UCRL 8896

# UNIVERSITY OF CALIFORNIA

## Ernest O. Lawrence

# Radiation Laboratory



BERKELEY, CALIFORNIA

**DISCLAIMER**

UCRL-8896

UNIVERSITY OF CALIFORNIA
Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

# THE GAUGE PROPERTIES OF PROPAGATORS IN QUANTUM ELECTRODYNAMICS

Bruno Zumino

September 8, 1959

Printed for the U. S. Atomic Energy Commission

UCRL-8896

# THE GAUGE PROPERTIES OF PROPAGATORS IN QUANTUM ELECTRODYNAMICS*

Bruno Zumino†

Lawrence Radiation Laboratory
University of California
Berkeley, California

September 8, 1959

## ABSTRACT

The effect of a change of gauge on the propagators is studied systematically for quantum electrodynamics. Various gauges are considered, among them the Coulomb, the Landau, the Feynman, and the Yennie gauges. The equivalence of the various formulations of the theory is demonstrated. For the relativistic gauges, the transformation of the wave function renormalization constant is described.

* This work was supported in part by the U.S. Atomic Energy Commission and in part by the Office of Naval Research. A preliminary account of it was given in the Bulletin of the American Physical Society, Series II, Vol. 4, No. 4, 280 (1959).

† Permanent address: Physics Department, New York University, New York, New York.

UCRL-8896

# THE GAUGE PROPERTIES OF PROPAGATORS IN QUANTUM ELECTRODYNAMICS

Bruno Zumino

Lawrence Radiation Laboratory
University of California
Berkeley, California

September 8, 1959

## 1. INTRODUCTION

The propagators of quantum electrodynamics are affected by ambiguities because the theory is invariant under gauge transformations. In this note we shall investigate systematically this ambiguity of the (unrenormalized) propagators and shall give the connection between the various gauges.

We use the Heisenberg equations of motion in the Coulomb gauge. In this gauge the longitudinal part of the magnetic vector potential is a c-number. The relativistic covariance of this formulation of the theory (briefly discussed in Appendix B) has been known for a long time. It is also known that the relativistic S-matrix theory of Feynman can be derived directly in the Coulomb gauge. In this note we show how the propagators in other gauges (including the relativistic ones) are connected to those in the Coulomb gauge. It is therefore clear that the Heisenberg equations in the Coulomb gauge provide a complete basis for quantum electrodynamics. The present formulation has the desirable feature that only physical states are considered, no supplementary condition and no indefinite product in Hilbert space are necessary.[1]

The study of the gauge transformation of the propagators becomes particularly simple and elegant if one employs the method of functional derivatives. This method, which has been largely used by Schwinger,[2] makes use of a generating functional (which we call Z) from which all propagators can be obtained by functional differentiation. The gauge ambiguity of the propagators will be shown here to arise from the gauge ambiguity in the functional Z itself. In previous work this ambiguity has been ignored to a large extent, with a consequent lack of clarity concerning the meaning of the operations to be performed, as for instance the differentiations with respect to the external sources.

The Heisenberg equations in the Coulomb gauge depend upon a c-number gauge function $\Lambda$ . As shown in Section 2, a change in the gauge function induces a gauge transformation in the generating functional Z. We call a quantity gauge-invariant if it is invariant with respect to this c-number gauge transformation. In Section 3 we derive the functional differential equations satisfied by Z and extend the definition to more general gauges, characterized by an operator four-vector $a_\mu$ . A suitable choice of $a_\mu$ gives the Coulomb gauge in any Lorentz frame. Another choice of $a_\mu$ gives a relativistic gauge in which the zero-order photon propagator has the form

$$D_{\mu\nu} = (g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2})D_c .$$

This gauge was widely used by Landau,[3] and we shall call it the Landau gauge. It is very convenient for the study of ultraviolet divergences. In Section 4, we proceed to a further generalization, introducing gauges depending in addition upon a function M. If we start from the Landau gauge $D_{\mu\nu}$ , we can obtain in this fashion all gauges where the zero-order photon propagator has the form[4]

$$D_{\mu\nu} + \partial_\mu \partial_\nu M .$$

In particular, for $M = \frac{1}{\partial^2} D_c$ , one can cancel the $\partial_\mu \partial_\nu$ term and obtain the ordinary Feynman form of the photon propagator. For $M = 3\frac{1}{\partial^2} D_c$ one obtains a gauge used by Fried and Yennie.[5] They have used it for the study of infrared divergences. Our general gauge is now characterized by the c-number gauge function $\Lambda$ plus $a_\mu$ and $M$ . Quantities invariant with respect to changes in $\Lambda$ are automatically invariant for changes of $a_\mu$ and $M$ . Therefore these more general gauge transformations do not correspond to new invariance properties of the theory. Rather they allow to establish a connection between different existing formulations of it.

