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Partial differential equations with gradient constraints arising in the optimal control of singular stochastic processes

by

Ryan C. Hynd

A dissertation submitted in partial satisfaction of the requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Lawrence C. Evans, Chair
Professor Fraydoun Rezakhanlou
Professor Andrew Packard

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Abstract

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This dissertation is a study of second order, elliptic partial differential equations (PDE) that subject solutions to pointwise gradient constraints. These equations fall into the broad class of scalar non-linear PDE, and therefore, we interpret solutions in the viscosity sense and use methods from the theory of viscosity solutions. These equations are also naturally associated to free boundary problems as the boundary of the region where the gradient constraint is strictly satisfied cannot, in general, be determined before a solution of the PDE has been obtained. Consequently, we also employ techniques from PDE theory developed for free boundary problems.

In addition, we identify connections with control theory. Each solution of the PDE we consider has a probabilistic interpretation as an optimal value of a stochastic control problem. A distinguishing feature of these optimization problems is that the controlled processes have sample paths of bounded variation and thus may be “singular” with respect to Lebesgue measure on the real line. The theory of stochastic singular control has been used to model spacecraft control, queueing systems, and financial markets in the presence of transaction costs. Our work makes considerable progress at rigorously interpreting the PDE that arise in these applications.

I dedicate this dissertation to Theodore P. Hill and Erika Rogers for their
continuous encouragement

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Acknowledgments

My path to obtaining a PhD in mathematics has been unlike many of my peers. I was not a great student in high school, nor did I attend a university upon graduation as many of my classmates did. A few years after I graduated from high school, I began taking courses at a community college where most students attended classes at night after working by day. Moreover, I did not become a serious student of mathematics until I was nearing the age of 23. I consider myself fortunate to be graduating from UC Berkeley with a PhD in mathematics, and I would like to express my deepest thanks to many of the people who helped me along the way. For the sake of brevity, I will only mention a few of these individuals below.

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Chapter 1

Introduction

In this work, we study partial differential equations (PDE) within the general class of equations

$$\max \{F(D^2u, Du, u, x), H(Du)\} = 0, \quad x \in O \quad (1.1)$$

where

$$\begin{cases} O \subset \mathbb{R}^n \\ F : \mathcal{S}(n) \times \mathbb{R}^n \times \mathbb{R} \times O \rightarrow \mathbb{R} \\ H : \mathbb{R}^n \rightarrow \mathbb{R} \end{cases} \quad (1.2)$$

are given and the unknown is a scalar function

$$u : O \rightarrow \mathbb{R}.$$

In equation (1.1), Du denotes the gradient vector of the function u , D^2u denotes the Hessian matrix of second derivatives of u , and u and its derivatives are evaluated at the point $x \in O$. In (1.2), $\mathcal{S}(n)$ denotes the collection of symmetric $n \times n$ matrices with real entries.

Observe that if u is a solution of (1.1), then

$$F(D^2u, Du, u, x) \leq 0 \quad \text{and} \quad H(Du) \leq 0, \quad x \in O.$$

The inequality $H(Du) \leq 0$ is interpreted as a *gradient constraint*. Also notice that

$$F(D^2u, Du, u, x) = 0 \quad \text{whenever} \quad H(Du) < 0, \quad x \in O.$$

Therefore, if the subset $\Omega \subset O$ determined by the inequality $H(Du) < 0$ was known a priori, we could attempt to solve the PDE

$$F(D^2u, Du, u, x) = 0 \quad x \in \Omega$$

and then solve $H(Du) = 0$ in the complement of Ω to obtain a solution. However, this is in general impossible as we would need a solution of (1.1) to determine Ω to begin with. In this sense, the problem of finding a solution can be interpreted as a *free boundary problem*.

We shall see that a good existence theory for solutions of (1.1) necessitates the assumption that the *non-linearity* F is continuous and *elliptic*:

$$F(X, p, r, x) \leq F(Y, p, s, x) \quad \text{whenever} \quad r \leq s, \quad Y \leq X \quad (1.3)$$

$r, s \in \mathbb{R}$, $x \in O$, $p \in \mathbb{R}^n$ and $X, Y \in \mathcal{S}(n)$.¹ In many cases, a good existence theory also requires that the gradient constraint function H satisfies the *monotonicity condition*:

$$H(sp) < H(tp) \quad \text{whenever} \quad 0 \leq s < t, \quad p \neq 0 \quad (1.4)$$

$s, t \in \mathbb{R}$, $p \in \mathbb{R}^n$. Accordingly, these will be standing assumptions in the problems we study below.

It is well known that equations of the form (1.1) need not have solutions that are twice continuously differentiable. That is, (1.1) may not have *classical solutions*. However, with the above assumptions, (1.1) is a non-linear, elliptic PDE for a scalar function u . Therefore, it is appropriate to interpret equation (1.1) in the sense of viscosity solutions. This point of view will allow us to establish comparison principles and obtain limited regularity (or smoothness) of solutions. As a result, this is the approach we will follow in this paper.

There has been much work on equations related to the general class of PDE (1.1). One of the first rigorous papers on this subject was written by L. C. Evans [10] (see also [11]), where it is assumed that L is linear and H is allowed to have dependence in the x variable

$$H(p, x) = |p| - g(x), \quad (x, p) \in O \times \mathbb{R}^n. \quad (1.5)$$

In [10], it was shown that solutions of the associated boundary value problem exist, are unique and under some technical assumptions, have locally Lipschitz continuous derivatives.

The main idea in [10] was to study solutions of an appropriate, “penalized” equation and deduce uniform estimates on solutions that allow one to pass to a limit in a strong sense and solve the original equation; this is a relatively standard approach for PDE free boundary problems [14]. The result of [10] was extended by M. Wiegner [23] who proved the regularity result in [10] without making the same technical assumptions. In turn, the result in [23] was extended by H. Ishii et. al. [16] who established the regularity of solutions all the way up to the boundary, assuming ∂O is sufficiently regular. The methods employed in these works were entirely based on analysis.

The next wave of research on these types of equations was driven by applications. Davis and Norman derived an equation of type (1.1) in their now classic paper [7] on optimal portfolio consumption in the presence of transaction costs. Shortly thereafter, Davis, Panas, and Zariphopoulou deduced an analogous equation in their model for pricing options in the presence of transaction costs [8]. Both mathematical models were based on the control of random processes that may be singular with respect to Lebesgue measure on the real line. This was no coincidence as for many choices of F and H , (1.1) is the dynamic programming

¹The monotonicity of $r \mapsto F(X, p, r, x)$ is sometimes isolated and termed *proper*.

equation for such a stochastic control problem. These connections are now well understood and have been studied further in [13].

Most analytical work on equations of type (1.1) assume that F is linear, H is closely related to (1.5) (with say $g \equiv 1$) and involves the classical boundary value or *Dirichlet* problem. The purpose of this dissertation is to investigate three problems that aim to depart from this familiar setup. The first involves the Dirichlet problem for a general gradient constraint function H ; the second involves a non-standard, non-linear eigenvalue problem; and the third involves asymptotic analysis for a parabolic version of (1.1) with a non-linear F .

1.1 Survey of results

We consider three problems regarding PDE of the form (1.1). First, we study the Dirichlet problem associated with the PDE

$$\max\{Lu - h(x), H(Du)\} = 0.$$

L is assumed to be linear and uniformly elliptic, h is a given smooth function and H is a given convex function satisfying (1.4). This is a model dynamic programming equation for many infinite horizon, stochastic singular control problems. We establish the existence of a unique viscosity solution of the Dirichlet problem, and show that if H is *uniformly* convex, this solution belongs to the function space $C_{loc}^{1,1}$. See Theorem 2.0.1 below.

Next, we consider the problem of finding a real number λ and a function u satisfying the PDE

$$\max\{\lambda - \Delta u - h(x), |Du| - 1\} = 0, \quad x \in \mathbb{R}^n.$$

Here h is a given convex function that grows superlinearly. We prove that there is a unique λ^* such that the above PDE has a viscosity solution u satisfying

$$\lim_{|x| \rightarrow +\infty} \frac{u(x)}{|x|} = 1.$$

Moreover, we show that associated to λ^* is a convex solution u^* belonging to the function space $C^{1,1}(\mathbb{R}^n)$; for a precise statement, see Theorem 3.0.1. We also formally argue that λ^* has a probabilistic interpretation as being the least, long-time averaged (“ergodic”) cost for a singular optimal control problem involving h .²

Finally, we consider the problem of pricing options on multiple assets in the large risk aversion, small transaction cost limit. In a relatively standard single-asset setting, G. Barles and H. Soner [2] showed that the limiting option price is a solution of a non-linear, Black-Scholes type PDE. In this paper, we establish an analogous result for a model of the problem

²In this dissertation, the term “formal” will always mean non-rigorous.

in a multi-asset framework. In particular, we study solutions z^ϵ of the initial value problem associated to the “mixed-type” parabolic equation

$$\max \left\{ z_t - \left(\Delta_x z + \frac{1}{\epsilon} |D_x z + y|^2 \right), |D_y z| - \sqrt{\epsilon} \right\} = 0$$

for small ϵ . Under some technical assumptions, we prove that, as ϵ tends to 0, appropriate limits of z^ϵ tend to solutions of the corresponding initial value problem for a non-linear, parabolic equation

$$\psi_t = \lambda (D^2 \psi).$$

Here, the non-linearity λ arises as a solution of a non-linear eigenvalue PDE problem. See Theorems 4.0.1 and 4.0.2 for precise statements of these results.

1.2 Technical preliminaries

Before we embark on our study, we pause to mention some key results from the theory of viscosity solutions and analysis that will be of great utility. We first give a definition of a viscosity solutions in terms of test functions and then give an equivalent characterization in terms of “jets.” Next, we present the basic tools for establishing comparison principles between viscosity sub- and supersolutions and tools for establishing the existence of viscosity solutions. Our references for standard results from the theory of viscosity solutions are [1], [5], and [6]. We conclude by presenting a construct that will be used to “penalize” various PDE with gradient constraints.

In this section, we assume that $O \subset \mathbb{R}^n$ is open and that

$$G : \mathcal{S}(n) \times \mathbb{R}^n \times \mathbb{R} \times O \rightarrow \mathbb{R}$$

is continuous and elliptic (i.e. satisfies condition (1.3)). We will use the ellipticity assumption to make a basic observation about solutions of the equation

$$G(D^2 u, Du, u, x) = 0, \quad x \in O. \tag{1.6}$$

Observe that if $u \in C^2(O)$ is a solution of (1.6) and $u - \varphi$ has a local maximum at $x_0 \in O$, where $\varphi \in C^2(O)$, then

$$G(D^2 \varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq 0$$

as

$$Du(x_0) = D\varphi(x_0) \quad \text{and} \quad D^2 u(x_0) \leq D^2 \varphi(x_0).$$

Likewise, if $u - \psi$ has a local minimum at $x_0 \in O$, where $\psi \in C^2(O)$, then

$$G(D^2 \psi(x_0), D\psi(x_0), u(x_0), x_0) \geq 0.$$

Further arguing along these lines leads to the comparison principle: if $u, v \in C^2(O)$ satisfy

$$\begin{cases} G(D^2u, Du, u, x) \leq 0, & x \in O \\ G(D^2v, Dv, v, x) \geq 0, & x \in O \\ u(x) \leq v(x) & x \in \partial O \end{cases},$$

then $u \leq v$ in \bar{O} . However, we are assuming that $u, v \in C^2(O)$, while solutions of (1.6) may not be twice continuously differentiable. Nevertheless, these basic calculus arguments motivate the definition of viscosity solutions which only requires the pointwise values of a function.

Definition 1.2.1. $u \in USC(\bar{O})$ is a *viscosity subsolution* of (1.6) if whenever $u - \varphi$ has a local maximum at $x_0 \in O$ where $\varphi \in C^2(O)$, then

$$G(D^2\varphi(x_0), D\varphi(x_0), u(x_0), x_0) \leq 0.$$

$v \in LSC(\bar{O})$ is a *viscosity supersolution* of (1.6) if whenever $v - \psi$ has a local minimum at $x_0 \in O$ where $\psi \in C^2(O)$, then

$$G(D^2\psi(x_0), D\psi(x_0), v(x_0), x_0) \geq 0.$$

w is a *viscosity solution* of (1.6) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 1.2.2. Viscosity solutions defined above are necessarily continuous.

A useful concept for us will be that of second order sub- and superjets.

Definition 1.2.3. (i) Let $x_0 \in O$. $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ belongs to the *second order superjet* of u at x_0 if

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + \frac{1}{2}X(x - x_0) \cdot (x - x_0) + o(|x - x_0|^2) \quad (1.7)$$

as $|x - x_0| \rightarrow 0, x - x_0 \in O$. The collection of all such pairs (p, X) is denoted $J^{2,+}u(x_0)$.

(ii) Let $x_0 \in O$. $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ belongs to the *second order subjet* of u at x_0 if

$$u(x) \geq u(x_0) + p \cdot (x - x_0) + \frac{1}{2}X(x - x_0) \cdot (x - x_0) + o(|x - x_0|^2) \quad (1.8)$$

as $|x - x_0| \rightarrow 0, x - x_0 \in O$. The collection of all such pairs (p, X) is denoted $J^{2,-}u(x_0)$.

Notice that if $u - \varphi$ has a local maximum [minimum] at x_0 and $\varphi \in C^2(O)$, then (1.7) [(1.8)] holds with

$$p = D\varphi(x_0) \quad \text{and} \quad X = D^2\varphi(x_0). \quad (1.9)$$

A converse to this fact is also true and we refer the reader to [5] for its proof.

Lemma 1.2.4. *Suppose that $(p, X) \in J^{2,+}u(x_0)$. Then there is an open set $U \ni x_0$ and $\varphi \in C^2(U)$ such that (1.9) holds.*

Therefore we can address the question of whether or not a function is a viscosity solution by studying the function's second order jets. In particular, if u is a subsolution of (1.6) and $(p, X) \in J^{2,+}u(x_0)$, then

$$G(X, p, u(x_0), x_0) \leq 0; \quad (1.10)$$

and if v is a supersolution of (1.6) and $(p, X) \in J^{2,-}v(x_0)$, then

$$G(X, p, v(x_0), x_0) \geq 0. \quad (1.11)$$

Remark 1.2.5. In what follows, it will be important for us to study the sets

$$\begin{aligned} \bar{J}^{2,+}u(x_0) &:= \{(p, X) \in \mathbb{R}^n \times \mathcal{S}(n) : \text{there exists } (x_n, p_n, X_n) \in O \times J^{2,+}u(x_n), \text{ for } n \in \mathbb{N}, \\ &\text{such that } (x_n, u(x_n), p_n, X_n) \rightarrow (x_0, u(x_0), p, X), \text{ as } n \rightarrow \infty\} \end{aligned}$$

$$\begin{aligned} \bar{J}^{2,-}v(x_0) &:= \{(p, X) \in \mathbb{R}^n \times \mathcal{S}(n) : \text{there exists } (x_n, p_n, X_n) \in O \times J^{2,-}v(x_n), \text{ for } n \in \mathbb{N}, \\ &\text{such that } (x_n, v(x_n), p_n, X_n) \rightarrow (x_0, v(x_0), p, X), \text{ as } n \rightarrow \infty\} \end{aligned}$$

for $x_0 \in O$. It is readily verified that if u is a viscosity subsolution of (1.6) and $(p, X) \in \bar{J}^{2,+}u(x_0)$, then (1.10) holds. Likewise, if v is a viscosity supersolution of (1.6) and $(p, X) \in \bar{J}^{2,-}v(x_0)$, then (1.11) holds.

Throughout this paper, all PDE and partial differential inequalities will be interpreted in the viscosity sense. Therefore, we may sometimes omit the term ‘‘viscosity’’ when we mention solutions, subsolutions, and supersolutions.

The next two lemmas are commonly used to establish comparison principles between sub- and supersolutions. They are both proved in [6].

Lemma 1.2.6. *Let $O \subset \mathbb{R}^n$. Assume that $u \in USC(O)$ and $v \in LSC(O)$ and set*

$$M_\delta = \sup_{O \times O} \left\{ u(x) - v(y) - \frac{1}{2\delta} |x - y|^2 \right\}$$

for $\delta > 0$. Assume that $M_\delta < +\infty$ for all δ positive and small and that there is $(x_\delta, y_\delta) \in O \times O$ such that

$$\lim_{\delta \rightarrow 0^+} \left\{ M_\delta - \left(u(x_\delta) - v(y_\delta) - \frac{1}{2\delta} |x_\delta - y_\delta|^2 \right) \right\} = 0.$$

Then

$$\lim_{\delta \rightarrow 0^+} \frac{1}{2\delta} |x_\delta - y_\delta|^2 = 0$$

and

$$\lim_{\delta \rightarrow 0^+} M_\delta = u(x_0) - v(x_0) = \sup_{x \in O} \{u(x) - v(x)\}$$

whenever x_0 is a limit point of x_δ as $\delta \rightarrow 0^+$.

Lemma 1.2.7. (*Theorem of Sums*) Let $O_i \in \mathbb{R}^{n_i}$ be open for $i = 1, \dots, k$ and

$$O := O_1 \times O_2 \times \dots \times O_k.$$

Assume $u_i \in USC(O_i)$ and that φ is twice continuously differentiable in a neighborhood of O . Set

$$w(x) := u_1(x_1) + \dots + u_k(x_k), \quad x = (x_1, \dots, x_k) \in O$$

and suppose that $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k) \in O$ is a point of local maximum of $w - \varphi$ relative to O . Then for each $\rho > 0$, there are $X_i \in \mathcal{S}(n_i)$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \overline{J}^{2,+} u_i(\hat{x}_i)$$

for $i = 1, \dots, k$, and

$$\begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_k \end{pmatrix} \leq A + \rho A^2$$

where $A = D^2\varphi(\hat{x}) \in \mathcal{S}(n_1 + n_2 + \dots + n_k)$.

Once a comparison principle for solutions of non-linear elliptic PDE has been established, the next goal typically is to exhibit a solution. This is often accomplished by Perron's method, which informally consists of showing the "largest" supersolution with the correct boundary conditions is a solution. This method hinges on the following lemmas. Again both are proved in [6].

Lemma 1.2.8. Let \mathcal{F} be a family of subsolutions of (1.6) and set $w(x) = \sup \{u(x) : u \in \mathcal{F}\}$. If w^* is finite for each $x \in O$, then w is a subsolution of (1.6).³

Lemma 1.2.9. Let u be a subsolution of (1.6) and suppose that u_* is not a supersolution at some point $x_0 \in O$.⁴ Then for all $\kappa > 0$ and small enough, there is a subsolution v such that

$$\begin{cases} v \geq u \\ \sup_O(v - u) > 0 \\ v(x) = u(x), \quad x \in O \text{ and } |x - x_0| \geq \kappa \end{cases}.$$

Informally, the above lemmas assert that a pointwise supremum of a family of subsolutions is again a subsolution, and if a subsolution is not a supersolution, it cannot be "maximal." An application of these results is the existence of solutions of the Dirichlet problem associated to (1.6). This is known as *Perron's method*.

³ $w^*(x) := \inf_{\delta > 0} \sup \{w(y) : |y - x| \leq \delta\}$ is the upper-semicontinuous envelope of w .

⁴ $u_*(x) := \sup_{\delta > 0} \inf \{u(y) : |y - x| \leq \delta\}$ is the lower-semicontinuous envelope of u .

Theorem 1.2.10. *Let $g \in C(\overline{O})$ and assume the equation*

$$\begin{cases} G(D^2u, Du, u, x) = 0, & x \in O \\ u = g, & x \in \partial O \end{cases} \quad (1.12)$$

admits a comparison principle i.e. if u is a subsolution, v is a supersolution, and $u \leq v$ on ∂O , then $u \leq v$ in \overline{O} . Suppose in addition that there is a subsolution \underline{u} and a supersolution \overline{u} that satisfy $\underline{u}_ = \overline{u}^* = g$ on ∂O . Then*

$$u(x) := \sup \{w(x) : \underline{u} \leq w \leq \overline{u}, \quad w \text{ is a subsolution of (1.6)}\}$$

is the unique solution of (1.12).

Proof. Uniqueness follows by assumption, so we only need to establish existence. To this end, we first notice that $\underline{u}_* \leq u_* \leq u \leq u^* \leq \overline{u}^*$, which by assumption implies $u = g$ on ∂O . Next, we have that u^* is a viscosity subsolution by Lemma 1.2.8 and thus $u^* \leq u$. Hence, $u^* = u$ is a viscosity subsolution. If u_* is not a viscosity supersolution, we would have a contradiction to Lemma 1.2.9. By the comparison of sub- and supersolutions, $u \leq u_*$. Hence, $u = u^* = u_*$ is a viscosity solution of equation (1.6). \square

We conclude this introduction by presenting a construct that will be of great use to us when confronting various PDE free boundary problems. This construct is a family of functions $(\beta_\epsilon)_{\epsilon>0}$ that we will call the *standard penalty function*. This family of functions satisfies

$$\begin{cases} \beta_\epsilon \in C^\infty(\mathbb{R}) \\ \beta_\epsilon = 0, \quad z \leq 0 \\ \beta_\epsilon > 0, \quad z > 0 \\ \beta'_\epsilon \geq 0 \\ \beta''_\epsilon \geq 0 \\ \beta_\epsilon(z) = \frac{z-\epsilon}{\epsilon}, \quad z \geq 2\epsilon \end{cases} \quad (1.13)$$

For each $\epsilon > 0$, we think of β_ϵ as a type of smoothing of $z \mapsto (z/\epsilon)^+$; for small ϵ , we think of β_ϵ as a smooth approximation of the set valued mapping

$$\beta_0(t) = \begin{cases} \{0\}, & t < 0 \\ [0, \infty], & t = 0 \end{cases}.$$

A basic result that we will assume is

Proposition 1.2.11. *A family of functions $(\beta_\epsilon)_{\epsilon>0}$ satisfying (1.13) exists.*

Chapter 2

An elliptic PDE with convex gradient constraint

In this chapter, we consider PDE associated with a general class of infinite horizon, stochastic singular control problems. This is a class of non-linear, second-order PDE that each have a free boundary determined by a convex gradient constraint. We show that the Dirichlet problem has a unique solution, and for uniformly convex gradient constraints, this solution has a locally Lipschitz continuous derivative. Our methods are entirely analytic. However, we provide an interpretation of solutions as the value function of appropriate singular control problems. Finally, we argue that the class of gradient constraint functions arising in applications can be replaced by an equivalent class of uniformly convex gradient constraints and show that our regularity result applies to some examples of the motivating singular control problems.

The PDE we focus on is

$$\begin{cases} \max\{Lu - h(x), H(Du)\} = 0, & x \in O \\ u = 0, & x \in \partial O \end{cases}, \quad (2.1)$$

where $O \subset \mathbb{R}^n$ is open and bounded with smooth boundary ∂O and h is a smooth, non-negative function on \bar{O} .

We assume that L is the linear differential operator

$$L\psi(x) := -a(x) \cdot D^2\psi + b(x) \cdot D\psi + c(x)\psi,^1 \quad \psi \in C^2(\bar{O})$$

with smooth coefficients $a : \bar{O} \rightarrow \mathcal{S}(n)$, $b : \bar{O} \rightarrow \mathbb{R}^n$ and $c : \bar{O} \rightarrow \mathbb{R}$. We shall further assume that L is (*uniformly*) *elliptic*:

$$a(x)\xi \cdot \xi \geq \gamma|\xi|^2, \quad \text{for all } x \in \bar{O}, \xi \in \mathbb{R}^n \quad (2.2)$$

¹For square matrices A and B of the same dimension, $A \cdot B := \text{tr}A^t B$.

for some $\gamma > 0$. The final assumption on L that we will make is that

$$c(x) \geq \delta, \quad x \in \overline{O}$$

where δ is a positive constant.

Our assumptions on $H : \mathbb{R}^n \rightarrow \mathbb{R}$ are that

$$\begin{cases} H \text{ is convex} \\ H(0) < 0 \\ [0, \infty) \ni t \mapsto H(tp) \text{ is increasing for } p \neq 0 \end{cases}. \quad (2.3)$$

Our central result is

Theorem 2.0.1. (i) *There is a unique continuous viscosity solution of (2.1).*
(ii) *If, in addition, H is uniformly convex, then $u \in C_{loc}^{1,1}(O)$.*

We use techniques from the theory of viscosity solutions of scalar non-linear elliptic PDE to prove the existence and uniqueness of solutions of (2.1). We use a penalization technique similar to the one introduced by L.C. Evans in [10] and refined by M. Wiegner [23] and H. Ishii et. al. [16] to establish the regularity result; we also believe that we have identified general structural conditions (2.3) on the type of gradient constraints for which penalization methods are successful at yielding regularity results. Finally, we discuss the motivating applications in singular control theory and give a probabilistic interpretation of solutions of (2.1).

2.1 Existence and uniqueness

Our main goal of this section is to establish a comparison principle among viscosity sub- and supersolutions. Here we *cannot* assume that sub- and supersolutions are smooth, we must rather use the definition of viscosity sub- and supersolutions and methods developed for this class of sub- and supersolutions. With such a comparison principle we will employ a routine application of Perron's method to establish the existence of solutions of (2.1).

Proposition 2.1.1. *Assume u is a viscosity subsolution of (2.1) and v is a viscosity supersolution of (2.1). If*

$$u \leq v \text{ on } \partial O \quad \text{and} \quad u \in L^\infty(\partial O),$$

then $u \leq v$ in \overline{O} .

Formal Proof. Before proving the above proposition, we give a formal proof (i.e. assuming $u, v \in C^2(O)$) that will help motivate a rigorous argument. Fix $\epsilon \in (0, 1)$ and set

$$w^\epsilon(x) = \epsilon u(x) - v(x), \quad x \in \overline{O}.$$

$w^\epsilon \in USC(\overline{O})$ and thus achieves its maximum at some $x_\epsilon \in \overline{O}$. If $x_\epsilon \in \partial O$, then

$$w^\epsilon(x_\epsilon) = -(1 - \epsilon)u(x_\epsilon) + u(x_\epsilon) - v(x_\epsilon) \leq -(1 - \epsilon)u(x_\epsilon) \leq (1 - \epsilon)|u|_{L^\infty(\partial O)}.$$

If $x_\epsilon \in O$, then by calculus

$$\begin{cases} 0 = Dw^\epsilon(x_\epsilon) = \epsilon Du(x_\epsilon) - Dv(x_\epsilon) \\ 0 \geq D^2w^\epsilon(x_\epsilon) = \epsilon D^2u(x_\epsilon) - D^2v(x_\epsilon) \end{cases}.$$

If $Du(x_\epsilon) = 0$, then $Dv(x_\epsilon) = 0$ and

$$H(Dv(x_\epsilon)) < 0. \quad (2.4)$$

Otherwise, by (2.3) we have $H(Dv(x_\epsilon)) = H(\epsilon Du(x_\epsilon)) < H(Du(x_\epsilon)) \leq 0$ and (2.4) still holds. In particular, since v is a supersolution, we have that

$$Lv(x_\epsilon) - h(x_\epsilon) \geq 0.$$

Therefore,

$$\begin{aligned} c(x_\epsilon)w(x_\epsilon) &\leq L(\epsilon u - v)(x_\epsilon) \\ &\leq -(1 - \epsilon)h(x_\epsilon) \\ &\leq 0 \end{aligned} \quad (2.5)$$

and hence $w^\epsilon(x_\epsilon) \leq 0$. In either case, $w^\epsilon \leq C(1 - \epsilon)$, and letting $\epsilon \rightarrow 1^-$ gives $u \leq v$. □

To make the above formal proof rigorous, we will use Lemma 1.2.6 and Lemma 1.2.7.

Proof. (of the proposition) 1. Fix $\epsilon \in (0, 1)$ and set

$$w^\eta(x, y) = \epsilon u(x) - v(y) - \frac{1}{2\eta}|x - y|^2, \quad x, y \in \overline{O}$$

for $\eta > 0$. $w^\eta \in USC(\overline{O} \times \overline{O})$ and so has a maximum at some point $(x_\eta, y_\eta) \in \overline{O} \times \overline{O}$. As \overline{O} is compact, Lemma 1.2.6 implies that (x_η, y_η) has a limit point of the form (x_ϵ, x_ϵ) through some sequence of $\eta \rightarrow 0^+$, where x_ϵ is a maximizing point of $x \mapsto \epsilon u(x) - v(x)$. If $x_\epsilon \in \partial O$, we have from the definition of w^η and our assumptions that

$$\epsilon u(x) - v(x) \leq \epsilon u(x_\epsilon) - v(x_\epsilon) = -(1 - \epsilon)u(x_\epsilon) + u(x_\epsilon) - v(x_\epsilon) \leq C(1 - \epsilon), \quad x \in \overline{O}.$$

2. Now we assume that $x_\epsilon \in O$ and without any loss of generality that $(x_\eta, y_\eta) \in O \times O$ for $\eta > 0$. According to the Theorem of Sums (Lemma 1.2.7), for each $\rho > 0$ there are $X, Y \in \mathcal{S}(\rho)$ such that

$$\left(\frac{x_\eta - y_\eta}{\eta}, X \right) \in \overline{J}^{2,+}(\epsilon u)(x_\eta)$$

$$\left(\frac{x_\eta - y_\eta}{\eta}, Y \right) \in \bar{J}^{2,-} v(y_\eta)$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \rho A^2. \quad (2.6)$$

Here

$$A = D^2 \frac{|x - y|^2}{2\eta} \Big|_{x=x_\eta, y=y_\eta} = \frac{1}{\eta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and I is the $n \times n$ identity matrix. In particular, choosing $\rho = \eta$ in (2.6) implies the matrix inequality

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\eta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \quad (2.7)$$

3. Since u is a viscosity subsolution

$$\max \left\{ c(x_\eta)u(x_\eta) - a(x_\eta) \cdot \frac{X}{\epsilon} - h(x_\eta), H \left(\frac{x_\eta - y_\eta}{\epsilon\eta} \right) \right\} \leq 0, \quad (2.8)$$

and since v is a viscosity supersolution

$$\max \left\{ c(y_\eta)v(y_\eta) - a(y_\eta) \cdot Y - h(y_\eta), H \left(\frac{x_\eta - y_\eta}{\eta} \right) \right\} \geq 0. \quad (2.9)$$

If $x_\eta = y_\eta$,

$$H \left(\frac{x_\eta - y_\eta}{\eta} \right) = H(0) < 0. \quad (2.10)$$

If $x_\eta \neq y_\eta$, we have $H \left(\frac{x_\eta - y_\eta}{\epsilon\eta} \right) \leq 0$ and again (2.10) holds as $H \left(\frac{x_\eta - y_\eta}{\eta} \right) = H \left(\epsilon \frac{x_\eta - y_\eta}{\epsilon\eta} \right) < 0$. By (2.9),

$$c(y_\eta)v(y_\eta) - a(y_\eta) \cdot Y - h(y_\eta) \geq 0. \quad (2.11)$$

Combining (2.8) and (2.11) gives,

$$\begin{aligned} \epsilon c(x_\eta)u(x_\eta) - c(y_\eta)v(y_\eta) &\leq a(x_\eta) \cdot X - a(y_\eta) \cdot Y + (b(x_\eta) - b(y_\eta)) \cdot \frac{x_\eta - y_\eta}{\eta} \\ &\quad + \epsilon h(x_\eta) - h(y_\eta) \\ &\leq a(x_\eta) \cdot X - a(y_\eta) \cdot Y + \text{Lip}(b) \frac{|x_\eta - y_\eta|^2}{\eta} + \text{Lip}(h) |x_\eta - y_\eta|. \end{aligned} \quad (2.12)$$

Note $x \mapsto a^{1/2}(x)$ is Lipschitz continuous since $x \mapsto a(x)$ is Lipschitz and $a \geq \gamma > 0$;² indeed

$$\text{Lip}(a^{1/2}) \leq \frac{\text{Lip}(a)}{2\gamma}.$$

²Here $a^{1/2}(x)$ is the *unique* positive square root of $a(x)$.

