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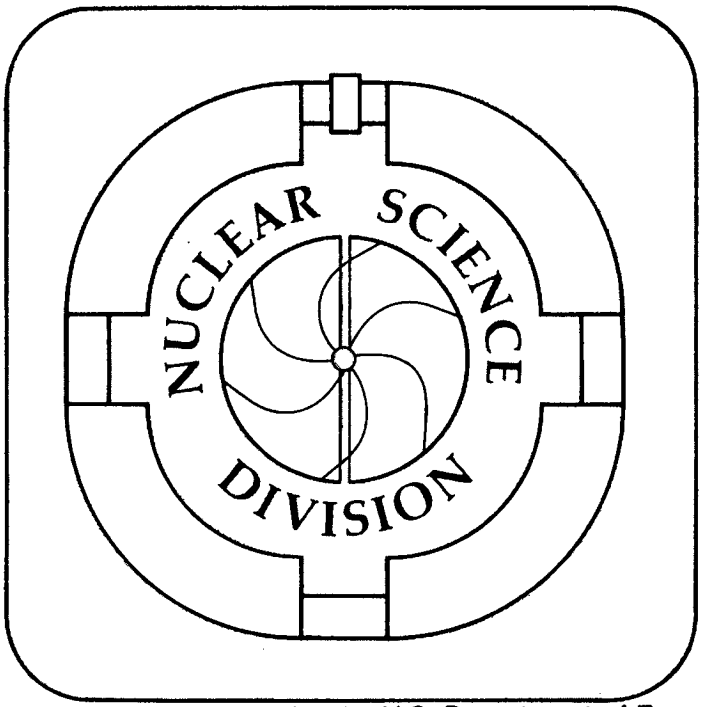




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# DAMPING OF THE WAVE PACKET MOTION IN A GENERAL TIME-DEPENDENT QUADRATIC FIELD

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ABSTRACT

We provide a framework for the study of a quantal time-dependent oscillator in the presence of a loss mechanism. Previous approaches to partial aspects of the problem are analyzed and cast into a unified global picture. The appearance of exact invariants associated with the damped motion of a gaussian wave packet is discussed. Several alternative descriptions of the situation are analyzed and it is shown that the proper generalization of the Hamiltonian formulation by Kostin is adequate for the case under consideration. Applications to a number of problems in which the mass of the oscillator is a given function of time are presented, including cases in which the mass becomes infinite.

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## I. INTRODUCTION

The problem of quantizing the damped motion of a particle in a quadratic field has received considerable attention. A good survey can be found in Ref. 1. These previous works usually deal with an oscillator with constant mass and stiffness placed in presence of a dissipative force $F = -\gamma\dot{X}$, $\gamma$ being also a constant. In addition, there exists a substantial body of work concerning the study of a classical, undamped harmonic oscillator with arbitrary time dependence in its parameters.[2,3,4] In those nonadiabatic oscillators, a problem of interest is the construction of invariants of the motion and this task has been successfully carried away by Lewis and Symon.[2,3]

The present paper aims at unifying these views in order to provide a treatment of a quantal oscillator in presence of a dissipation mechanism, in the most general situation in which the mass, the quadratic field and the damping are arbitrary functions of time. To achieve this goal, we will generalize the handling of the classical problem to the case in which the time-dependent oscillator is also damped (see Section II). We will see that it is possible to find exact invariants of the motion, in the spirit of Refs. 2,3. In Section III, we take advantage of standard quantization rules and derive equations of motion for both the first and second moments of the wave function. In Section IV, we introduce the proper generalization of already existing quantal descriptions of damping that do not possess a classical equivalent.[1] An analysis of the approaches presented in Sections III and IV allows us to choose one satisfactory

representation of the quantal problem, in the case in which the wave function happens to be a gaussian wave packet. This restriction is not a very dramatic one, since a number of applications can be derived on that assumption. Some typical examples will be presented and discussed in Section V.

## II. THE CLASSICAL PROBLEM

### II.1 The exact invariant for a damped oscillator.

In this chapter we will discuss the classical description of the motion of a damped oscillator. The firm basis for this study is, of course, the Newton equation:

$$\dot{P} = -\gamma(t)P - m(t)\Omega^2(t)X \,, \tag{2.1a}$$

$$\text{where } P = m(t)\dot{X}. \tag{2.1b}$$

In this equation, the friction parameter $\gamma$, the mass m and the frequency $\Omega$ may depend on time. The energy is not a constant in time; in such a case, it is a matter of great interest to build exact invariant quantities. The study of exact and adiabatic invariants for classical oscillators has received considerable attention from many authors (see, for example, papers from Lewis[2], Symon[3] and references quoted there).

The detailed treatment of the undamped oscillator by Symon may be generalized. Although his starting point is a very particular Hamiltonian representation, we will see that it is possible to find an exact invariant without any reference to a given Hamiltonian. We only need to observe that when $\gamma$, m, and $\Omega$ are constants, the solution of (2.1) is

$$X = w\, e^{-\gamma t/2} \cos\left(\sqrt{\Omega^2-\gamma^2/4}\; t + \phi_o\right) \,, \tag{2.2}$$

where w and $\phi_o$ are also constants. An ansatz for the general solution of (2.1), consistent with (2.2), could be

$$X = w(t)\, e^{-\int \frac{\gamma dt}{2}}\, e^{i\left(\int \Omega' dt + \phi_1(t)\right)}, \tag{2.3}$$

where $\Omega'(t) = \sqrt{\Omega^2(t) - \gamma^2(t)/4}$. (2.4)

The corresponding guess for the momentum would be

$$P = iz(t)\, e^{-\int \frac{\gamma dt}{2}}\, e^{i\left(\int \Omega' dt + \phi_2(t)\right)}, \tag{2.5}$$

Here w, z, $\phi_1$ and $\phi_2$ are real functions of time. Eq. (2.1) leads to equations of motion for w, z, $\phi_1$, $\phi_2$ and it can be seen that the quantity defined as

$$J = \frac{e^{\int \gamma dt}}{2} \left\{ \frac{X^2}{w^2} + \left[ wP - m\dot{X}\left(\dot{w} - \frac{\gamma}{2} w\right)\right]\right\}^2 \tag{2.6}$$

is an exact constant of motion. The details of the derivation are given in the Appendix. Notice that when $\gamma = 0$, expression (2.6) coincides with the one found by Symon.[3] We shall only quote here that $\overset{\circ}{w}$ satisfies the following non-linear second-order differential equation:

$$\ddot{w} + \dot{w}\frac{\dot{m}}{m} + w\left(\Omega'^2 - \frac{\dot{\gamma}}{2} - \frac{\gamma\dot{m}}{2m}\right) = \frac{1}{m^2 w^3}. \tag{2.7}$$

## II.2 General time dependent Hamiltonian

As already pointed out in the Introduction, we are interested in finding a description of a general damped oscillator that allows quantization. Although the Equation (2.1) cannot be derived from a Lagrangian containing a conservative potential, we can write a time-dependent Lagrangian that leads to the correct equations of motion; namely:

$$L(X,\dot{X},t) = (1/2)\, m_o \exp(f(t))\, [\dot{X}^2 - \Omega^2(t)\, X^2] \quad , \tag{2.8}$$

This is a generalization of the Lagrangian proposed by Kanai.[5] Here f(t) contains both the time-dependence of mass and friction, the only requirement being that f(t) and $\Omega(t)$ are differentiable functions of time.

The Equation (2.8) contains the following particular cases:

(i) $f(t) \equiv 0$, $\Omega^2 = c_o/m_o$, undamped oscillator with constant mass and stiffness,

(ii) $f(t) = \log(m/m_o)$, $\Omega^2(t) = c(t)/m(t)$, undamped oscillator with time dependent mass and stiffness,

(iii) $f(t) = \gamma t$, $\Omega^2(t) = c_o/m_o$, damped oscillator with constant parameters.

The canonical momentum is defined as

$$\bar{P} = \frac{\partial L}{\partial \dot{X}} = m_o \dot{X} \exp(f(t)) . \tag{2.9}$$

It differs from the kinetic momentum P given by (2.1b), their relationship being

$$\bar{P} = (m_o/m(t)) \exp(f(t))P . \tag{2.10}$$

Accordingly the Hamiltonian associated with the Lagrangian (2.8) reads,

$$H = (\bar{P}^2/2m_o) \exp(-f(t)) + (m_o/2) \Omega^2(t) X^2 \exp(f(t)) . \tag{2.11}$$

Notice that in general we cannot identify H as the energy E of the oscillator, since

$$E = T + V = (P^2/2m) + (1/2)m\Omega^2X^2 . \tag{2.12}$$

Only when P and $\bar{P}$ coincide we have $E \equiv H$, this happens in the absence of dissipation no matter which is the time-dependence of m and $\Omega$.

The Hamilton's equations of motion are:

$$\dot{X} = \bar{P}/m_o \exp(-f(t)) \tag{2.13a}$$

$$\dot{\bar{P}} = -m_o \Omega^2 X \exp(f(t)) \tag{2.13b}$$

If we introduce the kinetic momentum P(2.1b), we recognize the Newton's equation of motion:

$$\ddot{X} + \dot{f}\,\dot{X} + \Omega^2 X = 0 \,. \tag{2.14}$$

In addition, Eqs. (2.13) lead to the dissipation rate:

$$\dot{E} = (\dot{m}/m - 2\dot{f})\, P^2/2m + \frac{1}{2}(\frac{\dot{m}}{m} + 2\,\frac{\dot{\Omega}}{\Omega})m\Omega^2X^2 \,. \tag{2.15}$$

If we consider a damped harmonic oscillator with constant mass and frequency, (see case (iii) above), $\dot{E}$ takes the well-known form

$$\dot{E} = -\gamma(P^2/m) \,. \tag{2.16}$$

If we introduce a time-dependence for the mass, we have

$$f(t) = \log m/m_o + \gamma t \tag{2.17}$$

and,

$$\dot{E} = -(\gamma + \dot{m}/2m)P^2/m \,. \tag{2.18}$$

We can see that the mass variation acts like an extra damping term when $\dot{m}$ is positive. As E is not a constant in time we derive the

exact Invariant J. Here, as already mentioned, kinetic and canonical momenta are not identical; the method of Symon[3] to build J is more convenient since it is based upon the Hamiltonian description.

