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Improving Skills of Addition and Subtraction Involving Negative Numbers Based on Cognitive Task Analysis and Assessment of Mental Representations of Negative Numbers: A Case Study of a Seventh-Grade Student

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Abstract

The aim of this paper is to show how we can use cognitive task analysis and the SNARC effect (e.g., Fisher, 2003) to refine students' skills of addition and subtraction including negative numbers. We taught these problems to a seventh-grade student. She had frequently made errors in solving the problem. Several possible ways of calculation were identified based on a cognitive task analysis. Some were pictorial solutions involving a mature mental number line; others were purely algebraic solutions not requiring direct use of a mental number line. It also turned out that her errors were due to bug rules and the lack of a critical production when executing a purely algebraic solution. The direction of the SNARC effect suggested that the student had an immature mental number line, containing only positive entries. Considering the bug rules and her immature mental representation of numbers, we focused on refining her solution procedure without emphasizing the use of a mental number line. It worked very well. This case suggests that cognitive task analysis and assessment of the mental representation of numbers can provide a foundation for effective teaching of integer addition and subtraction.

Introduction

The authors run a social and academic activity under the supervision of the fourth author of this paper, called the Saturday classroom, in which we teach school subjects and play with students who suffer from a mild developmental disorder, such as autism, AD/HD, or a learning disorder. The goal of our research group is to understand these disorders from the viewpoints of psychology and brain science. The Saturday classroom gives us a chance to observe these disorders and to understand the difficulties the students are experiencing. Each student comes to our department twice a month, on Saturday.

From the beginning of the 2004 school year (from April 2004 to March 2005), the first three authors taught a 7th grade-student (initials AY) addition and subtraction incorporating negative numbers. In Japan, this calculation is the first topic of junior-high-school mathematics (7th to 9th grade). A doctor has diagnosed AY with a learning disorder, stemming from a car accident when she was an older preschooler. Her brain was severely damaged, and the difficulty she has in learning calculations involving negative integers may be due to her disorder. However, we believe that the errors she made can be observed in normal students, and what was done in AY's case is also useful for normal students. Moreover, the aim of this paper is to discuss usefulness of the information obtained from cognitive task analysis and the assessment of the mental representation of numbers in teaching calculations on integer numbers. For these reasons, hereafter we will not refer to her disorder.

It is surmised that the mental number line should be used as a critical base of calculation on real numbers $\{x \mid -\infty < x < +\infty\}$. The mental number line is a language-independent representation of number magnitude (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999). All the textbooks used in Japan, as far as we know, first teach this calculation on integer numbers as a movement on the (written) number line. For example, a textbook teaches $2-6$ as moving 6 to the negative direction (left) from the point of 2 along the number line. AY was initially taught to solve this kind of problem in the same way as the textbook does. When students reach a certain level of expertise, they no longer write a number line on paper. However, it is thought that an internal representation of the number line still plays a role in executing calculation. Several studies have shown that the mental number line is a critical source of mathematical thinking, including approximation (Dehaene, et al., 1999), elementary addition and subtraction (Griffin, Case, & Siegler, 1994), and solving word problems (Lewis, 1989; Paige & Simon, 1966; Terao, Koedinger, Sohn, Qin, Anderson, & Carter, 2004).

AY had often made errors in executing the integer calculation. For example, her answer to $2-5$ was 3. Using a written number line, she was taught that the answer to this problem should be a negative number (actually -3) because from the point of 2 we have to go over zero to reach the point of the answer on the number line. She seemed to understand the calculation and was able to solve similar problems correctly. But in the next Saturday classroom (two weeks later) she made the same errors again. Given her relapse, it became imperative to understand when and why these errors occurred, and to consider whether how she was being taught was in order.

Table 1: A LISP Function for a Pictorial Solution.

```
(defun pn_count (oper num1 num2)
   (cond ((< num2 0)
        (change_addend 'pn_count oper num1 num2))
            ;; If the second number is negative,
            ;; it will be changed to positive.
     ((equal oper '+) (count_up num1 num2))
     ((equal oper '-) (count_down num1 num2))))
```

The purpose of this paper is to describe the case of AY which illustrates how a cognitive task analysis and the assessment of the mental number line can refine and correct the skills of addition and subtraction involving negative numbers. A cognitive task analysis was done to understand the strategies she might employ in solving the addition-and-subtraction problems she had trouble with. Observing AY while she solved a series of problems, the pattern of correct and incorrect answers was traced in terms of a set of production rules. Further investigation into AY's mental number line using the SNARC effect (e.g., Fisher, 2003) yielded a targeted approach to correcting AY's erroneous strategies

Cognitive Task Analysis

Cognitive task analysis of a mathematical problem reveals how the problem could be solved. From the analysis, one can know what knowledge or representation is required to execute a solution and how the solution is organized. Once a solution is described in detail, it provides clues to understand where and why a student failed to solve the problem. For example, bug rules can be identified which lead the student to an incorrect answer (e.g., Koedinger & Terao, 2002).