The effect of changes in the function $M$ (within the class of relativistic gauges) on the wave function renormalization constant $Z_2$ has been studied in particular by Johnson and the present author.[6] They have given an exact transformation formula for $Z_2$ . Using it, one can easily verify that $Z_2$ (to order $e^2$) has no ultraviolet divergence in the Landau gauge and no infrared divergence in the Yennie gauge.

## 2. THE GENERATING FUNCTIONAL IN THE COULOMB GAUGE

We recall here first the equations for the Heisenberg operators of quantum electrodynamics in the Coulomb gauge.

The Hamiltonian is

$$\mathcal{H} = \int d\underline{x}\, T_{oo} \tag{1}$$

with

$$T_{oo} = \frac{1}{2}(\underline{E}^2 + \underline{H}^2) - \underline{j}\cdot\underline{A} + \frac{1}{2}[\psi^*, (\frac{-i}{2}\underline{\alpha}(\vec{\nabla} - \overleftarrow{\nabla}) + \beta m)\psi] , \tag{2}$$

and the commutation relations are

$$[E_r^{tr}(\underline{x}), A_s(\underline{x}')] = i(\delta_{rs} - \nabla_r \nabla_s \nabla^{-2})\, \delta(\underline{x} - \underline{x}') \tag{3}$$

and

$$\{\psi^*_\sigma(\underline{x}), \psi_\tau(\underline{x}')\} = \delta_{\sigma\tau}\, \delta(\underline{x} - \underline{x}') \tag{4}$$

while other commutators vanish. Here we have[7]

$$\underline{E} = \underline{E}^{tr} - \nabla\phi , \tag{5}$$

$$\phi = -\nabla^{-2} \rho , \tag{6}$$

$$\text{div}\, \underline{E}^{tr} = 0 , \tag{7}$$

$$\rho = \frac{e}{2}[\psi^*, \psi], \qquad \underline{j} = \frac{e}{2}[\psi^*, \underline{\alpha}\,\psi] , \tag{8}$$

$$\underline{H} = \text{curl}\, \underline{A} , \tag{9}$$

and

$$\underline{A} = \underline{A}^{tr} + \nabla\Lambda . \tag{10}$$

Since the longitudinal part of $\underline{A}$ commutes with all other operators, $\Lambda(\underline{x}, t)$ can be taken as an arbitrary c-number.

The above Hamiltonian and commutation relations give rise to the correct equations of motion:

$$i\dot{\psi} = \{m\beta - i\underline{\alpha}\cdot\nabla - e\underline{\alpha}\cdot\underline{A}\}\,\psi + \frac{e}{2}\{\phi, \psi\} \tag{11}$$

$$\dot{\underline{E}}^{tr} = \text{curl}\,\underline{H} - \underline{j}^{tr} \tag{12}$$

$$\dot{A}^{tr} = -\underline{E}^{tr}, \tag{13}$$

so that

$$\underline{E} = -\dot{\underline{A}}^{tr} - \nabla\phi\,. \tag{14}$$

If we set

$$A^{o} = \phi - \dot{\Lambda}\,, \tag{15}$$

we can write

$$\underline{E} = -\dot{\underline{A}} - \nabla A^{o}\,. \tag{16}$$

The set of equations presented above is invariant under the c-number gauge transformation:

$$\begin{aligned} \Lambda(\underline{x}, t) &\rightarrow \Lambda(\underline{x}, t) + \tilde{\Lambda}(\underline{x}, t) \\ \underline{A} &\rightarrow \underline{A} + \nabla\tilde{\Lambda} \\ A^{o} &\rightarrow A^{o} - \dot{\tilde{\Lambda}} \\ \psi &\rightarrow \psi \exp ie\tilde{\Lambda}\,. \end{aligned} \tag{17}$$

The choice $\Lambda = 0$ would fix the gauge (transverse Coulomb gauge). However, it is convenient to leave the gauge function arbitrary and to use the gauge transformation to define those operators that are gauge-invariant.

The operator condition (7) is consistent with the commutation relation (3). It is known that the formulation of quantum electrodynamics given above is relativistically covariant, in spite of the apparent asymmetry between space and time variables. The covariance is proven directly in Appendix B. An alternative proof results from the investigation carried out in Section 3 of the covariance of the equations for the propagators.

