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A Rearrangement Inequality for Diffusion Processes

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The dissertation of Teng Gao is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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DEDICATION

To my parents.

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ABSTRACT OF THE DISSERTATION

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Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on $[0,1)$, reflected at 0 and absorbed at 1. Let $x_i = i/n$, for $i = 0, 1, \cdots, n-1$, and $l_{T_1}^{x_i}$ be the local time of the process at x_i up to $T_1 = \inf\{t \geq 0 : \quad B_t = 1\}$. Given a positive sequence $\{\lambda_1, \cdots, \lambda_n\}$, $\{\lambda_1^*, \cdots, \lambda_n^*\}$ is its non-decreasing rearrangement. The main result of this thesis is the following local time rearrangement inequality:

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i l_{T_1}^{x_i}\right\}\right] \leq \mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i^* l_{T_1}^{x_i}\right\}\right].$$

Such an inequality holds true for a more general diffusion process $\{X_t\}_{t\geq 0}$ satisfying

$$dX_t = \sigma(X_t)dB_t,$$

where $\sigma(x) \geq \epsilon > 0$ for all $x \in [0,1]$.

Chapter 1

Introduction

We begin with a problem that motivates our main result. $\{B_t\}_{t\geq 0}$ is a Brownian motion traveling on a dangerous "alley way", denoted by the interval $[0, 1)$. Assuming the process has survived until time t, there is a small chance

$$k(B_t)dt + o(dt)$$

that the process will be annihilated in the time interval $(t, t+dt)$. Here, $k(x)$ is thought of as the risk associated with the location x, so it is assumed to be positive and Lebesgue integrable on the interval $[0, 1]$. For a traveler who starts at $x \in [0, 1)$, the probability of surviving-until-escape is

$$\mathbb{E}^x\left[\exp\left\{-\int_0^{T_1} k(B_t)dt\right\}\right], \tag{1.1}$$

where $T_1 = \inf\{t : B_t = 1\}$ is the usual hitting time of the process at level one.

If the process starts at 0, since it spend more time near the starting point, intuition will suggest that the likelihood of survival will increase if the dangers in the alley are arranged so as to be more concentrated near the exit. More precisely, let k^* denote the unique increasing right-continuous function from $[0, 1] \to [0, \infty)$ such that

$$\text{meas}\{\text{x} \in [0, 1) : \quad \text{k}^*(\text{x}) > \lambda\} = \text{meas}\{\text{x} \in [0, 1) : \quad \text{k}(\text{x}) > \lambda\}$$

for all $\lambda > 0$. (Here "meas" refers to Lebesgue measure.) We then have the following statement:

Theorem 1.

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(B_t)dt\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(B_t)dt\right\}\right]. \tag{1.2}$$

By an approximation argument, we can reduce the problem to the case in which k is continuous. It begins with the local time process $\{l_t^x : \quad 0 \le x < 1\}$ which tracks how much time the Brownian motion spends near a point $x \in [0, 1)$. There is the integral formula

$$\int_0^{T_1} k(B_t)dt = \int_0^1 k(x) l_{T_1}^x dx. \tag{1.3}$$

Because k is continuous, the integral on the right side of (1.3) can be approximated by Riemann sums

$$\frac{1}{n}\sum_{i=0}^{n-1} k(x_i) l_{T_1}^{x_i},$$

in which $x_i = i/n$. To prove (1.2), it therefore suffices to show that

$$\mathbb{E}^0\left[\exp\left(-\sum_{i=0}^{n-1}\lambda_i\xi_i\right)\right] \le \mathbb{E}^0\left[\exp\left(-\sum_{i=0}^{n-1}\lambda_i^*\xi_i\right)\right] \tag{1.4}$$

for any sequence $\{\lambda_0, \lambda_2, \cdots, \lambda_{n-1}\}$ of strictly positive constants and its non-decreasing rearrangement $\{\lambda_0^*, \lambda_2^*, \cdots, \lambda_{n-1}^*\}$. (Here we identify ξ_i with $l_{T_1}^{x_i}$.)

There are two results that are related to (1.2). Alexander R. Pruss in 1997 [10] proved an analogous rearrangement inequality in the discrete random walk setting (The detailed statement of Pruss' result can be found in the appendix). By the invariance principle, the scaled reflecting random walk will converge in distribution to a Brownian motion, so Pruss' result is naturally connected to ours.

By the Feynman-Kac equation, setting

$$\varphi(x) = \mathbb{E}^x\left[\exp\left\{-\int_0^{T_1} k(B_t)dt\right\}\right],$$

where k is a piece-wise constant, non-negative function, we have $\varphi(x)$ satisfies the boundary value problem

$$\frac{1}{2}\varphi'' = k\varphi, \quad \varphi'(0) = 0, \quad \text{and} \quad \varphi(1) = 1. \tag{1.5}$$

Then, (1.2) will imply

$$\varphi(0) \le \varphi^*(0),$$

where φ^* is the function obtained from replacing k in (1.5) by its non-decreasing rearrangement k^*.

A similar rearrangement result was first proved by M. Essén in 1975 [4]. It is worth pointing out that although (1.2) can be deduced from the results of Essén, they do not imply (1.4). However, (1.4) does imply Essén's result.

