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Minimal-positive-depth Representations of Groups of Relative Rank One over Nonarchimedean Local Fields

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MINIMAL-POSITIVE-DEPTH REPRESENTATIONS OF GROUPS OF RELATIVE RANK ONE OVER NONARCHIMEDEAN LOCAL FIELDS

A dissertation submitted in partial satisfaction
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

Philip Barron

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The Dissertation of Philip Barron is approved:

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Contents

Abstract

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by

Philip Barron

Reeder, Yu, and Gross have studied a class of representations of p-adic groups which they call *epipelagic* – those which are just slightly deeper than the surface (depth zero). In this thesis, we systematically study the representations of minimal positive depth for groups of relative rank one over nonarchimedean local fields. These are the groups for which the (reduced) Bruhat-Tits building is a tree. For such groups, we give a simplified proof that all irreducible minimal-positive-depth supercuspidal representations arise via compact induction. Furthermore, for certain classes of these groups we explicitly describe the orbits that provide such induction data.

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Introduction

Let K be a nonarchimedean local field with ring of integers $\mathcal{O}_K$, with maximal ideal $\mathfrak{p}_K = (\varpi_K)$. Let $\mathbb{F}_q$ be the residue field of $\mathcal{O}_K$. Let $\mathbf{G}$ be a connected semisimple group over K, and let $G = \mathbf{G}(K)$. Every smooth irreducible representation of G occurs in a representation parabolically induced from a supercuspidal irreducible representation of a Levi subgroup M of G. Thus representation theory is reduced to supercuspidal representations of Levi subgroups, which are groups of lesser relative rank. One can then view supercuspidal representations as the building blocks for the representation theory of p-adic groups.

Most well-known constructions of supercuspidal representations use compact induction, which led to the folklore conjecture that "every supercuspidal representation of a reductive p-adic group arises by induction from a compact-mod-center open subgroup."

For any G, Bruhat and Tits [2,3] define a cell complex called the reduced Bruhat-Tits building $\mathcal{B}(G) = X$. For each point $x \in X$, they construct a compact subgroup G_x of G, called a parahoric subgroup. X is the union of subcomplexes, called apartments, which G acts on transitively. To each maximal K-split torus $\mathbf{S}$ of $\mathbf{G}$, there is an associated apartment $\mathcal{A}(\mathbf{S}, K)$. Apartments in the building are metrically isomorphic to $\mathbb{R}^n$, where n is the relative rank of G.

Moy and Prasad [12,13] define a filtration of parahoric subgroups

$$G_x := G_{x,0} \rhd G_{x,r_1} \rhd G_{x,r_2} \dots$$

where the jumps $0 < r_1 < r_2 < \dots$ are real numbers depending on x. When $\mathbf{G}$ is a quasisplit over K, these filtration subgroups are defined in terms of filtrations of root subgroups and maximal tori. When $\mathbf{G}$ is not quasisplit over K, the Moy-Prasad filtration is defined by passing to an unramified extension, where **G** becomes quasisplit.

Let (π, V) be a smooth irreducible representation of G. Moy and Prasad define the depth $\varrho(\pi)$ of a (π, V) to be the smallest non-negative real number r such that $V^{G_{x,r+}}$ is nonzero for some point $x \in X$. In fact, the depth $r = \varrho(\pi)$ is a rational number. The possible depths of supercuspidal irreducible representations of G form a discrete subset of $\mathbb{R}$. Thus, there is a well-defined lowest positive depth that can occur. Gross, Reeder and Yu [9, 18] construct many such irreducible representations of G. These are the *simple supercuspidals* and, the more general, *epipelagic representations.*

Supercuspidal irreducible representations of higher depth almost all arise by compact induction from an intricate construction of Yu [27]. In fact, after work of Kim [10], we know that this is true for large enough residue characteristic. After a recent breakthrough of Fintzen, we know that all supercuspidal irreducible representations arise from Yu's construction, as long as the residue characteristic p does not divide the order of the Weyl group of G [8, Thm. 8.1]. This leaves the "wild" cases of the folklore conjecture still open.

To this end, Weissman [26] gives a less explicit proof of the folklore conjecture for groups of relative rank one, which holds in all residue characteristics. Motivated by this, this thesis is focused on these groups and their representations of minimal positive depth.

In aiming to make this thesis rather self-contained, Chapter 1 reviews many preliminary topics, such as reductive groups, the Bruhat-Tits building, Moy-Prasad filtration, and representation theory of groups defined over non-archimedean local fields. We also provide information on the groups of relative rank one whose representation theory we will study later in the thesis. These groups have been classified and tabulated by Tits [23] and Carbone [4].

Weissman [26], using technical theory from Schneider-Stuhler [20], proves that for all reductive groups of relative rank one irreducible supercuspidal representations arise via compact induction from the stabilizer of a vertex or edge in the Bruhat-Tits building. Let X denote the Bruhat-Tits building of G. In Chapter 2, motivated from Weissman's work, we use theory of Bestivina-Savin [1] to reprove Weissman's result in the setting where (π, V) is a representation of critical depth 35. By critical depth r, we mean a depth in which there exists a $z \in X$ such that $G_{z,r} \neq G_{z,r+}$, and for all $y \neq z$ in an open neighborhood around z, $G_{y,r+} = G_{y,r}$ and $G_{z,r+} \subseteq G_{y,r} \subseteq G_{z,r}$. These critical depths in the rank one setting include the minimal positive depths studied by Gross, Reeder, and Yu [9, 18].

In Chapter 3, we provide data about the Bruhat-Tits buildings for $SL_2(K)$, $SL_2(D)$, $SU_3^{L/K}(h)$, and $SU_3^{E/K}(h)$, describing the minimal positive depths, and the structure of the relevant Moy-Prasad filtration subgroups. We briefly mention that for the non-quasisplit groups, one can use unramified descent from $\mathbf{G}(K^{\mathrm{nr}})$ to describe the buildings and find the minimal positive depths. We do not provide the explicit data for these remaining groups, but one finds that the minimal positive depths for these groups are also critical depths. Using this data and the results of Chapter 2, one finds that all irreducible minimal-positive-depth supercuspidal representations of groups of relative rank one arise via compact induction.

In Chapter 4, we cover the theory of Gross, Reeder, and Yu [9, 18] on simple supercuspidals and epipelagic representations. We show that all minimal-positive-depth representations are contained in the construction of these representations.

In Chapter 5, we construct the irreducible minimal-positive-depth supercuspidal representations for $SL_2(K)$, $SL_2(D)$, $SU_3^{L/K}(h)$, and $SU_3^{E/K}(h)$ using information from Chapters 2 and 3. One can carry forward similar constructions for the remaining groups by means of unramified descent.

In the Appendix A, we describe the remaining groups of relative rank one. Here, we cover $SU_4^{L/K}(h)$ and the quaternionic unitary groups, following Prasad-Raghunathan [16].

Chapter 1

Groups of relative rank one

Let K be a nonarchimedean local field, with $\mathcal{O}_K$ its ring of integers and ϖ_K a uniformizing element of K. Let $\mathbb{F}_q = \mathcal{O}_K/\varpi_K\mathcal{O}_K$ be the residue field of K, where $q = p^f$ for some prime p and $f \in \mathbb{N}$. Let $\mathrm{val} : K^\times \to \mathbb{Z}$ be a valuation with $\mathrm{val}(K^\times) = \mathbb{Z}$. Let $\bar{K}$ denote a separable closure of K. Let $\mathbf{G}$ be a connected semisimple group defined over K, and let $G = \mathbf{G}(K)$ be the group of K-rational points of $\mathbf{G}$. Throughout this text, when saying "semisimple," we will mean "connected semisimple."

1.1 Reductive groups

1.1.1 Absolute and Relative rank

Let $\mathbf{G}_m = \mathbf{G}_{m/K}$ be the multiplicative group over K; then $\mathbf{G}_m(K) = K^\times$. A subgroup $\mathbf{T} \subseteq \mathbf{G}$ is called a torus if $\mathbf{T}_{\bar{K}} \cong \mathbf{G}^\ell_{m/\bar{K}}$. $\mathbf{G}$ is a finite dimensional group; hence, there exists a torus of maximal dimension, called a *maximal torus.* All maximal tori are conjugate over $\bar{K}$, thus they all have the same dimension, called the *absolute rank* of $\mathbf{G}$.

A *K-split torus* is a subgroup $\mathbf{S} \subset \mathbf{G}$, defined over K, with $\mathbf{S} \cong \mathbf{G}^r_{m/K}$ for some $r \in \mathbb{N}$. There exists a torus maximal amongst K-split tori. Call such a torus a *maximal K-split torus.* All maximal K-split tori are $\mathbf{G}(K)$-conjugate and hence have the same dimension. We call this dimension the *relative rank* of $\mathbf{G}$.

We say that $\mathbf{G}$ is *split* over K if $\mathbf{G}$ contains a split maximal torus; otherwise, we call

$\mathbf{G}$ non-split. If $\mathbf{G}$ is split over K, then the absolute rank and the relative rank of $\mathbf{G}$ are equal.

1.1.2 K-Forms

Let $\mathbf{G}$ and $\mathbf{G}'$ be linear algebraic groups defined over K. $\mathbf{G}'$ is called a K-form of $\mathbf{G}$ if $\mathbf{G} \cong \mathbf{G}'$ over $\bar{K}$ or a finite extension of K.

An *isogeny* of algebraic groups is a surjective homomorphism with finite kernel. We say that an isogeny is *central* if the kernel is central. We say that two groups $\mathbf{G}$ and $\mathbf{G}'$ are *strictly isogenous* if there is a group $\mathbf{H}$ and two central isogenies $\mathbf{H} \to \mathbf{G}$ and $\mathbf{H} \to \mathbf{G}'$.

Definition 1. *We say that* $\mathbf{G}$ *is almost simple over* K *if* $\mathbf{G}$ *contains no infinite normal algebraic* K*-subgroup, and we say* $\mathbf{G}$ *is absolutely almost simple if* $\mathbf{G}$ *is almost simple over* $\bar{K}$.

1.1.3 Relative and absolute roots of reductive groups

Let $X^\bullet(\mathbf{S}) = \mathrm{Hom}(\mathbf{S}, \mathbf{G}_m)$ and let $X_\bullet(\mathbf{S}) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{S})$ be the lattices of the characters and cocharacters of the split torus $\mathbf{S}$. We view $X_\bullet(\mathbf{S})$ as a lattice in the real vector space $V = X_\bullet(S) \otimes_{\mathbb{Z}} \mathbb{R}$, whose dimension is the relative rank of $\mathbf{G}$.

Let $\Phi = \Phi(\mathbf{G}, \mathbf{S}) \subset X^\bullet(\mathbf{S})$ be the set of roots for the adjoint action of $\mathbf{S}$ on the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. These are called the *relative roots* of $\mathbf{G}$ with respect to $\mathbf{S}$.

If a is a root then there are three possibilities for its multiples.

- $\mathbb{R}a \cap \Phi = \{\pm a\}$
- $\mathbb{R}a \cap \Phi = \{\pm a, \pm \frac{1}{2}a\}$, and we say that a is divisible.
- $\mathbb{R}a \cap \Phi = \{\pm a, \pm 2a\}$, and we say that a is multipliable.

For any $a \in \Phi$, we call $\mathbf{U}_a$ the root subgroup of $\mathbf{G}$ corresponding to a. $\mathbf{U}_a$ is the connected closed subgroup of $\mathbf{G}$ normalized by $\mathbf{S}$ whose Lie algebra is the sum of the root spaces corresponding to the roots that are positive integral multiples of a. Thus if a is multipliable, $\mathbf{U}_{2a} \subset \mathbf{U}_a$. Similarly, if a is divisible, $\mathbf{U}_{\frac{1}{2}a} \supset \mathbf{U}_a$.

Now let $\mathbf{T}$ be maximal torus of $\mathbf{G}$ defined over K. $\mathbf{G}$ becomes split over $\bar{K}$. Let $X^\bullet(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ and let $X_\bullet(\mathbf{T}) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ be the lattices of the characters and cocharacters of $\mathbf{T}$. Let $\Phi_{\bar{K}} = \Phi(\mathbf{G}, \mathbf{S}) \subset X^\bullet(\mathbf{T})$ be the set of roots for the adjoint action of $\mathbf{S}$ on the Lie algebra $\mathfrak{g}$ on $\mathbf{G}$. These are called the *absolute roots* of $\mathbf{G}$ with respect to $\mathbf{T}$.

Let $a \in \Phi_{\bar{K}}$ be a root. Then $\mathbb{R}a \cap \Phi = \{\pm a\}$, meaning the absolute root system is *reduced*. Furthermore for any root $a \in \Phi_{\bar{K}}$, $\mathbf{U}_a \cong \mathbf{G}_{a/\bar{K}}$, the additive group over $\bar{K}$.

A *Borel* subgroup $\mathbf{B} \subset \mathbf{G}$ is a maximal connected solvable algebraic subgroup. Borel subgroups have the form $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$ for a maximal torus $\mathbf{T}$ and a maximal unipotent subgroup of $\mathbf{G}$. The maximal unipotent subgroup $\mathbf{U}$, and is generated by $\mathbf{U}_a$ for some $a \in \Phi$. If $\mathbf{U}_a \subset \mathbf{U}$, then $\mathbf{U}_{-a} \not\subset \mathbf{U}$. In this way, choosing a Borel subgroup fixes a set of positive roots $\Phi^+_{\bar{K}} \subset \Phi_{\bar{K}}$. Similarly, in choosing a Borel subgroup, we choose and fix a set of positive roots $\Phi^+ \subset \Phi$.

Let Δ (resp. $\Delta_{\bar{K}}$) denote the set of relative (resp. absolute) simple roots. Every element of Φ (resp. $\Phi_{\bar{K}}$) can be expressed uniquely as a $\mathbb{Z}$-linear combination of relative (resp. absolute) simple roots.

For more details on theory of relative and absolute roots of reductive groups, refer to Springer [22].

1.1.4 Quasisplit groups

Definition 2. *Let* $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})' := [\mathbf{Z}_{\mathbf{G}}(\mathbf{S}), \mathbf{Z}_{\mathbf{G}}(\mathbf{S})]$ *be the derived subgroup of* $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})$*, called the anisotropic kernel of* $\mathbf{G}$*. If* $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})'$ *is trivial over* K*,* $\mathbf{G}$ *is called quasisplit over* K*. An equivalent condition to* $\mathbf{G}$ *being quasisplit is the existence of a Borel subgroup defined over* K*.*

Remark 3. $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})$ *always contains a maximal split torus; when* $\mathbf{G}$ *is split or quasisplit,* $\mathbf{Z}_{\mathbf{G}}(\mathbf{S})$ *is a maximal split torus.*

Proposition 4. [21, II.3.3, III.3.2] *Every algebraic group* $\mathbf{G}$ *defined over* K *becomes quasisplit over some finite unramified extension* L *of* K*. Furthermore, there exists some finite extension* E/L*, possibly ramified, such that* $\mathbf{G}$ *splits over* E*.*

1.2 Groups of relative rank one over nonarchimedean local fields

In [23] and [25], Tits classifies groups of relative rank one over nonarchimedean local fields. For this section, we will closely follow Carbone's notes [4], where she tabulates these groups.

1.2.1 Tits Index

Assume that $\mathbf{G}$ is absolutely almost simple. Denote by Γ the Galois group $Gal(\bar{K}/K)$. Let $\mathbf{S}$ be a maximal K-split torus of $\mathbf{G}$, and let $\mathbf{T}$ be a maximal torus containing $\mathbf{S}$, defined over K. Let $\mathbf{N}$ be the normalizer of $\mathbf{T}$, and let $\mathbf{W} = \mathbf{N}/\mathbf{T}$ be the Weyl group. Let $\Delta_{\bar{K}}$ be the simple roots of G relative to $\mathbf{T}$, and let $\delta_{\bar{K}}$ be the corresponding Dynkin diagram. Let $\Delta_0 \subseteq \Delta_{\overline{K}}$ denote the set of simple roots that vanish on $\mathbf{S}$. The Galois group $\Gamma = Gal(\bar{K}/K)$ acts on $\Delta_{\bar{K}}$ and on the Dynkin diagram $\delta_{\bar{K}}$.

A *Tits index consists of:*

(1) The simple roots $\Delta_{\bar{K}}$ and its corresponding Dynkin diagram $\delta_{\bar{K}}$.

(2) The action of Γ on $\Delta_{\bar{K}}$ and corresponding action on $\delta_{\bar{K}}$. This action is called the $*$-*action*.

We a call a vertex of the Dynkin diagram $\delta_{\bar{K}}$ *distinguished* if the corresponding simple root does not belong to Δ_0. These vertices will be circled. Vertices of the Dynkin diagram in the same orbit of Γ are drawn close together; if they are both distinguished, a common circle is drawn around them.

1.2.2 Classification

Theorem 5. *We have the following.*

- [23, Thm. 1]. *Over $\bar{K}$, G is characterized up to strict isogeny by its Dynkin diagram.*
- [23, Thm. 2.7.2(b)]. *G is determined up to strict isogeny by its strict $\bar{K}$-isogeny class, its Tits index, and the K-isogeny class of its semisimple anisotropic kernel.*

- [23]. *If G is quasisplit, G is determined up to strict isogeny by its Tits index.*

Using his theorems, Tits classifies all groups of relative rank 1 using the Tits index. Tits [23] uses the notation for a Tits index:

$$^{g}X^{t}_{n,r},$$

where

- $n =$ absolute rank $= \dim \mathbf{T}$,
- $r =$ relative rank $= \dim \mathbf{S}$,
- $g =$ the order of the quotient of Γ which acts faithfully on $\delta_{\bar{K}}$,
- $t =$ degree$(= \sqrt{\dim})$ of a division algebra which occurs in the definition of the considered form.
- $X =$ type of group over $\bar{K}$.

Remark 6. $t = 1$ *if and only if the group is quasisplit. Thus t is often omitted from the Tits index of quasisplit groups.*

Now that we have discussed the notation for the Tits index, here is a diagram, provided by Carbone [4], tabulating the possible groups up to strict isogeny, their Tits indices, relative Dynkin diagrams, and their forms over $\bar{K}$.

The relative local Dynkin diagram holds vital information about the reduced Bruhat-Tits tree $\mathcal{B}(G)$ of G. Vertices in the diagram labeled by the letters s or hs are special or hyperspecial points (vertices) in the tree. The number written above each vertex indicates the number of edges coming out of the vertex in $\mathcal{B}(G)$. If the letter d is written above at vertex v, this indicates that there are $q^d + 1$ edges coming out v. Here, q is the cardinality of the residue field of K. If the root system of G is non-reduced, vertices marked by x are vanishing loci for affine roots of gradients both $\pm a$ and $\pm 2a$. We will refer back to Bruhat-Tits building, relative Dynkin diagram, and affine roots in coming sections of this chapter.

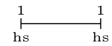
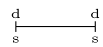
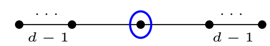
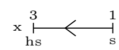
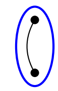
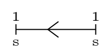
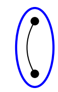
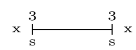
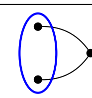
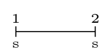
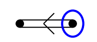
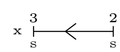
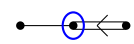
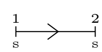
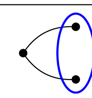
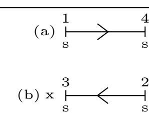
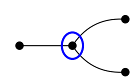
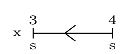
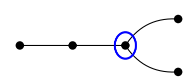
Index	Group	Over $\bar{K}$	Relative Local Dynkin Diagram	Tits Index
${}^{\mathbf{1}}\mathbf{A}^{\mathbf{1}}_{\mathbf{1,1}}$	$SL_2(K)$	$\mathbf{SL_2}$		
${}^{\mathbf{1}}\mathbf{A}^{\mathbf{d}}_{\mathbf{2d-1,1}}$	$SL_2(D)$	$\mathbf{SL_{2d}}$		
${}^{\mathbf{2}}\mathbf{A}^{\mathbf{1}}_{\mathbf{2,1}}$	$SU_3^{L/K}(h)$	$\mathbf{SL_3}$		
${}^{\mathbf{2}}\mathbf{A}^{\mathbf{1}}_{\mathbf{2,1}}$	$SU_3^{E/K}(h)$	$\mathbf{SL_3}$		
${}^{\mathbf{2}}\mathbf{A}^{\mathbf{1}}_{\mathbf{3,1}}$	$SU_4^{L/K}(h)$	$\mathbf{SL_4}$		
$\mathbf{C}^{\mathbf{2}}_{\mathbf{2},1}$	$SU_2^{D/K}(h')$	$\mathbf{Sp_4}$		
$\mathbf{C}^{\mathbf{2}}_{\mathbf{3},1}$	$SU_3^{D/K}(h')$	$\mathbf{Sp_6}$		
${}^{\mathbf{2}}\mathbf{D}^{\mathbf{2}}_{\mathbf{3,1}}$	$SU_3^{D/K}(h)$	$\mathbf{SO_6}$		
${}^{\mathbf{2}}\mathbf{D}^{\mathbf{2}}_{\mathbf{4,1}}$	$SU_4^{D/K}(h)$	$\mathbf{SO_8}$		
${}^{\mathbf{1}}\mathbf{D}^{\mathbf{2}}_{\mathbf{5,1}}$	$SU_5^{D/K}(h)$	$\mathbf{SO_{10}}$		

Figure 1.1: Table of groups of relative rank one [4]

Remark 7. [4, 25] *The Tits indices for ${}^2D^2_{3,1}$ and ${}^2A^1_{3,1}$ are the same, but Tits remarks that if a cyclic group of order 2 acts on the Dynkin diagram, then there is a quadratic extension of K fixed by the Galois group. In the case that this fixed field is ramified, the group is more naturally described as type D_3. For ${}^2A^1_{3,1}$, the fixed field of the Galois action in unramified. This is reflected in us only considering the $SU_4^{L/K}$ that splits over an unramified extension and the $SU_3^{D/K}(h)$ that splits over a ramified extension for the ${}^2A^1_{3,1}$ and ${}^2D^2_{3,1}$ groups, respectively.*

In the Bruhat-Tits tree, the edges protruding from x correspond to Borel subgroups of $\bar{\mathbf{M}}_x(\mathbb{F}_q) = G_{x,0}/G_{x,0+}$ (see 14). When working over an algebraically closed field or a finite field F, if B is a F-rational points of some fixed Borel subgroup, then $G = \mathbf{G}(F)$ acts transitively on the set of Borel subgroups by conjugation, and $N_G(B) = B$. This identifies G/B with the set of Borel subgroups of G. Thus, finding the cardinality of $\bar{M}_x$ and dividing by the cardinality of a Borel subgroup B_x of $\bar{M}_x$ will yield the number of edges protruding from x.

1.2.3 Descriptions of the occurring G

For unitary groups of relative rank one over K, we split into two situations: $G = SU_m^{L/K}(h)$ for a quadratic field extension L/K, and $G = SU_n^{D/K}(h)$ for a quaternion algebra D/K for some $m, n \in \mathbb{N}$.

Let L/K be a quadratic field extension, let σ be a generator of $Gal(L/K)$. Let h be a Hermitian form on L^m with respect to σ of Witt index 1. Then $SU_m^{L/K}(h)$ is a group of relative rank one. It will follow that $m = 3$ or 4 are the only possibilities.

Let D be a quaternion algebra over K. Let σ be an involution of D trivial on K, with $\dim_K(D^\sigma) = 3$. We will call this an involution of *first kind* and *first type*. We refer the reader to the Appendix A for details on these quaternionic unitary groups.