We consider now the generating functional[8]

$$Z_\Lambda[\eta, \bar{\eta}, J_\mu] = \langle 0 \mid T \exp i \int dx(\bar{\eta}\,\psi + \bar{\psi}\,\eta + J_\mu A^\mu) \mid 0 \rangle$$

$$\times \exp \frac{1}{2} \int dx\, J_o\, \nabla^{-2} J_o \quad . \tag{18}$$

All vacuum expectation values of time-ordered products of Heisenberg operators can be constructed from $Z$ by functional differentiation. Actually the form (18) is somewhat redundant, since $A^o$ is given through Eqs. (6) and (15) in terms of the spinor field. One could set $J_o = 0$ and work with the resulting functional. The form we have chosen, however, is more convenient for the investigation of transformation properties. We wish to emphasize here that it is <u>not</u> assumed that the various components of $J_\mu$ satisfy a continuity equation like

$$\partial_\mu J^\mu = 0 \ . \tag{19}$$

The complete arbitrariness is necessary in order to operate on $Z$ with functional derivatives. Only after all functional differentiations have been performed will one require that $J_\mu$ satisfies Eq. (19) (or even that it vanishes) and also, of course, that

$$\eta = \bar{\eta} = 0 \ . \tag{20}$$

The gauge ambiguity of Z and of the propagators is in fact connected with the necessity of extending (in an arbitrary way) the definition of Z to unphysical values of $J_\mu$ not satisfying Eq. (19). This fact can be illustrated by exhibiting the dependence of Z upon the gauge function Λ. It follows immediately from Eq. (17) that

$$Z_\Lambda[\eta, \bar{\eta}, J_\mu] = Z_0[\eta \exp(-ie\Lambda), \bar{\eta} \exp(ie\Lambda), J_\mu] \exp ie \int J_\mu \partial^\mu \Lambda \, dx \quad , \tag{21}$$

where the right-hand side refers to the value Λ = 0. This relation can be cast into the differential form

$$i \frac{\delta Z}{\delta \Lambda} = (\partial^\mu J_\mu + e \eta \frac{\delta}{\delta \eta} - e \bar{\eta} \frac{\delta}{\delta \bar{\eta}}) Z \ . \tag{22}$$

## 3. THE FUNCTIONAL EQUATIONS AND THEIR TRANSFORMATION PROPERTIES

Using the definition (18), the equations for the Heisenberg operators and the commutation relations, one can show that the generating functional satisfies the functional differential equations:

$$\{\partial^\lambda \left(\partial_\mu \frac{1}{i} \frac{\delta}{\delta J^\lambda} - \partial_\lambda \frac{1}{i} \frac{\delta}{\delta J^\mu}\right) + (\delta_\mu{}^\lambda + a_\mu \partial^\lambda)$$

$$\times \left(ie \frac{1}{i} \frac{\delta}{\delta \eta} \gamma_\lambda \frac{1}{i} \frac{\delta}{\delta \bar{\eta}} - J_\lambda\right) \}Z = 0 \tag{23}$$

$$\{ [\gamma^\mu(\partial_\mu - ie \frac{1}{i} \frac{\delta}{\delta J^\mu}) + m ] \frac{1}{i} \frac{\delta}{\delta \bar{\eta}} - \eta \}Z = 0 \tag{24}$$

$$\{ - \frac{1}{i} \frac{\delta}{\delta \eta} [ - \gamma^\mu(\partial_\mu + ie \frac{1}{i} \frac{\delta}{\delta J^\mu}) + m ] - \bar{\eta} \}Z = 0 \tag{25}$$

$$(a^\mu \frac{1}{i} \frac{\delta}{\delta J^\mu} + \Lambda)Z = 0 \ . \tag{26}$$

Equation (25) follows from the (Pauli) adjoint of Eq. (11). The source terms $J_\lambda$, $\eta$, and $\bar{\eta}$ arise from the time differentiation of the time-ordered products.

The operator vector $a_\mu$, introduced here at first for the sake of concise notation, is defined by

$$a_o = 0 \ , \qquad a_r = - \nabla_r \nabla^{-2} \tag{27}$$

in the Lorentz frame chosen to define the Coulomb gauge. The form of the equations suggests, however, that we consider more generally operator vectors $a_\mu$ (operating on functions of the four-dimensional variable $x$) satisfying

$$\partial^{\mu} a_{\mu} = -1 . \tag{28}$$

A suitable choice for $a_{\mu}$ will give, in particular, the Coulomb gauge in any Lorentz frame.[9] Another choice of interest is the limit as $\epsilon$ tends to zero of

$$a_{\mu} = \partial_{\mu}(-\partial^{2} - i\epsilon)^{-1} . \tag{29}$$

The different choices of $a_{\mu}$ give rise to different functionals $Z$ which are not related simply by changes in the gauge function $\Lambda$ and yet give rise to physically equivalent formulations of the theory. As shown in Eq. (31) below, a change $\delta a_{\mu}$ of $a_{\mu}$ which preserves Eq. (28)

$$\partial^{\mu}(\delta a_{\mu}) = 0 , \tag{30}$$

can be considered as a generalized type of gauge transformation and the various choices of $a_{\mu}$ as various possible gauges. The gauge given by Eq. (29) will be called the Landau gauge. Let us notice that the condition (28) is required for the consistency of the functional equation (23), as one can see by operating on this equation with $\partial^{\mu}$ .

