Title
ANOMALIES IN GAUGE THEORIES

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Publication Date
1987-03-01

LBL-23440 c.2

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Physics Division

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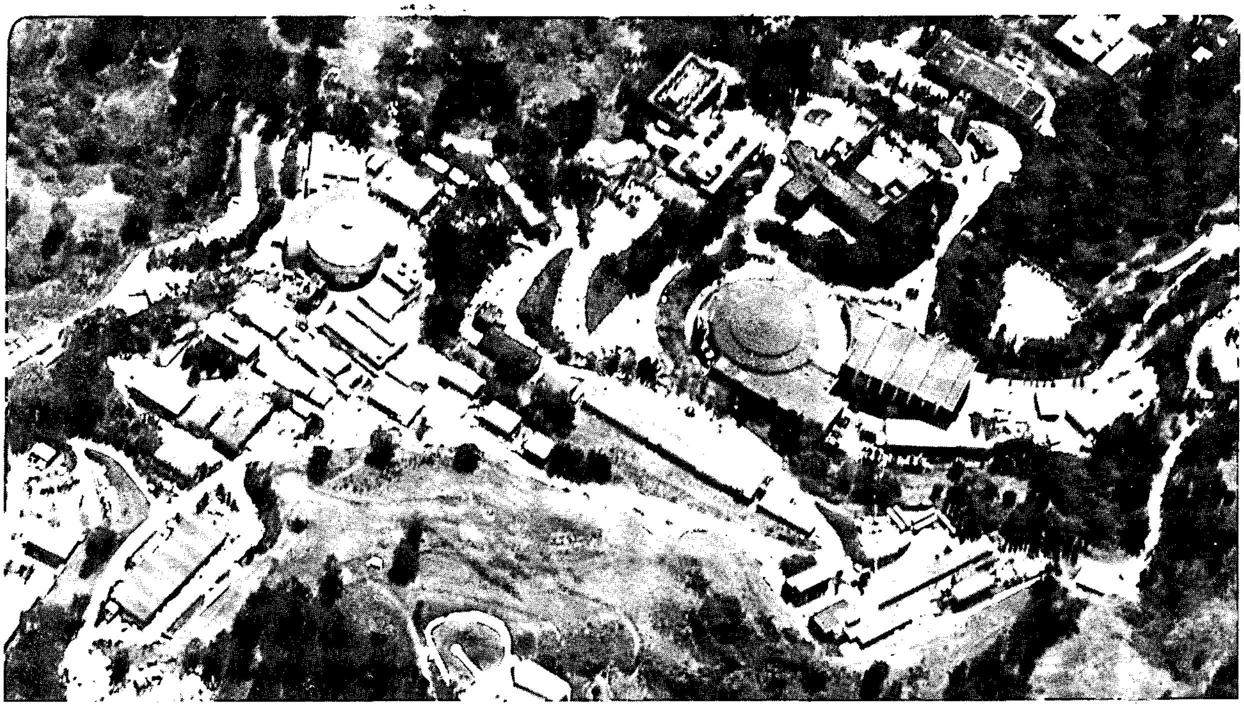
D.S. Hwang
(Ph.D. Thesis)

March 1987



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

UCB-PTH-87/20
LBL-23440

March 1987

ANOMALIES IN GAUGE THEORIES[1]

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Ph.D. Thesis

[1]This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and the National Science Foundation under Research Grant No. PHY-85-15857.

ACKNOWLEDGEMENTS

I wish to thank Professor Bruno Zumino for his guidance as my thesis advisor and many helps.

I am grateful to the professors who taught me during my graduate study, especially Professors Orlando Alvarez, Mary K. Gaillard, Gerson Goldhaber, Stanley Mandelstam, Jerrold E. Marsden, and Mahiko Suzuki.

I am also grateful to the members of the LBL theory group, especially Dr. Michael Chanowitz.

I would like to thank to fellow graduate students, Uwe Albertin, Zvi Bern, Hue-Sun Chan, Oren Cheyette, Mitch Golden, Yeong-Chuan Kao, Shobhit Mahajan, Sang-Jin Sin, Jon Yamron, and especially Randy Ingermanson and Zack Wolf for their discussions and friendship.

I thank my parents for so many things.

This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098, and the National Science Foundation under Research Grant No. PHY-85-15857.

CONTENTS

CHAPTER 1

INTRODUCTION

In field theory a symmetry at the classical level is sometimes not a symmetry at the quantum level. In such a situation the amount by which an effective action violates the symmetry, if it can not be removed by adding a local functional to the effective action, is called an anomaly. The gauge anomaly has been obtained by calculating Feynman diagrams [1-3], by the point-splitting method [4,5], and also by the path integral method by noticing that the measure of a fermion field is not invariant under a chiral transformation [6]. The anomaly was also understood in terms of differential geometry [7,8].

Anomalies themselves can be used for phenomenological applications since they give the amount of current non-conservation. Furthermore, by solving the anomaly equation $\delta W = Anomaly$, we can get the effective action W which gives the effective interactions among particles in the system. This was first done by Wess and Zumino for the case of $SU(3)_L \times SU(3)_R$ flavor symmetry and was applied to the interactions among the pseudo-scalar and vector particles successfully [9].

Another important application is as a criterion for the consistency of models of unified gauge theories. In order to have unitarity and renormalizability, a model should not have an anomaly of a dynamical symmetry [10,11]. For example, in the Weinberg-Salam model the gauge anomaly is canceled out for each generation when the lepton and the quark sectors are combined. A gravitational anomaly was found recently, and the cancellation of this anomaly became an important criterion for models which unify all the interactions including the gravitational interaction [12,13].

In this thesis we study various topics of anomalies in two dimensions. The reason for interest in two-dimensional anomalies is that they contain the main structures of higher dimensional anomalies and have their own interesting properties. The topics of this thesis consists of the anomalies of Yang-Mills gauge theory, gravitational theory, and supersymmetric gauge theory.

In chapter 2-4 we study the gauge anomaly. We obtain the solution of the anomaly equation (the Wess-Zumino term) with only gauge fields, without auxiliary fields. We show, up to the second non-trivial order, that this solution agrees with the result of Feynman diagram calculation. By studying the intimate relation between the anomaly and the Schwinger term, we find a method of obtaining the anomaly of $D_\mu J^\mu$ (chiral current) from the Schwinger terms of the equal time commutation relations. Through this procedure we can also understand easily the difficulties in quantizing anomalous gauge theories. By studying the Schwinger model, we show that the point-splitting method disagrees with the loop-diagram method by the sign and by the factor 1/2 for the anomaly of $\partial_\mu J_5^\mu$ and for the Schwinger term of $[J_5^0(x), J^0(y)]_{ETC}$ respectively.

In chapter 5 we study the gravitational anomaly. We calculate Feynman diagrams to get an effective action, and show, up to the second non-trivial order, that this effective action is in agreement with the anomaly obtained by the differential geometric method.

In chapter 6-7 we study the supersymmetry anomaly. We obtain a supersymmetric extension of the gauge anomaly, and find that this is the origin of a supersymmetry anomaly in the Wess-Zumino gauge. We obtain an effective action whose variations give rise to the gauge and supersymmetry anomalies in the Wess-Zumino gauge. We also find a supermultiplet which contains $\partial_\mu J_5^\mu$ and $\partial_\mu J^\mu$ as components, and a corresponding anomaly superfield. We confirm this anomaly superfield by diagram calculations.

In order to make comparisons with references easier, we use the metric of chapters 2-5 and that of chapters 6-7 differently. However, there will be no confusion since we specify the metric in each chapter.

CHAPTER 2

GAUGE ANOMALY

In this chapter we study Yang-Mills gauge fields coupled to chiral fermions. We obtain a solution to the anomaly equation with only gauge fields, without auxiliary fields. Similar problems were studied for the massless Dirac fermion case, and solutions were obtained in terms of gauge fields and auxiliary scalar fields, or in terms of auxiliary scalar fields alone [14-17]. However, we study the massless chiral fermion case and obtain the solution explicitly in terms of gauge fields alone, which is a power series of gauge fields for the non-Abelian theory. Then we can compare this solution with the result of Feynman diagram calculations.

In section 2.1 we obtain the gauge anomaly including the normalization factor up to the sign by using the differential geometric method. In section 2.2 we solve the anomaly equation to get the effective action which contains only gauge fields. In section 2.3 we calculate one-loop diagrams up to $O(A^3)$ in the effective action. We show that this calculation agrees with the results of sections 2.1 and 2.2.

2.1 Gauge Anomaly

Our system is composed of a multiplet of left-handed fermions and a multiplet of gauge fields. Its Lagrangian is given by

$$L = i\bar{\psi}\gamma^a(\partial_a + A_a)\psi \ , \tag{2.1}$$

where

$$\frac{1-\gamma_5}{2}\psi = \psi\ , \quad A_a = A_{ai}T_i\ , \quad [T_i, T_j] = f_{ijk}T_k\ .$$

We use $i,\ j,\ k,\ \cdots$ for group indicies, and $a,\ b,\ c,\ \cdots$ for Lorentz indicies. Our conventions for the metric and gamma matrices are given by

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\ , \qquad \eta^{00} = -\eta^{11} = 1\ ,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ , \quad \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\ . \tag{2.2}$$

We treat fermions as quantized fields and gauge fields as external classical fields. Their infinitesimal gauge transformations are given by

$$\begin{cases} \delta_\Lambda \psi = -\Lambda\psi \\ \delta_\Lambda A_a = D_a\Lambda = \partial_a\Lambda + [A_a, \Lambda]\ , \end{cases} \tag{2.3}$$

where

$$\Lambda = \Lambda_i T_i\ .$$

In this paper we study the consistent anomaly which is given by a variation of a connected vacuum functional. We call this variation an anomaly if we cannot make it vanish by adding a local functional to the connected vacuum functional. There is another kind of anomaly, a covariant anomaly (which transforms covariantly), but this anomaly is not given by a variation of a functional [13].

The consistent anomaly is defined by an equation which we will call an anomaly equation :

$$\delta_\Lambda W[A] = \Lambda \cdot G(A)\ , \tag{2.4}$$

where

$$\Lambda \cdot G(A) \equiv \int dx\, \Lambda_i G_i(A) \equiv [\text{ non-Abelian Anomaly }]\ , \tag{2.5}$$

$$\text{with} \int dx \equiv \int d(\text{Volume})\ .$$

In (2.4) $W[A]$ is the effective action, i.e., the connected vacuum functional. (2.4) tells us by how much the quantum effect of fermion fields causes the system not to be gauge invariant.

Since the consistent anomaly is given by a gauge variation of $W[A]$, we have "a consistency condition", i.e.,

$$(\delta_{\Lambda_1}\delta_{\Lambda_2} - \delta_{\Lambda_2}\delta_{\Lambda_1})W[A] = \delta_{[\Lambda_1,\Lambda_2]}W[A] \tag{2.6}$$

gives the consistency condition

$$\int dx\, (\Lambda_{2i}\delta_{\Lambda_1}G_i - \Lambda_{1i}\delta_{\Lambda_2}G_i) = \int dx\, ([\Lambda_1, \Lambda_2])_i G_i \tag{2.7}$$

which can also be used as a definition of the anomaly [9].

In order to obtain the two-dimensional non-Abelian anomaly which satisfies the consistency condition (2.7), let us follow briefly the differential geometric method given by Zumino [8]. The Atiyah-Singer index of the Dirac operator $\gamma^a D_a = \gamma^a(\partial_a + A_a)$ is given by the integral of the Chern character [18,19].

$$(n_+ - n_-) = \int_M Ch(V)\ , \tag{2.8}$$

where

$$Ch(V) = \mathrm{Tr}[\exp(\frac{i}{2\pi}F)]\ , \tag{2.9}$$

with

$$F = \frac{1}{2}\, F_{ab}\, dx^a \wedge dx^b\ , \quad F_{ab} = \partial_a A_b - \partial_b A_a + A_a A_b - A_b A_a\ . \qquad (2.10)$$

In order to obtain the two-dimensional non-Abelian anomaly, we start from the Chern character in four dimensions.

$$\begin{aligned} Ch(V)[\text{4-dim.}] &= \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \mathrm{Tr}(F^2) \\ &\equiv d\omega_3(A, F)\ , \end{aligned} \qquad (2.11)$$

where

$$\omega_3(A, F) = -\frac{1}{8\pi^2}\mathrm{Tr}(AF - \frac{1}{3}A^3)\ . \qquad (2.12)$$

Then the two-dimensional non-Abelian gauge anomaly is given by $\omega_2^1(v, A, F)$ which is first order in v when we expand $\omega_3(A + v, F)$ in powers of v, i.e.,

$$\omega_2^1(v, A, F) = -\frac{1}{8\pi^2}\mathrm{Tr}[v dA]\ . \qquad (2.13)$$

The non-Abelian anomaly is normalized with an additional factor of 2π in order to give the unique $Z = e^{iW}$, i.e.,

$$[\text{2-dim. non-Abelian Ano.}] = -\frac{1}{4\pi}\int_M \mathrm{Tr}[v dA]\ . \qquad (2.14)$$

By changing the form notation to the tensor notation, we have

$$[\text{2-dim. non-Abelian Ano.}] = -\frac{1}{4\pi}\int d^2x\, \mathrm{Tr}[\Lambda \partial_a A_b \epsilon^{ab}]\ , \qquad (2.15)$$

or $G_i(A)$ in (2.4) is given by

$$G_i(A) = -\frac{1}{4\pi}\mathrm{Tr}[T_i \partial_a A_b \epsilon^{ab}]\ . \qquad (2.16)$$

In the above derivation the overall sign of the anomaly is ambiguous. By chance it turns out that the above sign agrees with the Feynman diagram calculation in section 2.3.

In the following light-cone coordinates will be used often. The conventions and properties of the light-cone coordinates and ϵ^{ab} which we will use are given below.

$$\begin{aligned} x^{\pm} &= \frac{1}{\sqrt{2}}(x^0 \pm x^1)\ , \quad x_{\pm} = \frac{1}{\sqrt{2}}(x_0 \pm x_1)\ , \quad \gamma^{\pm} = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^1)\ , \\ &\gamma^+\gamma^- + \gamma^-\gamma^+ = 2\ , \quad \gamma^+\gamma^+ = \gamma^-\gamma^- = 0\ , \\ &\eta^{+-} = \eta^{-+} = \eta_{+-} = \eta_{-+} = 1 \ (\text{other } \eta\text{'s are zero})\ , \\ &\epsilon^{10} = -\epsilon^{01} = 1\ , \quad \epsilon^{+-} = -\epsilon^{-+} = 1\ . \end{aligned} \qquad (2.17)$$

Another form of the anomaly, which will be useful in the following analysis, is obtained by adding the gauge variation of a local functional $1/8\pi \mathrm{Tr} \int d^2x\, A_a A^a$ to (2.15), i.e.,

$$\begin{aligned} [\text{2-dim. non-Abelian Ano.}] &= -\frac{1}{4\pi}\int d^2x\, \mathrm{Tr}[\Lambda \partial_a A_b(\epsilon^{ab} + \eta^{ab})] \\ &= -\frac{1}{2\pi}\int d^2x\, \mathrm{Tr}[\Lambda \partial_+ A_-]\ . \end{aligned} \qquad (2.18)$$

2.2 Solution of the Anomaly Equation - Effective Action

The solution of the anomaly equation gives the effective action of the system. Wess and Zumino solved this equation in the following way [8,9].

Introduce ξ fields which transform non-linearly under a finite gauge transformation as

$$e^{\xi'} = e^{\xi} e^{\Lambda} \tag{2.19}$$

where

$$\Lambda = \Lambda_i T_i \ , \quad \xi = \xi_i T_i \ .$$

Then the solution of (2.4) is given in a compact form as

$$W[A,\xi] = \int dx \int_0^1 dt \ \xi_i G_i(A(t))(x) + W_C[A,\xi] \ , \tag{2.20}$$

where

$$A_a(t) = e^{t\xi} A_a e^{-t\xi} + e^{t\xi}\partial_a e^{-t\xi} \tag{2.21}$$

and $W_C[A,\xi]$ is an arbitrary gauge invariant functional.

In their original work, Wess and Zumino dealt with $SU(3)_V \times SU(3)_A$ flavor symmetry and treated the pseudo-scalar octet as non-linearly realized fields. Their solution describes the interactions among the pseudo-scalar and vector particles in good agreement with experiments.

In this section we will perform one more step to obtain a solution which is a functional of only gauge fields, without independent ξ fields. This solution will be useful since in section 2.3 it will be compared with Feynman diagram calculations which have only gauge fields as external fields. This solution is also interesting since it gives a system which contains only gauge fields. In (2.20) we observe that instead of independent ξ fields, we can use the functions $\xi(A)$ of gauge fields which transform as (2.19) when A transform as gauge fields, if we can find such functions.

In the Abelian case, we easily find such $\xi(A)$ as

$$\xi(A) = \frac{1}{\Box}\partial_a A^a \ , \qquad \text{where} \quad \Box = \partial_a\partial^a \ , \tag{2.22}$$

since

$$\xi'(A) = \frac{1}{\Box}\partial_a(A')^a = \frac{1}{\Box}\partial_a(A^a + \partial^a\Lambda) = \xi(A) + \frac{1}{\Box}\partial_a\partial^a\Lambda = \xi(A) + \Lambda \ , \tag{2.23}$$

which is the same as (2.19) for the Abelian case.

Then in two dimensions the solution of the anomaly equation can be given with only gauge fields as

$$\begin{aligned} W[A] &= \int d^2x \int_0^1 dt \, (\frac{1}{\Box}\partial_a A^a) \, G(A(t)) + W_C[A] \\ &= \int d^2x \int_0^1 dt \, (\frac{1}{\Box}\partial_c A^c)(\frac{i}{2\pi}\partial_a A_b(t)\epsilon^{ab}) + W_C[A] \ , \end{aligned} \tag{2.24}$$

where

$$A_b(t) = A_b + e^{t\xi(A)}\partial_b e^{-t\xi(A)} = A_b - t\partial_b\xi(A) \ . \tag{2.25}$$

Since the second term of (2.25) does not contribute in (2.24), we have

$$W[A] = \frac{i}{2\pi}\int d^2x \, (\frac{1}{\Box}\partial_c A^c)(\partial_a A_b \epsilon^{ab}) + W_C[A] \ . \tag{2.26}$$

Let us now consider the non-Abelian case in two dimensions. We notice first that when ξ transform as $e^{\xi'} = e^{\xi}e^{\Lambda}$,

$$A_a = e^{-\xi}\partial_a e^{\xi} \tag{2.27}$$

transforms like a gauge field as seen by

$$\begin{aligned} A'_a &= e^{-\xi'}\partial_a e^{\xi'} \\ &= e^{-\Lambda}e^{-\xi}\partial_a(e^{\xi}e^{\Lambda}) \\ &= e^{-\Lambda}(e^{-\xi}\partial_a e^{\xi})e^{\Lambda} + e^{-\Lambda}\partial_a e^{\Lambda} \\ &= e^{-\Lambda}A_a e^{\Lambda} + e^{-\Lambda}\partial_a e^{\Lambda} \,. \end{aligned} \tag{2.28}$$

Conversely, if we invert (2.27), we obtain $\xi(A)$ which transform as (2.19) when A transform as gauge fields.

We are going to obtain $W[A]$ as a functional of only A_- since in the next section we will compare this with Feynman diagram calculations which have only A_- as external fields. For this we will use the anomaly of the form (2.18) and invert (2.27) for $a = -$,

$$A_- = e^{-\xi}\partial_- e^{\xi} \,. \tag{2.29}$$

Note that in (2.29) we relate only A_- with ξ in this form, but A_+ is not related with ξ and arbitrary, therefore our gauge fields A_a's are not restricted to be pure gauge fields.

In the following procedure of inversion of (2.29), we will denote A_- by A and x^- by x for notational simplicity. Then (2.29) becomes

$$A = e^{-\xi}\frac{d}{dx}e^{\xi} \,. \tag{2.30}$$

Let us multiply both sides of (2.30) on the right by $e^{-\xi}$. Then we have

$$Ae^{-\xi} = e^{-\xi}(\frac{d}{dx}e^{\xi})e^{-\xi} = -\frac{d}{dx}e^{-\xi} \,. \tag{2.31}$$

By defining

$$\eta \equiv e^{-\xi} \,, \tag{2.32}$$

(2.31) becomes

$$\frac{d}{dx}\eta = -A\eta \,. \tag{2.33}$$

Solving (2.33) by iteration we get

$$\begin{aligned} \eta(x) &= \eta(-\infty) - \int_{-\infty}^{x} dx_1\, A(x_1)\eta(x_1) \\ &= \eta(-\infty) - \int_{-\infty}^{x} dx_1\, A(x_1)\eta(-\infty) + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2\, A(x_1)A(x_2)\eta(-\infty) \\ &\quad - \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2 \int_{-\infty}^{x_2} dx_3\, A(x_1)A(x_2)A(x_3)\eta(-\infty) + \quad \cdots \,. \end{aligned} \tag{2.34}$$

In (2.34), note that all x's in this equation are x^- component, however η and A are also dependent on x^+ even though we omitted writing this dependence explicitly.

Let us take the boundary condition

$$\xi(\text{at} \quad x = -\infty) = 0 \,, \quad \text{i.e.}, \quad \eta(-\infty) = 1 \,. \tag{2.35}$$

We also assume that $\lim_{x\to-\infty} A(x) = 0$ sufficiently rapidly.) Then (2.34) becomes

$$\begin{aligned} e^{-\xi(A)} &= 1 - \int_{-\infty}^{x} dx_1\, A(x_1) + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2\, A(x_1)A(x_2) \\ &\quad + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2 \int_{-\infty}^{x_2} dx_3\, A(x_1)A(x_2)A(x_3) + \cdots \,, \end{aligned} \tag{2.36}$$

which can be given as a compact expression

$$e^{-\xi(A)} = P[exp(-\int_{-\infty}^{x} dx'\, A(x'))] \tag{2.37}$$

by defining the path ordered product P to mean (2.36).

From (2.36) we can get $\xi(A)$ in a power series of A,

$$-\xi(A) = \ln\{1 - \int_{-\infty}^{x} dx_1\, A(x_1) + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2\, A(x_1)A(x_2) + \cdots\}\,. \quad (2.38)$$

Using $\ln(1+x) = x - \frac{1}{2}x^2 + \cdots$, (2.38) becomes

$$\xi(A) = \int_{-\infty}^{x} dx_1\, A(x_1) - \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2\, A(x_1)A(x_2) - \frac{1}{2}\Big(\int_{-\infty}^{x} dx_1\, A(x_1)\Big)^2 + O(A^3)\,, \quad (2.39)$$

where all the A's are A_- and they depend on both x^- and x^+.

We are now prepared to obtain $W[A]$ in terms of only gauge fields. Using the anomaly in (2.18), we get from (2.20)

$$W[A] = -\frac{1}{2\pi}\int d^2x \int_0^1 dt\, \mathrm{Tr}[\xi(A_-)\partial_+ A_-(t)]\,, \quad (2.40)$$

where

$$A_-(t) = e^{t\xi(A_-)}A_- e^{-t\xi(A_-)} + e^{t\xi(A_-)}\partial_- e^{-t\xi(A_-)}\,, \quad (2.41)$$

and $\xi(A_-)$ is given in (2.36), (2.37), or (2.39).

We can get $W[A]$ as a power series of A_- by the following procedure. First expand $A_-(t)$ of (2.41) in terms of t and integrate over t in (2.40). Then $W[A]$ becomes a sum of products of $\xi(A_-)$'s and A_-'s. Next expand the $\xi(A_-)$'s in $W[A]$ as power series of A_- using (2.39). Then $W[A]$ in (2.40) becomes a sum of products of power series of A_-. As a last step, expand these products to get $W[A]$ as one power series of A_-.