Also note that the $2n \times 2n$ matrix

$$\begin{pmatrix} a^{1/2}(x_\eta)a^{1/2}(x_\eta) & a^{1/2}(x_\eta)a^{1/2}(y_\eta) \\ a^{1/2}(y_\eta)a^{1/2}(x_\eta) & a^{1/2}(y_\eta)a^{1/2}(y_\eta) \end{pmatrix}$$

is non-negative definite, and by (2.7)

$$\begin{aligned} a(x_\eta) \cdot X - a(y_\eta) \cdot Y &= \operatorname{tr} [a(x_\eta)X - a(y_\eta)Y] \\ &= \operatorname{tr} \left[\begin{pmatrix} a^{1/2}(x_\eta)a^{1/2}(x_\eta) & a^{1/2}(x_\eta)a^{1/2}(y_\eta) \\ a^{1/2}(y_\eta)a^{1/2}(x_\eta) & a^{1/2}(y_\eta)a^{1/2}(y_\eta) \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \right] \\ &\leq \operatorname{tr} \left[\begin{pmatrix} a^{1/2}(x_\eta)a^{1/2}(x_\eta) & a^{1/2}(x_\eta)a^{1/2}(y_\eta) \\ a^{1/2}(y_\eta)a^{1/2}(x_\eta) & a^{1/2}(y_\eta)a^{1/2}(y_\eta) \end{pmatrix} \frac{3}{\eta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right] \\ &\leq \frac{3}{\eta} \operatorname{tr} [(a^{1/2}(x_\eta) - a^{1/2}(y_\eta))(a^{1/2}(x_\eta) - a^{1/2}(y_\eta))] \\ &\leq \frac{3\operatorname{Lip}(a)^2}{2\gamma^2} \frac{|x_\eta - y_\eta|^2}{2\eta}. \end{aligned}$$

By (2.12),

$$\epsilon c(x_\eta)u(x_\eta) - c(y_\eta)v(y_\eta) \leq \left(\frac{3\operatorname{Lip}(a)^2}{2\gamma^2} + 2\operatorname{Lip}(b) \right) \frac{|x_\eta - y_\eta|^2}{2\eta} + \operatorname{Lip}(h)|x_\eta - y_\eta|. \quad (2.13)$$

4. Let (x_ϵ, x_ϵ) be a limit point of (x_η, y_η) through as sequence of $\eta \rightarrow 0^+$. If $x_\epsilon \in \partial O$, we have from our remarks above that

$$\epsilon u(x_\epsilon) - v(x_\epsilon) \leq C(1 - \epsilon).$$

If $x_\epsilon \in O$, we let $\eta \rightarrow 0^+$ through the appropriate subsequence in (2.13) and arrive at

$$c(x_\epsilon)(\epsilon u(x_\epsilon) - v(x_\epsilon)) \leq 0.$$

This inequality implies $\epsilon u(x_\epsilon) - v(x_\epsilon) \leq 0$, and so in either case,

$$\epsilon u(x) - v(x) \leq \epsilon u(x_\epsilon) - v(x_\epsilon) \leq C(1 - \epsilon), \quad x \in \overline{O}.$$

We conclude by letting $\epsilon \rightarrow 1^-$. □

Remark 2.1.2. The purpose of “doubling the variables” was so that we could “put derivatives” on the smooth function $(x, y) \mapsto |x - y|^2/2\eta$ and use the definition of viscosity solutions to estimate w^η near its maximum value. This particular choice of test function forced x_η and y_η to be close so that $w(x_\eta, y_\eta)$ had to be close to the maximum value of $x \mapsto \epsilon u(x) - v(x)$ for η small.

With a comparison principle in hand, we can now employ a routine application of Perron's method to obtain the existence of solutions.

Proof. (of part (i) of Theorem 2.0.1) Note that

$$\underline{u} \equiv 0$$

is a viscosity subsolution of (2.1); and \bar{u} , the unique viscosity solution of

$$\begin{cases} Lv - h(x) = 0, & x \in O \\ v = 0, & x \in \partial O \end{cases}, \quad (2.14)$$

is a viscosity supersolution of (2.1). Therefore, Theorem (1.2.10) applies. \square

Remark 2.1.3. Perron's method is a simple and elegant way to prove existence of solutions, however it is not the only method. We will see other ways to prove existence via a "penalty" method and using a stochastic singular control interpretation of solutions.

Remark 2.1.4. The arguments we have used to prove existence and uniqueness can be generalized to large class of equations of the form

$$\max \{ F(D^2u, Du, u, x) - h(x), H(Du) \} = 0, \quad x \in O \quad (2.15)$$

where F is non-linear, elliptic and homogeneous: $F(tM, tp, tu, x) = tF(M, p, u, x)$, for all $t \geq 0$ and $(M, p, u, x) \in \mathcal{S}(n) \times \mathbb{R}^n \times \mathbb{R} \times O$. We did not, however, explore such equations as we were primarily interested in regularity and currently do not know how to establish an analogous regularity result for equations of the form (2.15).

2.2 Regularity

In this section, we prove part (ii) of Theorem 2.0.1, which we restate for the reader's convenience.

Theorem 2.2.1. *Let u be the unique viscosity solution of (2.1), and suppose that H is uniformly convex. Then $u \in C_{loc}^{1,1}(O)$.*

To this end, we will analyze solutions of the *penalized* equation

$$\begin{cases} Lu^\epsilon + \beta_\epsilon(H(Du^\epsilon)) = h(x), & x \in O \\ u^\epsilon = 0, & x \in \partial O \end{cases}, \quad (2.16)$$

where $(\beta_\epsilon)_{\epsilon>0}$ is the standard penalty function; see Proposition 1.13 for various properties of this family of functions. Since the values of $\beta_\epsilon(H(Du^\epsilon))$ can be large when $H(Du^\epsilon) > 0$

for small ϵ , solutions will seek to satisfy $H(Du^\epsilon) \leq 0$ and, in this sense, become closer to satisfying equation (2.1). Without any loss of generality, we assume

$$\begin{cases} H \in C^2(\mathbb{R}^n) \\ D^2H(p) \geq 1, p \in \mathbb{R}^n \end{cases} \quad (2.17)$$

If H is merely continuous, we can mollify H and argue as we do below without significant changes.

Notice that (2.16) is a semi-linear, uniformly elliptic PDE with smooth coefficients. By classical arguments, (2.16) has a unique, smooth solution u^ϵ [15]. Our goal is to derive a bound on $|u^\epsilon|_{W^{2,\infty}(O')}$, for each $O' \subset O$, that is independent of all $\epsilon > 0$ and small. Such an estimate would aid us in proving that a subsequence of u^ϵ converges to u , the solution of (2.1), in $C_{\text{loc}}^1(O)$ as $\epsilon \rightarrow 0^+$. Such a convergence result would necessarily provide a $W_{\text{loc}}^{2,\infty}$ estimate on u . We will obtain the desired bound on D^2u^ϵ through a sequence of lemmas.

Lemma 2.2.2. *There is a constant C such that*

$$|u^\epsilon(x)| \leq C, \quad x \in \bar{O}$$

for $\epsilon > 0$.

Proof. Let \bar{u} be the unique smooth solution of (2.14). As \bar{u} is a supersolution of equation (2.16), $u^\epsilon \leq \bar{u}$; while $u^\epsilon \geq 0$, since $\underline{u} : x \mapsto 0$ is a subsolution of (2.16). \square

An immediate corollary of the above proof is

Corollary 2.2.3. *There is a constant C such that*

$$|Du^\epsilon(x)| \leq C, \quad x \in \partial O$$

for $\epsilon > 0$.

Proof. Recall that we have assumed that ∂O is smooth. By the proof of the previous lemma, we have

$$\frac{\partial \bar{u}(x)}{\partial \nu} \leq \frac{\partial u^\epsilon(x)}{\partial \nu} \leq 0, \quad x \in \partial O$$

where ν is the outward normal on ∂O . \square

Lemma 2.2.4. *There is a constant C such that*

$$|Du^\epsilon(x)| \leq C, \quad x \in \bar{O}$$

for $0 < \epsilon < 1$.

Proof. 1. It suffices to bound the function

$$v^\epsilon(x) = |Du^\epsilon(x)|^2 - \lambda u^\epsilon(x), \quad x \in \bar{O}$$

from above, for some universal (that is, ϵ -independent) constant $\lambda > 0$. To this end, we suppress ϵ -dependence, function arguments and compute

$$\begin{cases} Dv = 2D^2uDu - \lambda Du \\ a \cdot D^2v = 2aD^2u \cdot D^2u + 2Du \cdot \left\{ D(a \cdot D^2u) - \sum_{i,j=1}^n u_{x_i x_j} Da_{ij} \right\} - \lambda a \cdot D^2u \end{cases} .$$

These quantities will help us study v near its maximum values.

2. Equation (2.16) may be rewritten as

$$a \cdot D^2u = c(x)u + b(x) \cdot Du + \beta_\epsilon(H(Du)) - h(x),$$

and for further ease of notation, we will write β for $\beta_\epsilon(H(Du))$. We have

$$\begin{aligned} a \cdot D^2v &\geq \gamma |D^2u|^2 + 2Du \cdot D(cu + b \cdot Du + \beta - h) - 2 \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} \\ &\quad - \lambda(cu + b \cdot Du + \beta - h) \\ &= \gamma |D^2u|^2 - 2 \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} + 2c|Du|^2 - 2Du \cdot Dh + 2uDu \cdot Dc - \lambda cu \\ &\quad - \lambda h + 2DbDu \cdot Du + 2\beta' Du \cdot D^2uDH + 2Du \cdot D^2ub - \lambda b \cdot Du - \lambda \beta \\ &\geq -C|Du|^2 - C + Dv \cdot (\beta' DH + b) + \lambda(\beta' Du \cdot DH - \beta) \\ &\geq -C|Du|^2 - C + Dv \cdot (\beta' DH + b) + \lambda\beta'(Du \cdot DH - H), \end{aligned}$$

as $\beta_\epsilon(z) \leq z\beta'_\epsilon(z)$ for all $z \in \mathbb{R}$. Since H is uniformly convex with $D^2H \geq 1$ and $H(0) \leq 0$,

$$p \cdot DH(p) - H(p) \geq |p|^2/2, \quad p \in \mathbb{R}^n.$$

The above inequality implies

$$a \cdot D^2v \geq -C|Du|^2 - C + Dv \cdot (\beta' DH + b) + \frac{\lambda\beta'}{2}|Du|^2 \quad (2.18)$$

for constants C independent of ϵ .

3. Let $x_0 \in \bar{O}$ be a maximizing point for v . If $x_0 \in \partial O$, a bound on $|Du(x_0)|^2$ that is independent of $\epsilon \in (0, 1)$ is immediate from the previous corollary. If $x_0 \in O$, then

$$Dv(x_0) = 0, \quad a(x_0) \cdot D^2v(x_0) \leq 0.$$

If $\beta' = \beta'(H(Du)) \leq 1 < 1/\epsilon$, then $\beta = \beta(H(Du)) \leq 1$. In particular, $H(Du) \leq 2\epsilon \leq 2$ which implies a bound on $|Du(x_0)|$ independent of $\epsilon \in (0, 1)$. If $\beta'(H(Du)) \geq 1$, (2.18) gives

$$0 \geq -C|Du|^2 - C + \frac{\lambda}{2}|Du|^2, \quad (2.19)$$

which implies a bound on $|Du(x_0)|^2$ independent of $\epsilon \in (0, 1)$, for $\lambda > 0$ chosen large enough. \square

Lemma 2.2.5. *For each $O' \Subset O$, there is a constant $C = C(O')$ such that*

$$0 \leq \beta_\epsilon(H(Du^\epsilon(x))) \leq C, \quad x \in O'$$

for $0 < \epsilon < 1$.

Remark 2.2.6. To simplify the arguments given below, we assume

$$b \equiv 0 \quad c \equiv \delta > 0.$$

We believe that incorporating more general coefficients b and c is merely technical and no new issues arise.

Proof. 1. It suffices to bound

$$x \mapsto \eta(x)\beta_\epsilon(H(Du^\epsilon(x))), \quad x \in \bar{O}$$

for each $\eta \in C_c^\infty(O)$, $0 \leq \eta \leq 1$. To this end, we will show that for each such η , there is a universal constant $\lambda > 0$ such that the function

$$v^\epsilon(x) := \eta(x)\beta_\epsilon(H(Du^\epsilon(x))) + \frac{\lambda}{2}|Du^\epsilon(x)|^2, \quad x \in \bar{O}$$

is bounded above independently of $\epsilon \in (0, 1)$. We remark this approach was introduced in [16]. As before, we will omit the ϵ dependence of u^ϵ and v^ϵ , arguments of functions and write β for $\beta_\epsilon(H(Du^\epsilon))$.

2. We perform several computations that will help us study v near its maximum values. Straightforward computations are

$$\begin{cases} Dv = D\eta\beta + \eta D\beta + \lambda D^2u Du \\ a \cdot D^2v = (a \cdot D^2\eta)\beta + 2aD\eta \cdot D\beta + \eta(a \cdot D^2\beta) \\ \quad + \lambda(aD^2u \cdot D^2u + \sum_{i,j=1}^n Du \cdot a_{ij} Du_{x_i x_j}) \end{cases}$$

We will do some further computations below to simplify the expression (above) that we have for $a \cdot D^2v$.

(a)

$$\begin{aligned}\eta((a \cdot D^2\eta)\beta + 2aD\eta \cdot D\beta) &= \eta(a \cdot D^2\eta)\beta + 2aD\eta \cdot \eta D\beta \\ &= (\eta a \cdot D^2\eta - 2aD\eta \cdot D\eta)\beta + 2aD\eta \cdot Dv - 2\lambda aD\eta \cdot D^2uDu\end{aligned}$$

(b)

$$\begin{aligned}a \cdot D^2\beta &= \beta'' a D^2u DH \cdot D^2u DH + \\ &\quad \beta' \left\{ a \cdot D^2u D^2H D^2u + DH \cdot D(a \cdot D^2u) - \sum_{ij=1}^n u_{x_i x_j} DH \cdot Da_{ij} \right\} \\ &= \beta'' a D^2u DH \cdot D^2u DH + \\ &\quad \beta' \left\{ a \cdot D^2u D^2H D^2u + DH \cdot D(\delta u - h) + D\beta \cdot DH - \sum_{ij=1}^n u_{x_i x_j} DH \cdot Da_{ij} \right\} \\ \eta(a \cdot D^2\beta) &= \eta\beta'' a D^2u DH \cdot D^2u DH + \\ &\quad \beta' \left\{ \eta a \cdot D^2u D^2H D^2u + \eta DH \cdot D(\delta u - h) + Dv \cdot DH \right. \\ &\quad \left. - \beta DH \cdot D\eta - \lambda DH \cdot D^2u Du - \eta \sum_{ij=1}^n u_{x_i x_j} DH \cdot Da_{ij} \right\}\end{aligned}$$

(c)

$$\begin{aligned}\sum_{i,j=1}^n Du \cdot a_{ij} Du_{x_i x_j} &= Du \cdot D(a \cdot D^2u) - \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} \\ &= Du \cdot D(\delta u - h) + Du \cdot D\beta - \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij}\end{aligned}$$

and so

$$\left\{ \begin{aligned} \eta \sum_{i,j=1}^n Du \cdot a_{ij} Du_{x_i x_j} &= \eta Du \cdot D(\delta u - h) + Du \cdot Dv \\ &\quad - \beta Du \cdot D\eta - \lambda D^2u Du \cdot D\eta - \eta \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij}. \end{aligned} \right.$$

Substituting the above computations (a) – (c) into our expression for $a \cdot D^2v$ gives

$$\begin{aligned}
\eta a \cdot D^2 v &= (\eta a \cdot D^2 \eta - 2a D\eta \cdot D\eta) \beta + 2a D\eta \cdot Dv \\
&\quad - 2\lambda a D\eta \cdot D^2 u Du + \eta^2 \beta'' a D^2 u DH \cdot D^2 u DH \\
&\quad + \eta \beta' \left\{ \eta a \cdot D^2 u D^2 H D^2 u + \eta DH \cdot D(\delta u - h) + Dv \cdot DH \right. \\
&\quad \left. - \beta DH \cdot D\eta - \lambda DH \cdot D^2 u Du - \eta \sum_{i,j=1}^n u_{x_i x_j} DH \cdot Da_{ij} \right\} \\
&\quad + \lambda \left\{ \eta a D^2 u \cdot D^2 u + \eta Du \cdot D(\delta u - h) + Du \cdot Dv \right. \\
&\quad \left. - \beta Du \cdot D\eta - \lambda D^2 u Du \cdot Du - \eta \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} \right\}. \tag{2.20}
\end{aligned}$$

3. Let $x_0 \in \overline{O}$ be a maximizing point for v . If $x_0 \in \partial O$ or $\eta(x_0) = 0$, then $v \leq v(x_0) \leq \lambda |Du(x_0)|^2 / 2 \leq C$, as desired. If $x_0 \in O$ and $\eta(x_0) > 0$, we have

$$Dv(x_0) = 0 \quad \text{and} \quad 0 \geq a(x_0) \cdot D^2 v(x_0);$$

and from (2.20),

$$\begin{aligned}
0 &\geq -C\beta - C|D^2 u| + \beta' \{ \eta |D^2 u|^2 - C - C\beta - C|D^2 u| \} \\
&\quad + \lambda \{ \eta \gamma |D^2 u|^2 - C - C|D^2 u| - \beta \},
\end{aligned}$$

where C denotes various constants that are independent of $\epsilon \in (0, 1)$. All functions above are evaluated at x_0 .

Recall that

$$\beta = a \cdot D^2 u - \delta u + h \leq C \{ 1 + |D^2 u| \}$$

and therefore we have

$$0 \geq \frac{\lambda \gamma \eta}{2} |D^2 u|^2 - C + \beta' \{ \eta |D^2 u|^2 - C \}$$

again for various constants C independent of ϵ .

If $\beta' = \beta'_\epsilon(H(Du^\epsilon)) \leq 1 < 1/\epsilon$, then $\beta = \beta_\epsilon(H(Du^\epsilon)) \leq 1$, and the claim follows for $v \leq 1 + \frac{\lambda}{2} |Du(x_0)|^2$. If $\beta' \geq 1$ (and without loss of generality $\eta |D^2 u(x_0)|^2 - C \geq 0$),

$$0 \geq \frac{\lambda \gamma \eta}{2} |D^2 u|^2 - C + \eta |D^2 u|^2.$$

For $\lambda > 0$ chosen large enough, we have $\eta(x_0) |D^2 u(x_0)|^2 \leq C$. Therefore,

$$\begin{aligned}
v &\leq v(x_0) \\
&\leq \eta(x_0) \beta(H(Du(x_0))) + \lambda |Du(x_0)|^2 / 2 \\
&\leq \eta(x_0) |D^2 u(x_0)|^2 + C \\
&\leq C
\end{aligned}$$

as desired. \square

With the estimates above, we are finally in a position to bound the second derivatives of u^ϵ . Our approach here was introduced by L. C. Evans [10] and later refined by M. Wiegner [23]; it should be noted that they studied the case of a convex gradient constraint function with x -dependence $H = H(p, x) = |p| - g(x)$. We suspect that our assumptions on H (2.3) are the most general for which this type of penalization technique yields regularity results.

Lemma 2.2.7. *For each $O' \Subset O$, there is a constant $C = C(O')$ such that*

$$|u^\epsilon|_{W^{2,\infty}(O')} \leq C$$

for each $0 < \epsilon < 1$.

Proof. 1. It is sufficient to bound, for each $\eta \in C_c^\infty(O)$ with $0 \leq \eta \leq 1$, the quantity

$$M_\epsilon := \max_{x \in \bar{O}} \sqrt{\eta(x)} |D^2 u^\epsilon(x)|$$

for all $0 < \epsilon < 1$. With this in mind, we shall bound the related quantity

$$v^\epsilon(x) = \eta(x) \left(\frac{1}{2} |D^2 u^\epsilon(x)|^2 + \lambda \beta(H(Du^\epsilon(x))) \right) + \frac{\mu}{2} |Du^\epsilon(x)|^2, \quad x \in \bar{O}$$

from above. Here λ, μ are constants that will be chosen below. As in previous proofs, we shall omit the ϵ dependence of u^ϵ, v^ϵ and their derivatives and many times we will write β for $\beta_\epsilon(H(Du^\epsilon))$.

2. We will perform various computations that will help us study v near its maximum values.

$$\left\{ \begin{array}{l} v_{x_i} = \eta_{x_i} \left(\frac{1}{2} |D^2 u|^2 + \lambda \beta \right) + \eta (D^2 u \cdot D^2 u_{x_i} + \lambda \beta' DH \cdot Du_{x_i}) + \mu Du \cdot Du_{x_i} \\ v_{x_i x_j} = \eta_{x_i x_j} \left(\frac{1}{2} |D^2 u|^2 + \lambda \beta \right) + \eta_{x_i} (D^2 u \cdot D^2 u_{x_j} + \lambda \beta' DH \cdot Du_{x_j}) + \\ \quad \eta_{x_j} (D^2 u \cdot D^2 u_{x_i} + \lambda \beta' DH \cdot Du_{x_i}) + \\ \quad \eta \{ D^2 u_{x_i} \cdot D^2 u_{x_j} + D^2 u \cdot D^2 u_{x_i x_j} \} + \\ \quad \eta \{ \lambda (\beta'' (DH \cdot Du_{x_i})(DH \cdot Du_{x_j}) + \beta' (D^2 H D u_{x_i} \cdot Du_{x_j} + DH \cdot Du_{x_i x_j})) \} \\ \quad + \mu \{ Du_{x_i} \cdot Du_{x_j} + Du \cdot Du_{x_i x_j} \} \end{array} \right. \quad (2.21)$$

Using the assumed matrix inequalities $a \geq \gamma$ (2.2) and $D^2H \geq 1$ (2.17), we have

$$\begin{aligned}
a \cdot D^2u &= \sum_{i,j=1}^n a_{ij} v_{x_i x_j} \\
&= (a \cdot D^2\eta) \left(\frac{1}{2} |D^2u|^2 + \lambda\beta \right) + 2 \sum_{i,j=1}^n a_{ij} \eta_{x_i} (D^2u \cdot D^2u_{x_j} + \lambda\beta' DH \cdot Du_{x_j}) + \\
&\quad \eta \left\{ \sum_{i,j=1}^n a_{ij} D^2u_{x_i} \cdot D^2u_{x_j} + \sum_{i,j=1}^n D^2u \cdot a_{ij} D^2u_{x_i x_j} + \right. \\
&\quad \left. \lambda \left(\beta'' a D^2 DH \cdot D^2u DH + \beta' (a \cdot D^2u D^2 H D^2u + \sum_{i,j=1}^n DH \cdot a_{ij} Du_{x_i x_j}) \right) \right\} \\
&\quad + \mu \left\{ a D^2u \cdot D^2u + \sum_{i,j=1}^n Du \cdot a_{ij} Du_{x_i x_j} \right\} \\
&\geq (a \cdot D^2\eta) \left(\frac{1}{2} |D^2u|^2 + \lambda\beta \right) + 2 \sum_{i,j=1}^n a_{ij} \eta_{x_i} (D^2u \cdot D^2u_{x_j} + \lambda\beta' DH \cdot Du_{x_j}) + \\
&\quad + \eta \left\{ \gamma |D^3u|^2 + \sum_{k,l}^n u_{x_k x_l} (a \cdot D^2u_{x_k x_l}) \right\} \\
&\quad + \eta \lambda \left(\beta'' \gamma |D^2u DH|^2 + \beta' \left[\gamma |D^2u|^2 + \sum_{k=1}^n H_{p_k} (a \cdot D^2u_{x_k}) \right] \right) \\
&\quad + \mu \left\{ \gamma |D^2u|^2 + \sum_{k=1}^n u_{x_k} (a \cdot D^2u_{x_k}) \right\}.
\end{aligned}$$

Below we will make further computations that will help us simplify the above expression for $a \cdot D^2v$.

Recall

$$a \cdot D^2u = \delta u - h + \beta.$$

Differentiating with respect to x_k gives

$$a \cdot D^2u_{x_k} = \delta u_{x_k} - h_{x_k} + \beta' DH \cdot Du_{x_k} - a_{x_k} \cdot D^2u,$$

and differentiating with respect to x_l gives

$$\left\{ \begin{aligned}
a \cdot D^2u_{x_k x_l} &= \delta u_{x_k x_l} - h_{x_k x_l} + \beta'' DH \cdot Du_{x_k} DH \cdot Du_{x_l} \\
&\quad + \beta' (D^2 H Du_{x_k} \cdot Du_{x_l} + DH \cdot Du_{x_k x_l}) \\
&\quad - a_{x_k} \cdot D^2u_{x_l} - a_{x_l} \cdot D^2u_{x_k} - a_{x_k x_l} \cdot D^2u
\end{aligned} \right. .$$

In particular,

$$\sum_{k=1}^n u_{x_k} (a \cdot D^2 u_{x_k}) = Du \cdot D(\delta u - h) - \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} + \beta' D^2 u DH \cdot Du,$$

$$\sum_{k=1}^n H_{p_k} (a \cdot D^2 u_{x_k}) = DH \cdot D(\delta u - h) - \sum_{i,j=1}^n u_{x_i x_j} DH \cdot Da_{ij} + \beta' D^2 u DH \cdot DH,$$

and

$$\left\{ \begin{array}{l} \sum_{k,l=1}^n u_{x_k x_l} (a \cdot D^2 u_{x_k x_l}) = D^2 u \cdot D^2(\delta u - h) \\ \quad - 2 \sum_{k,l=1}^n u_{x_k x_l} (a_{x_k} \cdot D^2 u_{x_l}) - \sum_{k,l=1}^n u_{x_k x_l} (a_{x_k x_l} \cdot D^2 u) \\ \quad + \beta'' D^2 u (D^2 u DH) \cdot (D^2 u DH) \\ \quad + \beta' \left(D^2 u \cdot D^2 u D^2 H D^2 u + \sum_{k,l=1}^n u_{x_k x_l} DH \cdot Du_{x_k x_l} \right) \end{array} \right. .$$

Substituting these equalities into our expression for $a \cdot D^2 v$ gives

$$\begin{aligned} a \cdot D^2 v &\geq (a \cdot D^2 \eta) \left(\frac{1}{2} |D^2 u|^2 + \lambda \beta \right) + 2 \sum_{i,j=1}^n a_{ij} \eta_{x_i} (D^2 u \cdot D^2 u_{x_j} + \lambda \beta' DH \cdot Du_{x_j}) + \\ &\quad \eta \left\{ \gamma |D^3 u|^2 + D^2 u \cdot D^2(\delta u - h) - \sum_{k,l=1}^n u_{x_k x_l} [2(a_{x_k} \cdot D^2 u_{x_l}) + (a_{x_k x_l} \cdot D^2 u)] \right. \\ &\quad + \beta'' [D^2 u (D^2 u DH) \cdot (D^2 u DH) + \lambda \gamma |D^2 u DH|^2] + \\ &\quad \left. \beta' \left[D^2 u \cdot D^2 u D^2 H D^2 u + \lambda \gamma |D^2 u|^2 - \frac{1}{\eta} \left(\frac{1}{2} |D^2 u|^2 + \lambda \beta \right) DH \cdot D\eta \right. \right. \\ &\quad \left. \left. + DH \cdot D(\delta u - h) - \sum_{i,j=1}^n u_{x_i x_j} DH \cdot Da_{ij} \right] \right\} + \\ &\quad \mu \left\{ \gamma |D^2 u|^2 + Du \cdot D(\delta u - h) - \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} \right\} + \\ &\quad \beta' \sum_{i=1}^n H_{p_i} \left[\eta_{x_i} \left(\frac{1}{2} |D^2 u|^2 + \lambda \beta \right) + \eta (D^2 u \cdot D^2 u_{x_i} + \lambda \beta' DH \cdot Du_{x_i}) + \mu Du \cdot Du_{x_i} \right], \end{aligned}$$

assuming that $\eta > 0$. Recalling our computation for v_{x_i} [given in (2.21)] gives our final expression for $a \cdot D^2 v$:

$$\begin{aligned}
\eta(a \cdot D^2v) &\geq (\eta a \cdot D^2\eta - 2aD\eta \cdot D\eta) \left(\frac{1}{2}|D^2u|^2 + \lambda\beta \right) + 2aD\eta \cdot (Dv - \mu D^2uDu) + \\
&\quad \eta \left\{ \gamma|D^3u|^2 + D^2u \cdot D^2(\delta u - h) - \sum_{k,l=1}^n u_{x_k x_l} [2(a_{x_k} \cdot D^2u_{x_l}) + (a_{x_k x_l} \cdot D^2u)] \right. \\
&\quad + \beta'' [D^2u(D^2uDH) \cdot (D^2uDH) + \lambda\gamma|D^2uDH|^2] + \\
&\quad \beta' \left[D^2u \cdot D^2u D^2H D^2u + \lambda\gamma|D^2u|^2 - \frac{1}{\eta} \left(\frac{1}{2}|D^2u|^2 + \lambda\beta \right) DH \cdot D\eta \right. \\
&\quad \left. \left. + DH \cdot D(\delta u - h) - \sum_{i,j=1}^n u_{x_i x_j} DH \cdot Da_{ij} \right] \right\} + \\
&\quad \mu \left\{ \gamma|D^2u|^2 + Du \cdot D(\delta u - h) - \sum_{i,j=1}^n u_{x_i x_j} Du \cdot Da_{ij} \right\} + \beta' DH \cdot Dv. \quad (2.22)
\end{aligned}$$

3. Let x_0 be a maximizing point for v . If $\eta(x_0) = 0$ or $x_0 \in \partial O$, then $v \leq v(x_0) \leq \mu|Du(x_0)|^2 \leq C$. This of course implies $M_\epsilon^2 \leq C$ as desired. Now suppose that $\eta(x_0) > 0$ and $x_0 \in O$, so that

$$Dv(x_0) = 0 \quad \text{and} \quad a(x_0) \cdot D^2v(x_0) \leq 0.$$

Inequality (2.22), evaluated at x_0 , gives

$$\begin{aligned}
0 &\geq -C\{1 + |D^2u|^2\} + \frac{1}{2}\mu\eta\gamma|D^2u|^2 + \\
&\quad \eta^2\beta''|D^2uDH|^2(\gamma\lambda - |D^2u|) + \eta\beta' \{(\lambda\eta\gamma - C|D^2u|)|D^2u|^2 - C|D^2u|^2 - C\} \quad (2.23)
\end{aligned}$$

where C denotes various constants that are independent of ϵ (but may depend on $\eta(x_0), D\eta(x_0), D^2\eta(x_0)$ etc).