We are now ready to describe the possible Tits indices for groups of relative rank one over K, with their corresponding semisimple groups, tabulated by Carbone [4], and seen in Figure 1.1. By Theorem 5, the corresponding group G is up to strict isogeny.

(1) Tits index ${}^1A_{1,1}^1$, corresponding group $G = SL_2(K)$. The group $\mathbf{G}$ is a split form of $\mathbf{SL_2}$, and is simply connected.

(2) Tits index ${}^1A_{2d-1,1}^d$, $d \geq 2$, corresponding group $G = SL_2(D)$, where D is a central division algebra of degree $d \geq 2$ over K. We have

$$\mathbf{G}(\bar{K}) = SL_2(D \otimes_K \bar{K}) \cong SL_2(M_d(\bar{K})) \cong SL_{2d}(\bar{K}).$$

$\mathbf{G}$ is a non-split form of $\mathbf{SL_{2d}}$, and is simply connected.

(3a) Tits index ${}^2A_{2,1}^1$, corresponding group $G = SU_3^{L/K}(h)$, L/K is an unramified quadratic extension, $\langle\sigma\rangle = Gal(L/K)$, h is a non-degenerate Hermitian form relative to σ. Over $\bar{K}$, $SU_3^{K/F} \cong SL_3$. The group $\mathbf{G}$ is a non-split form of $\mathbf{SL_3}$, and is simply connected.

(3b) Tits index ${}^2A^1_{2,1}$, corresponding group $G = SU_3^{E/K}(h)$, E/K is a ramified quadratic extension, $\langle\sigma\rangle = Gal(E/K)$, h is a non-degenerate Hermitian form relative to σ. Over $\bar{K}$, $SU_3^{E/K} \cong SL_3$. The group $\mathbf{G}$ is a non-split form of $\mathbf{SL_3}$, and is simply connected.

(4) Tits index ${}^2A^1_{3,1}$, corresponding group $G = SU_4^{L/K}(h)$, L/K is an unramified quadratic extension, $\langle\sigma\rangle = Gal(L/K)$, h is a non-degenerate Hermitian form relative to σ. Over $\bar{K}$, $SU_4^{L/K} \cong SL_4$. The group $\mathbf{G}$ is a non-split form of $\mathbf{SL_4}$, and is simply connected.

(5) Tits index $C^2_{2,1}$, corresponding group $G = SU_2^{D/K}(h')$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h' is a non-degenerate skew-Hermitian form relative to σ. The group $\mathbf{G}$ is a non-split form of $\mathbf{Sp_4}$, and is simply connected.

(6) Tits index $C^2_{3,1}$, corresponding group $G = SU_3^{D/K}(h')$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h' is a non-degenerate skew-Hermitian form relative to σ. The group $\mathbf{G}$ is a non-split form of $\mathbf{Sp_6}$, and is simply connected.

(7) Tits index ${}^2D^2_{3,1}$, corresponding group $G = SU_3^{D/K}(h)$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non-degenerate Hermitian form relative to σ. The group $\mathbf{G}$ is a non-split form of $\mathbf{SO_6}$, and is not simply connected.

(8) Tits index ${}^2D^2_{4,1}$, corresponding group $G = SU_4^{D/K}(h)$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non-degenerate Hermitian form relative to σ. The group $\mathbf{G}$ is a non-split form of $\mathbf{SO_8}$, and is not simply connected.

(9) Tits index ${}^2D^2_{5,1}$, corresponding group $G = SU_5^{D/K}(h)$, D is a quaternion division algebra of degree $d = 2$ over K, with an involution σ of the first kind, first type, h is a non-degenerate Hermitian form relative to σ. The group $\mathbf{G}$ is a non-split form of $\mathbf{SO_{10}}$, and is not simply connected.

1.3 Representation theory and the Bruhat-Tits building

A *representation* of G will mean a pair (π, V) where V is a complex vector space and $\pi : G \to GL(V)$ is a group homomorphism.

Definition 8. *A vector $v \in V$ is called smooth if*

$$stab_G(v) := \{g \in G : \pi(g)v = v\}$$

is open. If all vectors $v \in V$ are smooth, we call (π, V) a smooth representation.

If $H \leq G$ is an open compact subgroup, we denote the H-invariants by

$$V^H := \{v \in V : \pi(h)v = v \text{ for all } h \in H\}.$$

Definition 9. *A smooth representation (π, V) of G is called admissible if $dim_{\mathbb{C}}(V^H) < \infty$ for any compact open subgroup H of G.*

1.3.1 Supercuspidal representations

Let (π, V) be a smooth representation of G and $P = MU$ be a Levi decomposition of a parabolic subgroup $P < G$. Let (σ, W) be a smooth representation of L. We define the parabolic induction functor from P to G by

$$\mathrm{Ind}_P^G(\sigma) := \left\{ f \in C^\infty(G, W) : f(umg) = \delta_P^{\frac{1}{2}}(m)\sigma(m)f(g) \text{ for all } m \in M,\ \ u \in U,\ \ g \in G \right\}.$$

Here δ_P is the modular character of P.

Let $V(U) = \mathrm{span}_{\mathbb{C}}\{\pi(u)v - v : u \in U,\ \ v \in V\}$. Let $V_U := V/V(U)$ be the space of U-coinvariants, and consider the quotient action of M on V_U given by

$$\pi_U(m)(v + V(U)) = \pi(m)v + V(U).$$

The M-representation (π_U, V_U) is called the *Jacquet module* of (π, V) with respect to $P = MU$.

Let (π, V) be an admissible representation of G. We say that (π, V) is *supercuspidal* if for any proper parabolic subgroup $P = MU$ the Jacquet module $V_U = 0$.

Remark 10. *If (π, V) is irreducible, (π, V) is supercuspidal if and only if π is not equivalent to a subrepresentation of $Ind_P^G(\sigma)$ for any proper parabolic subgroup $P = MU$, and for any smooth representation (σ, W) of M.*

Thus, we can think of irreducible supercuspidal representations of G as those native to G; all other irreducible admissible representations come from representations of some proper Levi factor M. In fact, if (π, V) is a smooth irreducible representation of G, the π appears as a subrepresentation of $\mathrm{Ind}_P^G(\sigma)$ for some, not necessarily proper, parabolic $P = MU$ and supercuspidal M-representation σ.

Definition 11. [18, see 2.1] *Let $G = \mathbf{G}(K)$ with center Z. Let H be an open subgroup of G such that $Z \subseteq H$ and H/Z is compact, and let (σ, W) be a smooth finite-dimensional representation of H on a complex vector space W. The compactly induced representation* $\text{c-Ind}_H^G(\sigma, W)$ *is realized on the complex vector space of function $f : G \to W$ satisfying the following two conditions:*

- *$f(hg) = \sigma(h)f(g)$ for all $h \in H$, $g \in G$;*
- *f is supprted on only finitely many cosets of H in G.*

The group G acts on $\text{c-Ind}_H^G(\sigma, W)$ *by right translations: $[g \cdot f](x) = f(xg)$.*

1.3.2 The Bruhat-Tits building

For any $G = \mathbf{G}(K)$, Bruhat and Tits [2, 3] define a polysimplicial complex called the reduced building $\mathcal{B}(G) = X$. Let $\mathbf{S}$ be a maximal split torus of $\mathbf{G}$. Let $\dim(\mathbf{S}) = n$ be the relative rank. Then X is an n-dimensional polysimplicial complex, endowed with an action of G. Every maximal facet will be n-dimensional. For instance, if $\mathbf{S}$ has rank 1, then X is a tree.

To each point $x \in X$ corresponds to a compact subgroup $G_x \subset G$ contained in the stabilzer of x in G. The group G_x is referred to as the *parahoric subgroup* at x. If $x \in X^n$ – meaning x belongs to an n-dimensional facet – then G_x will be minimal amongst other parahoric subgroups; this class of parahoric subgroups is known as *Iwahori subgroups* of G. If $x \in X^0$, then G_x will be a maximal parahoric subgroup.

X is the union of subcomplexes, called apartments, on which G acts transitively. To each maximal K-split torus S of G there is an associated apartment. The apartment $\mathcal{A}(\mathbf{S}, K)$ associated with the torus S is an affine space under $X_\bullet(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, where n is the relative rank of G.

1.3.3 Valuations of root groups

Assume that $\mathbf{G}$ is quasisplit over K. And let E/K be a finite extension such that $\mathbf{G}_E$ is split. Let e denote the ramification index of E/K.

For this section and the next, we follow Fintzen's thesis [7, Section 2.2-2.4]. However, we skip over many details. Please refer to this text for more details and to Bruhat-Tits [3] for the non-quasisplit case.

Let $a \in \Phi_K$, and let $\mathbf{U}_a$ be the corresponding root subgroup of $\mathbf{G}$. Let $\mathbf{G}_{\pm a}$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{U}_a$ and $\mathbf{U}_{-a}$. Let $\pi : \mathbf{G}^{\pm a} \to \mathbf{G}_{\pm a}$ be a simply connected cover. π induces an isomorphism between a root subgroup $\mathbf{U}_+$ of $\mathbf{G}_{\pm a}$ and $\mathbf{U}_a$; we call $\mathbf{U}_+$ the positive root group of $\mathbf{G}^{\pm a}$. In order to describe the root group $\mathbf{U}_a$, one splits into two cases

1) $a \in \Phi_K$ is neither divisible nor multipliable.

2) $a \in \Phi_K$ is divisible or multipliable; i.e., $\frac{a}{2}$ or $2a \in \Phi$

Let $a' \in \Phi_E$ be root that equals a when restricted to $\mathbf{S}$. Note that $Gal(E/K)$ acts on Φ. We denote by $E_{a'}$ the fixed subfield of E of the stabilizer $\mathrm{stab}_{Gal(E/K)}(a')$. Let $e_{a'}$ denote the ramification index of $E_{a'}/K$.

1) If $a \in \Phi_K$ is neither divisible nor multipliable, let $a' \in \Phi_E$ be a root that equals a when restricted to $\mathbf{S}$. Then $\mathbf{G}^a \cong \mathrm{Res}_{E_{a'}/K}(SL_2)$, and $\mathbf{U}_a \cong \mathrm{Res}_{E_a/K}(\mathbf{U}^E_{a'})$, where $\mathbf{U}^E_{a'}$ is the root subgroup of $\mathbf{G}_E$ corresponding to a'. For $e \in \mathrm{Res}_{E_{a'}/K}\mathbf{G}_a(K) = E_{a'}$, one can define the valuation $\varphi_a : \mathbf{U}_a(K) \to \frac{1}{e_{a'}}\mathbb{Z} \cup \infty$ of $\mathbf{U}_a(K)$ by

$$\varphi_a(x_a(c)) = \mathrm{val}(c).$$

2) If $a \in \Phi_K$ is divisible or multipliable, let's assume that a is multipliable and describe $\mathbf{U}_a$ and $\mathbf{U}_{2a}$. Let $a', \tilde{a}' \in \Phi_E$ be roots that equal a when restricted to $\mathbf{S}$ such that $a' + \tilde{a}' \in \Phi_E$.

$a' + \tilde{a}'$ will then equal $2a$ when restricted to $\mathbf{S}$. $\mathbf{G}^{\pm a}$ is then isomorphic to $\mathrm{Res}_{E_{a'+\tilde{a}'}/K}(SU_3)$, where SU_3 is the special unitary group over $E_{a'+\tilde{a}'}$ defined by the Hermitian form $h : (x, y, z) \mapsto \sigma(x)z + \sigma(y)y + \sigma(z)x$ on $E_{a'}^3$, with σ the nontrivial element in $Gal(E_{a'}/E_{a'+\tilde{a}'})$. To simplify notation, write $L = E_{a'} = E_{\tilde{a}'}$, $L_2 = E_{a'+\tilde{a}'}$. Following [3], define the subset $H_0(L, L_2)$ of $L \times L$ by

$$H_0(L, L_2) = \{(c, d) \in L \times L : d + \sigma(d) = \sigma(c)c\}.$$

Viewing $L \times L$ as a four dimension vector space over L_2, and considering the corresponding scheme over L_2, we can view $H_0(L, L_2)$ as a closed subscheme of $L \times L$ over L_2, which we will again denote by $H_0(L, L_2)$. Then there exists an L_2-isomorphism $\mu : H_0(L, L_2) \to U_+$ given by

$$(c, d) \mapsto \begin{pmatrix} 1 & -\sigma(u) & -v \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix}.$$

Using this isomorphism, we can transfer the group structure of U_+ to $H_0(L, L_2)$ and thus turn the latter into a group scheme over L_2. Let $H(L, L_2)$ denote the restriction of scalars $\mathrm{Res}_{L_2/K} H_0(L, L_2)$. Then, identifying $\mathbf{G}^{\pm a}$ with $\mathrm{Res}_{E_{a'+\tilde{a}'}} SU_3$, we obtain an isomorphism

$$x_a : \pi \circ \mathrm{Res}_{E_{a'+\tilde{a}'}/K}\mu : H(L, L_2) \xrightarrow{\cong} \mathbf{U}_a \ ,$$

which we call the *parametrization* of $\mathbf{U}_a$.

The root subgroup $\mathbf{U}_{2a}$ corresponding to $2a$ is the subgroup of $\mathbf{U}_a$ given by the image of $x_a(0, d)$. Hence $\mathbf{U}_a(K)$ is identified with the group of elements in $E_{a'}$ of trace zero with respect to the quadratic extension $E_a'/E_{a'+\tilde{a}'}$, which we denote by $E_{a'}^0$.

Using the parametrization x_a, we define the *valuation* φ_a of $\mathbf{U}_a(K)$ and φ_{2a} of $\mathbf{U}_{2a}(K)$ by

$$\varphi_a(x_a(c, d)) = \frac{1}{2}\mathrm{val}(d),$$

$$\varphi_{2a}(x_a(0, d)) = \mathrm{val}(d).$$

1.3.4 Affine Roots

Again, for simplicity, assume that $\mathbf{G}$ is quasisplit. Let $\mathcal{A} = \mathcal{A}(\mathbf{S}, K)$ be the apartment corresponding to the maximal split torus $\mathbf{S}$ of $\mathbf{G}$ is an affine subspace of $X_\bullet(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$ by the coroots of $\mathbf{G}$. The apartment $\mathcal{A}$ can be defined as correponding to all valuations of $(\mathbf{T}(K), \mathbf{U}_a(K)_{a\in\Phi})$ in the sense of [2, Section 6.2] that are equipolent to the ones constructed in the previous section 1.3.3. In particular, the valuation defined in 1.3.3 corresponds to a special point in $\mathcal{A}$ that we denote by x_0. Then the set of affine roots Φ_{af} on $\mathcal{A}$ consists of the affine functions on $\mathcal{A}$ given by

$$\Phi_{\mathrm{af}} = \{y \mapsto a(y - x_0) + \gamma' \,:\, a \in \Phi, \gamma' \in \Gamma'_a\},$$

where

$$\Gamma'_a = \{\varphi_a(u) \,:\, u \in U_a - \{1\}, \varphi_a(u) = \sup(\varphi_a(uU_{2a}))\}.$$

For a more explicit definition of Γ'_a, we do the following. Let a be a multipliable root with $a' \in \Phi_E$ a root that equals a when restricted to $\mathbf{S}$. Define

$$(E_{a'})^0 = \left\{u \in E_{a'} \,:\, \mathrm{Tr}_{E_{a'}/E_{a'+\tilde{a}'}}(u) = 0\right\},$$

$$(E_{a'})^1 = \left\{u \in E_{a'} \,:\, \mathrm{Tr}_{E_{a'}/E_{a'+\tilde{a}'}}(u) = 1\right\},$$

$$(E_{a'})^1_{\max} = \left\{u \in E_{a'} \,:\, \mathrm{val}(u) = \sup\{\mathrm{val}(v) \,:\, v \in (E_{a'})^1\}\right\}.$$

By Bruhat-Tits [3, 4.2.20, 4.2.21], the set $(E_{a'})$ is nonempty, and, with λ any element of $(E_{a'})^1_{\max}$ and a taken to be multipliable, we have

$$\Gamma'_a = \frac{1}{2}\mathrm{val}(\lambda) + \mathrm{val}(E_{a'} - \{0\}),$$

$$\Gamma'_{2a} = \mathrm{val}((E_{a'})^0 = \{0\}) = \mathrm{val}(E_{a'} - \{0\}) - 2 \cdot \Gamma'_a.$$

For being neither multipliable nor divisible and $a' \in \Phi_E$ a root that equals a when restricted to $\mathbf{S}$, we have

$$\Gamma'_a = \mathrm{val}(E_{a'} - \{0\}).$$

1.3.5 Moy-Prasad filtration

As before, let $\mathbf{G}$ be a connected seimisimple, quasisplit group defined over K. Let $\mathcal{O}_K$ be its ring of integers, and ϖ_K a uniformizer. Let $\mathbf{S}$ be a K-split maximal torus. Let $\mathbf{T} := \mathbf{Z_G}(\mathbf{S})$ be a maximal torus defined over K. Let K^{nr} be the maximal unramified extension of K, and let E be the splitting field of $\mathbf{T}$.

Bruhat and Tits [2,3] associated to each point x in the Bruhat-Tits building $\mathcal{B}(G) = \mathcal{B}(\mathbf{G}(K))$ a parahoric group scheme over $\mathcal{O}_K$, which we denote by $\mathbf{G}_x$, whose generic fiber is isomorphic to $\mathbf{G}$. We recall the filtration of $G_x := \mathbf{G}_x(\mathcal{O}_K)$ introduced by Moy and Prasad in [12,13].

Define $T_0 = \mathbf{T}(K) \cap \mathbf{G}_x(\mathcal{O}_K)$. Then T_0 is a subgroup of finite order in the maximal bounded subgroup

$$\{t \in \mathbf{T}(K) : \operatorname{val}(\chi(t)) = 0 \text{ for all } \chi \in X^{\bullet}(\mathbf{T})\}$$

of $\mathbf{T}(K)$. For every positive $r \in \mathbb{R}$, define

$$T_r = \{t \in T_0 : \operatorname{val}(\chi(t) - 1) \geq r \text{ for all } \chi \in X^{\bullet}(\mathbf{T})\}.$$

For every affine root $\psi \in \Phi_{\mathrm{af}}$, we denote by $\dot{\psi}$ its gradient and define the subgroup U_ψ of $\mathbf{U}_{\dot{\psi}}(K)$ by

$$U_\psi = \left\{u \in \mathbf{U}_{\dot{\psi}}(K) : u = 1 \text{ or } \varphi_{\dot{\psi}}(u) \geq \psi(x_0)\right\}.$$

Then the Moy-Prasad filtration subgroups of $G_x := G_{x,0}$ are given by

$$G_{x,r} = \langle T_r, U_\psi : \psi \in \Phi_{\mathrm{af}},\ \ \psi(x) \geq r\rangle \ \text{ for } r \geq 0,$$

$$G_{x,r+} = \bigcup_{s>r} G_{x,s}.$$

Remark 12. *If $\mathbf{G}$ isn't quasisplit over K, one can extend to the K^{nr}-points of the group, yielding a quasisplit group, and define the filtration from there. Then, if $\Gamma = Gal(K^{nr}/K)$, we have the identification*

$$\mathcal{B}(\mathbf{G}(K^{nr}))^{\Gamma} = \mathcal{B}(\mathbf{G}(K)).$$

One then defines

$$(\mathbf{G}(K^{nr})_{x,r})^{\Gamma} = \mathbf{G}(K)_{x,r}.$$

We will now state some important properties of Moy-Prasad filtration.

Proposition 13. *For every $x \in \mathcal{B}(G)$, and every non-negative $r \in \mathbb{R}$, $G_{x,r}$ is a normal subgroup of G_x. This forms an exhaustive filtration of G_x by normal subgroups.*

Proposition 14. *Let $G = \mathbf{G}(K)$ be an n-dimensional group over K. Define the special fiber $\bar{\mathbf{G}}_x = \mathbf{G}_{x/\mathbb{F}_q}$. Then $\bar{\mathbf{G}}_x$ has dimension n by ftatness of parahoric group schemes. One has*

$$\bar{\mathbf{G}}_x(\mathbb{F}_q) = \mathbf{G}_x(\mathbb{F}_q) = G_{x,0}/G_{x,1}.$$

Define $\bar{\mathbf{M}}$ as the Levi quotient of $\bar{\mathbf{G}}_x$. One has

$$\bar{\mathbf{M}}_x(\mathbb{F}_q) = G_{x,0}/G_{x,0+}.$$

Similarly, let $\bar{\mathbf{U}}_x$ be the unipotent radical of $\mathbf{G}_x$. Then

$$\bar{\mathbf{U}}_x(\mathbb{F}_q) = G_{x,0+}/G_{x,1}.$$

Proposition 15. *The system of filtration is G-equivariant, i.e., if $g \in G$, then for all x, r as above, $gG_{x,r}g^{-1} = G_{gx,r}$.*

Definition 16. *An open compact subgroup H of G is nice if, for any smooth representation (π, V) of G generated by its H-fixed vectors, any subquotient of V is generated by its H-fixed vectors.*

Proposition 17. [1, Prop. 5.2] *For every $x \in \mathcal{B}(G)$ and $r > 0$ the Moy-Prasad group $G_{x,r}$ is nice.*

Proposition 18. [1, Prop. 3.1] *Let $x, y \in \mathcal{B}(G)$ and z be a point on the geodesic connecting x and y. If $r > 0$ then*

$$G_{z,r} \subseteq G_{x,r}G_{y,r}.$$

We finish with a warning about Moy-Prasad filtrations.

Remark 19. *It is not so clear that the Moy-Prasad filtrations are the most natural – and the best – filtrations to use in general. There exist many other filtration on parahoric subgroups. Notably, Yu* [28] *points out some problem when* $\mathbf{G}$ *is a group that does not split over a tamely extension.*

1.3.6 Depths of representations

Let (π, V) be an admissible irreducible representation of G. Moy and Prasad [12] define the depth of (π, V), labeled $\varrho(\pi)$ as the smallest depth r such that for some point $x \in \mathcal{B}(G)$,

$$V^{G_{x,r+}} \neq \{0\}.$$

Moy and Prasad [13, Prop. 6.6] show the method of constructing all depth-zero supercuspidal representations of a reductive group G. We briefly review this method. Let x be a vertex in $X = \mathcal{B}(G)$. Consider the group $\bar{M}_x = \bar{\mathbf{M}}_x(\mathbb{F}_q) := G_{x,0}/G_{x,0+}$. This group is the $\mathbb{F}_q$-points of some reductive group. Take an irreducible cuspidal representation (χ, W) of $\bar{M}_x$, and inflate this representation to $G_x = G_{x,0}$. Then compactly inducing this representation to G yields an irreducible supercuspidal representation (π, V).

$$V \cong \text{c-Ind}_{G_x}^{G}(\text{Inf}_{\bar{M}_x}^{G_x}(W)).$$

In this paper, we classify minimal-positive-depth representations for groups of relative rank one. That is, we consider representations (π, V) of G whose depth $r = \varrho(\pi)$ is the smallest possible positive number. This depth r will always be a rational number. This class of representations was studied by Gross, Reeder, and Yu [9, 18], and are known as *simple supercuspidals* [9], and more generally, *epipelagic representations* [18].

Chapter 2

Sheaves on trees

Let G be a group acting on a tree X, and let $\mathcal{S}$ be a G-equivariant sheaf of vector spaces on X. Its compactly supported cohomology is a representation of G. Assume X is now locally finite. Weissman [26, Thm. 2.4] proves that if $H_c^0(X, \mathcal{S})$ is an irreducible representation of G, then $H_c^0(X, \mathcal{S})$ is isomorphic to an irreducible representation induced from the stabilizer of vertex or edge of X.