After following this procedure, we get

$$W[A] = -\frac{1}{2\pi}\mathrm{Tr}\int d^2x\; [\,\frac{1}{2}f(x)\partial_+ A_-(x) + \frac{7}{12}f(x)\partial_+\{f(x)A_-(x)\} + \frac{5}{12}f(x)\partial_+\{A_-(x)f(x)\} - \frac{1}{2}g(x)\partial_+ A_-(x) - \frac{1}{4}(f(x))^2\partial_+ A_-(x)\,] + O(A^4)\,, \quad (2.42)$$

where

$$f(x) = f(x^-, x^+) \equiv \int_{-\infty}^{x^-} dx_1^-\, A_-(x_1^-, x^+)\,,$$

$$g(x) = g(x^-, x^+) \equiv \int_{-\infty}^{x^-} dx_1^- \int_{-\infty}^{x_1^-} dx_2^-\, A_-(x_1^-, x^+)A_-(x_2^-, x^+)\,. \quad (2.43)$$

In momentum space, (2.42) becomes

$$W[A] = \frac{1}{4\pi}\{\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2} d^2q\, \delta^2(p+q)\frac{p_+}{p_-}A_-(p)A_-(q) + \frac{2}{3}i\,\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r)\frac{p_+q_- - q_+p_-}{p_-q_-r_-}A_-(p)A_-(q)A_-(r) + O(A^4)\;\}\,. \quad (2.44)$$

As usual, (2.40) or (2.44) is ambiguous by a local functional of A_a.

When we obtained (2.44) from (2.42), we used the convolution properties

$$\int d^2x\, a(x)b(x) = \int \frac{d^2p}{(2\pi)^2} d^2q\, \delta^2(p+q)\, \hat{a}(p)\hat{b}(q)\,,$$

$$\int d^2x\, a(x)b(x)c(x) = \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r)\, \hat{a}(p)\hat{b}(q)\hat{c}(r) \quad (2.45)$$

(where $\hat{a}(p)$ is the Fourier transform of $a(x)$, etc.), and the properties of $f(x)$ and $g(x)$

$$\hat{f}(p) = \frac{A_-(p)}{ip_-}\,,$$

$$\int d^2x\, a(x)g(x) = \int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\, d^2r\, \delta^2(p+q+r)\, \hat{a}(p)\frac{A_-(q)}{-ip_-}\frac{A_-(r)}{ir_-} \,. \quad (2.46)$$

2.3 Comparison with Diagram Calculations

For our diagram calculations it is convenient to use the light-cone coordinates given in (2.17) [12]. In these coordinates the condition for a left-handed chiral fermion $(1+\gamma_5)\psi = \psi$ (where $\gamma_5 = \gamma^0\gamma^1$) becomes simply $\gamma^+\psi = 0$ or $\gamma_-\psi = 0$. Then, from (2.1) we have the interaction Lagrangian

$$L_{int.} = i\bar{\psi}\gamma_+ A_- \psi \,, \quad (2.47)$$

i.e., only the A_- component of the gauge field couples to the left-handed chiral fermion. Then we have the following Feynman rules.

For a vertex, from iL_{int},

$$-\gamma_+ A_- \,, \quad (2.48)$$

and for a propagator of a fermion,

$$\frac{ip_a\gamma^a}{p^2+i\varepsilon} = \frac{ip_+\gamma_- + ip_-\gamma_+}{2p_+p_- + i\varepsilon} = \frac{i}{2}\frac{\gamma_-}{p_- + i\varepsilon/p_+} \,, \quad (2.49)$$

where we have eliminated the γ_+ part since the vertex contains γ_+ and $(\gamma_+)^2 = 0$.

Using the fact $\mathrm{Tr}(\gamma_+\gamma_-)^n = 2^n$, we obtain the Feynman rule given in Fig.1 for one-loop diagrams.

Diagram [2.1] gives the ampltude

$$Amp. = -\int \frac{d^2k}{(2\pi)^2}\mathrm{Tr}[(-A_-(p))(-A_-(q))]\{\frac{i}{k_- + i\varepsilon/k_+}\}\{\frac{i}{(k+p)_- + i\varepsilon/(k+p)_+}\} \,. \quad (2.50)$$

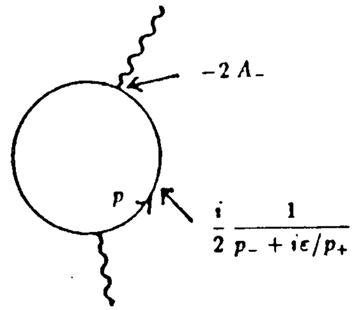


Fig.1 Feynman rule : Take $\mathrm{Tr}\int \frac{d^2k}{(2\pi)^2}$, and attach (-) sign for a fermion loop. Then multiply the symmetry factor $(1/n!)$ for an effective action.

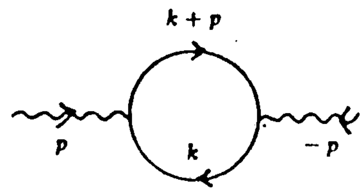
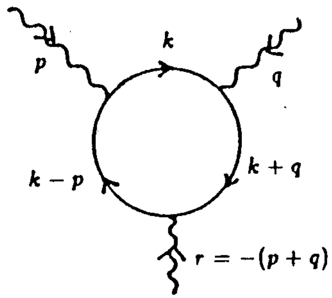


Fig.2 Diagram [2.1]



$+\ (p \longleftrightarrow q)$

Fig.3 Diagram [2.2]

We integrate first over k_- by using the residue method and then over k_+ to get the result [12]

$$Amp. = \frac{i}{2\pi}\frac{p_+}{p_-}\mathrm{Tr}[A_-(p)A_-(-p)]\,. \tag{2.51}$$

In order to obtain the effective action, we attach the symmetry factor (1/2!) and use the fact that the amplitude of a diagram calculation corresponds to $iW[A]$. Then we get for $O(A^2)$

$$W_1[A] = \frac{1}{4\pi}\int\frac{d^2p}{(2\pi)^2}d^2q\,\delta^2(p+q)\frac{p_+}{p_-}\mathrm{Tr}[A_-(p)A_-(q)]\,. \tag{2.52}$$

We call the lowest order of non-vanishing terms the first non-trivial order, and so on.

For the next order in $A(x)$, i.e., $O(A^3)$, we calculate Diagram [2.2].

$$\begin{aligned} Amp. &= -i\mathrm{Tr}[A_-(p)A_-(q)A_-(r)] \\ &\quad\times\int\frac{d^2k}{(2\pi)^2}\{\frac{1}{k_- + i\epsilon/k_+}\}\{\frac{1}{(k+q)_- + i\epsilon/(k+q)_+}\}\{\frac{1}{(k-p)_- + i\epsilon/(k-p)_+}\} \\ &\quad + (p \leftrightarrow q) \\ &= (-\frac{1}{2\pi})\frac{p_+q_- - q_+p_-}{p_-q_-r_-}\mathrm{Tr}[A_-(p)A_-(q)A_-(r)] \quad + (p \leftrightarrow q)\,, \end{aligned} \tag{2.53}$$

where we followed the same procedure as (2.50), (2.51).

We attach the symmetry factor (1/3!) and match this with $iW_2[A]$,

$$\begin{aligned} W_2[A] = i\frac{2}{2\pi}\frac{1}{3!}\int\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}d^2r\,\delta^2(p+q+r)\frac{(p_+q_- - q_+p_-)}{p_-q_-r_-} \\ \times\mathrm{Tr}[A_-(p)A_-(q)A_-(r)]\,. \end{aligned} \tag{2.54}$$

Adding (2.52) and (2.54) to have $W[A]$ up to $O(A^3)$,

$$\begin{aligned} W[A] = \frac{1}{4\pi}\{&\mathrm{Tr}\int\frac{d^2p}{(2\pi)^2}d^2q\,\delta^2(p+q)\frac{p_+}{p_-}A_-(p)A_-(q) \\ &+ i\frac{2}{3}\mathrm{Tr}\int\frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}d^2r\,\delta^2(p+q+r)\frac{(p_+q_- - q_+p_-)}{p_-q_-r_-}A_-(p)A_-(q)A_-(r) \\ &+ O(A^4)\}\,. \end{aligned} \tag{2.55}$$

This is the same as (2.44) which was obtained by expanding the solution (2.40) of the anomaly equation. Therefore it has been shown that (2.40) agrees with the diagram calculations up to the second non-trivial order, i.e., $O(A^3)$ in the effective action.

It is also interesting to see explicitly how (2.55) gives rise to the anomaly (2.18). We calculate $\delta_\Lambda W[A]$ in (2.4) by applying the gauge transformation (2.3) to (2.55). The derivative (∂_α) in (2.3) corresponds to (ip_α) in momentum space since we take external momenta as incoming as can be seen in Diagram

[2.1] and Diagram [2.2]. From the $O(A^2)$ term of $W[A]$, i.e., $W_1[A]$,

$$\begin{aligned}\delta_\Lambda W_1[A] = {}& \frac{1}{4\pi}\times 2\{\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2} d^2q\, \delta^2(p+q)\frac{p_+}{p_-}A_-(p)[iq_-\Lambda(q)] \\ & + \mathrm{Tr}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r)\frac{p_+}{p_-}A_-(p)[A_-(q)\Lambda(r)-\Lambda(r)A_-(q)]\} \\ = {}& \frac{1}{4\pi}\times 2\{\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2} d^2q\, \delta^2(p+q)(-ip_+)A_-(p)\Lambda(q) \\ & + \mathrm{Tr}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r)(\frac{p_+}{p_-}-\frac{q_+}{q_-})A_-(p)A_-(q)\Lambda(r)\}\ .\end{aligned} \tag{2.56}$$

From the $O(A^3)$ term of $W[A]$, i.e., $W_2[A]$,

$$\begin{aligned}\delta_\Lambda W_2[A] = {}& i\frac{1}{4\pi}\frac{2}{3}\times 3\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r)\frac{(p_+q_- - q_+p_-)}{p_-q_-r_-} \\ & A_-(p)A_-(q)[ir_-\Lambda(r)] \quad + O(A^3) \\ = {}& -\frac{2}{4\pi}\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2}\frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r)(\frac{p_+}{p_-}-\frac{q_+}{q_-})A_-(p)A_-(q)\Lambda(r) \\ & + O(A^3)\ .\end{aligned} \tag{2.57}$$

When we add (2.56) and (2.57), we see that the second term of (2.56) is cancelled by the first term of (2.57). We then expect that the second term of (2.57) will be cancelled by the first term of $\delta_\Lambda W_3[A]$, where $W_3[A]$ is the $O(A^4)$ term of $W[A]$, and so on. Therefore we expect that the direct contribution to the anomaly comes only from the two point vacuum functional. The reason behind this is that higher order diagrams are more finite and do not contribute to the anomaly. Then we have consistently with (2.18)

$$\begin{aligned}\delta_\Lambda W[A] &= \frac{1}{2\pi}\mathrm{Tr}\int \frac{d^2p}{(2\pi)^2} d^2q\, \delta^2(p+q)(-ip_+)A_-(p)\Lambda(q) \\ &= -\frac{1}{2\pi}\mathrm{Tr}\int d^2x\, \Lambda(x)\partial_+A_-(x)\ .\end{aligned} \tag{2.58}$$

CHAPTER 3

ANOMALY OF $D_\mu J^\mu$ FROM SCHWINGER TERMS

After the discovery of the anomaly of $\partial_\mu J_5^\mu$ (where $J_5^\mu = \bar\psi\gamma^\mu\gamma_5\psi$), it was soon understood that the Schwinger term of the equal time commutation relation is another face of the anomaly [5]. Recently this relation has been studied in the differential geometric method [20,21]. In this chapter we will show this relation clearly by obtaining the anomaly of $D_\mu J^\mu$ (where $J_a^\mu = \bar\psi\gamma^\mu\frac{1-\gamma_5}{2}\lambda_a\psi$) from the Schwinger terms of the equal time commutation relations.

We first derive a classical relation $\partial_0 G_a = (D_\mu J^\mu)_a$. Then we calculate the quantum version of this relation. By this procedure we obtain the anomaly of $D_\mu J^\mu$ from the Schwinger terms. This relation also suggests the situation that when $D_\mu J^\mu$ is anomalous, the constraint $G_a \mid \mathrm{phys}\rangle = 0$ does not propagate in time. We confirm this explicitly at the quantum level. This feature causes a difficulty in quantizing an anomalous gauge theory.

In section 3.1 we obtain a classical relation $\partial_0 G_a = (D_\mu J^\mu)_a$. In section 3.2 we obtain the anomaly of $\partial_\mu J^\mu$ from the Schwinger terms for the chiral Schwinger model. In section 3.3 we show that the result of section 3.2 agrees with that of the effective action method. In section 3.4 we show the difficulties in quantizing anomalous gauge theories, and study the non-Abelian case.

3.1 $\partial_0 G_a = (D_\mu J^\mu)_a$; A Classical Relation

In this section let us consider a four-dimensional chiral gauge theory which is described by the Lagrangian

$$L = -\frac{1}{4}F_a^{\mu\nu}F_{a\,\mu\nu} + i\overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu\frac{1-\gamma_5}{2})\psi\,, \tag{3.1}$$

where $[\lambda_a, \lambda_b] = if_{abc}\lambda_c\,,\ Tr(\lambda_a\lambda_b) = \frac{1}{2}\delta_{ab}\,,\ A_\mu = A_{a\,\mu}\lambda_a\,,$

$F^{\mu\nu} = F_a^{\mu\nu}\lambda_a = \partial^\mu A^\nu - \partial^\nu A^\mu - i[A^\mu, A^\nu]$, and we use $\eta_{\mu\nu} = (+,-,-,-)$.

(3.1) gives the equation of motion

$$D_\mu F^{\mu\nu} = -J^\nu\,,\ \text{where}\ \ J_a^\nu = \overline{\psi}\gamma^\nu\frac{1-\gamma_5}{2}\lambda_a\psi\,. \tag{3.2}$$

(3.2) can be written in components as

$$\vec{\nabla}\cdot\vec{E}_a - f_{abc}\vec{A}_b\cdot\vec{E}_c = -J_a^0\,, \tag{3.3}$$

$$\vec{\nabla}\times\vec{B}_a - \partial_0\vec{E}_a - f_{abc}(A_b^0\vec{E}_c + \vec{A}_b\times\vec{B}_c) = -\vec{J}_a\,, \tag{3.4}$$

where $E_a^k = F_a^{k0}\,,\ B_c^k = -\frac{1}{2}\epsilon^{kij}F_{ij}\ (\epsilon^{123} = 1)$, that is,

$$\vec{E}_a = -\vec{\nabla}A_a^0 - \partial_0\vec{A}_a + f_{abc}\vec{A}_bA_c^0\,, \tag{3.5}$$

$$\vec{B}_a = \vec{\nabla}\times\vec{A}_a - \frac{1}{2}f_{abc}\vec{A}_b\times\vec{A}_c\,. \tag{3.6}$$

In the following derivation, we treat (3.3) and (3.4) as satisfied only at the initial time, and derive how the Gauss law constraint $G_a(x)$ given in the following (3.7) propagates in time. That is, we do not treat the time derivatives of (3.3) and (3.4) as satisfied equations.

$$G_a(x) = J_a^0(x) + \vec{\nabla}\cdot\vec{E}_a(x) - f_{abc}\vec{A}_b(x)\cdot\vec{E}_c(x)\,, \tag{3.7}$$

then

$$\partial_0 G_a = \partial_0 J_a^0 + \vec{\nabla}\cdot\partial_0\vec{E}_a - f_{abc}\partial_0\vec{A}_b\cdot\vec{E}_c - f_{abc}\vec{A}_b\cdot\partial_0\vec{E}_c\,. \tag{3.8}$$

Using (3.4) and (3.5) for $\partial_0\vec{E}_a$ and $\partial_0\vec{A}_a$, we get

$$\begin{aligned}\partial_0 G_a = \partial_0 J_a^0 &\\ &+ \vec{\nabla}\cdot(\vec{J}_a + \vec{\nabla}\times\vec{B}_a - f_{abc}\vec{A}_b\times\vec{B}_c + f_{abc}\vec{E}_bA_c^0)\\ &- f_{abc}(-\vec{E}_b - \vec{\nabla}A_b^0 + f_{bde}\vec{A}_dA_e^0)\cdot\vec{E}_c\\ &- f_{abc}\vec{A}_b\cdot(\vec{J}_c + \vec{\nabla}\times\vec{B}_c - f_{cde}\vec{A}_d\times\vec{B}_e + f_{cde}\vec{E}_dA_e^0)\end{aligned} \tag{3.9}$$

Then after some calculations using (3.3)-(3.6), a vector identity and the Jacobi identity we obtain the following relation.

$$\partial_0 G_a(x) = (D_\mu J^\mu(x))_a\,. \tag{3.10}$$

That is, the time derivative of $G_a(x)$ is equal to the covariant derivative of the fermionic current. We can also write the left hand side of (3.10) as a gauge covariant form D_0G_a since $D_0G_a = \partial_0 G_a + f_{abc}A_bG_c = \partial_0 G_a$ by using (3.3). In the above we derived (3.10) in four dimensions. In two dimensions the corresponding derivation becomes simpler since there is no $\vec{B}_a$ in two dimensions, and the result is the same as (3.10). (3.10) is a classical relation. We will obtain a quantum version of this relation in two dimensions in sections 3.2 and 3.4.

3.2 Anomaly of $D_\mu J^\mu$ from Schwinger Terms

In this section we will obtain the anomaly of $\partial_\mu J^\mu$ (where $J^\mu = \overline{\psi}\gamma^\mu\frac{1-\gamma_5}{2}\psi$) for the chiral Schwinger model by using the Schwinger terms of the equal time commutation relations. When we calculate the Schwinger terms, we will use the BJL (Bjorken-Johnson-Low) limit method which is summarized below [1,5,22,23].

When we have a time ordered product of two operators

$$T(p) = \int d^2x e^{-ip\cdot x} \langle \alpha \mid T(A(x)B(0)) \mid \beta \rangle \,, \qquad (3.11)$$

the equal time commutation relation of these two operators is obtained by the following limiting procedure.

$$\lim_{p^0 \to \infty} p^0 T(p) = -i \int d\vec{x} e^{i\vec{p}\cdot\vec{x}} \langle \alpha \mid [A(0,\vec{x}), B(0,\vec{0})] \mid \beta \rangle \,. \qquad (3.12)$$

Then from (3.12) we have the following correspondence.

$$\begin{aligned} &[A(0,\vec{x}), B(0,\vec{0})] = i\delta(x) \,, \quad \text{if } \lim_{p^0\to\infty} p^0 T(p) = 1 \,, \\ &[A(0,\vec{x}), B(0,\vec{0})] = -\delta'(x) \,, \quad \text{if } \lim_{p^0\to\infty} p^0 T(p) = p^1 \,, \\ &\qquad \text{where } \delta(x) \equiv \delta(x^1) \,, \quad \delta'(x) \equiv \frac{\partial}{\partial x^1}\delta(x^1) \,. \end{aligned} \qquad (3.13)$$

If $T(p)$ has a polynomial in p^0 (that is, $1, p^0, (p^0)^2, \cdots$), we drop such terms since they do not contribute to the Schwinger terms.

The chiral Schwinger model is described by the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu (\partial_\mu - ieA_\mu \frac{1-\gamma_5}{2})\psi \,. \qquad (3.14)$$

Our conventions are given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \,, \qquad \eta^{00} = -\eta^{11} = 1 \,,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \; \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$
$$\epsilon^{01} = -\epsilon^{10} = 1 \,. \qquad (3.15)$$

From (3.14) we get the canonical conjugate momenta

$$\pi(A_\mu) = F^{\mu 0} \equiv E^\mu \,, \; \pi(\psi_\alpha) = i\psi_\alpha^* \,. \qquad (3.16)$$

Then we assign the Poisson brackets

$$\begin{aligned} &(A_\mu(t,x), E^\nu(t,x'))_{P.B.} = \eta_\mu{}^\nu \delta(x-x') \,, \\ &(i\psi_\alpha^*(t,x), \psi_\beta(t,x'))_{P.B.} = \delta_{\alpha\beta}\delta(x-x') \,. \end{aligned} \qquad (3.17)$$

The equation of motion is given by the Hamiltonian H as

$$\partial_0 f = (f, H)_{P.B.} \,, \qquad (3.18)$$

when f is not dependent explicitly on the time, which is satisfied in our case.

Using (3.16) we obtain the Hamiltonian

$$\begin{aligned} H_0 &= \int dx \{\pi(A_\mu)\partial_0 A_\mu + \pi(\psi_\alpha)\partial_0\psi_\alpha - L\} \\ &= \int dx \{\frac{1}{2}E^1E^1 + eA^1\bar{\psi}\gamma^1\frac{1-\gamma_5}{2}\psi - i\bar{\psi}\gamma^1\partial_1\psi - A_0(\partial_1 E^1 + e\bar{\psi}\gamma^0\frac{1-\gamma_5}{2}\psi)\} \,. \end{aligned} \qquad (3.19)$$

From (3.16) we see that $\pi(A_0) \approx 0$ is a primary constraint [24-26] (where $\approx$ means a weak condition in Dirac's sense), and we get a secondary constraint as

$$G \equiv \partial_0(\pi(A_0)) = (\pi(A_0), H_0)_{P.B.} = \partial_1 E^1 + e\bar{\psi}\gamma^0\frac{1-\gamma_5}{2}\psi \approx 0 \,. \qquad (3.20)$$

Since $\partial_0 G = (G, H_0)_{P.B.} = 0$, at the Poisson bracket level the chiral Schwinger model is consistent by having two first class constraints $\pi(A_0) \approx 0$ and

$G \approx 0$. Then by incorporating these constraints we have

$$H' = H_0 + a\pi(A_0) + bG\,, \tag{3.21}$$

where a, b are arbitrary functions of canonical coordinates and momenta. In (3.21) $\pi(A_0) \approx 0$ is always satisfied and $\partial_0 A_0 \approx a$ is arbitrary, so $\pi(A_0)$ and A_0 are not of interest. Therefore we neglect these two canonical variables [24]. Then we have the following Hamiltonian.

$$H = H_1 + H_2 + H_3\,,$$
$$\begin{cases} H_1 = \int dx(\frac{1}{2}E^1E^1 + eA^1\overline{\psi}\gamma^1(\frac{1-\gamma_5}{2})\psi) \\ H_2 = \int dx(uG) \\ H_3 = \int dx(-i\overline{\psi}\gamma^1\partial_1\psi)\,, \end{cases} \tag{3.22}$$

where u is an arbitrary function of the canonical variables A^1, E^1, ψ and ψ^*.

Now let us consider a quantum theory. We change (3.17) and (3.18) to the equal time commutation relations

$$[A^1(t,x), E^1(t,x')] = -i\delta(x-x')\,,$$
$$\{\psi^*_\alpha(t,x), \psi_\beta(t,x')\} = \delta_{\alpha\beta}\delta(x-x')\,, \tag{3.23}$$
$$\partial_0 O = i[H, O]\,.$$

If we calculate $[H,G]$ naively using (3.23), we get zero as in the case of using the Poisson brackets. However, we should be careful when there is an anomaly.

First let us calculate various basic equal time commutation relations using the BJL limit method which takes care of the quantum effect. Let us take the gauge $A_0(x) = 0$ in (3.14). Then

$$E^1 = -\partial_0 A^1\,, \tag{3.24}$$

and the propagator of the gauge field is given by

$$D_{\mu\nu}(p) = \frac{-i}{p^2+i\epsilon}[\eta_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{(p\cdot n)} + \frac{p_\mu p_\nu}{(p\cdot n)^2}]\,, \tag{3.25}$$

where $n_\mu = (1,0)$.

We quantized gauge fields as well as fermionic fields by considering our system as a sub-system of an anomaly-free larger system. Then we study whether this sub-system is anomalous or not. For example, our system which is composed of gauge fields and left-handed chiral fermions can be considered as a sub-system of a larger anomaly-free system which is composed of gauge fields and Dirac fermions.