Although many studies have been done as to how elementary addition and subtraction problems are solved (e.g., Carpenter, Noser, & Romberg, 1982), little is known about calculations incorporating negative numbers. We began by searching the major textbooks used in Japan for possible solution methods to the addition and subtraction problems which incorporate negative numbers.

Among several possible solutions that were found, two of them are described below. The first solution is an example of a "pictorial" solution, which in this paper means that the solution requires explicit use of a written or mental number line. This was the solution initially taught to AY during the early months of the school year. The second solution is an example of an algebraic solution, which does not need to use a written or mental number line. As it turned out, it was a buggy version of this method that AY was attempting to employ.

The Pictorial Solution

This solution requires the solver to use a written or mental number line because the calculation is equivalent to movement on a (mental) number line. This solution is an extension of a solution taught in the elementary arithmetic, where adding a positive number is equivalent to going right on the mental number line and subtracting it is going left. The elementary-math version of this solution has been called the counting-on strategy.

In this solution, the student initially looks at the second number (i.e., b in $a \pm b$) of the expression. If the number is a negative number, the student then tries to change it to a positive number. This can be done by changing the problem $a \pm b$ to $a \mp (-b)$. For example, if the problem is $2+(-5)$, the student will change it to $2-5$. If the second number is a positive number, the student does nothing at this stage. Then the student moves right (addition) or left (subtraction) along the number line from the point of the first number, and reaches the point of the answer.

To ensure that we grasp this solution in detail and there is no hole in our description of the solution, we implemented this solution with a set of LISP functions. Table 1 shows the main function for executing this solution. The subfunction count_up counts up num2 from num1; the subfunction count_down works in a similar way.

A more sophisticated version of this solution does not use a written number line. Students no longer do counting. This sophisticated solution totally depends upon a mental number line and makes use of a part-part-whole relation on a mental number line. For example, consider the problem $2-5$. First you move from 0 toward the point of the first addend 2 on your mental number lien. Then you imagine yourself to go 5 to the left. Now the whole length is 5, the length of the right part is 2, and the length of the left part is unknown. If you use a part-part-whole relation, you can find the unknown length is 3. The answer to this problem is -3, not +3, because you past zero when moving from the 2, a point in the positive side. We also implemented this more sophisticated solution with a set of LISP functions (not presented in this paper).

The Algebraic Solution

This solution is taught in all the textbooks we searched through. The description of the textbooks can be easily translated into a set of production rules shown in Table 2. First of all, you have to confirm the operator is addition. If not, you need to change the problem to the equivalent addition problem. The production P0 in Table 2 changes the problem from $a-b$ into $a+(-b)$. Then you look whether the signs of the two addends are the same or different. If the signs are in common, you add the absolute value of the second addend to the absolute value of the first addend (see production P1 in Table 2), and the polarity of the answer (positive or negative) is consistent with the common sign of the two addends (production P4). If the signs are different, you compare the absolute value of the first addend with the absolute value of the second addend, subtract the smaller absolute value from the larger absolute value (production P2), and the polarity of the answer is consistent with the sign of the addend of larger absolute value (production P5). This solution was also implemented with a set of LISP function (not presented in this paper).

Table 2: Production Rules for an Algebraic Solution

P0: Change-subtraction-to-addition If the operator is subtraction then change the operator to addition and change the sign of the second addend.
P1: Do-absolute-addition If the operator is addition and the two addends have the same sign and no absolute-value addition has been executed then add the absolute value of the each addend together.
P2: Do-absolute-subtraction If the operator is addition ;; A bug version of this production ;; lacks this part of the condition. and the two addends have different signs and no absolute-value subtraction has been executed then subtract the smaller absolute value from the larger absolute value of the two digits.
P3: Do-absolute-subtraction-same-sign If the operator is subtraction and the two digits have the same sign and no absolute-value subtraction has been made then subtract the smaller absolute value from the larger absolute value of the two digits.
P4: Attach-sign-to-common-sign-problem If the two digits have the same sign and an absolute-value calculation has been executed then attach the common sign to the result of the absolute calculation.
P5: Attach-sign-to-different-sign-problem If the two digits have different signs and an absolute-value calculation has been executed then attach the sign of the digit of larger absolute value to the result of the absolute calculation.