The idea is to construct a canonical transformation from $(X,\bar{P})$ to $(X',\bar{P}')$ such that the new Hamiltonian $H'$ takes the form

$$H' = (1/2)\,\Omega'(t)(X'^2 + P'^2). \tag{2.19}$$

Then the J quantity defined as:

$$J = 1/2(X'^2 + P'^2) \tag{2.20}$$

is an exact invariant ($\frac{dJ}{dt} \equiv 0$). The introduction of the damping and its time variation does not change basically the philosophy of the method; we find that the exact invariant for the motion described by the general Lagrangian of Eq. (2.2), is

$$J = 1/2\; w^{-2}X^2 + 1/2(w\,\bar{P} - m_o\,\dot{w}\,X\exp(f))^2 \tag{2.21}$$

where w is proportional to the time-dependent amplitude of the coordinate X and satisfies the second order non-linear differential equation.

$$\ddot{w} + \dot{f}\dot{w} + \Omega^2 w = \left[m_o^2\, w^3 \exp(2f)\right]^{-1}. \tag{2.22}$$

With a new definition of w these results are equivalent to those displayed in Eqs. 2.6 and 2.7.

## III. THE QUANTAL PROBLEM

### III.1 General form of the equations of motion

The problem of the quantization of a damped harmonic oscillator has been analyzed by several authors (see, for example, Hasse[1]) in the case of constant mass and stiffness. Two lines are usually followed: i) the straightforward quantization of the classical Hamiltonian enforcing the principle of correspondence and, ii) the construction of an ad-hoc Hamiltonian that does not possess a classical analogue.

Both approaches are legitimate if they satisfy the following constraints: i) the expectation values

$$x = \langle \hat{x} \rangle \text{ and } \bar{p} = \langle \hat{\bar{p}} \rangle \tag{3.1}$$

(where $\hat{x}$ and $\hat{\bar{p}}$ are respectively the quantal operators associated with the canonical variables) must satisfy the classical equations of motion; ii) the uncertainties in coordinate and momentum must fulfill the Heisenberg relation.

These are general constraints that are intrinsic to the procedure of quantization; additional constraints may be provided depending on the particular problem we want to analyze.

In addition to the expectation values x and $\bar{p}$ defined above, the quantal operators $\hat{x}$ and $\hat{\bar{p}}$ display fluctuations that do not have a classical equivalence. Following Hasse[6], we define $\chi,\phi$, the fluctuations of $\hat{x}$ and $\hat{\bar{p}}$, respectively, and the correlation $\sigma$.

$$\chi = \langle \hat{x}^2 \rangle - x^2 \tag{3.2a}$$

$$\phi = \langle \hat{\bar{p}}^2 \rangle - \bar{p}^2 \tag{3.2b}$$

$$\sigma = 1/2 \langle \hat{x}\hat{\bar{p}} + \hat{\bar{p}}\hat{x} \rangle - x\bar{p} . \tag{3.2c}$$

As $\hat{x}$ and $\hat{\bar{p}}$ are canonical conjugates, they satisfy the Heisenberg uncertainty principle:

$$\chi\phi \geq \hbar^2/4 . \tag{3.3}$$

But most physical situations we are interested in are expressed through gaussian wave packets and then, the inequality (3.3) becomes a strict equality binding the three fluctuations

$$\chi\phi = \sigma^2 + \hbar^2/4 . \tag{3.4}$$

The time evolution of the first (Eq. (3.1)) and second (Eqs. 3.2) moments of the wave packets can be derived using the definition of the total derivative of a given operator $\hat{A}$

$$\frac{d\hat{A}}{dt} = \frac{\partial \hat{A}}{\partial t} - \frac{1}{i\hbar} \left[\hat{A}, H\right] . \tag{3.5}$$

The first moments x and $\bar{p}$ are fixed by the Ehrenfest's limit, and the second ones are solutions of a set of three coupled first order differential equations. In most cases, these equations may be decoupled and provide closed-forms expressions for $\chi$, namely:

$$\chi\ddot{\chi} - 1/2(\dot{\chi})^2 + g(t)\chi\dot{\chi} + 2h(t)\chi^2 = \frac{\hbar^2}{2k(t)} \tag{3.6}$$

Here, g(t), h(t) and k(t) are well defined functions of time whose explicit form depends on the particular description of the damping and the time variation of the harmonic oscillator parameters.

This equation takes a simpler form, if we introduce the width u of the wave packet:

$$u(t) = (2/\hbar)^{1/2}\chi^{1/2} \tag{3.7}$$

We obtain

$$\ddot{u} + g(t)\dot{u} + h(t)u = \left[k(t)u^3\right]^{-1} \tag{3.8}$$

This is an equation describing a forced and damped oscillator. Although in the general case we do not dispose of analytical solutions, they are available in a few simple situations that we explore later.

It is interesting to notice that if we introduce the reduced width W(t) through the definition

$$u(t) = W(t) \exp\left(-1/2 \int g(t)dt\right). \tag{3.9}$$

We can rewrite the equation (3.8) in a compact form

$$\ddot{W} + \omega^2 W = d(t)W^{-3} \tag{3.10}$$

that proves to be useful when comparing different formalisms. The reduced frequency $\omega(t)$ and the function $d(t)$ can be shown to be:

$$\omega^2 = h - 1/4g^2 - 1/2\dot{g} \tag{3.11a}$$

$$d = k^{-1}\exp(2\int g(t)dt). \tag{3.11b}$$

## III.2 Quantal Invariant Operators

As the center of the quantal gaussian wave packets follows the classical equations of motion, we always can derive a quantal operator $J_o$ which is a constant of motion. Through the principle of correspondence, we replace in Eq. (2.6) the X and P variables by the associated quantal operators and we obtain:

$$J = \frac{e^{-\int\gamma dt}}{2}\left\{\frac{\hat{x}^2}{w^2} + w^2\hat{p}^2 - mw(\dot{w} - \frac{\gamma_w}{2})(\hat{x}\hat{p} + \hat{p}\hat{x})\right. \tag{3.12}$$

$$\left. + m^2\,(\dot{w} - \frac{\gamma_w}{2})^2\hat{x}^2\right\}$$

where $w(t)$ is the amplitude of the center of the wave packet, that fulfills the differential equation (2.7).

We can notice the formal similitude between the evolution equations for the width (3.8) and for the amplitude (2.7); this remarks may provide us another way of writing the invariant operator in terms of the width of the wave packet w(t). When the motion is governed by eq. (2.1a), the functions g(t), h(t) and k(t) in eq. (3.8) must satisfy the requirements.

$$g(t) = \dot{m}/m \tag{3.13a}$$

$$h(t) = \Omega^2 - \gamma^2/4 - \dot{\gamma}/2 - \frac{\gamma \dot{m}}{2m} \tag{3.13b}$$

$$\text{and } k(t) = m^2(t). \tag{3.13c}$$

This turns out to be the case for some pure quantal Hamiltonians (cf. following sections).

# IV. QUANTAL HAMILTONIANS

In this chapter we will present in a unified description the various approaches used in the literature to study damped oscillators and we shall extend them to the general time dependent oscillator. Since all this approaches have been chosen to provide the exact equations of motion for x and p, we will concentrate on the properties of the second moments.

## IV.1 PRESENTATION OF THE VARIOUS HAMILTONIANS

a) General time-dependent quantal Hamiltonian. Straightforward quantization of the equation (2.11) gives us:

$$\hat{H} = (\hbar^2/2m_o)\frac{\partial^2}{\partial x^2}(\exp(-f)) + (m_o/2)\Omega^2(t)\exp(f)\hat{x}^2. \qquad (4.1)$$

In order to write the Hamiltonian $\hat{H}$, we have quantized the canonical momentum $\overline{P}$ whose relationship with the kinetic one P is given by Eq. (2.10). This feature induces a difference between the fluctuation $\phi$ as defined in Eq. (3.2b) and the uncertainty in the kinetic momentum $\Delta p$; we have:

$$\Delta p^2 = (m/m_o)^2\exp(-2f(t))\,\phi. \qquad (4.2)$$

Thus, the uncertainty product of position and kinetic momentum follows the law:

$$\Delta x \Delta p \geq (\hbar/2)m/m_o \exp(-f(t)). \qquad (4.3)$$

It is interesting to notice that if we define the energy operator $\hat{E}$ as:

$$\hat{E} = (m/mo)\hat{H} \exp(-f(t)) \tag{4.4}$$

the mean value $\langle\hat{E}\rangle$ fulfills exactly the equation (2.15) for the classical dissipation rate (see Hasse[1] for the discussion of the oscillator with constant mass and stiffness).

Applying Eq. (3.5) to the quantal operators in Eq. (3.2), we find the set of coupled first-order differential equations:

$$\dot{\chi} = 2(m_o \exp(f))^{-1}\sigma \tag{4.5a}$$

$$\dot{\phi} = -2(m_o \exp(f))\Omega^2 \sigma \tag{4.5b}$$

$$\dot{\sigma} = -(m_o \exp(f))\Omega^2\chi + (m_o \exp(f))^{-1}\phi . \tag{4.5c}$$

We can easily check that the condition (3.4) defining a gaussian wave packet is preserved by the above equations of motion. Thus, we are ensured that if the wave packet is gaussian at initial time, it will keep its gaussian shape during its whole time evolution.