From (2.23), we have that if we set

$$\lambda = \lambda_\epsilon := \tau M_\epsilon,$$

where $\tau > 0$ is chosen large enough and independently of $\epsilon \in (0, 1)$,

$$0 \geq -C\{1 + |D^2u|^2\} + \frac{1}{2}\mu\eta\gamma|D^2u|^2.$$

This inequality implies a bound on $\eta(x_0)|D^2u(x_0)|^2$ for $\mu > 0$ chosen large enough and independent of $0 < \epsilon < 1$. Finally,

$$M_\epsilon^2 \leq \max_O v \leq \eta(x_0)|D^2u(x_0)|^2 + CM_\epsilon + C \leq C(M_\epsilon + 1),$$

and thus $M_\epsilon \leq C$. □

Proof. (of part (ii) of Theorem 2.0.1) We show here that there is a subsequence of $\epsilon \rightarrow 0^+$ such that $u^\epsilon \rightarrow u$ in $C_{\text{loc}}^1(O)$, where u is the solution of (2.1). This would in particular imply that $u \in W_{\text{loc}}^{2,\infty}(O) = C_{\text{loc}}^{1,1}(O)$.

1. Thus far, we have established that there is a constant $C > 0$ such that

$$|u^\epsilon|_{W^{1,\infty}(\bar{O})} \leq C, \quad \epsilon \in (0, 1),$$

and for each $O' \Subset O$, there is a constant C' such that

$$|u^\epsilon|_{W^{2,\infty}(O')} \leq C', \quad \epsilon \in (0, 1).$$

We claim that there is a function $v \in W^{1,\infty}(\bar{O}) \cap W_{\text{loc}}^{2,\infty}(O)$ and a sequence of ϵ tending to 0 such that as $\epsilon \rightarrow 0$

$$\begin{cases} u^\epsilon \rightarrow v & \text{uniformly in } \bar{O} \\ u^\epsilon \rightarrow v & \text{in } C_{\text{loc}}^1(O) \end{cases} \quad .^3$$

Set

$$O_j = \left\{ x \in O : \text{dist}(x, \partial O) \geq \frac{1}{j} \right\} \quad \text{for } j \in \mathbb{N},$$

and observe that the sequence of compact sets O_j is increasing and $O = \cup_{j \in \mathbb{N}} O_j$. Without loss of generality suppose $O_1 \neq \emptyset$. The above estimates and the Arzelà-Ascoli Theorem imply that there is a function $v^1 \in W^{1,\infty}(\bar{O}) \cap W^{2,\infty}(O_1)$ and a sequence $\epsilon_k^0 \rightarrow 0$ as $k \rightarrow \infty$ such that $u^{\epsilon_k^0} \rightarrow v^1$ uniformly in \bar{O} and $u^{\epsilon_k^0} \rightarrow v^1$ in $C^1(O_1)$ as $k \rightarrow \infty$.

The uniform bounds we have on the $W^{2,\infty}(O_2)$ norm of the sequence $u^{\epsilon_k^0}$ implies again with the Arzelà-Ascoli Theorem that there is a function $v^2 \in W^{1,\infty}(\bar{O}) \cap W^{2,\infty}(O_2)$ and a sub-sequence $(\epsilon_k^1)_{k \geq 1}$ of $(\epsilon_k^0)_{k \geq 1}$ such that $u^{\epsilon_k^1} \rightarrow v^2$ uniformly in \bar{O} and $u^{\epsilon_k^1} \rightarrow v^2$ in $C^1(O_2)$ as $k \rightarrow \infty$. By induction, we have for each $j \in \mathbb{N}$ a function $v^j \in W^{1,\infty}(\bar{O}) \cap W^{2,\infty}(O_j)$ and a sub-sequence $(\epsilon_k^j)_{k \geq 1}$ of $(\epsilon_k^{j-1})_{k \geq 1}$ such that $u^{\epsilon_k^j} \rightarrow v^j$ uniformly in \bar{O} and $u^{\epsilon_k^j} \rightarrow v^j$ in $C^1(O_j)$ as $k \rightarrow \infty$.

The diagonal sequence $(u^{\epsilon_k^k})_{k \in \mathbb{N}}$ is a subsequence of each $(u^{\epsilon_k^j})_{k \in \mathbb{N}}$ with j fixed. Hence, this diagonal sequence converges uniformly on \bar{O} to some $v \in W^{1,\infty}(\bar{O})$. Fix any $O' \Subset O$, and note that $O' \subset O_j$ for j fixed and large enough. $(u^{\epsilon_k^k})_{k \in \mathbb{N}}$ being a subsequence of $(u^{\epsilon_k^j})_{k \in \mathbb{N}}$ converges in $C^1(O') \subset C^1(O_j)$ to v as $k \rightarrow \infty$.

2. We now claim that v is a viscosity solution of (2.1) and therefore has to coincide with u by the uniqueness of viscosity solutions of (2.1). Suppose that $v - \varphi$ has a local maximum at $x_0 \in O$ and that $\varphi \in C^2(O)$. We must show

$$\max \left\{ \delta v(x_0) - a(x_0) \cdot D^2 \varphi(x_0) - h(x_0), H(D\varphi(x_0)) \right\} \leq 0. \quad (2.24)$$

By adding $x \mapsto \frac{\rho}{2}|x - x_0|^2$ to φ and later sending $\rho \rightarrow 0$, we may assume that $v - \varphi$ has a *strict* local maximum. Since u^{ϵ_k} converges to v uniformly (for some sequence $\epsilon_k \rightarrow 0$) as $k \rightarrow \infty$, there is a sequence of x_k such that

³That is, $u^\epsilon \rightarrow v$ uniformly in \bar{O} and $u^\epsilon \rightarrow v$ in $C^1(O')$ for each $O' \Subset O$ through a sequence of $\epsilon \rightarrow 0$.

$$\begin{cases} x_k \rightarrow x_0, & \text{as } k \rightarrow \infty \\ u^{\epsilon_k} - \varphi \text{ has a local maximum at } x_k \end{cases}.$$

As u^{ϵ_k} is a smooth solution of (2.16), we have

$$\delta u^{\epsilon_k}(x_k) - a(x_k) \cdot D^2\varphi(x_k) + \beta_\epsilon(H(D\varphi(x_k))) \leq h(x_k).$$

Since $\beta_\epsilon \geq 0$, we can send $k \rightarrow \infty$ to arrive at

$$\delta v(x_0) - a(x_0) \cdot D^2\varphi(x_0) \leq h(x_0).$$

By Lemma 2.2.5,

$$0 \leq \beta_{\epsilon_k}(H(D\varphi(x_k))) = \beta_{\epsilon_k}(H(Du^{\epsilon_k}(x_k))) \leq C,$$

which necessarily implies that when $k \rightarrow \infty$

$$H(D\varphi(x_0)) \leq 0.$$

Thus, (2.24) holds.

Now suppose that $v - \psi$ has a local minimum at $x_0 \in O$ and that $\psi \in C^2(O)$. We must show

$$\max \{ \delta v(x_0) - a(x_0) \cdot D^2\psi(x_0) - h(x_0), H(D\psi(x_0)) \} \geq 0. \quad (2.25)$$

Arguing as above, we discover there is a sequence $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, and x_k such that

$$\begin{cases} x_k \rightarrow x_0, & \text{as } k \rightarrow \infty \\ u^{\epsilon_k} - \psi \text{ has a local minimum at } x_k \end{cases}.$$

If

$$H(D\psi(x_0)) \geq 0,$$

then (2.25) holds. Suppose now that

$$H(D\psi(x_0)) < 0.$$

Since u^ϵ is a smooth solution of (2.16), we have

$$\delta u^{\epsilon_k}(x_k) - a(x_k) \cdot D^2\psi(x_k) + \beta_{\epsilon_k}(H(D\psi(x_k))) - h(x_k) \geq 0. \quad (2.26)$$

By the convergence established in part 1 of this proof, $H(D\psi(x_k)) = H(Du^{\epsilon_k}(x_k)) < 0$ for all large enough k . Hence,

$$\lim_{k \rightarrow \infty} \beta_{\epsilon_k}(H(D\psi(x_k))) = 0.$$

In this case, the above limit and (2.26) imply

$$\max \{ \delta v(x_0) - a(x_0) \cdot D^2\psi(x_0) - h(x_0), H(D\psi(x_0)) \} \geq \delta v(x_0) - a(x_0) \cdot D^2\psi(x_0) - h(x_0) \geq 0.$$

□

Remark 2.2.8. Any solution u of (2.1) satisfies $Lu \leq h$. Thus a (local) pointwise lower bound on Lu would imply a local $C^{1,\alpha}$ estimate on u by the Calderón-Zygmund estimate and the Sobolev inequality [15]. We suspect that such an estimate holds *without* assuming that H is uniformly convex.

Remark 2.2.9. In the case that H is not uniformly convex, we do not know whether or not u has locally bounded second derivatives. A small consolation is that u can be approximated uniformly by u^θ where u^θ is the $C_{\text{loc}}^{1,1}$ solution of

$$\begin{cases} \max \{Lu - h(x), \theta|Du|^2 + H(Du)\} = 0, & x \in O \\ u = 0, & x \in \partial O \end{cases}.$$

A close inspection of our methods show (a) $|Du^\theta(x)| \leq C$ for all $x \in \bar{O}$ and (b) for each $O' \Subset O$ there is a constant $C = C(O')$ such that

$$|D^2u^\theta(x)| \leq C/\theta$$

for almost every $x \in O'$.

2.3 Probabilistic interpretation

For a specific class of convex gradient constraint functions H , equation (2.1) is the Hamilton-Jacobi-Bellman (HJB) equation for the value function in a generic class of stochastic singular control problems. In this section, we briefly outline this correspondence and discuss how our regularity result is applicable for these value functions. For this class of gradient constraint functions H we also prove that, under the appropriate norm, the solution of the PDE is its own maximal Lipschitz extension.

2.3.1 Singular controls

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a standard n -dimensional Brownian motion $(W(t), t \geq 0)$. A *control process* is a pair (ξ, ρ) such that

$$\begin{cases} (\rho(t), \xi(t)) \in \mathbb{R}^n \times \mathbb{R}, & t \geq 0, \\ (\rho, \xi) \text{ is adapted to } W \\ |\rho(t)| = 1, & t \geq 0, \text{ a.s.} \\ \xi(0) = 0, & t \mapsto \xi(t) \text{ is non-decreasing and is left continuous with right hand limits a.s.} \end{cases}.$$

Now, let ℓ be a norm on \mathbb{R}^n and consider the stochastic control problem

$$v(x) := \inf_{\rho, \xi} \mathbb{E}^x \int_0^\tau e^{-\int_0^t c(X^{\rho, \xi}(s)) ds} [h(X^{\rho, \xi}(t)) dt + \ell(\rho(t)) d\xi(t)], \quad x \in \bar{O}. \quad (2.27)$$

Here $X^{\rho, \xi}$ satisfies the stochastic differential equation (SDE)

$$\begin{cases} dX(t) = -b(X(t))dt + \sigma(X(t))dW(t) - \rho(t)d\xi(t), & t \geq 0 \\ X(0) = x \end{cases}$$

and $\tau = \inf\{t \geq 0 : X^{\rho, \xi}(t) \notin O\}$. We are assuming that σ, b, c are smooth on \bar{O} and that the above SDE has as unique solution (in law) for each $x \in \bar{O}$ and control process (ρ, ξ) . In general, X will not have continuous sample paths and so it is regarded as a ‘‘singularly’’ controlled process. Therefore, we say that v is the value function of a problem of stochastic singular control.

W. Fleming and H. Soner [13] have shown that if the value function v satisfies a natural dynamic programming principle, then v is a viscosity solution of a HJB equation of the form (2.1). This result provides the connection between the PDE we studied in previous sections and stochastic singular control.

Theorem 2.3.1. *Assume that for each stopping time $\theta \geq 0$ and $x \in \bar{O}$,*⁴

$$v(x) = \inf_{\rho, \xi} \mathbb{E}^x \left\{ e^{-\int_0^{\tau \wedge \theta} c(X^{\rho, \xi}(s))ds} v(X^{\rho, \xi}(\tau \wedge \theta)) + \int_0^{\tau \wedge \theta} e^{-\int_0^t c(X^{\rho, \xi}(s))ds} [h(X^{\rho, \xi}(t))dt + \ell(\rho(t))d\xi(t)] \right\}.$$

Then the value function v is a viscosity solution of HJB equation

$$\begin{cases} \max \left\{ -\frac{1}{2}\sigma(x)\sigma^t(x) \cdot D^2u + b(x) \cdot Du + c(x)u - h(x), H(Du) \right\} = 0, & x \in O \\ u = 0, & x \in \partial O \end{cases} \quad (2.28)$$

where

$$H(p) = \max_{|v|=1} \{p \cdot v - \ell(v)\}, \quad p \in \mathbb{R}^n. \quad (2.29)$$

Remark 2.3.2. In particular, H defined by (2.29) satisfies (2.3), so v is the unique viscosity solution of (2.1) with

$$a(x) := \frac{1}{2}\sigma(x)\sigma^t(x), \quad x \in \bar{O}.$$

2.3.2 Regularity of the value function

Notice that in the case where ℓ is the standard Euclidean norm ($\ell(v) = |v|, v \in \mathbb{R}^n$)

$$H(p) = |p| - 1,$$

⁴ θ is a stopping time with respect to the filtration generated by W .

and (2.28) becomes

$$\max \left\{ -\frac{1}{2}\sigma(x)\sigma^t(x) \cdot D^2u + b(x) \cdot Du + c(x)u - h(x), |Du| - 1 \right\} = 0$$

which is equivalent to

$$\max \left\{ -\frac{1}{2}\sigma(x)\sigma^t(x) \cdot D^2u + b(x) \cdot Du + c(x)u - h(x), |Du|^2 - 1 \right\} = 0.$$

Since $p \mapsto |p|^2 - 1$ is uniformly convex, $v \in C_{\text{loc}}^{1,1}(O)$ while $H(p) = |p| - 1$ is not uniformly convex! We conjecture the reduction from the convex gradient constraint defined by (2.29) to a uniformly convex gradient constraint G can be achieved for a large class of norms ℓ . In fact, we always have a candidate for such a G .

Lemma 2.3.3. *Set*

$$G(p) := \max_{|v|=1} \{ (p \cdot v)^2 - (\ell(v))^2 \}, \quad p \in \mathbb{R}^n. \quad (2.30)$$

Then $H(p) \leq 0$ if and only if $G(p) \leq 0$.

Proof. Suppose $H(p) \leq 0$. Then $p \cdot v \leq \ell(v)$ and $-p \cdot v \leq \ell(-v) = \ell(v)$ for $|v| = 1$. Hence, $|p \cdot v| \leq \ell(v)$, and thus $G(p) \leq 0$. If $G(p) \leq 0$, then $\ell(v) \geq |p \cdot v| \geq p \cdot v$. Consequently, $H(p) \leq 0$. \square

Corollary 2.3.4. *The value function v is a viscosity solution of the PDE*

$$\begin{cases} \max \{ -\frac{1}{2}\sigma(x)\sigma^t(x) \cdot D^2u + b(x) \cdot Du + c(x)u - h(x), G(Du) \} = 0, & x \in O \\ u = 0, & x \in \partial O \end{cases}.$$

It remains to discover necessary and sufficient conditions on ℓ to ensure that G is uniformly convex. The example $\ell(v) = |v|$, for which $G(p) = |p|^2 - 1$, shows there are some norms ℓ for which G is uniformly convex.

Conjecture 2.3.5. *Assume that ℓ is twice continuously differentiable and that for each $|v| = 1$ the restriction of the linear map*

$$D^2\ell(v) - \ell(v)I_n \quad (2.31)$$

to v^\perp is non-positive definite. Then G is uniformly convex.

We now give some heuristic calculations that support Conjecture 2.3.5. Since

$$|p|^2 - \sup_{|v|=1} \ell(v)^2 \leq G(p) \leq |p|^2 - \inf_{|v|=1} \ell(v)^2$$

we expect

$$D^2G(p) \geq 2, \quad p \in \mathbb{R}^n. \quad (2.32)$$

Before we show why (2.31) formally implies (2.32), let us do some preliminary computations involving H .

We assume that for each $p \in \mathbb{R}^n$, there is a unique $v = v(p) \in \mathbb{S}^{n-1}$ such that

$$H(p) = p \cdot v - \ell(v). \quad (2.33)$$

We also assume that this v varies smoothly with p . Note that since ℓ is homogeneous of degree 1, and in particular $D\ell(v) \cdot v = \ell(v)$ for $v \neq 0$, $v(p)$ also a solution of the vector equation

$$p - D\ell(v) = H(p)v. \quad (2.34)$$

This can be also seen as a good heuristic by applying the theory of Lagrange multipliers to the maximization problem determined by H .

Notice that from (2.33) and (2.34)

$$\begin{aligned} DH(p) &= v(p) + Dv^t(p) (p - D\ell(v)) \\ &= v(p) + Dv^t(p) H(p)v(p) \\ &= v(p) + H(p)D(|v(p)|^2/2) \\ &= v(p). \end{aligned}$$

Hence, when $v = v(p)$

$$DH(p) = v. \quad (2.35)$$

Differentiating (2.34) gives

$$I_n - D^2\ell(v(p))Dv(p) = DH(p) \otimes v(p) + H(p)Dv(p).$$

Using (2.35) and rearranging gives an expression for the derivative of v

$$Dv(p) = (D^2\ell(v(p)) + H(p)I_n)^{-1} (I_n - v(p) \otimes v(p)). \quad (2.36)$$

The proof of the Lemma 2.3.3 shows that $v(p)$ will be a maximizer for the maximization problem determined by $G(p)$

$$G(p) = (p \cdot v)^2 - (\ell(v))^2;$$

and similarly to how we computed (2.35), we have

$$DG(p) = 2(p \cdot v(p))v(p).$$

A direct computation now yields an expression for the Hessian of G .

$$D^2G(p) = 2 \{v(p) \otimes v(p) + (v(p) \cdot p I_n + v(p) \otimes p) Dv(p)\}. \quad (2.37)$$

Now we will check that under the assumption (2.31), (2.32) holds.

First note that since $Dv(p)^t v(p) = (1/2)D|v(p)|^2 = 0$,

$$D^2G(p)v(p) \cdot v(p) = 2.$$

Now let $w \in \mathbb{S}^{n-1}$ be such that $w \cdot v(p) = 0$. (2.36) and (2.37) give

$$D^2G(p)w \cdot w = 2(p \cdot v(p)) (D^2\ell(v(p)) + H(p)I_n)^{-1} w \cdot w$$

$D^2G(p)w \cdot w \geq 2$ if and only if

$$(p \cdot v(p)) (D^2\ell(v(p)) + H(p)I_n)^{-1} \geq 1,$$

which by (2.33) holds if and only if

$$D^2\ell(v(p)) - \ell(v(p)) \leq 0.$$

This inequality is assumed by (2.31) and is the basis for our conjecture.

2.3.3 ℓ -Lipschitz extension formula

Equation (2.1) is a free boundary problem determined by a gradient constraint. In the case where H is given by (2.29) i.e.

$$H(p) = \max_{|v|=1} \{p \cdot v - \ell(v)\}, \quad p \in \mathbb{R}^n,$$

we shall see that there is convenient expression for the solution u outside of the constraint set. Before pursuing this matter, let us state a lemma that is of interest in its own right. This lemma generalizes the fact that the function defined as the distance to the boundary of a set Ω is a viscosity solution of the eikonal equation $|Du| = 1$ in Ω .

Lemma 2.3.6. *Let $\Omega \subset \mathbb{R}^n$ be non-empty and assume that $g : \partial\Omega \rightarrow \mathbb{R}$ satisfies*

$$|g(x) - g(y)| \leq \ell(x - y), \quad x, y \in \partial\Omega.$$

Then the function

$$u(x) := \inf_{y \in \partial\Omega} \{g(y) + \ell(x - y)\}, \quad x \in \bar{\Omega} \quad (2.38)$$

is the unique viscosity solution of the generalized eikonal equation

$$\begin{cases} H(Du(x)) = 0, & x \in \Omega \\ u(x) = g(x), & x \in \partial\Omega \end{cases}. \quad (2.39)$$

Proof. 1. We first show that u is a viscosity solution of (2.39). (2.38) implies

$$|u(x_1) - u(x_2)| \leq \ell(x_1 - x_2), \quad x_1, x_2 \in \Omega.$$

Now suppose that $p \in J^{1,+}u(x_0)$. That is, that

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + o(|x - x_0|)$$

as $\Omega \ni x - x_0 \rightarrow 0$. We set $x = x_0 - tv$, for $|v| = 1$ and t positive, and rearrange the above inequalities to get

$$p \cdot v - \ell(v) \leq o(1)$$

as $t \rightarrow 0^+$. Hence, $p \cdot v - \ell(v) \leq 0$ for all $|v| = 1$ and so $H(p) \leq 0$. Therefore, u is a viscosity subsolution of (2.39) satisfying $u(x) = g(x)$, $x \in \partial\Omega$. To see that u is a viscosity supersolution of (2.39), we show that any viscosity subsolution w with $w(x) = g(x)$, $x \in \partial\Omega$ is dominated by u . Lemma 1.2.9 then applies.

Let w be a subsolution of (2.39) satisfying $w(x) = g(x)$, $x \in \partial\Omega$. Similar to our arguments above, we find

$$|w(x_1) - w(x_2)| \leq \ell(x_1 - x_2), \quad x_1, x_2 \in \Omega.$$

Then it follows that

$$u(x) = \inf_{y \in \partial\Omega} \{w(y) + \ell(x - y)\} \geq w(x), \quad x \in \Omega.$$

2. Uniqueness follows from standard theorems in the theory of viscosity solutions of eikonal-type equations (see Theorem 5.9 in [1]). □

From the above lemma, we immediately have the following corollary which roughly asserts: once the free boundary is known, we only need to solve the PDE $Lu = h$ within the region $H(Du) < 0$ and then perform a certain Lipschitz extension of this solution over the whole domain O to obtain a solution of

$$\max\{Lu - h(x), H(Du)\} = 0, \quad x \in O.$$

Corollary 2.3.7. *Let u denote the unique solution of (2.1) with H given by (2.29). Moreover, assume that G given by (2.30) is uniformly convex. Then*

$$u(x) = \min_{y \in \bar{\Omega}} \{u(y) + \ell(x - y)\}, \quad x \in \bar{O}.$$

where $\Omega = \{x \in \bar{O} : H(Du(x)) < 0\}$.

2.4 Discussion

In this chapter, we studied (2.1) which is an elliptic PDE with a convex gradient constraint. We identified natural structural conditions on the gradient constraint function to ensure the uniqueness of a viscosity solution of the Dirichlet problem and to ensure some regularity of this solution. This work builds on previous efforts by L. C. Evans [10], M. Wiegner [23] and H. Ishii et.al. [16].

These types of equations arise naturally in singular control theory where the value functions of a broad class of control problems are viscosity solutions. Conveniently, the associated convex gradient constraints arising in these problems possess the structural properties needed for our analytic results. We also give an interesting characterization of the solution as being its own Lipschitz extension across the free boundary that seems to be new.

In further research, it would be of interest to deduce whether or not our regularity result is sharp and to extend our arguments to equations with fully non-linear operators. More important work would be to deduce some regularity properties of the free boundary as it is known that a sufficiently smooth free boundary allows for the construction of optimal controls (see, for instance, [21]).

Chapter 3

The eigenvalue problem of singular ergodic control

In this chapter, we address the following problem: find $\lambda \in \mathbb{R}$ such that the PDE

$$\max \{ \lambda - \Delta u - h(x), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n \quad (3.1)$$

has a solution $u : \mathbb{R}^n \rightarrow \mathbb{R}$. We call any such λ a (*non-linear*) *eigenvalue*. Here it is assumed that $h \in C^\infty(\mathbb{R}^n)$ is convex and satisfies the growth condition

$$\lim_{|x| \rightarrow \infty} \frac{h(x)}{|x|} = +\infty. \quad (3.2)$$

These assumptions imply that h is bounded from below and without any loss of generality it will also be assumed that h is non-negative.

Our main result is

Theorem 3.0.1. *There is a unique $\lambda^* \in \mathbb{R}$ such that (3.1) has a viscosity solution $u \in C(\mathbb{R}^n)$ satisfying*

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1. \quad (3.3)$$

Moreover, associated to this eigenvalue $\lambda^ \in \mathbb{R}$ is a convex solution u^* of (3.1) belonging to the space $C^{1,1}(\mathbb{R}^n)$ that satisfies (3.3).*

As far as we know, this work is the first to consider the question as posed above. However, a big part of our motivation was the work of Menaldi et. al. [20] who studied a very closely related problem arising in stochastic control theory. With regards to the framework we present, they used probabilistic arguments to build *an* eigenvalue. In this chapter, we establish the existence of a *unique* eigenvalue and obtain a better regularity result than what was obtained in [20] (as it does not require any special assumptions on h ; we only require

convexity and superlinear growth). Moreover, we have employed methods that are entirely analytic and use nothing from probability theory. These techniques, coming mostly from the theory of viscosity solutions, also generalize more naturally to a fully non-linear version of the PDE problem above that we address later in this work.

The organization of this chapter is as follows. In section 3.1, we show that there can be at most one λ such that there is a viscosity solution of equation (3.1) satisfying (3.3). In section 3.2, we present a PDE method for approximating the values of an eigenvalue. After obtaining the required estimates, we successfully pass to the limit in section 3.5 and build an eigenvalue. In section 3.6 we present two, new (approximate) min-max characterizations of the eigenvalue. Finally, in section 3.7 we discuss an extension of Theorem 3.0.1 regarding a fully non-linear, homogeneous operator.

Probabilistic interpretation of the eigenvalue. Equation (3.1) is related to the following stochastic optimal control problem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with n -dimensional Brownian motion $(W(t), t \geq 0)$. We set

$$X^\nu(t) := \sqrt{2}W(t) + \nu(t), \quad t \geq 0$$

where ν is an \mathbb{R}^n valued control process (adapted to the filtration generated by W) that satisfies

$$\begin{cases} \nu(0) = 0 \text{ a.s.} \\ t \mapsto \nu(t) \text{ is left continuous a.s.} \\ |\nu|(t) := TV_\nu[0, t] < \infty, \text{ for all } t > 0 \text{ a.s.} \end{cases} \quad .^1 \quad (3.4)$$

We say ν is a *singular control* as it may have sample paths that may not be absolutely continuous with respect to Lebesgue measure on $[0, \infty)$.

The optimization problem of interest is to find a singular control ν that minimizes the quantity

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \left\{ \mathbb{E} \int_0^t h(X^\nu(s)) ds + |\nu|(t) \right\}. \quad (3.5)$$

As (3.5) is a “long-time” average, we interpret this problem as one of *singular ergodic control*.