From our present more general point of view, the Eqs. (23) to (26) are obviously covariant, since we chose to treat $a_{\mu}$ as a four-vector. However, one must now investigate how the solution changes with an infinitesimal change $\delta a_{\mu}$ in $a_{\mu}$ . We show in the following that the corresponding change in $Z$ can be written as

$$\delta Z = i \int (\delta a_{\mu}) \frac{\delta}{\delta J_{\mu}} \frac{\delta}{\delta \Lambda} Z . \tag{31}$$

Clearly, if a functional $\mathcal{O}$ constructed from $Z$ by functional differentiation is gauge invariant in the sense that it does not change when one changes the gauge function $\Lambda$ , then we have

$$\frac{\delta \mathcal{O}}{\delta \Lambda} = 0 . \tag{32}$$

From Eq. (31) one then sees that such a functional also does not change in correspondence to a change $\delta a_\mu$ of $a_\mu$ .

We proceed now to prove our basic equation (31) by showing that the functional differential equations satisfied by $Z$ remain invariant if one performs simultaneous changes of $a_\mu$ and of $Z$ . An alternative proof proceeds directly from the explicit expression for $Z$ and is sketched in Appendix A. The invariance of Eqs. (24) and (25) is trivial, since they do not contain $J_\mu$ or $\Lambda$ . The invariance of Eq. (26) is also easily verified. To check the invariance of Eq. (23) one needs a simple identity satisfied by any solution of the functional equations. If one applies $-\frac{1}{i}\frac{\delta}{\delta\eta}$ to Eq. (24) and $-\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}$ to Eq. (25) and subtracts, one obtains[10]

$$\{ \partial^\lambda ( - \frac{1}{i} \frac{\delta}{\delta\eta} \gamma_\lambda \frac{1}{i} \frac{\delta}{\delta\bar{\eta}} ) - \eta \frac{1}{i} \frac{\delta}{\delta\eta} + \bar{\eta} \frac{1}{i} \frac{\delta}{\delta\bar{\eta}} \} Z = 0 . \tag{33}$$

Therefore, from Eq. (22), the terms containing $a_\mu$ and $J_\mu$ in Eq. (23) can be written as

$$- i\, a_\mu \frac{\delta Z}{\delta\Lambda} - J_\mu Z . \tag{34}$$

The invariance of Eq. (23) is now easily verified.

The basic transformation formula (31) allows one, at least in principle, to transform all propagators from one gauge $a_\mu$ to another. The transformation is particularly simple in the case $\eta = \bar{\eta} = 0$ . Remembering Eq. (22), one has, in this case,

$$\delta L = \int \partial^\rho J_\rho \, \delta a_\mu \frac{\delta}{\delta J_\mu} L \quad , \tag{35}$$

where we have defined

$$L[J_\mu] = Z[0, 0, J_\mu] \, . \tag{36}$$

One should observe here that we have

$$[\partial^\rho J_\rho, \ \delta a_\mu \frac{\delta}{\delta J_\mu}] = 0 \tag{37}$$

as a consequence of Eq. (30). Therefore the operation to be performed on L is a pure substitution,

$$L'[J_\mu] = L[J_\mu'] \, , \tag{38}$$

with

$$J_\mu' = J_\mu + \delta a_\mu \partial^\lambda J_\lambda \, . \tag{39}$$

We have indicated with a prime the functional in the new gauge. The transformation (38), (39) is correct for finite changes $\delta a_\mu$ also. The functionals in the two gauges coincide when Eq. (19) is satisfied.