For $[J^0(x), J^0(y)]_{ETC}$ (where ETC means equal time commutator), we consider

$$T(p) = \int d^2x e^{-ip\cdot x}\langle 0 \mid T(J^0(x)J^0(0)) \mid 0\rangle\,, \tag{3.26}$$

which corresponds to Fig.1. Then

$$T(p) = (-in_\mu)(-in_\nu)\Pi^{\mu\nu}(p)\,, \tag{3.27}$$

where

$$\Pi^{\mu\nu}(p) = -\int \frac{d^2k}{(2\pi)^2}\mathrm{Tr}[\frac{i}{\gamma\cdot k+\gamma\cdot p+i\epsilon}i\gamma^\mu\frac{1-\gamma_5}{2}\frac{i}{\gamma\cdot k+i\epsilon}i\gamma^\nu\frac{1-\gamma_5}{2}]$$
$$= -\frac{i}{4\pi}\frac{1}{p^2}(-\eta^{\mu\nu}p^2 + 2p^\mu p^\nu + \epsilon^{\mu\lambda}p_\lambda p^\nu + \epsilon^{\nu\lambda}p_\lambda p^\mu)\,. \tag{3.28}$$

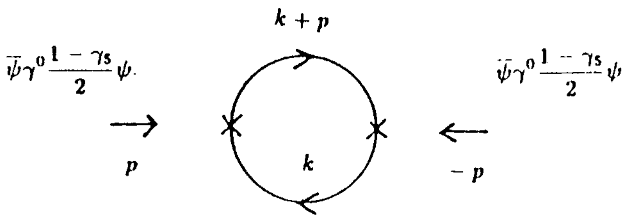


Fig.1

In the BJL limit,

$$\lim_{p^0\to\infty} p^0 T(p) = -\frac{i}{2\pi} p^1 ,$$

where we follow the prescription of the BJL limit method to drop a term which is proportional to p^0 in $p^0T(p)$. Then using the correspondence given in (3.13), we obtain

$$[J^0(x), J^0(0)]_{S.T.} = \frac{i}{2\pi}\delta'(x) , \tag{3.29}$$

where the commutator is an equal time commutator, and the subscript $S.T.$ means Schwinger term.

For other equal time commutation relations we follow the same procedure. For example, for $[J^0(x), \partial_1 E^1(y)]_{ETC}$ and $[\partial_1 E^1(x), \partial_1 E^1(y)]_{ETC}$, we consider the Feynman diagrams in Fig.2 and Fig.3 respectively. After similar calculations we obtain the following results.

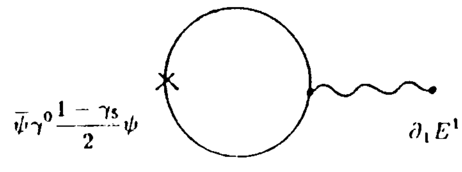


Fig.2

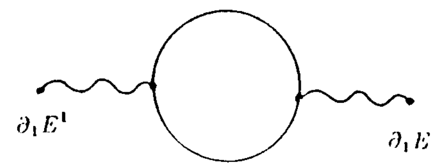


Fig.3

$$\begin{aligned}
&[J^0(x), J^0(y)]_{S.T.} = [J^0(x), J^1(y)]_{S.T.} = [J^1(x), J^1(y)]_{S.T.} = -2\delta'(x-y)k \\
&[J^0(x), \partial_1 E^1(y)]_{S.T.} = [J^1(x), \partial_1 E^1(y)]_{S.T.} = e\delta'(x-y)k \\
&[J^0(x), E^1(y)]_{S.T.} = [J^1(x), E^1(y)]_{S.T.} = -e\delta(x-y)k \\
&[J^0(x), A^1(y)]_{S.T.} = [J^1(x), A^1(y)]_{S.T.} = 0 \\
&[\text{ commutators among } \partial_1 E^1(x),\ E^1(x) \text{ and } A^1(x)\ ]_{S.T.} = 0 ,
\end{aligned} \tag{3.30}$$

where

$$k = -\frac{i}{4\pi},\ \delta(x-y) \equiv \delta(x^1-y^1),\ \delta'(x-y) \equiv \frac{\partial}{\partial x^1}\delta(x^1-y^1),$$

and all commutators are equal time commutators. We note the sign of the last term in

$$[J^1(x), J^0(y)]_{S.T.} = -2\delta'(x-y)k = +[J^0(x), J^1(y)]_{S.T.},\ \text{etc.},$$

in contrast to

$$[E^1(x), J^0(y)]_{S.T.} = \delta(x-y)k = -[J^0(x), E^1(y)]_{S.T.} .$$

Then let us calculate $[H, G(x)]$ using the results in (3.30). We first consider H_2 in (3.22).

$$
\begin{aligned}
[H_2, G(x)] &= \int dx' [u(x')G(x'), G(x)] \\
&= \int dx' \{u(x')[G(x'), G(x)] + [u(x'), G(x)]G(x')\} \qquad (3.31) \\
&\approx \int dx' u(x')[G(x'), G(x)] ,
\end{aligned}
$$

where we could decompose the commutator since a radiative correction does not give rise to an anomaly. By using the basic commutation relations in (3.30) we get

$$
[G(x), G(y)] = [\partial_1 E^1(x) + eJ^0(x), \partial_1 E^1(y) + eJ^0(y)] = 0 . \qquad (3.32)
$$

Therefore $[H_2, G(x)] = 0$, here and (3.33) we use $=$ instead of $\approx$ by restricting the Hilbert space to the physical space which satisfies $G(x) \mid \text{phys}\rangle = 0$. Therefore following relations are satisfied in the physical space.

For $[H_3, G(x)]$, we should calculate $[\overline{\psi}\gamma^1\partial_1\psi(x), \partial_1 E^1(y)]$ and $[\overline{\psi}\gamma^1\partial_1\psi(x), J^0(y)]$ which are not given in the table of (3.30). Since $\overline{\psi}\gamma^1\partial_1\psi$ has a derivative, corresponding diagrams are more divergent, so we should regularize. Using the Pauli-Villars regularization method, we obtain after a somewhat lengthy calculation the result that $T(p)$'s of those diagrams have no $(\frac{1}{p^n})$ term when we expand $T(p)$ in a Laurent series. Therefore H_3 does not give an anomalous

contribution. Then

$$
\begin{aligned}
-i\partial_0 G(x) &= [H, G(x)] = [H_1, G(x)] \\
&= \int dx' [\frac{1}{2} E^1(x')E^1(x') + eA^1(x')J^1(x'), \partial_1 E^1(x) + eJ^0(x)] \\
&= \int dx' \{\frac{1}{2} E^1(x')[E^1(x'), eJ^0(x)] + \frac{1}{2}[E^1(x'), eJ^0(x)]E^1(x') \\
&\qquad + eA^1(x')[J^1(x'), \partial_1 E^1(x)]\} \\
&= -\frac{i}{4\pi} e^2 \int dx' \{E^1(x')\delta(x - x') + A^1(x') \frac{\partial}{\partial x'} \delta(x - x')\} \\
&= -\frac{i}{4\pi} e^2 \int dx' \{E^1(x')\delta(x - x') + A^1(x') \frac{\partial}{\partial x'} \delta(x - x')\} \\
&= -\frac{i}{4\pi} e^2 \{E^1(x) - \partial_1 A^1(x)\} .
\end{aligned}
\qquad (3.33)
$$

Therefore we obtain a non-zero result for $[H, G(x)]$, whereas it is zero at the Poisson bracket level.

On the other hand, from the definition of G

$$
\partial_0 G = \partial_0(\partial_1 E^1 + eJ^0) = e\partial_0 J^0 + \partial_1(\partial_0 E^1) . \qquad (3.34)
$$

$\partial_0 E^1(x)$ in the right hand side of (3.34) is calculated as

$$
\begin{aligned}
\partial_0 E^1(x) &= i[H, E^1(x)] \\
&= \int dx' \{ie[A^1(x'), E^1(x)]J^1(x') + ieA^1(x')[J^1(x'), E^1(x)] \\
&\qquad + ieu(x')[J^0(x'), E^1(x)]\} \\
&= \int dx' \{ie(-i\delta(x' - x))J^1(x') + ieA^1(x')(e\frac{i}{4\pi}\delta(x' - x)) \\
&\qquad + ieu(x')(e\frac{i}{4\pi}\delta(x' - x))\} \\
&= eJ^1(x) - \frac{e^2}{4\pi}(A^1(x) + u(x)) .
\end{aligned}
\qquad (3.35)
$$

Then (3.34) becomes

$$\partial_0 G = e\partial_\mu J^\mu - \frac{e^2}{4\pi}\partial_1(A^1 + u)\,. \tag{3.36}$$

Therefore by combining (3.33) and (3.36) we have for $A_0 = 0$

$$\partial_\mu J^\mu(x) = \frac{e}{4\pi}\{E^1(x) + \partial_1 u(x)\} = \frac{e}{4\pi}\{\partial_0 A_1(x) + \partial_1 u(x)\}\,. \tag{3.37}$$

In (3.37) $u(x)$ should be dependent only on $A_1(x)$, since the anomaly is a local function and terms containing E^1, ψ or ψ^* would give rise to non-local functions in (3.37) because of their dimensionalities. Then

$$\partial_\mu J^\mu(x) = \frac{e}{4\pi}\{\partial_0 A_1(x) + c\partial_1 A_1(x)\}\,, \tag{3.38}$$

where c is an arbitrary constant. Therefore we calculated the anomaly of $\partial_\mu J^\mu$ using the Schwinger terms of the equal time commutators. In the next section we will show that (3.38) is consistent with the result of the effective action method.

It is sometimes allowed to add an arbitrary local function of the gauge field A to G. If we do it, the right hand side of (3.33) changes, but $\partial_\mu J^\mu$ in (3.37) does not change since the right hand side of (3.35) also changes by the same amount as that of (3.33). However, in this paper we do not consider such an ambiguity of G because the constraint G is given as (3.20) without ambiguity when we follow the Dirac's treatment as we did from (3.14) to (3.20). This is also true in the non-Abelian case in section 3.4.

3.3 Comparison with the Effective Action Method

Let us calculate the effective action for the chiral Schwinger model described by (3.14). Then we can calculate the anomaly of $\partial_\mu J^\mu$ following the familiar method [27]. The only diagram which gives an anomaly is given in Fig.4.

Then using $\Pi^{\mu\nu}(p)$ in (3.28)

$$\begin{aligned} iW[A] &= e^2 \int \frac{d^2p}{(2\pi)^2} d^2q\,\delta^2(p+q)\frac{1}{2}A_\mu(p)\Pi^{\mu\nu}(p)A_\nu(q) \\ &= -\frac{ie^2}{8\pi}\int \frac{d^2p}{(2\pi)^2} d^2q\,\delta^2(p+q)\frac{1}{p^2}\{-\eta^{\mu\nu}p^2 + 2p^\mu p^\nu + 2\epsilon^{\mu\lambda}p_\lambda p^\nu\}A_\mu(p)A_\nu(q)\,. \end{aligned} \tag{3.39}$$

Using $\delta_\Lambda A_\mu(p) = ip_\mu\Lambda(p)$ which corresponds to $\delta_\Lambda A_\mu(x) = \partial_\mu\Lambda(x)$ in coordinate space,

$$\begin{aligned} i\delta_\Lambda W[A] &= -\frac{ie^2}{4\pi}\int \frac{d^2p}{(2\pi)^2} d^2q\,\delta^2(p+q)\{ip_\mu A^\mu(p) - 2ip_\mu A^\mu(p) \\ &\qquad + \epsilon^{\mu\nu}ip_\mu A_\nu(p)\}\Lambda(q) \\ &= -\frac{ie^2}{4\pi}\int d^2x\,\Lambda(x)\{\epsilon^{\mu\nu}\partial_\mu A_\nu(x) - \partial_\mu A^\mu(x)\}\,. \end{aligned} \tag{3.40}$$

However, in (3.40) the second term which is proportional to $\partial_\mu A^\mu$ is a variation of a local term which is ambiguous in the loop calculation. When we take care of this ambiguity, (3.40) becomes

$$i\delta_\Lambda W[A] = -\frac{i}{4\pi}\int d^2x\,\Lambda(x)\{\epsilon^{\mu\nu}\partial_\mu A_\nu(x) + c'\partial_\mu A^\mu\}\,, \tag{3.41}$$

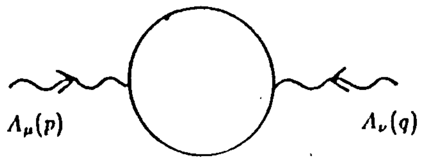


Fig.4

where c' is an arbitrary constant. On the other hand,

$$\begin{aligned}\delta_\Lambda W[A] &= \int d^2x \frac{\delta W}{\delta A_\mu} \delta_\Lambda A_\mu \\ &= \int d^2x e J^\mu(x) \partial_\mu \Lambda(x) \\ &= \int d^2x \Lambda(x) \{-e\partial_\mu J^\mu(x)\} .\end{aligned} \tag{3.42}$$

Then from (3.41) and (3.42)

$$\partial_\mu J^\mu = \frac{e}{4\pi} \{\epsilon^{\mu\nu} \partial_\mu A_\nu + c' \partial_\mu A^\mu\} . \tag{3.43}$$

In order to compare (3.43) with (3.37), let us take $A_0(x) = 0$ in (3.43). Then (3.43) becomes

$$\partial_\mu J^\mu(x) = \frac{e}{4\pi} \{\partial_0 A_1(x) - c' \partial_1 A_1(x)\} . \tag{3.44}$$

This is the same as (3.38) since both c and c' are arbitrary. Therefore we showed that the result of section 3.2 is consistent with that of the effective action method.

3.4 Difficulties in Quantization and the Non-Abelian Case

When we quantize a system with constraints $G_a \approx 0$ following Dirac's method, we should have the consistency conditions [24],

$$[G_a, G_b] \approx U_{ab}{}^c G_c , \tag{3.45}$$

$$[H, G_a] \approx V_a{}^b G_b , \tag{3.46}$$

where $\approx$ was introduced in section 3.2.

For the chiral Schwinger model in section 3.2 and 3.3, we have one constraint G given in (3.20). At the Poisson bracket level both (3.45) and (3.46) are satisfied. However, at the quantum level (3.45) is satisfied as shown in (3.32), but (3.46) is not satisfied as shown in (3.33). This means that when we impose on the physical state $|$ phys$\rangle$ the condition $G |$ phys$\rangle = 0$ at an initial time, this condition is not satisfied at a later time. Therefore we can not quantize the chiral Schwinger model consistently. References [28-30] discussed this difficulty of quantization in similar ways.

It is interesting that the classical relation (3.10) already suggests this difficulty when $D_\mu J^\mu$ is anomalous. Of course, this should be confirmed by calculations at the quantum level. We also note that we can not have a satisfactory situation by taking the right hand side of (3.33) as a new secondary constraint, since if we do that, the time derivative of this new constraint is again not zero and gives rise to another new secondary constraint, and so on.

Then we will have too many constraints.

Now let us consider a non-Abelian chiral gauge theory, which has constraints G_a's given in (3.7). At the Poisson bracket level both (3.45) and (3.46) are satisfied. However, at the quantum level the Schwinger terms can spoil this situation. Recently references [31-33] studied the condition (3.45) and presented the result that this has a Schwinger term.

Let us consider the condition (3.46) for the system described by the Lagrangian in (3.1) in two dimensions. In this non-Abelian case the Hamiltonian is given by

$$H = \int dx \{ (\frac{1}{2} E_a^1 E_a^1 + A_a^1 \bar{\psi} \gamma^1 (\frac{1-\gamma_5}{2}) \lambda_a \psi) + (u_a G_a) + (-i \bar{\psi} \gamma^1 \partial_1 \psi) \} \ , \quad (3.47)$$

which is similar to (3.22), and we have the following Schwinger terms like (3.30) of the Abelian case.

$$\begin{aligned}
&[J_a^0(x), J_b^0(y)]_{S.T.} = [J_a^0(x), J_b^1(y)]_{S.T.} = [J_a^1(x), J_b^1(y)]_{S.T.} = -\delta'(x-y) k \delta_{ab} \\
&[J_a^0(x), \partial_1 E_b^1(y)]_{S.T.} = [J_a^1(x), \partial_1 E_b^1(y)]_{S.T.} = \frac{1}{2} \delta'(x-y) k \delta_{ab} \\
&[J_a^0(x), E_b^1(y)]_{S.T.} = [J_a^1(x), E_b^1(y)]_{S.T.} = -\frac{1}{2} \delta(x-y) k \delta_{ab} \\
&[J_a^0(x), A_b^1(y)]_{S.T.} = [J_a^1(x), A_b^1(y)]_{S.T.} = 0 \\
&[\text{ commutators among } \partial_1 E^1(x) , \ E^1(x) \text{ and } A^1(x) \ ]_{S.T.} = 0 \ ,
\end{aligned} \quad (3.48)$$

where $k = -\frac{i}{4\pi}$. Then using these Schwinger terms we obtain the following result by the procedure which gave (3.33).

$$\partial_0 G_a = i[H, G_a] = \frac{1}{8\pi} \{ E_a^1 - \partial_1 A_a^1 \} \ . \quad (3.49)$$

(3.49) shows that the condition (3.46) is subject to the Schwinger term and this fact gives rise to a difficulty in quantization. Of course, when (3.45) has a Schwinger term, it also causes a difficulty in quantization [34].

Now let us calculate $(D_\mu J^\mu)_a$ using the procedure in section 3.2. From the definition of G_a

$$\partial_0 G_a = \partial_0 J_a^0 + \partial_1 (\partial_0 E_a^1) + f_{abc} A_{b1} (\partial_0 E_c^1) \ , \quad (3.50)$$

where $\partial_0 A_{b1} = E_{b1}$ is used since we are taking the gauge $A_{a0} = 0$. Using the same procedure as (3.35) $\partial_0 E_a^1$ in (3.50) is given by

$$\partial_0 E_a^1 = J_a^1 - \frac{1}{8\pi} (A_a^1 + u_a) \ . \quad (3.51)$$

Then (3.50) becomes

$$\partial_0 G_a = (D_\mu J^\mu)_a - \frac{1}{8\pi} \partial_1 (A_a^1 + u_a) \ , \quad (3.52)$$

where we used $f_{abc} A_{b1} u_c = 0$ since u_c is proportional to A_{c1} because of the reason explained below (3.37). Then from (3.49) and (3.52) we have

$$(D_\mu J^\mu)_a = \frac{1}{8\pi} \{ \partial_0 A_{a1} + c \partial_1 A_{a1} \} \ . \quad (3.53)$$

One can show that (3.53) agrees with the result of the effective action method in the same way as in section 3.3 [27].

CHAPTER 4

ASPECTS OF THE POINT-SPLITTING METHOD

As tools for calculating anomalies, the loop-diagram method and the point-splitting method have been important from the beginning of the discovery of the anomalies [1-5,35,36]. However, it is known that in four dimensions these two methods agree for the anomaly of $\partial_\mu J_5^\mu$, but disagree for the Schwinger term of $[\, J_5^0(x)\,,\, J^0(y)\,]_{ETC}$ (where "ETC" means "equal time commutator") [37-39]. When we calculate the Schwinger term [35] by the loop-diagram method, we use the Bjorken-Johnson-Low (BJL) limit method [22,23]. In this chapter we study the two-dimensional Abelian gauge theory (the Schwinger model) [40-42], and we find that in this case the two methods disagree for both the anomaly of $\partial_\mu J_5^\mu$ and the Schwinger term of $[\, J_5^0(x)\,,\, J^0(y)\,]_{ETC}$. This result shows that the disagreement of the two methods are more severe than it has been known.

In section 4.1 we calculate $\partial_\mu J_5^\mu$ and $[\, J_5^0(x)\,,\, J^0(y)\,]_{ETC}$ using the loop-diagram method. In section 4.2 we calculate the same quantities using the point-splitting method, and show that these two methods disagree.

4.1 Loop-Diagram Method

A. $\partial_\mu J_5^\mu$

We consider the two-dimensional Abelian gauge theory which is described

by the Lagrangian

$$L = i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi\,. \tag{4.1}$$

Our conventions and their properties are given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \qquad \eta^{00} = -\eta^{11} = 1\,,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,\ \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

$$\epsilon^{10} = -\epsilon^{01} = 1\,, \quad \gamma^\mu\gamma_5 = \epsilon^{\mu\nu}\gamma_\nu\,,\ Tr(\gamma_5\gamma^\mu\gamma^\nu) = -2\epsilon^{\mu\nu}\,. \tag{4.2}$$

In order to obtain $\partial_\mu J_5^\mu$, let us start by including a mass term $-m\bar{\psi}\psi$ in (4.1). Then using the equation of motion we get

$$\partial_\mu J_5^\mu = 2im\bar{\psi}\gamma_5\psi\,, \ \text{where } J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi\,. \tag{4.3}$$

Let us regularize (4.3) as

$$\partial_\mu J_5^\mu(reg.) = 2im\bar{\psi}\gamma_5\psi - 2iM\bar{\Psi}\gamma_5\Psi\,, \tag{4.4}$$

$$\text{where } J_5^\mu(reg.) = \bar{\psi}\gamma^\mu\gamma_5\psi - \bar{\Psi}\gamma^\mu\gamma_5\Psi\,.$$

In the above Ψ and M are the Pauli-Villars regulator field and its mass respectively [1]. Then for the massless fermion case ($m = 0$) given by (4.1), (4.4) becomes

$$\partial_\mu J_5^\mu(reg.) = -2iM\bar{\Psi}\gamma_5\Psi\,. \tag{4.5}$$

In terms of $R^{\mu\nu}$ and R^ν in Fig.1, (4.5) is expressed as

$$ip_\mu R^{\mu\nu}(reg.) = -2iMR^\nu\,, \tag{4.6}$$

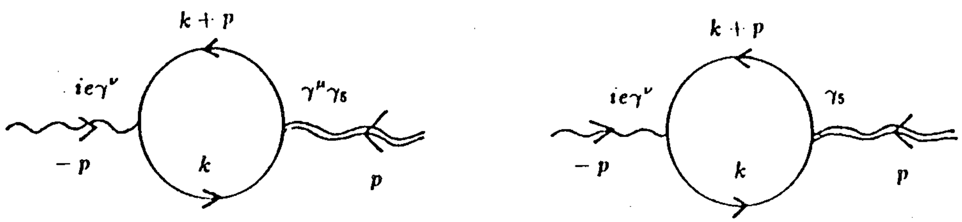


Fig.1 ; $R^{\mu\nu}$ and R^ν

where we used the correspondence $\partial_\mu \leftrightarrow ip_\mu$, since p_μ is an incoming momentum. Then the anomaly is given by

$$\partial_\mu J_5^\mu(renormalized) = Anomaly = \lim_{M\to\infty}\{-2iMR^\nu A_\nu\}. \qquad (4.7)$$

Let us calculate the right hand side of (4.7).

$$\begin{aligned} R^\nu &= -\int \frac{d^2k}{(2\pi)^2}\frac{i}{k^2-M^2}\frac{i}{(k+p)^2-M^2}Tr\{(\gamma\cdot k+\gamma\cdot p+M)\gamma_5(\gamma\cdot k+M)ie\gamma^\nu\} \\ &= 2iee^{\mu\nu}p_\mu MI, \end{aligned} \qquad (4.8)$$

$$\begin{aligned} \text{where } I &= \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2-M^2}\frac{1}{(k+p)^2-M^2} \\ &= \int_0^1 dx \int \frac{d^2l}{(2\pi)^2}\frac{1}{[l^2-M^2+p^2x(1-x)]^2}, \text{ with } l = k+p(1-x) \\ &= \int_0^1 dx\, i\int \frac{d^2l_E}{(2\pi)^2}\frac{1}{[l_E^2+M^2-p^2x(1-x)]^2} \\ &= \int_0^1 dx \frac{i\pi}{(2\pi)^2}\frac{1}{[M^2-p^2x(1-x)]}. \end{aligned} \qquad (4.9)$$

Then

$$\begin{aligned} Anomaly &= \lim_{M\to\infty}\{i\frac{e}{\pi}\epsilon^{\mu\nu}p_\mu A_\nu(p)M^2\int_0^1 dx\frac{1}{[M^2-p^2x(1-x)]}\} \\ &= \frac{e}{\pi}\epsilon^{\mu\nu}ip_\mu A_\nu(p). \end{aligned} \qquad (4.10)$$

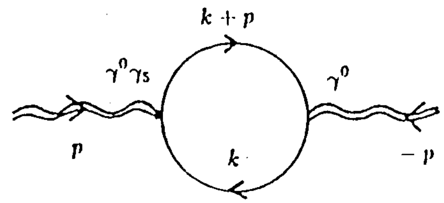


Fig.2

To get the coordinate space expression we use $\partial_\mu \leftrightarrow ip_\mu$ again, then

$$\partial_\mu J_5^\mu = \frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu. \qquad (4.11)$$

B. $[\ J_5^0(x)\ ,\ J^0(y)\ ]_{ETC}$

The BJL limit method says [1,5,43] that

$$\text{when } T(p) = \int d^2x\, e^{-ip\cdot x}\langle 0\mid T(J_5^0(x)J^0(0))\mid 0\rangle, \qquad (4.12)$$

$$\lim_{p_0\to\infty} p_0T(p) = -i\int dx^1\, e^{ip^1x^1}\langle 0\mid[\ J_5^0(0,x^1)\ ,\ J^0(0,0)\ ]\mid 0\rangle. \qquad (4.13)$$

Therefore if we get p^1 in the left hand side of (4.13), it means that $\langle 0 \mid [\ J_5^0(0,x^1)\ ,\ J^0(0,0)\ ]\mid 0\rangle$ in the right hand side is $-\frac{\partial}{\partial x^1}\delta(x^1)$, i.e.,

$$p^1 \leftrightarrow -\frac{\partial}{\partial x^1}\delta(x^1). \qquad (4.14)$$

$T(p)$ in (4.12) is given by Fig.2 as

$$\begin{aligned} T(p) &= -\int \frac{d^2k}{(2\pi)^2} \frac{i}{k^2} \frac{i}{(k+p)^2} Tr\{(\gamma \cdot k + \gamma \cdot p)\gamma_0\gamma_5\gamma \cdot k\gamma_0\} \\ &= -2\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} \frac{1}{(k+p)^2}(2k_0k_1 + k_0p_1 + k_1p_0) \\ &= -2\int_0^1 dx \int \frac{d^2l}{(2\pi)^2} \frac{-2p_0p_1x(1-x)}{[l^2+p^2x(1-x)]^2}, \text{ with } l = k + p(1-x) \\ &= 4p_0p_1 \int_0^1 dx\, i\int \frac{d^2l_E}{(2\pi)^2} \frac{x(1-x)}{[l_E^2 - p^2x(1-x)]^2} \\ &= -\frac{i}{\pi}\frac{p_0p_1}{p^2}. \end{aligned} \tag{4.15}$$

Then

$$\lim_{p_0\to\infty} p_0T(p) = -\frac{i}{\pi}\frac{p_0^2p_1}{p_0^2-p_1^2} = -\frac{i}{\pi}p_1 = \frac{i}{\pi}p^1. \tag{4.16}$$

Therefore from (4.14)

$$\langle 0 \mid [\ J_5^0(0,x^1)\,,\ J^0(0,0)\ ] \mid 0\rangle = -\frac{i}{\pi}\frac{\partial}{\partial x^1}\delta(x^1)\,,$$

or

$$\langle 0 \mid [\ J_5^0(x)\,,\ J^0(y)\ ]_{ETC} \mid 0\rangle = -\frac{i}{\pi}\frac{\partial}{\partial x^1}\delta(x^1-y^1)\,. \tag{4.17}$$

4.2 Point-Splitting Method

A. $\partial_\mu J_5^\mu$

From (4.1) we have the equations of motion

$$\begin{aligned} \gamma^\mu\partial_\mu\psi &= ie\gamma^\mu A_\mu\psi\,, \\ \overline{\psi}\overleftarrow{\partial}_\mu\gamma^\mu &= -ie\overline{\psi}\gamma^\mu A_\mu\,. \end{aligned} \tag{4.18}$$

Let us define the axial current in the following gauge invariant form [1,4,5].