In Table I, we list the actual form for the functions g(t), h(t), k(t), $\omega$(t) and d(t) as defined in the preceding section. In this particular case, it can be shown that the motion of $\phi$ can be decoupled; and in the same way as for $\chi$, we can define a width v(t) for the momentum distribution:

$$v(t) = (2/\hbar)^{1/2}\phi^{1/2}. \tag{4.6}$$

v(t) is solution of a second order differential equation:

$$\ddot{v} - (\dot{f} + 2\dot{\Omega}/\Omega)\dot{v} + \Omega^2 v = m_o^{\,2}\,\Omega^2 \exp(2f)/v^3. \tag{4.7}$$

From the formal similitude between eqs. (2.22) and (3.8) we can deduce an invariant $\hat{J}_1$ associated with the fluctuation:

$$\hat{J}_1 = (\hbar^2/4)\,\chi^{-1}\hat{x}^{\,2} + \chi\hat{p}^2 - (1/2)\,m_o \exp(f)\dot{\chi}(\hat{x}\hat{p} + \hat{p}\hat{x}) + \frac{1}{4}\,m_o^{\,2}\exp(2f)\dot{\chi}^{\,2}\chi^{-1}\hat{x}^2. \tag{4.8}$$

With the help of Eqs. (4.5), this reduces to

$$\hat{J}_1 = \phi\hat{x}^2 + \chi\hat{\bar{p}}^2 - \sigma(\hat{x}\hat{\bar{p}} + \hat{\bar{p}}\hat{x}) \tag{4.9}$$

Taking the expectation value of $\hat{J}_1$, we obtain the following time independent observable

$$J_1 = \hbar^2/2 + \phi x^2 + \chi\bar{p}^2 - 2\sigma\bar{p}x\,. \tag{4.10}$$

The various states of a gaussian wave packet in a harmonic oscillator well may be classified according to the eigenstates of $\hat{J}_1$. Using Eq. (3.4), we can see that the minimum value $J_1 = \hbar^2/2$ corresponding to the ground state is only reached when $x = \bar{p} = 0$, i.e., for motionless wave packets centered at the origin, whatever is their width.

b) Hasse's Hamiltonians.

Hasse[6] has shown that the Hamiltonian:

$$H = -\frac{\hbar^2}{2m(t)}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m(t)\Omega^2(t)\hat{x}^2 + \gamma(t)(\hat{x}-x)\left[\varepsilon\hat{p} + (1-\varepsilon)p\right] - \frac{i\hbar}{2}\varepsilon\gamma(t) \qquad (4.11)$$

synthesizes the several non-linear quantal frictional potentials presented in the literature, namely:

$\varepsilon = o$, gives Albrecht Hamiltonian[7],

$\varepsilon = 1$, gives Süssmann Hamiltonian[8],

$\varepsilon = \pm 1/2$ gives Hasse Hamiltonian[1].

We can assume any time-dependence for m, $\gamma$ and $\Omega$; the equations of motion for the fluctuations are the same as for the time-independent oscillator[6]

$$\dot{\chi} = 2\gamma'\chi + 2\sigma/m \qquad (4.12a)$$

$$\dot{\phi} = -2\gamma'\phi - 2m\Omega^2\sigma \qquad (4.12b)$$

$$\dot{\sigma} = -m\Omega^2\chi + \phi/m \qquad (4.12c)$$

with $\gamma' = \varepsilon\gamma$.

These expressions are consistent with the conservation of the gaussian shape of the wave packet (see Eq. (3.4)). Under this requirement, we can extract closed-form equations for the second moments. The results for $\chi$ and the width are displayed in Table I. Similar expressions can be found for $\phi$ and the momentum width v(t) as follows:

$$\ddot{v} - \left(\frac{\dot{m}}{m} + \frac{2\dot{\Omega}}{\Omega}\right)v + \left[\Omega^2 - \gamma'^2 - \gamma'\left(\frac{\dot{m}}{m} + 2\frac{\dot{\Omega}}{\Omega}\right) + \dot{\gamma}'\right] v = m^2\Omega^2/v^3 , \tag{4.13}$$

which is to be compared with Eq. (4.7).

The frictional term in the Hamiltonian (4.11) has been deviced to yield the correct Ehrenfest limit for the center of the wave packet. It does not possess a classical analogue. In fact, the expectation value of the operator $\hat{V}_{fr}$

$$\hat{V}_{fr} = \hat{H} - \hat{T} - \hat{V} \tag{4.14}$$

is

$$\langle \hat{V} \rangle_{fr} = \gamma\sigma . \tag{4.15}$$

Then the expectation value of the Hamiltonian is not the energy, in general.

It is possible to derive an exact invariant of the motion in terms of the amplitude w of the first moment (see Sec. III.2). We can observe that when $\gamma' = \gamma/2$ (Hasse Hamiltonian with $\varepsilon = 1/2$), w and

the width u satisfy the same equation of motion (see Eq. (2.7) and Table I respectively). Accordingly, we can write an invariant $\hat{J}_1$, whose expectation value in this case can be shown to be

$$\langle \hat{J}_1 \rangle = \frac{e^{\int \gamma dt}}{\hbar} \left[ \phi x^2 + \chi p^2 - 2\sigma xp + \hbar^2/2 \right] . \tag{4.16}$$

The reader can easily verify with the help of the equations for the first and second equations of motion that this observable is an exact invariant, i.e.,:

$$\frac{d\langle \hat{J}_1 \rangle}{dt} = 0 . \tag{4.17}$$

If at $t = 0$, the wave packet under consideration is a minimum uncertainty wave packet, the value of the invariant is simply:

$$\langle \hat{J}_1 \rangle = \frac{\hbar}{4} \left\{ \frac{\langle x^2 \rangle_o}{\chi_o} + \frac{\langle p^2 \rangle_o}{\phi_o} + 2 \right\} . \tag{4.18}$$

The obvious meaning is that the invariant of the motion for the quantal damped oscillator is essentially the sum of the potential and kinetic energies in units of the potential and kinetic energies contained in the initial coordinate and momentum width, respectively.

c) Kostin's Hamiltonian

It is also well-known that a former representation for friction on a quantal particle is due to Kostin.[9] The generalization for the general time-dependent oscillator reads:

$$H = -\frac{\hbar^2}{2m(t)} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m(t) \Omega^2(t) \hat{x}^2 - \frac{i\hbar}{2} \gamma \left[ \ln \frac{\psi}{\psi^*} - \left\langle \ln \frac{\psi}{\psi^*} \right\rangle \right] , \tag{4.19}$$

where $\psi$ and $\psi^*$ are the wave function and its complex-conjugate respectively. It is useful to write the gaussian wave packet as [1,6]:

$$\psi(x,t) = \frac{1}{(2\pi)}1/4\left(\frac{1}{\alpha} + \frac{1}{\alpha^*}\right)^{1/4}\exp\left\{\frac{(\hat{x}-x)^2}{2\alpha} + \frac{i}{\hbar}\left[p(\hat{x}-x) + \int L\,dt - \theta\right]\right\} \quad (4.20)$$

Here $\alpha$ is the complex time dependent width whose relationship with the real fluctuation $\chi$ is[6]

$$\frac{1}{2}\chi^{-1} = \mathrm{Re}(\alpha^{-1}) \quad (4.21)$$

In addition, L is the classical Lagrangian and $\theta$ is a real phase factor.

In presence of the wave packet, the frictional term in Kostin's Hamiltonian becomes:

$$\hat{V}_{fr} = \gamma p(\hat{x}-x) + \frac{i\hbar\gamma}{4}(\hat{x}-x)^2\left(\frac{1}{\alpha} - \frac{1}{\alpha^*}\right) \quad (4.22)$$

It is easy to verify that $\langle\hat{V}_{fr}\rangle$ is exactly zero and then the expectation value of the energy is identical to the expectation value of $\hat{H}$.

The equations of motion for the second moments are, in this case:

$$\dot{\chi} = 2\sigma/m(t) \quad (4.23a)$$

$$\dot{\phi} = -2m(t)\Omega^2(t)\sigma - 2\gamma(t)\phi + \frac{\hbar^2}{2}\frac{\gamma(t)}{\chi} \quad (4.23b)$$

$$\dot{\sigma} = -m(t)\Omega^2(t)\chi + \frac{\phi}{m(t)} - \gamma(t)\sigma. \quad (4.23c)$$

As in the previous examples, the motion of $\chi$ and $\phi$ can be decoupled (see Table I.).

The invariant operator we can write is $\hat{J}_o$ associated with the amplitude w of the coordinate. As seen in Table I, the functions g(t), h(t) and k(t) do not take the analytical form of Eqs. (3.13) and, as a consequence, we cannot write this exact invariant in terms of the fluctuations.

IV.2 Discussion

The aim of this section is to provide arguments for the choice of the best description of damping on a time-dependent oscillator. From inspection of Table I, it turns out that in the absence of damping the three representations are identical, as should be expected.

a) The general time-dependent Hamiltonian presents a serious shortcoming in presence of damping. If we examine the Equation (4.3) we realize that for sufficiently long time the uncertainty product $\Delta x \Delta p$ can be smaller than $\hbar/2$. This is a common feature to all descriptions that use different operators to represent classical and canonical moments. Senitzky[10] has shown that this effect is due to the neglection of the fluctuations in the loss mechanism itself. This limitation hampers the use of the Hamiltonian (4.1) in the quantal study of the damped harmonic oscillators.

b) A satisfactory description of damping should account for the fact that any wave packet containing excited states must decay to the ground-state. Accordingly, we would like to examine the solution for

the damped harmonic oscillator with constant parameters, since this analysis will provide a clue for the final choice of the best representation for our general problem.

From Table I, we see that with Hasse's Hamiltonians, the equation for the reduced width W(t) takes the following form:

$$\ddot{W} + \omega^2 W = W^{-3} \qquad (4.24a)$$

where $\omega^2$ is a constant of time: $\omega^2 = \Omega^2 - \gamma'^2$ . (4.24b)

The most general solution can be written as:

$$W(t) = \omega^{-1} (1 + A^2 + B^2)^{1/2} + A \cos 2\omega t + B \sin 2\omega t \qquad (4.25)$$

If we take the following initial conditions:

$$W(t=0) = W_o \text{ and } \dot{W}(t=0) = 0$$

we obtain

$$W(t) = \begin{cases} W_o & \text{if } W_o = \omega^{-1} \qquad (4.26a) \\ W_o + (\omega^2 W_o)^{-1} \sin^2 \omega t & \text{otherwise} \qquad (4.26b) \end{cases}$$

The constant solution corresponds to a fluctuation $\chi = \frac{\hbar}{2m\omega}$. In addition, if $W_o$ is different from $\omega^{-1}$, we see from Eq. (4.26b) that W will oscillate without decaying to the ground state as we

should expect. It is specially puzzling that even if $W_o$ takes its ground-state value, the wave packet width will undergo oscillations.