This algebraic solution does not require explicit use of a written or mental number line, although we can justify this solution by considering a pictorial solution similar to the sophisticated pictorial solution described in the previous subsection.

Identifying Solutions and Bug Rules

This section describes how we identified the details of the erroneous strategies that AY used. Observing AY while she solved a series of test questions, the pattern of correct and incorrect answers was noted as well as any intermediate results. These patterns were traced in terms of a set of production rules which were slightly different from the ones described in the previous section (Table 2). This production-rule trace was used to construct a second test, and AY's predicted performance was compared to her actual performance on the second test.

Production Trace of AY's Solution

One day, AY was given a test of eight addition problems and 12 subtraction problems. Of 12 subtraction problems, four problems were very simple ones in which the two numbers were positive and the answer was also positive, like (+8) - (+6). The other eight problems involved negative numbers like (+3) - (+6) or (-6) - (+3). The second number was always a positive number. AY made no error in the addition problems and the simple subtraction problems. Of eight subtraction problems involving negative numbers, AY gave wrong answers to seven problems.

Looking at the written calculation, it was suspected that she tried to execute the algebraic solution described in the previous section but failed in solving the problem because of lack of a critical production and use of bug rules which lack a critical test in the condition part. She showed a wrong solution like (+3) - (+6) = + (6 - 3) = +3 or (-6) - (+3) = - (6 - 3) = -3. It was also apparent that she did not have the production P0 in Table 2 because she did not change a subtraction problem to the equivalent addition problem. It was also suspected that she had a buggy version of the production P2 that lacks the test of operator (see Table 2). In order to make the set of AY's productions cover all types of addition and subtraction problems, one needs to consider productions for doing subtraction between two numbers of common sign. It was assumed that AY had the production P3 in Table 2 for this type of problem.

Putting all the productions together, we can describe the AY's strategy as follows:

> If the problem is an addition problem
> and the two addends have the same sign
> then add the absolute value of the each addend together
> and attach the common sign.
> Otherwise subtract the smaller absolute value from
> the larger absolute value
> and attach the sign of the digit of larger absolute value.

This strategy is almost the same as the correct algebraic solution described in the previous section, but it produces an incorrect answer to a certain types of problems. From the trace of AY's solution, we derived several predictions on her performance and tested these predictions with 56 problems.

Predictions of Performance

The production trace of AY's solution allows us to derive several predictions on her performance as follows:

(1) No error will occur for addition problems. Even if AY does not have the production P0 and does have a bug version of the production P2, this does not lead to any wrong answers to addition problems.

(2) For the problems which subtract a positive digit from another positive digit, an error was produced by the production P3 and the lack of P0 only if the second digit is larger than the first digit. In the case of a larger first digit, this problem is a simple subtraction problem and no error will occur.

(3) For the problems which do a subtraction between numbers of different signs, AY will always make an

error due to the buggy version of production P2 and the lack of P0.

(4) For the problems which subtract a negative number from another negative number, an error was produced by the production P3 and the lack of P0 only if the absolute value of the second digit is larger than the absolute value of the first digit. The production P3 is a bug rule. Nonetheless, if the absolute value of the first digit is larger than the absolute value of the second digit, the produced answer is consistent with the correct answer.

Test of Predictions

To test these predictions, we gave AY a wide variety of problems. Table 3 shows 28 pairs of the first and second digit of the problems (see the second and third column of the table). An addition problem and a subtraction problem were generated from each pair to yield 56 problems. The number pairs in Table 3 were also used to assess AY's mental representation of negative numbers as described in the next section. The names of the conditions in Table 3 are of this assessment.

These problems were presented on a screen one problem at a time. The operator (+ or -) was presented at the center of the screen for 1.5 seconds followed by two digits appeared to the left and right side of the operator. We asked AY to solve the problem in mind accurately but as quickly as possible and push a button when she had an answer. Then a pair of two numbers was presented on the screen after a short interval (1.5 seconds). She was asked to select the correct answer from the two numbers by pushing either of the two buttons. Six seconds after the button press, the next problem appeared.

Overall, the performance of AY supported our hypotheses, meaning that we successfully understood her strategy of calculation. The forth and fifth column of Table 3 show all the wrong answers AY gave. The blank means that she gave the correct answer to that problem.

(1) AY actually made no error for addition problems, except for an error (i.e., 9 + 4 = 5) which may be a careless mistake.

(2) Of four problems which subtract a larger positive digit from a smaller positive digit, AY gave wrong answers to three problems. The produced errors were consistent with the ones which were predicted from the production P3 and the lack of P0. She also gave a wrong answer, 7, to the problem of 0 - 7. This error can be explained by the production P3 and the lack of production P0 if she considered 0 as a positive number.