In this section we treat A_μ as an external field.

$$J_5^\mu(x;\epsilon) = \overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma_5\psi(x-\frac{\epsilon}{2})exp[ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} A_\nu(y)dy^\nu]\,. \tag{4.19}$$

Using (4.18)

$$\begin{aligned} \partial_\mu J_5^\mu(x;\epsilon) &= -ieJ_5^\mu(x;\epsilon)[A_\mu(x+\frac{\epsilon}{2}) - A_\mu(x-\frac{\epsilon}{2}) - \partial_\mu\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} A_\nu(y)dy^\nu] \\ &= -ieJ_5^\mu(x;\epsilon)\epsilon^\alpha[\partial_\alpha A_\mu(x) - \partial_\mu A_\alpha(x) + O(\epsilon)] \\ &= -ieJ_5^\mu(x;\epsilon)\epsilon^\alpha[F_{\alpha\mu}(x) + O(\epsilon)]\,, \\ &\text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \end{aligned} \tag{4.20}$$

Then

$$\begin{aligned} \langle 0 \mid \partial_\mu J_5^\mu(x) \mid 0\rangle &= -ieI^{\alpha\mu}[F_{\alpha\mu} + O(\epsilon)]\,, \\ \text{where } I^{\alpha\mu} &\equiv \epsilon^\alpha\langle 0 \mid J_5^\mu(x;\epsilon) \mid 0\rangle\,. \end{aligned} \tag{4.21}$$

From (4.19)

$$\begin{aligned} I^{\alpha\mu} &= \epsilon^\alpha\langle 0 \mid \overline{\psi}(x+\frac{\epsilon}{2})\gamma^\mu\gamma_5\psi(x-\frac{\epsilon}{2}) \mid 0\rangle exp[ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} A_\nu(y)dy^\nu] \\ &= -Tr\{\gamma^\mu\gamma_5\epsilon^\alpha\langle 0 \mid T(\psi(x-\frac{\epsilon}{2})\overline{\psi}(x+\frac{\epsilon}{2}) \mid 0\rangle\} exp[ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} A_\nu(y)dy^\nu] \\ &= -Tr\{\gamma^\mu\gamma_5\epsilon^\alpha\, S_A(x-\frac{\epsilon}{2}, x+\frac{\epsilon}{2})exp[ie\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} A_\nu(y)dy^\nu]\}\,, \end{aligned} \tag{4.22}$$

where $S_A(x,y)$ is a fermion propagator in the external field A_μ, and ϵ^0 is taken as positive [5]. $S_A(x,y)$ can be expanded in powers of A_μ as shown in Fig.3.

$$\begin{aligned} &S_A(x-\frac{\epsilon}{2}, x+\frac{\epsilon}{2}) \\ &= S_F(-\epsilon) + ie\int d^2y\, S_F(x-\frac{\epsilon}{2}-y)\gamma^\mu S_F(y-x-\frac{\epsilon}{2})A_\mu(y)\ \cdots\,, \end{aligned} \tag{4.23}$$

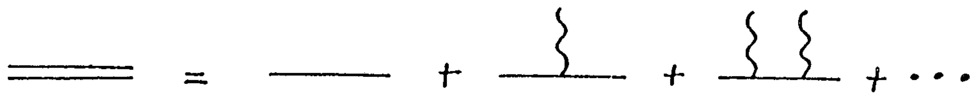


Fig.3 Fermion propagator $S_A(x)$ in the external field A_μ

where S_F is a free fermion propagator.

The first term $S_F(-\epsilon)$ in (4.23) or Fig.3 is as singular as $1/\epsilon$, and next terms are less singular. Therefore when ϵ goes to zero, only this first term contributes in (4.22)

Therefore from

$$\begin{aligned} S_F(x) &= i\int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x}\frac{\gamma\cdot p}{p^2}\,, \\ S_F(-\epsilon) &= i\int \frac{d^2p}{(2\pi)^2}\, e^{ip\cdot\epsilon}\frac{\gamma^\beta p_\beta}{p^2}\,, \end{aligned} \tag{4.24}$$

we have

$$\begin{aligned} I^{\alpha\mu} &= -iTr\{\gamma^\mu\gamma_5\gamma^\beta\}\int \frac{d^2p}{(2\pi)^2}\,\epsilon^\alpha e^{ip\cdot\epsilon}\frac{p_\beta}{p^2} \\ &= -2i\epsilon^{\mu\beta} i\int \frac{d^2p}{(2\pi)^2}\, e^{ip\cdot\epsilon}\frac{\partial}{\partial p_\alpha}(\frac{p_\beta}{p^2}) \\ &= 2\epsilon^\mu{}_\beta\int \frac{d^2p}{(2\pi)^2}\,\partial^\alpha(\frac{p^\beta}{p^2}). \end{aligned} \tag{4.25}$$

In (4.25) we apply the following property.

$$\int_{p=P} d^2p\,\partial^\alpha f(p) = i2\pi P^\alpha f(P)\,, \tag{4.26}$$

where P is the value of p at infinity which is the boundary of the volume

integral.

$$\int d^2p\,\partial^\alpha(\frac{p^\beta}{p^2}) = i2\pi\frac{P^\alpha P^\beta}{P^2} = i2\pi\frac{\eta^{\alpha\beta}}{2} = i\pi\eta^{\alpha\beta}\,, \tag{4.27}$$

where we applied the averaging procedure. Then

$$I^{\alpha\mu} = \frac{i}{2\pi}\epsilon^\mu{}_\beta\eta^{\alpha\beta} = -\frac{i}{2\pi}\epsilon^{\alpha\mu}\,. \tag{4.28}$$

Using (4.28), (4.21) becomes

$$\langle 0\mid \partial_\mu J_5^\mu(x)\mid 0\rangle = -\frac{e}{2\pi}\epsilon^{\alpha\mu}F_{\alpha\mu} = -\frac{e}{\pi}\epsilon^{\mu\nu}\partial_\mu A_\nu\,. \tag{4.29}$$

(4.29) disagrees with the result (4.11) of the loop-diagram method by the sign.

B. $[\ J_5^0(x)\,,\ J^0(y)\ ]_{ETC}$

In order to calculate the "equal time commutator" of two bilinears of fermion fields, we take the seperation ϵ^α as spatial [1,33]. The reason is that we will use the equal time canonical commutation relation

$$\{\,\psi^\dagger_\alpha(0,x^1)\,,\,\psi_\beta(0,y^1)\,\} = \delta_{\alpha\beta}\delta(x^1-y^1)\,. \tag{4.30}$$

Also, we take the point-splitting form (4.19) only for $J_5^0(x)$ since the result is the same as the case when we take the point-splitting form for both $J_5^0(x)$

and $J^0(y)$. Then

$$
\begin{aligned}
&[\,J_5^0(0,x^1;\epsilon^1)\,,\,J^0(0,y^1)\,] \\
&= [\,\psi_\alpha^\dagger(0,x^1+\frac{\epsilon^1}{2})\psi_\beta(0,x^1-\frac{\epsilon^1}{2})\,,\,\psi_\lambda^\dagger(0,y^1)\psi_\lambda(0,y^1)\,](\gamma_5)_{\alpha\beta}exp[ie\int_{x-\frac{\epsilon^1}{2}}^{x+\frac{\epsilon^1}{2}} A_1(z)dz^1] \\
&= \{\delta(x^1-y^1-\frac{\epsilon^1}{2})-\delta(x^1-y^1+\frac{\epsilon^1}{2})\}\psi^\dagger(0,x^1+\frac{\epsilon^1}{2})\gamma_5\psi(0,x^1-\frac{\epsilon^1}{2})exp[ie\int_{x-\frac{\epsilon^1}{2}}^{x+\frac{\epsilon^1}{2}} A_1(z)dz^1] \\
&= -\epsilon^1\frac{\partial}{\partial x^1}\delta(x^1-y^1)J_5^0(0,x^1;\epsilon^1)\,.
\end{aligned}
\tag{4.31}
$$

Therefore

$$
\begin{aligned}
&\langle 0\,|\,[\,J_5^0(0,x^1;\epsilon^1)\,,\,J^0(0,y^1)\,]\,|\,0\rangle \\
&= -\frac{\partial}{\partial x^1}\delta(x^1-y^1)\epsilon^1\langle 0\,|\,J_5^0(0,x^1;\epsilon^1)\,|\,0\rangle \\
&= -\frac{\partial}{\partial x^1}\delta(x^1-y^1)I^{10}, \text{ where } I^{\alpha\mu} \text{ is defined in (4.21)} \qquad (4.32) \\
&= -\frac{\partial}{\partial x^1}\delta(x^1-y^1)\frac{i}{2\pi}\epsilon^{10}, \text{ where we used (4.28)} \\
&= -\frac{i}{2\pi}\frac{\partial}{\partial x^1}\delta(x^1-y^1)\,.
\end{aligned}
$$

(4.32) is half of the result (4.17) of the loop-diagram method.

CHAPTER 5

GRAVITATIONAL ANOMALY

In this chapter we study the purely gravitational anomaly in the system of the gravitational field coupled to a chiral fermion [44-46]. We obtain an effective action by calculating Feynman diagrams in the light cone coordinates, and we show that the anomaly given by this effective action agrees with that given by the differential geometric method. We will call a general coordinate transformation an Einstein transformation and a general coordinate transformation anomaly an Einstein anomaly respectively [13].

In section 5.1 we obtain the anomaly up to the sign by using the differential geometric method. In section 5.2 we solve the anomaly equation. In section 5.3 we show that the result of section 5.1 agrees with the diagram calculations.

5.1 Einstein Anomaly

Let us consider a system of a left-handed chiral fermion interacting with an external gravitational field. It is described by a symmetrized Lagrangian

$$
L = \frac{i}{2}\, e\, e_a{}^\mu(\bar{\psi}\gamma^a D_\mu\psi - \overline{D_\mu\psi}\gamma^a\psi) \tag{5.1}
$$

where $e_a{}^\mu$ is a vierbein field and

$$
e = \det e_\mu{}^a\,, \quad e_a{}^\mu e_\mu{}^b = \delta_a{}^b\,, \quad D_\mu = \partial_\mu - \frac{1}{2}\omega_{\mu bc}\Sigma^{bc}\,, \quad \Sigma^{bc} = \frac{1}{4}[\gamma^b,\gamma^c]\,. \tag{5.2}
$$

Since in two dimensions we have only one independent Σ^{bc} which is proportional to γ_5, the term which contains the Cartan-Weyl connection $\omega_{\mu bc}$ in (5.1) is proportional to $\{\gamma^a, \gamma_5\}$ which vanishes. Therefore (5.1) becomes simply

$$L = \frac{i}{2}\, e\, e_a{}^\mu (\overline{\psi}\gamma^a \overleftrightarrow{\partial}_\mu \psi)\,, \quad \text{where } (a \overleftrightarrow{\partial}_\mu b) \equiv a(\partial_\mu b) - (\partial_\mu a)b\,. \tag{5.3}$$

Under the Einstein transformation, the vierbein and the connection transform as

$$\begin{cases} \delta_\xi\, e_\mu{}^a = \xi^\rho \partial_\rho e_\mu{}^a + \partial_\mu \xi^\rho e_\rho{}^a\,, \\ \delta_\xi \Gamma_{\lambda\mu}{}^\nu = \xi^\rho \partial_\rho \Gamma_{\lambda\mu}{}^\nu + \partial_\lambda \xi^\rho \Gamma_{\rho\mu}{}^\nu + \partial_\mu \xi^\rho \Gamma_{\lambda\rho}{}^\nu - \Gamma_{\lambda\mu}{}^\rho \partial_\rho \xi^\nu - \partial_\lambda \partial_\mu \xi^\nu\,. \end{cases} \tag{5.4}$$

Let us treat $e_\mu{}^a$, $\Gamma_{\lambda\mu}{}^\nu$, $-\partial_\mu \xi^\nu$ as matrices E , Γ_λ , and Λ respectively, i.e.,

$$(E)_\mu{}^a \equiv e_\mu{}^a\,, \quad (\Gamma_\lambda)_\mu{}^\nu \equiv \Gamma_{\lambda\mu}{}^\nu\,, \quad (\Lambda)_\mu{}^\nu \equiv -\partial_\mu \xi^\nu\,, \tag{5.5}$$

and decompose δ_ξ into two parts

$$\delta_\xi = \mathcal{L}_\xi + \delta_\Lambda\,, \tag{5.6}$$

such that

$$\begin{cases} \mathcal{L}_\xi E = \xi^\rho \partial_\rho E \\ \mathcal{L}_\xi \Gamma_\lambda = \xi^\rho \partial_\rho \Gamma_\lambda + \partial_\lambda \xi^\rho \Gamma_\rho\,, \end{cases} \tag{5.7}$$

and

$$\begin{cases} \delta_\Lambda E = -\Lambda E \\ \delta_\Lambda \Gamma_\lambda = D_\lambda \Lambda = \partial_\lambda \Lambda + \Gamma_\lambda \Lambda - \Lambda \Gamma_\lambda\,. \end{cases} \tag{5.8}$$

We notice that $\mathcal{L}_\xi$ is a Lie derivative with E as a scalar and Γ_λ as a covariant vector. δ_Λ is the same as a Yang-Mills gauge transformation with Γ_λ as a Yang-Mills gauge field. Repeated applications of $\delta_\Lambda E$ in (5.8) give

$$E' = e^{-\Lambda} E \tag{5.9}$$

for a finite transformation. (5.9) reminds us of (2.19) in chapter 2 and will be used when we solve an anomaly equation in section 5.2.

As in the Yang-Mills gauge theory case, we have an anomaly equation

$$\delta_\xi W_\xi = H_\xi\,, \tag{5.10}$$

where W_ξ is an effective action which gives rise to an Einstein anomaly H_ξ under the Einstein transformation δ_ξ. Then from

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1,\xi_2]}\,, \tag{5.11}$$

where

$$([\xi_1, \xi_2])^\mu = \xi_2^\rho \partial_\rho \xi_1^\mu - \xi_1^\rho \partial_\rho \xi_2^\mu\,, \tag{5.12}$$

we get a consistency condition

$$\delta_{\xi_1} H_{\xi_2} - \delta_{\xi_2} H_{\xi_1} = H_{[\xi_1,\xi_2]}\,. \tag{5.13}$$

Bardeen and Zumino showed that the Einstein anomaly which is the solution of (5.13) is given by the same function as that of a Yang-Mills gauge theory by replacing A, F by Γ, R respectively [13]. Therefore from (2.15) we have the two-dimensional Einstein anomaly as

$$[\text{2-dim. Einstein Ano.}] \quad \propto \quad -\frac{1}{4\pi} \mathrm{Tr} \int d^2x\, \Lambda \partial_\rho \Gamma_\lambda \epsilon^{\rho\lambda}\,. \tag{5.14}$$

However, the normalization factor is different from that of the Yang Mills gauge anomaly. The Atiyah-Singer index of the Dirac operator in the system (5.1) is given by the integration of the Dirac genus $\hat{A}(M)$ as [18,19]

$$n_+ - n_- = \int_M \hat{A}(M)\,, \tag{5.15}$$

where

$$\hat{A}(M) = \prod_{i=1}^{n/2} \frac{(x_i/2)}{\sinh(x_i/2)} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}\{7(p_1)^2 - 4p_2\} + \cdots . \qquad (5.16)$$

Then in four-dimensions, where we started to get the two-dimensional non-Abelian gauge anomaly,

$$\begin{aligned} n_+ - n_- &= -\frac{1}{24}P_1 = -\frac{1}{24}\int_M p_1(T(M)) \\ &= \frac{1}{24\cdot 8\pi^2}\,\mathrm{Tr}\int_M (R\wedge R)\,. \end{aligned} \qquad (5.17)$$

In (5.17) we have an additional factor of $(-1/24)$ compared with $(-1/8\pi^2)\mathrm{Tr}(F^2)$ in (2.11). Therefore, with the correct normalization factor we obtain the two-dimensional Einstein anomaly as

$$\begin{aligned} [\text{2-dim. Einstein Ano.}] &= (-\frac{1}{24})(-\frac{1}{4\pi})\mathrm{Tr}\int d^2x\,\Lambda\partial_\rho\Gamma_\lambda\epsilon^{\rho\lambda} \\ &= \frac{1}{96\pi}\int d^2x\,(\Lambda)_\mu{}^\nu\partial_\rho(\Gamma_\lambda)_\nu{}^\mu\epsilon^{\rho\lambda} \\ &= -\frac{1}{96\pi}\int d^2x\,\partial_\mu\xi^\nu\partial_\rho\Gamma_{\lambda\nu}{}^\mu\epsilon^{\rho\lambda}\,. \end{aligned} \qquad (5.18)$$

5.2 Solution of the Anomaly Equation - Effective Action

The anomaly equation for the Einstein anomaly is given by

$$\begin{cases} \delta_\xi W_\xi = H_\xi \\ H_\xi = -\frac{1}{96\pi}\int d^2x\,\partial_\mu\xi^\nu\partial_\rho\Gamma_{\lambda\nu}{}^\mu\epsilon^{\rho\lambda}\,. \end{cases} \qquad (5.19)$$

First let us consider only the δ_Λ part of $\delta_\xi = \mathcal{L}_\xi + \delta_\Lambda$ in (5.6). From (5.8) and (5.9) we have the transformations caused by δ_Λ as

$$\begin{cases} \delta_\Lambda\Gamma_\lambda = \partial_\lambda\Lambda + \Gamma_\lambda\Lambda - \Lambda\Gamma_\lambda \\ e^{H'} = e^{-\Lambda}\,e^H\,, \end{cases} \qquad (5.20)$$

where we used the matrix notation of (5.5) and

$$e^H \equiv E\,. \qquad (5.21)$$

Let us also write the anomaly equation (5.19) using only δ_Λ. It will be shown later that the solution of this modified anomaly equation has zero Lie derivative.

$$\begin{cases} \delta_\Lambda W_\xi = H_\xi \\ H_\xi = \frac{1}{96\pi}\int d^2x\,Tr(\Lambda\partial_\rho\Gamma_\lambda\epsilon^{\rho\lambda})\,. \end{cases} \qquad (5.22)$$

We notice that the first equation in (5.20) is the same as the Yang-Mills gauge transformation, and the second equation in (5.20) is a non-linear transformation like (2.19). Therefore, we obtain a solution of (5.22) in analogy with (2.20) [13],

$$W_\xi[\Gamma, H] = \frac{1}{96\pi}\int d^2x\int_0^1 dt\,Tr[(-H)\partial_\rho\Gamma_\lambda(t)\epsilon^{\rho\lambda}]\,, \qquad (5.23)$$

where

$$\Gamma_\lambda(t) = e^{-tH}\Gamma_\lambda e^{tH} + e^{-tH}\partial_\lambda e^{tH}\,. \qquad (5.24)$$

Let us show that the Lie derivative of $W_\xi[\Gamma,\ H]$ in (5.23) is zero. (5.7) says that E is a scalar under a Lie derivative, so H is also a scalar. (5.7) also says that Γ_λ is a covariant vector, then $\Gamma_\lambda(t)$ in (5.24) is also a covariant vector, and then $\partial_\rho \Gamma_\lambda(t)\epsilon^{\rho\lambda}$ in (5.23) is a scalar density. Therefore the integrand of $W_\xi[\Gamma,\ H]$ in (5.23) is a product of a scalar and a scalar density, i.e., a scalar density. Therefore,

$$W_\xi[\Gamma,\ H] = \int d^2x\, M\ ,$$

where $M = 1/96\pi \int_0^1 dt\, Tr[(-H)\partial_\rho \Gamma_\lambda(t)\epsilon^{\rho\lambda}]$ is a scalar density. Then

$$\mathcal{L}_\xi W_\xi[\Gamma,\ H] = \int d^2x\, \{\xi^\mu \partial_\mu M + (\partial_\mu \xi^\mu) M\} = \int d^2x\, \partial_\mu(\eta^\mu M) = 0\ . \quad (5.25)$$

Combining (5.22) and (5.25), we have

$$\delta_\xi W_\xi = (\mathcal{L}_\xi + \delta_\Lambda) W_\xi = \delta_\Lambda W_\xi = H_\xi\ . \quad (5.26)$$

Thus it has been shown that (5.23) is a solution of the original anomaly equation (5.19).

5.3 Comparison with Diagram Calculations

As we have shown in section 5.1, our system is described by the Lagrangian

$$L = \frac{i}{2}\, e\, e^{a\mu} \bar{\psi} \gamma_a \overleftrightarrow{\partial_\mu}\, \psi\ , \quad (5.27)$$

where $(1+\gamma_5)\psi = 0$, i.e., $\gamma_- \psi = 0$. Let us linearize the vierbein with the symmetrized $h_{\mu a}$ as

$$e_\mu{}^a \equiv \delta_\mu{}^a + h_\mu{}^a\ . \quad (5.28)$$

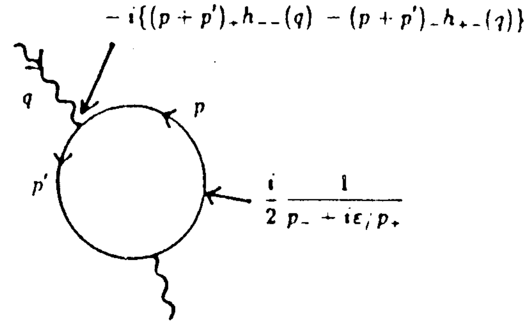


Fig.1 Feynman rule : Take $\mathrm{Tr}\int \frac{d^2k}{(2\pi)^2}$, and attach (-) sign for a fermion loop.