These observations lead us to the conclusion that Hasse's Hamiltonians are not completely adequate for the treatment of the damped harmonic oscillator.

c) If we want to initiate the same study with the Kostin's Hamiltonian, we recall the equation for the reduced width for the time independent oscillator (in Table I):

$$\ddot{W} + \omega^2 W = W^{-3}\exp(\gamma t) \tag{4.27a}$$

$$\text{where } \omega^2 = \Omega^2 - \gamma^2/4 . \tag{4.27b}$$

In this case, it is not easy to write down W(t) in the way of Eqs. (4.26). Instead we will find useful to discuss the corresponding equation for the width u:

$$\ddot{u} + \gamma\dot{u} + \Omega^2 u = m^{-2}u^{-3} . \tag{4.28}$$

We can easily realize that there is a constant solution:

$$u = (m\Omega)^{-1/2} . \tag{4.29}$$

The corresponding fluctuation $\chi$ turns out to be

$$\chi = \hbar/(2m\Omega) \tag{4.30}$$

and this is the actual dispersion of the ground-state. It is illustrative to try a solution of Eq. (4.28) that differs only slightly from the constant value (4.29).

Let us write u(t) as:

$$u(t) = (m\Omega)^{-1/2} + \delta(t) \tag{4.31}$$

with $\delta(t)$ such that $|\delta(t)| << (m\Omega)^{-1/2}$ for all t .

Straightforward substitution of Eq. (4.31) into Eq. (4.28) yields

$$\ddot{\delta} + \gamma\dot{\delta} + (2\Omega)^2\delta = 0 \tag{4.32}$$

It means that:

$$\delta = \delta_o \exp(-\gamma t/2)\sin(2\Omega t + \phi_o) \cdot \tag{4.33}$$

This shows that any state that differs only slightly from the ground state will decay to it.

We observe from Eq. (4.28) there do not exist steady solutions with large deviations from the constant value (Eq. 4.29); since in that case, the right-hand side becomes negligible and u behaves like the position of a damped oscillator.

This is illustrated in Fig. 1. The time evolution of $\chi$ and the total energy are displayed for two different initial conditions, namely, $\chi_o = 0.2$ and $\chi_o = 4$, in units of the ground-state fluctuation. We observe that $\chi$ reaches the ground-state value in less

than three periods. The same is true for x, although it should be remarked that the number of oscillations undergone by the fluctuation is twice that of the coordinate. It is also interesting to look at the evolution of the energy. The slight initial difference between the two curves can be traced back to the contribution of the fluctuations $\chi$ and $\phi$ to the potential and kinetic terms, respectively.

In view of these considerations, we believe that Kostin's Hamiltonian description is adequate to face the study of a general time-dependent Gaussian wave packet. In the following section, we will present several applications of this formalism.

V. Applications.

In the two next sections we are going to study some specific interplays between mass variation and damping, that might be connected with a number of physical situations. In subsection (V.1) we will present typical results for systems initially in the ground state, that undergo a displacement and different mass variations. In subsection (V.2) we deal with a specific example inspired in situations appearing in heavy-ion physics.

V.1 Departures from the ground state

The system we will consider is a displaced ground-state wave packet, that presents at $t = 0^-$ the following characteristics: $m_o = 1$, $c_o = 1$, $\chi_o = \hbar/2$, $\phi_o = \hbar/2$, $\sigma_o = 0$ and $x_o = 1$, $p_o = 0$; we consider that at $t = 0$ a mass variation law $m(t)$ is superimposed on $m_o$.

a) Exponentially increasing mass. We choose the following variation law for $m(t)$:

$$\begin{cases} m = 1 \ ; \ t<0 \\ \\ m = e^{\lambda t} ; \ t\geq 0, \lambda \geq 0 \end{cases} \quad . \qquad (5.1)$$

In this example, the mass presents a singularity at infinite time. Small values of $\lambda$ provide us a way of studying adiabatic behaviors. As $\frac{\dot{m}}{m} = \lambda$, the functions $g(t)$ and $\omega(t)$ (see Table 1) take the following simple form:

$$g(t) = \gamma + \lambda \tag{5.2a}$$

$$\omega^2(t) = e^{-\lambda t} - \frac{1}{4}(\gamma+\lambda)^2 \frac{\lambda^2}{2} \tag{5.2b}$$

Eq. (3.8) then becomes,

$$\ddot{u} + (\gamma+\lambda)\dot{u} + e^{-\lambda t}u = e^{-2\lambda t}u^{-3} \tag{5.3}$$

One can solve this equation for sufficiently large time under the assumption that u remains finite. We find

$$u(t) = A + B(\gamma+\lambda)^{-1}\exp(-(\gamma+\lambda)t) \tag{5.4}$$

and the same result holds for the displacement x(t). Since the increasing mass will cause the kinetic energy to vanish, the asymptotic energy would be only potential and depending on the final constant values of x and u.

Typical results are shown in Fig. 2. We see that irrespectively of the value of $\lambda$, the fluctuation $\chi$ remains almost identical to the adiabatic value $\hbar/(2m(t)\Omega(t))$. It is insensitive to changes in the strength of the damping parameter when $\lambda$ is small enough ($\lambda = 0.1$ in units of the unperturbed frequency). For larger $\lambda$, a slight deviation from the adiabatic trend is observed according to different $\gamma$'s. In contrast, damping has large effects on the evolution of the coordinate and momentum: this is reflected in the energy curves.

We observe that for large damping ($\gamma$ = 1.) it takes half a period to dissipate most of the initial energy; while for small $\gamma$ ($\gamma$ =0.1), dissipation is slower and presents smooth oscillations. Both regimes converge asymptotically towards the adiabatic trend E = 0.5 $\hbar$ $\omega(t)$.

b) Singularity at finite time.

To illustrate the case when the mass becomes infinite at a finite time we choose the representation:

$$\begin{cases} m = 1 \text{ for } t<0 \\ \\ m = \exp(t/(T_1 - t))^2, \ t\geqslant 0 \end{cases} . \qquad (5.5)$$

$T_1$ is a parameter that fixes both the position of the singularity and the rate of increase of the mass.

The striking result of Fig. 3 is the fact that the final energy presents an inversion as a function of the damping parameter $\gamma$. As in the preceding case, the final energy is purely potential. For values of $\gamma$ close to the critical one (i.e., $\gamma = 2\Omega$), the fluctuation $\chi$ overcompensates the attenuation of the position x, providing a large potential energy. In addition, we notice that the final displacement is not zero for small damping. The actual value of this displacement depends on the initial phase; accordingly we may expect the final energy to present some range of variations for small $\gamma$'s.

In contrast with the first example, we see that the evolution of $\chi$ presents large departure from the adiabatic trend. During the process, there is an interplay between the role of damping and mass

variation. For small times when $\dot{m}/m$ is close to zero, the damping parameter $\gamma$ determines the slope with which $\chi$ comes apart from the adiabatic curve. Afterwards, the term $\dot{m}/m$ overcomes $\gamma$ and accounts for the final state of the system.

c ) Periodic mass variation.

A variable mass provides a mean of simulating an input or removal of energy into the system. An explicit representation that accounts for several interesting features is a periodic perturbation on a constant mass, i.e.

$$\begin{cases} m = 1 , & t<0 \\ m = 1 + \alpha \sin\lambda t , & t\geq 0 \end{cases} \qquad (5.6)$$

with $\alpha < 1$.

This time variation in the mass can be traced to some external oscillating field; consequently, we can expect some resonant behavior. A search through a wide range of parameters $\alpha$ and $\lambda$ allowed us to obtain the results displayed in Fig. 4. In this case, the frequency of the mass is twice that of the unperturbed oscillator. The first point to notice is that both the energy and the fluctuation oscillate with the frequency of the mass; the deformation in the peaks of the energy curve can be associated with the fact that the position x is oscillating with the shifted frequency of the damped oscillator. Second, the resonant behavior disappears when the strength of the damping becomes close to the critical value. In this case, $\chi$ and E perform constant amplitude oscillations around the ground-state values.

Selection of the parameters $\alpha$ and $\lambda$ other than those corresponding to Fig. 4, yields modulation of the displayed curves and a much smoother increase of the amplitudes.

V.2 Infinite mass system at t = 0.

It has been suggested[11,12] that the charge equilibration process during heavy-ion reactions may be pictured as the relaxation of a collective coordinate placed in a quadratic potential. As this equilibration is impossible when the ions are far apart (before and after the reaction), this suggests that the collective coordinate corresponds to an oscillator whose variable mass is infinite at t = 0, reaches a finite value during the interaction time and becomes infinite again when the two ions split apart. Not willing to get into the physical details, we are going to take a simplified representation of the above mentioned situation; namely, an oscillator with constant stiffness c = 1, a constant damping $\gamma$ and a mass given by the law

$$m(t) = \begin{cases} \exp\left(\frac{T_1}{t}-1\right)^2 & \text{if } 0 \leq t \leq T_1 \\ 1 & \text{if } T_1 \leq t \leq T_1 + T_2 \\ \exp\left(\frac{t_3}{T_1+T_2+T_3-t}-1\right)^2 & \text{if } T_1+T_2 \leq t \leq T_1+T_2+T_3 . \end{cases} \tag{5.7}$$

A typical pattern for m(t) is presented in Fig. 5 for a particular selection of the intervals $T_1, T_2$ and $T_3$.

When m(t) is very large, we can simplify the equations of motions for both first and second moments (Eqs. 2.1 and 4.23) provided that the initial p is finite and the initial $\chi$ is not zero.