To see how (3.1) is related to the control problem described above, we suppose that there is $\lambda \in \mathbb{R}$ such that (3.1) has a convex solution $u \in C^2(\mathbb{R}^n)$. Let ν be a singular control process. According to Ito’s rule for semi-martingales [17],

¹ $TV_f[a, b]$ denotes the total variation of f on the interval $[a, b)$.

$$\begin{aligned}
\mathbb{E}u(X^\nu(t)) &= u(x) + \mathbb{E} \int_0^t \Delta u(X^\nu(s)) ds + \mathbb{E} \int_0^t Du(X^\nu(s)) \cdot d\nu(s) \\
&\quad + \sum_{0 \leq s < t} \mathbb{E} \int_0^t [u(X^\nu(s+)) - u(X^\nu(s)) - Du(X^\nu(s)) \cdot (X^\nu(s+) - X^\nu(s))] \\
&\geq u(x) + t\lambda - \mathbb{E} \int_0^t h(X^\nu(s)) ds - \mathbb{E} \int_0^t |Du(X^\nu(s))| d|\nu|(s) \\
&\geq u(x) + t\lambda - \mathbb{E} \int_0^t h(X^\nu(s)) ds - |\nu|(t).
\end{aligned}$$

Thus

$$\lambda \leq \frac{1}{t} \left\{ \mathbb{E} \int_0^t h(X^\nu(s)) ds + |\nu|(t) \right\} + \frac{\mathbb{E}u(X^\nu(t)) - u(x)}{t}, \quad t > 0. \quad (3.6)$$

We would like to conclude that

$$\lambda \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \left\{ \mathbb{E} \int_0^t h(X^\nu(s)) ds + |\nu|(t) \right\}. \quad (3.7)$$

Suppose that the right hand side of the inequality (3.7) is finite or else (3.7) clearly holds. In this case,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{E}h(X^\nu(s)) ds < \infty$$

which implies that there is a sequence of positive numbers $t_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$\limsup_{k \rightarrow \infty} \mathbb{E}h(X^\nu(t_k)) < \infty.$$

As h grows superlinearly and as u grows at most as fast as $|x|$, as $|x| \rightarrow \infty$,

$$\limsup_{k \rightarrow \infty} \mathbb{E}u(X^\nu(t_k)) < \infty.$$

Choosing $t = t_k$ in (3.6) and sending $k \rightarrow \infty$ establishes (3.7). In particular,

$$\lambda \leq \lambda^* := \inf_{\nu} \limsup_{t \rightarrow \infty} \frac{1}{t} \left\{ \mathbb{E} \int_0^t h(X^\nu(s)) ds + |\nu|(t) \right\}. \quad (3.8)$$

If there is a control ν^* such that

$$\lambda - \Delta u(X^{\nu^*}(t)) - h(X^{\nu^*}(t)) = 0, \text{ for } t \in (0, \infty) \text{ a.s.}, \quad (3.9)$$

$$\int_0^t Du(X^{\nu^*}(s)) \cdot d\nu^*(s) = -\nu^*(t), \text{ for } t \in (0, \infty) \text{ a.s.}, \quad (3.10)$$

and

$$u(X^{\nu^*}(t+)) - u(X^{\nu^*}(t)) - Du(X^{\nu^*}(t)) \cdot (X^{\nu^*}(t+) - X^{\nu^*}(t)) = 0, \text{ for } t \in (0, \infty) \text{ a.s.}, \quad (3.11)$$

then equality holds in (3.8). In this case, we have λ^* as a probabilistic formula for the eigenvalue appearing in (3.1).

Remark 3.0.2. ν^* satisfying (3.9), (3.10), and (3.11) is a good candidate for an optimal control. Designing such an optimal control can be done via reflected diffusions if enough regularity is assumed on u and on the boundary of the set of points x such that $|Du(x)| < 1$. This procedure is discussed in detail in [20] and in [21].

3.1 Comparison principle

The purpose of this section is to prove that there can be at most one eigenvalue for which the PDE (3.1) has a viscosity solution satisfying (3.3). In order to clearly state our results, we make the following definition.

Definition 3.1.1. $u \in USC(\mathbb{R}^n)$ is a *viscosity subsolution* of (3.1) with *eigenvalue* $\lambda \in \mathbb{R}$ if for each $x_0 \in \mathbb{R}^n$,

$$\max \{ \lambda - \Delta\varphi(x_0) - h(x_0), |D\varphi(x_0)| - 1 \} \leq 0$$

whenever $u - \varphi$ has a local maximum at x_0 and $\varphi \in C^2(\mathbb{R}^n)$. $v \in LSC(\mathbb{R}^n)$ is a *viscosity supersolution* of (3.1) with *eigenvalue* $\mu \in \mathbb{R}$ if for each $y_0 \in \mathbb{R}^n$,

$$\max \{ \mu - \Delta\psi(y_0) - h(y_0), |D\psi(y_0)| - 1 \} \geq 0$$

whenever $v - \psi$ has a local minimum at y_0 and $\psi \in C^2(\mathbb{R}^n)$. $u \in C(\mathbb{R}^n)$ is a *viscosity solution* of (3.1) with *eigenvalue* $\lambda \in \mathbb{R}$ if it is both a viscosity sub- and supersolution of (3.1) with eigenvalue λ .

Towards establishing a uniqueness result, we first establish a comparison principle for eigenvalues with sub- and supersolutions. We first present a formal argument to convey the motivating ideas.

Proposition 3.1.2. *Suppose u is a subsolution of (3.1) with eigenvalue λ and that v is a supersolution of (3.1) with eigenvalue μ . If in addition*

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|}, \quad (3.12)$$

then $\lambda \leq \mu$.

Formal proof. Here we assume that $u, v \in C^2(\mathbb{R}^n)$. Fix $0 < \epsilon < 1$ and set

$$w^\epsilon(x) = \epsilon u(x) - v(x), \quad x \in \mathbb{R}^n.$$

By (3.12), we have $\lim_{|x| \rightarrow \infty} w^\epsilon(x) = -\infty$, so there is $x_\epsilon \in \mathbb{R}^n$ such that

$$w^\epsilon(x_\epsilon) = \sup_{x \in \mathbb{R}^n} w^\epsilon(x).$$

Basic calculus gives

$$\begin{cases} 0 = Dw^\epsilon(x_\epsilon) = \epsilon Du(x_\epsilon) - Dv(x_\epsilon) \\ 0 \geq \Delta w^\epsilon(x_\epsilon) = \epsilon \Delta u(x_\epsilon) - \Delta v(x_\epsilon) \end{cases}.$$

Note in particular that

$$|Dv(x_\epsilon)| = \epsilon |Du(x_\epsilon)| \leq \epsilon < 1,$$

and since v is a supersolution of (3.1) with eigenvalue μ ,

$$0 \leq \mu - \Delta v(x_\epsilon) - h(x_\epsilon).$$

As u is a subsolution of (3.1) with eigenvalue λ

$$\begin{aligned} \epsilon \lambda - \mu &\leq \epsilon \Delta u(x_\epsilon) - \Delta v(x_\epsilon) - (1 - \epsilon)h(x_\epsilon) \\ &\leq -(1 - \epsilon)h(x_\epsilon) \\ &\leq 0. \end{aligned}$$

Here we have used that h is non-negative. Letting $\epsilon \rightarrow 1^-$, gives $\lambda \leq \mu$.

□

We now make this rigorous by using a “doubling the variables” type of argument.

Proof. (of the proposition) 1. Fix $0 < \epsilon < 1$ and set

$$w^\epsilon(x, y) = \epsilon u(x) - v(y), \quad x, y \in \mathbb{R}^n.$$

For $\delta > 0$, we also set

$$\varphi_\delta(x, y) = \frac{1}{2\delta} |x - y|^2, \quad x, y \in \mathbb{R}^n.$$

The inequality

$$\begin{aligned} w^\epsilon(x, y) - \varphi_\delta(x, y) &= \epsilon(u(x) - u(y)) - \frac{1}{2\delta} |x - y|^2 + \epsilon u(y) - v(y) \\ &\leq \left(|x - y| - \frac{1}{2\delta} |x - y|^2 \right) + \epsilon u(y) - v(y) \end{aligned}$$

implies

$$\lim_{|(x,y)| \rightarrow \infty} \{w^\epsilon(x,y) - \varphi_\delta(x,y)\} = -\infty.$$

Therefore, $w^\epsilon - \varphi_\delta$ achieves a global maximum at a point $(x_\delta, y_\delta) \in \mathbb{R}^n \times \mathbb{R}^n$.

2. According to the Theorem of Sums (Lemma 1.2.7), for each $\rho > 0$, there are $X, Y \in \mathcal{S}(n)$ such that

$$\begin{aligned} \left(\frac{x_\delta - y_\delta}{\delta}, X \right) &= (D_x \varphi_\delta(x_\delta, y_\delta), X) \in \bar{J}^{2,+}(\epsilon u)(x_\delta), \\ \left(\frac{x_\delta - y_\delta}{\delta}, Y \right) &= (-D_y \varphi_\delta(x_\delta, y_\delta), Y) \in \bar{J}^{2,-}v(y_\delta), \end{aligned} \quad (3.13)$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \rho A^2. \quad (3.14)$$

Here

$$A = D^2 \varphi_\delta(x_\delta, y_\delta) = \frac{1}{\delta} \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix}.$$

Applying both sides of the matrix inequality (3.14) to the vector $(\xi, \xi)^t \in \mathbb{R}^{2n}$ and then taking the dot product with $(\xi, \xi)^t$ yields

$$X\xi \cdot \xi - Y\xi \cdot \xi \leq 0.$$

As $\xi \in \mathbb{R}^n$ is arbitrary, $X \leq Y$.

3. We also have

$$\frac{1}{\epsilon} \frac{x_\delta - y_\delta}{\delta} \in \bar{J}^{1,+}u(x_\delta),$$

and since $|Du| \leq 1$ (in the sense of viscosity solutions),

$$\left| \frac{x_\delta - y_\delta}{\delta} \right| \leq \epsilon < 1.$$

Since v is a viscosity supersolution of (3.1) with eigenvalue μ , we have

$$0 \leq \mu - \text{tr}Y - h(y_\delta)$$

by (3.13). As u is a viscosity subsolution of (3.1) with eigenvalue λ ,

$$\lambda - \frac{\text{tr}X}{\epsilon} - h(x_\delta) \leq 0.$$

Therefore,

$$\epsilon\lambda - \mu \leq \text{tr}[X - Y] + \epsilon h(x_\delta) - h(y_\delta) \leq h(x_\delta) - h(y_\delta). \quad (3.15)$$

4. We now claim that $x_\delta \in \mathbb{R}^n$ is bounded for all small enough $\delta > 0$. If not, then there is a sequence of $\delta \rightarrow 0$, for which $(w^\epsilon - \varphi_\delta)(x_\delta, y_\delta)$ tends to $-\infty$ as this sequence of δ tends to 0. Indeed

$$\begin{aligned} (w^\epsilon - \varphi_\delta)(x_\delta, y_\delta) &= (\epsilon u(x_\delta) - v(x_\delta)) + v(x_\delta) - v(y_\delta) - \frac{|x_\delta - y_\delta|^2}{2\delta} \\ &\leq (\epsilon u(x_\delta) - v(x_\delta)) + |x_\delta - y_\delta| - \frac{|x_\delta - y_\delta|^2}{2\delta} \\ &\leq \epsilon u(x_\delta) - v(x_\delta) + \frac{\delta}{2} \end{aligned}$$

which tends to $-\infty$ as $\delta \rightarrow 0$ provided $\lim_{\delta \rightarrow 0^+} |x_\delta| = +\infty$. This would be the case for some sequence of $\delta \rightarrow 0$, provided x_δ is unbounded.

However,

$$\begin{aligned} (w^\epsilon - \varphi_\delta)(x_\delta, y_\delta) &= \max_{x, y \in \mathbb{R}^n} \left\{ \epsilon u(x) - v(y) - \frac{|x - y|^2}{2\delta} \right\} \\ &\geq \epsilon u(0) - v(0) \\ &> -\infty, \end{aligned}$$

and thus x_δ lies in a bounded subset of \mathbb{R}^n . $|x_\delta - y_\delta|^2/2\delta \rightarrow 0$ by Lemma 1.2.6, and thus y_δ is also bounded for all $\delta > 0$ and small. Again by Lemma 1.2.6, we have that the sequence $((x_\delta, y_\delta))_{\delta > 0}$ has a cluster point (x_ϵ, x_ϵ) for a sequence of $\delta \rightarrow 0^+$. Passing to this limit in (3.15) along this such a sequence gives

$$\epsilon \lambda - \mu \leq 0.$$

We conclude by letting $\epsilon \rightarrow 1^-$. □

Uniqueness of eigenvalues with solutions having the appropriate growth for large values of $|x|$ is now immediate.

Corollary 3.1.3. *There can be at most one $\lambda \in \mathbb{R}$ such that (3.1) has a viscosity solution u satisfying the growth condition (3.3).*

3.2 Approximation

Another interesting corollary of Proposition 3.1.2 is

Corollary 3.2.1. *Suppose there exists an eigenvalue λ^* as described in Theorem (3.0.1). Then*

$$\lambda^* = \sup \left\{ \lambda \in \mathbb{R} : \text{there exists a subsolution } u \text{ of (3.1) with eigenvalue } \lambda, \right. \\ \left. \text{satisfying } \limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1. \right\} \quad (3.16)$$

and

$$\lambda^* = \inf \left\{ \mu \in \mathbb{R} : \text{there exists a supersolution } v \text{ of (3.1) with eigenvalue } \mu, \right. \\ \left. \text{satisfying } \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|} \geq 1. \right\} \quad (3.17)$$

It would be of great interest to show both expressions on the right hand sides of (3.16) and (3.17) are equal and that this number is the unique eigenvalue of equation (3.1). Such a procedure for producing eigenvalues would be reminiscent of Perron's method for exhibiting viscosity solutions of PDE enjoying a comparison principle. Unfortunately, this method does not work so directly in our context as it is not clear that if, say, the right hand side of (3.16) is not an eigenvalue we can find a strictly bigger number with a corresponding u that is a subsolution of (3.1). Therefore, we are led to an alternative procedure of approximating an eigenvalue.

The method we propose is a PDE version of the probabilistic approach used by Menaldi et.al [20]. However, we believe the earliest application of this method appears in periodic homogenization of viscosity solutions of PDE in [19] and [10]. This approach essentially amounts to replacing λ in (3.1) with " δu " and studying the resulting PDE for $\delta > 0$ and small. If this resulting PDE has a unique solution u_δ , the hope is that there is a sequence of δ tending to 0 such that δu_δ tends to λ .

To this end, we will study solutions of the PDE

$$\max \{ \delta u - \Delta u - h(x), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n. \quad (3.18)$$

In particular, we seek a viscosity solution u satisfying (3.3)

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

Proposition 3.2.2. *Suppose u is a subsolution of (3.18) and that v is a supersolution of (3.18). If in addition*

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|},$$

then $u \leq v$.

Proof. We omit the proof as it is almost identical to the proof of Proposition 3.1.2. \square

Corollary 3.2.3. *There can be at most one viscosity solution u (3.18) satisfying the growth condition (3.3).*

With a comparison principle in hand, we can now employ a routine application of Perron's method to obtain existence of solutions once we have appropriate sub and supersolutions.

Lemma 3.2.4. Fix $0 < \delta < 1$.

(i) There is a universal constant $K > 0$ such that

$$\underline{u}(x) = (|x| - K)^+, \quad x \in \mathbb{R}^n \quad (3.19)$$

is a viscosity subsolution of (3.18) satisfying the growth condition (3.3).

(ii) There is a universal constant $K > 0$ such that

$$\bar{u}(x) = \frac{K}{\delta} + \begin{cases} \frac{1}{2}|x|^2, & |x| \leq 1 \\ |x| - \frac{1}{2}, & |x| \geq 1 \end{cases}, \quad x \in \mathbb{R}^n \quad (3.20)$$

is a viscosity supersolution of (3.18) satisfying the growth condition (3.3).

Proof. (i) Choose $K > 0$ such that

$$\underline{u}(x) \leq h(x), \quad x \in \mathbb{R}^n.$$

Such a K can be chosen due to the superlinear growth of h .

As \bar{u} is convex and as $\text{Lip}(\underline{u})=1$, if $(p, X) \in J^{2,+}\underline{u}(x_0)$

$$|p| \leq 1 \quad \text{and} \quad X \geq 0.$$

Hence,

$$\max \{ \delta \underline{u}(x_0) - \text{tr}X - h(x_0), |p| - 1 \} \leq \max \{ \underline{u}(x_0) - h(x_0), |p| - 1 \} \leq 0.$$

Thus \underline{u} is a viscosity subsolution.

(ii) Choose

$$K := n + \max_{|x| \leq 1} h(x)$$

and assume that $(p, X) \in J^{2,-}\underline{u}(x_0)$. If $|x_0| < 1$, \bar{u} is smooth in a neighborhood of x_0 and

$$\begin{cases} \bar{u}(x_0) = \frac{K}{\delta} + \frac{|x_0|^2}{2} \\ D\bar{u}(x_0) = x_0 = p \\ \Delta\bar{u}(x_0) = n = \text{tr}X \end{cases}.$$

Therefore,

$$\delta \bar{u}(x_0) - \Delta\bar{u}(x_0) - h(x_0) \geq K - n - h(x_0) \geq 0,$$

which implies

$$\max \{ \delta \bar{u}(x_0) - \Delta\bar{u}(x_0) - h(x_0), |D\bar{u}(x_0)| - 1 \} \geq 0. \quad (3.21)$$

Now suppose $|x_0| \geq 1$. $\bar{u} \in C^1(\mathbb{R}^n)$, so $p = D\bar{u}(x_0) = x_0/|x_0|$ and in particular $|D\bar{u}(x_0)| = 1$. Thus (3.21) still holds, and consequently, \bar{u} is a viscosity supersolution. \square

The following result follows directly from Theorem 1.2.10 using \underline{u} and \bar{u} above.

Theorem 3.2.5. Fix $0 < \delta < 1$. There is a unique viscosity solution $u = u_\delta$ of the (3.18) satisfying (3.3).

3.3 Basic estimates

Before we attempt to pass to the limit as $\delta \rightarrow 0$, we will obtain some better estimates on u_δ that will help us build an eigenvalue λ^* and establish estimates on a solution u^* of (3.1) corresponding to this eigenvalue. So far we have shown that (3.18) has a unique solution u_δ that satisfies the growth condition (3.3). Moreover, from the sub- and supersolutions (3.19) and (3.20) above, we have for each $0 < \delta < 1$

$$(|x| - K)^+ \leq u_\delta(x) \leq \frac{K}{\delta} + |x|, \quad x \in \mathbb{R}^n$$

and

$$|u_\delta(x) - u_\delta(y)| \leq |x - y|, \quad x, y \in \mathbb{R}^n.$$

Our goal now is to obtain *second derivative estimates* on u_δ . We first prove

Proposition 3.3.1. *There is a constant $C > 0$ such that for all $0 < \delta < 1$ and (Lebesgue) almost every $x \in \mathbb{R}^n$, the following estimate holds*

$$0 \leq D^2 u_\delta(x) \leq \frac{1}{\delta} \max_{|y| \leq C} |D^2 h(y)|. \quad (3.22)$$

The above proposition follows from the following two lemmas. In the first lemma, we show that u_δ is convex by adapting the classical “convexity maximum principle” argument of Korevaar [18]; in the second lemma, we estimate the second-order difference quotient of u_δ from above to obtain the upper bound in (3.22).

Lemma 3.3.2. *u_δ is convex.*

Proof. 1. We first assume $u \in C^2(\mathbb{R}^n)$ and for ease of notation, we write u for u_δ . Fix $0 < \epsilon < 1$ and set

$$\mathcal{C}^\epsilon(x, y) = \epsilon u\left(\frac{x + y}{2}\right) - \frac{u(x) + u(y)}{2}, \quad x, y \in \mathbb{R}^n.$$

We aim to bound \mathcal{C}^ϵ from above and later send $\epsilon \rightarrow 1^-$.

We first claim that \mathcal{C}^ϵ achieves its maximum value at some point $(x_\epsilon, y_\epsilon) \in \mathbb{R}^n \times \mathbb{R}^n$; it suffices to show

$$\lim_{|(x, y)| \rightarrow \infty} \mathcal{C}^\epsilon(x, y) = -\infty. \quad (3.23)$$

Let $(x_k, y_k) \in \mathbb{R}^n \times \mathbb{R}^n$ be such that

$$|x_k| + |y_k| \rightarrow \infty$$

as $k \rightarrow \infty$. Let N be large enough so that $|x_k| + |y_k| > 0$ for $k \geq N$. Note that for $k \geq N$

$$\begin{aligned} \frac{\mathcal{C}^\epsilon(x_k, y_k)}{|x_k| + |y_k|} &= \epsilon \frac{u\left(\frac{x_k + y_k}{2}\right)}{|x_k| + |y_k|} - \frac{1}{2} \left\{ \left(\frac{|x_k|}{|x_k| + |y_k|} \right) \frac{u(x_k)}{|x_k|} + \left(\frac{|y_k|}{|x_k| + |y_k|} \right) \frac{u(y_k)}{|y_k|} \right\} \\ &\leq \frac{\epsilon}{2} \frac{u\left(\frac{x_k + y_k}{2}\right)}{\left|\frac{x_k + y_k}{2}\right|} - \frac{1}{2} \left\{ \left(\frac{|x_k|}{|x_k| + |y_k|} \right) \frac{u(x_k)}{|x_k|} + \left(\frac{|y_k|}{|x_k| + |y_k|} \right) \frac{u(y_k)}{|y_k|} \right\} \end{aligned}$$

when of course $|x_k + y_k| > 0$.

If $|x_k + y_k|$ happens to be bounded, then

$$\limsup_{k \rightarrow \infty} \frac{\mathcal{C}^\epsilon(x_k, y_k)}{|x_k| + |y_k|} \leq -\frac{1}{2} < 0.$$

while if $|x_k + y_k| \rightarrow \infty$, as $k \rightarrow \infty$, we still have

$$\limsup_{k \rightarrow \infty} \frac{\mathcal{C}^\epsilon(x_k, y_k)}{|x_k| + |y_k|} \leq \frac{-(\epsilon - 1)}{2} < 0.$$

Consequently, $\limsup_{k \rightarrow \infty} \mathcal{C}^\epsilon(x_k, y_k) = -\infty$. The claim (3.23) follows since (x_k, y_k) was an arbitrary unbounded sequence.

2. As (x_ϵ, y_ϵ) is a maximizing point for \mathcal{C}^ϵ ,

$$0 = D_x \mathcal{C}^\epsilon(x_\epsilon, y_\epsilon) = \frac{\epsilon}{2} Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{1}{2} Du(x_\epsilon)$$

and

$$0 = D_y \mathcal{C}^\epsilon(x_\epsilon, y_\epsilon) = \frac{\epsilon}{2} Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{1}{2} Du(y_\epsilon).$$

Thus,

$$\epsilon Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) = Du(x_\epsilon) = Du(y_\epsilon).$$

The function $v \mapsto \mathcal{C}^\epsilon(x_\epsilon + v, y_\epsilon + v)$ has a maximum at $v = 0$ which implies

$$0 \geq \epsilon \Delta u\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{\Delta u(x_\epsilon) + \Delta u(y_\epsilon)}{2}.$$

Since,

$$|Du(x_\epsilon)| = |Du(y_\epsilon)| = \epsilon \left| Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) \right| \leq \epsilon < 1.$$

we have

$$\delta u(z) - \Delta u(z) - h(z) = 0, \quad z = x_\epsilon, y_\epsilon.$$

Combining the above inequalities gives

$$\begin{aligned}
\delta\mathcal{C}^\epsilon(x, y) &\leq \delta\mathcal{C}^\epsilon(x_\epsilon, y_\epsilon) \\
&= \epsilon\delta u\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{\delta u(x_\epsilon) + \delta u(y_\epsilon)}{2} \\
&\leq \epsilon\Delta u\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{\Delta u(x_\epsilon) + \Delta u(y_\epsilon)}{2} \\
&\quad \epsilon h\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{h(x_\epsilon) + h(y_\epsilon)}{2} \\
&\leq h\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{h(x_\epsilon) + h(y_\epsilon)}{2} \\
&\leq 0
\end{aligned}$$

by the convexity of h , for each $x, y \in \mathbb{R}^n$.

3. To make this formal argument rigorous, we employ a doubling the variables type of argument. Moreover, since \mathcal{C}^ϵ above is a type of doubling the variables function, it is appropriate going to “quadruple the variables.” This can be done by fixing $0 < \epsilon < 1$ and setting

$$w^\epsilon(x, y, x', y') = \epsilon u\left(\frac{x + y}{2}\right) - \frac{u(x') + u(y')}{2}, \quad x, y, x', y' \in \mathbb{R}^n,$$

and for $\eta > 0$, setting

$$\varphi_\eta(x, y, x', y') = \frac{1}{2\eta} \{|x - x'|^2 + |y - y'|^2\}, \quad x, y, x', y' \in \mathbb{R}^n.$$

Notice that

$$\begin{aligned}
(w^\epsilon - \varphi_\eta)(x, y, x', y') &= \epsilon \left\{ u\left(\frac{x + y}{2}\right) - u\left(\frac{x' + y'}{2}\right) \right\} - \frac{1}{2\eta} \{|x - x'|^2 + |y - y'|^2\} \\
&\quad + \epsilon u\left(\frac{x' + y'}{2}\right) - \frac{u(x') + u(y')}{2} \\
&\leq \left(\frac{|x - x'|}{2} + \frac{|y - y'|}{2} - \frac{1}{2\eta} \{|x - x'|^2 + |y - y'|^2\} \right) \\
&\quad + \epsilon u\left(\frac{x' + y'}{2}\right) - \frac{u(x') + u(y')}{2}.
\end{aligned}$$

From our computations above, it follows that

$$\lim_{|(x, y, x', y')| \rightarrow \infty} (w^\epsilon - \varphi_\eta)(x, y, x', y') = -\infty$$

and, in particular, that there is $(x_\eta, y_\eta, x'_\eta, y'_\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ maximizing $w^\epsilon - \varphi_\eta$. By the Theorem of Sums, for each $\rho > 0$ there are $X, Y \in \mathcal{S}(2n)$ such that

$$(D_x \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta), D_y \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta), X) \in \bar{J}^{2,+} \left((x, y) \mapsto \epsilon u \left(\frac{x+y}{2} \right) \right) \Big|_{x=x_\eta, y=y_\eta},$$

$$(-D_{x'} \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta), -D_{y'} \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta), Y) \in \bar{J}^{2,-} \left((x', y') \mapsto \frac{u(x') + u(y')}{2} \right) \Big|_{x'=x'_\eta, y'=y'_\eta},$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \rho A^2. \quad (3.24)$$

Here

$$A = D^2 \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta) = \frac{1}{\eta} \begin{pmatrix} I_{2n} & -I_{2n} \\ -I_{2n} & I_{2n} \end{pmatrix}.$$

Note that the matrix inequality (3.24) implies $X \leq Y$.

Set

$$p_\eta := D_x \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta) = -D_{x'} \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta) = \frac{x_\eta - x'_\eta}{\eta},$$

$$q_\eta := D_y \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta) = -D_{y'} \varphi_\eta(x_\eta, y_\eta, x'_\eta, y'_\eta) = \frac{y_\eta - y'_\eta}{\eta},$$

and also write

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

for appropriate $n \times n$ matrices X_i, Y_i $i = 1, \dots, 4$. As $X, Y \in \mathcal{S}(2n)$, $X_1, X_4, Y_1, Y_4 \in \mathcal{S}(n)$ and $X_2^t = X_3, Y_2^t = Y_3$.

By direct verification, we have

$$\begin{cases} (p_\eta, X_1) \in J^{2,+} \left(x \mapsto \frac{\epsilon}{2} u \left(\frac{x+y_\eta}{2} \right) \right) \Big|_{x=x_\eta} \\ (q_\eta, X_4) \in J^{2,+} \left(y \mapsto \frac{\epsilon}{2} u \left(\frac{x_\eta+y}{2} \right) \right) \Big|_{y=y_\eta} \\ (p_\eta, Y_1) \in J^{2,-} \left(\frac{1}{2} u \right) (x'_\eta) \\ (q_\eta, Y_4) \in J^{2,-} \left(\frac{1}{2} u \right) (y'_\eta) \end{cases}. \quad (3.25)$$

Since the Lipschitz constant of the function $x \mapsto \epsilon u((x + y_\eta)/2)$ is less than or equal $\epsilon/2$, $|p_\eta| \leq \epsilon/2 < 1/2$. Since $p_\eta \in J^{1,-} \left(\frac{1}{2} u \right) (x'_\eta)$ and u is a viscosity solution of (3.18),

$$\delta u(x'_\eta) - \text{tr} Y_1 - h(x'_\eta) = 0.$$

Likewise, we conclude that

$$\delta u(y'_\eta) - \text{tr} Y_4 - h(y'_\eta) = 0.$$

As u is a viscosity solution of (3.18), we have from the first two inclusions in (3.25)

$$\delta u \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{\text{tr} X_1}{\epsilon} - h \left(\frac{x_\eta + y_\eta}{2} \right) \leq 0$$

and

$$\delta u \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{\text{tr} X_4}{\epsilon} - h \left(\frac{x_\eta + y_\eta}{2} \right) \leq 0.$$

Averaging the two above inequalities gives

$$\delta u \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{\text{tr}[X_1 + X_4]}{2\epsilon} - h \left(\frac{x_\eta + y_\eta}{2} \right) \leq 0.$$

Altogether we have

$$\begin{aligned} \delta \left\{ \epsilon u \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{u(x'_\eta) + u(y'_\eta)}{2} \right\} &\leq \frac{1}{2} \text{tr}[X_1 - Y_1] + \frac{1}{2} \text{tr}[X_4 - Y_4] \\ &\quad + \epsilon h \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{h(x'_\eta) + h(y'_\eta)}{2} \\ &= \frac{1}{2} \text{tr}[X - Y] + h \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{h(x'_\eta) + h(y'_\eta)}{2} \\ &\leq h \left(\frac{x_\eta + y_\eta}{2} \right) - \frac{h(x'_\eta) + h(y'_\eta)}{2}. \end{aligned} \quad (3.26)$$

4. Another simple estimate for $w^\epsilon - \varphi_\eta$ is

$$\begin{aligned} (w^\epsilon - \varphi_\eta)(x, y, x', y') &= \epsilon u \left(\frac{x + y}{2} \right) - \frac{u(x) + u(y)}{2} + \\ &\quad \frac{u(x) - u(x') + u(y) - u(y')}{2} - \frac{1}{2\eta} \{|x - x'|^2 + |y - y'|^2\} \\ &\leq \epsilon u \left(\frac{x + y}{2} \right) - \frac{u(x) + u(y)}{2} + \\ &\quad \frac{|x - x'| + |y - y'|}{2} - \frac{1}{2\eta} \{|x - x'|^2 + |y - y'|^2\} \\ &\leq \epsilon u \left(\frac{x + y}{2} \right) - \frac{u(x) + u(y)}{2} + \frac{\eta}{2}. \end{aligned}$$

This estimate implies that (x_η, y_η) is a bounded sequence. For otherwise, $(w^\epsilon - \varphi_\eta)(x_\eta, y_\eta, x'_\eta, y'_\eta)$

tends to $-\infty$ (by the above estimate on $w^\epsilon - \varphi_\eta$) while

$$\begin{aligned} (w^\epsilon - \varphi_\eta)(x_\eta, y_\eta, x'_\eta, y'_\eta) &= \max_{x, y, x', y'} (w^\epsilon - \varphi_\eta)(x, y, x', y') \\ &\geq (w^\epsilon - \varphi_\eta)(0, 0, 0, 0) \\ &= (\epsilon - 1)u(0) \\ &> -\infty, \end{aligned}$$

for each $\eta > 0$. By Lemma 1.2.6, there is a cluster point $(x_\epsilon, y_\epsilon, x'_\epsilon, y'_\epsilon)$ of the sequence $((x_\eta, y_\eta, x'_\eta, y'_\eta))_{\eta > 0}$ through some sequence of $\eta \rightarrow 0$ that maximizes the function

$$(x, y) \mapsto \epsilon u \left(\frac{x + y}{2} \right) - \frac{u(x) + u(y)}{2}.$$

Passing to the limit through this sequence of η tending to 0 in (3.26) gives for any $x, y \in \mathbb{R}^n$

$$\epsilon u \left(\frac{x + y}{2} \right) - \frac{u(x) + u(y)}{2} \leq h \left(\frac{x_\epsilon + y_\epsilon}{2} \right) - \frac{h(x_\epsilon) + h(y_\epsilon)}{2} \leq 0$$

due to the convexity of h . Finally, sending $\epsilon \rightarrow 1^-$ establishes the claim. \square

Aleksandrov's Theorem [12] now implies the following corollary.