Then multiply the symmetry factor $\prod_i (1/n_i!)$ for an effective action.

Then

$$e_a{}^\mu = \delta_a{}^\mu - h_a{}^\mu + O(h^2) \quad \text{from} \quad e_a{}^\mu e_\mu{}^b = \delta_a{}^b\ , \quad (5.29)$$

and

$$e = \det(e_\mu{}^a) = \det(I+h) = \exp\{\mathrm{Tr}[\ln(I+h)]\} = 1 + h_a{}^a + O(h^2)\ . \quad (5.30)$$

Using these expansions, we have the interaction Lagrangian

$$L_{int.} = \frac{i}{2}(h_{+-}\bar{\psi}\gamma_+ \overleftrightarrow{\partial_-}\, \psi - h_{--}\bar{\psi}\gamma_+ \overleftrightarrow{\partial_+}\, \psi) + O(h^2)\ . \quad (5.31)$$

Then we obtain the Feynman rule given in Fig.1 for one-loop diagrams in the same way as in section 2.3 of chapter 2.

Using the Feynman rule in Fig.1 we get the following amplitude for Dia-

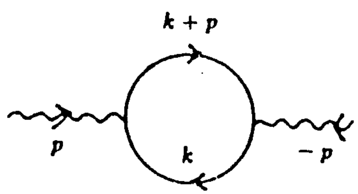


Fig.2 Diagram [5.1]

gram [5.1].

$$\begin{aligned} Amp. = &-\int \frac{dk_+\,dk_-}{(2\pi)^2}\{(2k+p)_+h_{--}(p)-(2k+p)_-h_{+-}(p)\} \\ &\times\{(2k+p)_+h_{--}(-p)-(2k+p)_-h_{+-}(-p)\} \\ &\times\frac{1}{4}\{\frac{1}{k_-+i\epsilon/k_+}\}\{\frac{1}{(k+p)_-+i\epsilon/(k+p)_+}\} \\ = &-\frac{i}{24\pi}\frac{p_+^3}{p_-}h_{--}(p)h_{--}(-p)\ +(\text{local terms})\,. \end{aligned} \tag{5.32}$$

In the above calculation we followed the same procedure as that for (2.50) in chapter 2. We attach the symmetry factor (1/2!) to (5.32) and match this to iW_1,

$$\begin{aligned} W_1 = &-\frac{1}{48\pi}\int\frac{d^2p}{(2\pi)^2}\,d^2q\,\delta^2(p+q)\,\{\frac{p_+^3}{p_-}h_{--}(p)h_{--}(q) \\ &+a\,p_+^2h_{--}(p)h_{-+}(q)+b\,p_-p_+h_{--}(p)h_{++}(q) \\ &+c\,p_-p_+h_{-+}(p)h_{-+}(q)+d\,p_-^2h_{-+}(p)h_{++}(q)\}\,. \end{aligned} \tag{5.33}$$

In (5.33) we added a general local functional which is Lorentz invariant [12].

In order to show that the diagram calculation gives the same anomaly as that obtained by the differential geometric method, let us try to adjust the coefficients a, b, c and d such that the variation of (5.33) gives rise to the

$O(h)$ term of (5.18). First we expand (5.18) for comparison,

[2-dim.Einstein Ano.]

$$\begin{aligned} &= -\frac{1}{96\pi}\int d^2x\,\partial_\mu\xi^\nu\partial_\rho\Gamma_{\lambda\nu}{}^\mu\epsilon^{\rho\lambda} \\ &= -\frac{i}{96\pi}\int\frac{d^2p}{(2\pi)^2}\,d^2q\,\delta^2(p+q)\,\{\xi_-(q)p_+-\xi_+(q)p_-\} \\ &\qquad\times\{p_-^2h_{++}(p)-2p_+p_-h_{+-}(p)+p_+^2h_{--}(p)\}\ +O(h^2)\,. \end{aligned} \tag{5.34}$$

When the system is assumed to have local Lorentz invariance, we can use the local Lorentz transformation as a restoring transformation to keep the symmetrized $h_{\mu\alpha}$ symmetric under the Einstein transformation in the following way. From $e_{\mu a}\equiv\eta_{\mu a}+h_{\mu a}$, under the genuine Einstein transformation plus the local Lorentz transformation,

$$\delta_\xi h_{\mu a}=\xi^\rho\partial_\rho h_{\mu a}+\partial_\mu\xi^\rho(\eta_{\rho a}+h_{\rho a})-\theta_a{}^b(\eta_{\mu b}+h_{\mu b})\,, \tag{5.35}$$

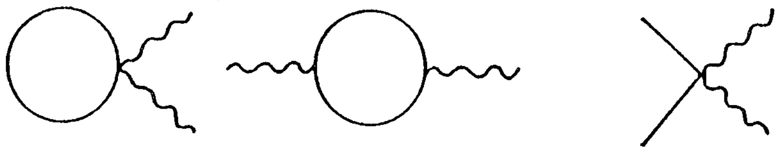
where $\theta_{ab}(x)$ is an antisymmetric parameter function for the local Lorentz transformation. Then by choosing θ_{ab} as

$$\theta_{ab}=-\frac{1}{2}(\partial_a\xi_b-\partial_b\xi_a)\ +O(h)\,, \tag{5.36}$$

we have

$$\delta_\xi h_{\mu a}=\frac{1}{2}(\partial_\mu\xi_a+\partial_a\xi_\mu)\ +O(h)\,. \tag{5.37}$$

Since the variation in (5.37) is symmetric, $h_{\mu a}$ is kept symmetric under the transformation if we started with a symmetric $h_{\mu a}$. Each component of $h_{\mu a}$



(a) (b) (c)

Fig.3 Diagrams of the order of $O(h^2)$

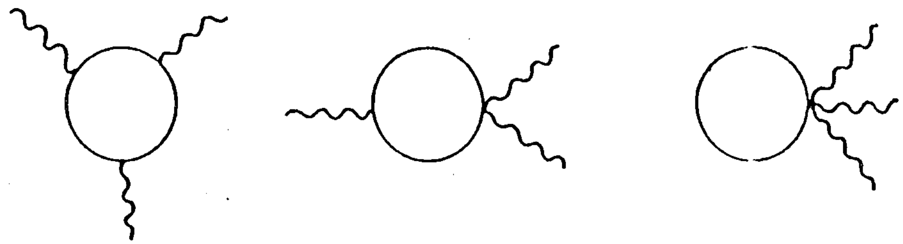
transforms as follows up to the lowest order in h, i.e., $O(h^0)$:

$$\begin{cases} \delta_\xi h_{++}(p) = ip_+\xi_+(p) \\ \delta_\xi h_{-+}(p) = \frac{i}{2}(p_+\xi_-(p) + p_-\xi_+(p)) \\ \delta_\xi h_{--}(p) = ip_-\xi_-(p)\,. \end{cases} \tag{5.38}$$

It can be shown that W_1 in (5.33) with the following assignment of a, b, c and d gives rise to (5.34) under δ_ξ given by (5.38),

$$\begin{aligned} W_1 = -\frac{1}{48\pi}\int \frac{d^2p}{(2\pi)^2}\, d^2q\,\delta^2(p+q)\,\{&\frac{p_+^3}{p_-}h_{--}(p)h_{--}(q) \\ &- 3p_+^2 h_{--}(p)h_{-+}(q) + p_-p_+h_{--}(p)h_{++}(q) \\ &+ 2p_-p_+h_{-+}(p)h_{-+}(q) - p_-^2 h_{-+}(p)h_{++}(q)\}\,. \end{aligned} \tag{5.39}$$

Therefore, it has been shown up to $O(h)$ in the anomaly that the anomaly (5.18) which was obtained by the differential geometric method agrees with the result of the Feynman diagram calculation. In the above diagram calculation we did not include the diagram in Fig.3(a) which is the same order in h as the diagram in Fig.3(b) since this diagram would give a local functional. Actually in our system the vertex in Fig.3(c) does not exist when we expand the Lagrangian (5.27) using (5.28), (5.29) and (5.30). Then the diagram in Fig.3(a) does not exist.



(a) (b) (c)

Fig.4 Diagrams of the order of $O(h^3)$

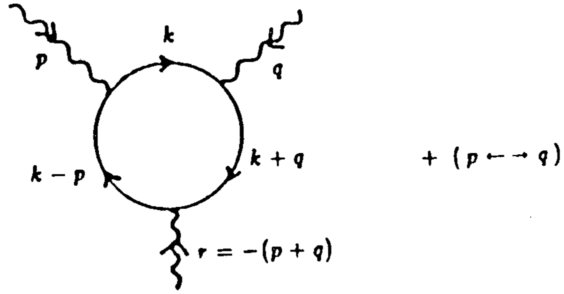


Fig.5 Diagram [5.2]

Let us now show that diagram calculations agree with (5.18) up the next order, i.e., $O(h^2)$ in the anomaly. For this we need to calculate only the diagram in Fig.4(a), since the diagram in Fig.4(b) does not exist because of the absence of the vertex in Fig.3(c), and the diagram in Fig.4(c) would give rise to a local functional to the effective action.

Applying the Feynman rule in Fig.1, we have the amplitude for Diagram

[5.2].

$$
\begin{aligned}
Amp. = -(\frac{1}{2})^3 \int \frac{d^2k}{(2\pi)^2} & [(2k-p)_+ h_{--}(p) - (2k-p)_- h_{+-}(p)] \\
& \times [(2k+q)_+ h_{--}(q) - (2k+q)_- h_{+-}(q)] \\
& \times [(2k-p+q)_+ h_{--}(r) - (2k-p+q)_- h_{+-}(r)] \\
& \times \{\frac{1}{k_- + i\epsilon/k_+}\}\{\frac{1}{(k+q)_- + i\epsilon/(k+q)_+}\}\{\frac{1}{(k-p)_- + i\epsilon/(k-p)_+}\} \\
& + (p \longleftrightarrow q) \, .
\end{aligned}
\tag{5.40}
$$

(5.40) contains the following four cases for the combinations of the external h fields.

$$
\begin{cases}
(case\ 1) \ : \ (h_{--} \, , \ h_{--} \, , \ h_{--}) \\
(case\ 2) \ : \ (h_{--} \, , \ h_{--} \, , \ h_{+-}) \\
(case\ 3) \ : \ (h_{--} \, , \ h_{+-} \, , \ h_{+-}) \\
(case\ 4) \ : \ (h_{+-} \, , \ h_{+-} \, , \ h_{+-}) \, .
\end{cases}
\tag{5.41}
$$

As we did before, we integrate (5.40) first over k_- by using the residue method and then over k_+. After these integrations we find that (*case* 3) and (*case* 4) give rise to local functionals for the effective action which can be ignored since the effective action is ambiguous by a local functional. (*case* 1)

and (*case* 2) produce the following effective action W_2 of $O(h^3)$.

$$
\begin{aligned}
W_2 = \frac{1}{24\pi} \int & \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} d^2r \, \delta^2(p+q+r) \\
& \times \{-\frac{1}{3}\frac{1}{r_-}\frac{p_+^3(2r_+ + p_+)}{p_-} h_{--}(p)h_{--}(q)h_{--}(r) \\
& + \frac{p_+^3}{p_-} h_{--}(p)h_{--}(q)h_{+-}(r)\} \, .
\end{aligned}
\tag{5.42}
$$

Now we want to show that the $O(h^2)$ terms of $\delta_\xi(W_1 + W_2)$ agree with the $O(h^2)$ terms of (5.18). In order to calculate $O(h^2)$ terms of $\delta_\xi W_1$, we need $O(h)$ terms of $\delta_\xi h_{\mu\nu}$, i.e., one higher order than $\delta_\xi h_{\mu\nu}$ of (5.37) or (5.38).

Following a similar procedure as from (5.35) to (5.37) we obtain

$$
\begin{aligned}
\delta_\xi h_{mn} = & \frac{1}{2}(\partial_m \xi_n + \partial_n \xi_m) \\
& + \xi^l \partial_l h_{mn} - (\xi^l \partial_m h_{ln} + \xi^l \partial_n h_{lm}) \\
& - \frac{1}{4}(\partial_m \xi^l h_{ln} + \partial_n \xi^l h_{lm}) - \frac{1}{4}(\partial^l \xi_m h_{ln} + \partial^l \xi_n h_{lm}) \\
& + O(h^2) \, .
\end{aligned}
\tag{5.43}
$$

Using (5.43), (2.45) and the correspondence between (∂_μ) and (ip_μ) as explained above (2.56) in chapter 2, we have the following $O(h^2)$ terms of

$\delta_\xi(W_1 + W_2)$ which we will call $\delta_\xi(W_1 + W_2)[O(h^2)]$.

$$\begin{aligned}
\delta_\xi(W_1 + W_2)[O(h^2)] = -\frac{i}{96\pi} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r) \\
\times \{ & h_{--}(p)h_{--}(q)\xi_+(r)(p_+^3 + 3p_+^2 q_+) \\
& + h_{--}(p)h_{++}(q)\xi_-(r)(-2p_+^2 p_- - p_+p_-q_+ + 3p_+^2 q_- + 3p_+q_+q_- + q_+^2 q_-) \\
& + h_{--}(p)h_{++}(q)\xi_+(r)(p_+p_-^2 + 3p_+p_-q_- - p_-q_+q_- + p_+q_-^2) \\
& + h_{++}(p)h_{++}(q)\xi_-(r)(p_-q_-^2 - q_-^3) \\
& + h_{--}(p)h_{-+}(q)\xi_-(r)(9p_+^3 + 9p_+^2 q_+ + 3p_+q_+^2 + q_+^3) \\
& + h_{--}(p)h_{-+}(q)\xi_+(r)(-p_-p_+^2 - 2p_-p_+q_+ - 3p_+^2 q_- + 3p_-q_+^2 - 4p_+q_-q_+ + q_-q_+^2) \\
& + h_{-+}(p)h_{++}(q)\xi_-(r)(3p_-^2 p_+ + p_-^2 q_+ - 4p_-p_+q_- - 2p_-q_-q_+ - p_+q_-^2 + q_-^2 q_+) \\
& + h_{-+}(p)h_{++}(q)\xi_+(r)(-p_-^3 - 3p_-^2 q_- - p_-q_-^2 - q_-^3) \\
& + h_{-+}(p)h_{-+}(q)\xi_-(r)(-2p_-p_+^2 + 4p_-p_+q_+ + 6p_+^2 q_-) \\
& + h_{-+}(p)h_{-+}(q)\xi_+(r)(4p_-^2 p_+ + 4p_-p_+q_-)\} \, .
\end{aligned} \tag{5.44}$$

Since we used the light-cone coordinates for the diagram calculations, these calculations are not covariant and fairly complicated. After lengthy calculations it can be shown that (5.44) becomes the same as the $O(h^2)$ terms of (5.18) by adding the following Lorentz invariant local functional W_C to

$(W_1 + W_2)$.

$$\begin{aligned}
W_C = \frac{i}{96\pi} \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} d^2r\, \delta^2(p+q+r) \\
\times \{ & h_{--}(p)h_{--}(q)h_{++}(r)(-\frac{1}{2})(p_+^2 + 8p_+q_+ + q_+^2) \\
& + h_{++}(p)h_{++}(q)h_{--}(r)\, \frac{1}{2}\, (p_-^2 + 4p_-q_- + q_-^2) \\
& + h_{--}(p)h_{++}(q)h_{-+}(r)\, 2\, (-3p_-p_+ - 4p_+q_- + q_-q_+) \\
& + h_{--}(p)h_{-+}(q)h_{-+}(r)(-2)(3p_+^2 + p_+q_+ + q_+^2) \\
& + h_{-+}(p)h_{++}(q)h_{-+}(r)\, 2\, (p_-^2 + p_-q_- - q_-^2) \\
& + h_{-+}(p)h_{-+}(q)h_{-+}(r)\, \frac{4}{3}\, (2p_-p_+ + p_-q_+ + p_+q_- + 2q_-q_+)\} \, .
\end{aligned} \tag{5.45}$$

Therefore it has been shown that (5.18) agrees with the diagram calculations up to the second non-trivial order, i.e., $O(h^2)$ in the anomaly. That is, in this section we showed, up to the second non-trivial order, that the purely gravitational anomaly obtained by the differential geometric method agrees with the variation of W obtained by diagram calculations by adding appropriate local counter terms.

CHAPTER 6

SUPERSYMMETRY ANOMALY

In this chapter we study supersymmetric Yang-Mills gauge theory in which the supersymmetry is a rigid symmetry [52]. We can think of two kinds of supersymmetry anomalies. The first is an anomaly of a supersymmetry transformation in a superfield formulation without fixing a specific gauge. We will call this transformation a genuine supersymmetry transformation, and this anomaly a genuine supersymmetry anomaly respectively. The second is an anomaly of a supersymmetry transformation in the Wess-Zumino gauge which is composed of two steps of transformations, i.e. a genuine supersymmetry transformation and a restoring supersymmetric gauge transformation. We will call this anomaly a supersymmetry anomaly in the Wess-Zumino gauge.

We find a supersymmetric extension of a gauge anomaly which we will call a supersymmetric gauge anomaly. This anomaly is then used to obtain a gauge anomaly and a supersymmetry anomaly in the Wess-Zumino gauge, which satisfy the mixed consistency conditions. In this derivation it is transparent that the supersymmetry anomaly in the Wess-Zumino gauge originates only from a restoring supersymmetric gauge transformation, not from a genuine supersymmetry transformation. This indicates that there is no genuine supersymmetry anomaly [48]. This situation can be guessed from the fact that the genuine supersymmetry transformation is a rigid transformation. This also shows that when the gauge anomaly is canceled, the supersymmetry anomaly in the Wess-Zumino gauge is also canceled automatically.

Furthermore, we obtain the supersymmetric extension of the Wess-Zumino term following Wess and Zumino's original method in superspace [9,53]. We modify this extension such that it depends only on the vector multiplet. This extended Wess-Zumino term's gauge and supersymmetry variations give rise to the gauge and supersymmetry anomalies in the Wess-Zumino gauge respectively.

In section 6.1 we present two-dimensional superfields and their supersymmetry and gauge transformations. In section 6.2 we obtain a supersymmetric gauge anomaly and gauge and supersymmetry anomalies in the Wess-Zumino gauge. In section 6.3 we obtain the supersymmetric extension of the Wess-Zumino term.

6.1 Two-dimensional Superspace and Superfields

In two-dimensional superspace we have two real space-time coordinates x^0, x^1 and two real spinorial coordinates θ_1, θ_2. The conventions which we will use are given by

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}, \qquad -\eta^{00} = \eta^{11} = 1, \quad \eta^{01} = \eta^{10} = 0,$$
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.1}$$

The rest of our conventions and their properties are given in Appendix A.

A scalar superfield is given by

$$S = A + i\bar{\theta}\psi + \frac{i}{2}\bar{\theta}\theta F \ , \tag{6.2}$$

where $\bar{\theta}_a = \theta_b \gamma^0_{ba}$ is a linear combination of θ_a's and is not independent of the θ_a's. The supersymmetry transformation of S is given by using a generator

$$Q_a = -\frac{\partial}{\partial \bar{\theta}_a} + i\gamma^{\mu}_{ab}\theta_b \partial_{\mu} \tag{6.3}$$

as

$$\delta_{\alpha} S = [S, \bar{\alpha} Q] :$$

$$\begin{cases} \delta A = i\bar{\alpha}\psi \\ \delta\psi = \partial_{\mu} A \gamma^{\mu}\alpha + F\alpha \\ \delta F = i\bar{\alpha}\gamma \cdot \partial\psi \ . \end{cases} \tag{6.4}$$

A spinor superfield or a vector multiplet V_a, which is real and contains a gauge field A_{μ} as one component field, is given by [52]

$$V_a = \xi_a + \gamma^{\mu}_{ab}\theta_b A_{\mu} + \gamma^5_{ab}\theta_b M + \theta_a N + \frac{i}{2}\bar{\theta}\theta\zeta_a \ . \tag{6.5}$$

Its supersymmetry transformation is given by

$$\delta_{\alpha} V_a = [V_a, \bar{\alpha} Q] :$$

$$\begin{cases} \delta\xi = \gamma^{\mu}\alpha A_{\mu} + \gamma^5\alpha M + \alpha N \\ \delta A_{\mu} = \frac{i}{2}\bar{\alpha}\gamma_{\nu}\gamma_{\mu}\partial^{\nu}\xi - \frac{i}{2}\bar{\alpha}\gamma_{\mu}\zeta \\ \delta M = -\frac{i}{2}\bar{\alpha}\gamma^5\gamma \cdot \partial\xi - \frac{i}{2}\bar{\alpha}\gamma^5\zeta \\ \delta N = \frac{i}{2}\bar{\alpha}\gamma \cdot \partial\xi - \frac{i}{2}\bar{\alpha}\zeta \\ \delta\zeta = -\gamma^{\nu}\gamma^{\mu}\alpha\partial_{\mu}A_{\nu} - \gamma^5\gamma^{\mu}\alpha\partial_{\mu}M - \gamma^{\mu}\alpha\partial_{\mu}N \ . \end{cases} \tag{6.6}$$

In order to have a gauge structure, we let a set of scalar superfields form a representation of a gauge group such that $S = \{S_i\}$ transforms under a finite gauge transformation as

$$S' = e^{-\Lambda} S \tag{6.7}$$

or under an infinitesimal transformation as

$$\delta_{\Lambda} S = -\Lambda S \ , \tag{6.8}$$

where $\Lambda = \Lambda_i T_i$, Λ_i's are real scalar superfields which are supersymmetric gauge transformation parameters, i.e., $\Lambda_i = a_i + i\bar{\theta}\chi_i + \frac{i}{2}\bar{\theta}\theta f_i$ and T_i's are anti-hermitian gauge group generators which satisfy $[T_i, T_j] = f_{ijk} T_k$

We gauge covariantize

$$D_a S = (-\frac{\partial}{\partial\bar{\theta}} - i\gamma^{\mu}\theta\partial_{\mu})_a S \tag{6.9}$$

to

$$\nabla_a S = (D_a - iV_a)S = -i(iD_a + V_a)S \ , \tag{6.10}$$

by requiring $V_a = V_{ai} T_i$ to transform under a gauge transformation as $(iD_a + V_a)' = e^{-\Lambda}(iD_a + V_a)e^{\Lambda}$ in order to have $(\nabla_a S)' = e^{-\Lambda}(\nabla_a S)$ That is, under a finite transformation

$$V'_a = (e^{-\Lambda} i D_a e^{\Lambda}) + e^{-\Lambda} V_a e^{\Lambda} \tag{6.11}$$

or under an infinitesimal trasformation

$$\delta_\Lambda V_a = i D_a \Lambda + [V_a, \Lambda]\,. \tag{6.12}$$

In terms of the component fields, (6.12) becomes

$$\begin{cases} \delta\xi = \chi + [\xi, a] \\ \delta A_\mu = \partial_\mu a + [A_\mu, a] + \frac{i}{2}(\bar\xi\gamma_\mu\chi + \bar\chi\gamma_\mu\xi) \\ \delta M = [M, a] + \frac{i}{2}(\bar\xi\gamma^5\chi + \bar\chi\gamma^5\xi) \\ \delta N = f + [N, a] + \frac{i}{2}(-\bar\xi\chi + \bar\chi\xi) \\ \delta\zeta = -\gamma\cdot\partial\chi + [\zeta, a] + [\xi, f] - [A_\mu, \gamma^\mu\chi] - [M, \gamma^5\chi] - [N, \chi]\,. \end{cases} \tag{6.13}$$

When we have the gauge symmetry (6.12) or (6.13), we can choose the Wess-Zumino gauge in which $\xi = 0$, $N = 0$ in the following way. Let us start with $\xi = 0$, $N = 0$, then we have the following transformations of ξ and N.

