauss map G_{Ω_I} of the surface $\partial\Omega_I$, as in [GH, Sect. 2.2 and 2.3]: the torus $\mu^{-1}(p)$ is periodic if

$$G_{\Omega_I}(p) = \frac{\vec{q}}{\|\vec{q}\|} \quad (6.14)$$

for some integer vector $\vec{q}$. The minimal period of such a torus is given by

$$T = \tau_I(\hat{p}) \|\vec{q}\| \quad (6.15)$$

While the formula for the period is for points p in the interior of $\partial\Omega_I$, the map τ_I is defined on an open neighborhood U_I of $S_{\geq 0}^{d-1}$ for a smooth toric domain.

Lastly, one can show that invariant torus L_p is morse-bott non-degenerate if and only if p is regular point of G_I by calculating Jacobian matrices of φ in the mixed

coordinate defined at the of the chapter.

6.2 Proof of Theorem 1.1: the Smooth Case

The strategy of the proof is to count the number of critical tori of a semi-admissible Hamiltonian H with slope less than s . Then we perturb H to make it non-degenerate; each torus will then separate into 2^d non-degenerate Hamiltonian orbits, where $d \leq n$. These Hamiltonian orbits are the generators of $CF(H)$, and the number of generators is a bound for the number of ϵ -bars starting before s .

Theorem 6.1. *Let W_f , for $f : S_{\geq 0}^{n-1} \rightarrow \mathbb{R}_+$, be a smooth convex or concave toric domain or a real analytic toric domain. Then the number of generators with action less than s is bounded from above by*

$$C_n(f) s^n + C_0(f), \quad (6.16)$$

where $C_n(f)$ and $C_0(f)$ are independent of the auxiliary structures and s , and depend C^1 -continuously on f .

Proof. We will first assume that Ω satisfies the following conditions:

1. $G_{\Omega, I} : \partial\Omega_I \rightarrow S^{d-1}$ has no rational critical value for all $I \subset \{1, \dots, n\}$.
2. $G_{\Omega, I}^{-1}(p)$ is a finite set with $|G_{\Omega, I}^{-1}(p)| \leq N$ for some integer N , for all $p \in \partial\Omega_I$ such that $G_{\Omega, I}(p)$ is a rational point.

Let H be a semi-admissible Hamiltonian for W_Ω ; thus every periodic orbit of H comes from a Reeb orbit with period less than s . We will start by counting the number of periodic invariant tori. For an invariant torus $L_x = \mu^{-1}(x)$, where $x \in \partial\Omega_I$, to be periodic, $G_I(x)$ must be a rational point (a scalar multiple of an integer vector):

$$G_I(x) = \frac{\vec{q}}{\|\vec{q}\|} \quad (6.17)$$

for some $\vec{q} \in \mathbb{Z}^d$. Then the period of the family of periodic Reeb orbits in L_x is given by

$$T = \tau_I(x) \|\vec{q}\| \quad (6.18)$$

Define $\tau_I(\vec{q}) := \min_{x \in G_I^{-1}(\vec{q})} \tau_I(x)$. If $\tau_I(\vec{q}) \|\vec{q}\| < s$ for $\vec{q} \in \mathbb{Z}^d$, then at least one of the inverse images has period less than s . Set

$$m_I := \inf_{x \in S_{>0}^{d-1}} \tau_I(x) \geq \min_{x \in S_{\geq 0}^{d-1}} \tau_I(x) > 0 \quad (6.19)$$

The number of such integer vectors is bounded by the number of integers satisfying $m_I \|\vec{q}\| < s$, which is the number of integer points in the d -dimensional ball of radius $\frac{s}{m_I}$, and each integer vector has at most N corresponding inverse images. Thus the number of periodic invariant tori with period less than s in the coordinate subspace I is bounded by

$$N \frac{V_d}{m_I^d} s^d + D_d \quad (6.20)$$

for any V_d that is larger than the volume of the d -dimensional unit ball, where D_d is determined by the choice of V_d and the constant m_I . Now let V be a constant that is larger than the volumes of all unit balls of dimension $d \leq n$, and let $m := \min_I m_I$. Accounting for all coordinate subspaces, the total number of periodic tori is bounded by

$$(2^n - 1) N \frac{V}{m^n} s^n + D \quad (6.21)$$

Again, the constant D is determined by V and m .