The change induced in

$$\mathcal{A}_\mu = \frac{1}{L} \frac{1}{i} \frac{\delta}{\delta J^\mu} L \tag{40}$$

is a simple gauge transformation. Indicating only the dependence on $J_\mu$, one has

$$\mathcal{A}_\mu'[J_\rho] = \mathcal{A}_\mu[J_\rho'] + \partial_\mu \delta a^\lambda \mathcal{A}_\lambda[J_\rho'] \, . \tag{41}$$

Similarly one obtains, for the exact photon propagator

$$\mathcal{G}_{\mu\nu} = \frac{\delta \mathcal{A}_\nu}{\delta J^\mu} = \frac{\delta \mathcal{A}_\mu}{\delta J^\nu} , \tag{42}$$

the relation

$$\mathcal{G}'_{\mu\nu}[J] = (\delta_\mu{}^\lambda + \partial_\mu \delta a^\lambda)(\delta_\nu{}^\rho + \partial_\nu \delta a^\rho) \mathcal{G}_{\lambda\rho}[J'] . \tag{43}$$

The relations (41) and (43) are of course also valid for finite gauge transformations. An obvious simplification occurs when Eq. (19) is satisfied.

When differentiations with respect to $\eta$ or $\overline{\eta}$ occur, it does not seem possible to obtain formulas of simplicity comparable to the above. Thus, for the exact electron propagator,

$$G(x, y) = \frac{-i}{Z} \frac{\delta^2 Z}{\delta \eta(y) \, \delta \overline{\eta}(x)} , \tag{44}$$

one obtains from Eqs. (21) or (22), after setting $\eta = \overline{\eta} = 0$ ,

$$L_\Lambda G_\Lambda(x, y) = \exp[ie(\Lambda(x) - \Lambda(y) + i \int J_\mu \partial^\mu \Lambda] L_0 G_0(x, y) \tag{45}$$

or

$$i \frac{\delta}{\delta\Lambda(z)} [ L G(x, y)] = [ \partial^\mu J_\mu(z) - e \delta(x - z) + e \delta(y - z)] L G(x, y) . \tag{46}$$

Finally, from Eq. (31), we obtain

$$L' G'[x, y; J_\mu(z)] = L G[x, y; J_\mu'(z) - \delta a_\mu(e \delta(x - z) - e \delta(y - z) ) ] , \tag{47}$$

where $J_\mu{}'$ is given by Eq. (39). This formula, which was given first by Schwinger and Johnson,[11] shows clearly what an intricate connection exists between the electron propagators in two different gauges of the type considered here. In particular, set $J_\mu = 0$; one sees that to obtain $G'$ it is necessary to know $G$ for nonvanishing values of the current equal to

$$\tilde{J}_\mu(z) = - e \, \delta \, a_\mu(\delta(x - z) - \delta(y - z)) \, . \tag{48}$$

Notice however that, because of Eq. (30), one has

$$\partial^\mu \tilde{J}_\mu(z) = 0 \, . \tag{49}$$

## 4. TRANSITION TO MORE GENERAL GAUGES

In the preceding section, we have considered the generating functional and the propagators in the various gauges specified by different choices of $\Lambda$ and of $a_\mu$ . More general types of gauges can also be considered. Of particular interest are changes of the basic functional given by

$$\delta Z = \frac{i}{2} \int \int \frac{\delta}{\delta\Lambda} (\delta M) \frac{\delta}{\delta\Lambda} Z \quad , \tag{50}$$

where $\delta M(x - y)$ is an arbitrary infinitesimal function symmetric in $x - y$. The generating functional can be considered now as dependent upon a new (symmetric) function $M(x - y)$ in such a way that the infinitesimal change $\delta M$ induces in $Z$ the change given by Eq. (50). The explicit expression for $Z$ as a function of $\Lambda$ , $a_\mu$ , and $M$ is given in Appendix A.

Clearly a functional $\mathcal{O}$ which is invariant under the original c-number gauge transformations (17), and which therefore satisfies Eq. (32), will also not be affected by the change (50).

We can easily deduce the effect of the change (50) on the first few propagators. First, setting $\eta = \bar{\eta} = 0$, we see from Eq. (22) that

$$\delta L = -\frac{i}{2} \int \int \partial^\mu J_\mu (\delta M) \partial^\rho J_\rho L \, , \tag{51}$$

or, in finite form

$$L'[J_\mu] = \exp \{ -\frac{i}{2} \int \int \partial^\mu J_\mu (\delta M) \partial^\rho J_\rho \} L \, . \tag{52}$$

From this it follows that

$$\mathcal{A}'_\mu[J] = \mathcal{A}_\mu[J] + \partial_\mu \int (\delta M) \partial^\rho J_\rho \, . \tag{53}$$

and

$$\mathcal{G}'_{\mu\nu} = \mathcal{G}_{\mu\nu} + \partial_\mu \partial_\nu (\delta M) \quad . \tag{54}$$