In that case, we have

$$x = x_o \tag{5.8a}$$

$$\rho = \rho_o - \frac{x_o}{\gamma}(1 - \exp(-\gamma t)) \tag{5.8b}$$

$$\chi = \chi_o \tag{5.8c}$$

$$\phi = \hbar^2/4\chi_o + \frac{\chi_o}{\gamma^2}(1-\exp(-\gamma t))^2 \tag{5.8d}$$

$$\sigma = \frac{\chi_o}{\gamma}\left[1-\exp(-\gamma t)\right] \tag{5.8e}$$

These are the equations governing the initial motion when the initial correlation $\sigma$ is zero, and it is interesting that they are independent of the actual value of the mass. A similar feature is expected in the vicinity of the second singular point $t = T_1+T_2+T_3$.

In Fig. 6 we display the energy time dependence for various damping parameters and typical value of the time intervals. In that case, the equation governing the time-evolution of E is

$$\dot{E} = -(\gamma+\dot{m}/2m)(\phi+p^2)/m + \hbar^2\gamma/(4\chi m) \tag{5.9}$$

For this particular mass law $\dot{E} = 0$ for $t = 0$ and $t = T_1+T_2+T_3$. This feature is clearly represented in the Figure 6.

Due to the action of the damping on $\phi$ and $p^2$, dramatic effects are induced on the dissipation rate even when $\dot{m}/m$ is large: the height of the plateau is strongly dependent on the value of $\gamma$; in addition, the larger the $\gamma$ parameter the shorter the elapsed time before reaching this plateau.

As in a preceeding example (Sec. V.1b), the final energy is only potential and depends upon the final value of fluctuations and displacements. In Fig. 6, we see that when $\gamma = 0$, there is a larger residual displacement that is determined by the history of the system (initial displacement, length of the plateau, etc.). A sensible amount of damping on a sufficiently long plateau causes x to vanish, accordingly the final energy is given by the fluctuations only. These are displayed in Fig. 7. We have assumed a small initial width that intends to represent a well-localized wave packet. This actual value is critical to determine the amplitude of the fluctuation oscillations. The wide oscillations of the undamped case are smoothed away by finite $\gamma$ values and disappear completely when $\gamma$ is close to the critical damping.

For sufficiently (not too) large value of damping, the system looses memory of its past history in a couple of periods. In that case, the whole problem reduces to the study of the evolution of an initial ground state when a pertubation in the form of an increasing mass is superimposed.

For smaller amounts of damping, the final results are much more sensitive to the particular selection of the time intervals and initial conditions.

For the sake of completeness, in Fig. 8 we display the time evolution of the coordinate x. It is worthwhile to remark that although the frequency of the width oscillations is twice as large as that of the coordinate, both $\chi$ and x follow essentially the same pattern.

## VI. CONCLUSIONS

We have generalized both the existing procedures for the study of a classical, undamped, time-dependent oscillator and the methods for describing a quantal, damped, time-independent oscillator. We have been able to provide tools for the analysis of a general quantal, damped, time-dependent oscillator in the case in which the quantal state can be represented by a gaussian wave packet. We have shown that although the conservation of energy is destroyed under the circumstances enforced by our definition of the oscillator, it is possible to find an exact invariant of the motion that is related to the amplitude of the coordinate. Under some particular conditions, it may be also connected with the fluctuations.

We have found criteria to select, among those that are available to us, the only correct description for damping of a gaussian wave packet in a quadratic field. This description is correct in the sense that it both preserves the uncertainty principle and yields the expected asympotic behavior. We have shown that this formalism holds even in the case of a mass reaching an infinite value, Kostin's frictional potential provides the right decay of any gaussian wave packet to the ground state, in the presence of dissipation. We have illustrated in a number of examples possible applications of this method. One particular case that deserved our attention was the time evolution of the ground-state when a perturbation, i.e., a displacement and a time-variation in the mass, is applied. There is a definite interplay between damping and mass variation; some important

features are: a) when the logarithmic derivative $\dot{m}/m$ is smaller than unity, the fluctuation $\chi$ remains close to the adiabatic limit $\hbar/2m(t)\Omega(t)$ and damping is a second-order effect in its evolution. In this situation, damping is mainly used to dissipate the energy of the motion; b) when $\dot{m}/m$ becomes large (for example in the vicinity of a singularity at a finite time) strong deviations from this adiabatic trend can be observed. Nevertheless, the initial slope of the departure is determined by the value of the damping parameter. We have also seen that a perturbation on the ground-state can account for a resonant behavior; in this case the presence of dissipation provides attenuations of the motion and of the increase in the amplitude of the fluctuation $\chi$; the critical damping ensures steady oscillations in both the fluctuation and the energy.

An interesting problem bearing some connection to the excitation of collective modes in some nuclear (i.e., heavy ion) reactions is that of an initially infinite mass, lowering to a finite constant value and rising to infinite afterwards (Section V.2). Such a situation involves a number of parameters, namely, initial width and displacement of the wave packet, duration of the mass decrease, length of the plateau, duration of the mass increase, and damping. Only a detailed treatment of the physics here contained can fix their actual values. However, we have illustrated the expected behavior of the wave packet for a particular selection of the parameters. We believe that the most important point to be remarked is that damping is essential in determining the evolution pattern. From the exhibited

curves we see that it takes some time for the motion to start. This is due to the fact that initially the mass is very heavy. When the mass takes a reasonable finite value, time evolution in x, $\chi$ and E can be observed. However, the slope of this evolution is completely different according to the actual value of $\gamma$. Even a small amount of damping is able to cause a sensitive departure from the undamped motion. In presence of large damping, the whole motion should not be critically dependent on the details of the mass evolution.

As a final statement, we should remark that since there is no unique quantal description of friction, any particular application of dissipation models should be preceded by a critical examination as we have intended here for the case of a gaussian wave packet in the most general harmonic field.

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Appendix The invariants of the motion for a classical damped oscillator.

As we introduce the ansatz (2.3) and (2.5) in the Newtonian equations (2.1) (2.2), splitting the resulting expressions into real and imaginary parts we obtain the following system:

$$\dot{w} - \frac{\gamma}{2} w = \frac{z}{m} \sin(\phi_1 - \phi_2) \tag{A.1}$$

$$\dot{z} + \frac{\gamma}{2} z = -m\Omega^2 w \sin(\phi_1 - \phi_2) \tag{A.2}$$

$$\dot{\phi}_1 = \frac{z}{m}\cos(\phi_1 - \phi_2) - \Omega' \tag{A.3}$$

$$\dot{\phi}_2 = m \Omega^2 \frac{w}{z}\cos(\phi_1 - \phi_2) - \Omega' \tag{A.4}$$

The quantity

$$U = e^{\int \gamma dt \frac{X^* P - X P^*}{2i}} = z\, w \cos(\phi_1 - \phi_2) \tag{A.5}$$

can be shown to be a constant of the motion and can be fixed equal to unity with a suitable normalization of X and P. Its meaning will become evident below since we shall see that the ansatz (2.3), (2.5) leads to a canonical transformation to new variables $X'$, $P'$, in terms of which an invariant can be constructed.

As we square Eq. (A.1), the normalization $U = 1$ (A.6) allows us to write the expression

$$z^2 = \frac{1}{w^2} + m^2(\dot{w} - \frac{\gamma}{2}w)^2 \tag{A.7}$$

If we now differentiate Eq. (A.6) and use (A.1), (A.2), we find the following equation for the time evolution of w(t):

$$\ddot{w} + \dot{w}\frac{\dot{m}}{m} + w\left(\Omega'^2 - \frac{\dot{\gamma}}{2} - \frac{\dot{\gamma}m}{2m}\right) = \frac{1}{m^2 w^2} \tag{A.8}$$

An equivalent procedure leads to a similar equation for z,

$$\ddot{z} + \dot{z}\left(\frac{\dot{m}}{m} + \frac{2\dot{\Omega}}{\Omega} - \frac{\gamma}{2}\right) + z\left(\Omega'^2 + \frac{\dot{\gamma}}{2} - \frac{\gamma\dot{m}}{2m} - \frac{\dot{\gamma}\Omega}{\Omega}\right) = \frac{m^2\Omega^2}{z^2} \tag{A.9}$$

The normalization (A.6) requires that we multiply X, P by a convenient constant $Ae^{i\phi_o}$. Accordingly, the most general (real) solutions of Newton equation for a time-dependent, damped oscillator can be sampled as shown below:

$$X(t) = e^{-\int\frac{\gamma dt}{2}}\left[A\omega\cos\left(\phi_o + \int\Omega' dt\right)\cos\phi_1 - A\omega\sin\left(\phi_o + \int\Omega' dt\right)\sin\phi_2\right] \tag{A.10}$$

$$P(t) = e^{-\int\frac{\gamma dt}{2}}\left[-Az\cos\left(\phi_o + \int\Omega' dt\right)\sin\phi_2 - Az\sin\left(\phi_o + \int\Omega' dt\right)\cos\phi_2\right] \tag{A.11}$$

If we adopt the definition

$$X'(t) = A\cos\left(\phi_o + \int\Omega' dt\right) \tag{A.12}$$

$$P'(t) = A\sin\left(\phi_o + \int\Omega' dt\right) \tag{A.13}$$

we realize that Eqs. (A.10), (A.11) represent a linear transformation from $(X',P')$ into $(X,P)$. The determinant is, precisely,

$$\Delta = -e^{-\int\frac{\gamma dt}{2}} z\, w \cos(\phi_1-\phi_2) =$$

$$= -e^{-\int\frac{\gamma dt}{2}} \tag{A.14}$$

From (A.12), (A.13), it becomes obvious that the quantity

$$J = \frac{1}{2}(X'^2+P'^2) = \frac{1}{2}A^2 \tag{A.15}$$

is a constant. Inversion of the system (A.10), (A.11) and substitution into (A.15) allows us to reach the alternative forms,

$$J = \frac{e^{\int\gamma dt}}{2}\left\{\frac{X^2}{w^2} + \left[w\,P - mX(\dot{w}-\frac{\gamma}{2}w)\right]^2\right\} \tag{A.16}$$

or

$$J = \frac{e^{\int\gamma dt}}{2}\left\{\frac{P^2}{z^2} + \left[zX + \frac{P}{m\Omega^2}(\dot{z}+\frac{\gamma}{2}z)\right]^2\right\} \tag{A.17}$$

This quantity is the invariant for a general time-dependent, damped oscillator and it generalizes the one obtained by Symon[3] for a general quadratic, non-dissipative time-dependent Hamiltonian.