By condition 1 and [GH, p. 3555], every periodic torus is Morse–Bott non-degenerate. The semi-admissible Hamiltonian can be perturbed in such a way that each torus of dimension d gives rise to 2^d non-degenerate Hamiltonian orbits. Thus the number of generators of $CF(sH)$ is bounded by

$$2^n (2^n - 1) N \frac{V}{m^n} s^n + D \quad (6.22)$$

If Ω is smooth and strictly convex/concave, the Ω_I are strictly convex/concave. Therefore the G_I are all injective, and thus condition 2 holds with $N = 1$. Since strictly convex/concave domains have strictly negative/positive definite second fundamental form, the G_I 's do not have any critical values, and thus condition 1 is satisfied as well.

If Ω is merely convex/concave, we can take a C^2 -small perturbation of f that is strictly convex/concave. Now assume that W is a real analytic toric domain. Extend f analytically to an open neighborhood U of $S_{\geq 0}^{n-1}$ in S^{n-1} , and still denote the surface defined by the extension by $\partial\tilde{\Omega}$. Let $B \subset SO(n)$ be an open neighborhood of the identity, sufficiently small that $RS_{\geq 0}^{n-1}$ stays inside U for all $R \in B$. Define $f_R := f \circ R : S_{\geq 0}^{n-1} \rightarrow \mathbb{R}_+$. Denote the surface defined by f_R by $\partial\Omega_R$, and the respective coordinate subspace surfaces by $\partial\Omega_{I,R}$. We will show that Ω_R satisfies condition 1 for a residual set of elements of B .

Let's fix a coordinate subspace $\mathbb{R}^I$. Let $B_I \subset B$ be rotations that act as identity on the orthogonal complement of $\mathbb{R}^I$. Then

$$G_{I,R} = R^{-1}G_I \circ R \quad (6.23)$$

where R on the right is applied to the original Gauss map considered as a rotation of $\mathbb{R}^k$. Let $\mathcal{R}$ denote the set of regular values of $G_I : S_{\geq 0}^{k-1} \rightarrow S^{k-1}$; by Sard's theorem $\mathcal{R}$ is dense.

Let $p \in S^{k-1}$. The subset $V_I(p)$ of B_I for which p is not a critical value of $G_{I,R}$ is the inverse image of $\mathcal{R}$ under the map $R \mapsto Rp$, thus a dense subset. This implies that the set $V(p)$ of rotations in B such that p is a regular value of $G_{I,R}$ is dense as well: if for $R \in B$, p is not a regular value of $G_{I,R}$, then we can find $R_k \in B_I$ converging to the identity with $R_k Rp \in \mathcal{R}$, and then $R_k R \in V(p)$ converges to R .

Now we want to show that $V(p)$ is open. For that, view the Gauss map as a map

from $S_{\geq 0}^{k-1}$. The following set is closed:

$$\{(R, x) \in B \times S_{\geq 0}^{k-1} \mid G_I(Rx) = Rp \text{ and } \det(dG_I)_{Rx} = 0\}. \quad (6.24)$$

Since $S_{\geq 0}^{k-1}$ is compact, its projection to the first factor is closed, and $V(p)$ is the complement of this projection; thus $V(p)$ is open.

Now since the rational points on the sphere are countable and we have finite number of subcoordinates, the set of $R \in B$ such that f_R satisfies the requirement is a countable intersection of open dense subsets.

In particular, we can choose R arbitrarily close to the identity, and thus f_R can be made arbitrarily C^∞ -close to f . Condition 2 holds automatically once condition 1 is achieved. Set of regular values are open due to compactness and Number of inverse image of a regular point is locally constant and for analytic maps it has finitely many connected component. The map $(R, p) \mapsto G_{I,R}(p)$ is real analytic and (R, p) is regular value if and only if p is regular value of $G_{I,R}$. Thus inverse image of regular value is bounded by some integer N that is independent of R and p .

Remark 6.2. *We could replace m here by any constant in the range $(0, m]$. This plays a central role in the proof of Theorem 1.2.*

□

6.3 Proof of Theorem 1.2: the Non-smooth Case

The strategy of the proof is to construct a sequence $f_n : S_{\geq 0}^{n-1} \rightarrow (0, \infty)$ of smooth functions defining smooth strictly convex domains W_{f_n} such that $f_n \rightarrow f$ uniformly and

$$\frac{f_n}{\sqrt{1 + \|\nabla_\theta f_n\|^2 / f_n^2}} > m_0 \quad (6.25)$$

for some constant $m_0 > 0$ that only depends on f . By Remark 6.2,

$$\mathbf{b}_\epsilon(W_{f_n}, s) \leq C m_0^{-n} s^n + C'' \quad (6.26)$$

and $d_{\text{SBM}}(W_{f_n}, W_f) \rightarrow 0$. Hence, by (5.38),

$$\mathbf{b}_\epsilon(W_f, s) \leq \liminf_{n \rightarrow \infty} \mathbf{b}_\epsilon(W_{f_n}, s) \leq C_n m_0^{-n} s^n + C \quad (6.27)$$