Corollary 3.3.3. *u_δ is twice differentiable at (Lebesgue) almost every point in \mathbb{R}^n .*

Since u_δ is convex and h grows superlinearly, we expect

$$\delta u_\delta(x) - \Delta u_\delta(x) - h(x) \leq \delta u_\delta(x) - h(x) \leq K + |x| - h(x) < 0$$

for all x large enough and all $0 < \delta < 1$. Here K is the constant in the (3.20). In other words, if $|Du_\delta(x)| < 1$, then $|x| \leq C$ for some C independent of $0 < \delta < 1$. We give a proof of this in terms of jets.

Corollary 3.3.4. *There is a constant $C > 0$, independent of $0 < \delta < 1$, such that if $|x| \geq C$ and $p \in J^{1,-}u_\delta(x)$, then $|p| = 1$.*

Proof. Let K be the constant in the (3.20) and choose C so large that

$$K + |z| < h(z), \quad |z| \geq C.$$

Recall that $J^{1,-}u_\delta(x) = \underline{\partial}u_\delta(x)$ by the convexity of u_δ (see Proposition 4.7 in [1]). Here

$$\underline{\partial}u(x) = \{p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x) \text{ for all } y \in \mathbb{R}^n\}$$

is the *subdifferential* of u at the point x .

Moreover, $(p, 0) \in J^{2,-}u_\delta(x)$, and so

$$\max\{\delta u_\delta(x) - h(x), |p| - 1\} \geq 0.$$

As

$$\delta u_\delta(x) - h(x) \leq K + |x| - h(x) < 0,$$

$$|p| = 1. \quad \square$$

Lemma 3.3.5. *Let $C_1 > C$, where C is the constant in the previous corollary. For almost every $x \in \mathbb{R}^n$,*

$$D^2u_\delta(x) \leq \frac{1}{\delta} \max_{|y| \leq C_1} |D^2h(y)|$$

for all $0 < \delta < 1$.

Proof. 1. Fix $0 < \epsilon < 1$, $0 < |z| < C_1 - C$, and set

$$\mathcal{C}(x) := \epsilon u_\delta(x + z) - 2u_\delta(x) + \epsilon u_\delta(x - z), \quad x \in \mathbb{R}^n.$$

As in previous arguments, we will give a formal proof (i.e. assuming $u \in C^2(\mathbb{R}^n)$) first and then later describe how to our justify arguments. For ease of notation, we write u for u_δ .

As $\lim_{|x| \rightarrow \infty} u(x)/|x| = 1$,

$$\lim_{|x| \rightarrow \infty} \mathcal{C}(x) = -\infty.$$

Thus, there is $\hat{x} \in \mathbb{R}^n$ such that

$$\mathcal{C}(\hat{x}) = \max_{x \in \mathbb{R}^n} \mathcal{C}(x).$$

At \hat{x} , we have

$$\begin{cases} 0 = D\mathcal{C}(\hat{x}) = \epsilon Du(\hat{x} + z) - Du(\hat{x}) + \epsilon Du(\hat{x} - z) \\ 0 \geq \Delta\mathcal{C}(\hat{x}) = \epsilon \Delta u(\hat{x} + z) - \Delta u(\hat{x}) + \epsilon \Delta u(\hat{x} - z) \end{cases}.$$

Thus,

$$|Du(\hat{x})| = \frac{\epsilon}{2} |Du(\hat{x} + z) + Du(\hat{x} - z)| \leq \epsilon < 1$$

and in particular

$$\begin{cases} |\hat{x}| \leq C \text{ (from the previous corollary)} \\ \delta u(\hat{x}) - \Delta u(\hat{x}) - h(\hat{x}) = 0 \end{cases}.$$

Hence, for $x \in \mathbb{R}^n$

$$\begin{aligned}
\delta\mathcal{C}(x) &\leq \delta\mathcal{C}(\hat{x}) \\
&= \epsilon(\delta u(\hat{x} + z) + \delta u(\hat{x} - z)) - 2u(\hat{x}) \\
&\leq \epsilon\Delta u(\hat{x} + z) - \Delta u(\hat{x}) + \epsilon\Delta u(\hat{x} - z) + \\
&\quad + \epsilon(h(\hat{x} + z) + h(\hat{x} - z)) - 2h(\hat{x}) \\
&\leq h(\hat{x} + z) - 2h(\hat{x}) + h(\hat{x} - z) \\
&\leq \max_{-1 \leq \xi \leq 1} D^2h(\hat{x} + \xi z)z \cdot z \\
&\leq \max_{|y| \leq C_1} |D^2h(y)||z|^2.
\end{aligned}$$

As the last expression is independent of ϵ , we send $\epsilon \rightarrow 1^-$ and arrive at the inequality

$$\frac{u(x+z) - 2u(x) + u(x-z)}{|z|^2} \leq \frac{1}{\delta} \max_{|y| \leq C_1} |D^2h(y)|, \quad 0 < |z| < C_1 - C.$$

The claim now follows as D^2u exists a.e. in \mathbb{R}^n .

2. Similar to previous proofs, we will “triple the variables.” Again we fix $0 < \epsilon < 1$ and $0 < |z| < C_1 - C$. Set

$$\begin{cases} w(x_1, x_2, x_3) := \epsilon(u(x_1 + z) + u(x_2 - z)) - 2u(x_3) \\ \varphi_\eta(x_1, x_2, x_3) := \frac{1}{2\eta} \{|x_1 - x_3|^2 + |x_2 - x_3|^2\} \end{cases},$$

for $x_1, x_2, x_3 \in \mathbb{R}^n$ and $\eta > 0$. Notice that

$$\begin{aligned}
(w - \varphi_\eta)(x_1, x_2, x_3) &= \epsilon(u(x_1 + z) - u(x_3 - z) + u(x_2 + z) - u(x_3 - z)) \\
&\quad + \mathcal{C}(x_3) - \frac{1}{2\eta} \{|x_1 - x_3|^2 + |x_2 - x_3|^2\} \\
&\leq \left(|x_1 - x_3| - \frac{1}{2\eta} |x_1 - x_3|^3 \right) + \left(|x_2 - x_3| - \frac{1}{2\eta} |x_2 - x_3|^3 \right) + \mathcal{C}(x_3),
\end{aligned}$$

which immediately implies

$$\lim_{|(x_1, x_2, x_3)| \rightarrow \infty} (w - \varphi_\eta)(x_1, x_2, x_3) = -\infty.$$

In particular, there is $(x_1^\eta, x_2^\eta, x_3^\eta)$ globally maximizing $w - \varphi_\eta$. Now we can invoke the Theorem of Sums and argue very similarly to how we did in the proof of the convexity of solutions of (3.18). We leave the details to the reader. □

Corollary 3.3.6. *We have the following:*

(i) $u_\delta \in C^{1,1}(\mathbb{R}^n)$.

(ii)

$$\Omega_\delta := \{x \in \mathbb{R}^n : |Du_\delta(x)| < 1\}$$

is open and bounded independently of all $0 < \delta < 1$.

(iii) $u_\delta \in C^{k+2,\alpha}(\Omega_\delta)$, provided $h \in C^{k,\alpha}(\mathbb{R}^n)$ for some $0 < \alpha < 1$.

(iv) There is L (independent of $0 < \delta < 1$) such that

$$D^2u_\delta(x) \leq L, \quad x \in \Omega_\delta.$$

Proof. As usual we write u for u_δ . (i) is immediate from Proposition 3.3.1. (ii) follows from Corollary 3.3.4 and (i), since $x \mapsto |Du(x)|$ is continuous on \mathbb{R}^n . (iii) follows from basic elliptic regularity theory since u satisfies the linear elliptic PDE

$$\delta u(x) - \Delta u(x) = h(x), \quad x \in \Omega_\delta$$

(see Theorem 6.17 [15]). As for (iv), we have by convexity that if $x \in \Omega_\delta$ and $|\xi| = 1$

$$\begin{aligned} D^2u(x)\xi \cdot \xi &\leq \Delta u(x) \\ &= \delta u(x) - h(x) \\ &\leq K + \delta|x| \\ &\leq K + C =: L. \end{aligned} \tag{3.27}$$

□

We conclude this section with a statement that u_δ is a Lipschitz extension of its values in $\overline{\Omega_\delta}$.

Proposition 3.3.7.

$$u_\delta(x) = \min_{y \in \overline{\Omega_\delta}} \{u_\delta(y) + |x - y|\}, \quad x \in \mathbb{R}^n. \tag{3.28}$$

Proof. It is simple to check that, since $\text{Lip}[u_\delta] \leq 1$, the formula above holds for $x \in \overline{\Omega_\delta}$. We now proceed to show that the formula above also holds in the complement of Ω_δ .

As easy argument using the convexity of u_δ establishes that the minimum in (3.28) is achieved on $\partial\overline{\Omega_\delta} = \partial\Omega_\delta$ for $x \notin \Omega_\delta$. So we are left to show

$$u_\delta(x) = \min_{y \in \partial\Omega_\delta} \{u_\delta(y) + |x - y|\}, \quad x \notin \Omega_\delta. \tag{3.29}$$

To this end, we first notice that u_δ satisfies the eikonal equation

$$\begin{cases} |Dv(x)| = 1, & x \in \overline{\Omega_\delta}^c \\ v(x) = u_\delta(x), & x \in \partial\Omega_\delta \end{cases} \quad (3.30)$$

and we claim this PDE has a unique solution given by the right hand side of (3.29). It is not hard to see that the right hand side (RHS) above is a solution of (3.30). RHS clearly defines a function with Lipschitz constant at most 1 and hence is a subsolution of (3.30). The RHS also dominates *every* subsolution of the eikonal equation that is equal to u_δ on $\partial\Omega_\delta$ and therefore is a supersolution of (3.30) by Lemma 1.2.9. The proof of uniqueness is a straightforward adaptation of the proof of comparison of sub- and supersolutions of (3.18) (see also Theorem 5.9 of [1]). \square

Corollary 3.3.8. *There is a universal constant $C > 0$, such that the estimate*

$$D^2u_\delta(x) \leq \frac{1}{|x| - C}, \quad \text{a.e. } |x| > C$$

holds for all $0 < \delta < 1$.

Proof. Recall that Ω_δ is bounded independently of $0 < \delta < 1$; let C be chosen so large that if $x \in \Omega_\delta$, then $|x| \leq C$. Also recall that $x \mapsto |x|$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and that

$$D^2|x| = \frac{1}{|x|} \left(I_n - \frac{x \otimes x}{|x|^2} \right) \leq \frac{1}{|x|} I_n, \quad x \neq 0.$$

Let $x \in \mathbb{R}^n$ with $|x| > C$. From (3.28), there is $y \in \partial\Omega_\delta$ such that $u_\delta(x) = u_\delta(y) + |x - y|$; moreover $|y| \leq C$. Also from (3.28) and the above computation, we have that as $|z| \rightarrow 0$

$$\begin{aligned} \frac{u_\delta(x+z) - 2u_\delta(x) + u_\delta(x-z)}{|z|^2} &\leq \frac{|x+z-y| - 2|x-y| + |x-z-y|}{|z|^2} \\ &\leq \frac{1}{|x-y|} + o(1) \\ &\leq \frac{1}{|x|-|y|} + o(1) \\ &\leq \frac{1}{|x|-C} + o(1). \end{aligned}$$

The corollary now follows as u_δ is twice differentiable almost everywhere in \mathbb{R}^n . \square

3.4 A uniform second derivative estimate

According to Corollary 3.3.8, D^2u_δ is bounded from above for all x large enough independently of all δ positive and small. However, the upper bound we have in the whole

space

$$\frac{1}{\delta} \max_{|y| \leq C_1} |D^2 h(y)|,$$

blows up as $\delta \rightarrow 0^+$. Our aim in this section is to obtain an estimate on the second derivative of u_δ that is *uniform* in all small $\delta > 0$. In fact, we prove

Lemma 3.4.1. *For each ball $B \subset \mathbb{R}^n$, there is a constant $C = C(B)$ such that*

$$|Du_\delta(x) - Du_\delta(y)| \leq C|x - y|, \quad x, y \in B$$

for each $0 < \delta < 1$.

Having established the above lemma, we would immediately have from Corollary 3.3.8 the following uniform second derivative estimate.

Theorem 3.4.2. *There is a universal constant L such that*

$$0 \leq D^2 u_\delta(x) \leq L, \quad \text{a.e. } x \in \mathbb{R}^n$$

for all $0 < \delta < 1$.

Towards proving Lemma 3.4.1, we fix $0 < \delta < 1$ and for ϵ positive and small study the solutions of the PDE

$$\begin{cases} \delta v - \Delta v + \beta_\epsilon (|Dv|^2 - 1) = h(x), & x \in B \\ v = u_\delta, & x \in \partial B \end{cases}, \quad (3.31)$$

where $(\beta_\epsilon)_{\epsilon > 0}$ is the standard penalty function and $B \subset \mathbb{R}^n$ is a fixed ball. As (3.31) is a semi-linear elliptic PDE with smooth coefficients, it has a unique smooth solution v_ϵ for each $\epsilon > 0$ [15]. Our goal is to deduce a pointwise bound $D^2 v_\epsilon$ that is independent of all ϵ (and δ) positive and small. With such an estimate we would be in a good position to pass to the limit and show $v_\epsilon \rightarrow u_\delta$ in $C^1(B)$ and in particular that $u_\delta \in W^{2,\infty}(B)$.

This is a very similar approximation to the one used in the previous chapter; see equation (2.16). The primary difference between equations (3.31) and (2.16) is that equation (3.31) has a non-zero boundary condition. However, as u_δ is a subsolution of (3.31), the arguments go through just the same. Using these methods, we obtain the following bounds.

Theorem 3.4.3. (i) *There is a constant C such that the following estimate holds*

$$|Dv_\epsilon(x)| \leq C, \quad x \in \bar{B}$$

for all $0 < \epsilon < 1$.

(ii) *For each $B' \Subset B$, there is a constant C' such that the following estimate holds*

$$|D^2 v_\epsilon(x)| \leq C' \left(1 + |\delta v_\epsilon|_{L^\infty(\bar{B})} + |Dv_\epsilon|_{L^\infty(\bar{B})}^2 \right), \quad x \in B' \quad (3.32)$$

for all $0 < \epsilon < 1$.

With the above estimates, we are now ready to establish Lemma 3.4.1.

Proof. (of Lemma 3.4.1) 1. Let $B \subset \mathbb{R}^n$ be a ball. Theorem (3.4.3) asserts that there is a constant $C > 0$ such that

$$|v_\epsilon|_{W^{1,\infty}(\overline{B})} \leq C, \quad \epsilon \in (0, 1),$$

and for each $B' \Subset B$, there is a constant C' such that

$$|v_\epsilon|_{W^{2,\infty}(B')} \leq C', \quad \epsilon \in (0, 1).$$

As in the proof of Theorem (2.0.1), it follows that there is a function $v \in W^{1,\infty}(\overline{B}) \cap W_{\text{loc}}^{2,\infty}(B)$ and a sequence ϵ_k tending to 0, as $k \rightarrow \infty$, such that

$$\begin{cases} v_{\epsilon_k} \rightarrow v & \text{uniformly in } \overline{B} \\ v_{\epsilon_k} \rightarrow v & \text{in } C_{\text{loc}}^1(B) \end{cases}$$

as $k \rightarrow \infty$.

2. A similar argument to the one presented in the proof of Theorem 2.0.1 shows that v is a viscosity solution of (3.18). Therefore, v has to coincide with u_δ , the unique viscosity solution of the PDE

$$\begin{cases} \max \{ \delta v - \Delta v - h(x), |Dv| - 1 \} = 0, & x \in B \\ v = u_\delta, & x \in \partial B \end{cases},$$

by a variant of the uniqueness assertion of Theorem 2.0.1.

3. From estimate (3.32), we have that for $x, y \in B' \Subset B$, there is a constant C' such that

$$|Dv_{\epsilon_k}(x) - Dv_{\epsilon_k}(y)| \leq C' (1 + |\delta v_{\epsilon_k}|_{L^\infty(B)} + |Dv_{\epsilon_k}|_{L^\infty(B)}^2) |x - y|$$

for all k sufficiently large. As $v_{\epsilon_k} \rightarrow u_\delta$ in $C_{\text{loc}}^1(B)$, and as

$$|\delta u_\delta|_{L^\infty(B)} + |Du_\delta|_{L^\infty(B)}^2 \text{ is bounded for } 0 < \delta < 1,$$

we let $k \rightarrow \infty$ to discover that there is a constant L such that

$$|Du_\delta(x) - Du_\delta(y)| \leq L|x - y|, \quad x, y \in B', \quad 0 < \delta < 1.$$

□

3.5 Passing to the limit

We now have the following estimates on u_δ ($0 < \delta < 1$)

$$\begin{cases} (|x| - K)^+ \leq u_\delta(x) \leq \frac{K}{\delta} + |x|, & x \in \mathbb{R}^n \\ |Du_\delta(x)| \leq 1, & x \in \mathbb{R}^n \\ |Du_\delta(x) - Du_\delta(y)| \leq L|x - y|, & x, y \in \mathbb{R}^n. \end{cases}$$

Our aim is to pass to limit as $\delta \rightarrow 0^+$ and prove there is an eigenvalue λ^* as stated in Theorem 3.0.1. To this end, we define

$$\begin{cases} \lambda_\delta := \delta u_\delta(x_\delta) \\ v_\delta(x) := u_\delta(x) - u_\delta(x_\delta) \end{cases}$$

where x_δ is a global minimizer of u_δ . Of course $Du_\delta(x_\delta) = 0$, and in particular $x_\delta \in \Omega_\delta$. Moreover, Corollary 3.3.4 asserts that $|x_\delta| \leq C$ for some constant C independent of all $0 < \delta < 1$.

For this constant C , we have that

$$0 \leq \lambda_\delta \leq K + C$$

and that v_δ satisfies

$$\begin{cases} |v_\delta(x)| \leq |x| + C \\ |Dv_\delta(x)| \leq 1 \\ |Dv_\delta(x) - Dv_\delta(y)| \leq L|x - y| \end{cases},$$

for all $x, y \in \mathbb{R}^n$, $0 < \delta < 1$. We will now use the above estimates to prove the following lemma which completes the proof of Theorem 3.0.1.

Lemma 3.5.1. *There is a sequence $\delta_k > 0$ tending to 0 as $k \rightarrow \infty$, $\lambda^* \in \mathbb{R}$, and $u^* \in C^{1,1}(\mathbb{R}^n)$ such that*

$$\begin{cases} \lambda^* = \lim_{k \rightarrow \infty} \lambda_{\delta_k} \\ v_{\delta_k} \rightarrow u^* \text{ in } C_{loc}^1(\mathbb{R}^n), \text{ as } k \rightarrow \infty \end{cases}. \quad (3.33)$$

Moreover, u^* is a convex solution of (3.1) satisfying the growth condition (3.3) with eigenvalue λ^* .

Proof. Routine compactness and diagonalization arguments establishes the convergence (3.33); similar arguments were used to prove Lemma 3.4.1.

It is immediate from the definition that viscosity solutions pass to the limit under local uniform convergence. It follows that u^* satisfies the PDE

$$\max\{\lambda^* - \Delta u^* - h, |Du^*| - 1\} = 0, \quad x \in \mathbb{R}^n$$

in the sense of viscosity solutions. As $|u^*(x)| \leq |x| + C$ for all $x \in \mathbb{R}^n$,

$$\limsup_{|x| \rightarrow \infty} \frac{u^*(x)}{|x|} \leq 1.$$

By the Lipschitz extension formula (3.28) and Corollary 3.3.4, we also have for all $|x|$ sufficiently large,

$$v_\delta(x) = u_\delta(x) - u_\delta(x_\delta) \geq |x| - C$$

for some C independent of $0 < \delta < 1$. Thus,

$$\liminf_{|x| \rightarrow \infty} \frac{u^*(x)}{|x|} \geq 1,$$

and so u^* satisfies (3.3). □

Remark 3.5.2. As we established for u_δ , u^* is its own Lipschitz extension

$$u^*(x) = \min_{y \in \Omega_0} \{u^*(y) + |x - y|\}, \quad x \in \mathbb{R}^n$$

where $\Omega_0 = \{x \in \mathbb{R}^n : |Du^*(x)| < 1\}$. Therefore, it suffices only to know u^* within Ω_0 to know it everywhere in space.

3.6 Min-max formulae

Recall formula (3.16)

$$\lambda^* = \sup \left\{ \lambda \in \mathbb{R} : \text{there exists a subsolution } u \text{ of (3.1) with eigenvalue } \lambda, \right. \\ \left. \text{satisfying } \limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1. \right\}$$

and formula (3.17)

$$\lambda^* = \inf \left\{ \mu \in \mathbb{R} : \text{there exists a supersolution } v \text{ of (3.1) with eigenvalue } \mu, \right. \\ \left. \text{satisfying } \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|} \geq 1. \right\},$$

which are consequences of the comparison principle established in Proposition 3.1.2. We shall use these characterizations to establish the following “min-max formulae” approximations of λ^* .

Proposition 3.6.1. *Define*

$$\lambda_- = \sup \left\{ \inf_{x \in \mathbb{R}^n} \{ \Delta \phi(x) + h(x) \} : \phi \in C^2(\mathbb{R}^n), |D\phi| \leq 1 \right\} \quad (3.34)$$

and

$$\lambda_+ = \inf \left\{ \sup_{|D\psi(x)| < 1} \{ \Delta \psi(x) + h(x) \} : \psi \in C^2(\mathbb{R}^n), \liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{|x|} \geq 1 \right\}. \quad (3.35)$$

Then (i)

$$\lambda_- = \lambda^* \leq \lambda_+,$$

and (ii) if there is a $C^2(\mathbb{R}^n)$ supersolution ψ^* of (3.1) with eigenvalue λ^* , such that

$$\liminf_{|x| \rightarrow \infty} \frac{\psi^*(x)}{|x|} \geq 1,$$

then $\lambda^* = \lambda_+$.

Proof. 1. ($\lambda^* = \lambda_-$) For $\phi \in C^2(\mathbb{R}^n)$ with $|D\phi| \leq 1$, set

$$\mu^\phi := \inf_{x \in \mathbb{R}^n} \{ \Delta \phi(x) + h(x) \}.$$

If $\mu^\phi = -\infty$, then $\mu^\phi \leq \lambda^*$. If $\mu^\phi > -\infty$, then

$$\max\{\mu^\phi - \Delta \phi(x) - h(x), |D\phi(x)| - 1\} \leq 0, \quad x \in \mathbb{R}^n.$$

(3.16) implies $\mu^\phi \leq \lambda^*$. Consequently, $\lambda_- = \sup \mu^\phi \leq \lambda^*$.

Now let u^* be a convex, $C^{1,1}(\mathbb{R}^n)$ solution associated to λ^* and $u^\epsilon := \eta^\epsilon * u^*$ be the standard mollification of u^* for $\epsilon > 0$. Note that as $|Du^*| \leq 1$ and $0 \leq D^2u^* \leq L$, we have

$$|Du^\epsilon| \leq 1 \quad \text{and} \quad 0 \leq D^2u^\epsilon \leq L, \quad \text{for all } \epsilon > 0.$$

Also note that as $u^* \in C^{1,1}$

$$\Delta u^\epsilon = \eta^\epsilon * \Delta u^* \geq \lambda^* - h^\epsilon,$$

where h^ϵ is the standard mollification of h .

As h grows superlinear and D^2u is bounded, there is $R > 0$ such that $x \mapsto \Delta u^\epsilon(x) + h(x)$ achieves its minimum value for an $x \in B_R$, for all $\epsilon > 0$. Hence, as $\epsilon \rightarrow 0^+$

$$\begin{aligned} \lambda^* &\leq \inf_{|x| \leq R} \{ \Delta u^\epsilon(x) + h^\epsilon(x) \} \\ &\leq \inf_{|x| \leq R} \{ \Delta u^\epsilon(x) + h(x) \} + o(1) \\ &\leq \inf_{x \in \mathbb{R}^n} \{ \Delta u^\epsilon(x) + h(x) \} + o(1) \\ &\leq \lambda_- + o(1). \end{aligned}$$

2. ($\lambda^* = \lambda_+$) Assume that $\psi \in C^2(\mathbb{R}^n)$ and that $\liminf_{|x| \rightarrow \infty} \psi(x)/|x| \geq 1$. Similar to our argument above, we set

$$\tau^\psi := \sup_{|D\psi(x)| < 1} \{\Delta\psi(x) + h(x)\}.$$

If $\tau^\psi = +\infty$, then $\lambda^* \leq \tau^\psi$. If $\tau^\psi < \infty$, then

$$\max\{\tau^\psi - \Delta\psi(x) - h(x), |D\psi(x)| - 1\} \geq 0, \quad x \in \mathbb{R}^n.$$

(3.17) implies $\lambda^* \leq \tau^\psi$. Consequently, $\lambda^* \leq \inf \tau^\psi = \lambda_+$. This proves (i).

If there is a $C^2(\mathbb{R}^n)$ supersolution ψ^* of (3.1) with eigenvalue λ^* , such that

$$\liminf_{|x| \rightarrow \infty} \frac{\psi^*(x)}{|x|} \geq 1,$$

then

$$\lambda_+ \leq \sup_{|D\psi^*(x)| < 1} \{\Delta\psi^*(x) + h(x)\} \leq \lambda^*$$

which proves assertion (ii). □

We believe that the assumption on the existence of ψ^* is not needed. Our intuition is that the solution u^* we constructed in Lemma 3.5.1 is twice continuously differentiable on the set of points that $|Du^*| < 1$, and therefore, should be amongst the class of ψ in the infimum defining λ_+ ; in this case

$$\lambda^* = \sup_{|Du^*(x)| < 1} \{\Delta u^*(x) + h(x)\} \geq \lambda_+.$$

Conjecture 3.6.2. $\lambda^* = \lambda_+$.

3.7 Generalizations

We conclude this chapter with an interesting question: what are appropriate assumptions on an elliptic non-linearity F to obtain a result analogous to Theorem 3.0.1 for the PDE

$$\max\{\lambda + F(D^2u, Du, x), |Du| - 1\} = 0, \quad x \in \mathbb{R}^n?$$

While we are not in a position to answer this question, we claim that the method of proof we used actually implies the following

Theorem 3.7.1. *Let $F : \mathcal{S}(n) \rightarrow \mathbb{R}$ satisfy*

$$\begin{cases} F(M) \leq F(N), & N \leq M \\ F(tM) = tF(M), \\ F(M + N) \leq F(M) + F(N) \end{cases}$$

for each $N, M \in \mathcal{S}(n)$, $t \geq 0$. That is F is elliptic, homogeneous, and concave. Then there is a unique $\lambda^* \in \mathbb{R}$ such that the PDE

$$\max \{ \lambda + F(D^2u) - h(x), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n$$

has a viscosity solution u satisfying (3.3)

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

Associated with λ^* is a convex viscosity solution u^* satisfying (3.3).

Note however that we do not make any further claims about the regularity of u^* ; perhaps if F is uniformly elliptic, more regularity of u^* can be obtained. The method of the proof of the above theorem is essentially the same as the proof of Theorem 3.0.1. The comparison principle for eigenvalues is virtually unchanged and to approximate λ^* , one studies solutions of the equation

$$\max \{ \delta u + F(D^2u) - h(x), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n$$

satisfying the growth condition (3.3). A comparison principle holds in this case as before and analogous sub and supersolutions (to \underline{u} and \bar{u}) can be written down to establish the existence of solutions via Perron's method. The *concavity* of F is what is used to obtain the convexity of the solution u_δ ; and as u^* will (essentially) be a pointwise limit of u_δ as $\delta \rightarrow 0$, u^* will be also be convex. We hope to settle more general matters in forthcoming work.

Chapter 4

Asymptotic analysis of parabolic PDE with applications to option pricing

In their celebrated paper [3], F. Black and M. Scholes derived a formula for the fair price of a European call option on a single stock in an arbitrage free market. They also presented a “replication portfolio” that enabled the issuer of the option to hedge his position upon selling the option. The Black-Scholes model presented the first rational method for valuing options, and consequently, this model has been used in a large number of industrial applications.

Aside from the financial implications, interesting mathematics also came out of this work as the Black Scholes formula is a solution of a certain *linear*, parabolic PDE

$$\psi_t + \frac{1}{2}\sigma^2 p^2 \psi_{pp} + rp\psi_p - r\psi = 0,$$

now known as the Black-Scholes equation. The purpose of this chapter is to discuss an extension of the Black-Scholes model and further connections between *non-linear* PDE and option pricing.

The Black-Scholes model, being the first of its kind, has various shortcomings. One such shortcoming is the assumption that there are no costs for making transactions; in fact, in the Black-Scholes model, the issuer of an option is trying to hedge his position at each moment of time and thus transaction costs would be ruinous. This fact has been formalized and proved rigorously [22]. Another shortcoming of the model, is that it does not account for risk preferences of option issuers or purchasers; option prices are the same for buyers and sellers and each price is completely determined by known market parameters and the option’s payoff.