Genuine supersymmetry transformation for ξ and N :

$$\begin{cases} \delta\xi = \gamma^\mu\alpha A_\mu + \gamma^5\alpha M \\ \delta N = -\frac{i}{2}\bar\alpha\zeta\,. \end{cases} \tag{6.14}$$

Supersymmetric gauge transformation for ξ and N :

$$\begin{cases} \delta\xi = \chi \\ \delta N = f\,. \end{cases} \tag{6.15}$$

As we see in (6.14), even though we start with $\xi = 0$, $N = 0$, these component fields become non-zero after a genuine supersymmetry transformation. But we can come back to $\xi = 0$, $N = 0$ by performing a restoring

gauge transformation which is given by the following gauge transformation parameter Λ_{RG} as can be seen in (6.15).

$$\delta_{RG} V = i D\Lambda_{RG} + [V, \Lambda_{RG}]$$

with Λ_{RG} :

$$\begin{cases} a = 0 \\ \chi = -\gamma^\mu\alpha A_\mu - \gamma^5\alpha M \\ f = \frac{i}{2}\bar\alpha\lambda \end{cases} \tag{6.16}$$

where

$$\lambda = \zeta + \gamma\cdot\partial\xi\,.$$

Therefore the supersymmetry and gauge transformations in the Wess-Zumino gauge are given by

$$\begin{cases} \delta_{S(WZ)} = \delta_{GEN.\,SUSY} + \delta_{RG} \\ \delta_{G(WZ)} = \delta_{SUP.\,GAUGE} \quad \text{with} \quad \Lambda_G: \quad a = a,\ \chi = 0,\ f = 0\,, \end{cases} \tag{6.17}$$

where $\delta_{GEN.\,SUSY}$, δ_{RG} and $\delta_{SUP.\,GAUGE}$ mean genuine supersymmetry, restoring gauge and supersymmetric gauge transformations respectively. Afterwards, we will write $\delta_{S(WZ)}$ and $\delta_{G(WZ)}$ simply as δ_S and δ_G. Under these transformations, the component fields A_μ, M, λ in the Wess-Zumino gauge transform as

$$\begin{cases} \delta_S A_\mu = -\frac{i}{2}\bar\alpha\gamma_\mu\lambda \\ \delta_S M = -\frac{i}{2}\bar\alpha\gamma^5\lambda \\ \delta_S\lambda = \gamma^\mu\gamma^\nu\alpha F_{\mu\nu} + 2\gamma^\mu\gamma^5\alpha(\partial_\mu M + [A_\mu, M])\,, \end{cases} \tag{6.18}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$,

$$\begin{cases} \delta_G A_\mu = \partial_\mu a + [A_\mu, a] \\ \delta_G M = \qquad [M, a] \\ \delta_G \lambda = \qquad [\lambda, a] \,. \end{cases} \tag{6.19}$$

Note that in two dimensions the Wess-Zumino gauge has a pseudo-scalar field M as well as A_μ and λ, in contrast to the four-dimensional case in which there is no M [52].

For reference we write down the following supersymmetry and gauge transformations of a scalar multiplet in the Wess-Zumino gauge.

$$\begin{cases} \delta_S \phi = i\bar{\alpha}\psi \\ \delta_S \psi = \gamma^\mu \alpha D_\mu \phi + \gamma^5 \alpha M \phi + \alpha F \\ \delta_S F = i\bar{\alpha}\gamma^\mu D_\mu \psi + i\bar{\alpha}\gamma^5 M \psi - \frac{i}{2}\bar{\alpha}\lambda\phi \end{cases} \tag{6.20}$$

where

$$D_\mu = \partial_\mu + A_\mu \,,$$

$$\begin{cases} \delta_G \phi = -a\phi \\ \delta_G \psi = -a\psi \\ \delta_G F = -aF \,. \end{cases} \tag{6.21}$$

6.2 Anomalies

First let us find a supersymmetric extension of a gauge anomaly which we will call a supersymmetric gauge anomaly. A vector multiplet given by (6.5) gives rise to

$$\bar{D}\gamma^5 V = \bar{D}_a \gamma^5_{ab} V_b = 2M - i\bar{\theta}\gamma^5\lambda + i\bar{\theta}\theta\epsilon^{\mu\nu}\partial_\mu A_\nu \,. \tag{6.22}$$

Then with $\Lambda = a + i\bar{\theta}\chi + \frac{i}{2}\bar{\theta}\theta f$, we have

$$Tr(\Lambda\bar{D}\gamma^5 V)|_{\bar{\theta}\theta} = i\bar{\theta}\theta\, Tr(a\,\epsilon_{\mu\nu}\partial^\mu A^\nu + \frac{i}{2}\bar{\chi}\gamma^5\lambda + fM) \,. \tag{6.23}$$

Since the first term on the right hand side of (6.23) is just an ordinary non-supersymmetric gauge anomaly, it seems plausible that (6.23) is a supersymmetric gauge anomaly. In order to confirm this we should show that

$$\triangle(\Lambda) = Tr\int d^2x\, d\bar{\theta}\, d\theta\, (\Lambda\bar{D}\gamma^5 V) \tag{6.24}$$

satisfies the consistency condition

$$\delta_{\Lambda_2}\triangle(\Lambda_1) - \delta_{\Lambda_1}\triangle(\Lambda_2) = \triangle([\Lambda_2, \Lambda_1]) \,, \tag{6.25}$$

where $\delta_\Lambda V_a$ is given in (6.12).

Let us show this.

$$\begin{aligned} \delta_{\Lambda_2}(\Lambda_1\bar{D}\gamma^5 V) &= \Lambda_1\bar{D}\gamma^5(iD\Lambda_2 + [V, \Lambda_2]) \\ &= \Lambda_1(\bar{D}U)\Lambda_2 - \Lambda_1 U(\bar{D}\Lambda_2) - \Lambda_1(\bar{D}\Lambda_2)U - \Lambda_1\Lambda_2(\bar{D}U) \,, \end{aligned} \tag{6.26}$$

where $U \equiv \gamma^5 V$, and $\bar{D}\gamma^5 D = 0$ was used.

From (6.26) we have

$$Tr\{\delta_{\Lambda_2}(\Lambda_1\bar{D}\gamma^5 V) - \delta_{\Lambda_1}(\Lambda_2\bar{D}\gamma^5 V)\} = Tr\{[\Lambda_2, \Lambda_1]\bar{D}\gamma^5 V + \bar{D}([\Lambda_2, \Lambda_1]\gamma^5 V)\} \,. \tag{6.27}$$

Since the second term on the right hand side of (6.27) is a total derivative, it vanishes under the integration $\int d^2x\, d\bar{\theta}\, d\theta$. Thus by taking $\int d^2x\, d\bar{\theta}\, d\theta$ on both sides of (6.27) we obtain (6.25). Of course, (6.25) is also satisfied with an arbitrary normalization factor in (6.24). We can show the above in a more elegant, but equivalent way which is given in Appendix B.

The supersymmetry and gauge transformations in the Wess-Zumino gauge, i.e., (6.18) and (6.19) satisfy the algebra:

$$\begin{cases} [\delta_S(\beta),\, \delta_S(\alpha)] = \delta_G(-2i(\overline{\alpha}\gamma^\mu\beta)A_\mu - 2i(\overline{\alpha}\gamma^5\beta)M) + 2(\overline{\alpha}\gamma^\mu\beta)i\partial_\mu \\ [\delta_G(b),\, \delta_G(a)] = \delta_G([b,\, a]) \\ [\delta_G(a),\, \delta_S(\alpha)] = 0\,. \end{cases} \tag{6.28}$$

Then a supersymmetry anomaly $\triangle_S(\alpha)$ and a gauge anomaly $\triangle_G(a)$ in the Wess-Zumino gauge satisfy the consistency conditions:

$$\begin{cases} \delta_S(\beta)\triangle_S(\alpha) - \delta_S(\alpha)\triangle_S(\beta) = \triangle_G(-2i(\overline{\alpha}\gamma^\mu\beta)A_\mu - 2i(\overline{\alpha}\gamma^5\beta)M) \\ \delta_G(b)\triangle_G(a) - \delta_G(a)\triangle_G(b) = \triangle_G([b,\, a]) \\ \delta_G(a)\triangle_S(\alpha) - \delta_S(\alpha)\triangle_G(a) = 0\,. \end{cases} \tag{6.29}$$

The term $2(\overline{\alpha}\gamma^\mu\beta)i\partial_\mu$ in (6.28) did not contribute to (6.29), since the vacuum functional is invariant under translation if we impose the condition that a surface integral vanishes.

The interesting thing is that we can obtain $\triangle_S(\alpha)$, $\triangle_G(a)$ which satisfy (6.29) by using the supersymmetric gauge anomaly (6.24) in the following way. Let us rewrite (6.24) with an arbitrary normalization factor as

$$\begin{aligned} \triangle(\Lambda) &= -icTr\int d^2x\, d^2\theta\,(\Lambda\overline{D}\gamma^5 V) \\ &= cTr\int d^2x\,(a\,\epsilon^{\mu\nu}\partial_\mu A_\nu + \frac{i}{2}\overline{\chi}\gamma^5\lambda + fM) \end{aligned} \tag{6.30}$$

$$\text{where} \quad \int d^2\theta \equiv -\frac{1}{4}\int d\overline{\theta}\, d\theta \qquad \text{such that} \quad \int d^2\theta\; \overline{\theta}\theta = 1\,.$$

At first we obtain $\triangle_G(a)$ from (6.30) by taking $a = a$, $\chi = 0$, $f = 0$ since δ_G in (6.17) or (6.19) was given by this assignment of Λ, i.e., Λ_G. Next, in order to obtain $\triangle_S(\alpha)$ we observe that δ_S in (6.17) or (6.18) is composed of two steps, i.e., $\delta_{GEN.\,SUSY}$ and δ_{RG} . But we expect that the $\delta_{GEN.\,SUSY}$ step will not produce any anomaly since this transformation is a rigid transformation. Then we expect that $\triangle(\Lambda)$ with $\Lambda = \Lambda_{RG}$ in (6.16) will give rise to $\triangle_S(\alpha)$ [48]. That is, we expect the following to be the solution of (6.29).

$$\begin{aligned} \triangle_G(a) &= \triangle(\Lambda_G:\; a = a,\, \chi = 0,\, f = 0) \\ &= c\,Tr\int d^2x\, a\,\epsilon^{\mu\nu}\partial_\mu A_\nu\,, \end{aligned} \tag{6.31}$$

$$\begin{aligned} \triangle_S(\alpha) &= \triangle(\Lambda_{RG}:\; a = 0,\, \chi = -\gamma^\mu\alpha A_\mu - \gamma^5\alpha M,\, f = \frac{i}{2}\overline{\alpha}\lambda) \\ &= ic\,Tr\int d^2x(\frac{1}{2}A_\mu\overline{\alpha}\gamma^\mu\gamma^5\lambda + M\overline{\alpha}\lambda)\,. \end{aligned} \tag{6.32}$$

We have confirmed that these $\triangle_G(a)$ and $\triangle_S(\alpha)$ satisfy (6.29) by explicit application of (6.18) and (6.19). Therefore we have found that the supersymmetry anomaly in the Wess-Zumino gauge originates from the supersymmetric gauge anomaly. This indicates that there is no genuine supersymmetry anomaly. This also shows that when the gauge anomaly is canceled, the supersymmetry anomaly in the Wess-Zumino gauge is also canceled automatically.

6.3 Supersymmetric Extension of the Wess-Zumino Term

We will find a supersymmetric extension of the Wess-Zumino term which depends only on component fields of a vector multiplet. This vacuum functional gives rise to gauge and supersymmetry anomalies by gauge and supersymmetry variations respectively. In order to understand the derivation better, let us review briefly the familiar non-supersymmetric gauge theory case [8,9].

An effective action can be obtained by solving an anomaly equation

$$\delta_a W = \int d^2x \; a_i G_i \,, \tag{6.33}$$

where G_i's are anomalies. Wess and Zumino solved this equation and obtained the solution

$$W[A,\pi] = \int d^2x \int_0^1 dt \; \pi_i G_i(A(t))(x) \,, \tag{6.34}$$

where the π_i's are a set of fields which transforms as

$$e^{\pi'} = e^{\pi} e^{a} \tag{6.35}$$

and

$$A_\mu(t) = e^{t\pi} A_\mu e^{-t\pi} + e^{t\pi}\partial_\mu e^{-t\pi} \,, \tag{6.36}$$

where

$$a = a_i T_i, \; \pi = \pi_i T_i, \; A_\mu = A_{\mu i} T_i \,.$$

Any gauge invariant functional can be added in (6.34), so (6.34) is a particular solution and is called the Wess-Zumino term.

Now we are interested in having a solution $W[A]$ as a functional of only A_μ without the independent π. This can be achieved by replacing the independent π by a function $\pi(A)$ which transforms as (6.35) when A_μ transforms as a gauge field, if we can find such a function [27].

In the Abelian case we find such a $\pi(A)$ easily as

$$\pi(A) = \frac{1}{\Box}\partial_\mu A^\mu \,, \quad \text{where } \Box \equiv \partial_\mu \partial^\mu \tag{6.37}$$

$$\text{since} \qquad \delta_a A^\mu = \partial^\mu a \,, \quad \delta_a \pi(A) = \frac{1}{\Box}\partial_\mu(\partial^\mu a) = a \,.$$

Then

$$\begin{aligned} W[A] &= \int d^2x \int dt \, (\frac{1}{\Box}\partial_\mu A^\mu) G(A(t)) \\ &= \int d^2x \int_0^1 dt \, (\frac{1}{\Box}\partial_\mu A^\mu) \frac{i}{2\pi}\partial_\nu A_\lambda(t) \varepsilon^{\nu\lambda} \\ &= \frac{i}{2\pi}\int d^2x \int_0^1 dt \, (\frac{1}{\Box}\partial_\mu A^\mu) \partial_\nu (A_\lambda - t\partial_\lambda \pi(A)) \varepsilon^{\nu\lambda} \\ &= \frac{i}{2\pi}\int d^2x \, (\frac{1}{\Box}\partial_\mu A^\mu) \partial_\nu A_\lambda \varepsilon^{\nu\lambda} \,, \end{aligned} \tag{6.38}$$

for the quantum effect of a left-handed chiral fermion. We use the convention $\varepsilon^{10} = -\varepsilon^{01} = 1$, $\varepsilon^{+-} = -\varepsilon^{-+} = -1$. Our conventions are summarized in Appendix A.

In the light-cone coordinates, let us use the anomaly in the form $-(i/\pi)\partial_+ A_-$ which is equivalent to $(i/2\pi)\partial_\nu A_\lambda \varepsilon^{\nu\lambda}$ in (6.38), since they differ by a variation of a local functional $(i/2\pi)\int d^2x \, A_\mu A^\mu$. (∂_μ) in the coordinate space corresponds to $(-ip_\mu)$ in the momentum space, since we will take external momenta as out-going. Then $W[A]$ can be written as

$$W[A] = \frac{i}{2\pi} \int \frac{d^2p}{(2\pi)^2} \frac{p_+}{p_-} A_-(p) A_-(-p) \,, \tag{6.39}$$

after adding an appropriate local functional to (6.38) which is allowed since $W[A]$ is ambiguous by a local functional.

In the non-Abelian case we can get $\pi(A)$ which transforms as (6.35) by inverting

$$A_\mu = e^{-\pi} \partial_\mu e^{\pi} \,. \tag{6.40}$$

This inversion can be done as a power series of A_μ and the lowest order term has the same form as (6.37). Note that even though we are inverting the pure gauge form (6.40), $\pi(A)$ obtained by this procedure transforms as (6.35) for a general A_μ. That is, (6.40) is just a guide for obtaining $\pi(A)$ for a general A_μ [27]. Using this $\pi(A)$ we can obtain $W[A]$ as a power series of A_μ which starts with the lowest order term similar to (6.38) or (6.39) as

$$W[A] = c' Tr \int \frac{d^2p}{(2\pi)^2} \frac{p_+}{p_-} A_-(p) A_-(-p) \quad + O(A^3) \,. \tag{6.41}$$

We can use the above procedure to get a supersymmetric extension of the Wess-Zumino term which gives rise to $\triangle_G(a)$, $\triangle_S(\alpha)$ in (6.31), (6.32) by $\delta_G(a)$, $\delta_S(\alpha)$ in (6.18), (6.19) respectively [53]. First we will treat the Abelian case in detail.

The consistency conditions for the Abelian case are the same as (6.29) except that the second condition is replaced by

$$\delta_G(b)\triangle_G(a) - \delta_G(a)\triangle_G(b) = 0 \,. \tag{6.42}$$

The solution of the consistency conditions is given, in analogy with (6.31) and (6.32), by

$$\triangle_G(a) = i \int d^2x \, a \, \epsilon^{\mu\nu} \partial_\mu A_\nu \,, \tag{6.43}$$

$$\triangle_S(\alpha) = - \int d^2x \, (\frac{1}{2} A_\mu \bar{\alpha} \gamma^\mu \gamma^5 \lambda + M \bar{\alpha} \lambda) \,, \tag{6.44}$$

which can be obtained from a supersymmetric Abelian gauge anomaly

$$\begin{aligned} \triangle_{Abel.}(\Lambda) &= \int d^2x \, d^2\theta \, (\Lambda \overline{D} \gamma^5 V) \\ &= i \int d^2x \, (a \epsilon^{\mu\nu} \partial_\mu A_\nu + \frac{i}{2} \bar{\chi} \gamma^5 \lambda + f M) \end{aligned} \tag{6.45}$$

through the same procedure as that used in section 6.2 for the non-Abelian case. In (6.45) we take such a normalization factor for convenience.

In the present two-dimensional supersymmetric case, (6.34) is replaced by

$$W[V, \Pi] = \int d^2x \, d^2\theta \int_0^1 dt \, \Pi_i (\overline{D} \gamma^5 V_i(t)) \tag{6.46}$$

where the Π_i's transform under a supersymmetric gauge transformation as

$$e^{\Pi'} = e^{\Pi} e^{\Lambda} \tag{6.47}$$

and

$$V_a(t) = e^{t\Pi} V_a e^{-t\Pi} + e^{t\Pi} (i D_a) e^{-t\Pi} \,. \tag{6.48}$$

The above formulas (6.46), (6.47) and (6.48) are also valid for the non-Abelian case where $\Pi = \Pi_i T_i$, $\Lambda = \Lambda_i T_i$ are Lie algebra valued scalar superfields and $V_a = V_{ai} T_i$. The formulas for the Abelian case are simply given by omitting the sum over the subscript i in (6.46) and using the Abelian nature of Π, Λ and V_a.

In the Abelian case the gauge transformation given in (6.12) becomes

$$\delta V_a = i D_a \Lambda \ . \tag{6.49}$$

Then we find easily that

$$\Pi(V) = -i \frac{1}{\overline{D}D} \overline{D} V \tag{6.50}$$

transforms as (6.47) which is the same as $\Pi' = \Pi + \Lambda$ in the Abelian case. The expression (6.50) means

$$\begin{aligned} \Pi(V) &= -i \frac{1}{\overline{D}_a D_a} \overline{D}_b V_b \\ &= -i \frac{1}{(\overline{D}_a D_a)(\overline{D}_c D_c)} (\overline{D}_d D_d)(\overline{D}_b V_b) \\ &= \frac{i}{4} \frac{1}{\Box} (\overline{D}_d D_d)(\overline{D}_b V_b) \end{aligned} \tag{6.51}$$

since

$$(\overline{D}D)^2 = (\overline{D}_a D_a)(\overline{D}_c D_c) = -4 \partial_\mu \partial^\mu = -4 \Box \ .$$

Then from (6.46) we get $W[V]$ as a functional of V_a only

$$\begin{aligned} W[V] &= -i \int d^2x d^2\theta \int_0^1 dt (\frac{1}{\overline{D}D} \overline{D} V)(\overline{D} \gamma^5 V(t)) \\ &= -i \int d^2x d^2\theta \int_0^1 dt (\frac{1}{\overline{D}D} \overline{D} V) \{ \overline{D} \gamma^5 (V - t i D \Pi) \} \\ &= -i \int d^2x d^2\theta (\frac{1}{\overline{D}D} \overline{D} V)(\overline{D} \gamma^5 V) \end{aligned} \tag{6.52}$$

since $\quad \overline{D} \gamma^5 D = 0$.

Let us express (6.52) in terms of component fields in the Wess-Zumino gauge.

$$V = \gamma^\mu \theta A_\mu + \gamma^5 \theta M + \frac{i}{2} \bar{\theta} \theta \lambda$$

$$\overline{D} V = -i \bar{\theta} \lambda + i \bar{\theta} \theta \partial_\mu A^\mu$$

$$\frac{1}{\overline{D}D} \overline{D} V = i \frac{\partial_\mu A^\mu}{\Box} + i \bar{\theta} (-\frac{i}{2} \frac{\gamma \cdot \partial \lambda}{\Box}) \tag{6.53}$$

and

$$\overline{D} \gamma^5 V = 2M + i \bar{\theta} (-\gamma^5 \lambda) + \frac{i}{2} \bar{\theta} \theta (2 \epsilon^{\mu\nu} \partial_\mu A_\nu) \ . \tag{6.54}$$

Using (6.53) and (6.54) we get

$$W[V] = i \epsilon^{\mu\nu} \int d^2x (\frac{\partial_\lambda A^\lambda}{\Box} \partial_\mu A_\nu - \frac{i}{4} \bar{\lambda} \gamma_\nu \frac{\partial_\mu \lambda}{\Box}) \ . \tag{6.55}$$

We have checked explicitly that the variations of (6.55) give the anomalies (6.43) and (6.44).

When we use the light-cone coordinates in (6.55), the terms from ϵ^{+-} and ϵ^{-+} are equivalent to each other, since their variations give rise to anomalies which differ by variations of a local functional. Therefore we can replace (6.55) by twice the ϵ^{+-} term in (6.55). Then we have $W[V]$ as

$$W[V] = i \int \frac{d^2p}{(2\pi)^2} [\frac{p_+}{p_-} A_-(-p) A_-(p) - \frac{1}{4} \frac{1}{p_-} \bar{\lambda}(-p) \gamma_- \lambda(p)] \tag{6.56}$$

by adding an appropriate local functional. Variations of (6.56) give rise to anomalies of the form

$$\begin{cases} \Delta_G(a) = -2i \int d^2x \, a \, \partial_+ A_- \\ \Delta_S(\alpha) = \int d^2x \, (A_+ \bar{\alpha} \gamma_- \lambda - M \bar{\alpha} \lambda) \end{cases} \tag{6.57}$$

which are equivalent to (6.43) and (6.44) since they differ by variations of a local functional.