Furthermore, if $\mathbf{G}$ is a reductive group over a local field K, and (π, V) is a supercuspidal representation of $G = \mathbf{G}(K)$, then Schneider and Stuhler [20, IV.4.17] construct an isomorphism $V \cong H_c^0(X, \mathcal{S}^{(e)})$ for a particular G-equivariant sheaf $\mathcal{S}^{(e)}$ depending on the depth of (π, V). As an immediate consequence to [26, Thm. 2.4] and [20, IV.4.17], Weissman arrives at a Corollary [26, Cor. 2.5] that when G has relative rank 1, every irreducible supercuspidal representation arises by induction from a compact-mod-center subgroup.

In this chapter, we will touch on the theory of sheaves on trees used by Weissman. We will then turn to Bestvina-Savin [1] to construct an equivariant contraction of tree, and then will prove a similar statement to [20, IV.4.17] in the case where $X = \mathcal{B}(G)$ is a tree and (π, V) is a supercuspidal irreducible representation of G of minimal positive depth. This proof will use cosheaves rather than sheaves, but one can easily translate between the two languages to arrive at the necessary results needed for [26, Cor. 2.5]. These results are essential for this thesis, where we provide a construction for, and prove that all minimal-positive-depth supercuspidal representations arise via compact induction from an Iwahori subgroup.

2.1 Notation and conventions

In this section, we closely follow Weissman [26]. Let X be a tree with vertex set X^0 and edge set X^1. If $v \in X^0$ and $e \in X^1$, then we write $v < e$ to mean that v is an endpoint of e.

Definition 20. *Fixing a field k, a sheaf on X will mean a cellular sheaf of k-vector spaces on X. Such a sheaf consists of k-vector spaces $\mathcal{S}_v$ and $\mathcal{S}_e$ for every vertex $v \in X^0$ and edge $e \in X^1$, respectively. Such sheaves are equipped with linear maps*

$$\mathrm{Res}_{v,e} : \mathcal{S}_v \to \mathcal{S}_e$$

for all $v < e$. $Res_{v,e}$ are called restriction maps, and the vector spaces $\mathcal{S}_v$ and $\mathcal{S}_e$ are called the stalks of $\mathcal{S}$.

Definition 21. *A cosheaf on X will mean a cellular cosheaf of k-vector spaces on X. Such a cosheaf consists of k-vector spaces $\hat{\mathcal{S}}_v$ and $\hat{\mathcal{S}}_e$ for every vertex $v \in X^0$ and edge $e \in X^1$, respectively. Cosheaves are equipped with linear maps*

$$\mathrm{Cor}_{e,v} : \hat{\mathcal{S}}_e \to \hat{\mathcal{S}}_v$$

for all $v < e$. $\mathrm{Cor}_{e,v}$ are called corestriction maps, and the vector spaces $\hat{\mathcal{S}}_v$ and $\hat{\mathcal{S}}_e$ are called the stalks of $\hat{\mathcal{S}}$

Definition 22. *Let G be a group acting on X. A G-equivariant structure on a sheaf $(\mathcal{S}, \mathrm{Res})$ consists of linear maps*

$$\eta_{g,v} : \mathcal{S}_v \to \mathcal{S}_{gv}, \qquad \eta_{g,e} : \mathcal{S}_e \to \mathcal{S}_{ge}$$

for all $g \in G$, $v \in X^0$, $e \in X^1$, satisfying the axioms:

- *For all $v \in X^0$, $e \in X^1$, the linear maps $\eta_{1,v}$ and $\eta_{1,e}$ are the identity.*
- *For all $g, h \in G$ $v \in X^0$, and $e \in X^1$, $\eta_{g,hv} \circ \eta_{h,v} = \eta_{gh,v}$ and $\eta_{g,he} \circ \eta_{h,e} = \eta_{gh,e}$.*
- *For all $g \in G$, and $v < e$, we have $\mathrm{Res}_{gv,ge} \circ \eta_{g,v} = \eta_{g,e} \circ \mathrm{Res}_{v,e}$.*

A G-equivariant sheaf on X will mean a sheaf $(\mathcal{S}, \mathrm{Res})$ endowed with a G-equivariant structure.

Definition 23. *Let G be a group acting on X. A G-equivariant structure on a cosheaf $(\hat{\mathcal{S}}, \mathrm{Cor})$ consists of linear maps*

$$\eta_{g,v} : \hat{\mathcal{S}}_v \to \hat{\mathcal{S}}_{gv}, \qquad \eta_{g,e} : \hat{\mathcal{S}}_e \to \hat{\mathcal{S}}_{ge}$$

for all $g \in G$, $v \in X^0$, $e \in X^1$, satisfying the axioms:

- *For all $v \in X^0$, $e \in X^1$, the linear maps $\eta_{1,v}$ and $\eta_{1,e}$ are the identity.*
- *For all $g, h \in G$ $v \in X^0$, and $e \in X^1$, $\eta_{g,hv} \circ \eta_{h,v} = \eta_{gh,v}$ and $\eta_{g,he} \circ \eta_{h,e} = \eta_{gh,e}$.*
- *For all $g \in G$, and $v < e$, we have $\mathrm{Cor}_{ge,gv} \circ \eta_{g,e} = \eta_{g,v} \circ \mathrm{Cor}_{e,v}$.*

A G-equivariant cosheaf on X will mean a cosheaf $(\hat{\mathcal{S}}, \mathrm{Cor})$ endowed with a G-equivariant structure.

Fix a vertex $v_0 \in X^0$ as a base point. We use v_0 to fix an orientation on the tree as follows. Let $x_e, y_e < e$ be the vertices of an edge $e \in X^1$. Label x_e and y_e such that $\mathrm{dist}(x_e, v_0) < \mathrm{dist}(y_e, v_0)$, where $\mathrm{dist}(x, y)$ is the unique distance number of edges in the unique simple path from x to y. We fix the orientation $\mathrm{or}(x_e, e) = 1$, $\mathrm{or}(y_e, e) = -1$. We say with this orientation that every edge of X is oriented towards v_0.

Fix a sheaf $(\mathcal{S}, \mathrm{Res})$. If $v \in X^0$, $s \in \mathcal{S}_v$, then we define

$$\mathrm{d}s = \sum_{e>v} \mathrm{or}(v, e) \cdot \mathrm{Res}_{v,e}(s) \in \bigoplus_{e>v} \mathcal{S}_e.$$

The compactly-supported cohomology of $\mathcal{S}$ is then computed by complex

$$0 \to \bigoplus_{v \in X^0} \mathcal{S}_v \xrightarrow{\mathrm{d}} \bigoplus_{e \in X^1} \mathcal{S}_e \to 0.$$

Thus, $H^0_c(X, \mathcal{S}) = \mathrm{Ker}\ \mathrm{d}$, $H^1_c(X, \mathcal{S}) = \mathrm{Coker}\ \mathrm{d}$. When $(\mathcal{S}, \mathrm{Res})$ is G-equivariant, the complex above and its cohomology inherit actions of G. In particular $H^i_c(X, \mathcal{S})$ is a representation of G on a k-vector space for $i = 0, 1$.

Similarly, fixing a cosheaf $(\hat{\mathcal{S}}, \mathrm{Cor})$, for $e \in X^1$, $s \in \hat{\mathcal{S}}_e$, we define

$$\hat{\mathrm{d}}s = \sum_{v<e} \mathrm{or}(v, e) \cdot \mathrm{Cor}_{v,e}(e) \in \bigoplus_{v<e} \hat{\mathcal{S}}_v.$$

The homology of $\hat{\mathcal{S}}$ is computed by the complex

$$0 \to \bigoplus_{e \in X^1} \hat{\mathcal{S}}_e \xrightarrow{\hat{\mathrm{d}}} \bigoplus_{v \in X^{(0)}} \hat{\mathcal{S}}_v \to 0.$$

Thus, $H_0(X, \hat{\mathcal{S}}) = \mathrm{Coker}\ \hat{\mathrm{d}}$, $H_1(X, \hat{\mathcal{S}}) = \mathrm{Ker}\ \hat{\mathrm{d}}$. When $(\hat{\mathcal{S}}, \mathrm{Cor})$ is G-equivariant, the complex above and its homology inherit actions of G. In particular $H_i(X, \hat{\mathcal{S}})$ is a representation of G on a k-vector space for $i = 0, 1$.

For more details on theory of cellular sheaves and cosheaves, refer to Justin Curry's thesis [5].

2.2 Weissman's induction theorem

Suppose that $\mathcal{S}$ is a G-equivariant sheaf on X. We define its 0-*rank* to be the cardinal number

$$\mathrm{Rank}^0(\mathcal{S}) = \sum_{G \cdot v \in G \backslash X^{(0)}} \dim(\mathcal{S}_v).$$

One sees that if $G \backslash X^0$ is finite, and $\mathcal{S}_v$ has finite-dimensional stalks, then $\mathrm{Rank}^0(\mathcal{S})$ will be finite.

Theorem 24. [26, Thm. 2.4] *Assume that $\mathrm{Rank}^0(\mathcal{S})$ is finite. If $H_c^0(X, \mathcal{S}) = 0$ or $H_c^0(X, \mathcal{S})$ is an irreducible representation of G, then $H_c^0(X, \mathcal{S})$ is isomorphic to a representation induced from the stabilizer of a vertex or edge of X.*

Let $\mathbf{G}$ be a reductive group over a nonarchimedean local field K. Let $G = \mathbf{G}(K)$. (π, V) be an irreducible supercuspidal representation of G. Recall that the reduced Bruhat-Tits building $\mathcal{B}(G)$ is a polysimplicial complex on which G acts [2, 3]. Schneider and Stuhler [20, IV.1] construct a G-equivariant sheaf $\mathcal{S}$ on $\mathcal{B}(G)$, with finite-dimensional stalks, such that $V \cong H_c^0(X, \mathcal{S})$ [20, IV.4.17].

Combining [20, IV.4.17] and Theorem 24, Weissman arrives at the corollary:

Corollary 25. [26, Cor. 2.5] *Let $\mathbf{G}$ be a reductive group over a nonarchimedean local field K, whose derived subgroup has relative rank one. Let $G = \mathbf{G}(K)$. Then every irreducible* *supercuspidal representation of G is isomorphic to* $\text{c-Ind}_H^G(\sigma)$ *for some compact-mod-center open subgroup $H \subset G$ and some irreducible representation (σ, W) of H.*

In his proof, Weissman uses results of Schneider-Stuhler as a black box; as a result he is not able to say very much about the inducing data (σ, W) for supercuspidals. In what follows, we reprove a foundational result of Schneider-Stuhler in some simple cases, which allows us to reprove the compact induction theorem and explicitly describe the resulting supercuspidals.

2.3 Schneider-Stuhler: Filtrations and cosheaves

Let G be the K-rational points of a reductive group $\mathbf{G}$ defined over K. Schneider and Stuhler [20, I.2] develop their own filtration of parahorics $U_\sigma^{(e)}$ attached to the reduced Bruhat-Tits building $X := \mathcal{B}(G)$ for all facets $\sigma \subseteq X$. Unlike the Moy-Prasad filtration [12, 13], this filtration is constant on all facets of the building. To each smooth representation (π, V) of G, Schneider and Stuhler [20, II.2] construct a G-equivariant cosheaf $\hat{\mathcal{S}}^{(e)}$ with stalks $\hat{\mathcal{S}}_\sigma^{(e)} \cong V^{U_\sigma^{(e)}}$.

For the remainder of the section, assume that G is a group of relative rank one, so X is a tree. For convenience, we would like to use Moy-Prasad filtrations in place of Schneider-Stuhler filtrations. One can do so by working with a refined building. For every $r \geq 0$, Moy and Prasad attach a subgroup $G_{x,r}$ such that $G_x := G_{x,0}$ is the parahoric subgroup of G attached to x. Let $G_{x,r+} = \bigcup\limits_{s>r} G_{x,s}$. For all rational numbers $r > 0$, we can refine X by adding vertices at certain points along edges such that the function $x \mapsto G_{x,r+}$ is constant on refined facets. For example, when $G = SL_2(K)$, and we fix r to be any half-integer, then by adding a vertex at the midpoint of each edge in X, then the function $x \mapsto G_{x,r+}$ is constant on edges. A general construction exists for any Bruhat-Tits building X of rank n.

For rank one groups, let c/d be the minimal-positive depth that can occur for a smooth representation of G. Then dividing each edge into d components will be sufficient. To help visualize this phenomenon, here is the Moy-Prasad-DeBacker diagram for $SU_3^{L/K}(h)$, where L/K is an unramified quadratic extension. Here, x, y are vertices and z_1, z_2 lie on edges in the building, but z_1 and z_2 are vertices in the refined building. In this diagram, the horizontal axis corresponds to points in an apartment of $\mathcal{B}(SU_3^{L/K})$, and the vertical axis corresponds to depths. Take a point z in the apartment, and move vertically upward; whenever we pass a line,

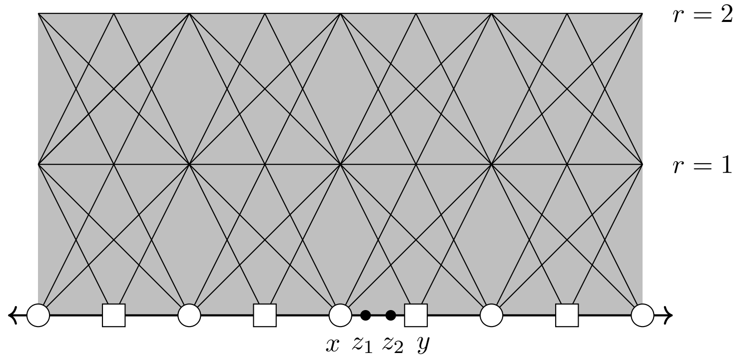


Figure 2.1: The Moy-Prasad-DeBacker diagram of $SU_3^{L/K}(h)$ unramified

this represents a jump in the Moy-Prasad filtration for the G_z parahoric subgroup. Note that this trisection of edges makes the function $z \mapsto G_{z,r}$ constant for any z on an edge. We will revisit these diagrams in more detail for other relative rank 1 groups in Chapter 3.

We will now define an adaptation of the Schneider-Stuhler cosheaf $\hat{\mathcal{S}}^{(r)} := \hat{\mathcal{S}}$, where $r \geq 0$, is fixed, using the Moy-Prasad filtration and our subdivision of X.

Definition 26. *Let (π, V) be a smooth irreducible representation of G. Define the cellular cosheaf $\hat{\mathcal{S}}$ by the stalks*

$$\hat{\mathcal{S}}_x = V^{G_{x,r+}} \text{ and } \hat{\mathcal{S}}_e = V^{G_{e,r+}}$$

for vertices x and edges e. As $e \mapsto G_{e,r+}$ is constant on the refined tree, $\hat{\mathcal{S}}_e$ is well-defined. Furthermore, $G_{x,r+} \subseteq G_{e,r+}$ whenever $x < e$. Thus we define corestriction maps by inclusion

$$\mathrm{Cor}_{e,x} : \hat{\mathcal{S}}_e \to \hat{\mathcal{S}}_x.$$

Definition 27. *One similarly defines the cellular sheaf $\mathcal{S}^{(r)} := \mathcal{S}$, with stalks*

$$\mathcal{S}_x = V^{G_{x,r+}} \text{ and } \mathcal{S}_e = V^{G_{e,r+}}$$

for vertices x and edges e. As $e \mapsto G_{e,r+}$ is constant on our refined tree, $\mathcal{S}_e$ is well-defined. Furthermore, $G_{x,r+} \subseteq G_{e,r+}$ whenever $x < e$. Thus we define restriction maps by projecting *onto invariants*

$$\mathrm{Res}_{x,e} : \mathcal{S}_x \to \mathcal{S}_e.$$

Remark 28. *Let (π, V) be a smooth irreducible representation of G, we see that in this setup that the stalks of $\mathcal{S}$ and $\hat{\mathcal{S}}$ agree. However, Schneider and Stuhler prove that the cosheaf provides a homological resolution of V, for all irreducible representations V. That is, V is recovered in $H_0(X, \hat{\mathcal{S}})$ and the higher homologies vanish. But, they prove that the cohomology of the sheaf $H_c^j(X, \mathcal{S})$ is nonzero in degree 0 or 1, where it recovers V. The degree of the cohomology depends on whether the irreducible is supercuspidal or not.*

2.4 Contractions of Bruhat-Tits trees

Let $\mathbf{G}$ be a reductive group over a nonarchimedean local field K, whose derived subgroup has relative rank one. Let $G = \mathbf{G}(K)$. Then $X := \mathcal{B}(G)$ is a tree.

Fix a rational number $r \geq 0$, and refine X so that the function $x \mapsto G_{x,r+}$ is constant on refined facets. Let $C_i(X)$ be the free abelian group with basis consisting of the i-dimensional facets of X. Let $C_{-1}(X) = \mathbb{Z}$. Fix an orientation on X. Then $C_\bullet(X)$ is realized as the complex

$$0 \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\Sigma} \mathbb{Z} \longrightarrow 0 \ ,$$

where for an edge $e \in X^1$, $\partial e = \sum\limits_{v<e} \mathrm{or}(v,e)v$. Σ is the sum of the coefficients map. This sequence exact because $H_0(X) = \mathbb{Z}$ and $H_1(X) = 0$ when X is a tree.

Let (π, V) be a smooth irreducible representation of G of depth r. We tensor the above exact complex with the vector space V, yielding the exact sequence

$$0 \longrightarrow C_1(X) \otimes_{\mathbb{Z}} V \xrightarrow{\partial} C_0(X) \otimes_{\mathbb{Z}} V \xrightarrow{\Sigma} V \longrightarrow 0 \ .$$

Let $\hat{\mathcal{S}}$ be the associated cosheaf of (π, V). The above complex has a subcomplex,

$$0 \longrightarrow C_1(X, \hat{\mathcal{S}}) \longrightarrow C_0(X, \hat{\mathcal{S}}) \xrightarrow{\Sigma} V \longrightarrow 0 \ ,$$

or equivalently,

$$0 \longrightarrow \bigoplus_{e \in X^1} \hat{\mathcal{S}}_e \longrightarrow \bigoplus_{x \in X^0} \hat{\mathcal{S}}_x \xrightarrow{\Sigma} V \longrightarrow 0.$$

The goal of this section is to prove

Theorem 29. [20, II.3.1],[1, Cor. 5.3] *Let (π, V) be a smooth irreducible representation of G of depth r, and let $\hat{\mathcal{S}}$ be the associated cosheaf. Then the resolution*

$$0 \longrightarrow C_1(X, \hat{\mathcal{S}}) \longrightarrow C_0(X, \hat{\mathcal{S}}) \longrightarrow V \longrightarrow 0$$

is exact. In other words, $H_0(X, \hat{\mathcal{S}}) = V$, $H_1(X, \hat{\mathcal{S}}) = 0$.

Bestvina and Savin [1, Cor. 5.3] prove this result, by showing how it follows from the existence of a family of equivariant contractions of the Bruhat-Tits building. These contractions are a bit mysterious in general, but for trees we give a direct construction.

Definition 30. *A contraction c of $C_\bullet(X)$ based at x is a sequence of homomorphisms*

$$c_i : C_i(X) \to C_{i+1}(X), \text{ for } i = -1, 0, 1, \ldots$$

such that $c_{-1}(1) = x$ and $c_{i-1}\partial + \partial c_i = \mathrm{Id}$.

Since we take X to be a tree, we have $C_i(X) = 0$ for $i \notin \{-1, 0, 1\}$. Thus, a contraction of X based at x is defined by a single map

$$c : C_0(X) \to C_1(X)$$

with conditions

$$c\partial e = e \text{ and } x + \partial c y = y,$$

for all edges e and vertices y.

Proposition 31. *Let X be any tree, and let x be any vertex of X. Assume that every edge of X is oriented towards x. For any vertex $y \in X$, let $P(x, y)$ denote the set of edges on the unique path from x to y. Define a $\mathbb{Z}$-linear map $c : C_0(X) \to C_1(X)$ by*

$$c(y) = -\sum_{e \in P(x,y)} e.$$

Then c defines a contraction of X based at x.

Proof. First we prove that $c\partial e = e$ for all edges e. Let v and v' be the vertices adjoined by the edge e, with v closer to x. Then $\partial e = v - v'$ We have

$$c\partial e = c(v - v') = c(v) - c(v') = -\sum_{e' \in P(x,v)} e' + \sum_{e' \in P(x,v')} e' = e.$$

Second, we prove that $x + \partial cy = y$ for all vertices y. Let $v_0, v_1, \ldots, v_k$ be the vertices along the path from x to y, with $x = v_0$ and $y = v_k$. Let e_i be the edge joining v_{i-1} to v_i, for $1 \leq i \leq k$. Thus

$$c(y) = -\sum_{i=1}^{k} e_i.$$

For each e_i, $\partial e_i = v_i - v_{i+1}$. Thus,

$$x + \partial c(y) = v_0 + \partial c(v_k) = x - (v_0 - v_1 + v_1 - v_2 + \cdots - v_{k-1} + v_{k-1} - v_k)$$

$$= v_0 - (v_0 - v_k) = v_k = y.$$

Thus we have established that the map $c(y) = -\sum_{e \in P(x,y)} e$ defines a contraction on X based at x. $\square$

Proposition 32. *Keep the notation from the previous theorem, and now suppose that a group G acts on the tree X, and let G_x be the stabilizer of the vertex x. Then the contraction c based at x is G_x-equivariant, i.e.,*

$$c_{-1}(g \cdot n) = g \cdot c_{-1}(n) \textit{ for all } n \in \mathbb{Z}, \textit{ where } \mathbb{Z} \textit{ has the trivial } G_x\textit{-action}$$

$$c(g \cdot y) = g \cdot c(y) \textit{ for all vertices } y.$$

Proof. We see that as G_x fixes x,

$$c_{-1}(g \cdot 1) = c_{-1}(1) = x = g \cdot x.$$

Thus the first equation is satisfied.

Let $P(x, y)$ be the set of edges along the path from a vertex x to y. Then

$$c(g \cdot y) = -\sum_{e \in P(g \cdot x, g \cdot y)} e = -\sum_{e \in P(x, g \cdot y)} e = -\sum_{e \in P(v, v_k)} g \cdot e = g \cdot c(y).$$

Thus, we have established that c is G_x-equivariant. $\square$

Remark 33. *Let c be the contraction we constructed above based at x with σ a vertex in X. Using Bestiva-Savin's notions, we write $c(\sigma) = \sum_{\tau \in X^1} c(\sigma, \tau)\tau$ for constants $c(\sigma, \tau) \in \mathbb{Z}$. In our contraction, we found that $c(\sigma, \tau) = -1$ when τ is an edge in the path from x to σ, and $c(\sigma, \tau) = 0$ otherwise.*

Using our contraction $c(y) = -\sum_{e \in P(x,y)} e$, we have reproven [1, Prop. 2.5] in an explicit manner for trees.

Proposition 34. [1, Prop. 2.5] *For every vertex x in the refined building X, there exists a G_x-invariant contraction c such that*

(1) c is G_x-equivariant.

(2) if $c(\sigma, \tau) \neq 0$ then τ lies on the path connecting the vertex x to σ.

Fix a non-negative rational number r, and refine X in such a way that $x \mapsto G_{x,r+}$ is constant for all points x along an edge. Let (π, V) be a smooth representation of G. For any facet $\sigma \subset X$ with interior point x, let $V_\sigma = V^{G_{x,r+}}$. The complex $C_\bullet(X) \otimes_{\mathbb{Z}} V$ admits a natural representation of G defined by $g(\sigma \otimes v) = g(\sigma) \otimes g(v)$, for all $g \in G$. Let $C_\bullet(X, \hat{\mathcal{S}})$ be the subcomplex spanned by $\tau \otimes v$, where $v \in V^{G_{\tau,r+}}$. The boundary ∂ preserves $C_\bullet(X, \hat{\mathcal{S}})$ because $V^{G_{\sigma,r+}} \subseteq V^{G_{\rho,r+}}$ whenever ρ is in the boundary of σ. The action of G on $C_\bullet(X) \otimes_{\mathbb{Z}} V$ preserves the subcomplex $C_\bullet(X, \hat{\mathcal{S}})$.