Since W_f is a convex domain, we have an extension of f to a neighborhood U of $S_{\geq 0}^{n-1}$ such that

$$\{rx \in \mathbb{R}^n \mid r \leq f(x), x \in U\} \quad (6.28)$$

is convex. The homogeneous function of degree one defined by $F := \frac{r}{f}$ is then a convex function, defined on the cone spanned by U ; we will denote the cone spanned by U by C . Thus both F and f are uniformly Lipschitz if U is a sufficiently small neighborhood of $S_{\geq 0}^{n-1}$, with the same Lipschitz constant (see [Ro]).

We will approximate F by a standard procedure (cf. [Az]). First, fix a smooth bump function δ on $\mathbb{R}^n$ with compact support and integral equal to 1. Then the sequence $\delta_k = k^n \delta(kx)$ is an approximate identity. Define

$$F_k(x) := (F * \delta_k)(x) = \int F(x - y) \delta_k(y) dy \quad (6.29)$$

The function F_k is well defined for every x such that $x - \text{supp } \delta_k \subset C$. Let U_0 be a smaller neighborhood of $S_{\geq 0}^{n-1}$ and define $C_0 := \mathbb{R}_+ U_0 \setminus \{x \in \mathbb{R}^n \mid \sum_i x_i < \epsilon\}$. The set C_0 is the cone spanned by U_0 with the tip of the cone cut off. If U_0 is sufficiently small, the sequence of functions

$$F_k := F * \delta_k, \quad (6.30)$$

defined on the convex set C_0 , consists of smooth convex functions (cf. [Az, Ro]).

Since F is uniformly Lipschitz on C , the functions $F_k : C_0 \rightarrow (0, \infty)$ are uniformly Lipschitz with the same Lipschitz constant as F , and $F_k \rightarrow F$ uniformly on C_0 . In view of $\partial\Omega_f = F^{-1}(1)$, we define

$$\Sigma_k := F_k^{-1}(1) \subset C_0 \quad (6.31)$$

We will show that Σ_k is a smooth hypersurface by showing that 1 is a regular value of F_k for all $k \in \mathbb{N}$. This is a consequence of the following lemma.

Lemma 6.3. *Let $x \in S_{\geq 0}^{n-1}$ and let $r_2 > r_1$ be in a compact interval I . Then*

$$F_k(r_2x) - F_k(r_1x) \geq \xi(r_2 - r_1) \quad (6.32)$$

for some $\xi > 0$ depending only on f and I , for all sufficiently large k .

Proof. Set $x_2 = r_2x - y$ and $x_1 = r_1x - y$, and write $\hat{x}_i = x_i/\|x_i\|$. Since $F(x) = \|x\|/f(\hat{x})$,

$$\begin{aligned} F(x_2) - F(x_1) &= \frac{\|x_2\| - \|x_1\|}{f(\hat{x}_2)} - \|x_1\| \frac{f(\hat{x}_2) - f(\hat{x}_1)}{f(\hat{x}_1)f(\hat{x}_2)} \\ &\geq \frac{\|x_2\| - \|x_1\|}{f(\hat{x}_2)} - \left\| \|x_1\| \frac{f(\hat{x}_2) - f(\hat{x}_1)}{f(\hat{x}_1)f(\hat{x}_2)} \right\| \end{aligned} \quad (6.33)$$

By the triangle inequality, $\|x_2\| - \|x_1\| \geq (r_2 - r_1) - 2\|y\|$. Setting $\xi_0 = \inf_U \frac{1}{f} > 0$, the first term satisfies

$$\frac{\|x_2\| - \|x_1\|}{f(\hat{x}_2)} \geq \xi_0((r_2 - r_1) - 2\|y\|) \quad (6.34)$$

For the second term, assume that $\|y\|$ is small enough that x_1 and x_2 form an acute angle with x . Writing $\hat{x}_i = g(y/r_i)$, where $g(z) = (x - z)/\|x - z\|$ is smooth near $z = 0$, we have

$$\|\hat{x}_2 - \hat{x}_1\| \leq \frac{C(r_2 - r_1)\|y\|}{\|x_1\|\|x_2\|} \quad (6.35)$$

for some constant C depending only on f . Hence, for a constant K depending only on f and I ,

$$\left| \|x_1\| \frac{f(\hat{x}_2) - f(\hat{x}_1)}{f(\hat{x}_1) f(\hat{x}_2)} \right| \leq K \|y\| (r_2 - r_1) \quad (6.36)$$