In the non-Abelian case, in order to get a $W[V]$ depending only on V_a from (6.46), we need a function $\Pi(V)$ which transforms as (6.47) when V_a

transforms as (6.12). This can be obtained by inverting

$$V_a = e^{-\Pi} i D_a e^{\Pi} \tag{6.58}$$

in analogy with (6.40) in the non-supersymmetric non-Abelian case. $\Pi(V)$ can be expanded in a power series of V and the lowest order term has the same form as (6.50) in the Abelian case. Using this $\Pi(V)$ we can get $W[V]$ as a power series of V which starts with the lowest order term similar to (6.55) or (6.56) as

$$W[V] = c'' Tr \int \frac{d^2p}{(2\pi)^2}[\frac{p_+}{p_-}A_-(-p)A_-(p) - \frac{1}{4}\frac{1}{p_-}\bar{\lambda}(-p)\gamma_-\lambda(p)] \quad + O(V^3) . \tag{6.59}$$

Appendix A

Let us summarize our conventions and their properties. We use μ, ν, λ, $\cdots$ for space-time indices, and a, b, c, $\cdots$ for spinorial indices.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad -\eta^{00} = \eta^{11} = 1, \quad \eta^{01} = \eta^{10} = 0,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

$$\epsilon^{10} = -\epsilon^{01} = 1, \quad \epsilon^{00} = \epsilon^{11} = 0.$$

$$\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu , \quad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu}\gamma^5 , \quad \gamma^\mu\gamma^\nu\gamma^5 = \eta^{\mu\nu}\gamma^5 - \epsilon^{\mu\nu}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\rho\mu}\gamma^\nu ,$$

$$\epsilon^{\mu\nu}\epsilon^{\rho\lambda} = -(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho}) .$$

$$x^\pm = \frac{1}{\sqrt{2}}(\pm x^0 + x^1) , \; x_\pm = \frac{1}{\sqrt{2}}(\pm x_0 + x_1) ,$$

$$x_- = x^+ , \; x_+ = x^- .$$

$$\eta^{+-} = \eta^{-+} = \eta_{+-} = \eta_{-+} = 1 , \text{ other } \eta\text{'s} = 0 .$$

$$a^\mu b_\mu = a^+ b_+ + a^- b_- = a_+ b_- + a_- b_+ ,$$

$$a^\mu a_\mu = 2a_+ a_- .$$

$$\epsilon^{+-} = -\epsilon^{-+} = -\epsilon_{+-} = \epsilon_{-+} = -1 , \text{ other } \epsilon\text{'s} = 0 .$$

$$\gamma^+ = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^1) = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} , \; \gamma^- = \frac{1}{\sqrt{2}}(-\gamma^0 + \gamma^1) = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix} .$$

$$\gamma^+\gamma^+ = \gamma^-\gamma^- = 0 , \quad \gamma^+\gamma^- + \gamma^-\gamma^+ = 2 .$$

$$\bar{\theta}_a = \theta_b\gamma^0_{ba} = -\epsilon_{ab}\theta_b \; (\gamma^0_{ab} = \epsilon_{ab} , \; \epsilon_{12} = 1) , \; \theta_a = -\bar{\theta}_b\gamma^0_{ba} = \epsilon_{ab}\bar{\theta}_b .$$

$$\theta_a = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \bar{\theta}_2 \\ -\bar{\theta}_1 \end{pmatrix} , \; \bar{\theta}_a = \begin{pmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{pmatrix} = \begin{pmatrix} -\theta_2 \\ \theta_1 \end{pmatrix} .$$

$$\frac{\partial}{\partial\theta_a}\bar{\theta}_b = \gamma^0_{ab} , \quad \frac{\partial}{\partial\bar{\theta}_a}\theta_b = -\gamma^0_{ab} ,$$

$$\frac{\partial}{\partial\theta_a} = -\frac{\partial}{\partial\bar{\theta}_b}\gamma^0_{ba} , \quad \frac{\partial}{\partial\bar{\theta}_a} = \frac{\partial}{\partial\theta_b}\gamma^0_{ba} .$$

$$\bar{\theta}_a\theta_b = \frac{1}{2}\bar{\theta}\theta\delta_{ab} \quad (\bar{\theta}\theta = \bar{\theta}_a\theta_a) ,$$

$$\frac{\partial}{\partial\bar{\theta}_a}(\bar{\theta}\theta) = 2\theta_a , \; \frac{\partial}{\partial\theta_a}(\bar{\theta}\theta) = -2\bar{\theta}_a .$$

$$D_a = -\frac{\partial}{\partial \bar{\theta}_a} - i\gamma^\mu_{ab}\theta_b\partial_\mu \, , \; \overline{D}_a = D_b\gamma^0_{ba} = \frac{\partial}{\partial \theta_a} + i\bar{\theta}_b\gamma^\mu_{ba}\partial_\mu \, ,$$

$$Q_a = -\frac{\partial}{\partial \bar{\theta}_a} + i\gamma^\mu_{ab}\theta_b\partial_\mu \, , \; \overline{Q}_a = Q_b\gamma^0_{ba} = \frac{\partial}{\partial \theta_a} - i\bar{\theta}_b\gamma^0_{ba}\partial_\mu \, .$$

Fierz rearrangement:

$$(\bar{\alpha}\psi)\beta_a = -\frac{1}{2}\{(\bar{\alpha}\beta)\psi_a + (\bar{\alpha}\gamma^5\beta)(\gamma^5\psi)_a + (\bar{\alpha}\gamma_\mu\beta)(\gamma^\mu\psi)_a\} \, .$$

$$\bar{\alpha}\gamma_A\beta = \bar{\beta}\,\tilde{\gamma}_A\,\alpha \quad (\; \alpha, \beta \text{ are real spinors }) \; :$$

Using $(\gamma^\mu)^T = \gamma^0\gamma^\mu\gamma^0$, $(\gamma^5)^T = \gamma^0\gamma^5\gamma^0 = \gamma^5$ (where superscript T means *Transpose*),

$$\tilde{1} = 1 \, , \quad \tilde{\gamma^\mu} = -\gamma^\mu \, , \quad \tilde{\gamma^5} = -\gamma^5 \, , \quad (\widetilde{\gamma^5\gamma^\mu}) = -(\gamma^5\gamma^\mu) \, ,$$

$$(\widetilde{\gamma^\mu\gamma^\nu}) = (\gamma^\nu\gamma^\mu) \, , \quad (\widetilde{\gamma^5\gamma^\mu\gamma^\nu}) = -(\gamma^5\gamma^\nu\gamma^\mu) \, .$$

We take external momenta as out-going, therefore (∂_μ) in the coordinate space corresponds to $(-ip_\mu)$ in the momentum space.

Appendix B

Let us show in another way that (6.24) satisfies (6.25). Here we treat Λ as a ghost and we take the following BRS transformation.

$$\begin{cases} S\Lambda = -\Lambda^2 \\ SV_a = -iD_a\Lambda - \Lambda V_a - V_a\Lambda \, . \end{cases} \tag{B.1}$$

Then (6.25) can be expressed simply as [8]

$$S\triangle(\Lambda) = 0 \, . \tag{B.2}$$

Let us show that (6.24), i.e.,

$$\triangle(\Lambda) = Tr \int d^2x \, d\bar{\theta} \, d\theta \, (\Lambda\overline{D}\gamma^5 V) \tag{B.3}$$

satisfies (B.2). In the following expression, every term is to have Tr in front, i.e., we omit Tr in front of every term for notational simplicity.

$$\begin{aligned} S(\Lambda\overline{D}\gamma^5V) &= (S\Lambda)\overline{D}\gamma^5V + (-1)^2\Lambda\overline{D}\gamma^5(SV) \\ &= (-\Lambda^2)\overline{D}\gamma^5V + \Lambda\overline{D}\gamma^5(-iD\Lambda - \Lambda V - V\Lambda) \\ &= -\Lambda^2\overline{D}U - \Lambda\overline{D}(\Lambda U) - \Lambda\overline{D}(U\Lambda) \; (\text{where } U \equiv \gamma^5 V \, , \quad \overline{D}\gamma^5 D = 0 \text{ were used} \\ &= -\Lambda^2\overline{D}U - \Lambda(\overline{D}\Lambda)U + \Lambda\Lambda(\overline{D}U) - \Lambda(\overline{D}U)\Lambda + \Lambda U(\overline{D}\Lambda) \\ &= (\overline{D}\Lambda)\Lambda U - \Lambda(\overline{D}\Lambda)U + \Lambda\Lambda(\overline{D}U) \\ &= \overline{D}(\Lambda^2 U) \, . \end{aligned}$$

That is,

$$S\triangle(\Lambda) = \int d^2x \, d\bar{\theta} \, d\theta \, \overline{D}\{Tr(\Lambda^2 U)\} = 0 \, ,$$

since the integrand is a total derivative.

Appendix C Three-dimensional Superspace

We summarize the three-dimensional supersymmetry for reference because of its similarity to the two-dimensional case. The three-dimensional superspace is described by three real space-time coordinates x^0, x^1, x^2 and two real spinorial coordinates θ_1, θ_2. Therefore the structure of the spinorial coordinates is the same as the two-dimensional one. Q_a and D_a have the same forms as (6.3) and (6.9),

$$Q_a = -\frac{\partial}{\partial\bar{\theta}_a} + i\gamma^\mu_{ab}\theta_b\partial_\mu \ , \ D_a = -\frac{\partial}{\partial\bar{\theta}_a} - i\gamma^\mu_{ab}\theta_b\partial_\mu \ , \ \mu = 0,\, 1,\, 2\ .$$

Our conventions and their properties are as follows.

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad -\eta^{00} = \eta^{11} = \eta^{22} = 1,$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 == \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\epsilon^{012} = \epsilon^{120} = -\epsilon^{102} = 1, \ etc.$$

$$[\, \gamma^\mu \, , \, \gamma^\nu \,] = 2\epsilon^{\mu\nu\lambda}\gamma_\lambda$$

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\lambda}\gamma_\lambda \ , \quad Tr(\gamma^\mu\gamma^\nu\gamma^\lambda) = 2\epsilon^{\mu\nu\lambda}\ ,$$

$$\epsilon^{\lambda\alpha\beta}\epsilon_{\lambda\mu\nu} = -\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu\ .$$

A vector multiplet is given by

$$V_a = \xi_a + \gamma^\mu_{ab}\theta_b A_\mu + \theta_a N + \frac{i}{2}\bar{\theta}\theta\zeta_a\ .$$

Genuine supersymmetry transformation : $\delta_\alpha V_a = [V_a, \bar{\alpha}Q]$

$$\begin{cases} \delta\xi = \gamma^\mu\alpha A_\mu + \alpha N \\ \delta A_\mu = \frac{i}{2}\epsilon_{\mu\nu\lambda}\bar{\alpha}\gamma^\nu\partial^\lambda\xi + \frac{i}{2}\bar{\alpha}\partial_\mu\xi - \frac{i}{2}\bar{\alpha}\gamma_\mu\zeta \\ \delta N = \frac{i}{2}\bar{\alpha}\gamma\cdot\partial\xi - \frac{i}{2}\bar{\alpha}\zeta \\ \delta\zeta = \epsilon_{\mu\nu\lambda}\gamma^\lambda\alpha\partial^\mu A^\nu - \alpha\partial_\mu A^\mu - \gamma^\mu\alpha\partial_\mu N\ . \end{cases}$$

Supersymmetric gauge transformation : $\delta_\Lambda V_a = iD_a\Lambda + [V_a, \Lambda]$

$$\begin{cases} \delta\xi = \chi + \quad [\xi, a] \\ \delta A_\mu = \partial_\mu a + [A_\mu, a] + \frac{i}{2}(\bar{\xi}\gamma_\mu\chi + \bar{\chi}\gamma_\mu\xi) \\ \delta N = f + \quad [N, a] + \frac{i}{2}(-\bar{\xi}\chi + \bar{\chi}\xi) \\ \delta\zeta = -\gamma\cdot\partial\chi + [\zeta, a] + [\xi, f] - [A_\mu, \gamma^\mu\chi] - [N, \chi]\ . \end{cases}$$

In the above $V_a = V_{ai}T_i$, $\Lambda = \Lambda_i T_i$ (T_i's are anti-hermitian), $\Lambda_i = a_i + i\bar{\theta}\chi_i + \frac{i}{2}\bar{\theta}\theta f_i$.

In the Wess-Zumino gauge, $\xi = 0$, $N = 0$, $\lambda = \zeta$,

$$\begin{cases} \delta_{S(WZ)} = \delta_{GEN.\,SUSY} + \delta_{RG} \\ \qquad \text{with } \Lambda_{RG} :\ a = 0\ ,\ \chi = -\gamma^\mu\alpha A_\mu\ ,\ f = \frac{i}{2}\bar{\alpha}\lambda\ , \\ \delta_{G(WZ)} = \delta_{SUP.\,GAUGE} \\ \qquad \text{with } \Lambda_G :\ a = a\ ,\ \chi = 0\ ,\ f = 0\ . \end{cases}$$

$$\begin{cases} \delta_S(\alpha)A_\mu = -\frac{i}{2}\bar{\alpha}\gamma_\mu\lambda \\ \delta_S(\alpha)\lambda = \gamma^\mu\gamma^\nu\alpha F_{\mu\nu}\ , \end{cases}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$,

$$\begin{cases} \delta_G(a)A_\mu = \partial_\mu a + [A_\mu, a] \\ \delta_G(a)\lambda = \qquad [\lambda, a]\ , \end{cases}$$

where δ_S, δ_G mean $\delta_{S(WZ)}$, $\delta_{G(WZ)}$ respectively. They satisfy the following algebra.

$$\begin{cases} [\delta_S(\beta)\ ,\ \delta_S(\alpha)] = \delta_G(-2i\bar{\alpha}\gamma^\mu\beta A_\mu) + 2(\bar{\alpha}\gamma^\mu\beta)i\partial_\mu \\ [\delta_G(b)\ ,\ \delta_G(a)] = \delta_G([b\ ,\ a]) \\ [\delta_G(a)\ ,\ \delta_S(\alpha)] = 0\ . \end{cases}$$

An interesting feature of the three-dimensional gauge theory is that there is a gauge invariant topological mass term [54]. In the three-dimensional supersymmetric gauge theory, we have the following supersymmetric topological mass term [55].

$$W = Tr \int d^3x \, \{\lambda_a \gamma^0_{ab} \lambda_b + 2i\epsilon^{\mu\nu\lambda}(A_\mu F_{\nu\lambda} - \frac{2}{3} A_\mu A_\nu A_\lambda)\}\ .$$

Under $\delta_G(a)$ and $\delta_S(\alpha)$,

$$\begin{cases} \delta_G(a) W = 4i\, Tr \int d^3x \, \epsilon^{\mu\nu\lambda} \partial_\mu (a \partial_\nu A_\lambda) \\ \delta_S(\alpha) W = -2\, Tr \int d^3x \, \epsilon^{\mu\nu\lambda} \partial_\mu (A_\nu (\bar{\alpha}\gamma_\lambda \lambda))\ . \end{cases}$$

Therefore when we assume that a surface integral is zero, W is invariant under $\delta_G(a)$ and $\delta_S(\alpha)$.

CHAPTER 7

ANOMALY SUPERFIELD FOR BOTH $\partial_\mu J^\mu_5$ AND $\partial_\mu J^\mu$

In four dimensions Ferrara and Zumino [56] showed that $\gamma^\mu S_\mu$, $\theta_\mu{}^\mu$ and $\partial_\mu J^\mu_5$ are components of a Wess-Zumino multiplet [57] (where S_μ is an improved supercurrent, and $\theta_{\mu\nu}$ is an improved energy-momentum tensor, and J^μ_5 is an axial vector current). The corresponding anomalies have been studied in references [58-62].

In this chapter we find in two dimensions a vector multiplet which contains $\partial_\mu J^\mu_5$ and $\partial_\mu J^\mu$ as components (where J^μ_5 is an axial vector current and J^μ is a vector current), and we find a corresponding anomaly superfield. Then we confirm that this anomaly superfield is realized by Feynman diagram calculations. These calculations show clearly how corresponding anomalies form a superfield. We study an Abelian case, but the extension to a non-Abelian case is not difficult.

In section 7.1 we review the gauge anomaly for a non-supersymmetric theory. In section 7.2 we find a vector multiplet which contains $\partial_\mu J^\mu_5$ and $\partial_\mu J^\mu$ as components, and a corresponding anomaly superfield. In section 7.3 we confirm this anomaly superfield by Feynman diagram calculations

7.1 Review of Non-Supersymmetric Case

In this section we review the gauge anomaly of a non-supersymmetric theory. We consider a system of a massive Dirac fermion and a gauge field, which is described by the following Lagrangian. (We omit writing the kinetic energy term of the gauge field.)

$$L = -i\overline{\psi}\gamma^\mu(\partial_\mu + ieB_\mu)\psi - im\overline{\psi}\psi \,. \qquad (7.1)$$

Our conventions are given by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \qquad -\eta^{00} = \eta^{11} = 1\,,$$
$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\epsilon^{10} = -\epsilon^{01} = 1\,. \qquad (7.2)$$

And their useful properties are

$$\gamma^\mu\gamma^5 = \epsilon^{\mu\nu}\gamma_\nu\,, \quad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu}\gamma^5\,, \quad \gamma^\mu\gamma^\nu\gamma^5 = \eta^{\mu\nu}\gamma^5 - \epsilon^{\mu\nu}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho + \eta^{\nu\rho}\gamma^\mu - \eta^{\rho\mu}\gamma^\nu\,, \quad \epsilon^{\mu\nu}\epsilon^{\rho\lambda} = -(\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho})\,. \quad (7.3)$$

(7.1) gives the following equations of motion.

$$(\gamma^\mu\partial_\mu + m + ie\gamma^\mu B_\mu)\psi = 0\,,$$
$$\overline{\psi}(\gamma^\mu\overleftarrow{\partial}_\mu - m - ie\gamma^\mu B_\mu) = 0\,. \qquad (7.4)$$

When we use the equations of motion naively, we get

$$\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) = 2m\overline{\psi}\gamma_5\psi\,. \qquad (7.5)$$

However, (7.5) is true only classically, and the quantum correction modifies it. This modification is called an anomaly. We can obtain the anomaly in the following way [1,5].

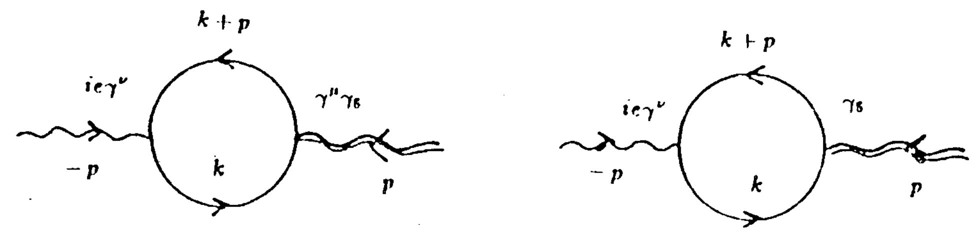


Fig.1 $R^{\mu\nu}$ and R^ν

In Fig.1 the double wiggled line in $R^{\mu\nu}$ corresponds to the axial vector current $\overline{\psi}\gamma^\mu\gamma_5\psi$, and that in R^ν corresponds to $\overline{\psi}\gamma_5\psi$. Then the left hand side of (7.5) is realized by $R^{\mu\nu}$ as

$$ip_\mu R^{\mu\nu} = ip_\mu(-1)\int\frac{d^2k}{(2\pi)^2}Tr\{\frac{-i(\gamma\cdot k + \gamma\cdot p - im)}{(k+p)^2+m^2}\gamma^\mu\gamma_5\frac{-i(\gamma\cdot k - im)}{k^2+m^2}ie\gamma^\nu\}$$
$$= -2e\epsilon^{\mu\nu}\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2}\{-2m^2p_\mu + (k^2+m^2)(k_\mu + p_\mu)$$
$$- ((k+p)^2+m^2)k_\mu\}$$
$$= -2e\epsilon^{\mu\nu}\int\frac{d^2k}{(2\pi)^2}\{\frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2}(-2m^2p_\mu) + [\frac{k_\mu+p_\mu}{(k+p)^2+m^2} - \frac{k_\mu}{k^2+m^2}]$$
$$= -2e\epsilon^{\mu\nu}\{-2m^2p_\mu\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2} + i\frac{p_\mu}{4\pi}\}\,, \qquad (7.6)$$

where we used the correspondence $\partial_\mu \leftrightarrow ip_\mu$ since p_μ is an incoming momentum for the wiggled line. In the last step of (7.6), we used the fact [5]

$$\Delta_\mu(a) = \int\frac{d^2k}{(2\pi)^2}[\frac{(k+a)_\mu}{(k+a)^2+m^2} - \frac{k_\mu}{k^2+m^2}] = i\frac{a_\mu}{4\pi}\,. \qquad (7.7)$$

Let us regularize (7.6).

$$ip_\mu R^{\mu\nu}(reg.) = \{RHS \text{ of } (7.6)\} - \{RHS \text{ of } (7.6) \text{ with } m \to M\}$$

$$= 4m^2\varepsilon^{\mu\nu}p_\mu e \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+m^2}\frac{1}{(k+p)^2+m^2} \qquad (7.8)$$
$$- 4M^2\varepsilon^{\mu\nu}p_\mu e \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M^2}\frac{1}{(k+p)^2+M^2}.$$

The first term in the RHS (right hand side) of (7.8) is nothing but $2mR^\nu$ in Fig.1, which corresponds to the RHS of (7.5), i.e., the naive classical result. Therefore the anomaly is given by the second term of the RHS of (7.8) when we renormalize (7.8), that is, when we take the limit $M \to \infty$.

Using $\frac{1}{k^2+M^2}\frac{1}{(k+p)^2+M^2} = \int_0^1 dx \frac{1}{[l^2+M^2+p^2x(1-x)]^2}$, $l = k+p(1-x)$,

$$-4M^2\varepsilon^{\mu\nu}p_\mu e \int \frac{d^2k}{(2\pi)^2}\frac{1}{k^2+M^2}\frac{1}{(k+p)^2+M^2}$$
$$= -4M^2\varepsilon^{\mu\nu}p_\mu e \int_0^1 dx \int \frac{d^2l}{(2\pi)^2}\frac{1}{[l^2+M^2+p^2x(1-x)]^2}$$
$$= -4M^2\varepsilon^{\mu\nu}p_\mu e \int_0^1 dx\, i \int \frac{d^2l_E}{(2\pi)^2}\frac{1}{[l_E^2+M^2+p^2x(1-x)]^2}$$
$$= -4M^2\varepsilon^{\mu\nu}p_\mu e \int_0^1 dx \frac{i\pi}{(2\pi)^2}\frac{1}{M^2+p^2x(1-x)}$$
$$\to -\frac{e}{\pi}\varepsilon^{\mu\nu}ip_\mu, \text{ when } M \to \infty. \qquad (7.9)$$

Then using the correspondence $\partial_\mu \leftrightarrow ip_\mu$, we have from (7.8) and (7.9)

$$\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) = 2m\overline{\psi}\gamma_5\psi - \frac{e}{\pi}\varepsilon^{\mu\nu}\partial_\mu B_\nu. \qquad (7.10)$$

Therefore we obtain the anomaly as $-(e/\pi)\varepsilon^{\mu\nu}\partial_\mu B_\nu$. We will generalize this to a superfield in the next section.

7.2 Superfield Extension

In this section we extend the anomaly in (7.10) to an anomaly superfield, and find a superfield which contains $\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi)$ and $\partial_\mu(\overline{\psi}\gamma^\mu\psi)$ as components. We consider a vector multiplet (a spinorial superfield) V_a which contains a gauge field B_μ as a component, i.e., a supersymmetric extension of a gauge field [52,63]. We use μ, ν, λ, $\cdots$ for space-time indices, and a, b, c, $\cdots$ for spinorial indices.

$$V_a = \xi_a + \gamma^\mu_{ab}\theta_b B_\mu + \gamma^5_{ab}\theta_b M + \theta_a N + \frac{i}{2}\overline{\theta}\theta\zeta_a. \qquad (7.11)$$

The conventions in this chapter are the same as those in chapter 6 and summarized in Appendix A of that chapter.

Using the supersymmetric derivative

$$D_a = -\frac{\partial}{\partial\overline{\theta}_a} - i\gamma^\mu_{ab}\theta_b\partial_\mu,\ \overline{D}_a = D_b\gamma^0_{ba}, \qquad (7.12)$$

we have

$$\overline{D}\gamma_5 V = \overline{D}_a\gamma^5_{ab}V_b = 2M - i\overline{\theta}\gamma^5\lambda + i\overline{\theta}\theta\varepsilon^{\mu\nu}\partial_\mu B_\nu, \qquad (7.13)$$

$$\frac{i}{2}(\gamma_5 D)_a(\overline{D}\gamma_5 V) = -\frac{1}{2}\lambda_a + \gamma^\mu_{ab}\theta_b(\varepsilon_{\mu\nu}\partial^\nu M) + \gamma^5_{ab}\theta_b(\varepsilon_{\mu\nu}\partial^\mu B^\nu) + \theta_a(0) + \frac{i}{2}\overline{\theta}\theta(-\frac{1}{2}\gamma\cdot\partial\lambda)_a, \qquad (7.14)$$

where $\lambda = \zeta + \gamma\cdot\partial\xi$.

That is, $\frac{i}{2}(\gamma_5 D)_a(\overline{D}\gamma_5 V)$ composes a spinorial superfield whose components are given by

$$\begin{cases} \xi_a \ : \ -\frac{1}{2}\lambda_a \\ \mathrm{B}_\mu \ : \ \epsilon_{\mu\nu}\partial^\nu M \\ \mathrm{M} \ : \ \epsilon_{\mu\nu}\partial^\mu B^\nu \\ \mathrm{N} \ : \ 0 \\ \zeta_a \ : \ -\frac{1}{2}(\gamma\cdot\partial\lambda)_a \,. \end{cases} \tag{7.15}$$

Let us study this supersymmetric extension of the anomaly. An interesting property of (7.15) is that the M component is the ordinary gauge anomaly, and the N component is zero. Also, we observe in (7.11) that when M is a pseudoscalar, N is a scalar. Therefore we are led to expect that there exists a superfield which has $\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi)$ as the M component and $\partial_\mu(\overline{\psi}\gamma^\mu\psi)$ as the N component. Indeed, there exists such a superfield, which is given in the following.