We are now ready to prove

Theorem 29. [20, II.3.1],[1, Cor. 5.3] *Let (π, V) be a smooth irreducible representation of G of depth r, and let $\hat{\mathcal{S}}$ be the associated cosheaf. Then the resolution*

$$0 \longrightarrow C_1(X, \hat{\mathcal{S}}) \longrightarrow C_0(X, \hat{\mathcal{S}}) \longrightarrow V \longrightarrow 0$$

is exact. In other words, $H_0(X, \hat{\mathcal{S}}) = V$, $H_1(X, \hat{\mathcal{S}}) = 0$.

Proof. We use the proof of Bestvina-Savin. It suffices to prove that the complex is exact in every Bernstein component. The complex $C_\bullet(X, \hat{\mathcal{S}})$ is a direct sum of G-modules isomorphic to $\text{c-Ind}_{G_\tau}^G(V^{G_{\tau,r+}})$, where τ is a facet in the refined building – an edge or a vertex. This module is generated by $V^{G_{\tau,r+}}$.

We now recall Definition 16 that an open compact subgroup $H \subset G$ is called *nice* if for any smooth representation V of G by generated by its H-fixed vectors, any subquotient of V is generated by its H-fixed vectors. Recall further that for any $x \in X$ and $r > 0$ the Moy-Prasad subgroup $G_{x,r}$ is nice 17.

Let $x \in \bar{\tau}$. Since $G_{x,r+}$ is nice, and $G_{x,r+} \subseteq G_{\tau,r+}$, it follows that $\text{c-Ind}_{G_\tau}^G(V^{G_{\tau,r+}})$ is generated by $G_{x,r+}$-fixed vectors. Thus any Bernstein summand of $C_\bullet(X, \hat{\mathcal{S}})$ is generated by $G_{x,r+}$-fixed vectors, for some vertex x and exactness can be checked by passing to $G_{x,r+}$-fixed vectors.

Bestvina and Savin [1, Thm 4.1] show that whenever $x \in X$ is a vertex and c is an x-based contraction of $C_\bullet(X)$ satisfying (1) and (2) in Proposition 34, then $C_\bullet(X, \hat{\mathcal{S}})^{G_{x,r+}}$, given by

$$0 \longrightarrow C_1(X, \hat{\mathcal{S}})^{G_{x,r+}} \longrightarrow C_0(X, \hat{\mathcal{S}})^{G_{x,r+}} \longrightarrow V^{G_{x,r+}} \longrightarrow 0$$

is exact. Thus we have established the exactness of

$$0 \longrightarrow C_1(X, \hat{\mathcal{S}}) \longrightarrow C_0(X, \hat{\mathcal{S}}) \longrightarrow V \longrightarrow 0, \ ,$$

giving $H_0(X, \hat{\mathcal{S}}) = V$, $H_1(X, \hat{\mathcal{S}}) = 0$.

$\square$

2.5 Critical depths

Definition 35. *A positive real number r is called a critical depth if the following condition is satisfied. Let z be a point in X such that $G_{z,r} \neq G_{z,r+}$. Then there exists an open neighborhood U of z in X such that if $y \neq z$ and $y \in U$, then $G_{y,r} = G_{y,r+}$.*

In other words, all jump points at depth r are isolated. Consequently, if $z \in X$ such that $G_{z,r} \neq G_{z,r+}$ for a critical depth r, then z is a vertex in the refined tree.

Proposition 36. *Suppose that (π, V) is an irreducible representation of G of critical depth r. Then there exists a vertex z of the refined tree such that $V \cong \text{c-Ind}_{G_z}^{G}(V^{G_{z,r+}})$.*

Proof. The homological resolution

$$0 \longrightarrow C_1(X, \hat{\mathcal{S}}) \longrightarrow C_0(X, \hat{\mathcal{S}}) \longrightarrow V \longrightarrow 0$$

can be rewritten as

$$0 \longrightarrow \bigoplus_{e \in X^1} \hat{\mathcal{S}}_e \hookrightarrow \bigoplus_{x \in X^0} \hat{\mathcal{S}}_x \xrightarrow{\Sigma} V \longrightarrow 0.$$

Or equivalently,

$$0 \longrightarrow \bigoplus_{e \in X^1} V^{G_{e,r+}} \hookrightarrow \bigoplus_{x \in X^0} V^{G_{x,r+}} \xrightarrow{\Sigma} V \longrightarrow 0.$$

As r is a critical depth, $V^{G_{e,r+}} = 0$ for all edges e in the refined tree X. Thus exactness yields an isomorphism of G-representations

$$\bigoplus_{x \in X^0} V^{G_{x,r}} \to V.$$

Recall that G acts on X^0 with a finite number of orbits, say $G \cdot \nu_1, \ldots G \cdot \nu_k$ for some $k \in \mathbb{N}$. Then

$$\bigoplus_{x \in X^0} V^{G_{x,r+}} \cong \bigoplus_{x \in G \cdot \nu_1} V^{G_{\nu_1,r+}} \oplus \cdots \oplus \bigoplus_{x \in G \cdot \nu_k} V^{G_{\nu_k,r+}}.$$

As V is irreducible, without loss of generality, and setting $z = \nu_1$, we have

$$V \cong \bigoplus_{x \in G \cdot z} V^{G_{z,r+}} \cong \bigoplus_{\bar{g} \in G/G_z} V^{G_{g \cdot z, r+}} \cong \text{c-Ind}_{G_z}^{G}(V^{G_{z,r+}}),$$

thus completing the proof. □

Remark 37. *When z is not a vertex of the original Bruhat-Tits tree, the group G_z is an Iwahori subgroup.*

Chapter 3

Moy-Prasad-DeBacker Facets

In this chapter, we will use what we call *Moy-Prasad-DeBacker Facets*, a diagram attributed to Stephen DeBacker [6] to visualize the depths of jumps in rank 1 buildings. Using Moy-Prasad-DeBacker facets, we will see the unique point z along an edge whose first positive jump in the Moy-Prasad filtration occurs at depth r – which will turn out to be the lowest positive depth a G-representation can be to be supercuspidal. Furthermore, these diagrams we will help provide us explicit descriptions for the group $\bar{M}_z = G_{z,0}/G_{z,0+}$ and vector space $V_z = G_{z,r}/G_{z,r+}$. These computations will allow us to construct depth-r irreducible supercuspidal representations; we reserve these constructions for Chapter 5.

As before, let K denote a nonarchimedean local local field. Let $\mathcal{O}_K$ be its ring of integers and $\mathfrak{p}_K = \varpi_K \mathcal{O}_K$ the maximal ideal of $\mathcal{O}_K$. In Chapter 1, we tabulated and provided descriptions for all semisimple groups over K of relative rank one.

In this chapter, we provide explicit constructions for $SL_2(K), SL_2(D), SU_3^{L/K}(h)$ and $SU_3^{E/K}(h)$ where D is a central simple algebra over K, and L/K (resp. E/K) is an unramified (resp. ramified) quadratic field extension. We will briefly touch upon the theory used to apply similar constructions for the remaining groups of relative rank one.

3.1 $SL_2(K)$ Moy-Prasad filtration and quotients

Let $G = \mathbf{G}(K) = SL_2(K)$. $\mathbf{G}$ is split over K and has a maximal torus $\mathbf{S}$ such that

$$S := \mathbf{S}(K) = \left\{ h(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \,:\, t \in K^\times \right\}.$$

With respect to this torus, $\mathbf{G}$ has root system $\Phi = \{\pm a\}$, with root subgroups

$$\mathbf{U}_a(K) = \left\{ u_+(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \,:\, x \in K \right\},$$

$$\mathbf{U}_{-a}(K) = \left\{ u_-(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \,:\, x \in K \right\}.$$

3.1.1 Affine roots and Root subgroup filtration

Define $U_{a+\gamma} = \{u_+(x) : \mathrm{val}(x) \geq \gamma\}$, and similarly $U_{-a+\gamma} = \{u_-(x) : \mathrm{val}(x) \geq \gamma\}$. We say that $\pm a + \gamma$ is an affine root if $U_{\pm a+\gamma+} \subsetneq U_{\pm a+\gamma}$. Since $U_{\pm a+\gamma+} \neq U_{\pm a+\gamma}$ precisely when $\gamma \in \mathrm{val}(K)$, one finds that $\pm a + \gamma$ is an affine root if and only if γ is an integer.

Thus, we have affine root system $\Phi_{\mathrm{af}} = \{\pm a + \gamma : \gamma \in \mathbb{Z}\}$.

Let $\gamma \in \mathbb{Z}$. One finds that $U_{a+\gamma}/U_{a+\gamma+} \cong \begin{pmatrix} 0 & \mathbb{F}_q \\ 0 & 0 \end{pmatrix}$ and $U_{-a+\gamma}/U_{-a+\gamma+} \cong \begin{pmatrix} 0 & 0 \\ \mathbb{F}_q & 0 \end{pmatrix}$. Both of these quotients are one-dimensional vector space over the residue field $\mathbb{F}_q$ of K.

3.1.2 Filtration of the torus

Define

$$T_0 := \left\{ h(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \,:\, t \in \mathcal{O}_K^\times \right\},$$

the maximal compact subgroup of S. We define a filtration on the T_0 by

$$T_r := \left\{ h(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \in T_0 \,:\, \mathrm{val}(t-1) \geq r \right\}.$$

$T_r \neq T_{r+}$ if and only if $r \in \mathrm{val}(K)$. One finds that

$$T_0/T_1 \cong \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} : t \in \mathbb{F}_q^\times \right\},$$

$$T_r/T_{r+1} \cong \left\{ \begin{pmatrix} t & \\ & -t \end{pmatrix} : t \in (\mathbb{F}_q, +) \right\}.$$

T_0/T_1 is the $\mathbb{F}_q$-points of a reductive group, isomorphic to $GL_1(\mathbb{F}_q)$, and T_r/T_{r+1} for $r > 0$ is a 1-dimensional $\mathbb{F}_q$-vector space.

3.1.3 Special points and their filtration subgroups

Let $x \in \mathcal{B}(G)$ be a vertex in the standard apartment. One can make x a base point in the apartment $\mathcal{A}(\mathbf{S}, K)$ so that $a(x) = 0$. One can then choose $y \in \mathcal{A}(\mathbf{S}, K)$, an adjacent vertex to x, so that $a(y) = \mathrm{val}(\varpi_K) = 1$.

Now

$$G_x := G_{x,0} = \langle T_0, U_{a+0}, U_{-a+0} \rangle \cap G,$$

$$G_{x,0+} = G_{x,1} = \langle T_1, U_{a+1}, U_{-a+1} \rangle \cap G.$$

Similarly,

$$G_y := G_{y,0} = \langle T_0, U_{a-1}, U_{-a+1} \rangle \cap G,$$

$$G_{y,0+} = G_{y,1} = \langle T_1, U_{a+0}, U_{-a+2} \rangle \cap G.$$

Taking z to be the midpoint of the edge adjoining x and y, one finds,

$$G_z := G_{z,0} = G_x \cap G_y = \langle T_0, U_{a+0}, U_{-a+1} \rangle \cap G,$$

$$G_{z,0+} = G_{z,\frac{1}{2}} = \langle T_1, U_{a+0}, U_{-a+1} \rangle \cap G,$$

$$G_{z,\frac{1}{2}+} = G_{z,1} = \langle T_1, U_{a+1}, U_{-a+2} \rangle \cap G.$$

3.1.4 Moy-Prasad-DeBacker Facets

Before explaining the figure in the next section, which is a Moy-Prasad-DeBacker facet for the group $SL_2(K)$, we describe general Moy-Prasad-DeBacker facets for groups of relative rank one. For shorthand notation, we will write *MPD facet* in place of Moy-Prasad-DeBacker facet.

In MPD facet diagrams, the horizontal axis represents points along an apartment. White circles and squares represent vertices in the Bruhat-Tits building. Depending on the group G, under the action of the building on G, there are one or two orbits of vertices. Circles and squares distinguish these orbits, when applicable. Each vertex is a zero of an affine root; ones draws black lines emitting from each vertex of slope equal to the gradient of such affine roots. In the case of relative rank one groups, the possible slopes are ± 1 and ± 2 as a group of relative rank one has roots $\Phi = \{\pm a, \pm 2a\}$ or $\Phi = \{\pm a\}$.

Moving up and down vertically represents a change of depth. Thus, $G_{x,r}$ is visualized as the point in the diagram with horizontal coordinate x and vertical coordinate r. If (x_1, r_1) and (x_2, r_2) belong to the same facet – gray polyhedron – then this tells us that G_{x_1,r_1} and G_{x_2,r_2} are the same group. As briefly mentioned above, the sloped black lines represent affine roots; the slope of these lines is the gradient of the corresponding affine root which it represents. The horizontal black lines represent jumps in the filtration of the torus. If G_{x,r_1} and G_{x,r_2} belong to different facets, then one can find the generators of the group G_{x,r_2} given the generators of G_{x,r_1} by counting the number of lines of each slope which separate the two facets containing the points (x, r_1) and (x, r_2). One can do the same for points (x_1, r_1) and (x_2, r_2) to compare G_{x_1,r_1} and G_{x_2,r_2}.

3.1.5 $SL_2(K)$ Moy-Prasad-DeBacker facets

Below is the Moy-Prasad-DeBacker diagram for $SL_2(K)$. I have labeled points x, y, and z corresponding to the points in $\mathcal{B}(G)$ described earlier in this section.

I have labeled by a green circle, the first positive depth jump of G_z. We see this jump

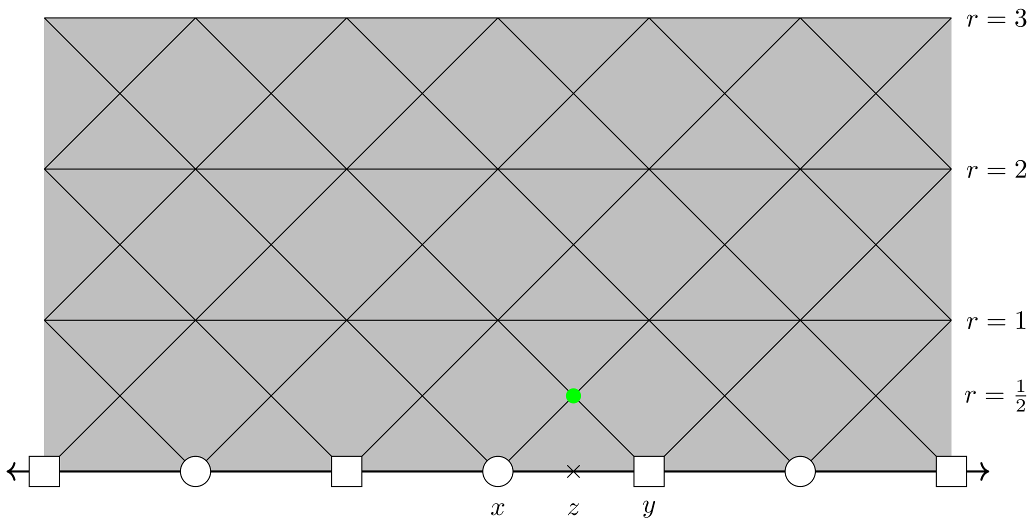


Figure 3.1: The Moy-Prasad-DeBacker diagram of $SL_2(K)$

occurs at depth $\frac{1}{2}$, and it is a jump in valuation of $\mathbf{U}_a(K)$ and $\mathbf{U}_{-a}(K)$. Thus,

$$\bar{G}_z = G_{z,0}/G_{z,0+} \cong GL_1(K), \;\; V_z = G_{z,\frac{1}{2}}/G_{z,\frac{1}{2}+} \cong \begin{pmatrix} & \mathbb{F}_q \\ \mathbb{F}_q & \end{pmatrix}.$$

3.2 $SL_2(D)$ Moy-Prasad Filtration and Quotients

3.2.1 Central simple algebras over K

Let D be central simple algebra of degree d over K. Let $\mathcal{O}_D$ be its ring of integers, and let ϖ_D be a uniformizing element of D. Let L be a maximal commutative K-subfield of D which is an unramified Galois extension of K. Then $[L : K] = d$.

One can write

$$D = L \oplus L\varpi_D \oplus \cdots \oplus L\varpi_D^{d-1},$$

where $\varpi_D^d = \varpi_K$, a uniformizing element of K. For all $\lambda \in L$, we also have $\varpi_L \lambda \varpi_L^{-1} = \phi(\lambda)$ for some generator ϕ of $\Gamma = Gal(L/K)$. Using the L basis of D, write $x \in D$ as $x = \sum_{i=0}^{d-1} x_i \varpi_d^i$ for $x_i \in L$. This L-basis gives an embedding $D \hookrightarrow M_d(L)$ of K-algebras, which sends $x \in D$ to the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ \phi(x_{d-1})\varpi_K & \phi(x_0) & \dots & \phi(x_{d-2}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{d-1}(x_1)\varpi_F & \phi^{d-1}(x_2)\varpi_D & \dots & \phi^{d-1}(x_0) \end{pmatrix}.$$

Let $\mathrm{Nrd} : D \to K$ be the determinant map on D with elements of D viewed as matrices over L. Nrd is known as the *reduced norm* of D over K.

Let val_K be a valuation on K with $\mathrm{val}(\varpi_K) = 1$. For $x \in D$, let

$$\mathrm{val}(x) := \frac{1}{d}\mathrm{val}_F(\mathrm{Nrd}(x)).$$

$\mathrm{val} : D^\times \to \frac{1}{d}\mathbb{Z}$ is the unique valuation on D extending val_K.

3.2.2 $SL_2(D)$ tori and root subgroups

Let $\mathbf{G}$ be the algebraic group over K whose group of C-rational points (for C a commutative K-algebra) is $SL_2(D \otimes_K C)$. Then $G = \mathbf{G}(K) = SL_2(D \otimes_K K) = SL_2(D)$. Note $\mathbf{G}$ is non-quasisplit over K, but splits (and quasisplits) over L, as $\mathbf{G}(L) = SL_2(D \otimes_K L) \cong SL_{2d}(L)$.

Let $\mathbf{S}$ be the K-split torus of $\mathbf{G}$ such that

$$S = \mathbf{S}(K) = \left\{ h(s) = \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} \ : \ s \in K^\times \right\}.$$

Let $\mathbf{Z_G}(\mathbf{S})$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. Then,

$$M := \mathbf{Z_G}(\mathbf{S})(K) = \left\{ m(t_1, t_2) = \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \ : \ t_1, t_2 \in D^\times,\ \ \mathrm{Nrd}(t_1 t_2) = 1 \right\}.$$

There exists a maximal torus $\mathbf{T}$ defined over K, with $\mathbf{T} \subset \mathbf{Z_G}(\mathbf{S})$, such that

$$T := \mathbf{T}(K) = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \ : \ t_1, t_2 \in L^\times,\ \ \mathrm{N}_{L/K}(t_1 t_2) = 1 \right\}.$$

By extension of scalars

$$\mathbf{T}(L) \cong \left\{ \begin{pmatrix} \vec{t_1} & \\ & \vec{t_2} \end{pmatrix} : \vec{t_1}, \vec{t_2} \in (L^\times)^d \text{ and } \prod_{i=1}^{d} \vec{t_1}\vec{t_2} = 1 \right\}.$$

The embedding $\mathbf{T}(K) \hookrightarrow \mathbf{T}(L)$ sends a pair (t_1, t_2) to a pair of vectors $(\vec{t_1}, \vec{t_2})$ via

$$t_1 \mapsto (t_1, \phi(t_1), \ldots, \phi^{d-1}(t_1)), \quad t_2 \mapsto (t_2, \phi(t_2), \ldots, \phi^{d-1}(t_2)).$$

Here ϕ denotes Frobenius, i.e., a generator of $Gal(L/K)$.

With respect to $\mathbf{S}$, $\mathbf{G}$ has the root system $\Phi = \{\pm a\}$, with root subgroups

$$\mathbf{U}_a(K) = \left\{ u_+(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in D \right\},$$

$$\mathbf{U}_{-a}(K) = \left\{ u_-(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} : x \in D \right\}.$$

3.2.3 Affine roots and root subgroup filtrations

Recall that we extend the valuation val_K from K to D, so that $\mathrm{val}(K) = \mathbb{Z}$ yields $\mathrm{val}(D) = \frac{1}{d}\mathbb{Z}$. That is, $\mathrm{val}(\varpi_D) = \frac{1}{d}$.

Just as in the $SL_2(K)$ case, we define

$$U_{a+\gamma} = \{u_+(x) : \mathrm{val}(x) \geq \gamma\}, \;\; U_{-a+\gamma} = \{u_-(x) : \mathrm{val}(x) \geq \gamma\}.$$

One finds that $\pm a + \gamma$ is an affine root if $U_{\pm a+\gamma+} \subsetneq U_{\pm a+\gamma}$. As $U_{\pm a+\gamma+} \neq U_{\pm a+\gamma}$ precisely when $\gamma \in \frac{1}{d}\mathbb{Z}$, one finds that $\Phi_{\mathrm{af}} = \{\pm a + \frac{1}{d}\mathbb{Z}\}$.