(3) AY gave wrong answers to five of eight problems in which she had to do subtraction between numbers of different signs. These errors can be explained by executing the buggy version of production P2 and the lack of the production P0.

(4) Differing from our prediction, AY gave the correct answers to all the four problems which subtract a negative number from another negative number and in which the absolute value of the second digit is larger than the absolute value of the first digit. However, this could be due to the fact that we failed to include the predicted wrong answer as one of the two candidates appearing on the screen. The correct answer and the predicted wrong answer have the same absolute value differing only in polarity. Thus, AY might have had the incorrect answer (e.g., (-4) - (-9) = - (9 - 4) = -5), but chose the opposite polarity when presented with the two candidates (i.e., selecting 5 in the previous example). It is worth mentioning that about one month before this test, she solved a similar problem, (-3) - (-8), and gave the predicted answer, -5.

Table 3: Number pairs used for calculation and comparison

Condition	1st digit	2nd digit	Addition	Subtraction
A+N+	4	9		5
	2	8		
	0	7		7
	−2	6		4
	3	5		2
	1	4		3
	−1	3		2
A−N+	−9	−4		
	−8	−3		
	−7	−2		
	−6	−1		
	−5	0		
	−4	1		−3
	−3	2		−1
A+N−	−4	−9		
	−3	−8		
	−2	−7		
	−1	−6		
	0	−5		
	1	−4		−3
	2	−3		
A−N−	9	4	5	
	8	2		
	7	0		
	6	−2		
	5	3		
	4	1		
	3	−1		

Assessment of Mental Number Line

Besides knowing AY's calculation strategy in detail, knowing how AY represented numbers would also be useful in deciding what kind of corrective teaching should be employed with the student. For example, after knowing that a student tried to execute an algebraic solution but failed in solving the problem due to a lack of production and/or use of bug rules, should we focused on correcting it within the algebraic solution? Or we should emphasize use of a mental number line because it is a central concept for mathematical thinking? We think it depends on the nature of his or her mental number line. On one hand, if the student has a mature mental number line, a pictorial solution can be taught and the student should have no difficulty in understanding and using that solution. On the other hand, if his or her mental number line is immature, the teacher could either try to develop the number line first, or switch to another method that does not require a mature mental number line.

The SNARC Effect

To assess AY's mental number line, the authors made use of the SNARC effect (for spatial-numerical association of response codes). It has been shown that the mental number line represents spatial positional codes, with small numbers on the left side and larger numbers on the right. Any effects due to this property of the mental number line are called the SNARC effects. This effect has been replicated with various tasks (e.g., Fias, Brysbaert, Geypens, & d'Ydewalle, 1996; Fisher, 2001, 2003).

Fisher (2003) used the SNARC effect to examine whether negative numbers become associated with the left side of space as a result of experience with them. He presented fourteen students (age range: 20-38 years) with pairs of digits and asked them to press the button near the smaller (or larger) number. The response times in two critical conditions were compared: the A+N- condition and the A-N+ condition (see Table 3). The notation here indicates whether the left-right ordering of the digits' absolute magnitudes was congruent (A+) or incongruent (A-) with the left-right orientation of the mental number line, and whether the left-right ordering of the digits' numerical magnitudes was congruent (N+) or incongruent (N-) with the left-right orientation of the mental number line. It is predicted that the comparison speed will be faster in the A-N+ condition than in the A+N- condition if students have a mature mental number line extending to both left and right from zero. On the other hand, if students' mental number line is immature and contains only positive entries, the response time should be faster in the A+N- condition.

Fisher found that his participants have a mature mental number line, on which negative numbers are associated with the left side of the space, indicating a numerical continuum from $-\infty$ (left) to $+\infty$ (right).

Assessment

Using Fisher's experiment as a basis, the same set of digit pairs in the two critical conditions (the A+N- condition and the A-N+ condition) was used in the assessment of AY's mental number line. However, for the other two conditions a different set of number pairs was used instead. The reason for this change was that the same number pairs were used in the calculation task described in the previous section. Fisher kept the split between the digits in all pairs constant at 5 to avoid being contaminated by the distance effect (increasing comparison time with the distance between the two digits). If we had used exactly the same set of number pairs as used in Fisher's experiment, the answers to subtraction problems would have been always 5 or -5.

The procedure was almost the same as in the calculation task. Instead of executing the calculation, AY was asked to compare the magnitudes of two digits. In the first block, 28 pairs of numbers shown in Table 3 were presented, and AY selected the larger number from the two digits. In the second block, we asked her to select the smaller number. The response was made by pushing either of two buttons with her right had: the Left choice was her index finger and, and the Right choice was her middle finger. The total number of trials was 56.