We consider a complex scalar superfield

$$\begin{aligned} S &= A + i\overline{\theta}\psi + \frac{i}{2}\overline{\theta}\theta F \,, \\ S^* &= A^* + i\overline{\theta}\psi^* + \frac{i}{2}\overline{\theta}\theta F^* \,. \end{aligned} \tag{7.16}$$

After studying the structure of the superfield and using trial and error, we obtain the following spinorial superfield which we expected to exist.

$$i(D_a S \overline{D}_b D_b S^* - D_a S^* \overline{D}_b D_b S) =$$

$$\begin{cases} \xi_a \ : \ -2i(\psi F^* - \psi^* F) \\ \mathrm{B}_\mu \ : \ -[\overline{\psi}\overleftrightarrow{\partial}_\mu\psi - \epsilon_{\mu\nu}\partial^\nu(\overline{\psi}\gamma_5\psi) + 2i(\partial_\mu A F^* - \partial_\mu A^* F)] \\ \mathrm{M} \ : \ \partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) \\ \mathrm{N} \ : \ \partial_\mu(\overline{\psi}\gamma^\mu\psi) \\ \zeta_a \ : \ -2i[\epsilon^{\mu\nu}\partial_\mu(\gamma_5\psi\partial_\nu A^* - \gamma_5\psi^*\partial_\nu A) + \partial_\mu(\psi\partial^\mu A^* - \psi^*\partial^\mu A)] \,. \end{cases} \tag{7.17}$$

Then we anticipate that the superfield (7.17) is subject to the quantum correction which gives rise to the anomaly superfield (7.15). In the next section we will confirm this fact by diagram calculations similar to that in section 7.1.

7.3 Realization by Diagram Calculations

We have the following supersymmetric extension of the Lagrangian (7.1) [52]. (We omit writing the kinetic energy term of V_a.)

$$L = [-\frac{i}{2}\overline{\nabla}_a S^* \nabla_a S + mS^* S]_F \,, \tag{7.18}$$

where the subscript F means the F component of a scalar superfield of the form (7.16). Since L in (7.18) is the last component of a superfield, it is invariant under a supersymmetry transformation up to a total derivative. In (7.18) ∇_a and $\overline{\nabla}_a$ are covariant derivatives

$$\nabla_a S = (D_a + eV_a)S \,, \ \ \overline{\nabla}_a S^* = (\overline{D}_a - e\overline{V}_a)S^* \,. \tag{7.19}$$

Then (7.18) is invariant under the supersymmetric gauge transformation

$$\begin{cases} S' = e^{ie\Lambda} S \\ V_a' = V_a - iD_a\Lambda \,, \end{cases} \tag{7.20}$$

where Λ is a scalar superfield which is a supersymmetric gauge transformation parameter

$$\Lambda = a + i\bar{\theta}\chi + \frac{i}{2}\bar{\theta}\theta f \,. \tag{7.21}$$

(7.18) gives the equation of motion

$$\overline{\nabla}\nabla S - 2imS = 0\,, \text{ i.e.,}$$

$$(\overline{D} + e\overline{V})(D + eV)S - 2imS = 0\,,$$

$$\overline{D}DS - 2imS + e^2\overline{V}VS + e\overline{V}DS + e(\overline{D}V)S + e(\overline{D}S)V = 0\,. \tag{7.22}$$

The Lagrangian in (7.18) is expressed in terms of component fields of S and V_a as

$$\begin{aligned} L = &- i\overline{\psi}\gamma^\mu(\partial_\mu + ieB_\mu)A - im\overline{\psi}\psi \\ &- (\partial_\mu - ieB_\mu)A^*(\partial^\mu + ieB^\mu)A - m^2A^*A \\ &- e^2M^2A^*A + eM\overline{\psi}\gamma_5\psi - \frac{e}{2}(A\overline{\psi}\lambda - A^*\overline{\lambda}\psi)\,. \end{aligned} \tag{7.23}$$

When we obtained (7.23), we took the Wess-Zumino gauge, i.e., $\xi = 0$, $N = 0$. We can anticipate the realization of the anomalies (7.15) in this special gauge, since (7.15) depends only on the component fields in the Wess-Zumino gauge.

As we obtained (7.5) for the ordinary gauge theory, we obtain the corresponding supersymmetric equation from (7.17) by using the equation of motion (7.22) for $\overline{D}_bD_bS$ and the equation of motion for $\overline{D}_bD_bS^*$ which is given by modifying (7.22) by the replacement of $(e \to -e)$.

$$i[D_aS(\overline{D}DS^*) - D_aS^*(\overline{D}DS)] = \{\text{Normal Terms}\} + c\frac{i}{2}(\gamma_5 D)_a(\overline{D}\gamma_5 V)\,, \tag{7.24}$$

where {Normal Terms} means the terms which are given by the naive use of the equation of motion such as $2m\overline{\psi}\gamma_5\psi$ in (7.5). In (7.24) we added the anomaly term like in (7.10) with a normalization factor c which will be determined by diagram calculations.

In components (7.24) is given by the following equations.

$$\left\{\begin{aligned} &\xi_a \;:\; -2i(\psi F^* - \psi^* F) = 2im(\psi A^* - \psi^* A) + c(-\tfrac{1}{2}\lambda) \\ &\mathrm{B}_\mu \;:\; -[\overline{\psi}\overleftrightarrow{\partial}_\mu\psi - \epsilon_{\mu\nu}\partial^\nu(\overline{\psi}\gamma_5\psi) + 2i(\partial_\mu AF^* - \partial_\mu A^*F)] \\ &\qquad\qquad = (\text{normal terms}) + c(\epsilon_{\mu\nu}\partial^\nu M) \\ &\mathrm{M} \;:\; \partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) = 2m\overline{\psi}\gamma_5\psi + 2ieM\overline{\psi}\psi + \tfrac{1}{2}ie(A^*\overline{\lambda}\gamma_5\psi - A\overline{\psi}\gamma_5\lambda) \\ &\qquad\qquad + c(\epsilon_{\mu\nu}\partial^\mu B^\nu) \\ &\mathrm{N} \;:\; \partial_\mu(\overline{\psi}\gamma^\mu\psi) = \tfrac{1}{2}ie(A\overline{\psi}\lambda + A^*\overline{\lambda}\psi) + c(0) \\ &\zeta_a \;:\; -2i[\epsilon^{\mu\nu}\partial_\mu(\gamma_5\psi\partial_\nu A^* - \gamma_5\psi^*\partial_\nu A) + \partial_\mu(\psi\partial^\mu A^* - \psi^*\partial^\mu A)] \\ &\qquad\qquad = (\text{normal terms}) + c(-\tfrac{1}{2}\gamma\cdot\partial\lambda)\,. \end{aligned}\right. \tag{7.25}$$

In (7.25) we did not write down the normal terms for the B_μ and ζ_a component equations since they are long and not interesting in our study.

Now let us show that the anomalies in (7.25) are realized by diagram calculations. The Lagrangian in (7.23) gives us the Feynman rules of Fig.2. Let us study (7.25) component by component.

First for the ξ_a component equation, we use the equations of motion

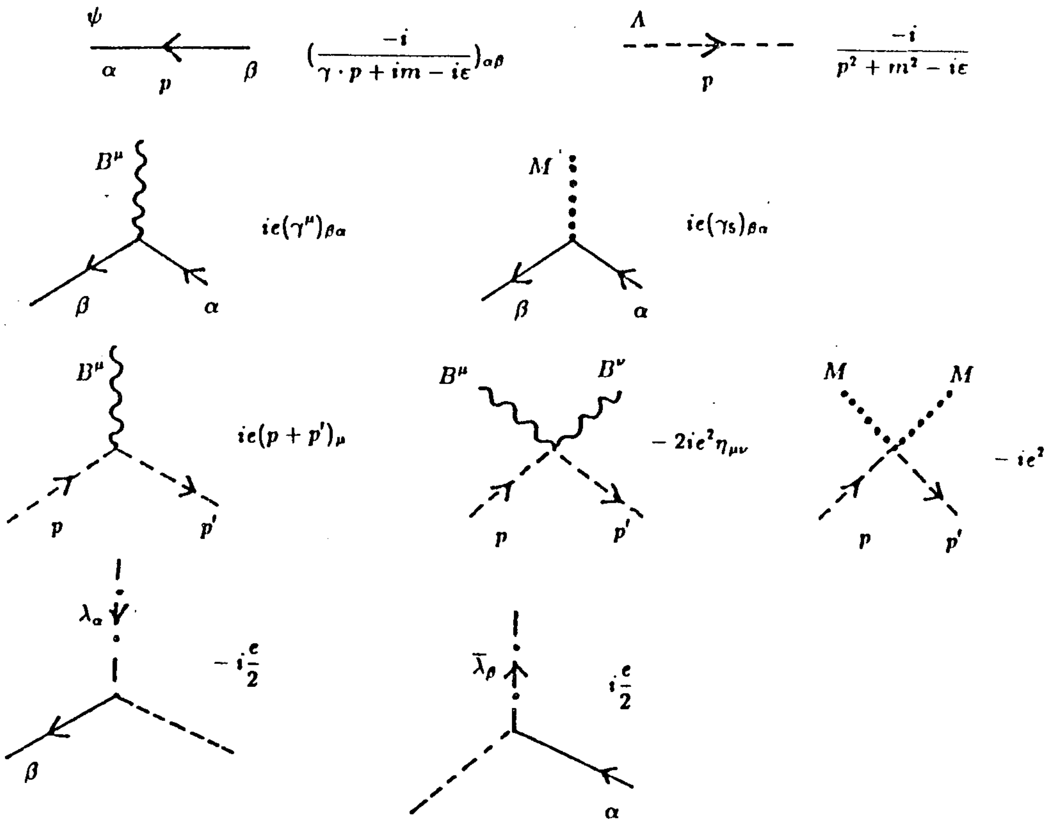


Fig.2 Feynman Rules

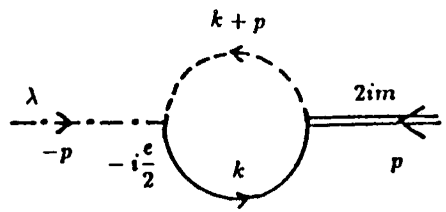


Fig.3 $-2i\bar\psi F^*$

$F = -mA$ and $F^* = -mA^*$ for F and F^* in the LHS (left hand side). Then the diagram in Fig.3 is potentially anomalous. In Fig.3 the double line represents the first term of the LHS of the ξ_n component equation, which will be denoted by $\xi_n(m)$. Then from the Feynman rules in Fig.2 we have

$$\begin{aligned}\xi_n(m) &= 2\times 2im\int\frac{d^2k}{(2\pi)^2}\,\frac{-i(\gamma\cdot k - im)}{k^2+m^2}\left(-i\frac{c}{2}\right)\lambda\frac{-i}{(k+p)^2+m^2}\\ &= -2cm\int\frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2+m^2}\,\frac{1}{(k+p)^2+m^2}(\gamma\cdot k - im)\lambda \\ &= -2cm\frac{i}{4\pi}\int_0^1 dx\,\frac{-\gamma\cdot p(1-x)-im}{[m^2+p^2x(1-x)]}\lambda\,.\end{aligned} \tag{7.26}$$

We regularize this by subtracting the Pauli-Villars term.

$$\xi_n(reg.) = \xi_n(m) - \xi_n(M)\,. \tag{7.27}$$

The first term in the RHS of (7.27) is a normal term. We renormalize (7.27) by taking $M\to\infty$.

$$\begin{aligned}\xi_n(ren.) &= \lim_{M\to\infty}\xi_n(reg.)\\ &= (\text{normal term}) - \frac{c}{\pi}\left(-\frac{1}{2}\lambda\right).\end{aligned} \tag{7.28}$$

For the $\mathbf{B}_\mu$ component equation, the diagrams in Fig.4 are potentially anomalous. Diag.4A and 4B come from the first term, Diag.4C and 4D from the second term, and Diag.4E from the third term of the LHS of the $\mathbf{B}_\mu$ component equation. However, Diag.4D turns out to produce no anomaly and anomalies from Diag.4B and 4E cancel each other. Then only Diag.4A

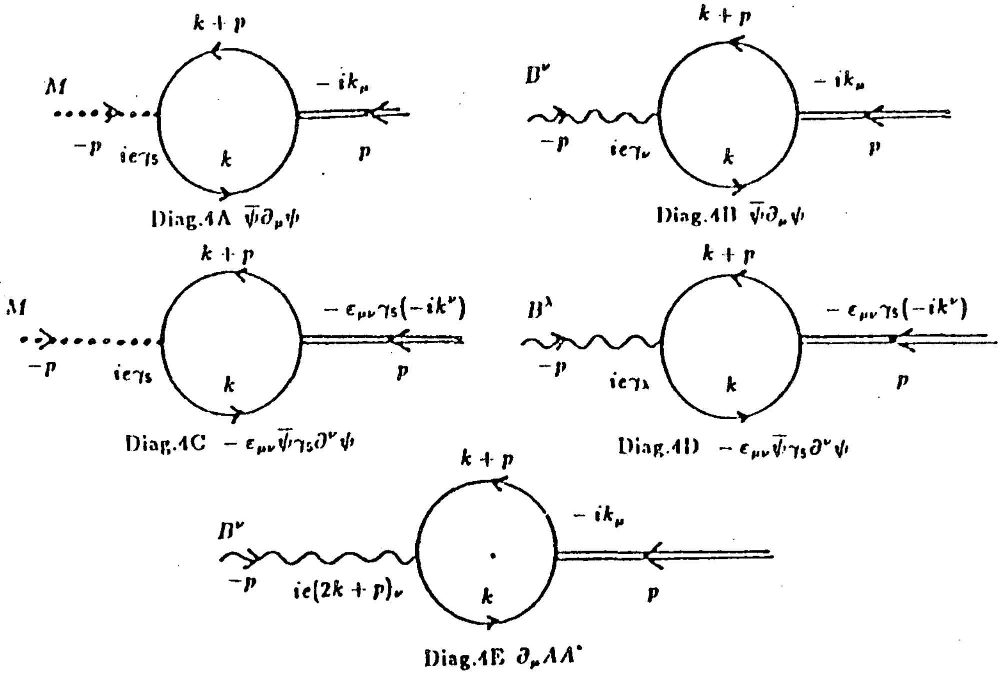


Fig.4

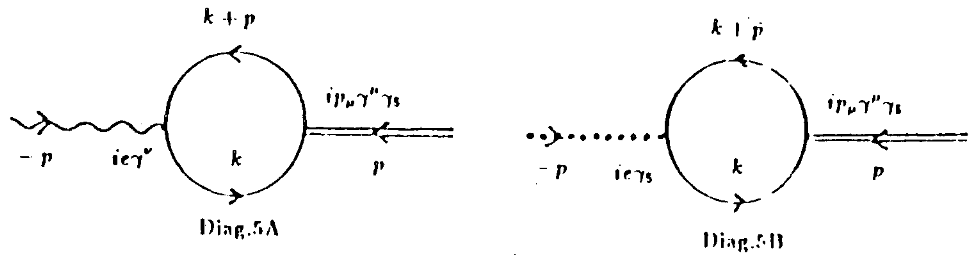


Fig.5 $\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi)$

and 4C combine to give rise to an anomaly.

$$B_\mu(m) = (-2)\times(-1)\int\frac{d^2k}{(2\pi)^2}\left[Tr\left\{ie\gamma_5\frac{-i(\gamma\cdot k+\gamma\cdot p-im)}{(k+p)^2+m^2}\right.\right.$$
$$\left.\left.\times\left[(-ik_\mu)+(-\epsilon_{\mu\nu}\gamma_5)(-ik^\nu)\right]\frac{-i(\gamma\cdot k-im)}{k^2+m^2}\right\}\right]. \qquad (7.29)$$

Then after some calculations, we have

$$B_\mu(ren.) = \lim_{M\to\infty}\{B_\mu(m)-B_\mu(M)\}$$
$$= (\text{normal term}) - \frac{e}{\pi}\epsilon_{\mu\nu}ip^\nu. \qquad (7.30)$$

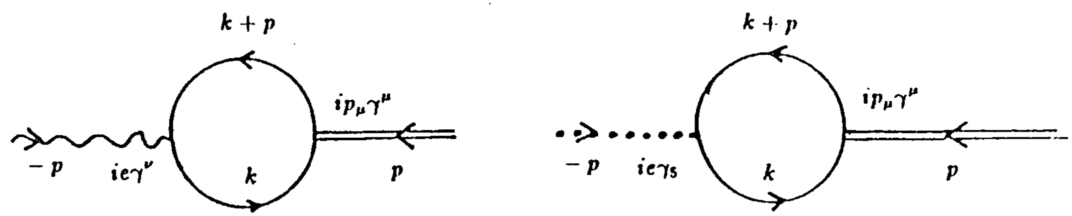
Then using the correspondence $\partial_\mu \leftrightarrow ip_\mu$,

$$B_\mu(ren.) = (\text{normal term}) - \frac{e}{\pi}(\epsilon_{\mu\nu}\partial^\nu M). \qquad (7.31)$$

For the M component equation, diagrams in Fig.5 are potentially anomalous. However, Diag.5B turns out to produce no anomaly. Diag.5A is the same as that in Fig.1 in section 7.1 which gave the result (7.10) Therefore we have

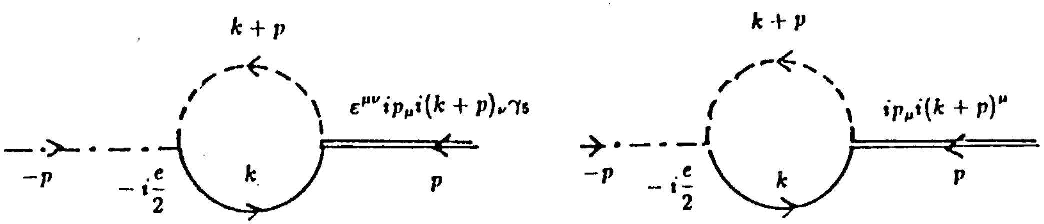
$$M(ren.) = (\text{normal term}) - \frac{e}{\pi}(\epsilon_{\mu\nu}\partial^\mu B^\nu). \qquad (7.32)$$

For the N component equation, the calculations are similar to those for the M component equation. Diag.6B turns out to produce no anomaly as



Diag.6A   Diag.6B

Fig.6 $\partial_\mu(\bar\psi\gamma^\mu\psi)$



Diag.7A $\epsilon^{\mu\nu}\partial_\mu(\gamma_5\psi\partial_\nu A^*)$ Diag.7B $\partial_\mu(\psi\partial^\mu A^*)$

Fig.7

Diag.5B, and Diag.6A gives

$$\begin{aligned} N(m) &= -B_\nu\int\frac{d^2k}{(2\pi)^2}Tr\{ie\gamma^\nu\frac{-i(\gamma\cdot k+\gamma\cdot p-im)}{(k+p)^2+m^2}i\gamma\cdot p\frac{-i(\gamma\cdot k-im)}{k^2+m^2}\} \\ &= 2eB_\nu\int\frac{d^2k}{(2\pi)^2}[\frac{(k+p)^\nu}{(k+p)^2+m^2}-\frac{k^\nu}{k^2+m^2}] \\ &= \frac{e}{2\pi}ip^\nu B_\nu, \quad \text{using (7.7).} \end{aligned} \tag{7.33}$$

Then

$$N(ren.) = \lim_{M\to\infty}\{N(m)-N(M)\} = 0\,. \tag{7.34}$$

Therefore

$$N(ren.) = (\text{normal term}) - \frac{e}{\pi}(0)\,. \tag{7.35}$$

For the ζ_a component equation, diagrams in Fig.7 give rise to an anomaly.

$$\begin{aligned} \zeta_a(m) = (-4i)\int\frac{d^2k}{(2\pi)^2}&[\epsilon^{\mu\nu}ip_\mu i(k+p)_\nu\gamma_5 + ip_\mu i(k+p)^\mu] \\ &\times\frac{-i(\gamma\cdot k-im)}{k^2+m^2}(-i\frac{e}{2})\lambda\frac{-i}{(k+p)^2+m^2}\,. \end{aligned} \tag{7.36}$$

Then after some calculations similar to those for (7.29), we have

$$\begin{aligned} \zeta_a(ren.) &= \lim_{M\to\infty}\{\zeta_a(m)-\zeta_a(M)\} \\ &= (\text{normal term}) - \frac{e}{\pi}(-\frac{1}{2}\gamma\cdot\partial\lambda)\,. \end{aligned} \tag{7.37}$$

Therefore by (7.28, 31, 32, 35) and (7.37) we have confirmed (7.24) or (7.25) with $c = -e/\pi$, that is,

$$i[D_aS(\bar DDS^*) - D_aS^*(\bar DDS)] = \{\text{Normal Terms}\} - \frac{ie}{2\pi}(\gamma_5 D)_a(\bar D\gamma_5 V)\,. \tag{7.38}$$

CHAPTER 8

CONCLUSION

We studied various topics of anomalies in two dimensions. In chapter 2 we obtained the gauge anomaly including the normalization factor up to the sign by using the differential geometric method. We obtained the solution of the anomaly eqution (the Wess-Zumino term) only in terms of the gauge fields, without auxiliary fields. Then we showed, up to the second non-trivial order, that this solution agrees with the Feynman diagram calculations. This solution is interesting because it may be applied as another approach to the effective theory.

In chapter 3 we obtained the anomaly of $D_\mu J^\mu$ from the Schwinger terms for the chiral Schwinger model and the non-Abelian chiral gauge theory. This method provides a new way of calculating the anomaly of $D_\mu J^\mu$ and shows clearly the intimate relation between the anomaly of $D_\mu J^\mu$ and the Schwinger terms. Through this study we could also understand the difficulties in quantizing anomalous gauge theories.

In chapter 4 we showed in the Schwinger model that the point-splitting method disagrees with the loop-diagram method by the sign and by the factor 1/2 for the anomaly of $\partial_\mu J_5^\mu$ and for the Schwinger term of $[J_5^0(x), J^0(y)]_{ETC}$ respectively. When we calculated the Schwinger term by the point-splitting method, we used a spatial splitting which is not covariant. This may be part of the reasons why the two methods disagree, since if we had used a spatial

splitting instead of a covariant splitting for $\partial_\mu J_5^\mu$ in section 3.2, we would have got half of the result of section 3.2 for $\partial_\mu J_5^\mu$. However, this does not explain the disagreements completely, because the two methods disagreed by the sign for $\partial_\mu J_5^\mu$ even though we used a covariant splitting.

In chapter 5 we obtained the gravitational anomaly including the normalization factor up to the sign by using the differential geometric method. Using the light-cone coordinates, we calculated the Feynman diagrams and showed, up to the second non-trivial order, that the anomaly obtained by the differential geometric method agrees with the Feynman diagram calculations.

In chapter 6 we showed that the origin of the supersymmetry anomaly in the Wess-Zumino gauge is the supersymmetric gauge anomaly. This indicates that there is no genuine supersymmetry anomaly. This also shows that when the gauge anomaly is canceled, the supersymmetry anomaly in the Wess-Zumino gauge is also canceled automatically. We expect that the situation is the same in other supersymmetric gauge theories which have superfield formulations. However, in a theory which has no superfield formulation, a different analysis may be necessary. We have also obtained the supersymmetric extension of the Wess-Zumino term in the form which depends only on the external vector multiplet.

In chapter 7 we obtained a spinorial superfield which contains $\partial_\mu J_5^\mu$ and $\partial_\mu J^\mu$ as M (pseudoscalar) and N (scalar) components respectively. This superfield is subject to the quantum correction which gives rise to an anomaly superfield, which has the ordinary gauge anomaly and zero as M and N components respectively. We confirmed this anomaly superfield by diagram calculations. This result could be expected since the Pauli-Villars regulator terms constitute a superfield.

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