Similarly, for the electron propagator, we obtain

$$\delta(L\,G) = \frac{i}{2} \int \int \frac{\delta}{\delta\Lambda}(\delta M) \frac{\delta}{\delta\Lambda}(L\,G) \ , \tag{55}$$

and, using Eqs. (46) and (52), the result in finite form,

$$G'(x, y; J) = \exp\{ ie^2 (\delta M(x - y) - \delta M(0)) \cdot$$

$$+ ie \int (\delta M(x - z) - \delta M(y - z)) \partial^\rho J_\rho(z) dz \} \cdot G(x, y; J) \ . \tag{56}$$

For $J = 0$, of course, we have

$$G'(x, y; 0) = \exp\{ ie^2 (\delta M(x - y) - \delta M(0)) \} G(x, y; 0) \ . \tag{57}$$

Finally, Eq. (56) can be used to obtain the change in the propagator

$$C_\mu(x, y; z) = \frac{\delta}{\delta J^\mu(z)} G(x, y; J) \ , \tag{58}$$

which is closely related to the vertex part. Setting $J_\mu = 0$ after the differentiation, one has

$$C_\mu'(x, y; z) = \exp\{ ie^2 (\delta M(x - y) - \delta M(0)) \}$$

$$\times [C_\mu(x, y; z) - ie \frac{\partial}{\partial z^\mu} (\delta M(x - z) - \delta M(y - z))] \ . \tag{59}$$

As seen in Eq. (54), the gauge transformations considered in this section permit us to operate the transition from what we have called in the INTRODUCTION the Landau form of the photon propagator to the Feynman and the Yennie forms. To achieve this, one has only to chose

$$\delta M = \gamma(-\partial^2 - i\epsilon)^{-2}\delta(x-y) \qquad (60)$$

with a suitable constant $\gamma$. Since the choice (60) for $\delta M$ gives rise to a rather singular function, a regularization procedure is necessary before evaluating consequences of the gauge transformation. This has been done by Johnson and Zumino.[6] From the transformation formula, they have deduced information about the infrared structure of the electron propagator.

If the function $\delta M$ has a reasonable Fourier transform and vanishes at large distances in momentum space, one can use Eq. (57) to give the change induced in the wave-function renormalization constant $Z_2$ by going from one relativistic gauge to another. It is sufficient to remember that, in a relativistic gauge, $Z_2$ can be defined from

$$G(x-y) \approx Z_2\, G_m(x-y) \qquad (61)$$

for large coordinate separation. Here $G_m$ is the zero-order, Feynman propagator for a Dirac particle of mass $m$. Comparing with Eq. (57), we obtain

$$Z_2' = \exp\{-e^2\,\delta M(0)\}Z_2\,. \qquad (62)$$

In conclusion the author wishes to thank Dr. Kenneth Johnson for many illuminating discussions on the topics treated in this note.

# APPENDICES

## Appendix A: Explicit Form of the Generating Functional

The physically interesting solution of the functional equations (23) to (26) can be obtained by following methods that were developed first without giving particular consideration to questions of gauge invariance.[2] We shall exhibit here the solution, so as to see its explicit dependence on the gauge.

Consider the electron propagator in an external field $B_\mu$ ,

$$\tilde{G}[B_\mu ; x, y] = \{\gamma^\mu(\partial_\mu - ie B_\mu) + m - i\epsilon\}^{-1} \delta(x - y). \tag{A1}$$

The vacuum polarization (closed loops) due to the external field can be expressed by the functional

$$F[B_\mu] = \exp \{ - \mathrm{Tr} \log( \tilde{G}[B] (\tilde{G}[0])^{-1} ) \} , \tag{A2}$$

where we have used an obvious notation of multiplication for integral kernels, and the symbol Tr means the trace taken with respect to space time as well as spinor indices. Notice that

$$\tilde{G}[B_\mu + \partial_\mu \Lambda; x, y] = \exp[ie(\Lambda(x) - \Lambda(y))]\tilde{G}[B_\mu ; x, y] , \tag{A3}$$

and therefore

$$F[\bar{B}_\mu + \partial_\mu \Lambda] = F[B_\mu] . \tag{A4}$$

It can also be verified, by direct evaluation, that

$$\left(\frac{\delta}{\delta B_\mu} - e \frac{1}{i} \frac{\delta}{\delta\eta} \gamma^\mu \frac{1}{i} \frac{\delta}{\delta\bar{\eta}}\right) \left( \exp \{ i \bar{\eta} \tilde{G}[B_\mu] \eta \} F[B_\mu] \right) = 0 \tag{A5}$$