An alternative model, that addresses the aforementioned modeling issues, was presented by Davis, Panas, and Zariphopoulou [8]. This model (which we will call the DPZ model) uses the principle of certainty equivalent amount to define option prices and poses the option valuation problem as a problem of stochastic control theory. Within the DPZ model, G. Barles and H. Soner [2] discovered that in markets with small transaction costs $\approx \sqrt{\epsilon}$, the

asking price z^ϵ of a European option by a very risk averse $\approx \frac{1}{\epsilon}$ seller is approximately given by

$$z^\epsilon(t, p, y) \approx \psi(t, p) + \epsilon u \left(p \frac{\psi_p(t, p) - y}{\sqrt{\epsilon}} \right),$$

as ϵ tends to 0. Here ψ is a solution of a PDE resembling a non-linear version of the Black-Scholes equation

$$\psi_t + e^{-r(T-t)} \lambda (e^{r(T-t)} p^2 \psi_{pp}) + rp\psi_p - r\psi = 0,$$

and $\lambda = \lambda(A)$ and $x \mapsto u = u(x; A)$ satisfy the ODE

$$\max \left\{ \lambda - \frac{\sigma^2}{2} (A + A^2 u'' + (x + Au')^2), |u'| - 1 \right\} = 0, \quad x \in \mathbb{R}$$

for each $A \in \mathbb{R}$. We shall see below how this ODE is naturally associated to a *non-linear eigenvalue problem*. Establishing the convergence of z^ϵ to ψ , as ϵ tends to 0, is a problem of asymptotic analysis of parabolic PDE as z^ϵ is a solution of a PDE of the form

$$\max \left\{ -z_t - \frac{1}{2} \sigma^2 p^2 \left(z_{pp} + \frac{1}{\epsilon} (z_p - y)^2 \right), |z_y| - \sqrt{\epsilon} p \right\} = 0.$$

In this chapter, we discuss a model problem for the generalization of the Barles and Soner result to options on *multiple assets*. We do not directly address the multi-asset problem as formidable technical problems arise. The most notable problem is that the associated eigenvalue problem amounts to solving a non-linear PDE and does not seem able to be resolved by trivial means. Moreover, understanding basic properties of the eigenvalue is also non-trivial. Nevertheless, we found the model problem presented below to be both interesting and instructive. We believe this work sheds light on the open and difficult problem of deducing the large risk aversion, small transaction cost limit for options on multiple assets.

4.0.1 A model problem

We consider solutions $z^\epsilon = z^\epsilon(t, x, y)$ of the initial value problem

$$\begin{cases} \max \{ z_t - (\Delta_x z + \frac{1}{\epsilon} |D_x z + y|^2), |D_y z| - \sqrt{\epsilon} \} = 0, & (0, T) \times O \times \mathbb{R}^n \\ z = g(x), & \{t = 0\} \times O \times \mathbb{R}^n, \\ z = 0, & (0, T) \times \partial O \times \mathbb{R}^n \end{cases} \quad (4.1)$$

where $O \subset \mathbb{R}^n$ is open, bounded and has smooth boundary ∂O . It is assumed that ϵ and T are positive and $g \in C(\bar{O})$ is non-negative. We do not establish the existence of solutions of (4.1), although we believe that this can be accomplished by standard methods. It will

be a standing assumption that T and g are chosen such that there is a continuous viscosity solution z^ϵ of (4.1) for all ϵ positive and small. We also assume

$$\lim_{|y| \rightarrow \infty} \frac{z^\epsilon(t, x, y)}{\sqrt{\epsilon}|y|} = 1 \quad (4.2)$$

uniformly for all ϵ positive and sufficiently small and for all x in a compact subset of O . Our goal is to understand the behavior of solutions when ϵ tends to 0.

In analogy with the aforementioned work of G. Barles and H. Soner [2], we shall formally see that

$$z^\epsilon(t, x, y) \approx \psi(t, x) + \epsilon u \left(\frac{D\psi(t, x) + y}{\sqrt{\epsilon}} \right)$$

as $\epsilon \rightarrow 0^+$, where $\psi = \psi(t, x)$ is a solution of the Cauchy problem for a non-linear, diffusion equation

$$\begin{cases} \psi_t = \lambda(D^2\psi), & (0, T) \times O \\ \psi = g, & \{t = 0\} \times O. \\ \psi = 0, & (0, T) \times \partial O \end{cases} \quad (4.3)$$

We shall also see that $\lambda = \lambda(A)$ and $x \mapsto u(x; A)$ together satisfy the PDE

$$\max \{ \lambda - \text{tr} (A + AD^2uA + (x + ADu) \otimes (x + ADu)), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n \quad (4.4)$$

for each $A \in \mathcal{S}(n)$.

Our first theorem is

Theorem 4.0.1. *(Solution of the eigenvalue problem) For each $A \in \mathcal{S}(n)$, there is a unique $\lambda = \lambda(A)$ such that (4.4) has a viscosity solution satisfying*

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1. \quad (4.5)$$

Moreover, associated to $\lambda(A)$ is a convex viscosity solution u satisfying (4.5).

It follows from Theorem 4.0.1 that the eigenvalue problem associated to the PDE (4.4) has a well defined solution $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$. Using this function λ and making some technical assumptions, we establish the following theorem, which is the main result of this chapter.

Theorem 4.0.2. *(Convergence of solutions) Assume the following technical conditions:*

(i) *There is a function $\psi \in C([0, T] \times \overline{O})$ such that for each closed interval $I' \subset (0, T)$ and each compact $O' \subset O$*

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{\substack{(t, x) \in I' \times O' \\ \sqrt{\epsilon}|y| \leq \eta}} |z^\epsilon(t, x, y) - \psi(t, x)| = o(1), \quad \text{as } \eta \rightarrow 0^+,$$

(ii) λ is continuous and monotone non-decreasing, and

(iii) when $\det A \neq 0$, there is a convex solution $u \in C^2(\mathbb{R}^n)$ of (4.4) with eigenvalue $\lambda(A)$ that satisfies (4.5),

Then ψ is the unique viscosity solution of (4.3).

In section 4.1, we study the eigenvalue problem in detail. After showing the eigenvalue problem has a unique solution, we deduce important properties of λ in section 4.2. In section 4.3, we prove Theorem 4.0.1. Finally, in section 4.4, we present a multi-asset option pricing model and pose a problem analogous to the single-asset problem solved by G. Barles and H. Soner [2]. Before we undertake this work, we shall perform some important formal computations that will guide our intuition for analyzing z^ϵ for ϵ small.

4.0.2 Formal asymptotics

Here we give a step-by-step formal derivation of how we arrived at the PDE [equation (4.3)] for the limit ψ and the PDE [equation (4.4)] arising in the eigenvalue problem. These heuristic calculations are arguably the most important part of our work since the techniques we later use are founded on these results. These computations are based largely on section 3.2 of [2].

Step 1. $|D_y z^\epsilon| \leq \sqrt{\epsilon}$, so we expect $\lim_{\epsilon \rightarrow 0^+} z^\epsilon$ to be independent of y . This observation leads to the choice of *ansatz*

$$z^\epsilon(t, x, y) \approx \psi(t, x) + \epsilon u(x^\epsilon(t, x, y)),$$

for ϵ small. Here ψ , u and x^ϵ are yet to be determined. Using this *ansatz*, we formally calculate

$$\begin{aligned} z_t^\epsilon &\approx \psi_t + \epsilon D u(x^\epsilon) \cdot x_t^\epsilon \\ D_y z^\epsilon &\approx \epsilon (D_y x^\epsilon)^t D u(x^\epsilon) \\ D_x z^\epsilon &\approx D \psi + \epsilon (D_x x^\epsilon)^t D u(x^\epsilon) \\ D_x^2 z^\epsilon &\approx D^2 \psi + \epsilon \left((D_x x^\epsilon)^t D^2 u(x^\epsilon) D_x x^\epsilon + D_x^2 x^\epsilon \cdot D u(x^\epsilon) \right) \end{aligned}$$

where $(D_x^2 x^\epsilon \cdot D u(x^\epsilon))_{ij} := x_{x_i x_j}^\epsilon \cdot D u(x^\epsilon)$, $i, j = 1, \dots, n$.

Step 2. We also observe that since

$$\epsilon |(D_y x^\epsilon)^t D u(x^\epsilon)| \approx |D_y z^\epsilon| \leq \sqrt{\epsilon},$$

x^ϵ (and its derivatives) should probably scale at worst like $1/\sqrt{\epsilon}$. With this assumption, we calculate

$$\begin{aligned}
I^\epsilon &:= z_t^\epsilon - \left(\Delta_x z^\epsilon + \frac{1}{\epsilon} |D_x z^\epsilon + y|^2 \right) \\
&\approx \psi_t - \Delta \psi - \left[\text{tr} \left(\sqrt{\epsilon} D_x x^\epsilon \right)^t D^2 u(x^\epsilon) \sqrt{\epsilon} D_x x^\epsilon + \left| \frac{D\psi + y}{\sqrt{\epsilon}} + (\sqrt{\epsilon} D_x x^\epsilon)^t Du(x^\epsilon) \right|^2 \right] \\
&\approx \psi_t - \text{tr} \left[D^2 \psi + (\sqrt{\epsilon} D_x x^\epsilon)^t D^2 u(x^\epsilon) \sqrt{\epsilon} D_x x^\epsilon + \right. \\
&\quad \left. \left(\frac{D\psi + y}{\sqrt{\epsilon}} + (\sqrt{\epsilon} D_x x^\epsilon)^t Du(x^\epsilon) \right) \otimes \left(\frac{D\psi + y}{\sqrt{\epsilon}} + (\sqrt{\epsilon} D_x x^\epsilon)^t Du(x^\epsilon) \right) \right]
\end{aligned}$$

Step 3. Notice that

$$\sqrt{\epsilon} D_x \left(\frac{D\psi + y}{\sqrt{\epsilon}} \right) = D^2 \psi.$$

This basic observation and the above computations lead us to choose the new “variable”

$$x^\epsilon := \frac{D\psi + y}{\sqrt{\epsilon}}$$

and the new “parameter”

$$A := D^2 \psi.$$

We further *postulate* that there is a function λ such that

$$\psi_t = \lambda(A).$$

Step 4. With the above choices and postulate,

$$I^\epsilon \approx \lambda(A) - \text{tr} \left(A + AD^2 u(x^\epsilon) A + (x^\epsilon + ADu(x^\epsilon)) \otimes (x^\epsilon + ADu(x^\epsilon)) \right)$$

and also

$$|Du(x^\epsilon)| \lesssim 1.$$

Since

$$\max\{I^\epsilon, |D_y z| - \sqrt{\epsilon}\} = 0,$$

we will require that u and $\lambda(A)$ satisfy

$$\max \left\{ \lambda - \text{tr} \left(A + AD^2 u A + (x + ADu) \otimes (x + ADu) \right), |Du| - 1 \right\} = 0$$

for $x \in \mathbb{R}^n$. Since $\lim_{|y| \rightarrow \infty} z^\epsilon(t, x, y)/|y| = \sqrt{\epsilon}$, we additionally require

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

In summary, we have the following **(non-linear) eigenvalue problem**:

For $A \in \mathcal{S}(n)$, find $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} \max \{ \lambda - \text{tr} (A + AD^2uA + (x + ADu) \otimes (x + ADu)), |Du| - 1 \} = 0, & x \in \mathbb{R}^n \\ \lim_{|x| \rightarrow \infty} u(x)/|x| = 1 \end{cases}.$$

If we can solve the above eigenvalue problem *uniquely for a monotone function* λ , we have the solution of the PDE (4.3) as a candidate for the limit of z^ϵ as $\epsilon \rightarrow 0^+$. We remark that, philosophically, the procedure we described above is similar to the formal asymptotics of periodic homogenization. In analogy with that context, λ plays the role of the effective Hamiltonian, and the eigenvalue problem plays the role of the cell problem [19, 10].

4.1 Analysis of the eigenvalue problem

In this section, we prove Theorem 4.0.1 which we restate for the reader's convenience.

Theorem 4.1.1. *For each $A \in \mathcal{S}(n)$, there is a unique $\lambda = \lambda(A)$ such that equation (4.4)*

$$\max \{ \lambda - \text{tr} (A + AD^2uA + (x + ADu) \otimes (x + ADu)), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n$$

has a viscosity solution satisfying (4.5)

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

Moreover, associated to $\lambda(A)$ is a convex viscosity solution u satisfying (4.5).

Notice that the above statement is a non-linear version of Theorem 3.0.1, the main result of the previous chapter. To our good fortune, many of the methods we used in Chapter 3, can be used for the above eigenvalue problem. We highlight the differences and omit proofs where similar arguments have already been made. First, we give a definition that will allow for clear statements below.

Definition 4.1.2. $u \in USC(\mathbb{R}^n)$ is a *viscosity subsolution* of (4.4) with *eigenvalue* $\lambda \in \mathbb{R}$ if for each $x_0 \in \mathbb{R}^n$,

$$\max \{ \lambda - \text{tr} (A + AD^2\varphi(x_0)A + (x_0 + AD\varphi(x_0)) \otimes (x_0 + AD\varphi(x_0))), |D\varphi(x_0)| - 1 \} \leq 0,$$

whenever $u - \varphi$ has a local maximum at x_0 and $\varphi \in C^2(\mathbb{R}^n)$. $v \in LSC(\mathbb{R}^n)$ is a *viscosity supersolution* of (4.4) with *eigenvalue* $\mu \in \mathbb{R}$ if for each $y_0 \in \mathbb{R}^n$,

$$\max \left\{ \lambda - \operatorname{tr} \left(A + AD^2\psi(y_0)A + (y_0 + AD\psi(y_0)) \otimes (y_0 + AD\psi(y_0)) \right), |D\psi(y_0)| - 1 \right\} \geq 0,$$

whenever $v - \psi$ has a local minimum at y_0 and $\psi \in C^2(\mathbb{R}^n)$. $u \in C(\mathbb{R}^n)$ is a *viscosity solution* of (4.4) with *eigenvalue* $\lambda \in \mathbb{R}$ if its both a viscosity sub- and supersolution of (4.4) with eigenvalue λ .

4.1.1 Comparison principle

We start our treatment of the eigenvalue problem by establishing a fundamental comparison principle that will allow us to compare eigenvalues associated to sub- and supersolutions of (4.4).

Proposition 4.1.3. *Suppose u is a subsolution of (4.4) with eigenvalue λ and that v is a supersolution of (4.4) with eigenvalue μ . If in addition*

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|}, \quad (4.6)$$

then $\lambda \leq \mu$.

Proof. Let us first assume that $u, v \in C^2(\mathbb{R}^n)$. Fix $0 < \epsilon < 1$ and set

$$w^\epsilon(x) = \epsilon u(x) - v(x), \quad x \in \mathbb{R}^n.$$

By (4.6), we have that $\lim_{|x| \rightarrow \infty} w^\epsilon(x) = -\infty$, so there is $x_\epsilon \in \mathbb{R}^n$ such that

$$w^\epsilon(x_\epsilon) = \sup_{x \in \mathbb{R}^n} w^\epsilon(x).$$

Basic calculus gives

$$\begin{cases} 0 = Dw^\epsilon(x_\epsilon) = \epsilon Du(x_\epsilon) - Dv(x_\epsilon) \\ 0 \geq D^2w^\epsilon(x_\epsilon) = \epsilon D^2u(x_\epsilon) - D^2v(x_\epsilon) \end{cases}.$$

Note in particular that

$$|Dv(x_\epsilon)| = \epsilon |Du(x_\epsilon)| \leq \epsilon < 1,$$

and since v is a supersolution of (4.4) with eigenvalue μ

$$\mu - \operatorname{tr} \left(A + AD^2v(x_\epsilon)A + (x_\epsilon + ADv(x_\epsilon)) \otimes (x_\epsilon + ADv(x_\epsilon)) \right) \geq 0.$$

As u is a subsolution of (4.4) with eigenvalue λ ,

$$\begin{aligned}
\epsilon\lambda - \mu &\leq \operatorname{tr} [(\epsilon - 1)A + A(\epsilon D^2 u(x_\epsilon) - D^2 v(x_\epsilon))A \\
&\quad + \epsilon(x_\epsilon + ADu(x_\epsilon)) \otimes (x_\epsilon + ADu(x_\epsilon)) - (x_\epsilon + ADv(x_\epsilon)) \otimes (x_\epsilon + ADv(x_\epsilon))] \\
&\leq \operatorname{tr} [(\epsilon - 1)A + \epsilon(x_\epsilon + ADu(x_\epsilon)) \otimes (x_\epsilon + ADu(x_\epsilon)) \\
&\quad - (x_\epsilon + \epsilon ADu(x_\epsilon)) \otimes (x_\epsilon + \epsilon ADu(x_\epsilon))] \\
&= \operatorname{tr} [(\epsilon - 1)(A + x_\epsilon \otimes x_\epsilon) + \epsilon(1 - \epsilon)ADu(x_\epsilon) \otimes ADu(x_\epsilon)] \\
&\leq (\epsilon - 1)\operatorname{tr} A + \epsilon(1 - \epsilon)|ADu(x_\epsilon)|^2 \\
&\leq (\epsilon - 1)\operatorname{tr} A + \epsilon(1 - \epsilon)|A|^2.
\end{aligned}$$

We conclude by letting $\epsilon \rightarrow 1^-$. We can now argue as we have previously (in, say, Proposition 3.1.2) to make the formal argument above rigorous. \square

Corollary 4.1.4. *For each $A \in S(n)$, there can be at most one λ such that (4.4) has a solution u satisfying (4.5).*

4.1.2 Approximation

To approximate the values of a potential eigenvalue, we study the PDE

$$\max \{ \delta u - \operatorname{tr} (A + AD^2 u A + (x + ADu) \otimes (x + ADu)), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n \quad (4.7)$$

for $\delta > 0$ and small, and seek solutions that satisfy growth condition (4.5)

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

The goal is to show that the above PDE has a unique solution u_δ and that there is a sequence of $\delta \rightarrow 0^+$ such that $\delta u_\delta(0) \rightarrow \lambda(A)$. Moreover, that we hope that $u_\delta - u_\delta(0)$ converges to a solution u of (4.4). First, we address the question of uniqueness of solutions of (4.7). As this can be handled similar to the comparison principle for eigenvalues, we omit the proof.

Proposition 4.1.5. *Suppose u is a subsolution of (4.7) and that v is a supersolution of (4.7). If in addition*

$$\limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1 \leq \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|},$$

then $u \leq v$.

Corollary 4.1.6. *For each $A \in S(n)$, there can be at most one solution of (4.7) satisfying (4.5).*

To establish uniqueness, we need sub- and supersolutions with the appropriate growth as $|x| \rightarrow \infty$.

Lemma 4.1.7. Fix $0 < \delta < 1$.

(i) There is a universal constant $K > 0$ such that

$$\underline{u}(x) = (|x| - K)^+ + \frac{\text{tr}A}{\delta}, \quad x \in \mathbb{R}^n \quad (4.8)$$

is a viscosity subsolution of (4.7) satisfying the growth condition (4.5).

(ii) There is universal constant $K > 0$ such that

$$\bar{u}(x) = \frac{K}{\delta} + \begin{cases} \frac{1}{2}|x|^2, & |x| \leq 1 \\ |x| - \frac{1}{2}, & |x| \geq 1 \end{cases}, \quad x \in \mathbb{R}^n \quad (4.9)$$

is a viscosity supersolution of (4.7) satisfying the growth condition (4.5).

Proof. (i) Choose $K > 0$ such that

$$(|x| - K)^+ \leq (|x| - |A|)^2, \quad x \in \mathbb{R}^n.$$

As \bar{u} is convex and as $\text{Lip}(\underline{u})=1$, if $(p, X) \in J^{2,+}\underline{u}(x_0)$

$$|p| \leq 1 \quad \text{and} \quad X \geq 0.$$

Hence,

$$\begin{aligned} \delta \underline{u}(x_0) - \text{tr}(A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)) &\leq (|x_0| - K)^+ - \text{tr}A^2X - |x_0 + Ap|^2 \\ &\leq (|x_0| - K)^+ - (|x_0| - |A|)^2 \\ &\leq 0. \end{aligned}$$

Thus \underline{u} is a viscosity subsolution.

(ii) Choose

$$K := \max_{|x| \leq 1} \text{tr}[A + A^2 + (I_n + A)x \otimes (I_n + A)x]$$

and assume that $(p, X) \in J^{2,-}\underline{u}(x_0)$. If $|x_0| < 1$, \bar{u} is smooth in a neighborhood of x_0 and

$$\begin{cases} \bar{u}(x_0) = \frac{K}{\delta} + \frac{|x_0|^2}{2} \\ D\bar{u}(x_0) = x_0 = p \\ D^2\bar{u}(x_0) = I_n = X \end{cases}.$$

Therefore,

$$\delta \bar{u}(x_0) - \text{tr}(A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)) \geq K - \text{tr}[A + A^2 + (I_n + A)x_0 \otimes (I_n + A)x_0] \geq 0,$$

which implies

$$\max\{\delta \bar{u}(x_0) - \text{tr}(A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)), |D\bar{u}(x_0)| - 1\} \geq 0. \quad (4.10)$$

Now suppose $|x_0| \geq 1$. $\bar{u} \in C^1(\mathbb{R}^n)$, so $p = D\bar{u}(x_0) = x_0/|x_0|$ and in particular $|D\bar{u}(x_0)| = 1$. Thus (4.10) still holds, and consequently, \bar{u} is a viscosity supersolution. \square

As the existence of a unique viscosity solution now follows directly from Theorem 1.2.10, we also omit the proof.

Theorem 4.1.8. *Fix $0 < \delta < 1$. There is a unique viscosity solution $u = u_\delta$ of the (4.7) satisfying (4.5).*

A fundamental property of the solution u_δ that we deduce below is that it is convex. Many other properties of u_δ will be derived directly from this property. The method proof is virtually the same as Lemma 3.3.2 in the previous chapter, so we only give a formal argument.

Proposition 4.1.9. *u_δ is convex.*

Proof. 1. We assume $u \in C^2(\mathbb{R}^n)$ and for ease of notation, we write u for u_δ . Fix $0 < \epsilon < 1$ and set

$$\mathcal{C}^\epsilon(x, y) = \epsilon u\left(\frac{x+y}{2}\right) - \frac{u(x) + u(y)}{2}, \quad x, y \in \mathbb{R}^n.$$

We aim to bound \mathcal{C}^ϵ from above and later send $\epsilon \rightarrow 1^-$.

2. As in the proof of Lemma 3.3.2, there exists (x_ϵ, y_ϵ) maximizing \mathcal{C}^ϵ . At this point,

$$0 = D_x \mathcal{C}^\epsilon(x_\epsilon, y_\epsilon) = \frac{\epsilon}{2} Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{1}{2} Du(x_\epsilon)$$

and

$$0 = D_y \mathcal{C}^\epsilon(x_\epsilon, y_\epsilon) = \frac{\epsilon}{2} Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{1}{2} Du(y_\epsilon).$$

Thus,

$$\epsilon Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) = Du(x_\epsilon) = Du(y_\epsilon).$$

Also observe that $v \mapsto \mathcal{C}^\epsilon(x_\epsilon + v, y_\epsilon + v)$ has a maximum at $v = 0$ which implies

$$0 \geq \epsilon D^2 u\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{D^2 u(x_\epsilon) + D^2 u(y_\epsilon)}{2}.$$

Since,

$$|Du(x_\epsilon)| = |Du(y_\epsilon)| = \epsilon \left| Du\left(\frac{x_\epsilon + y_\epsilon}{2}\right) \right| \leq \epsilon < 1.$$

we have

$$\delta u(z) - \text{tr}\left(A + AD^2 u(z)A + (z + ADu(z)) \otimes (z + ADu(z))\right) = 0, \quad z = x_\epsilon, y_\epsilon.$$

Set $z_\epsilon = (x_\epsilon + y_\epsilon)/2$, $p_\epsilon = Du(z_\epsilon)$, and notice

$$\begin{aligned}
\delta\mathcal{C}^\epsilon(x, y) &\leq \delta\mathcal{C}^\epsilon(x_\epsilon, y_\epsilon) \\
&= \epsilon\delta u\left(\frac{x_\epsilon + y_\epsilon}{2}\right) - \frac{\delta u(x_\epsilon) + \delta u(y_\epsilon)}{2} \\
&\leq \epsilon\text{tr}\left(A + AD^2u(z_\epsilon)A + (z_\epsilon + ADu(z_\epsilon)) \otimes (z_\epsilon + ADu(z_\epsilon))\right) \\
&\quad - \frac{1}{2}\text{tr}\left(A + AD^2u(x_\epsilon)A + (x_\epsilon + ADu(x_\epsilon)) \otimes (x_\epsilon + ADu(x_\epsilon))\right) \\
&\quad - \frac{1}{2}\text{tr}\left(A + AD^2u(y_\epsilon)A + (y_\epsilon + ADu(y_\epsilon)) \otimes (y_\epsilon + ADu(y_\epsilon))\right) \\
&= (\epsilon - 1)\text{tr}A + \text{tr}\left[A\left(\epsilon D^2u(z_\epsilon) - \frac{D^2u(x_\epsilon) + D^2u(y_\epsilon)}{2}\right)A\right] \\
&\quad + \epsilon|z_\epsilon + ADu(z_\epsilon)|^2 - \frac{1}{2}|x_\epsilon + ADu(x_\epsilon)|^2 - \frac{1}{2}|y_\epsilon + ADu(y_\epsilon)|^2 \\
&\leq (\epsilon - 1)\text{tr}A + \epsilon|z_\epsilon + Ap_\epsilon|^2 - \frac{1}{2}|x_\epsilon + \epsilon Ap_\epsilon|^2 - \frac{1}{2}|y_\epsilon + \epsilon Ap_\epsilon|^2 \\
&= (\epsilon - 1)\text{tr}A + \frac{(\epsilon - 1)}{2}(|x_\epsilon|^2 + |y_\epsilon|^2) + \epsilon(1 - \epsilon)|Ap_\epsilon|^2 \\
&\leq (\epsilon - 1)\text{tr}A + \epsilon(1 - \epsilon)|A|^2,
\end{aligned}$$

for each $x, y \in \mathbb{R}^n$. We conclude by sending $\epsilon \rightarrow 1^-$. □

Aleksandrov's Theorem [12] now implies the following corollary.

Corollary 4.1.10. *u_δ is twice differentiable at (Lebesgue) almost every point in \mathbb{R}^n .*

Since u_δ is convex and $u_\delta \leq \bar{u}$ given by (4.9), we expect

$$\begin{aligned}
\delta u_\delta - \text{tr}\left(A + AD^2u_\delta A + (x + ADu_\delta) \otimes (x + ADu_\delta)\right) &\leq \delta u_\delta(x) - \text{tr}A - |x + ADu_\delta(x)|^2 \\
&\leq K + |x| - \text{tr}A - (|x| - |A|)^2 \\
&< 0
\end{aligned}$$

for all x large enough and $\delta \in (0, 1)$, where K is the constant in (4.9). In other words, if $|Du_\delta(x)| < 1$, then $|x| \leq C$ for some C independent of $\delta \in (0, 1)$. The appropriate statement in terms of jets is given below, and the proof is omitted as it is very similar to Corollary 3.3.4.

Corollary 4.1.11. *There is a constant $C = C(A) > 0$, independent of $0 < \delta < 1$, such that if $|x| \geq C$ and $p \in J^{1,-}u_\delta(x)$, then $|p| = 1$.*

Corollary 4.1.12. *There is a constant $C = C(A) > 0$, independent of $0 < \delta < 1$, such that*

$$u_\delta(x) = \min_{|y| \leq C} \{u_\delta(y) + |x - y|\}, \quad x \in \mathbb{R}^n. \quad (4.11)$$

Proof. Choose C such that

$$K + |x| - \operatorname{tr}A - (|x| - |A|)^2 \leq 0 \quad \text{for } |x| \geq C,$$

where K is the constant appearing in the definition of \bar{u} in equation (4.9). Also set v to be the right hand side of (4.11). As $\operatorname{Lip}[u_\delta] \leq 1$,

$$u_\delta \leq v$$

and $v = u_\delta$ for $|x| \leq C$.

It is clear that $\operatorname{Lip}[v] \leq 1$, and it is also straightforward to verify that as u_δ is convex, v is convex, as well. Now let $(p, X) \in J^{2,+}v(x_0)$. If $|x_0| < C$, the $v = u_\delta$ is a neighborhood of x_0 and so

$$\max \{ \delta v(x_0) - \operatorname{tr}(A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)), |Dv(x_0)| - 1 \} \leq 0.$$

If $|x_0| \geq C$, then by the convexity of v

$$\begin{aligned} \delta v(x_0) - \operatorname{tr}(A + AXA + (x_0 + Ap) \otimes (x_0 + Ap)) &\leq \delta v(x_0) - \operatorname{tr}A - |x_0 + ADp|^2 \\ &\leq \delta(u(0) + |x_0|) - \operatorname{tr}A - (|x_0| - |A|)^2 \\ &\leq K + |x_0| - \operatorname{tr}A - (|x_0| - |A|)^2 \\ &\leq 0, \end{aligned}$$

while we always have $|p| \leq 1$. Therefore, v is a subsolution of (4.7), and consequently

$$v \leq u_\delta.$$

□

Towards establishing an important lower bound on u_δ , we first observe that u_δ has its global minimum value at $x = 0$.

Proposition 4.1.13. $0 \in \partial u_\delta(0)$, and in particular u_δ achieves its minimum value at $x = 0$.

Proof. By Theorem 4.1.8,

$$u_\delta(x) = u_\delta(-x), \quad x \in \mathbb{R}^n$$

as $x \mapsto u_\delta(-x)$ satisfies (4.7) and (4.5). If u_δ is differentiable at $x = 0$, then $Du_\delta(0) = 0$. By convexity,

$$u_\delta(x) \geq u_\delta(0) + Du_\delta(0) \cdot x = u_\delta(0), \quad x \in \mathbb{R}^n.$$

In general (not assuming differentiability at $x = 0$), we write $u = u_\delta$ and set

$$u^\epsilon(x) = \eta^\epsilon * u(x) = \int_{\mathbb{R}^n} \eta^\epsilon(y) u(x - y) dy, \quad x \in \mathbb{R}^n$$

where $\eta^\epsilon \in C^\infty$ is the standard mollifier. Recall η^ϵ is radially symmetric, is supported in the ball B_ϵ , and satisfies $\int \eta^\epsilon = 1$ for all $\epsilon > 0$. As u is continuous, $u^\epsilon \rightarrow u$ locally uniformly as $\epsilon \rightarrow 0^+$. One checks that u^ϵ is convex and also that $u^\epsilon(x) = u^\epsilon(-x)$. From our remarks above, we conclude $u^\epsilon(x) \geq u^\epsilon(0)$ for $x \in \mathbb{R}^n$. Sending $\epsilon \rightarrow 0^+$, gives $u(x) \geq u(0)$ for all $x \in \mathbb{R}^n$. \square

We conclude this subsection by establishing an crucial lower bound on u_δ ; this lower bound is key to establishing the existence of an eigenvalue.