3.2.4 Filtrations of the centralizer of the split torus

$\mathbf{G}$ isn't quasisplit over K, but it becomes split (and quasisplit) over L. Recall that one can define the Moy-Prasad filtration subgroups for $\mathbf{G}(L) = SL_2(D \otimes_K L) \cong SL_{2d}(L)$, and one has

$$(\mathbf{G}(L)_{x,r+})^{\Gamma} = \mathbf{G}(K)_{x,r+}$$

for all $x \in \mathcal{B}(\mathbf{G}(L))$. These filtration subgroups are provided by Lanksy-Raghuram [11, Section 4]. Note that the filtration will be applied to $M = \mathbf{Z_G}(K)$, rather than on $T = \mathbf{T}(K)$. Let

$$M_0 = \left\{m(t_1, t_2) \in M : u, v \in \mathcal{O}_D^{\times},\ \mathrm{Nrd}(t_1 t_2) = 1\right\}.$$

We define a filtration on M_0 by

$$M_r := \{m(t_1, t_2) \in M_0 : \mathrm{val}(t_i - 1) \geq r\}.$$

One sees $M_r \neq M_{r+}$ if and only if $r \in \frac{1}{d}$. We have a filtration

$$M_0 \rhd M_{\frac{1}{d}} \rhd M_{\frac{2}{d}} \rhd M_{\frac{3}{d}} \ldots$$

with

$$M_0/M_{\frac{1}{d}} \cong \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} : t_1, t_2 \in \mathbb{F}_{q^d}^{\times},\ \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(t_1 t_2) = 1 \right\}.$$

3.2.5 Special points and their filtration subgroups

Let $x \in \mathcal{B}(G)$ be a vertex in the apartment $\mathcal{A}(\mathbf{S}, K) = X_{\bullet}(\mathbf{S}) \otimes \mathbb{R}$ so that $a(x) = 0$. One can then choose $y \in \mathcal{A}(\mathbf{S}, K)$, an adjacent vertex to x, so that $a(y) = \mathrm{val}(\varpi_D) = \frac{1}{d}$. Now

$$G_x := G_{x,0} = \langle M_0, U_{a+0}, U_{-a+0} \rangle \cap G,$$

$$G_{x,0+} = G_{x,\frac{1}{d}} = \langle M_{\frac{1}{d}}, U_{a+\frac{1}{d}}, U_{-a+\frac{1}{d}} \rangle \cap G,$$

$$G_{x,\frac{1}{d}+} = G_{x,\frac{2}{d}} = \langle M_{\frac{2}{d}}, U_{a+\frac{2}{d}}, U_{-a+\frac{2}{d}} \rangle \cap G, \ldots$$

$$G_{x,\frac{d-1}{d}+} = G_{x,1} = \langle M_1, U_{a+1}, U_{-a+1} \rangle \cap G.$$

Similarly,

$$G_y := G_{y,0} = \langle M_0, U_{a-\frac{1}{d}}, U_{-a+\frac{1}{d}} \rangle \cap G$$

$$G_{y,0}+ = G_{y,\frac{1}{d}} = \langle M_{\frac{1}{d}}, U_{a+0}, U_{-a+\frac{2}{d}} \rangle \cap G,$$

$$G_{y,\frac{1}{d}+} = G_{y,\frac{2}{d}} = \langle M_{\frac{2}{d}}, U_{a+\frac{1}{d}}, U_{-a+\frac{3}{d}} \rangle \cap G, \ldots$$

$$G_{y,1} = G_{y,\frac{d-1}{d}+} = \langle M_1, U_{a+\frac{d-1}{d}}, U_{-a+\frac{d+1}{d}} \rangle \cap G.$$

One finds that

$$G_{x,0}/G_{x,0+} \cong G_{y,0}/G_{y,0+} \cong \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_{q^d}) \, : \, N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(\det(g)) = 1 \right\}.$$

Taking z to be the midpoint of the edge adjoining x and y, one finds,

$$G_z := G_{z,0} = G_x \cap G_y = \langle M_0, U_{a+0}, U_{-a+\frac{1}{d}} \rangle \cap G,$$

$$G_{z,0+} = G_{z,\frac{1}{2d}} = \langle M_{\frac{1}{d}}, U_{a+0}, U_{-a+\frac{1}{d}} \rangle \cap G,$$

$$G_{z,\frac{1}{2d}+} = G_{z,\frac{1}{d}} = \langle M_{\frac{1}{d}}, U_{a+\frac{1}{d}}, U_{-a+\frac{2}{d}} \rangle \cap G.$$

This gives us

$$\bar{M}_z = G_{z,0}/G_{z,0+} = M_0/M_{\frac{1}{d}}$$

$$= \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \, : \, t_1, t_2 \in \mathbb{F}_{q^d}^{\times}, \;\; \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(t_1 t_2) = 1 \right\} = \left\{ \begin{pmatrix} tn & \\ & t^{-1} \end{pmatrix} \, : \, t, n \in \mathbb{F}_{q^d}^{\times}, \;\; \mathrm{N}_{\mathbb{F}_{q^d}/\mathbb{F}_q}(n) = 1 \right\},$$

a $2d-1$-dimensional group over $\mathbb{F}_q$.

$$V_z = G_{z,\frac{1}{2d}}/G_{z,\frac{2}{d}+} \cong \begin{pmatrix} & \mathbb{F}_{q^d} \\ \mathbb{F}_{q^d} & \end{pmatrix},$$

a $2d$-dimensional vector space over $\mathbb{F}_q$.

3.2.6 $SL_2(D)$ Moy-Prasad-DeBacker facets

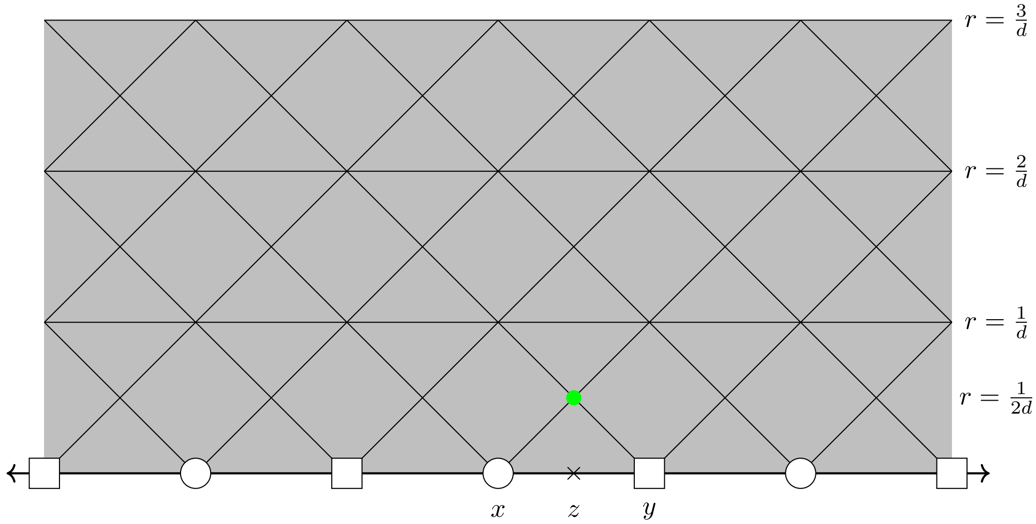


Figure 3.2: The Moy-Prasad-DeBacker diagram of $SL_2(D)$

Here is the Moy-Prasad-DeBacker facets diagram for $SL_2(D)$. I have labeled points x, y, and z corresponding to the points in $\mathcal{B}(G)$ described earlier in this section. The small green dot represents $G_{z,\frac{1}{2d}}$. We will find in Chapter 5 that $SL_2(D)$ has depth-$\frac{1}{2d}$ supercuspidal representations.

3.3 Unramified $SU_3^{L/K}(h)$ – Moy-Prasad filtrations and quotients

Let L/K be an unramified quadratic extension. Denote by $\mathcal{O}_L$ the ring of integers of L, and denote by ϖ_L a uniformizing element. Let σ be generator of $Gal(L/K)$. If $\ell \in L$, let $\bar{\ell}$ denote the image of ℓ under σ. Let $h : L^3 \times L^3 \to K$ be a Hermitian form with Witt Index 1. Without loss of generality, we take

$$h((x_1, x_2, x_3), (y_1, y_2, y_3)) = \bar{x}_1 y_3 + \bar{x}_2 y_2 + \bar{x}_3 y_1.$$

Then h corresponds to the Hermitian matrix

$$H = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let $G = SU_3^{L/K}(h)$. G is then realized as

$$G = \left\{ g \in GL_n(L) : {}^t\bar{g}Hg = H, \text{ and } \det(g) = 1 \right\}.$$

We fix a maximal K-split torus $\mathbf{S}$ of $\mathbf{G}$, such that

$$S := \mathbf{S}(K) = \left\{ m(s) = \begin{pmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{pmatrix} : s \in K^\times \right\}.$$

Since $\mathbf{G}$ is quasisplit over K, $\mathbf{T} = \mathbf{Z}_{\mathbf{G}}(\mathbf{S})$ is maximal torus of $\mathbf{G}$. Let

$$T := \mathbf{T}(K) = \left\{ m(t) = \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} : t \in L^\times \right\}.$$

With respect to S, G has the root system $\Phi = \{\pm a, \pm 2a\}$, with root subgroups

$$\mathbf{U}_a(K) = \left\{ u_+(c,d) = \begin{pmatrix} 1 & -\bar{c} & d \\ & 1 & c \\ & & 1 \end{pmatrix} : c, d \in L, \;\; \mathrm{N}(c) + \mathrm{Tr}(d) = 0 \right\},$$

$$\mathbf{U}_{2a}(K) = \{u_+(0,d) : \mathrm{Tr}(d) = 0\},$$

$$\mathbf{U}_{-a}(K) = \left\{ u_-(c,d) = \begin{pmatrix} 1 & & \\ c & 1 & \\ d & -\bar{c} & 1 \end{pmatrix} : c, d \in L, \;\; \mathrm{N}(c) + \mathrm{Tr}(d) = 0 \right\},$$

$$\mathbf{U}_{-2a}(K) = \{u_-(0,d) : \mathrm{Tr}(d) = 0\}.$$

3.3.1 Affine roots and root subgroups filtrations

Following [25, Sections 1.16, 3.11], we define $\delta = \sup\{\mathrm{val}(\gamma) : \mathrm{Tr}(\gamma) = -1 \ \ \lambda \in L\}$. Choose a $\lambda \in L$ such that $\mathrm{val}(\lambda) = \delta$ and choose a uniformizing element ϖ_L of L such that $\mathrm{Tr}(\lambda\varpi_L) = 0$. Let $L^\circ = \{d \in L : d + \bar{d} = 0\}$ denote the traceless elements of L.

For general $SU_3^{L/K}(h)$, the set of affine roots is realized as

$$\Phi_{\mathrm{af}} = \left\{\pm a + \mathrm{val}(L) + \frac{\delta}{2}\right\} \cup \{\pm 2a + \mathrm{val}(L^\circ)\}.$$

We are working in the case where L/K is unramified. It follows that $\mathrm{val}(L^\circ) = \mathrm{val}(L) = \mathbb{Z}$ and $\delta = 0$. Thus, the affine roots of G are

$$\Phi_{\mathrm{af}} = \{\pm a + \mathbb{Z}\} \cup \{\pm 2a + \mathbb{Z}\}.$$

Let γ be an integer. We have affine root subgroups

$$U_{\pm a+\gamma} := \{u_\pm(c,d) : \mathrm{val}(d) \geq 2\gamma\},$$

$$U_{\pm 2a+\gamma} := \{u_\pm(0,d) : \mathrm{val}(d) \geq \gamma\}.$$

In the unramified setting, $\mathrm{val}(d) \geq 2\gamma$ implies $\mathrm{val}(c) \geq \gamma$.

3.3.2 Filtration of the torus

We define a filtration on the maximal bounded subgroup T_0 of T. We set

$$T_0 = \left\{m(t) : t \in \mathcal{O}_L^\times\right\}.$$

For any $r \in \mathbb{R}$, let

$$T_r = \{m(t) : \mathrm{val}(t-1) \geq r\}.$$

One finds that

$$T_0/T_1 \cong \left\{ \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} : t \in \mathbb{F}_{q^2}^{\times} \right\} \cong \mathbb{F}_{q^2}^{\times},$$

and for all $r > 0$,

$$T_r/T_{r+1} \cong \left\{ \begin{pmatrix} a & & \\ & \bar{a} - a & \\ & & -\bar{a} \end{pmatrix} : a \in (\mathbb{F}_{q^2}, +) \right\} \cong (\mathbb{F}_{q^2}, +).$$

3.3.3 Special points and their filtration subgroups

Identifying $\mathcal{A}(\mathbf{S}, K)$ with $X_{\bullet}(\mathbf{S}) \otimes \mathbb{R}$, the root a can be evaluated at the points along the apartment. One can choose adjacent special points x and y such that

$$a(x) = \frac{\delta}{2} = 0, \quad a(y) = \frac{1}{2}(\mathrm{val}(\varpi_L + \delta)) = \frac{1}{2}.$$

In this setting, x is a hyperspecial point and y is a special point of $\mathcal{B}(G)$.

Set z be the barycenter of the alcove adjoining x and y. That is, we take z the unique satisfying $a(z) = [-2a+1](z) = \frac{1}{3}$. One finds,

$$G_x := G_{x,0} = \langle T_0,\ U_{a+0},\ U_{2a+0},\ U_{-a+0},\ U_{-2a+0} \rangle \cap G,$$

$$G_{x,0+} = G_{x,1} = \langle T_1,\ U_{a+1},\ U_{2a+1},\ U_{-a+1},\ U_{-2a+1} \rangle \cap G.$$

$$G_y := G_{y,0} = \langle T_0,\ U_{a+0},\ U_{2a-1},\ U_{-a+1},\ U_{-2a+1} \rangle \cap G,$$

$$G_{y,0+} = G_{y,\frac{1}{2}} = \langle T_1,\ U_{a+0},\ U_{-2a+0},\ U_{-a+1},\ U_{-2a+2} \rangle \cap G,$$

$$G_{y,\frac{1}{2}+} = G_{y,1} = \langle T_1,\ U_{a+1},\ U_{-2a+0},\ U_{-a+2},\ U_{-2a+2} \rangle \cap G.$$

$$G_z := G_{z,0} = G_x \cap G_y = \langle T_0\ \ U_{a+0},\ U_{2a+0},\ U_{-a+1},\ U_{-2a+1} \rangle \cap G,$$

$$G_{z,0+} = G_{z,\frac{1}{3}} = \langle T_1\ \ U_{a+0},\ U_{2a+0},\ U_{-a+1},\ U_{-2a+1} \rangle \cap G,$$

$$G_{z,\frac{1}{3}+} = G_{z,\frac{2}{3}} = \langle T_1 \;\; U_{a+1}, \;\; U_{2a+0}, \;\; U_{-a+1}, \;\; U_{-2a+2} \rangle \cap G.$$

$$G_{z,\frac{2}{3}+} = G_{z,1} = \langle T_1 \;\; U_{a+1}, \;\; U_{-2a+1}, \;\; U_{-a+2}, \;\; U_{-2a+2} \rangle \cap G.$$

As for some quotients, one finds that

$$G_{x,0}/G_{x,0+} \cong SU_3^{\mathbb{F}_{q^2}/\mathbb{F}_q}(h),$$

$$G_{y,0}/G_{y,0+} \cong U(1,1)^{\mathbb{F}_{q^2}/\mathbb{F}_q}(h),$$

$$\bar{M}_z = G_{z,0}/G_{z,0+} \cong T_0/T_1 \cong \left\{ \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} \;:\; t \in \mathbb{F}_{q^2}^{\times} \right\} \cong \mathbb{F}_{q^2}^{\times},$$

$$V_z = G_{z,\frac{1}{3}}/G_{z,\frac{1}{3}+} \cong \left\{ \begin{pmatrix} & -\bar{u} & \\ & & u \\ v & & \end{pmatrix} \;:\; u \in \mathbb{F}_{q^2}, v \in \mathbb{F}_{q^2}^{\circ} \right\} \cong \mathbb{F}_{q^2} \oplus \mathbb{F}_q,$$

where $\mathbb{F}_{q^2}^{\circ} = \left\{ d \in \mathbb{F}_{q^2} \;:\; d + \bar{d} = 0 \right\}$. Here $\bar{d} = d^q$ denotes a Frobenius generator of $Gal(\mathbb{F}_{q^2}/\mathbb{F}_q)$.

3.3.4 Unramified $SU_3^{L/K}(h)$ – Moy-Prasad-DeBacker facets

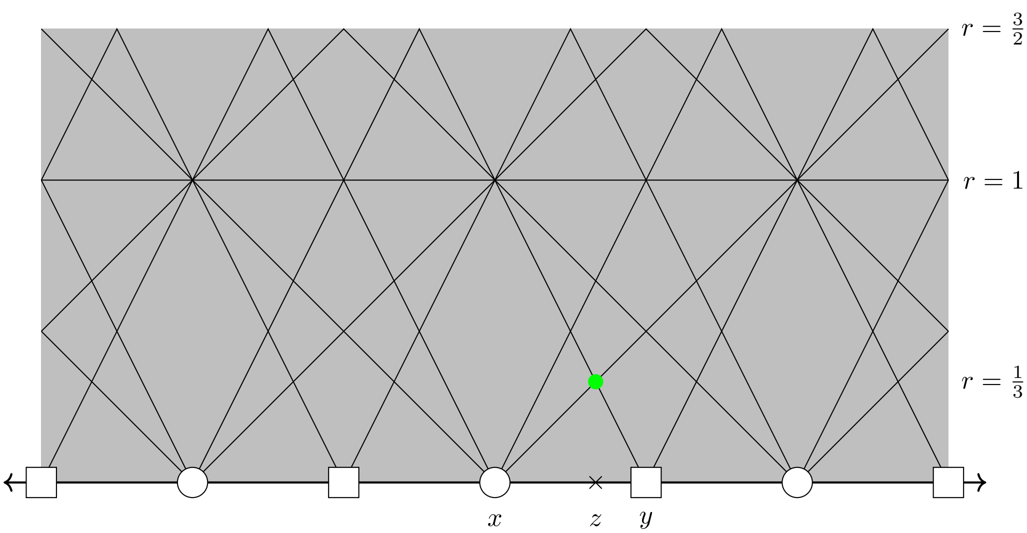


Figure 3.3: MPD facet diagram of unramified $G = SU_3^{L/K}(h)$

One sees that the first non-trivial jump of G_z occurs at depth $r = \frac{1}{3}$, from a jump in valuation of the $\mathbf{U}_a(K)$ and $\mathbf{U}_{-2a}(K)$ root subgroups. In Chapter 5, we will study $\bar{M}_z = G_{z,0}/G_{z,\frac{1}{3}}$ acting on $V_z = G_{z,\frac{1}{3}}/G_{z,\frac{1}{3}+}$ to construct depth-$\frac{1}{3}$ representations of G.

3.4 Ramified $SU_3^{E/K}(h)$ – Moy-Prasad filtrations and quotients

Let E/K be a ramified quadratic extension. Denote by $\mathcal{O}_E$ the ring of integers of E, and denote by ϖ_E a uniformizing element. Let σ be generator of $Gal(E/K)$. If $e \in E$, let $\bar{e}$ denote the image of k under σ. Let $h : E^3 \times E^3 \to K$ be a Hermitian form with Witt Index 1. Without loss of generality, we take

$$h((x_1, x_2, x_3), (y_1, y_2, y_3)) = \bar{x}_1 y_3 + \bar{x}_2 y_2 + \bar{x}_3 y_1.$$

Then h corresponds to the Hermitian matrix

$$H = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Let $G = \mathbf{G}(K) = SU_3^{E/K}(h)$. G is then realized as

$$G = \left\{ g \in GL_n(E) : {}^t\bar{g} H g = H, \text{ and } \det(g) = 1 \right\}.$$

We fix a maximal K-split torus $\mathbf{S}$ of $\mathbf{G}$ such that

$$S := \mathbf{S}(K) = \left\{ \begin{pmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{pmatrix} : s \in K^\times \right\}.$$

$\mathbf{G}$ is quasisplit over K, thus $\mathbf{T} = \mathbf{Z_G}(\mathbf{S})$ a maximal torus of $\mathbf{G}$. Let

$$T := \mathbf{T}(K) = \left\{ m(t) = \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} : t \in E^\times \right\}.$$

With respect to $\mathbf{S}$, $\mathbf{G}$ has root system $\Phi = \{\pm a, \pm 2a\}$, with root subgroups

$$\mathbf{U}_a(K) = \left\{ u_+(c,d) = \begin{pmatrix} 1 & -\bar{c} & d \\ & 1 & c \\ & & 1 \end{pmatrix} : c, d \in E, \;\; \mathrm{N}(c) + \mathrm{Tr}(d) = 0 \right\},$$

$$\mathbf{U}_{2a}(K) = \{ u_+(0,d) : \mathrm{Tr}(d) = 0 \},$$

$$\mathbf{U}_{-a}(K) = \left\{ u_-(c,d) = \begin{pmatrix} 1 & & \\ c & 1 & \\ d & -\bar{c} & 1 \end{pmatrix} : c, d \in E, \;\; \mathrm{N}(c) + \mathrm{Tr}(d) = 0 \right\},$$

$$\mathbf{U}_{-2a}(K) = \{ u_-(0,d) : \mathrm{Tr}(d) = 0 \}.$$

3.4.1 Affine roots and root subgroups filtrations

Following [25, Sections 1.16, 3.11], we define $\delta = \sup\{\mathrm{val}(\gamma) : \mathrm{Tr}(\gamma) = -1, \;\; \lambda \in E\}$. Choose a $\lambda \in E$ such that $\mathrm{val}(\lambda) = \delta$ and choose a uniformizing element ϖ_E of E such that $\mathrm{Tr}(\lambda \varpi_E) = 0$. Let $E^\circ = \{d \in E : d + \bar{d} = 0\}$ denote the traceless elements of E.

For general $SU_3^{E/K}(h)$, the set of affine roots is realized as

$$\Phi_{\mathrm{af}} = \left\{ \pm a + \mathrm{val}(E) + \frac{\delta}{2} \right\} \cup \{ \pm 2a + \mathrm{val}(E^\circ) \}.$$

In the ramified setting, when the residue characteristic $p \neq 2$, we have $\delta = 0$. When $p = 2$, $\delta < 0$. Furthermore, when E/K is ramified, $\delta \notin \mathrm{val}(E^\circ)$ [25, 1.15 (6)].

Thus, when $p \neq 2$, $\delta = 0$, making $0 \notin \mathrm{val}(E^\circ)$. It then follows that $\mathrm{val}(E^\circ) = \frac{1}{2} + \mathbb{Z}$. This gives the set of affine roots when $p \neq 2$,

$$\Phi_{\mathrm{af}} = \left\{ \pm a + \frac{1}{2}\mathbb{Z} \right\} \cup \left\{ \pm 2a + \frac{1}{2} + \mathbb{Z} \right\}.$$

Let $\gamma \in \mathbb{Z}$. One has

$$U_{\pm a+\frac{1}{2}\gamma} = \{u_{\pm}(c,d) : \mathrm{val}(d) \geq \gamma\},$$

$$U_{\pm 2a+\frac{1}{2}+\gamma} = \left\{u_{\pm}(0,d) : \mathrm{val}(d) \geq \frac{1}{2} + \gamma\right\}.$$

In this setting, $\mathrm{val}(d) \geq \gamma$ implies $\mathrm{val}(c) \geq \frac{1}{2}\gamma$.

We now cover the case when $p = 2$, where $\delta < 0$. As $\delta \notin \mathrm{val}(E^{\circ})$, we can write $\mathrm{val}(E^{\circ}) = \frac{1}{2} + \delta + \mathbb{Z}$. Thus,

$$\Phi_{\mathrm{af}} = \left\{\pm + \frac{\delta}{2} + \frac{1}{2}\mathbb{Z}\right\} \cup \left\{\pm 2a + \frac{1}{2} + \delta + \mathbb{Z}\right\}.$$

Let $\gamma \in \mathbb{Z}$. Then

$$U_{\pm a+\frac{1}{2}(\delta+\gamma)} = \{u_{\pm}(c,d) : \mathrm{val}(d) \geq \delta + \gamma\}.$$

But what does this say about the valuation of c?

Consider $U_{\pm a-\frac{\delta}{2}} := \left\{u_{\pm}(c,d) : \bar{c}c + d + \bar{d} = 0,\ \ \mathrm{val}(d) \geq -\delta\right\}$. Let d be taken such that $\mathrm{val}(d) = -\gamma$. Such a d has the form

$$d = \varpi_E^{-4\delta} d_0,$$

for some uniformizer ϖ_E of E and some d_0 satisfying $\mathrm{val}(d_0) = \delta$, $\mathrm{Tr}(d_0) = -1$. We now see that the requirement of $\bar{c}c + d + \bar{d} = 0$, using the d from above, gives us

$$\bar{c}c + \varpi_E^{-4\delta}(-1) = 0.$$

Equivalently,

$$\bar{c}c = \varpi_E^{-4\delta},$$

showing that $\mathrm{val}(c) \geq -\delta$. Similarly, one sees that $U_{\pm a+\frac{\delta}{2}}$ gives that requirement that $\mathrm{val}(d) \geq \delta$, $\mathrm{val}(c) \geq 0$. One then arrives at,

$$U_{\pm a+\frac{1}{2}(\delta+\gamma)} = \{u_{\pm}(c,d) : \mathrm{val}(d) \geq \delta + \gamma\}$$

implies that $\mathrm{val}(c) \geq \frac{\gamma}{2}$.