REFERENCES

1. R. W. Hasse, J. Math. Phys. 16 (1975) 2005.

2. H. R. Lewis, Phys. Rev. Let., 18 (1967) 510.

3. K. R. Symon, J. Math. Phys. 11 (1970) 1320.

4. J. E. Howard, Phys Fluids, 13 (1970) 2407.

5. E. Kanai, Prog. Theor. Phys. 3 (1970) 440.

6. R. W. Hasse, J. of Phys. A11 (1978) 1245.

7. K. Albrecht, Phys. Let. 56B (1975) 127.

8. G. Süssman, Los Alamos seminar talk 1973, unpublished.

9. M. D. Kostin, J. Stat. Phys. 12 (1975) 145.

10. I. R. Senitzky, Phys. Rev. 119 (1960) 670.

11. W. D. Myers, private communication.

12. M. Berlanger, A. Gobbi, F. Hanappe, H. Lynen, C. Ngô, A. Olmi, H. Sann, H. Stelzer, H. Richel and M. F. Rivet, Zeif. f. Phys. A291 (1979) 151.

TABLE CAPTIONS

Table 1. Comparison among the functions g(t), h(t) k(t), $\omega^2(t)$, and d(t) appearing in Eqs. (3.6) and (3.10) in text for the various Hamiltonians analyzed in the paper. Notice that if $f(t) \equiv \ln(m/mo) + \int \gamma(t)dt$, the reduced frequencies $\omega^2$ for the GTD and Hasse (i.e., $\gamma' = \gamma/2$) Hamiltonians are identical.

FIGURE CAPTIONS

Fig. 1. Time evolutions of energy, fluctuation and position of a gaussian wave packet in units of $\hbar\Omega_o/2$, $\hbar/(2m_o\Omega_o)$ and $\hbar^{1/2}/(m_o\Omega_o)^{1/2}$ respectively. The time unit is the natural period. These calculations correspond to Kostin's potential with constant mass and stiffness; the damping parameter is 0.5 in units of the inverse period. The full lines correspond to an initial fluctuation $\chi_o = 0.2$, the dashed lines are for an initial $\chi_o = 4$. The initial displacement $x_o$ is always equal to 1.

Fig. 2. The same as Fig. 1, but for an exponentially increasing mass. Full lines correspond to the damping parameter $\gamma = 0.1$, dashed-dotted lines correspond to $\gamma = 1$. Both lines coincide in the time evolution of $\chi$ when $\dot{m}/m = 0.1$. The initial conditions are that of a displaced wave packet with the ground state width.

Fig. 3. The same as Fig. 1, but with a mass going to infinity at $t = 2$. The full, dashed and dashed-dotted lines correspond respectively to $\gamma = 1.99$, 0.5 and 0.. The initial conditions are the same as in Fig. 2.

Fig. 4. The same as Fig. 3, but with a periodic perturbation $m = 1 + 0.5\sin(2\Omega t)$. The dashed and full lines correspond to $\gamma = 0.25$ and $\gamma = 1.99$ respectively. The initial conditions are the same as in Fig. 2.

Fig. 5. Typical evolution of the inverse of a mass that goes to infinity at t = 0 and t = 5 (see Eq. (5.7) in text). The time intervals are $T_1 = 1$ and $T_2 = T_3 = 2$.

Fig. 6. Time evolution of the total energy for the oscillation whose mass evolution is displayed in Fig. 5. The full, dashed and dashed-dotted lines correspond to damping parameters $\gamma = 1.99, 0.25, 0$, respectively. The initial values are $\chi_o = 0.2$ and $x_o = 1$.

Fig. 7. Time evolution of the fluctuation $\chi$ corresponding to the situation of Fig. 6.

Fig. 8. Time evolution of the position x corresponding to the situation of Fig. 7.

TABLE I.

| | GENERAL TIME-DEPENDENT (GTD) HAMILTONIAN | HASSE HAMILTONIANS | KOSTIN HAMILTONIAN |
|---|---|---|---|
| $g(t)$ | $\dot{f}$ | $\dot{m}/m$ | $\gamma + \dot{m}/m$ |
| $h(t)$ | $\Omega^2$ | $\Omega^2 - \gamma'^2 - \gamma' \dot{m}/m - \dot{\gamma}'$ | $\Omega^2$ |
| $k(t)$ | $m_o^2 \exp(2f)$ | $m^2$ | $m^2$ |
| $\omega^2(t)$ | $\Omega^2 - 1/4(\dot{f})^2 - 1/2\ddot{f}$ | $h(t) + (\frac{\dot{m}}{2m})^2 - \frac{\ddot{m}}{2m}$ | $h(t) + (\frac{\dot{m}}{2m})^2 - \frac{\ddot{m}}{2m}$ |
| $d(t)$ | 1 | 1 | $\exp(\gamma t)$ |