Corollary 4.1.14. *There is a constant $C = C(A) > 0$, independent of $0 < \delta < 1$, such that*

$$u_\delta(x) \geq u_\delta(0) + (|x| - C)^+, \quad x \in \mathbb{R}^n.$$

Proof. By above proposition $u_\delta(x) \geq u_\delta(0)$ for all $x \in \mathbb{R}^n$ and so the claim follows directly from Corollary 4.1.12. \square

4.1.3 Convergence of scheme

We assume that A is a fixed $n \times n$ symmetric matrix and will now establish the existence of a unique eigenvalue $\lambda(A)$. Proposition 4.1.3 asserts uniqueness, so all that is left to prove is the existence of an eigenvalue. To this end, we will use the estimates we have obtained on the sequence of solutions u_δ :

$$\begin{cases} (|x| - K)^+ + \frac{\text{tr}A}{\delta} \leq u_\delta(x) \leq \frac{K}{\delta} + |x| \\ |u_\delta(x) - u_\delta(y)| \leq |x - y| \\ u_\delta((x+y)/2) \leq (u_\delta(x) + u_\delta(y))/2, \end{cases}$$

for $x, y \in \mathbb{R}^n$ and $0 < \delta < 1$.

Define

$$\begin{cases} \lambda_\delta := \delta u_\delta(0) \\ v_\delta(x) := u_\delta(x) - u_\delta(0) \end{cases} .$$

Notice that

$$\text{tr}A \leq \lambda_\delta \leq K$$

and v_δ satisfies

$$\begin{cases} |v_\delta(x)| \leq |x| \\ |Dv_\delta(x)| \leq 1 \end{cases}$$

for $x \in \mathbb{R}^n$. We are now in a good position to prove the following lemma, which will complete the proof of Theorem 4.0.1.

Lemma 4.1.15. *There is a sequence $\delta_k > 0$ tending to 0 as $k \rightarrow \infty$, $\lambda(A) \in \mathbb{R}$, and $u^* \in C(\mathbb{R}^n)$ with $\text{Lip}[u^*] \leq 1$ such that*

$$\begin{cases} \lambda(A) = \lim_{k \rightarrow \infty} \lambda_{\delta_k} \\ v_{\delta_k} \rightarrow u^* \text{ in locally uniformly as } k \rightarrow \infty \end{cases} . \quad (4.12)$$

Moreover, u^* is a convex solution of (4.4) with eigenvalue $\lambda(A)$ that satisfies the growth condition (4.5).

Proof. The convergence assertion follows from an argument very similar to the proof of Lemma 3.5.1. It also follows easily from convergence assertion that u^* satisfies the PDE

$$\max \left\{ \lambda(A) - \text{tr} \left(A + AD^2u^*A + (x + ADu^*) \otimes (x + ADu^*) \right), |Du^*| - 1 \right\} = 0, \quad x \in \mathbb{R}^n$$

in the sense of viscosity solutions. As $|u^*(x)| \leq |x|$ for all $x \in \mathbb{R}^n$,

$$\limsup_{|x| \rightarrow \infty} \frac{u^*(x)}{|x|} \leq 1.$$

By Corollary 4.1.14, for all $|x|$ sufficiently large

$$v_\delta(x) = u_\delta(x) - u_\delta(0) \geq |x| - C,$$

for some C independent of $0 < \delta < 1$. Thus,

$$\liminf_{|x| \rightarrow \infty} \frac{u^*(x)}{|x|} \geq 1,$$

and so u^* satisfies (4.5). □

4.2 Properties of the eigenvalue

In view of Theorem 4.0.1, the solution of the eigenvalue problem defines a function that we shall denote $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$. We will establish a few important properties of this function. Our basic tool will be the comparison principle. Our first result is a direct consequence of this principle.

Proposition 4.2.1. *Let $A \in \mathcal{S}(n)$ and assume that $\lambda(A)$ is the solution of the eigenvalue problem associated to equation (4.4). Then*

$$\lambda(A) = \sup \left\{ \lambda \in \mathbb{R} : \text{there exists a subsolution } u \text{ of (4.4) with eigenvalue } \lambda, \right. \\ \left. \text{satisfying } \limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \leq 1. \right\} \quad (4.13)$$

and

$$\lambda(A) = \inf \left\{ \mu \in \mathbb{R} : \text{there exists a supersolution } v \text{ of (4.4) with eigenvalue } \mu, \right. \\ \left. \text{satisfying } \liminf_{|x| \rightarrow \infty} \frac{v(x)}{|x|} \geq 1. \right\} \quad (4.14)$$

The above formulae, manifestations of the comparison principle, will be used below to establish bounds on the eigenvalue by what we call “min-max” formulae. Our hope is that the eigenvalue itself is given by these min-max formulae, and in the future, they will help us deduce the behavior of the eigenvalue function λ as A varies. We also use formulae (4.13) and (4.14) to show that there are monotone non-decreasing upper and lower bounds for λ ; we believe that λ itself is monotone non-decreasing and view these bounds as positive evidence for this conjecture.

4.2.1 Min-max formulae

In this subsection, we prove

Proposition 4.2.2. *For $A \in \mathcal{S}(n)$, set*

$$\lambda_-(A) = \sup \left\{ \inf_{x \in \mathbb{R}^n} \text{tr} [A + AD^2\phi(x)A + (x + AD\phi(x)) \otimes (x + AD\phi(x))] : \right. \\ \left. \phi \in C^2(\mathbb{R}^n), |D\phi| \leq 1 \right\} \quad (4.15)$$

and

$$\lambda_+(A) = \inf \left\{ \sup_{|D\psi(x)| < 1} \text{tr} [A + AD^2\psi(x)A + (x + AD\psi(x)) \otimes (x + AD\psi(x))] : \right. \\ \left. \psi \in C^2(\mathbb{R}^n), \liminf_{|x| \rightarrow \infty} \frac{\psi(x)}{|x|} \geq 1 \right\}. \quad (4.16)$$

Then

$$\lambda_-(A) \leq \lambda(A) \leq \lambda_+(A).$$

Moreover, if there is a $C^2(\mathbb{R}^n)$ subsolution of (4.4) with eigenvalue $\lambda(A)$, then $\lambda_-(A) = \lambda(A)$; and if there is a $C^2(\mathbb{R}^n)$ supersolution u of (4.4) with eigenvalue $\lambda(A)$ satisfying

$$\liminf_{|x| \rightarrow \infty} \frac{u(x)}{|x|} \geq 1 \quad (4.17)$$

then $\lambda(A) = \lambda_+(A)$.

Proof. ($\lambda_- \equiv \lambda$) Fix $A \in \mathcal{S}(n)$, let $\phi \in C^2$ and suppose that $|D\phi| \leq 1$. Now set

$$\mu^\phi(A) := \inf_{x \in \mathbb{R}^n} \operatorname{tr} [A + AD^2\phi(x)A + (x + AD\phi(x)) \otimes (x + AD\phi(x))].$$

If $\mu^\phi(A) = -\infty$, then $\mu^\phi(A) \leq \lambda(A)$; if $\mu^\phi(A) > -\infty$, by the assumptions on ϕ and the definition of $\mu^\phi(A)$

$$\max\{\mu^\phi(A) - \operatorname{tr} [A + AD^2\phi(x)A + (x + AD\phi(x)) \otimes (x + AD\phi(x))], |D\phi(x)| - 1\} \leq 0, \quad x \in \mathbb{R}^n.$$

By (4.13), we still have $\mu^\phi(A) \leq \lambda(A)$. Thus,

$$\lambda_-(A) = \sup \mu^\phi(A) \leq \lambda(A).$$

If there is $C^2(\mathbb{R}^n)$ subsolution u of equation (4.4) with eigenvalue $\lambda(A)$, then

$$\lambda(A) \leq \mu^u(A) \leq \lambda_-(A).$$

($\lambda_+ \equiv \lambda$) Again fix $A \in \mathcal{S}(n)$. Now let $\psi \in C^2$ satisfy $\liminf_{|x| \rightarrow \infty} \psi(x)/|x| \geq 1$ and set

$$\tau^\psi(A) := \sup_{|D\psi(x)| < 1} \operatorname{tr} [A + AD^2\psi(x)A + (x + AD\psi(x)) \otimes (x + AD\psi(x))].$$

If $\tau^\psi(A) = +\infty$, then $\tau^\psi(A) \geq \lambda(A)$; if $\tau^\psi(A) < +\infty$, by the assumptions on ψ and the definition of $\tau^\psi(A)$

$$\max\{\tau^\psi(A) - \operatorname{tr} [A + AD^2\psi(x)A + (x + AD\psi(x)) \otimes (x + AD\psi(x))], |D\psi(x)| - 1\} \geq 0, \quad x \in \mathbb{R}^n.$$

By (4.14), we still have $\tau^\psi(A) \geq \lambda(A)$. Hence,

$$\lambda_+(A) = \inf \tau^\psi(A) \geq \lambda(A).$$

If there is $C^2(\mathbb{R}^n)$ supersolution u of equation (4.4) with eigenvalue $\lambda(A)$ satisfying (4.17), then

$$\lambda(A) \geq \tau^u(A) \geq \lambda_+(A).$$

□

4.2.2 Monotone upper and lower bounds

In this subsection, we establish the following proposition.

Proposition 4.2.3. *There are monotone non-decreasing functions $\underline{\lambda}, \bar{\lambda} : \mathcal{S}(n) \rightarrow \mathbb{R}$ such that*

$$\underline{\lambda}(A) \leq \lambda(A) \leq \bar{\lambda}(A), \quad A \in \mathcal{S}(n).$$

A much sharper result has already been established in dimension $n = 1$ by G. Barles and H. Soner [2]. Specifically, they showed that the eigenvalue problem associated to the ODE

$$\max\{\lambda - (A + A^2u'' + (x + Au')^2), |u'| - 1\} = 0, \quad x \in \mathbb{R} \quad (4.18)$$

has a unique solution $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ that is *monotone increasing* and continuous. Moreover, associated to $\lambda(A)$ is a solution $u = u(\cdot ; A) \in C(\mathbb{R})$ that satisfies

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = 1.$$

Furthermore, when $A \neq 0$, $u(\cdot ; A) \in C^2(\mathbb{R})$. We will need the following variant of this result for our purposes.

Lemma 4.2.4. (*solution of the 1D eigenvalue problem*)

(i) For each $A \in \mathbb{R}$ and $\alpha > 0$, there is a unique $\lambda = \lambda_1(A, \alpha) \in \mathbb{R}$ such that the ODE

$$\max\{\lambda - (A + A^2u'' + (x + Au')^2), |u'| - \alpha\} = 0, \quad x \in \mathbb{R} \quad (4.19)$$

has a solution $u = u_1(\cdot ; A, \alpha) \in C(\mathbb{R})$ satisfying

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|} = \alpha. \quad (4.20)$$

When $A \neq 0$, $u_1(\cdot ; A, \alpha) \in C^2(\mathbb{R})$.

(ii) The function $A \mapsto \lambda_1(A, \alpha)$ is continuous and monotone non-decreasing for each $\alpha > 0$.

Proof. Let $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be the solution of the eigenvalue problem associated to the ODE (4.18), with solution $u = u(\cdot ; A)$ for each $A \in \mathbb{R}$ as described above. It is easy to check that

$$u_1(x; A, \alpha) := u(\alpha x; \alpha^2 A), \quad x \in \mathbb{R}$$

is a solution of (4.19) with eigenvalue

$$\lambda_1(A, \alpha) := \frac{\lambda(\alpha^2 A)}{\alpha^2}$$

that satisfies (4.20). The uniqueness of λ_1 follows from the same ideas used to prove Proposition (4.1.3). \square

We shall use λ_1 to design $\underline{\lambda}$ and $\bar{\lambda}$ in Proposition (4.2.3). Our main tool will be the comparison principle, and in particular, formulae (4.13) and (4.14). First, however, we will perform a change of variables and rewrite (4.4). For a given $A \in \mathcal{S}(n)$, we may write

$$A = PAP^t$$

where $P^t P = I_n$ and

$$\Lambda = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}. \quad (4.21)$$

Note that

$$\begin{aligned} \operatorname{tr} A &= \sum_{i=1}^n a_i, \\ \operatorname{tr} AD^2 u A &= \operatorname{tr}[P \Lambda P^t D^2 u P \Lambda P^t] \\ &= \operatorname{tr}[\Lambda P^t D^2 u P \Lambda] \\ &= \operatorname{tr}[\Lambda^2 P^t D^2 u P] \\ &= \sum_{i=1}^n a_i^2 (P^t D^2 u P) e_i \cdot e_i \\ &= \sum_{i=1}^n a_i^2 D^2 u P e_i \cdot P e_i \\ &= \sum_{i=1}^n a_i^2 u_{P e_i, P e_i}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr}(x + ADu)(x + ADu) &= |x + ADu|^2 \\ &= |PP^t x + P \Lambda P^t Du|^2 \\ &= |P^t x + \Lambda P^t Du|^2 \\ &= \sum_{i=1}^n (x \cdot P e_i + a_i u_{P e_i})^2. \end{aligned}$$

Making the change of (independent) variables

$$y = P^t x, \quad x = P y$$

and the change of (dependent) variables

$$v(y) = u(Py), \quad u(x) = v(P^t x), \quad x, y \in \mathbb{R}^n,$$

we have that if

$$\max\{\lambda - \operatorname{tr}[A + AD^2 u + (x + ADu) \otimes (x + ADu)], |Du| - 1\} = 0, \quad x \in \mathbb{R}^n$$

then

$$\max\left\{\lambda - \sum_{i=1}^n (a_i + a_i^2 v_{y_i y_i} + (y_i + a_i v_{y_i})^2), |Dv| - 1\right\} = 0, \quad y \in \mathbb{R}^n.$$

The above PDE is closely related to the equation

$$\max_{1 \leq i \leq n} \left\{ \lambda - \sum_{i=1}^n (a_i + a_i^2 v_{y_i y_i} + (y_i + a_i v_{y_i})^2), |v_{y_i}| - 1 \right\} = 0, \quad y \in \mathbb{R}^n,$$

which has the *separation of variables* solution

$$\lambda = \sum_{i=1}^n \lambda_1(a_i, 1) \quad \text{and} \quad v(y) = \sum_{i=1}^n u_1(y_i; a_i, 1), \quad y \in \mathbb{R}^n.$$

These computations motivate the following lemma.

Lemma 4.2.5. *Let $A \in \mathcal{S}(n)$ and assume that $A = P\Lambda P^t$ where $P^t P = I_n$ and Λ is given by (4.21).*

(i) *Set*

$$\underline{\lambda} := \sum_{i=1}^n \lambda_1(a_i, 1/\sqrt{n}) \quad \text{and} \quad \underline{v}(y) := \sum_{i=1}^n u_1(y_i; a_i, 1/\sqrt{n})$$

where λ_1 and u_1 are the solutions of (4.19) as described in Lemma 4.2.4. Then $\underline{u} : x \mapsto \underline{v}(P^t x)$ is a subsolution of (4.4) with eigenvalue $\underline{\lambda}$.

(ii) *Set*

$$\bar{\lambda} := \sum_{i=1}^n \lambda_1(a_i, \sqrt{n}) \quad \text{and} \quad \bar{v}(y) := \sum_{i=1}^n u_1(y_i; a_i, \sqrt{n})$$

where λ_1 and u_1 are the solutions of (4.19) as described in Lemma 4.2.4. Then $\bar{u} : x \mapsto \bar{v}(P^t x)$ is a supersolution of (4.4) with eigenvalue $\bar{\lambda}$ satisfying

$$\liminf_{|x| \rightarrow \infty} \frac{\bar{u}(x)}{|x|} \geq 1. \quad (4.22)$$

Proof. We prove the case where $\det A \neq 0$, so that $u_1(\cdot; a_i, \cdot) \in C^2(\mathbb{R})$ for $i = 1, \dots, n$. The general case follows analogously.

(i) By assumption, \underline{v} is a solution of the equation

$$\max_{1 \leq i \leq n} \left\{ \lambda - \sum_{i=1}^n (a_i + a_i^2 v_{y_i y_i} + (y_i + a_i v_{y_i})^2), |v_{y_i}| - \frac{1}{\sqrt{n}} \right\} = 0, \quad y \in \mathbb{R}^n.$$

As

$$\text{tr} [A + AD^2 \underline{u} + (x + AD\underline{u}) \otimes (x + AD\underline{u})] = \sum_{i=1}^n (a_i + a_i^2 \underline{v}_{y_i y_i} + (y_i + a_i \underline{v}_{y_i})^2),$$

and

$$\begin{aligned}
|D\underline{u}(x)| &= |PD\underline{v}(P^t x)| \\
&= |D\underline{v}(P^t x)| \\
&\leq \sqrt{n} \max_{1 \leq i \leq n} |\underline{v}_{y_i}(P^t x)| \\
&\leq \sqrt{n} \frac{1}{\sqrt{n}} \\
&= 1,
\end{aligned}$$

we have that

$$\max\{\underline{\lambda} - \text{tr}[A + AD^2\underline{u} + (x + AD\underline{u}) \otimes (x + AD\underline{u})], |D\underline{u}| - 1\} \leq 0, \quad x \in \mathbb{R}^n.$$

Thus \underline{u} is a subsolution of (4.4) with eigenvalue $\underline{\lambda}$.

(ii) By assumption, \bar{v} is a solution of the equation

$$\max_{1 \leq i \leq n} \left\{ \lambda - \sum_{i=1}^n (a_i + a_i^2 v_{y_i y_i} + (y_i + a_i v_{y_i})^2), |v_{y_i}| - \sqrt{n} \right\} = 0, \quad y \in \mathbb{R}^n.$$

Notice that

$$\text{tr}[A + AD^2\bar{u} + (x + AD\bar{u}) \otimes (x + AD\bar{u})] = \sum_{i=1}^n (a_i + a_i^2 \bar{v}_{y_i y_i} + (y_i + a_i \bar{v}_{y_i})^2)$$

and

$$\begin{aligned}
|D\bar{u}(x)| &= |PD\bar{v}(P^t x)| \\
&= |D\bar{v}(P^t x)| \\
&\geq \frac{1}{\sqrt{n}} \max_{1 \leq i \leq n} |\bar{v}_{y_i}(P^t x)|,
\end{aligned}$$

which of course implies

$$\sqrt{n} (|D\bar{u}(x)| - 1) \geq \max_{1 \leq i \leq n} |\bar{v}_{y_i}(P^t x)| - \sqrt{n}.$$

It follows that

$$\max\{\bar{\lambda} - \text{tr}[A + AD^2\bar{u} + (x + AD\bar{u}) \otimes (x + AD\bar{u})], |D\bar{u}| - 1\} \geq 0, \quad x \in \mathbb{R}^n.$$

Since,

$$\frac{\bar{u}(x)}{|x|} = \frac{\bar{v}(y)}{|y|} \geq \frac{\sum_{i=1}^n u_1(y_i; a_i)}{\sqrt{n} \max_{1 \leq i \leq n} |y_i|},$$

where $y = P^t x$, and

$$\lim_{|t| \rightarrow \infty} \frac{u_1(t; a_i, \sqrt{n})}{|t|} = \sqrt{n},$$

we have

$$\liminf_{|x| \rightarrow \infty} \frac{\bar{u}(x)}{|x|} \geq 1.$$

Hence, \bar{u} is a supersolution of (4.4) with eigenvalue $\bar{\lambda}$ that satisfies (4.22). \square

Corollary 4.2.6. *Set*

$$\underline{\lambda}(A) := \text{tr} \lambda_1(A, 1/\sqrt{n})$$

and

$$\bar{\lambda}(A) := \text{tr} \lambda_1(A, \sqrt{n}).$$

for $A \in \mathcal{S}(n)$. Then

$$\underline{\lambda} \leq \lambda \leq \bar{\lambda}.$$

Proof. For $A = P\Lambda P^t$ where $P^t P = I_n$ and Λ is given by (4.21), we have

$$\text{tr} \lambda_1(A, 1/\sqrt{n}) = \sum_{i=1}^n \lambda_1(a_i, 1/\sqrt{n})$$

and

$$\text{tr} \lambda_1(A, \sqrt{n}) = \sum_{i=1}^n \lambda_1(a_i, \sqrt{n}).$$

Therefore, this corollary follows directly from the above lemma and formulae (4.13) and (4.14). \square

Finally, Proposition 4.2.3 is established by the following proposition which is proved in Appendix A.

Proposition 4.2.7. *Let $f \in C(\mathbb{R})$ be monotone non-decreasing. Then $\mathcal{S}(n) \ni A \mapsto \text{tr} f(A)$ is monotone non-decreasing with respect to the partial ordering on $\mathcal{S}(n)$.*

4.3 Passing to the limit

This short section is dedicated to the proof of Theorem 4.0.2 which we restate below.

Theorem 4.3.1. *Assume the following technical conditions:*

(i) There is a function $\psi \in C([0, T] \times \overline{O})$ such that for each closed interval $I' \subset (0, T)$ and each compact $O' \subset O$

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{\substack{(t,x) \in I' \times O' \\ \sqrt{\epsilon}|y| \leq \eta}} |z^\epsilon(t, x, y) - \psi(t, x)| = o(1), \quad \text{as } \eta \rightarrow 0^+,$$

(ii) λ is continuous and monotone non-decreasing, and

(iii) when $\det A \neq 0$, there is a convex solution $u \in C^2(\mathbb{R}^n)$ of (4.4) with eigenvalue $\lambda(A)$ that satisfies (4.5),

Then ψ is the unique viscosity solution of (4.3).

We adapt the perturbed test function method of L. C. Evans [9] and some aspects of the convergence proof given in the work of G. Barles and H. Soner [2]. We will make use of the following technical lemma, which is proved in Appendix B.

Lemma 4.3.2. Assume $m \geq 1$, $U \subset \mathbb{R}^m$ is open, and for each $0 < \epsilon \leq 1$, $u^\epsilon \in C(U \times \mathbb{R}^n)$. Moreover, suppose that for each compact $U' \subset U$

$$\lim_{|y| \rightarrow \infty} \frac{u^\epsilon(x, y)}{|y|} = -1$$

uniformly in $0 < \epsilon \leq 1$ and in $x \in U'$ and that

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{\substack{x \in U' \\ \sqrt{\epsilon}|y| \leq \eta}} |u^\epsilon(x, y)| = o(1)$$

as $\eta \rightarrow 0^+$. Then there is a sequence of positive numbers ϵ_k tending to 0, as $k \rightarrow \infty$, such that

$$U \ni x \mapsto \sup_{y \in \mathbb{R}^n} u^{\epsilon_k}(x, y)$$

converges to 0, as $k \rightarrow \infty$, locally uniformly in U .

Proof. (of Theorem 4.0.2) 1. First assume that $\psi - \phi$ has a local maximum at some point $(t_0, x_0) \in (0, T) \times O$. We must show

$$\phi_t(t_0, x_0) - \lambda(D^2\phi(t_0, x_0)) \leq 0. \quad (4.23)$$

By adding $x \mapsto \frac{\rho}{2}|x - x_0|^2$ to ϕ and later sending $\rho \rightarrow 0^+$, we may assume that (t_0, x_0) is a strict local maximum point for $\psi - \phi$ and that

$$\det D^2\phi(t_0, x_0) \neq 0.$$

We fix $\delta > 0$ and set

$$\begin{cases} A^\delta(t, x) := (1 + \delta)^2 D^2 \phi(t, x) \\ A_0 := A^\delta(t_0, x_0) \\ x^{\epsilon, \delta}(t, x, y) := (1 + \delta) \frac{D\phi(t, x) + y}{\sqrt{\epsilon}} \\ \phi^{\epsilon, \delta}(t, x, y) := \phi(t, x) + \epsilon u(x^{\epsilon, \delta}(t, x, y); A_0) \end{cases}$$

for $(t, x, y) \in (0, T) \times O \times \mathbb{R}^n$. We are assuming that u is a convex, $C^2(\mathbb{R}^n)$ solution of (4.4) with eigenvalue $\lambda(A_0)$ that satisfies (4.5).

2. It is easily verified that

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{\substack{(t, x) \in (0, T) \times O \\ \sqrt{\epsilon}|y| \leq \eta}} |\phi^{\epsilon, \delta}(t, x, y) - \phi(t, x)| \leq (1 + \delta)\eta$$

and

$$\lim_{|y| \rightarrow \infty} \frac{\phi^{\epsilon, \delta}(t, x, y) - \phi(t, x)}{\sqrt{\epsilon}|y|} = 1 + \delta$$

uniformly in $\epsilon \in (0, 1]$ and in $(t, x) \in [0, T] \times O$. Therefore, the hypotheses of Lemma 4.3.2 are satisfied with

$$\begin{cases} m := n + 1 \\ U := (0, T) \times O \\ u^\epsilon(t, x, y) := \frac{1}{\delta} \{ (z^\epsilon - \phi^{\epsilon, \delta})(t, x, y) - (\psi - \phi)(t, x) \}, \quad (t, x, y) \in U \times \mathbb{R}^n \end{cases}$$

Consequently, there is a sequence $\epsilon \rightarrow 0^+$ tending to 0 such that

$$\sup_{y \in \mathbb{R}^n} (z^\epsilon - \phi^{\epsilon, \delta}) \rightarrow \psi - \phi$$

locally uniformly on $(0, T) \times O$. As $\psi - \phi$ has a *strict* local maximum at (t_0, x_0) , there is a sequence of $\epsilon_k > 0$ tending to 0, as $k \rightarrow \infty$, and a sequence $(t_k, x_k) \in (0, T) \times O$ such that

$$\begin{cases} (t_k, x_k) \rightarrow (t_0, x_0), \text{ as } k \rightarrow \infty \\ \sup_{y \in \mathbb{R}^n} (z^{\epsilon_k} - \phi^{\epsilon_k, \delta}) \text{ has a local maximum at } (t_k, x_k), k \in \mathbb{N} \end{cases}$$

Moreover,

$$\lim_{|y| \rightarrow \infty} \frac{(z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t_k, x_k, y)}{\sqrt{\epsilon_k}|y|} = -\delta,$$

which implies there is a sequence $(y_k)_{k \geq 1} \subset \mathbb{R}^n$ such that

$$y \mapsto (z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t_k, x_k, y)$$

has a maximum at y_k for each $k \in \mathbb{N}$. It is readily verified that (t_k, x_k, y_k) is point of local maximum for $z^{\epsilon_k} - \phi^{\epsilon_k, \delta}$, for $k \in \mathbb{N}$.

3. Since, z^ϵ is a subsolution of the eikonal equation

$$|D_y z| = \sqrt{\epsilon}$$

and $\phi^{\epsilon_k, \delta} \in C^2((0, T) \times O \times \mathbb{R}^n)$, we have at the point (t_k, x_k, y_k)

$$|D_y \phi^{\epsilon_k, \delta}| \leq \sqrt{\epsilon_k}.$$

Thus

$$\begin{aligned} |Du(x^{\epsilon_k, \delta})| &< |(1 + \delta)Du(x^{\epsilon_k, \delta})| \\ &= \frac{1}{\sqrt{\epsilon_k}} |\sqrt{\epsilon_k}(1 + \delta)Du(x^{\epsilon_k, \delta})| \\ &= \frac{1}{\sqrt{\epsilon_k}} |D_y \phi^{\epsilon_k, \delta}| \\ &\leq 1. \end{aligned}$$

Hence,

$$\begin{cases} |x^{\epsilon_k, \delta}| \leq C \\ \lambda(A_0) - \text{tr}[A_0 + A_0 D^2 u(x^{\epsilon_k, \delta}) A_0 + (x^{\epsilon_k, \delta} + A_0 Du(x^{\epsilon_k, \delta})) \otimes (x^{\epsilon_k, \delta} + A_0 Du(x^{\epsilon_k, \delta}))] = 0 \end{cases}$$

for all $k \geq 1$. Computing as we did in subsection 4.0.2 we arrive at

$$\begin{aligned} 0 &\geq \phi_t^{\epsilon_k, \delta} - \left(\Delta_x \phi^{\epsilon_k, \delta} + \frac{1}{\epsilon_k} |D_x \phi^{\epsilon_k, \delta} + y_k|^2 \right) \\ &\geq \phi_t(t_k, x_k) + o(1) \\ &\quad - \frac{1}{(1 + \delta)^2} \text{tr}[A_0 + A_0 D^2 u(x^{\epsilon_k, \delta}) A_0 + (x^{\epsilon_k, \delta} + A_0 Du(x^{\epsilon_k, \delta})) \otimes (x^{\epsilon_k, \delta} + A_0 Du(x^{\epsilon_k, \delta}))] \\ &= \phi_t(t_0, x_0) - \frac{1}{(1 + \delta)^2} \lambda(A_0) + o(1) \\ &= \phi_t(t_0, x_0) - \frac{1}{(1 + \delta)^2} \lambda((1 + \delta)^2 D^2 \phi(t_0, x_0)) + o(1) \end{aligned}$$

as $k \rightarrow \infty$. Therefore, we let $k \rightarrow \infty$ and then $\delta \rightarrow 0^+$ to conclude (4.23).

4. Now assume that $\psi - \phi$ has a local minimum at some point $(t_0, x_0) \in (0, T) \times O$. We must show

$$\phi_t(t_0, x_0) - \lambda(D^2 \phi(t_0, x_0)) \geq 0. \quad (4.24)$$

By subtracting $x \mapsto \frac{\rho}{2}|x - x_0|^2$ from ϕ and later sending $\rho \rightarrow 0^+$, we may assume that (t_0, x_0) is a *strict* local minimum point for $\psi - \phi$ and that

$$\det D^2\phi(t_0, x_0) \neq 0.$$

We fix $\delta \in (0, 1)$ and set

$$\begin{cases} A^\delta(t, x) := (1 - \delta)^2 D^2\phi(t, x) \\ A_0 := A^\delta(t_0, x_0) \\ x^{\epsilon, \delta}(t, x, y) := (1 - \delta) \frac{D\phi(t, x) + y}{\sqrt{\epsilon}} \\ \phi^{\epsilon, \delta}(t, x, y) := \phi(t, x) + \epsilon u(x^{\epsilon, \delta}(t, x, y); A_0) \end{cases}$$

for $(t, x, y) \in (0, T) \times O \times \mathbb{R}^n$. We are assuming that u is a convex, $C^2(\mathbb{R}^n)$ solution of (4.4) with eigenvalue $\lambda(A_0)$ that satisfies (4.5).