As for the $\pm 2a$ subgroups, we have, like before,

$$U_{\pm 2a+\frac{1}{2}+\delta+\gamma} = \left\{ u_{\pm}(0,d) : \mathrm{val}(d) \geq \frac{1}{2} + \delta + \gamma \right\}.$$

3.4.2 Filtration of the torus

We define a filtration on the maximal bounded subgroup T_0 of T. We set

$$T_0 = \left\{ m(t) : t \in \mathcal{O}_E^{\times} \right\}.$$

For any nonnegative $r \in \mathbb{R}$, let

$$T_r = \{ m(t) : \mathrm{val}(t-1) \geq r \}.$$

One finds that

$$T_0/T_{\frac{1}{2}} \cong \left\{ \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} : t \in \mathbb{F}_q^{\times} \right\} = \left\{ \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} : t \in \mathbb{F}_q^{\times} \right\} \cong \mathbb{F}_q^{\times},$$

and for all $r > 0$,

$$T_r/T_{r+\frac{1}{2}} \cong \left\{ \begin{pmatrix} a & & \\ & -2a & \\ & & a \end{pmatrix} : a \in (\mathbb{F}_q, +) \right\} \cong (\mathbb{F}_q, +).$$

3.4.3 Special points and their filtration subgroups

Identifying $\mathcal{A}(\mathbf{S}, K)$ with $X_{\bullet}(\mathbf{S}) \otimes \mathbb{R}$, the root a can be evaluated at the points along the apartment. One can choose adjacent special points x and y such that

$$a(x) = \frac{\delta}{2}, \quad a(y) = \frac{1}{2}(\mathrm{val}(\varpi_E + \delta)) = \frac{1}{4} + \frac{\delta}{2}.$$

In this setting, x and and y are special points of $\mathcal{B}(G)$, and neither are hyperspecial.

Set z to be the point on the edge adjoining x and y such that $a(z) = \frac{1}{6} + \frac{\delta}{2}$.

When $p \neq 2$, we simply have $a(x) = 0$, $a(y) = \frac{1}{4}$, $a(z) = \frac{1}{6}$. Let's list some Moy-Prasad filtration subgroups for the case $p \neq 2$. Then we will follow with the general case.

$$G_x := G_{x,0} = \langle T_0,\ U_{a+0},\ U_{2a+\frac{1}{2}},\ U_{-a+0},\ U_{-2a+\frac{1}{2}} \rangle,$$

$$G_{x,0+} = G_{x,\frac{1}{2}} = \langle T_{\frac{1}{2}},\ U_{a+\frac{1}{2}},\ U_{2a+\frac{1}{2}},\ U_{-a+\frac{1}{2}},\ U_{-2a+\frac{1}{2}} \rangle,$$

$$G_{x,\frac{1}{2}+} = G_{x,1} = \langle T_1,\ U_{a+1},\ U_{2a+\frac{3}{2}},\ U_{-a+1},\ U_{-2a+\frac{3}{2}} \rangle.$$

$$G_y := G_{y,0} = \langle T_0,\ U_{a-\frac{1}{2}},\ U_{2a-\frac{1}{2}},\ U_{-a+\frac{1}{2}} \rangle,$$

$$G_{y,0+} = G_{y,\frac{1}{4}} = \langle T_{\frac{1}{2}},\ U_{a-\frac{1}{2}},\ U_{2a+\frac{1}{2}},\ U_{-a+\frac{1}{2}},\ U_{-2a+\frac{3}{2}} \rangle,$$

$$G_{y,\frac{1}{4}+} = G_{y,\frac{1}{2}} = \langle T_{\frac{1}{2}},\ U_{a+0},\ U_{2a+\frac{1}{2}},\ U_{-a+1},\ U_{-2a+\frac{3}{2}} \rangle,$$

$$G_{y,\frac{1}{2}+} = G_{y,\frac{3}{4}} = \langle T_1,\ U_{a+0},\ U_{2a+\frac{1}{2}},\ U_{-a+1},\ U_{-2a+\frac{3}{2}} \rangle,$$

$$G_{y,\frac{3}{4}+} = G_{y,1} = \langle T_1,\ U_{a+\frac{1}{2}},\ U_{2a+\frac{1}{2}},\ U_{-a+\frac{3}{2}},\ U_{-2a+\frac{3}{2}} \rangle.$$

$$G_z := G_{z,0} = G_x \cap G_y = \langle T_0,\ U_{a+0},\ U_{2a+\frac{1}{2}},\ U_{-a+\frac{1}{2}},\ U_{-2a+\frac{1}{2}} \rangle,$$

$$G_{z,0+} = G_{z,\frac{1}{6}} = \langle T_{\frac{1}{2}},\ U_{a+0},\ U_{2a+\frac{1}{2}},\ U_{-a+\frac{1}{2}},\ U_{-2a+\frac{1}{2}} \rangle,$$

$$G_{z,\frac{1}{6}+} = G_{z,\frac{1}{3}} = \langle T_{\frac{1}{2}},\ U_{a+\frac{1}{2}},\ U_{2a+\frac{1}{2}},\ U_{-a+\frac{1}{2}},\ U_{-2a+\frac{3}{2}} \rangle, \text{ etc.}$$

Let's write out a few of the filtration subgroups in the general case, and then write some quotients. Note that the quotients in the $p = 2$ and $p \neq 2$ cases are compatible.

Some Moy-Prasad filtration subgroups for $p = 2$ case:

$$G_x := G_{x,0} = \langle T_0,\ U_{a-\frac{\delta}{2}},\ U_{2a-\delta+\frac{1}{2}},\ U_{-a+\frac{\delta}{2}},\ U_{-2a+\delta+\frac{1}{2}} \rangle,$$

$$G_{x,0+} = G_{x,\frac{1}{2}} = \langle T_{\frac{1}{2}},\ U_{a-\frac{\delta}{2}+\frac{1}{2}},\ U_{2a+\frac{1}{2}-\delta},\ U_{-a+\frac{\delta}{2}+\frac{1}{2}},\ U_{-2a+\frac{1}{2}+\delta} \rangle.$$

$$G_y := G_{y,0} = \langle T_0, U_{a-\frac{\delta}{2}},\ U_{2a-\frac{\delta}{2}-\frac{1}{2}},\ U_{-a+\frac{\delta}{2}+\frac{1}{2}},\ U_{-2a+\delta+\frac{1}{2}} \rangle,$$

$$G_{y,0+} = G_{y,\frac{1}{4}} = \langle T_{\frac{1}{2}}, U_{a-\frac{\delta}{2}},\ U_{2a-\frac{\delta}{2}+\frac{1}{2}},\ U_{-a+\frac{\delta}{2}+\frac{1}{2}},\ U_{-2a+\delta+\frac{3}{2}}\rangle.$$

$$G_z := G_{z,0} = \langle T_0,\ U_{a-\frac{\delta}{2}},\ U_{2a-\delta+\frac{1}{2}},\ U_{-a+\frac{\delta}{2}+\frac{1}{2}},\ U_{-2a+\delta+\frac{1}{2}}\rangle,$$

$$G_{z,0+} = G_{z,\frac{1}{6}} = \langle T_{\frac{1}{2}},\ U_{a-\frac{\delta}{2}},\ U_{2a+\frac{1}{2}-\delta},\ U_{-a+\frac{\delta}{2}+\frac{1}{2}},\ U_{-2a+\delta+\frac{1}{2}}\rangle,$$

$$G_{z,\frac{1}{6}+} = G_{z,\frac{1}{3}} = \langle T_{\frac{1}{2}},\ U_{a-\frac{\delta}{2}+\frac{1}{2}},\ U_{2a+\frac{1}{2}-\delta},\ U_{-a+\frac{\delta}{2}+\frac{1}{2}},\ U_{-2a+\delta+\frac{3}{2}}\rangle, \text{ etc.}$$

One finds $G_{x,0}/G_{x,0+} \cong SO_3(\mathbb{F}_q)$, $G_{y,0}/G_{y,0+} \cong Sp_2(\mathbb{F}_q)$. Additionally,

$$\bar{M}_z := G_{z,0}/G_{z,0+} \cong T_0/T_1 \cong GL_1(\mathbb{F}_q),$$

$$V_z = G_{z,\frac{1}{6}}/G_{z,\frac{1}{6}+} \cong \left\{ \begin{pmatrix} & -u & \\ & & u \\ v & & \end{pmatrix} \, : \, u, v \in \mathbb{F}_q \right\} \cong \mathbb{F}_q \oplus \mathbb{F}_q.$$

3.4.4 Ramified $SU_3^{E/K}(h)$, Moy-Prasad-DeBacker facets

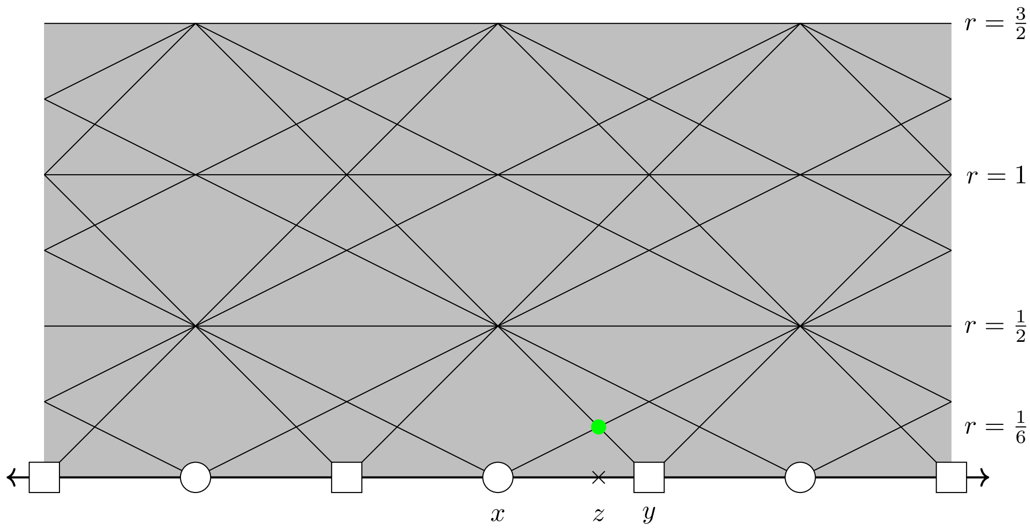


Figure 3.4: MPD facet diagram of ramified $G = SU_3^{E/K}(h)$

One sees that that the jumps are compatible with our data from the previous section. Indeed, in the following chapters, we'll see that G has depth-$\frac{1}{6}$ supercuspidal representations.

3.5 Notes on other groups

All of the remaining groups become quasisplit over an unramified quadratic extension. For general reductive groups $G = \mathbf{G}(K)$, $\mathbf{G}(K^{\mathrm{nr}})$ is quasisplit, where K^{nr} is the maximal unramified extension of K. For example, $G = \mathbf{G}(K) = SU_2^{D/K}(h')$ has $\mathbf{G}(L) \cong Sp_4(L)$ a split group, $SU_3^{D/K}(h') \cong SO_6^{\mu}$, a quasisplit group. Refer to Oi's work [14] for details on the simple supercuspidal representations of these quasisplit groups. See Chapter 1 for details on the split forms of these groups.

Bruhat-Tits [2,3] (and Rousseau [19]) prove that when L/K is unramified (resp. tamely ramified), $\mathcal{B}(\mathbf{G}(L))^{\Gamma} = \mathcal{B}(\mathbf{G}(K))$. Furthermore, it follows from later work that the Moy-Prasad filtration groups match up as well, giving $(\mathbf{G}(L)_{x,r})^{\Gamma} = (\mathbf{G}(K)_{x,r})$ for $x \in \mathcal{B}(\mathbf{G}(K))$. One can then obtain the depths of jumps of root subgroups and the centralizer of torus components.

Chapter 4

The Induction Theorem at the Lowest Positive Depth

Let K be a nonarchimedean local field, let $\mathbf{G}$ be a reductive group defined over K, and let $G = \mathbf{G}(K)$ a group of relative rank 1. Let $z \in \mathcal{A}(\mathbf{S}, K)$ be the unique point along an edge in the standard apartment of $\mathcal{B}(G) = X$ such that z is not a vertex, and the first positive-depth jump of G_z is a jump in valuation of two distinct affine root subgroups. Define r to be the depth of this jump.

The goal of this chapter is to prove the following theorem:

Theorem 38. *Let (π, V) be an irreducible representation of G of depth r. Then (π, V) is supercuspidal, and it is compactly induced from an irreducible representation (σ, W) of the Iwahori subgroup $G_z = G_{z,0}$, such that σ factors through $G_{z,0}/G_{z,r+}$. (σ, W) is obtained by restricting (domain and range) of π: Namely, $W := V^{G_{x,r+}}$ and $\sigma(g)w = \pi(g)w$ for all $g \in G_z$ and $w \in W$. Moreover, restricting σ to $V_z = G_{z,r}/G_{z,r+}$, one finds only affine-generic characters.*

In particular, when $\mathbf{G}$ satisfies the assumptions of Reeder-Yu [18]*, these are precisely the epipelagic representations constructed there.*

In Chapter 2, we have established that all irreducible representations of critical depth of the K-rational points G of a group $\mathbf{G}$ of relative rank 1 arise via compact induction from the stabilizer of a vertex or edge of $\mathcal{B}(G) = X$. Furthermore, we will use the assumption that the minimal-positive depths r we are considering are critical depths.

Remark 39. *This assumption can be verified using unramified descent.*

We will now provide details on how to verify this assumption. Note that $\mathbf{G}$ is quasisplit over L, a finite unramified extension of K. Let z be the barycenter of an alcove $\mathcal{C} \subset \mathcal{B}(\mathbf{G}(K^{\mathrm{nr}}))$ invariant under $\Gamma = Gal(L/K)$. Let r be the depth of the first nontrivial jump of z – this depth is a critical jump. For each of these remaining cases where $\mathbf{G}$ non-quasisplit over K, one uses unramified descent identifying

$$\mathcal{B}(\mathbf{G}(L))^{\Gamma} = \mathcal{B}(G),$$

and the filtration subgroups

$$(\mathbf{G}(L)_{x,s})^{\Gamma} = G_{x,s},$$

for all $s > 0$ and $x \in \mathcal{B}(G)$.

One will find that in each case the critical jump r of $z \in \mathcal{B}(\mathbf{G}(L)$ is a critical jump of $z \in \mathcal{B}(G)$. That is, $(\mathbf{G}(L)_{z,r+})^{\Gamma} = G_{z,r+}$, with $G_{z,r} \neq G_{z,r+}$, and there exists an open neighborhood U of z such that when $y \in U - \{z\}$, $G_{y,r+} = G_{y,r}$. Unramified descent is a laborious task, but going through this process with these remaining groups will show that the first positive jump at the barycenter of an alcove in $\mathcal{B}(G) = X$ occurs before the first positive jump in the centralizer of the torus component. Refer to Oi [14] for details on the filtration subgroups and depths of jumps at the barycenter of an alcove for the various $\mathbf{G}(L)$.

This assumption gives us that all irreducible minimal-positive-depth representations of groups of relative rank one arise via compact induction by the stabilizer of edge of X by Remark 37. We are concerned with supercuspidal representations of minimal positive depth. Such representations have been classified by Gross, Reeder, and Yu [9, 18], under the names of *simple supercuspidals* and *epipelagic representations*. To establish Theorem 38, we shall describe the construction of these representations, and show that no other depth-r supercuspidal representations exist outside of their construction.

In Chapter 5, we will include specific data on the construction of such representations for some of the groups of relative rank 1.

4.1 Simple supercuspidals and epipelagic representations

Let G be the K-rational points of a reductive group $\mathbf{G}$. Let $x \in \mathcal{B}(G)$. Denote the filtration subgroups of G_x by

$$G_x := G_{x,0} \rhd G_{x,r_1} \rhd G_{x,r_2} \rhd G_{x,r_3} \dots$$

Let $\bar{M}_x = G_{x,0}/G_{x,r_1}$, $V_x = G_{x,r_1}/G_{x,r_2}$. V_x is naturally an $\mathbb{F}_q$ vector space. Conjugation gives an action of $\bar{M}_x$ on V_x; this is an algebraic representation.

Remark 40. [28, Section 9.1] *For non-tame groups, quotients $G_{x,r}/G_{x,r+}$ may not always be $\mathbb{F}_q$ vector spaces. However, in our case, for z the barycenter alcove, $V_z = G_{z,r}/G_{z,r+}$ will be.*

Here's an argument based on Yu [27]. Take r to be positive. By étale descent [27, 9.1], there's a unique smooth group scheme $\mathbf{G}$ over $\mathcal{O}$ such that $\mathbf{G}(\mathcal{O}') = \mathbf{G}(K')_{x,r}$ for every unramified extension K'/K (with ring of integers $\mathcal{O}'/\mathcal{O}$). The special fiber of $\mathbf{G}$ is a group scheme $\bar{\mathbf{G}}$ over the residue field k of K, and $\bar{\mathbf{G}}(k)$ can be identified with $G_{x,r}/G_{x,r+1}$ (similarly for finite extensions of k, using finite unramified extensions of K). This group scheme $\bar{\mathbf{G}}$ decomposes as a direct product of schemes over k: multiplication gives an isomorphism from $\bar{\mathbf{T}} \times \bar{\mathbf{U}}^+ \times \bar{\mathbf{U}}^-$ to $\bar{\mathbf{G}}$, as in [27, 8.3 (iii)].

There exists an algebraic group $\mathbf{V}_z$ over k such that $\mathbf{V}_z(k) = G_{z,r}/G_{z,r+}$, and similarly for finite extensions of k. This $\mathbf{V}_z$ is naturally a quotient of the special fiber $\bar{\mathbf{G}}$. In particular, if $\mathbf{T}$ does not jump at depth r, then $\mathbf{V}_z$ is a quotient of $\bar{\mathbf{U}}^+ \times \bar{\mathbf{U}}^-$, as groups over k. As a quotient of a connected unipotent group, $\mathbf{V}_z$ is a connected unipotent group; it's abelian too, and our residue fields are finite – hence perfect – so $\mathbf{V}$ is isomorphic to a product of additive groups. In other words $\mathbf{V}_z$ is a vector group, and $V_z = \mathbf{V}_z(k) = G_{z,r}/G_{z,r+}$ is a vector space over k.

Let $\check{V}_x := \mathrm{Hom}_{\mathbb{F}_q}(V_x, \mathbb{F}_q)$. A linear functional $\lambda \in \check{V}_x$ is called *stable* if its orbit is Zariski closed and its stabilizer in $\bar{M}_x$ is finite. Fix a nontrivial character $\chi : (\mathbb{F}_q, +) \to \mathbb{C}^\times$.

Let $\lambda \in \check{V}_x$ be a stable functional. The composition

$$\chi_\lambda := \chi \circ \lambda : V_x \to \mathbb{C}^\times$$

is a character of G_{x,r_1} which is trivial on G_{x,r_2}. Reeder and Yu prove [18, Prop 2.4] that

$$\pi_x(\lambda) := \text{c-Ind}_{G_{x,r_1}}^{G}(\chi_\lambda)$$

is a finite direct sum of irreducible supercuspidal representations. These representations are supercuspidal of depth-r_1. That is, these representations are of minimal positive depth, and thus, are aptly named *epipelagic* supercuspidal representations.

4.1.1 Simple supercuspidals

An *alcove* is a facet of maximal dimension in the Bruhat-Tits building X. Thus in a rank one building, an alcove will be an edge. For every affine function $\psi \in \Phi_{\text{af}}$ denote by A_ψ the set $\psi^{-1}([0,\infty))$, with boundary ∂A_ψ given by $\alpha^{-1}(0)$. Let $\mathcal{C}$ be an alcove in X and let $L_0, \dots, L_\ell$ be the walls bounding $\mathcal{C}$. For $i \in \{0, \dots, \ell\}$, let ψ_i be the unique affine root such that $L_i = \partial A_{\psi_i}$, and $\frac{1}{2}\psi_i \notin \Phi_{\text{af}}$. The set $\Pi_{\mathcal{C}} = \{\psi_i : i = 0, \dots, \ell\}$ is called the basis of Φ_{af} associated to $\mathcal{C}$. There exists a unique point $z \in \mathcal{C}$ with value $m \in \mathbb{R}_+$ such that for all α_i in the basis Φ_{af} associated to $\mathcal{C}$, $\psi_i(z) = m$. This unique point $z \in C$ is called the *barycenter* of C. By definition of the Moy-Prasad filtration subgroups $G_{z,r}$, we have $m = r_1$.

When G is semisimple and split and $z \in \mathcal{B}(G)$ is taken to be the barycenter of an alcove, the constructed representations from the previous section yield the simple supercuspidal representations of G. Gross and Reeder [9] use *affine-generic characters* in their construction. A complex character χ of G_{z,r_1} is called affine-generic if it is trivial on G_{z,r_2} and the restriction of χ to each eigenspace for $\bar{M}_z$ in $G_{z,r_1}/G_{z,r_2}$ is non-trivial. Affine-generic characters are always stable – they are of the form $\chi_\lambda := \chi \circ \lambda : V_x \to \mathbb{C}^\times$ for λ a stable functional, $\chi : (\mathbb{F}_q, +) \to \mathbb{C}^\times$ a nontrivial character.

4.1.2 Generalized simple supercuspidals

We follow Reeder-Yu [18, Section 2.6]. Please refer to this text, as we leave out details.

Let $\mathbf{G}$ be a reductive group, and let $G = \mathbf{G}(K)$. $\mathbf{G}$ is quasisplit over K^{nr}. Let $G' = \mathbf{G}(K^{\text{nr}})$. Assume that $\mathbf{G}$ splits over a tamely ramified extension E_0/K^{nr} of degree e; let $\Gamma_0 = Gal(E_0/K^{\text{nr}})$. Let $\mathbf{S}$ be a maximal K-split torus, and let $\mathbf{T}$ be a maximal K^{nr}-split torus containing $\mathbf{S}$. Since $\mathbf{G}$ is quasisplit over K^{nr}, a generator γ of Γ_0 acts on the based root datum $(X, \Delta, \check{R}, \check{\Delta})$ of $\mathbf{G}$ via an automorphism $\vartheta \in \mathrm{Aut}(R, \Delta)$ of order e.

Let $\mathcal{A}(\mathbf{T}, K^{\mathrm{nr}})$ be the apartment in $\mathcal{B}(G')$ associated to $\mathbf{T}$. Let F be a generator of $Gal(K^{\mathrm{nr}}/K)$. Let $\mathcal{C}$ be an alcove of $\mathcal{A}(\mathbf{T}, K^{\mathrm{nr}})$; we can choose $\mathcal{C}$ such that $F(\mathcal{C}) = \mathcal{C}$. Recall that in an alcove of G', no affine K^{nr}-root vanishes. Each hyperplane bounding $\mathcal{C}$ is the zero locus of a unique affine K^{nr}-root which is positive on $\mathcal{C}$. Let $\Pi_{\mathcal{C}} = \{\psi_0, \psi_1, \ldots, \psi_{\ell_\vartheta}\}$ be the set of these affine K^{nr}-roots, where $\ell_\vartheta = \dim(\mathcal{A}(\mathbf{T}, K^{\mathrm{nr}}))$.

The barycenter of $\mathcal{C}$ is the unique point $z \in \mathcal{C}$ for which all $\psi_i \in \Pi_{\mathcal{C}}$ take the same value; this common value is $1/h_\vartheta$, where h_ϑ the twisted Coxeter number of (R, ϑ) [17]. By the uniqueness of the barycenter of an alcove, it follows that $F(z) = z$, so in fact $z \in \mathcal{A}(\mathbf{S}, K)$. It then follows that the minimal positive depth for G_z is is $1/h_\vartheta$.