Data from the two number pairs in the A-N+ condition were discarded because of the wrong response and a program error. Comparing the two critical conditions, the mean reaction time in the A+N- condition was 1115 ms; the mean reaction time in the A-N+ condition was 1275 ms. The difference between the two conditions was marginally significant by the paired t-test, $t(11) = 2.15, p = .055,$ in which the stimuli composed of the same digits (e.g., [-9,-4] and [-4, -9]) were paired. This difference reached a significant level by the Wilcoxon's test, $p = .05$.

From the assessment of AY's mental representation of numbers, it appeared that AY did not have a mature mental number line. The direction of the SNARC effect was just opposite to the one observed in the Fisher's (2003) study, implying that her mental number line was a numerical continuum from 0 (left) to $+\infty$ (right), not from $-\infty$ to $+\infty$. Since her mental number line was immature, it is reasonable that AY did not notice she gave a wrong answer to the problem in which movement on the mental number line clearly goes over zero from the positive side (e.g., 4 - 9 = 5). She cannot feel going over zero if her mental number line does not have the negative side.

Correcting AY's Calculation Strategy

Having determined the deficiencies in AY's calculation strategy and considering her immature mental number line, we decided to correct her existing calculation strategy without emphasizing the use of the mental number line. She was taught that the subtraction problem has to be changed to the equivalent addition problem and all the other manipulations are executed only for addition problems, thus teaching the production P0 and correcting errors in AY's production P2. It is worth mentioning that this kind of correction is the basis of intelligent tutoring systems: tracing students' solutions and fixing buggy rules.

This correction worked very well for AY, and she now gives the correct answer to the integer problems she would have previously missed. The success of this intervention is not temporary one. Six month after the intervention, we gave AY a retest of the 56 problems and she still gave the correct answers to all problems.

We assume that acquiring a mature mental number line is a prerequisite for teaching a pictorial solution for addition and subtraction problems involving negative numbers. AY gave a wrong answer to $2-5$, +3, even after we taught her that it is a stupid answer because we have to go over zero from the positive side to reach the correct answer along the number line. We thought that she was not able to imagine going over zero because of her immature mental number line. As mentioned above, if a student does not have a mature mental number line, the teacher could either try to develop the number line first, or switch to another method that does not require a mature mental number line. We chose the second option in AY's case. It never means that we despaired of developing her mental number line. We admit that a correct, mathematically full-fledged solution strategy for such addition and subtraction problems necessarily implies a good understanding of the number line. Our intention is first to give her the ability of giving the correct answer to any addition and subtraction problems and then to develop her mental number line gradually. If she can always give the correct answer, this must relieve her. If we tried to develop her number line first, it could take longer time, and it could then disquiet her because she has to spend a long time without being able to give the correct answer to such a problem. Instead, we decided to correct her wrong algebraic solution then try to develop her mental number line, which can be use to verify the corrected algebraic solution. We can find this kind of "first-can-do-then-think-why" teaching strategy in some textbooks used in Japan at any grade, and we think this way of teaching is sometimes useful.

Discussion

The case of AY suggests that cognitive task analysis and assessment of the mental number line provide useful information for deciding how to refine the student's calculation skills. The cognitive task analysis revealed possible solutions for addition and subtraction on integer numbers. Based on this analysis, AY's solution was traced and where and why she failed in solving problems was determined. Furthermore, making use of the SNARC effect, it turned out that AY's mental number line was immature. Given the above information, it was determined that AY would be most responsive to a correction that did not emphasize the use of the mental number line. AY responded well to this targeted teaching, and has retained the correction to date, which is far longer than previous attempts at teaching other (pictorial) methods.

The case of AY raises several questions for additional research. First, we need to determine a more comprehensive set of the kinds of errors that can be encountered in solving addition and subtraction problems involving negative numbers. The errors AY made should be only a small part of the set of all possible errors, and correcting the erroneous solutions was relatively easy in the case of AY. But we do not know if this is true for other types of errors. Second, we need to know a sufficient number of trials in a SNARC experiment. Fisher (2003) required his participants to go through more than 560 trials. In the case of AY only 56 trials were enough to yield a reliable SNARC effect, but more trials may be needed for another student. Third, it should be examined whether seventh-grade students have generally acquired a mature mental number line. We did not emphasize the use of the mental number line when we tried to correct AY's calculation strategy. However, if seventh-grade students generally have an immature mental number line but it is normally developed quickly soon after, it would be better to focus on training her mental number line.

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