Thus we get

$$F(x_2) - F(x_1) \geq \xi_0(r_2 - r_1) - K'\|y\| \quad (6.37)$$

for a constant K' depending only on f and I . Now multiply by δ_k and integrate over y :

$$F_k(r_2x) - F_k(r_1x) \geq \xi_0(r_2 - r_1) - K' \int \|y\| \delta_k(y) dy \quad (6.38)$$

Since $\int \|y\| \delta_k(y) dy \rightarrow 0$ as $k \rightarrow \infty$, we can choose any $\xi < \xi_0$ for k large enough.

□

The first implication of the lemma is

$$\frac{\partial F_k}{\partial r} \geq \xi \quad (6.39)$$

which also implies that the Σ_k are smooth hypersurfaces. Since the F_k are smooth convex functions, lines passing through the origin intersect Σ_k exactly once. Hence there exist functions $f_k : U_0 \rightarrow (0, \infty)$ satisfying

$$F_k(f_k(x) x) = 1 \quad (6.40)$$

The functions f_k are smooth by the implicit function theorem, and they define smooth convex toric domains, since f_k is defined on U_0 and the extended surface is also convex. Moreover, by adding a sufficiently small positive definite quadratic form to F_k , we can make the domains strictly convex. The constant m would also change arbitrarily small, leaving the upper bounds essentially unchanged.

$f_k \rightarrow f$ uniformly; thus $d_{\text{SBM}}(W_{f_k}, W_f) \rightarrow 0$ as $k \rightarrow \infty$. Differentiating the

identity (6.40), we get

$$\left. \frac{\partial F_k}{\partial r} \right|_{r=f_k} \nabla f_k + \nabla_\theta F_k = 0 \quad (6.41)$$

Thus, by the lemma,

$$\|\nabla f_k\| \leq \frac{L}{\xi} \quad (6.42)$$

where L is the Lipschitz constant of F . Since $f_k \rightarrow f$ uniformly, we have $\min f_k \geq \frac{1}{2} \min f$ for all sufficiently large k , and hence

$$\frac{f_k}{\sqrt{1 + \|\nabla_\theta f_k\|^2 / f_k^2}} \geq m_0 := \frac{\min f}{2\sqrt{1 + \frac{4L^2}{\xi^2(\min f)^2}}} \quad (6.43)$$

and this concludes the proof.

The argument goes through for concave domains with straightforward modifications to the argument. As above, let U be a small open neighborhood of $S_{\geq 0}^{n-1}$ and let f extend to it, so that the set $\{rx \in \mathbb{R}^n \mid r \geq f(x), x \in U\}$ is convex. The function $F = r/f$ is then a homogeneous concave function of degree one, and the sets $\{F \geq c\} = c \{rx \mid r \geq f(x)\}$ are convex. Hence $G := -F$ is convex, and both F and G are uniformly Lipschitz on a possibly smaller cone [Ro]. The role of the truncated cone C_0 is played by the set $\{rx \in \mathbb{R}^n \mid r_0 \geq r \geq f(x), x \in U_0\}$ for sufficiently large $r_0 \in \mathbb{R}_+$, which is a compact convex set. Applying same approximation procedure we get smooth convex functions G_k , and the functions $F_k := -G_k = F * \delta_k$ are smooth and concave. Note that we did not use the convexity of F in the proof of Lemma 6.3: its proof uses only that $F(x) = \|x\|/f(\hat{x})$ and that f is Lipschitz — and hence $\partial F_k / \partial r \geq \xi > 0$. Therefore 1 is a regular value of F_k , and the level set $\Sigma_k = F_k^{-1}(1)$ is a smooth hypersurface meeting every ray through the origin exactly once, and the functions f_k defined by $F_k(f_k(x)x) = 1$ are smooth by the implicit function theorem. The sets $\{F_k \geq 1\} = \{rx \mid r \geq f_k(x)\}$ are convex since F_k is concave, so the functions f_k define smooth concave toric domains; the bound $\|\nabla f_k\| \leq L/\xi$ follows as before. Finally, adding a small negative definite quadratic

form $-\epsilon_k \|x\|^2$ to F_k makes the domains strictly concave without essentially changing the lower bound on m , if need be. The proof is then concluded as in the convex case.

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