The generating functional considered in the text is now given, in its dependence upon $\Lambda$, $a_\mu$, and $M$, by

$$Z[\eta, \bar{\eta}, J_\mu] = \exp \{ i \bar{\eta} \tilde{G}[ \frac{1}{i} \frac{\delta}{\delta J_\mu} ] \eta \} F[ \frac{1}{i} \frac{\delta}{\delta J_\mu} ]$$

$$\times \exp \{ \frac{i}{2} (J_\mu + a_\mu \partial^\rho J_\rho) D_c (J^\mu + a^\mu \partial^\rho J_\rho)$$

$$- i \partial^\rho J_\rho \Lambda - \frac{i}{2} \partial^\rho J_\rho M \partial^\lambda J_\lambda \} , \qquad (A6)$$

where $D_c$ is the Feynman function,

$$D_c(x - y) = (-\partial^2 - i\epsilon)^{-1} \delta(x - y) . \qquad (A7)$$

The expression (A6) satisfies Eqs. (23) to (25). It does not satisfy Eq. (26) unless one sets $M = 0$. On the other hand, the dependence on $M$ in Eq. (A6) clearly agrees with Eq. (50).

We indicate now briefly how one can verify that the explicit formula given actually satisfies the functional equations, without however going into the question of the boundary conditions that ensure the uniqueness of the solution. The only equation that is not trivially satisfied is Eq. (23). If we operate with

$$\partial^\lambda (\partial_\mu \frac{1}{i} \frac{\delta}{\delta J^\lambda} - \partial_\lambda \frac{1}{i} \frac{\delta}{\delta J^\mu} ) \qquad (A8)$$

on the last exponential in Eq. (A6) we obtain, after simplifications involving the use of Eq. (28), a factor

$$(J_\mu + a_\mu \partial^\rho J_\rho) . \qquad (A9)$$

The terms containing $\Lambda$ and $M$ give no contribution. Now one must pass the factor to the left of the terms in Eq. (A6) which contain the functional derivatives $\frac{1}{i}\frac{\delta}{\delta J_\mu}$. If one uses Eq. (A5), one sees that the net effect is to replace $J_\mu$ with

$$J_\mu - ie\, \frac{1}{i}\frac{\delta}{\delta \eta}\, \gamma_\mu\, \frac{1}{i}\frac{\delta}{\delta \bar{\eta}}\ , \tag{A10}$$

so that Eq. (23) obtains.

Using the explicit form (A6), one can prove again the formulas given in the text for the various changes of gauge. The basic tools are now the relations

$$F\left[\frac{1}{i}\frac{\delta}{\delta J_\mu}\right]\partial^\rho J_\rho = \partial^\rho J_\rho\, F\left[\frac{1}{i}\frac{\delta}{\delta J_\mu}\right] \tag{A11}$$

and

$$\tilde{G}\left[\frac{1}{i}\frac{\delta}{\delta J_\mu}\,;\, x,\, y\right]\partial^\rho J_\rho(z) = (\partial^\rho J_\rho(z) - e\,\delta(x - z) + e\,\delta(y - z)\,)\,\tilde{G}\left[\frac{1}{i}\frac{\delta}{\delta J_\mu}\,;\, x,\, y\right] \tag{A12}$$

which are immediate consequences of (A3) and (A4). The expression for $L[J_\mu]$ is obtained directly from Eq. (A6) by setting $\eta = \bar{\eta} = 0$. The expression for the electron propagator (44) is then given by the equation

$$L[J_\mu]\, G[x,\, y;\, J_\mu] = \tilde{G}[\,\frac{1}{i}\frac{\delta}{\delta J_\mu}\,;\, x,\, y\,]\, L[J_\mu]\ . \tag{A13}$$

In this form the evaluation of changes induced by a change in the function $M$ becomes particularly simple.

## Appendix B: Covariance of the Operator Formalism

The covariance of the equations for the Heisenberg operators in the Coulomb gauge can be proven by exhibiting the ten fundamental generators of infinitesimal Lorentz transformations $P_\mu$ and $M_{\mu\nu}$ and by verifying that they satisfy the correct structure relations. Since the covariance of the operator equations under space-time translations and space rotations is obvious, we shall restrict ourselves to a very brief discussion for the case of actual Lorentz transformations.

The corresponding generators are given by

$$M_{or} = x_o P_r + \int x_r T_{oo} \, d\underline{x} \tag{B1}$$

where $T_{oo}$ is the component of the energy-momentum tensor given in Eq. (2). The form (B1) is obtained by analogy from the classical theory. The change induced by an infinitesimal Lorentz transformation in any operator $\xi$ is given by

$$\delta \xi = i \left[ \frac{1}{2} \epsilon_{\mu\nu} M^{\mu\nu}, \xi \right] , \tag{B2}$$

where the antisymmetric infinitesimal tensor $\epsilon_{\mu\nu}$ characterizes the Lorentz transformation in question.