Let z be the barycenter of an alcove in $\mathcal{B}(G)$. Then there exists a stable linear functional $\lambda \in \check{V}_z$. Fix a nontrivial character $\chi : (\mathbb{F}_q, +) \to \mathbb{C}^\times$. The composition

$$\chi_\lambda := \chi \circ \lambda : V_z \to \mathbb{C}^\times$$

is a character of $G_{z,r}/G_{z,r+}$.

Theorem 41. [18, Prop. 2.4]

$$\pi_z(\lambda) := \text{c-Ind}_{G_{z,r}}^{G}(\chi_\lambda)$$

is a finite direct sum of irreducible supercuspidal representations. These representations are supercuspidal of depth-r.

There is no tameness condition required in Theorem 41.

Remark 42. [18, Lemma 2.1] *Let H be an open subgroup of G containing the center Z of G, with H/Z compact, and let τ be an irreducible smooth representation of H. Then the representation* $\text{c-Ind}_H^G(\tau)$ *is irreducible for G if and only if $I(G, H, \tau) = H$. Here,*

$$I(G, H, \tau) := \{g \in G : \tau \cong {}^g\tau \text{ on } H \cap {}^gH\}.$$

Let Z denote the center of G. Note that $Z \subseteq G_{z,0}$. Let $H := Z \cdot G_{z,r}$. Let $m =$

$[Z \cdot G_{z,r} : G_{z,r}]$. As before, let χ_λ be an affine generic representation of $G_{z,r}/G_{z,r+}$. By Clifford theory, $\mathrm{Ind}_{G_{z,r}}^{H}(\chi_\lambda)$ splits into a sum of m non-isomorphic irreducible representations, each distinguished by their central characters. Let $\chi_{\lambda,i}$ for $i \in \{1, 2, \ldots, m\}$ denote these irreducible H-representations. Let $\tau = \chi_{\lambda_i}$ for some $i \in \{1, 2, \ldots, m\}$. Then Reeder and Yu [18, Lemma 2.2, Prop. 2.4] show that $I(G, H, \tau) = H$, thus showing that c-$\mathrm{Ind}_H^G(\tau)$ is irreducible. Furthermore, for each i,

$$\pi_z(\lambda_i) := \text{c-Ind}_H^G(\chi_{\lambda,i})$$

is an irreducible supercuspidal representation of depth r.

4.1.3 Missing cases

Recall that in this thesis, we are only concerned about the representations of groups of relative rank one. In this rank one setting, the only point at which stable functionals can occur is at the barycenter of an alcove $z \in \mathcal{C}$ in $\mathcal{B}(G)$. However, for buildings of higher rank, stable functionals can exist outside of the barycenter of an alcove (see [18]), such as along codimension one facets of the building. Therefore, the epipelagic representations we will construct are these generalized simple supercupsidal representations from [18, Section 2.6].

Reeder and Yu's construction covers the construction of epipelagic representations at the barycenter of the alcove $z \in \mathcal{C}$ when G is a group that splits over a tamely ramified extension. As relative rank one groups have Weyl groups of order 2, non-tame epipelagic representations in this setting only occur when working with groups over K of residue characteristic $p = 2$, which are not split over K^{nr}.

This leaves the following groups outside of the construction of Reeder and Yu.

1) $G = \mathbf{G}(K) = SU_3^{E/K}(h)$, where E/K is a ramified quadratic extension and $p = 2$,

2) $G = \mathbf{G}(K) = SU_3^{D/K}(h)$, where $p = 2$,

3) $G = \mathbf{G}(K) = SU_4^{D/K}(h)$, where $p = 2$,

4) $G = \mathbf{G}(K) = SU_5^{D/K}(h)$, where $p = 2$.

For these $\mathbf{G}$, the Moy-Prasad filtration and structure of $\mathbf{G}(K^{\mathrm{nr}})$ is well-understood.

Thus we can employ the method of unramified descent [3, Section 4; 15], giving us stable vectors at the barycenter of an alcove. Then we can construct epipelagic representations of $G = \mathbf{G}(K)$.

In 5 we describe the the stable vectors for the missing case, 1) $G = SU_3^{E/K}(h)$, $p = 2$. As this group is quasisplit over K, we don't require the treatment of unramified descent to construct its epipelagic representations.

4.2 Exhaustion of depth-r representations

Let $H \subseteq G$ be a compact open subgroup. Let π be a smooth admissible representation of G. Recall the set of H-invariants of π is defined by

$$V^H := \{v \in V : \pi(h)v = v \text{ for all } h \in H\}.$$

Further, recall that the depth of an irreducible supercuspidal representation π of $G = \mathbf{G}(K)$, labeled $\varrho(\pi)$ is defined to be the smallest depth r such that for some point $x \in \mathcal{B}(G)$,

$$V^{G_{x,r+}} \neq \{0\}.$$

We will now prove the theorem stated in beginning of this chapter. Let $G = \mathbf{G}(K)$ be a group of relative rank one.

Theorem 38. *Let (π, V) be an irreducible representation of G of depth r. Then (π, V) is supercuspidal, and it is compactly induced from an irreducible representation (σ, W) of the Iwahori subgroup $G_z = G_{z,0}$, such that σ factors through $G_{z,0}/G_{z,r+}$. (σ, W) is obtained by restricting (domain and range) of π: Namely, $W := V^{G_{x,r+}}$ and $\sigma(g)w = \pi(g)w$ for all $g \in G_z$ and $w \in W$. Moreover, restricting σ to $V_z = G_{z,r}/G_{z,r+}$, one finds only affine-generic characters.*

Proof. Let (π, V) be an irreducible representation of G of depth r. Here r is the minimal positive depth, by our assumption, this is a critical depth. By 36,

$$(\pi, V) \cong \text{c-Ind}_J^G(\sigma, W),$$

where $W := V^{G_{z,r+}}$ and σ is the action of the Iwahori $J = G_z$ via π. An equivalent condition to being supercuspidal is having matrix coefficients that are compactly supported modulo the center Z of G. It follows that if an irreducible representation occurs via compact induction from a compact-open subgroup, then it is supercuspidal.

By construction, (σ, W) factors through the quotient $G_z/G_{z,r+}$. Recall that the jump $G_{z,r}$ to $G_{z,r+}$ is a jump in valuation of two affine root subgroups of different gradients. Say these jumps are U_α and U_β to $U_{\alpha+}$ and $U_{\beta+}$ for affine roots α and β. Let x and y be adjacent vertices in the boundary of the edge containing z. We recall that $G_{x,0} \cap G_{y,0} = G_{z,0}$. And recall $G_{x,0+} \subset G_{z,0+}$ and $G_{y,0+} \subset G_{z,0+}$, and these containments differ in the valuation of one root subgroup. Recall also that

$$G_{x,0+} = G_{x,r} = G_{x,r+},$$

$$G_{y,0+} = G_{y,r} = G_{y,r+},$$

$$G_{z,0+} = G_{z,r}.$$

For π a depth-r irreducible representation, we have $V^{G_{z,r+}} \neq \{0\}$, but $V^{G_{x',r+}} = \{0\}$ for all other points $x' \in \overline{\mathcal{C}} - \{z\}$.Thus one finds that π must be nontrivial on both U_α and U_β. Otherwise, if π is trivial on one of these affine root subgroups then $V^{G_{x,0+}} = \{0\}$ or $V^{G_{y,0+}} = \{0\}$, making π a depth-0 representation. Thus π is a representation that is trivial on $U_{\alpha+}$ and $U_{\beta+}$, but nontrivial on U_α and U_β.

$\square$

Chapter 5

Description of Minimal Positive Depth Irreducible Representations

Let K be a nonarchimedean local field, and let G be the K-points of a reductive group, such that the K-rank of G is one.

Let (π, V) be an irreducible representation of G of minimal-positive-depth r. We showed in Theorem 38 that (π, V) is supercuspidal, and it is compactly induced from an irreducible representation (σ, W) of the Iwahori subgroup $G_z = G_{z,0}$, such that σ factors through $G_{z,0}/G_{z,r+}$. Furthermore, we showed that when restricting σ to $V_z = G_{z,r}/G_{z,r+}$, one finds only affine-generic characters.

On the other hand, let (σ, W) be a representation of $G_z/G_{z,r+}$, nontrivial on $G_{z,r}$ that compactly induces to a depth-r supercuspidal irreducible representation of G. Let W_χ be the χ-isotypic subspace of W. That is,

$$W_\chi = \{w \in W : \sigma(v)w = \chi(v) \cdot w \text{ for all } v \in V\}.$$

Then we have

$$W \cong \bigoplus_{\chi \in \hat{V}_z} W_\chi,$$

by decomposing (σ, W) after restricting to V_z. Mackey theory gives a bijection between the

- irreducible representations of $G_{z,0}/G_{z,r+}$ that are affine-generic on $G_{z,r}$.
- G_z-orbits on $\mathrm{Hom}((Z \cdot G_{z,r})/G_{z,r+}, \mathbb{C}^\times)$ that are affine-generic on $G_{z,r}$.

Minimal-positive-depth irreducible supercsupidals (π, V) arise as c-Ind(σ, W), for some irreducible representation (σ, W) of $G_{z,0}/G_{z,r+}$ that is affine-generic on $G_{z,r}$. Any such irreducible representation W arises as c-$\mathrm{Ind}_{ZG_{z,r}}^{G_z}(\chi)$ for some affine-generic character of $G_{z,r}$ extended to $ZG_{z,r}$. This is because if χ is an affine-generic character of $G_{z,r}$, then its stabilizer in G_z will be $ZG_{z,r}$ – at least for all of the groups in Chapter 5.

So the inducing data consists of:

- A G_z-orbit –the same as a $\bar{M}_z$-orbit – of affine generic characters of $G_{z,r}$.
- A character of Z (in the case when $Z \not\subseteq G_{z,r}$).

In this chapter, using the data from Chapter 3 on the filtrations subgroups of the relative rank one groups $SL_2(K)$, $SL_2(D)$, $SU_3^{L/K}(h)$, and $SU_3^{E/K}(h)$, we provide the inducting data used to construct the depth-r irreducible supercuspidal representations for these groups.

5.1 $SL_2(K)$ simple supercuspidals

Take x, y, and z as in Section 3.1.3. Recall that

$$\bar{M}_z = G_{z,0}/G_{z,0+} = T_0/T_1 \cong \mathbb{F}_q^\times,$$

$$V_z = G_{z,\frac{1}{2}}/G_{z,\frac{1}{2}+} = (U_{a+0}/U_{a+1}) \oplus (U_{-a+1}/U_{-a+2}) \cong \begin{pmatrix} & \mathbb{F}_q \\ \mathbb{F}_q & \end{pmatrix}.$$

5.1.1 Affine-generic characters

Let $\star$ denote the action of conjugation. We have $\bar{M}_z \star V_z$, with

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \star \begin{pmatrix} & u \\ v & \end{pmatrix} = \begin{pmatrix} & t^2u \\ t^{-2}v & \end{pmatrix}$$

for $t \in \mathbb{F}_q^\times$, $u, v \in \mathbb{F}_q$. We use shorthand-notation $t \star (u, v) = (t^2u, t^{-2}v)$.

Let $\chi_{a,b}$ denote the character of $V_z = G_{z,\frac{1}{2}}/G_{z,\frac{1}{2}+}$ such that $\chi_{a,b}((u,v)) = e^{\frac{2\pi i}{p}\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(au+bv)}$. To be affine-generic, we require $\chi_{a,b}|_{U_{a+0}/U_{a+1}} \neq 1$, and $\chi_{a,b}|_{U_{-a+1}/U_{-a+2}} \neq 1$. Thus, we require $a, b \in \mathbb{F}_q^{\times}$, giving us a total of $(q-1)^2$ affine-generic characters. Now, define

$$\psi_{a,b} := \mathrm{Inf}_{V_z}^{G_{z,\frac{1}{2}}}(\chi_{a,b}).$$

This is again an irreducible character, but now of $J = G_{z,\frac{1}{2}}$.

When the residue characteristic $p \neq 2$, $t^2 = 1$ has two distinct solutions in $\mathbb{F}_q$, thus giving $(q-1)/2$ affine-generics in each orbit of $\bar{M}_z$ which induce to the same G-representation.

In other words, when $p \neq 2$, there are $\frac{(q-1)^2}{(q-1)/2} = 2(q-1)$ unique G-representations obtained by inducing each of the various affine-generic characters.

When $p = 2$, every element of $\mathbb{F}_q$ is a square, thus giving $q-1$ affine-generics in each orbit of $\bar{M}_z$ which induce to the same G-representation.

That is, when $p = 2$, there are $\frac{(q-1)^2}{q-1} = q-1$ unique G-representations obtained by inducing the various affine-generic characters.

5.1.2 Extending to the center

Let $Z = \pm\mathrm{Id}$ be the center of G. Let $H = Z \cdot G_{z,r}$, let $J = G_{z,r}$. One finds that $[H : J] = 2$ when the residue characteristic $p \neq 2$. But when $p = 2$, $-1 \in 1 + \mathfrak{p}_K$, making $H = J$.

For any affine generic character ψ of J, and any $h \in J\backslash H/J$, $\psi \cong \psi^h$. In other words,

$$\mathrm{End}_H(\mathrm{Ind}_J^H(\psi)) \cong \bigoplus_{h \in J\backslash H/J} \mathrm{Hom}_J(\psi, \psi^h) \cong \mathbb{C}^{[H:J]}.$$

Inducing ψ to H yields a representations that splits into $[H : J]$ total irreducible representations.

When $p \neq 2$, we have

$$\mathrm{Ind}_J^H(\psi_{a,b}) = \psi_{a,b}^+ \oplus \psi_{a,b}^-,$$

where

$$\psi^{\pm}_{a,b}\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}\right) = \pm 1;$$

these two representations are distinguished by their central characters. When $p = 2$, $Z \leq G_{z,r}$, so

$$\mathrm{Ind}_J^H(\psi_{a,b}) = \mathrm{Ind}_J^J(\psi_{a,b}) = \psi_{a,b}.$$

5.1.3 Simple supercuspidals

Now, let ψ denote one of the various $Z \cdot G_{z,r}$ irreducible representations induced from an affine-generic. By Reeder-Yu [18, Lemma 2.1, Prop. 2.4],

$$\text{c-Ind}_H^G(\psi) = \text{c-Ind}_{G_z}^G(Ind_H^{G_z}(\psi)) = \pi$$

is a depth-$\frac{1}{2}$ irreducible simple supercuspidal representation of G compactly induced from a the stabilizer of an edge G_z in the Bruhat-Tits building X.

Let $p \neq 2$. Since we have $2(q-1)$ orbits of affine-generic characters, and $\mathrm{Ind}_J^H(\psi_{a,b}) = \psi^+_{a,b} \oplus \psi^-_{a,b}$, we find that there are $4(q-1)$ irreducible simple supercuspidal representations of G.

Let $p = 2$. Since we have $q-1$ orbits of affine-generic characters, and $J = H$, so we find that there are $q-1$ simple supercuspidal representations of G.

5.2 $SL_2(D)$ simple supercuspidals

Let D be a central-simple algebra of degree d over K. Take z as in Section 3.2.5. Recall that

$$\bar{M}_z = T_0/T_{\frac{1}{d}} \cong \mathbb{F}^{\times}_{q^d},$$

$$V_z \cong \begin{pmatrix} & \mathbb{F}_{q^d} \\ \mathbb{F}_{q^d} & \end{pmatrix}.$$

5.2.1 Affine generic characters

Let $\star$ denote the conjugation action of $\bar{M}_z$ on V_z. Then, we have

$$\bar{M}_z \star V_z,$$

with

$$\begin{pmatrix} tn & \\ & t^{-1} \end{pmatrix} \star \begin{pmatrix} & u \\ v & \end{pmatrix} = \begin{pmatrix} & t^2nu \\ t^{-2}nv & \end{pmatrix},$$

for $t, n \in \mathbb{F}_{q^d}^\times$, $N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(n) = 1$, and $u, v \in \mathbb{F}_{q^d}$. Or, in shorthand notation,

$$(tn, t^{-1}) \star (u, v) = (t^2nu, t^{-2}n^{-1}v).$$

For $a, b \in \mathbb{F}_{q^d}$, let $\chi_{a,b}$ be the character of V_z such that

$$\chi_{a,b}((u, v)) = e^{\frac{2\pi i}{p}\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_{q^d}}(au+bv)}.$$

$\chi_{a,b}$ is affine-generic when $a, b \in \mathbb{F}_{q^d}^\times$. Thus, we have $(q^d-1)^2$ affine-generic characters. Let $\psi_{a,b} = \mathrm{Inf}_{V_z}^{G_{z,\frac{1}{2d}}}(\chi_{a,b})$; we will now refer to these $G_{z,\frac{1}{2d}}$-representations as affine-generic.

In order to calculate the size of the orbits of the affine-generic characters, we need to find the number of elements of $\mathbb{F}_{q^d}$ representable by t^2n here $t, n \in \mathbb{F}_{q^d}^\times$, $N_{\mathbb{F}_{q^d}/\mathbb{F}_q}(n) = 1$.

If the residue characteristic $p \neq 2$, then we know that there are $(q^d-1)/2$ squares in $\mathbb{F}_{q^d}^\times$, and we know that there are $(q^d-1)/(q-1)$ norm-one elements in $\mathbb{F}_{q^d}$.

Claim 43. *If $N(n) = 1$, then $n \in (\mathbb{F}_{q^d}^\times)^2$.*

Proof. Let s^2 be a square in $\mathbb{F}_{q^d}^\times$. Then $N(s^2) = N(s)^2$. As the norm map for extensions of finite fields is surjective, one finds that there are $q-1$ possible norms for elements of $\mathbb{F}_{q^d}^\times$, and there are $(q-1)/2$ possible norms for the squares in $\mathbb{F}_{q^d}^\times$. In other words, the $(q^d-1)/2$ squares of $\mathbb{F}_{q^d}^\times$ map to $(q-1)/2$ elements of $\mathbb{F}_q$ under the norm map. Since $N(1) = 1$, and 1 is a square in $\mathbb{F}_{q^d}^\times$, 1 is one of the $(q-1)/2$ possible norms for the squares in $\mathbb{F}_{q^d}^\times$. Similarly, since there are $(q-1)/2$ possible norms for non-squares, one finds that all norm-one elements of $\mathbb{F}_{q^d}^\times$ have to be squares. $\square$

When $p \neq 2$, norm-one elements of $\mathbb{F}_{q^d}^\times$ are squares, one finds that there are $(q-1)/2$ elements in the orbit of a $(u, v) \in V_z$, with $(u, v) \neq 0$. Thus, this gives precisely,

$$\frac{(q^d-1)^2}{(q^d-1)/2} = 2(q^d-1)$$

orbits of affine generic characters. Now let $p = 2$. Since every element of $\mathbb{F}_{q^d}^\times$ is a square, all norm-one elements of $\mathbb{F}_{q^d}^\times$ are squares. It follows that (u, v) is in the same orbit as $(c \cdot u, c^{-1} \cdot v)$ for all $k \in \mathbb{F}_{q^d}^\times$.

We have $(q^d-1)^2$ affine-generic characters, and each orbit has size $q^d - 1$, giving us $q^d - 1$ elements in each orbit.

5.2.2 Extending to the center

Let Z be the center of $SL_2(D)$. Since the center of D is K, we know that elements of the center have the form

$$Z = \left\{ \begin{pmatrix} k & \\ & k \end{pmatrix} : k \in K,\ \ \mathrm{Nrd}(k^2) = 1 \right\} = \left\{ \begin{pmatrix} k & \\ & k \end{pmatrix} : k \in K,\ \ k^{2d} = 1 \right\}.$$

So the center consists of the $2d^{\text{th}}$-roots of unity of K. Recall, $J = Z \cdot G_{z, \frac{1}{2d}}$. By Clifford theory, $\mathrm{Ind}_{G_{z,\frac{1}{2d}}}^{J}(\psi_{a,b})$ splits into $m := [J : G_{z,\frac{1}{2d}}]$ irreducible H-representations, distinguished by their central characters.

Giving a precise value for m is complicated; m depends on the residue characteristic, the roots of unity in the ground field, and the degree d. However, we can say for certain that $\pm 1 \in 1 + \varpi_D \mathcal{O}_D$ when $p = 2$.

Remark 44. *The roots of unity for $\mathbb{Q}_p$ are the $(p-1)^{st}$ roots of unity when $p \neq 2$; when $p = 2$, they are ± 1 – the second roots of unity. We can adjoin more roots of unity to unramified extensions of $\mathbb{Q}_p$.*

5.2.3 Simple supercuspidals

Let ψ be one of the $\psi^{\pm}_{a,b}$, then c-Ind$^G_H(\psi) = \pi$ is a depth-$\frac{1}{2d}$ irreducible supercuspidal representation – a simple supercuspidal representation.

- When $p \neq 2$, we have $2(q^d - 1)$ orbits of affine-generic characters, and $[J : G_{z,\frac{1}{2d}}] = m$, giving $2m(q^d - 1)$ non-isomorphic simple supercuspidal representations of G.
- When $p = 2$, we have $m(q^d - 1)$ non-isomorphic simple supercuspidal representations of G.

5.3 $SU_3^{L/K}(h)$ unramified, simple supercuspidals

Let L/K be a unramfied quadratic extension. Let $\varpi_K = \varpi_L$ be a uniformizing element of K and L, and let $\mathbb{F}_q$ (resp. $\mathbb{F}_{q^2}$) be the residue field of K (resp. L). Let σ be a generator of $Gal(L/K)$. Denote by $\bar{\ell} = \sigma(\ell)$ for $\ell \in L$.

Let $h : L^3 \times L^3 \to L$ be the Hermitian form

$$h(\vec{x}, \vec{y}) = \bar{x}_3 y_1 + \bar{x_2} x_2 + \bar{x_1} y_3.$$

Let $G = SU_3^{L/K}(h)$. Take z as in Section 3.3.3. Recall,

$$\bar{M}_z = T_0/T_1 \cong \left\{ \begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} : t \in \mathbb{F}_{q^2}^{\times} \right\} \cong \mathbb{F}_{q^2}^{\times},$$

$$V_z = G_{z,\frac{1}{3}}/G_{z,\frac{1}{3}+} \cong \left\{ \begin{pmatrix} & -\bar{u} & \\ & & u \\ v & & \end{pmatrix} : u \in \mathbb{F}_{q^2}, v \in \mathbb{F}_{q^2}^{\circ} \right\} \cong \mathbb{F}_{q^2} \oplus \mathbb{F}_q.$$

5.3.1 Affine-generic characters

Let $\star$ denote the conjugation action of $\bar{M}_z$ on V_z. Then

$$\begin{pmatrix} t & & \\ & \bar{t}/t & \\ & & \bar{t}^{-1} \end{pmatrix} \star \begin{pmatrix} & -\bar{u} & \\ & & u \\ v & & \end{pmatrix} = \begin{pmatrix} & -(t^2/\bar{t})\bar{u} & \\ & & (\bar{t}^2/t)u \\ N(t)^{-1}v & & \end{pmatrix};$$

let's shorthand this by writing $t \star (u,v) = (\bar{t}^2/t \cdot u, N(t)^{-1}v)$.

For $a \in \mathbb{F}_q^2$, $b \in \mathbb{F}_q$, let $\chi_{a,b}$ denote the character of V_z given by

$$\chi_{a,b}((u,v)) = e^{\frac{2\pi i}{p}\left(\mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_{q^2}}(au) + \mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(bv)\right)}.$$

$\chi_{a,b}$ is affine-generic when $a, b \neq 0$. Thus, there are $(q^2-1)(q-1)$ affine-generic characters of V_z. Let $\psi_{a,b} = \mathrm{Inf}_{V_z}^{G_{z,r}}(\chi_{a,b})$; we now refer to the $\psi_{a,b}$ as affine-generic characters of $G_{z,r}$.