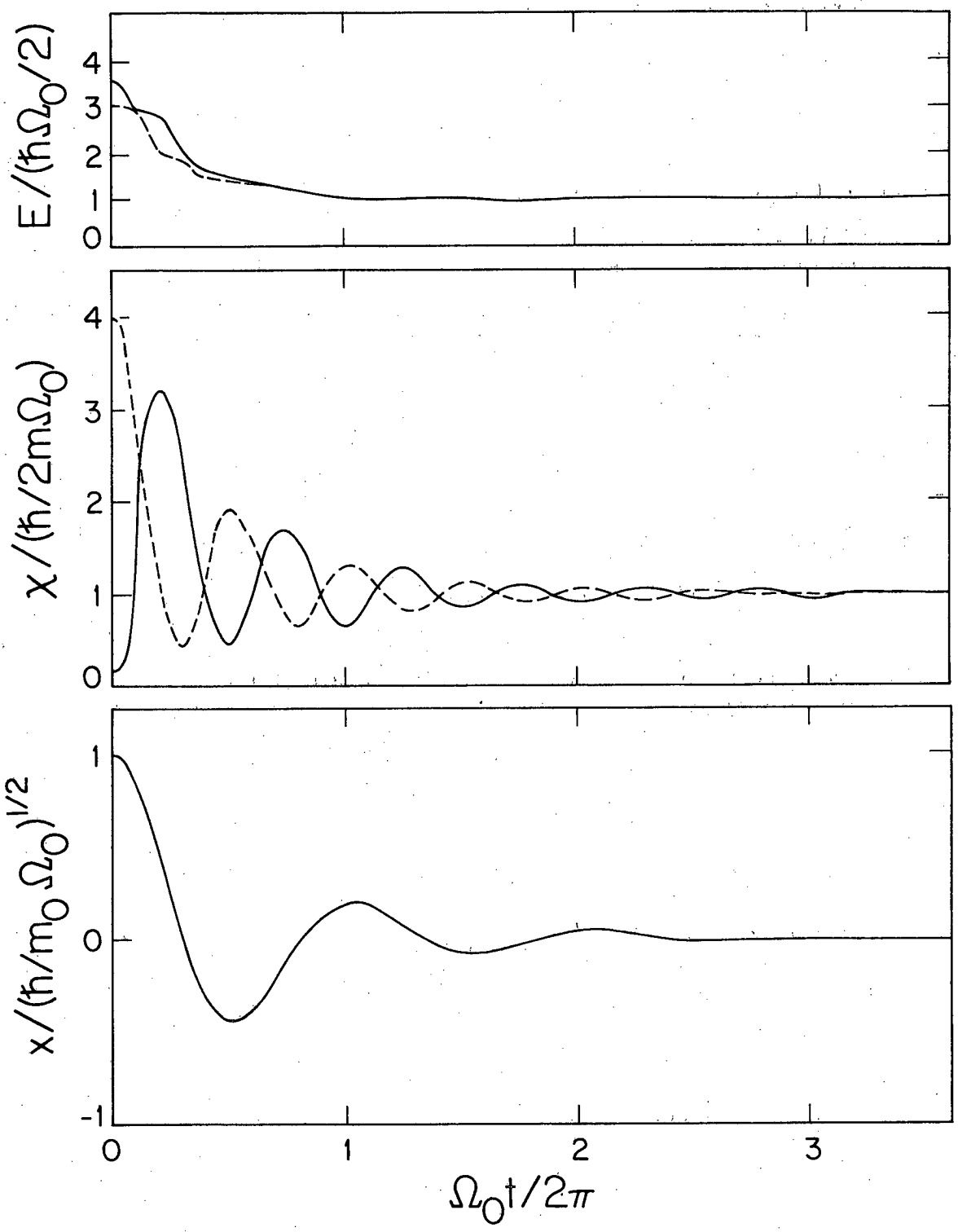


Fig. 1

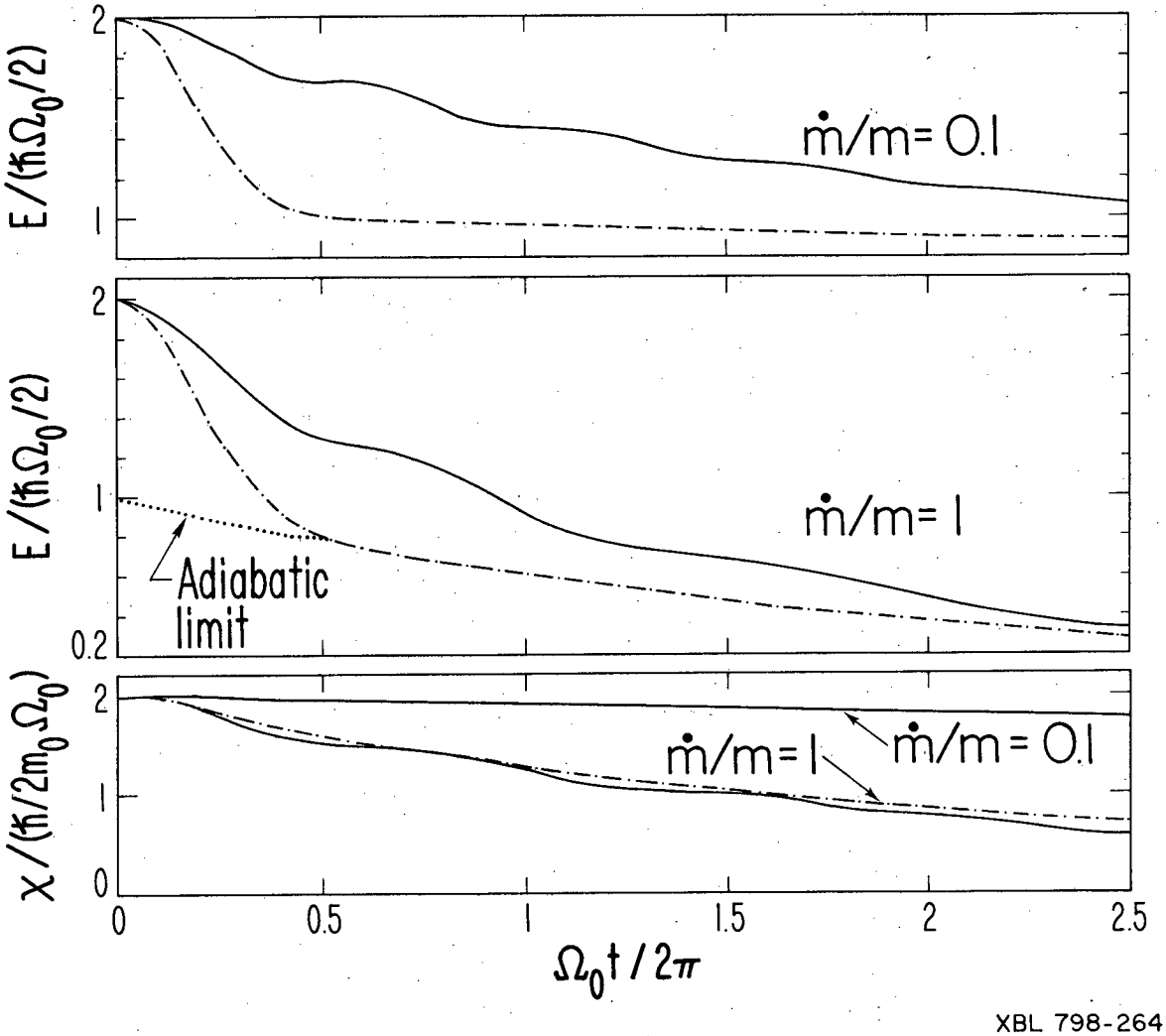


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Fig. 2

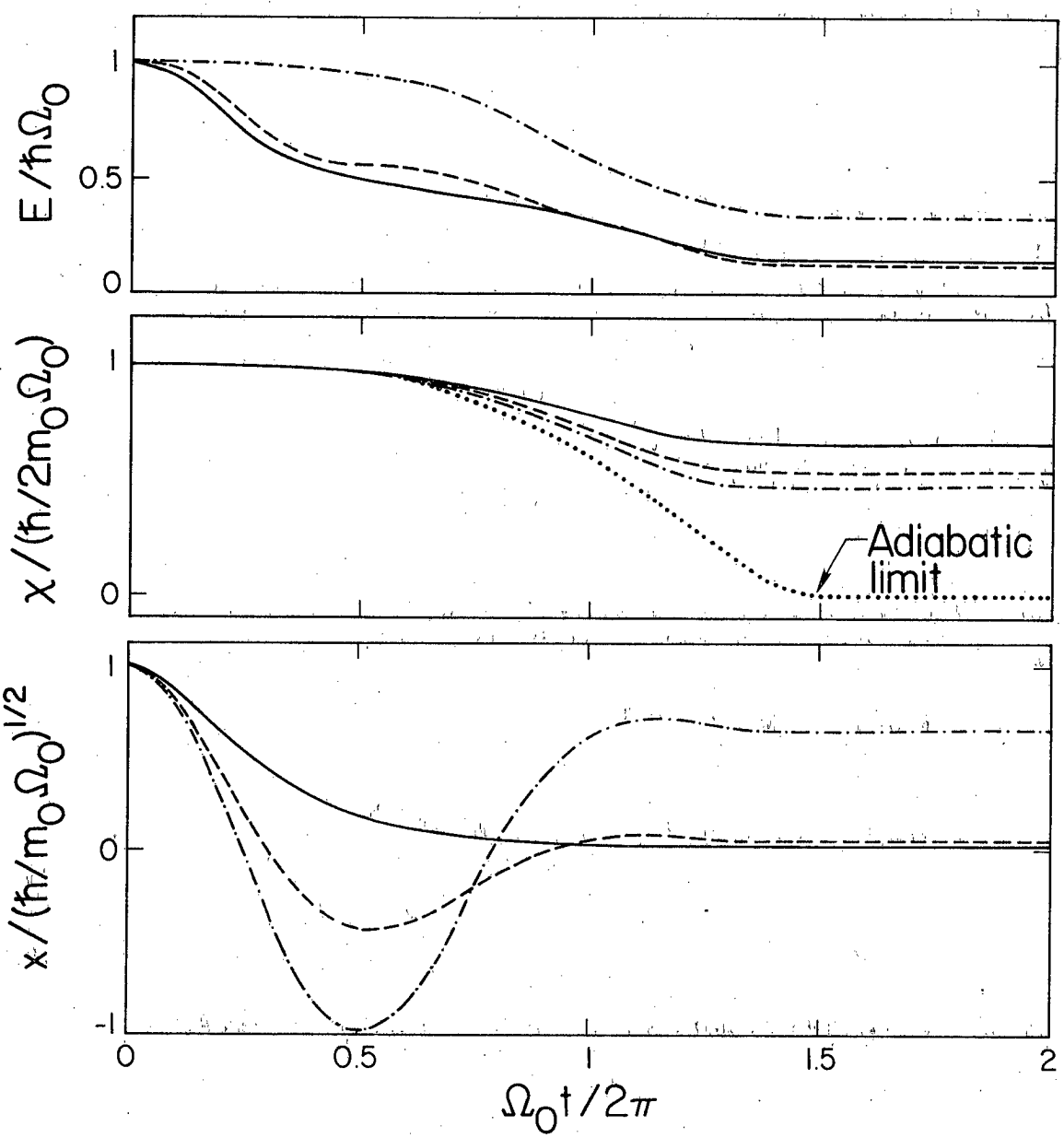


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Fig. 3

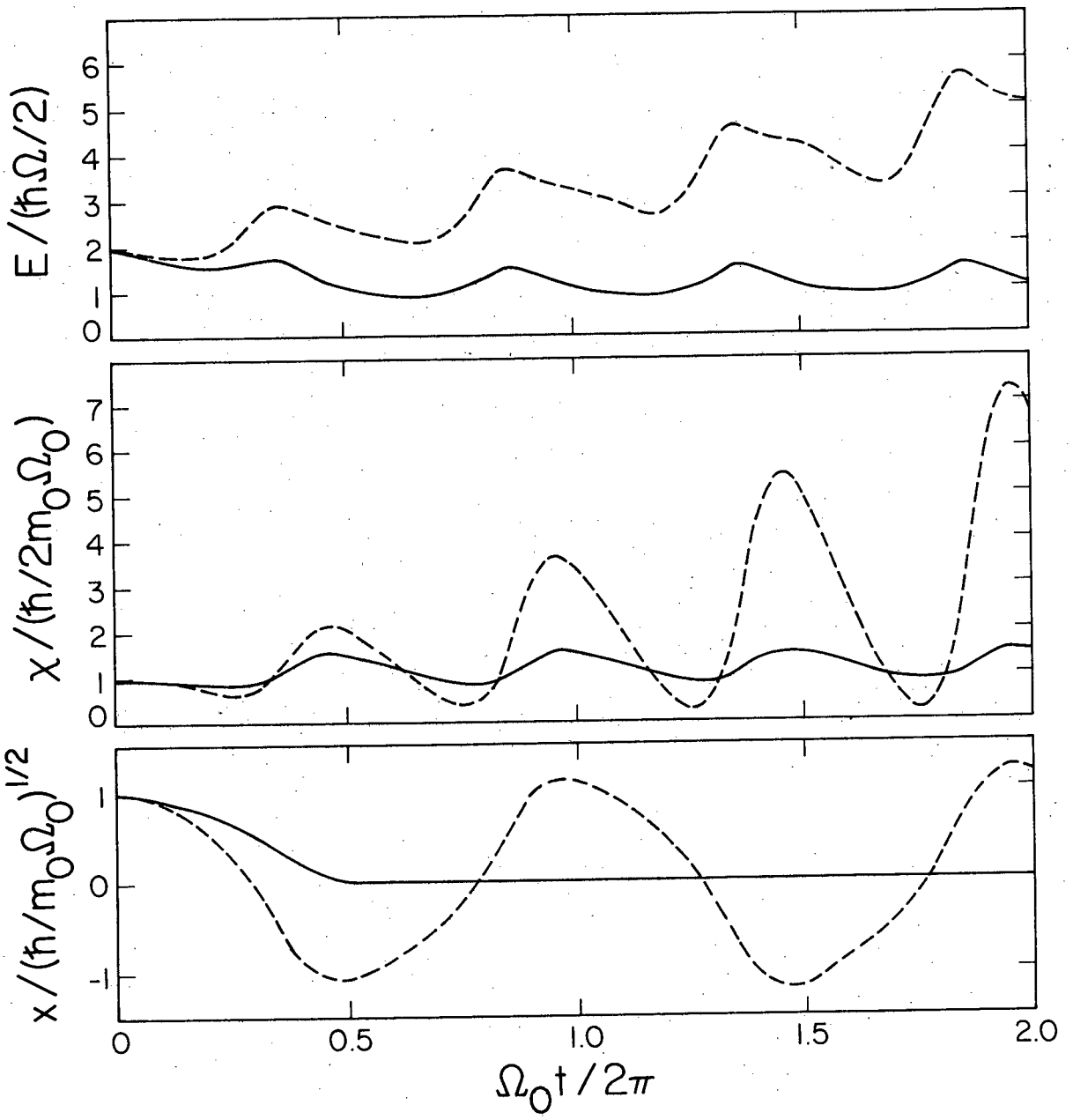


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Fig. 4