The changes induced in the basic operators $\underline{A}$ and $\psi$ are easily obtained. It can be shown from the commutation relations (3) and (4) that

$$i[M_{or}, A_\ell] = -(x_o \partial_r - x_r \partial_o)A_\ell + \delta_{r\ell} A_o + \partial_\ell B_r \tag{B3}$$

and

$$i[M_{or}, \psi] = -(x_o \partial_r - x_r \partial_o)\psi - \frac{1}{2} \alpha_r \psi + \frac{ie}{2} \{ \psi, B_r \} , \tag{B4}$$

where

$$B_r = \nabla_s \nabla^{-2}(x_r E_s) - x_r \nabla_s \nabla^{-2} E_s \quad . \tag{B5}$$

Equations (B3) and (B4) show that, in going to a new Lorentz frame, not only do $A_\mu$ and $\psi$ transform like a four-vector and a spinor, respectively, but that they also undergo an operator gauge transformation which reestablishes the Coulomb gauge in the new Lorentz frame. It is easy to verify that the operator gauge transformation leaves invariant symmetrized expressions like (8), so that the current and charge densities, for instance, transform like a four-vector. It is worth noticing that the symmetrization, necessary for charge conjugation invariance, also appears necessary to ensure the relativistic covariance of the theory. One can give a more convenient form to the gauge operator B, in which the transverse and the longitudinal parts of the electric field are separated. We give only the result

$$B_r = \nabla^{-2} E_r^{tr} - \frac{1}{2} [ x_r \nabla^{-2} \rho - \nabla^{-2}(x_r \rho)] \quad . \tag{B6}$$

One can now proceed to verify the structure relations, the expression for the space components $M_{rs}$ and for $P_r$ being well known. This will not be done here. A simpler check on the covariance of the theory is the direct substitution of the transformed quantities obtained from Eqs. (B3) and (B4) into the differential equations. Obviously only the invariance of the Dirac equation under the gauge part of Eq. (B4) requires detailed examination, since the Lorentz covariance of the unquantized theory is well known. Both procedures result in proving the covariance.[11]

FOOTNOTES

1. The Coulomb gauge has been used recently by Schwinger and Johnson (Kenneth Johnson, Massachusetts Institute of Technology, private communication, 1959). They have arrived independently at several of the results described in the present note.
2. J. Schwinger, Proc. Natl. Acad. Sci. U. S. 37, 452 (1951). For a more detailed discussion see, e.g., K. Symanzik, Z. Naturforsch. 9a, 809 (1954) and E. S. Fradkin, Doklady Akad. Nauk. S.S.S.R. 98, 47 (1954) and 100, 897 (1955).
3. L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Doklady Akad. Nauk. S.S.S.R. 95, 773 (1954).
4. The M transformation has been given first by L. D. Landau and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) 29, 89 (1955); English translation in Soviet Physics JETP 2, 69 (1956). Their derivation, however, is based on an operator gauge transformation, the validity of which appears rather questionable.
5. H. M. Fried and D. R. Yennie, Phys. Rev. 112, 1391 (1958).
6. K. Johnson and B. Zumino, The Gauge Dependence of the Wave-Function Renormalization Constant, UCRL-8866, August 1959.
7. The transverse part of a vector $\underline{D}$ is of course defined by

$$\underline{D}^{tr} = \underline{D} - \nabla \nabla^{-2} \operatorname{div} \underline{D} .$$

8. In order to avoid formal difficulties in connection with the use of the anticommuting spinor sources $\eta$ and $\bar{\eta}$, it is best not to interpret the bar as a relation of hermitian conjugation between $\eta$ and $\bar{\eta}$. Rather, one should consider $\bar{\eta}(x)$ and $\eta(x)$ as independent anticommuting symbols

8. (Cont.)

and carry out all the formal operations from this point of view. In the final expression one always sets $\eta = \bar{\eta} = 0$ , or, more correctly, one takes that part of the expression which is independent of $\eta$ and of $\bar{\eta}$ .

9. A covariant expression for $a_\mu$ corresponding to the Coulomb gauge in the Lorentz frame characterized by the unit time-like vector $n_\mu$ is

$$a_\mu = - \frac{\partial_\mu + n_\mu(n\cdot\partial)}{\partial^2 + (n\cdot\partial)^2} .$$

10. For $\eta = \bar{\eta} = 0$, Eq. (33) gives the conservation of the vacuum currents.

11. It has been pointed out by Schwinger that the analogous covariance test fails if an anomalous Pauli moment is introduced into the theory. Glashow and Gilbert have shown how the covariance of the theory can be saved by the further introduction of a term describing the self-interaction of the magnetic-moment density. The author would like to thank Dr. Glashow for an illuminating correspondence on this question of covariance.