Let's now consider the orbits of these representations. One finds that $\psi_{a,b}$ and $\psi_{c,d}$ induce to the same G-representation if $(c,d) = (\bar{t}^2/t \cdot a, N(t)^{-1}b)$ for some $t \in \mathbb{F}_{q^2}^\times$.

If the residue characteristic $p \neq 2$, one finds that there are q^2-1 unique $(\bar{t}^2/t, N(t)^{-1})$ for various $t \in \mathbb{F}_{q^2}^\times$, thus giving $(q^2-1)(q-1)/(q^2-1) = q-1$ orbits.

If $p = 2$, we need to split into two cases. $q = 2$, and $q \neq 2$.

- When $q = 2$, $\bar{t} = t^2$, so $\mathrm{Ind}_{G_{z,r}}^G(\psi_{a,b}) = \mathrm{Ind}_{G_{z,r}}^G(\psi_{a,N(t)^{-1}b})$ for varying $t \in \mathbb{F}_{q^2}$. This gives us $(q^2-1)(q-1)/(q-1) = q^2-1$ total orbits.
- When $p = 2$, $q \neq 2$, we have $\bar{t}^2/t$ representing the various squares in $\mathbb{F}_{q^2}$. However, every element of $\mathbb{F}_{q^2}$ is a square. Thus, we have $(q^2-1)(q-1)/(q^2-1) = q-1$ total orbits.

5.3.2 Extending to the center

Let Z denote the center of $G = SU_3^{L/K}(h)$, for L/K an unramified quadratic extension. Note that elements of the center have to be diagonal matrices of determinant one. This leaves us with possibilities, $Z = \mathrm{Id}$ or $Z = \langle \zeta \cdot \mathrm{Id} \rangle$, where ζ is a third root of unity. However, for $\zeta \cdot \mathrm{Id}$

to be in the center, we require $\zeta \in L \setminus K$, as

$$\begin{pmatrix} \sigma(\zeta) & & \\ & \sigma(\zeta) & \\ & & \sigma(\zeta) \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix} = \begin{pmatrix} & & \sigma(\zeta)\zeta \\ & \sigma(\zeta)\zeta & \\ \sigma(\zeta)\zeta & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Thus, we need $\sigma(\zeta) = \zeta^2$ to preserve our Hermitian matrix $H = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$. So we need $L = K[\zeta]$ in order for G to have a non-trivial center.

Note that $\zeta \in \mathbb{Q}_p$ when $3 \mid (p-1)$. So when the residue characteristic of K has $p \equiv 1 \bmod 3$, we can always say that the center of G is trivial. Furthermore, adding third roots of unity to an extension of $\mathbb{Q}_3$, results in a ramified extension. Thus, when $p = 3$, we are also guaranteed a trivial center.

Let $J = G_{z,r}$, $H = Z \cdot G_{z,r}$. Let $[H : J] = m$; then $m = 1$ or 3. Let ψ be an affine-generic character of J. When $m = 1$,

$$\mathrm{Ind}_J^H(\psi) = \mathrm{Ind}_J^J(\psi) = \psi.$$

While when $m = 3$,

$$\mathrm{Ind}_J^H(\psi) = \psi_0 + \psi_1 + \psi_2,$$

where $\psi_i\left(\begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix}\right) = \zeta^i$.

5.3.3 Simple supercuspidals

Let ψ_i denote one of the summands of $\mathrm{Ind}_J^H(\psi)$. By Reeder-Yu [18, Prop. 2.4], c-$\mathrm{Ind}_H^G(\psi_i)$ is an irreducible representation – a supercuspidal representation of depth-$\frac{1}{3}$. It follows that

- when $p = 2$, $q = 2$, G has $m(q^2 - 1)$ total simple supercuspidal irreducible representations.

- when $p = 2$, $q \neq 2$, or when $p \neq 2$, G has $m(q-1)$ total simple supercuspidal irreducible representations.

5.4 $SU_3^{E/K}(h)$ ramified, simple supercuspidals

Let E be a quadratic ramified extension of K, a non-archimedean local field. Let ϖ_E be a uniformizing element of E, and let the residue field of E and K be $\mathbb{F}_q$, with characteristic p. Recall from Section 3.4.3 that

$$\bar{M}_z = G_{z,0}/G_{z,\frac{1}{6}} \cong \left\{ \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} : t \in \mathbb{F}_q^\times \right\},$$

$$V_z = G_{z,\frac{1}{6}}/G_{z,\frac{1}{6}+} \cong \left\{ \begin{pmatrix} & -u & \\ & & u \\ v & & \end{pmatrix} u, v \in \mathbb{F}_q \right\}.$$

5.4.1 Affine-generic characters

Let $\star$ denote the action of conjugation. We have $\bar{M}_z \star V_z$, with

$$\begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} \star \begin{pmatrix} & -u & \\ & & u \\ v & & \end{pmatrix} = \begin{pmatrix} & -tu & \\ & & tu \\ t^{-2}v & & \end{pmatrix}.$$

We use the shorthand-notation $t \star (u, v) = (tu, t^{-2}v)$. Let $\chi_{a,b}$ denote the character of V_z such that

$$\chi_{a,b}((u, v)) = e^{\frac{2\pi i}{p} \mathrm{Tr}_{\mathbb{F}_p}^{\mathbb{F}_q}(au+bv)}.$$

To be affine generic, we require $a, b \neq 0$. Thus, there are $(q-1)^2$ affine-generic characters of V_z. Define $\psi_{a,b} = \mathrm{Inf}_{V_z}^{G_{z,\frac{1}{6}}}(\chi_{a,b})$; from now, on we'll refer to these $G_{z,\frac{1}{6}}$-representations as affine-generic. The affine-generic characters $\psi_{a,b}$ and $\psi_{ta,t^{-2}v}$ are in the same G orbit when $t \in \mathbb{F}_q^\times$. Thus, each orbit has $q-1$ affine-generics. Therefore, we have $\frac{(q-1)^2}{(q-1)} = q-1$ orbits of affine-generic characters.

5.4.2 Extending to the center

Let Z denote the center of $G = SU_3^{E/K}(h)$. Note that elements of the center have to be diagonal matrices of determinant one. This leaves us with possibilities, $Z = \mathrm{Id}$ or $Z = \langle \zeta \cdot \mathrm{Id} \rangle$, where ζ is a third root of unity.

However, for $\zeta \cdot \mathrm{Id}$ to be in the center, we require $\zeta \in E \setminus K$, since

$$\begin{pmatrix} \sigma(\zeta) & & \\ & \sigma(\zeta) & \\ & & \sigma(\zeta) \end{pmatrix} \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix} = \begin{pmatrix} & & \sigma(\zeta)\zeta \\ & \sigma(\zeta)\zeta & \\ \sigma(\zeta)\zeta & & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

Thus, we need $\sigma(\zeta) = \zeta^2$ to preserve our Hermitian matrix $H = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$.

To have a nontrivial center, we require $E = K[\zeta]$. The only possible setting where adding third roots of unity to K can yield a ramified extension is when the residue characteristic $p = 3$. So when $E = K[\zeta]$ is an extension of the field K of residue characteristic $p = 3$,

$$[Z \cdot G_{z,\frac{1}{6}} : G_{z,\frac{1}{6}}] = 3.$$

Thus, in this setting, for an affine-generic character ψ of $G_{z,\frac{1}{6}}$,

$$\mathrm{Ind}_{G_{z,\frac{1}{6}}}^{Z \cdot G_{z,\frac{1}{6}}}(\psi) = \psi_0 \oplus \psi_1 \oplus \psi_2,$$

where

$$\psi_i\left(\begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix}\right) = \zeta^i.$$

5.4.3 Simple supercuspidals

Let ψ_i denote a character appearing in $\mathrm{Ind}_{G_{z,\frac{1}{6}}}^{Z\cdot G_{z,\frac{1}{6}}}(\psi)$ for an affine generic character ψ. By [18, Prop. 2.4],

$$\text{c-Ind}_{Z\cdot G_{z,\frac{1}{6}}}^{G}(\psi_i) = \pi$$

is an irreducible supercuspidal representation of depth-$\frac{1}{6}$.

- When $L = K[\mu_3]$ is a ramified extension, which can only occur when $p = 3$, each orbit of affine-generic characters contributes three simple supercuspidal representation of G. Thus, we have $3(q-1)$ simple supercuspidal representations in this setting.

- In all other settings, each orbit of affine-generic characters contributes one simple supercuspidal representation of G. Thus, giving $q-1$ simple supercuspidal representations.

5.5 Table of inducing data

Here, we provide a table summarizing the computations in this chapter for the inducing data of minimal-positive-depth representations of the groups $SL_2(K)$, $SL_2(D)$, $SU_3^{L/K}(h)$, and $SU_3^{E/K}(h)$.

Group	$\bar{M}_z$	$Z \cap \bar{M}_z$	$\check{V}_z^{\text{aff.gen.}}$	Action	# Irreps
$SL_2(K)$, $p \neq 2$	$\mathbb{F}_q^\times$	μ_2	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$t \star (u,v) = (t^2 u, t^{-2} v)$	$4(q-1)$
$SL_2(K)$, $p = 2$	$\mathbb{F}_q^\times$	Id	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$t \star (u,v) = (t^2 u, t^{-2} v)$	$q-1$
$SL_2(D)$, $p \neq 2$	$\mathbb{F}_{q^d}^\times$	μ_m	$\mathbb{F}_{q^d}^\times \times \mathbb{F}_{q^d}^\times$	$t \star (u,v) = (t^2 nu, t^{-2} n^{-1} v)$	$2m(q-1)$
$SL_2(D)$, $p = 2$	$\mathbb{F}_{q^d}^\times$	μ_m	$\mathbb{F}_{q^d}^\times \times \mathbb{F}_{q^d}^\times$	$t \star (u,v) = (t^2 nu, t^{-2} n^{-1} v)$	$m(q-1)$
$SU_3^{L/K}(h)$, $p \neq 2$ or $p = 2, q \neq 2$ $L = K[\mu_3]$	$\mathbb{F}_{q^2}^\times$	μ_3	$\mathbb{F}_{q^2}^\times \times (\mathbb{F}_{q^2}^\circ)^\times$	$t \star (u,v) = (\bar{t}^2 u/t, N(t)^{-1} v)$	$3(q-1)$
$SU_3^{L/K}(h)$, $p \neq 2$ or $p = 2, q \neq 2$, $L \neq K[\mu_3]$	$\mathbb{F}_{q^2}^\times$	Id	$\mathbb{F}_{q^2}^\times \times (\mathbb{F}_{q^2}^\circ)^\times$	$t \star (u,v) = (\bar{t}^2 u/t, N(t)^{-1} v)$	$q-1$
$SU_3^{L/K}(h)$, $p = 2$ $L = K[\mu_3]$	$\mathbb{F}_{q^2}^\times$	μ_3	$\mathbb{F}_{q^2}^\times \times (\mathbb{F}_{q^2}^\circ)^\times$	$t \star (u,v) = (\bar{t}^2 u/t, N(t)^{-1} v)$ $= (u, N(t)^{-1} v)$	$3(q^2-1)$
$SU_3^{L/K}(h)$, $p = 2$ $L \neq K[\mu_3]$	$\mathbb{F}_{q^2}^\times$	Id	$\mathbb{F}_{q^2}^\times \times (\mathbb{F}_{q^2}^\circ)^\times$	$t \star (u,v) = (\bar{t}^2 u/t, N(t)^{-1} v)$ $= (u, N(t)^{-1} v)$	q^2-1
$SU_3^{E/K}(h)$, $p = 3$ $E = K[\mu_3]$	$\mathbb{F}_q^\times$	μ_3	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$t \star (u,v) = (tu, t^{-2} v)$	$3(q-1)$
$SU_3^{E/K}(h)$, $E \neq K[\mu_3]$	$\mathbb{F}_q^\times$	Id	$\mathbb{F}_q^\times \times \mathbb{F}_q^\times$	$t \star (u,v) = (tu, t^{-2} v)$	$q-1$

Figure 5.1: Inducing data for groups in Chapter 5

Remark 45. *For $SL_2(D)$, m is the number of $2d$ roots of unity in $G_{z,0}/G_{z,0+}$. This is the index $[Z \cdot G_{z,\frac{1}{2d}} : G_{z,\frac{1}{2d}}]$.*

Appendix A

Quaternionic unitary groups

We will briefly describe the quaternionic special unitary groups, which we call $SU_m^{D/K}(h')$ and $SU_n^{D/K}(h)$ where $m = 2$ or 3 and $n = 3, 4,$ or 5. We follow Prasad-Raghunathan [16, Sections 1.3-1.4, 1.7-1.9]. Refer the reader to this text or to Tits [24], where Tits first introduces these groups, for further details.

A.1 $SU_2^{D/K}(h')$ and $SU_3^{D/K}(h')$

Let D be a quaternion division algebra over K, and let σ be an involution of D of the first kind and first type. Let $V = e_{-1} \cdot D \oplus e_1 \cdot D$ be a right vector space over D of dimension 2 and let h' be a σ-skew-Hermitian form such that $h'(w, v) = -h'(v, w)^\sigma$ for all $v, w \in V$, determined by:

$$h'(e_{-1}, e_{-1}) = 0 = h'(e_1, e_1),$$

$$h'(e_{-1}, e_1) = 1 = -h'(e_1, e_{-1}).$$

Let $G = \mathbf{G}(K) = SU_2^{D/K}(h')$. Then $\mathbf{G}$ is an absolutely almost simple, simply connected K-group of relative rank 1, and it's of type C_2. The relative root system of G is reduced, having roots $\{\pm a\}$.

Let $\{e_{-1}, e_1\}$ be a basis of V. Let $\mathbf{S}$ be the 1-dimensional maximal K-split torus such that

$$S := \mathbf{S}(K) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \; : \; t \in K^\times \right\}.$$

Let L/K be a quadratic unramified extension; L is a splitting field of D. Let $\mathbf{T}$ be the two dimensional residually K-split torus such that

$$T := \mathbf{T}(K) = \left\{ \begin{pmatrix} t & \\ & \sigma(t)^{-1} \end{pmatrix} \; : \; t \in L^\times \right\}.$$

Let

$$M = \mathbf{Z_G}(\mathbf{S})(K) := \left\{ \begin{pmatrix} t & \\ & \sigma(t)^{-1} \end{pmatrix} \; : \; t \in D^\times \right\},$$

and note that $T \subset M$. Let $D^\sigma = \{d \in D : d^\sigma + d = 0\}$. Let

$$u_+(d) = \begin{pmatrix} 1 & d \\ & 1 \end{pmatrix}, \; u_-(d) = \begin{pmatrix} 1 & \\ d & 1 \end{pmatrix}, \text{ where } d \in D^\sigma.$$

Then

$$\mathbf{U}_a(K) = \{u_+(d) : d \in D^\sigma\}, \; \mathbf{U}_{-a}(K) = \{u_-(d) : d \in D^\sigma\}.$$

The group $\mathbf{G}$ splits over the quadratic unramified extension L of K.

We now move onto $G = \mathbf{G}(K) = SU_3^{D/K}(h')$. Let D be a quaternion division algebra over K, and let σ be an involution of D of the first kind and first type. Let V be a right vector space over D of dimension 3, and let $h' : V \times V \to D$ be a non-degenerate σ-skew-Hermitian form on V of Witt index 1. Assume that h' is trace-valued. That is, for all $v \in V$, $h'(v, v) \in \{d - d^\sigma : d \in D\}$. Then we can find $e_{-1}, e_0, e_1 \in V$ such that

$$V = e_{-1} \cdot D \oplus e_0 \cdot D \oplus e_1 \cdot D,$$

$$h'(e_{-1}, e_{-1}) = 0 = h'(e_1, e_1),$$

$$h'(e_{-1}, e_0) = 0 = h'(e_1, e_0),$$

$$h'(e_{-1}, e_1) = 1 = -h'(e_1, e_{-1}),$$

$$h'(e_0, e_0) \neq 0.$$

For instance, h' can be realized as the σ-skew-Hermitian matrix

$$H' = \begin{pmatrix} & & 1 \\ & b & \\ -1 & & \end{pmatrix},$$

where $b \in D$, $\sigma(b) = -b$.

G is the special unitary group with respect to h'. So one can realize

$$G = \{g \in SL_3(D) : tg^{\sigma} H' g = H'\}.$$

Let $V' = e_{-1} \cdot D \oplus e_1 \cdot D$, and let f' denote the restriction of h' to V'. Then $SU_2^{D/K}(f')$ is the group from above, and it embeds into $SU_3^{D/K}(h')$.

$$S := \mathbf{S}(K) = \left\{ \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix} : t \in K^{\times} \right\}$$

is maximal K-split torus. Let L/K be a quadratic unramified extension; L is a splitting field of D. Let $\mathbf{T}$ be the three dimensional residually K-split torus such that

$$T := \mathbf{T}(K) = \left\{ \begin{pmatrix} t & & \\ & n & \\ & & \sigma(t)^{-1} \end{pmatrix} : t, n \in L^{\times}, \;\; N_{L/K}(n) = 1 \right\};$$

we find that this is a maximal torus of G. T is contained in

$$M := Z_G(S) = \left\{ \begin{pmatrix} t & & \\ & n & \\ & & \sigma(t)^{-1} \end{pmatrix} : t, n \in D^{\times}, \;\; \mathrm{Nrd}_{\sigma}(n) = 1 \right\}.$$

Let $u_+(c,d) = \begin{pmatrix} 1 & \sigma(c)b & d \\ & 1 & c \\ & & 1 \end{pmatrix}$, where $-d+\sigma(d)+\sigma(c)bc = 0$, and let $u_-(c,d) = \begin{pmatrix} 1 & & \\ c & 1 & \\ d & -\sigma(c)b & 1 \end{pmatrix}$, where $-d+\sigma(d)-\sigma(c)bc = 0$. Then we have root subgroups,

$$\mathbf{U}_a(K) = \{u_+(c,d) : c,d \in D\}\,, \ \mathbf{U}_{-a}(K) = \{u_-(c,d) : c,d \in D\}\,,$$

$$\mathbf{U}_{2a}(K) = \{u_+(0,d) : d \in D^\sigma\}\,, \ \mathbf{U}_{-2a}(K) = \{u_-(0,d) : d \in D^\sigma\}\,.$$

The group $\mathbf{G}$ splits over the quadratic unramified extension L of K.

A.2 $SU_n^{D/K}(h)$ for $n = 3,4,5$

Here we describe the K-rank 1 forms of type A_3 which do not split over the maximal unramified extension of K, and the K-rank 1 forms of type D_4 and D_5.

Let D be a quaternion algebra over K, with σ an involution of D of first type and first kind. Let L/K be a splitting field of D (an unramified quadratic extension of F), where $L = K[u]$. Furthermore, let ϖ_D be a uniformizing element of D such that $u\varpi_D = \varpi_D\gamma(u)$, where γ is the generator of $Gal(L/K)$. Let V be a finite dimensional right vector space over D. Set

$$D_\sigma = \{d - d^\sigma\varepsilon : d \in D\}.$$

Let V be a right vector space over D of dimension $n = 3, 4$, or 5. Let $q : V \to D/D_\sigma$ be a σ-quadratic form and let $h : V \times V \to D$ be the associated σ-Hermitian form. In other words, h is the σ-Hermitian form such that

$$q(v+w) = q(v) + q(w) + (h(v,w) + D_\sigma)$$

for $v, w \in V$. We choose q to be non-degenerate in the following sense: For $v \in V$, $h(v,V) = \{0\}$ and $q(v) = 0$ implies that $v = 0$. Furthermore, we assume that (h,q) is of Witt index 1. This gives V a direct sum decomposition

$$V = e_{-1} \cdot D \oplus V_0 \oplus e_1 \cdot D,$$

where $e_{-1}, e_1 \in V$, and V_0 is a subspace of V such that

$$q(e_{-1}) = 0 = q(e_1),$$

$$h(e_{-1}, e_1) = 1 = h(e_1, e_{-1}),$$

$$h(e_i, V_0) = \{0\} \text{ for } i = \pm 1, \text{ and}$$

$$0 \notin q(V_0 = \{0\}).$$

When char $K \neq 2$, $h(v, v) \neq 0$ for all nonzero $v \in V_0$. When char $K = 2$, we assume further that q is non-defective, meaning h is non-degenerate. Set

$$V_0' = \{v \in V_0 : h(v, v) \neq 0\}.$$

Then when char $K \neq 2$, $V_0' = V_0 - \{0\}$, and if char $K = 2$, then $V_0' = V_0 = \{0\}$ if $\dim(V_0) = 1$, and if $\dim(X_0) > 1$, V_0' is a non-empty Zariski-open subset of X_0.

Let $G = SU_n^{D/K}(h)$. Then $\mathbf{G}$ is an absolutely almost simple K-group of K-rank 1. It is an outer form of type $D_3 = A_3$ or D_4, or an inner form of type D_5 according as $n = 3, 4,$ or 5. Let $\mathbf{G}'$ be the simply connected cover of $\mathbf{G}$ defined over K, and let $\pi : \mathbf{G}' \to \mathbf{G}$ be the canonical central K-isogeny. π is central, so it is an isomorphism when restricted to any unipotent K-subgroup $\mathbf{U}'$ of $\mathbf{G}'$; thus we shall identify $\mathbf{U}'$ with $\pi(\mathbf{U}')$.

For $t \in K^\times$, let $m(t)$ be the linear transformation of V defined by

$$m(t) : \begin{cases} e_{-1} \mapsto e_{-1} \cdot t & \\ v_0 \mapsto v_0 & (v_0 \in V_0) \\ e_1 \mapsto e_1 \cdot \sigma(t)^{-1} & \end{cases}$$

and let $\mathbf{S}'$ be the 1-dimensional maximal K-split torus of G' such that

$$S' := \mathbf{S}'(K) = \{m(t) : t \in K^\times\}.$$

Let $\mathbf{S}$ be the corresponding K-split torus in G, with $S := \mathbf{S}(K)$. The K-roots of G with respect to $\mathbf{S}$ are $\{\pm a, \pm 2a\}$. From the Tits index one finds that $\mathbf{U}_{2a}(K)$ is 1-dimensional over K.

Let $Z = \{(z,d) : z \in V_0,\ \ d \in D,\ \ q(z) = d + D_\sigma\}$, and let

$$Z' = \{(z,d) : (z,d) \in Z,\ \ z \in V_0'\}\,.$$

For $(z,d) \in Z$, let $u_+(z,d)$ and $u_-(z,d)$ be the linear transformations of V defined by

$$u_+(z,d) : \begin{cases} e_{-1} \mapsto e_{-1} \\ v_0 \mapsto v_0 - e_{-1} \cdot h(z, v_0) \quad (v_0 \in V_0) \\ e_1 \mapsto e_1 + z - e_{-1} \cdot d \end{cases}$$

$$u_-(z,d) : \begin{cases} e_{-1} \mapsto e_{-1} + z - e_1 \cdot d \\ v_0 \mapsto v_0 - e_1 \cdot h(z, v_0) \quad (v_0 \in V_0) \\ e_1 \mapsto e_1 \end{cases}$$

Then

$$\mathbf{U}_a(K) = \{u_+(z,d) : (z,d) \in Z\}\,,\ \ \mathbf{U}_{-a}(K) = \{u_-(z,d) : (z,d) \in Z\}\,,$$

$$\mathbf{U}_{2a}(K) = \{u_+(0,d) : d \in D_\sigma\}\,,\ \ \mathbf{U}_{-2a}(K) = \{u_-(0,d) : d \in D_\sigma\}\,.$$

Please consult the text for more details.

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