5. It is easily verified that

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{\substack{(t, x) \in (0, T) \times O \\ \sqrt{\epsilon}|y| \leq \eta}} |\phi^{\epsilon, \delta}(t, x, y) - \phi(t, x)| \leq (1 - \delta)\eta$$

and

$$\lim_{|y| \rightarrow \infty} \frac{\phi^{\epsilon, \delta}(t, x, y) - \phi(t, x)}{\sqrt{\epsilon}|y|} = 1 - \delta$$

uniformly in $\epsilon \in (0, 1]$ and in $(t, x) \in [0, T] \times O$. Therefore, the hypotheses of Lemma 4.3.2 are satisfied with

$$\begin{cases} m := n + 1 \\ U := (0, T) \times O \\ u^\epsilon(t, x, y) := \frac{1}{\delta} \{ -(z^\epsilon - \phi^{\epsilon, \delta})(t, x, y) + (\psi - \phi)(t, x) \}, \quad (t, x, y) \in U \times \mathbb{R}^n \end{cases}.$$

Consequently, there is a sequence of $\epsilon \rightarrow 0^+$ such that

$$\inf_{y \in \mathbb{R}^n} (z^\epsilon - \phi^{\epsilon, \delta}) \rightarrow \psi - \phi$$

locally uniformly on $(0, T) \times O$. As $\psi - \phi$ has a *strict* local minimum at (t_0, x_0) , there is a sequence of positive numbers ϵ_k tending to 0, as $k \rightarrow \infty$, and a sequence $(t_k, x_k) \in (0, T) \times O$ such that

$$\begin{cases} (t_k, x_k) \rightarrow (t_0, x_0), \text{ as } k \rightarrow \infty \\ \inf_{y \in \mathbb{R}^n} (z^{\epsilon_k} - \phi^{\epsilon_k, \delta}) \text{ has a local minimum at } (t_k, x_k), \text{ } k \in \mathbb{N} \end{cases}.$$

Moreover,

$$\lim_{|y| \rightarrow \infty} \frac{(z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t_k, x_k, y)}{\sqrt{\epsilon_k}|y|} = +\delta,$$

which implies there is a sequence $(y_k)_{k \geq 1} \subset \mathbb{R}^n$ such that

$$y \mapsto (z^{\epsilon_k} - \phi^{\epsilon_k, \delta})(t_k, x_k, y)$$

has a minimum at y_k for each $k \in \mathbb{N}$. It is readily verified that (t_k, x_k, y_k) is point of local minimum for $z^{\epsilon_k} - \phi^{\epsilon_k, \delta}$, for $k \in \mathbb{N}$.

6. As $u \in C^2(\mathbb{R}^n)$, we have at the point (t_k, x_k, y_k)

$$\begin{aligned} |D_y \phi^{\epsilon_k, \delta}| &= |(1 - \delta) \sqrt{\epsilon_k} Du(x^{\epsilon_k, \delta})| \\ &\leq (1 - \delta) \sqrt{\epsilon_k} \\ &< \sqrt{\epsilon_k}. \end{aligned}$$

Since z^{ϵ_k} is a viscosity solution of (4.1) and $\phi^{\epsilon_k, \delta} \in C^2((0, T) \times O \times \mathbb{R}^n)$, we compute as in subsection 4.0.2 to get

$$\begin{aligned} 0 &\leq \phi_t^{\epsilon_k, \delta} - \left(\Delta_x \phi^{\epsilon_k, \delta} + \frac{1}{\epsilon_k} |D_x \phi^{\epsilon_k, \delta} + y_k|^2 \right) \\ &\leq \phi_t(t_k, x_k) + o(1) \\ &\quad - \frac{1}{(1 - \delta)^2} \text{tr}[A_0 + A_0 D^2 u(x^{\epsilon_k, \delta}) A_0 + (x^{\epsilon_k, \delta} + A_0 Du(x^{\epsilon_k, \delta})) \otimes (x^{\epsilon_k, \delta} + A_0 Du(x^{\epsilon_k, \delta}))] \\ &\leq \phi_t(t_k, x_k) - \frac{1}{(1 - \delta)^2} \lambda(A_0) + o(1) \\ &\leq \phi_t(t_0, x_0) - \frac{1}{(1 - \delta)^2} \lambda(A_0) + o(1) \\ &= \phi_t(t_0, x_0) - \frac{1}{(1 - \delta)^2} \lambda((1 - \delta)^2 D^2 \phi(t_0, x_0)) + o(1) \end{aligned}$$

as $k \rightarrow \infty$. We conclude (4.24) by first letting $k \rightarrow \infty$ and then $\delta \rightarrow 0$.

7. Uniqueness of solutions of the PDE (4.3) follow from our assumptions that λ is continuous and monotone (elliptic). See section 8 of the reference [6] for a proof of this fact. \square

4.4 Financial application

We close this chapter by presenting a multi-asset version of the Davis, Panas, Zariphopoulou (DPZ) option pricing model. In this setting, we consider the analog of the limit discovered by Barles and Soner. We show that the analogous limit again amounts to understanding a limit of a sequence of solutions of a non-linear PDE very similar to the one considered in this paper. However, new technical issues arise and our methods seem to be just short of handling these issues. Nevertheless, we formulate a very reasonable conjecture.

4.4.1 Option pricing model

We consider a Brownian motion based financial market consisting of n stocks and a money market account (a “bond”) with interest rate $r \geq 0$. The stock is modeled as a stochastic process satisfying the SDE

$$dP^i(s) = \sum_{j=1}^n \sigma_{ij} P^i(s) dW^j(s), \quad s \geq 0, \quad i = 1, \dots, n$$

where $(W(t), t \geq 0)$ is a standard n -dimensional Brownian motion and σ is a non-singular $n \times n$ matrix. We assume each participant in the market assumes a trading strategy which is simply a way of purchasing and selling shares of stock and the money market account. Furthermore, in this model we assume that participants pay transaction costs that are proportional to the amount of the underlying stock; the proportionality constant we use is $\sqrt{\epsilon}$.

On a time interval $[t, T]$, a trading strategy will be modeled by a pair of vector processes $(L, M) = ((L^1, \dots, L^n), (M^1, \dots, M^n))$. Here $L^i(s)$ represents the cumulative purchases of the i th stock and $M^i(s)$ represents the cumulative sales of the i th stock at time $s \in [t, T]$; we assume L^i, M^i are non-decreasing processes, adapted to the filtration generated by W , that satisfy $L^i(t) = M^i(t) = 0$ for $i = 1, \dots, n$. Associated to a given trading strategy (L, M) is a process X , the amount of dollars held in the money market, and processes Y^i , the number of shares of the i th stock held, for $i = 1, \dots, n$. These processes are modeled by the SDE

$$\begin{cases} dX(s) = rX(s)ds + \sum_{i=1}^n (-(1 + \sqrt{\epsilon})P^i(s)dL^i(s) + (1 - \sqrt{\epsilon})P^i(s)dM^i(s)) \\ dY^i(s) = dL^i(s) - dM^i(s) \quad i = 1, \dots, n \end{cases} \quad t \leq s \leq T.$$

We assume that for a given amount of wealth $w \in \mathbb{R}$, a seller of a European option with maturity T and payoff $g(P(T)) \geq 0$ has the *utility*

$$U_\epsilon(w) = 1 - e^{-w/\epsilon}$$

and in particular has constant *risk aversion*

$$\frac{-U_\epsilon''(w)}{U_\epsilon'(w)} = \frac{1}{\epsilon}.$$

If the seller does not sell the option, his expected utility from final wealth is

$$v^{\epsilon, f}(t, x, y, p) = \sup_{L, M} \mathbb{E} U_\epsilon(X(T) + Y(T) \cdot P(T)).^1$$

¹Here, and below, we are assuming that $X(t) = x, Y(t) = y$ and $P(t) = p$.

If he does sell the option, he will have to payout $g(P(T))$ at time T , so in this case his expected utility from final wealth is

$$v^\epsilon(t, x, y, p) = \sup_{L, M} \mathbb{E}U_\epsilon(X(T) + Y(T) \cdot P(T) - g(P(T))).$$

Note that since U_ϵ is monotone increasing $v^\epsilon \leq v^{\epsilon, f}$. We define the seller's price Λ_ϵ as the amount which offsets this difference (and makes the seller "indifferent" to selling the option or not)

$$v^\epsilon(t, x + \Lambda_\epsilon, y, p) = v^{\epsilon, f}(t, x, y, p).$$

As in the single asset case [8], we expect the following proposition

Proposition 4.4.1. *Suppose that $v = v^\epsilon, v^{\epsilon, f}$ satisfy the dynamic programming principle: for each W stopping time $\tau \in [t, T]$*

$$v(t, x, y, p) = \sup_{L, M} \mathbb{E}v(\tau, X(\tau), Y(\tau), P(\tau)),$$

for $t \in (0, T), x \in \mathbb{R}, y \in \mathbb{R}^n, p \in (0, \infty)^n$. Then $v^\epsilon, v^{\epsilon, f}$ are viscosity solutions of the PDE

$$\max_{1 \leq i \leq n} \left\{ v_{y_i} - (1 + \sqrt{\epsilon})p_i v_x, -v_{y_i} + (1 - \sqrt{\epsilon})p_i v_x, v_t + \frac{1}{2}d(p)\sigma\sigma^t d(p) \cdot D_p^2 v + rp \cdot D_p v + rxv_x \right\} = 0, \quad (4.25)$$

in the region $t \in (0, T), x \in \mathbb{R}, y \in \mathbb{R}^n, p \in (0, \infty)^n$ with

$$v^\epsilon(T, x, y, p) = 1 - \exp(-(x + y \cdot p - g(p))/\epsilon) \quad \text{and} \quad v^{\epsilon, f}(T, x, y, p) = 1 - \exp(-(x + y \cdot p)/\epsilon).$$

Important reductions: (a) To simplify the presentation, we set $r = 0$. However, this is done without any loss of generality as the function

$$\tilde{v}(t, x, y, p) := v(t, e^{-r(T-t)}x, y, e^{-r(T-t)}p)$$

satisfies the PDE (4.25) with $r = 0$, provided of course that v is a solution of (4.25). Moreover, $\tilde{v}(T, x, y, p) = v(T, x, y, p)$.

(b) The main virtue of working with the exponential utility function is that the value functions typically depend on the x variable in a simple way. Notice that (upon setting $r = 0$)

$$X^{t,x}(s) = x + \int_t^s \left\{ -(1 + \sqrt{\epsilon})P(s) \cdot dL(s) + (1 - \sqrt{\epsilon})P(s) \cdot dM(s) \right\}, \quad t \leq s \leq T$$

and so $v = v^\epsilon, v^{\epsilon, f}$ satisfy

$$v(t, x, y, p) = 1 + e^{-x/\epsilon}(v(t, 0, y, p) - 1). \quad (4.26)$$

This is convenient as it reduces the variable dependence of solutions of (4.25).

4.4.2 The large risk aversion, small transaction cost limit

In order to use the PDE methods to study Λ_ϵ , we define $z^\epsilon, z^{\epsilon,f}$ implicitly via

$$v^\epsilon = U_\epsilon(x + y \cdot p - z^\epsilon) = 1 - \exp(-(x + y \cdot p - z^\epsilon)/\epsilon)$$

and

$$v^{\epsilon,f} = U_\epsilon(x + y \cdot p - z^{\epsilon,f}) = 1 - \exp(-(x + y \cdot p - z^{\epsilon,f})/\epsilon).$$

In particular, notice that

$$\Lambda_\epsilon = z^\epsilon - z^{\epsilon,f}.$$

Moreover, using (4.26), it is straightforward to check that

$$z = z^\epsilon, z^{\epsilon,f} \text{ are independent of } x.$$

Another consequence of this change of variable is that as $v^\epsilon, v^{\epsilon,f}$ are expected to be viscosity solutions of (4.25), $z^\epsilon, z^{\epsilon,f}$ are expected to be viscosity solutions of

$$\max_{1 \leq i \leq n} \left\{ -z_t - \frac{1}{2} d(p) \sigma \sigma^t d(p) \cdot \left(D_p^2 z + \frac{1}{\epsilon} (D_p z - y) \otimes (D_p z - y) \right), |z_{y_i}| - \sqrt{\epsilon} p_i \right\} = 0 \quad (4.27)$$

for $t < T, p \in \mathbb{R}_+^n, y \in \mathbb{R}^n$ with

$$z^\epsilon(T, p, y) = g(p), \quad \text{and} \quad z^{\epsilon,f}(T, p, y) = 0.$$

Note the resemblance of equation (4.27) with (4.1), the PDE we focused on in this chapter.

A formal application of the maximum principle has

$$0 \leq z^{\epsilon,f}(t, y, p) \leq \sqrt{\epsilon} \sum_{i=1}^n p_i |y_i|$$

for all $(t, y, p) \in (0, T) \times \mathbb{R}^n \times (0, \infty)^n$. Therefore, we expect that the option price is approximately

$$\Lambda_\epsilon = z^\epsilon - z^{\epsilon,f} \approx z^\epsilon$$

as $\epsilon \rightarrow 0^+$.

Performing heuristic computations as we did for solutions of PDE (4.1), we find

$$z^\epsilon(t, p, y) \approx \psi(t, p) + \epsilon u \left(d(p) \frac{D_p \psi(t, p) - y}{\sqrt{\epsilon}} \right), \text{ as } \epsilon \rightarrow 0^+$$

where

$$d(p) = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_n \end{pmatrix}$$

and ψ is a solution of the *non-linear Black-Scholes equation* (when $r = 0$)

$$\psi_t + \lambda (d(p)D^2\psi d(p)) = 0.^2 \quad (4.28)$$

What's more is that the non-linearity and function u in the error term for z^ϵ together solve the following *corrector problem*: for each $A \in \mathcal{S}(n)$, find $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfy the PDE

$$\max_{1 \leq i \leq n} \left\{ \lambda - \frac{1}{2} \text{tr} [\sigma \sigma^t (A + AD^2uA + (x + ADu) \otimes (x + ADu))] , |u_{x_i}| - 1 \right\} = 0, \quad x \in \mathbb{R}^n. \quad (4.29)$$

Using the methods we presented in this paper, it can be shown that for each $A \in \mathcal{S}(n)$, there is a unique $\lambda(A)$ such that there is a continuous viscosity solution u of the (4.29) satisfying

$$\lim_{|x| \rightarrow \infty} \frac{u(x)}{\sum_{i=1}^n |x_i|} = 1.$$

This defines the function that appears in (4.4) and will be crucial to rigorously establish the limit described above.

The methods we have employed must be augmented and strengthened to establish the analog of the limit established by G. Barles and H. Soner. Two technical issues that now arise are that 1) the PDE (4.27) has a unbounded domain for the p variable, so appropriate growth conditions must be required of solutions; and 2) the function determining the gradient constraint

$$H(p) = \max_{1 \leq i \leq n} |p_i| - 1, \quad p \in \mathbb{R}^n$$

is not uniformly convex (recall the assumptions of Theorem 2.0.1), so we will expect to have less regularity of solutions of equation (4.29). We hope to resolve these issues in future research and prove the following conjecture.

Conjecture 4.4.2. *Solutions z^ϵ of (4.1) converge locally uniformly to a viscosity solution of the PDE (4.28) satisfying the terminal condition $\psi(T, p) = g(p)$, $p \in (0, \infty)^n$, as $\epsilon \rightarrow 0^+$.*

²When $r \neq 0$, $\psi_t + e^{-r(T-t)}\lambda (e^{r(T-t)}d(p)D_p^2\psi d(p)) + rp \cdot D_p\psi - r\psi = 0$.

Chapter 5

Concluding remarks

In this dissertation, we considered three problems involving PDE of the form (1.1)

$$\max \{F(D^2u, Du, u, x), H(Du)\} = 0$$

where F is elliptic and H satisfies a monotonicity condition. Regarding the first problem, we showed the Dirichlet problem associated with the PDE

$$\max\{Lu - h(x), H(Du)\} = 0$$

is well-posed, where L is a linear, uniformly elliptic operator. Moreover, if H is *uniformly convex*, the solution of the Dirichlet problem belongs to $C_{\text{loc}}^{1,1}$.

For the second problem, we proved that there is unique $\lambda^* \in \mathbb{R}$ such that the PDE

$$\max \{ \lambda - \Delta u - h(x), |Du| - 1 \} = 0, \quad x \in \mathbb{R}^n$$

has a solution u satisfying

$$\lim_{|x| \rightarrow +\infty} \frac{u(x)}{|x|} = 1.$$

Moreover, we showed that associated to λ^* is a convex solution u^* belonging to $C^{1,1}(\mathbb{R}^n)$.

In the third problem, we established under technical assumptions that appropriate limits (as ϵ tends to 0) of solutions z^ϵ of the PDE

$$\max \left\{ z_t - \left(\Delta_x z + \frac{1}{\epsilon} |D_x z + y|^2 \right), |D_y z| - \sqrt{\epsilon} \right\} = 0$$

must satisfy a non-linear, parabolic equation of the form

$$\psi_t = \lambda (D^2 \psi).$$

Moreover, we find that the non-linearity $\lambda : \mathcal{S}(n) \rightarrow \mathbb{R}$ is the solution of an eigenvalue problem associated with the PDE

$$\max \{ \lambda - \text{tr} (A + AD^2uA + (x + ADu) \otimes (x + ADu)), |Du| - 1 \} = 0$$

where $A \in \mathcal{S}(n)$.

These PDE arose naturally in mathematical models involving the optimal control of singular stochastic processes. We regard this work as progress in rigorously interpreting the PDE that arise in these applications. However, many fundamental questions remain unanswered about these equations and in general about the PDE in the class (1.1). We close by discussing some interesting, yet largely unexplored research directions.

Consider the Dirichlet problem associated to equation (1.1):

$$\begin{cases} \max \{F(D^2u, Du, u, x), H(Du)\} = 0, & x \in O \\ u = g, & x \in \partial O \end{cases}$$

where $g \in C(\overline{O})$, F is fully non-linear and elliptic, and H may satisfy (1.4). What are the appropriate structural conditions on g , F , and H so that the above equation has a unique (viscosity) solution? In particular, is there an analog for Theorem 2.0.1 in the case where F is uniformly elliptic and H is uniformly convex? In general, we are asking: how do solutions of fully non-linear, elliptic PDE behave when a pointwise gradient constraint is imposed?

Next, we inquire about the local geometry of the *free boundary* associated to (1.1). This is defined to be the boundary of the set

$$\Omega = \{x \in \Omega : H(Du(x)) < 0\}.$$

We ask: what are the local regularity properties of $\partial\Omega$? This question is motivated by results for the PDE

$$\max\{Lv - h(x), v - g(x)\} = 0.$$

Here is known that if h and g are sufficiently regular functions, the boundary of the set

$$\Omega' = \{x \in \Omega : v(x) < g(x)\}$$

is locally the graph of a smooth function (at most points) [4]. A regularity result about $\partial\Omega$ may have modeling implications, as well, as optimal controls for singular control problems can be constructed via reflected diffusions, provide $\partial\Omega$ is smooth enough [20] [21].

As a final remark, we mention that it would also be of great interest to design a general problem involving asymptotic analysis of PDE to which our results on Chapter 3 on eigenvalue problems can be applied. We presented a very involved example of this in Chapter 4, and we now seek a simple yet interesting model problem for such phenomena. The motivation for this question is from the theory of periodic homogenization, where the fundamental cell problem is a non-linear eigenvalue problem and is used to understand non-trivial limits of solutions of PDE [9].

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Appendix A

Monotonicity of $A \mapsto \operatorname{tr} f(A)$

In this appendix, we prove Proposition 4.2.7, which we restate for the reader's convenience.

Proposition A.0.3. *Let $f \in C(\mathbb{R})$ be monotone non-decreasing. Then $\mathcal{S}(n) \ni A \mapsto \operatorname{tr} f(A)$ is monotone non-decreasing with respect to the partial ordering on $\mathcal{S}(n)$.*

We first prove a version of this proposition for smooth functions.

Proposition A.0.4. *Let $f \in C^\infty(\mathbb{R})$ and $N \in \mathcal{S}(n)$. For each $A \in \mathcal{S}(n)$, we have*

$$D\operatorname{tr}[f(A)]N := \lim_{t \rightarrow 0} \frac{\operatorname{tr} f(A + tN) - \operatorname{tr} f(A)}{t} = \operatorname{tr}[f'(A)N]. \quad (\text{A.1})$$

Proof. $A = O\Lambda O^t$, where $O^t O = I_n$ and Λ is diagonal; we shall also write $A = (a_{ij})_{1 \leq i, j \leq n}$ and $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. With this notation,

$$\operatorname{tr} f(A) = \sum_{i=1}^n f(\lambda_i).$$

Now $\lambda_k = AOe_k \cdot Oe_k$ and $A = \sum_{i,j=1}^n a_{ij}e_i e_j^t$. Therefore

$$\begin{aligned} \frac{\partial \lambda_k}{\partial a_{ij}} &= e_i e_j^t Oe_k \cdot Oe_k + A \frac{\partial}{\partial a_{ij}} Oe_k \cdot Oe_k + AOe_k \cdot \frac{\partial}{\partial a_{ij}} Oe_k \\ &= e_j^t Oe_k \cdot e_i^t Oe_k + 2AOe_k \cdot \frac{\partial}{\partial a_{ij}} Oe_k \\ &= O_{ik} O_{jk} + 2\lambda_k Oe_k \cdot \frac{\partial}{\partial a_{ij}} Oe_k \\ &= O_{ik} O_{jk} + \lambda_k \frac{\partial}{\partial a_{ij}} |Oe_k|^2 \\ &= O_{ik} O_{jk}, \end{aligned}$$

since $O^t O = I_n$.

Let $N \in \mathcal{S}(n)$. We have from the definition of $D\text{tr}[f(A)]N$ in (A.1), the chain rule, and the above computations that

$$\begin{aligned}
D\text{tr}[f(A)]N &= \sum_{i,j=1}^n \frac{\partial}{\partial a_{ij}} \text{tr} f(A) N_{ij} \\
&= \sum_{i,j=1}^n \sum_{k=1}^n f'(\lambda_k) \frac{\partial \lambda_k}{\partial a_{ij}} N_{ij} \\
&= \sum_{k=1}^n f'(\lambda_k) \sum_{i,j=1}^n O_{ik} O_{jk} N_{ij} \\
&= \sum_{k=1}^n f'(\lambda_k) (O^t N O)_{kk} \\
&= \text{tr}[f'(A)N].
\end{aligned}$$

□

Lemma A.0.5. *Let $f \in C^\infty(\mathbb{R})$ be non-decreasing. Then $\mathcal{S}(n) \ni A \mapsto \text{tr} f(A)$ is non-decreasing.*

Proof. Let $A, B \in \mathcal{S}(n)$ with $B \geq A$. From the above proposition,

$$\begin{aligned}
\text{tr} f(B) - \text{tr} f(A) &= \int_0^1 \frac{d}{dt} \text{tr} f(A + t(B - A)) dt \\
&= \int_0^1 D\text{tr} f(A + t(B - A))(B - A) dt \\
&= \int_0^1 \text{tr}[f'(A + t(B - A))(B - A)] dt \\
&\geq 0,
\end{aligned}$$

as the matrix $f'(A + t(B - A)) \geq 0$ for all $t \in [0, 1]$ and $B - A \geq 0$. □

Proof. (Proposition (4.2.7)) Let f^ϵ denote the standard mollifier of f and suppose $A, B \in \mathcal{S}(n)$ with $B \geq A$. By the above lemma, $\text{tr} f^\epsilon(B) \geq \text{tr} f^\epsilon(A)$ for all $\epsilon > 0$ as f^ϵ is non-decreasing. Letting $\epsilon \rightarrow 0^+$ implies $\text{tr} f(B) \geq \text{tr} f(A)$. □

Appendix B

Technical convergence lemma

Here, we verify Lemma 4.3.2, which we restate for the reader's convenience.

Lemma B.0.6. *Assume $m \geq 1$, $U \subset \mathbb{R}^m$ is open, and for each $0 < \epsilon \leq 1$, $u^\epsilon \in C(U \times \mathbb{R}^n)$. Moreover, suppose that for each compact $U' \subset U$*

$$\lim_{|y| \rightarrow \infty} \frac{u^\epsilon(x, y)}{|y|} = -1 \quad (\text{B.1})$$

uniformly in $0 < \epsilon \leq 1$ and in $x \in U'$ and that

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{\substack{x \in U' \\ \sqrt{\epsilon}|y| \leq \eta}} |u^\epsilon(x, y)| = o(1) \quad (\text{B.2})$$

as $\eta \rightarrow 0^+$. Then there is a sequence of positive numbers ϵ_k tending to 0, as $k \rightarrow \infty$, such that

$$U \ni x \mapsto \sup_{y \in \mathbb{R}^n} u^{\epsilon_k}(x, y)$$

converges to 0, as $k \rightarrow \infty$, locally uniformly in U .

Proof. 1. Fix a compact $U' \subset U$. We first aim to show there is a sequence of $\epsilon \rightarrow 0^+$ such that

$$\overline{\lim}_{\epsilon \rightarrow 0^+} \max_{x \in U'} \left| \sup_{y \in \mathbb{R}^n} u^\epsilon(x, y) \right| = 0. \quad (\text{B.3})$$

Note that (B.1) implies

$$\lim_{|y| \rightarrow \infty} u^\epsilon(x, y) = -\infty$$

for each $\epsilon \in (0, 1]$ and $x \in U'$, and (B.2) implies

$$u^\epsilon \rightarrow 0 \text{ locally uniformly in } U' \times \mathbb{R}^n.$$

In particular, for each $\epsilon \in (0, 1]$ and $x \in U'$ there is $y^{\epsilon, x}$ such that

$$\max_{y \in \mathbb{R}^n} u^\epsilon(x, y) = u^\epsilon(x, y^{\epsilon, x}).$$

As

$$x \mapsto \max_{y \in \mathbb{R}^n} u^\epsilon(x, y)$$

is continuous and U' is compact, there is $x^\epsilon \in U'$ such that

$$\max_{x \in U'} \left| \max_{y \in \mathbb{R}^n} u^\epsilon(x, y) \right| = |u^\epsilon(x^\epsilon, y^{\epsilon, x^\epsilon})|$$

for $0 < \epsilon \leq 1$.

2. The sequence $(\sqrt{\epsilon}y^{\epsilon, x^\epsilon}, 0 < \epsilon \leq 1)$ must be bounded for all $\epsilon > 0$ and small enough. Otherwise, there exists a sequence $(\sqrt{\epsilon_k}y_k)_{k \in \mathbb{N}}$ such that

$$\begin{cases} x_k = x^{\epsilon_k}, k \in \mathbb{N} \\ y_k := y^{\epsilon_k, x_k}, k \in \mathbb{N} \\ \lim_{k \rightarrow \infty} \epsilon_k = 0 \\ \lim_{k \rightarrow \infty} \sqrt{\epsilon_k} |y_k| = \infty \end{cases}.$$

Necessarily $\lim_{k \rightarrow \infty} |y_k| = \infty$, and by assumption (B.1)

$$\lim_{k \rightarrow \infty} \frac{u^{\epsilon_k}(x_k, y_k)}{\sqrt{\epsilon_k} |y_k|} = -1. \quad (\text{B.4})$$

It follows that

$$\lim_{k \rightarrow \infty} u^{\epsilon_k}(x_k, y_k) = -\infty;$$

however, this contradicts

$$\underline{\lim}_{k \rightarrow \infty} u^{\epsilon_k}(x_k, y_k) \geq \underline{\lim}_{k \rightarrow \infty} u^{\epsilon_k}(x_k, 0) = 0, \quad (\text{B.5})$$

which holds by local uniform convergence.

3. Let $(\sqrt{\epsilon_k}y_k)_{k \in \mathbb{N}}$ be a convergent subsequence of $(\sqrt{\epsilon}y^{\epsilon, x^\epsilon}, 0 < \epsilon \leq 1)$ and set

$$\eta_k := \sqrt{\epsilon_k} |y_k|, \quad k \in N.$$

Without loss of generality, we assume the sequence $(\eta_k)_{k \in \mathbb{N}}$ converges to some limit η_0 . If $\eta_0 = 0$, then we conclude (B.3) by noting

$$\begin{aligned} \overline{\lim}_{k \rightarrow \infty} \max_{x \in U'} \left| \max_{y \in \mathbb{R}^n} u^{\epsilon_k}(x, y) \right| &= \overline{\lim}_{k \rightarrow \infty} \max_{x \in U'} \left| \max_{\sqrt{\epsilon_k} y \leq \eta_k} u^{\epsilon_k}(x, y) \right| \\ &\leq \overline{\lim}_{k \rightarrow \infty} \max_{\substack{x \in U' \\ \sqrt{\epsilon_k} y \leq \eta_k}} |u^{\epsilon_k}(x, y)| \\ &= 0. \end{aligned}$$

The last equality above follows from assumption (B.2). If $\eta_0 > 0$, $|y_k|$ must tends to $+\infty$. Again we would have (B.4), which implies

$$\overline{\lim}_{k \rightarrow \infty} u^{\epsilon_k}(x_k, y_k) \leq 0;$$

As (B.5) still holds, we have

$$\lim_{k \rightarrow \infty} u^{\epsilon_k}(x_k, y_k) = 0,$$

and in particular,

$$\lim_{k \rightarrow \infty} \max_{x \in U'} \left| \max_{y \in \mathbb{R}^n} u^{\epsilon_k}(x, y) \right| = \lim_{k \rightarrow \infty} |u^{\epsilon_k}(x_k, y_k)| = 0.$$

Therefore, we have established (B.3).

4. Since U' was an arbitrary compact subset of the open set U , we can employ a standard diagonalization argument (see the proof of Theorem 2.0.1) to conclude that there is a sequence $\epsilon_k \rightarrow 0$, as $k \rightarrow \infty$, such that

$$U \ni x \mapsto \sup_{y \in \mathbb{R}^n} u^{\epsilon_k}(x, y)$$

converges to 0 locally uniformly on U , as $k \rightarrow \infty$. □