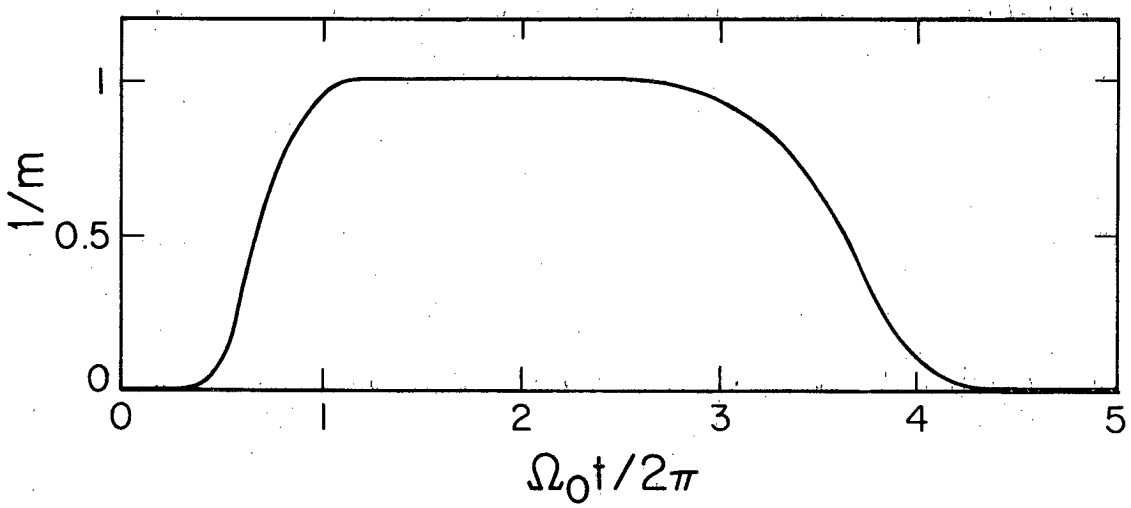


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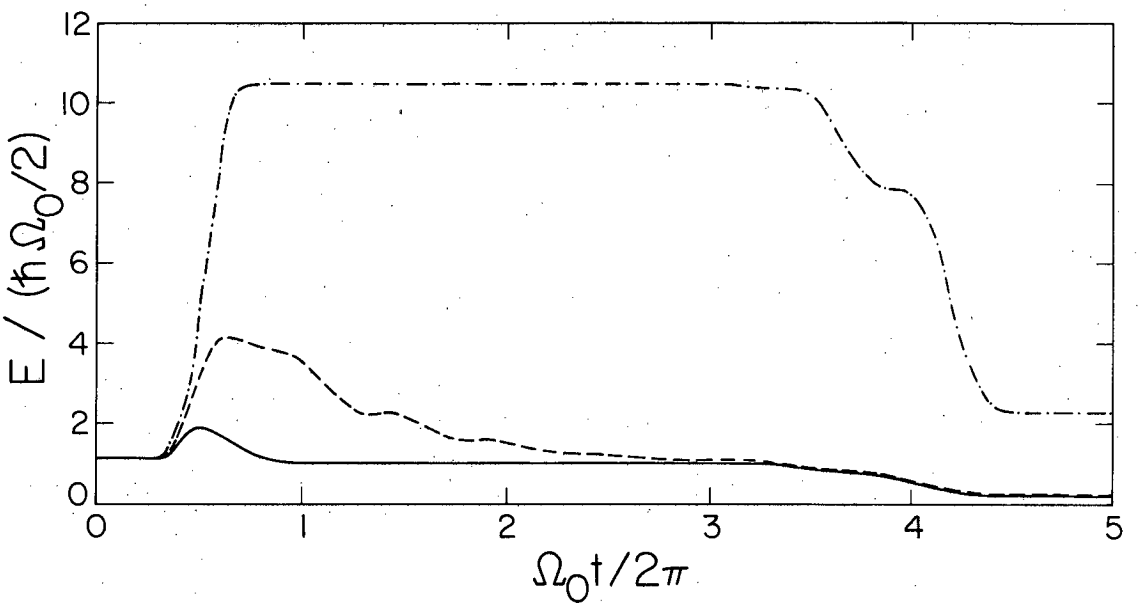
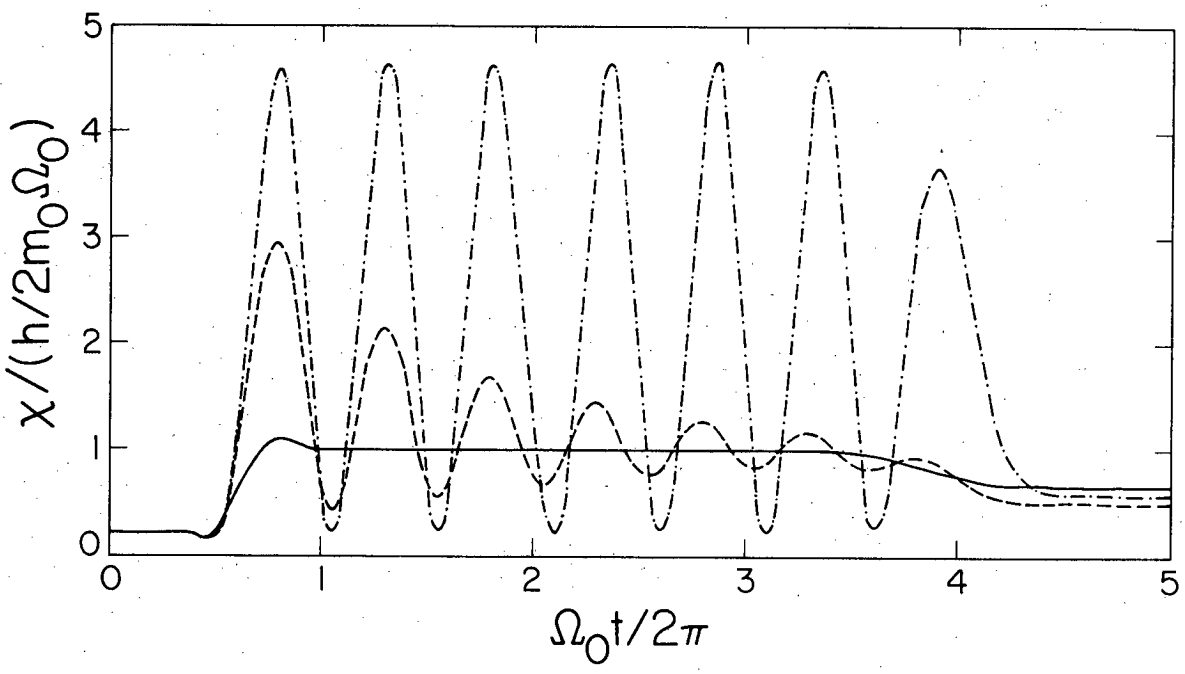
Fig. 5



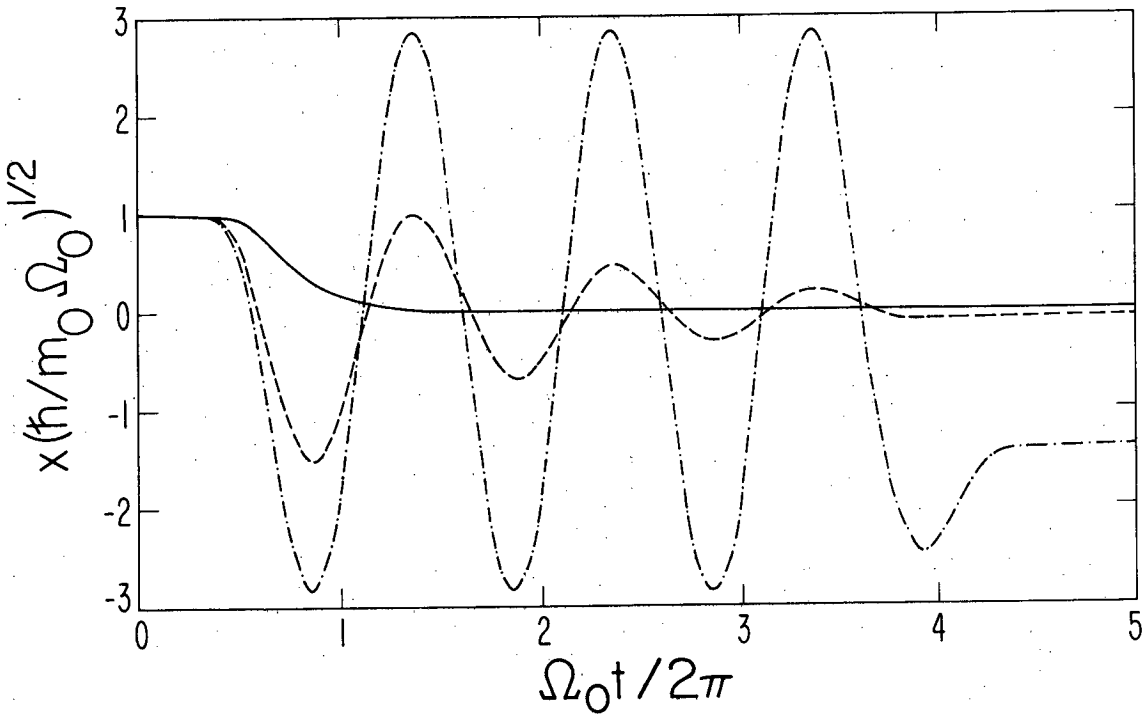

XBL 798-2637

Fig. 6




XBL 798-2638

Fig. 7




XBL 798-2643

Fig. 8

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