The tool that allows us to prove (1.4) is a result derived from a paper of J. Rosen and M. Marcus [9] to show

$$\mathbb{E}^0\left[\exp\left(-\sum_{i=0}^{n-1}\lambda_i\xi_i\right)\right] = \frac{1}{\det(I+\Sigma\Lambda)},$$

where Λ is the n by n diagonal matrix with

$$\Lambda_{i,j} = \lambda_i\delta_{i,j},$$

and Σ is the n by n matrix with the entries

$$\Sigma_{i,j} = G(x_i, x_j),$$

where G is the Green function for the process.

Thus the proof of our main result reduces to proving the following determinant inequality:

$$\det(I+\Sigma\Lambda) \geq \det(I+\Sigma\Lambda^*),$$

where

$$\Lambda^*_{i,j} = \lambda^*_i\delta_{i,j}.$$

Our result can be extended to a bigger class of diffusion process $\{X_t\}_{t\geq 0}$ with some restriction on its infinitesimal parameters σ and μ , where

$$\mu(x) = \lim_{h\downarrow 0}\mathbb{E}^x[X(h) - X(0)],$$

and

$$\sigma^2(x) = \lim_{h\downarrow 0}\mathbb{E}^x[(X(h) - X(0))^2].$$

The outline of this thesis is as follow. In chapter 2, we will go over the set-up. We will give the definition of local time, and list some relevant results used in our proof.

We will state the result of M. Marcus and J. Rosen, and give a detailed calculation of the Green function with boundary conditions appropriate for our case.

The proof of our main result is given in Chapter 3. In Chapter 4, we will derive the identity

$$\begin{aligned}&\mathbb{E}^x\left[\exp\left\{-\int_0^{T_1}(k1_{[a,b)}(B_s)+k1_{[b,c)}(B_s))ds\right\}\right]\\&=\mathbb{E}^x\left[\exp\left\{-\int_0^{T_b}k1_{[a,b)}(B_s)ds\right\}\right]\cdot\mathbb{E}^b\left[\exp\left\{-\varphi'(b)l^b_{T_1}-\int_0^{T_1}k1_{[b,c)}(B_s)ds\right\}\right],\end{aligned}$$

where

$$\varphi(x)=\mathbb{E}^x\left[\exp\left\{-\int_0^{T_1}k1_{[a,b)}(B_s)ds\right\}\right],$$

for $x\in[a,b)$ and $a<b<c$. We will use this idenity to give an alternative proof of (1.2) that is based on Essén's method. Some interesting probabalistic conclusions can be drawn from such an approach.

In chapter 5, we give a proof that extends Essén's result to the class of postive integrable functions, and we show that our main result will hold for Brownian motion with constant drift.

In chapter 6, we will give an application of the main result. By the method of time change, we contruct a birth-death process on the state space $\{0,1,\cdots,N\}$, reflected at 0, and absorbed at N, with equal birth and death rate on each states except the end points. Applying our main result gives a holding rate rearrangement inequality for such a process.

Chapter 2

Marcus and Rosen Identity

Let $\{X_t\}_{t\geq 0}$ be a diffusion process on the interval $[0, 1)$ with a reflecting boundary at 0, and an absorbing boundary at 1, satisfying the stochastic differential equation:

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt, \tag{2.1}$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion.

To ensure the existence of a weak solution, here and in what follows, we assume the continuity of $\mu(x)$ and $\sigma^2(x)$. In addition, $\sigma^2(x) \geq \epsilon > 0$ for all $x \in [0, 1]$.

We have

$$P^x(T_1 < \infty) = 1, \qquad \forall x \in [0, 1).$$

We will show in this chapter that the diffusion process $\{X_t\}_{t\geq 0}$ satisfies the identity:

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i\xi_i\right\}\right] = \frac{1}{\det(I + \Sigma\Lambda)}, \quad \text{where} \quad \xi_i = l^{x_i}_{T_1}. \tag{2.2}$$

Here, Σ is a matrix whose entries are given by the Green function of the process as $\Sigma_{i,j} = G(x_i, x_j)$, Λ is a diagonal matrix whose entries is $\Sigma_{i,j} = \lambda_i\delta_{i,j}$, and $l^{x_i}_t$ is the local time of the process at $x_i = i/n$ up to time t.

Additionally, we will show that given two process $\{X^{(1)}_t\}_{t\geq 0}$ and $\{X^{(2)}_t\}_{t\geq 0}$, such that their corresponding infinitesimal generators satisfying the inequality

$$\frac{\mu_1(x)}{\sigma^2_1(x)} \geq \frac{\mu_2(x)}{\sigma^2_2(x)}, \quad \text{for all} \quad x \in [0, 1],$$

then

$$\mathbb{E}_1^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i l_{T_1}^{x_i}\right\}\right] \geq \mathbb{E}_2^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i l_{T_1}^{x_i}\right\}\right].$$

We will begin this section with a brief discussion on local time and Green functions.

2.1 Local Time

In this section, we will review the definition of local time and give a list of results that will be used in the ensuing sections. The main goal is to show that for a positive continuous function k, and an equal space partition of the interval $[0, 1]$, we have

$$\int_0^{T_1} k(X_s)ds = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} k(x_i)l_{T_1}^{x_i} m(x_i) \qquad \text{a.s.} \tag{2.3}$$

where $x_i = i/n$.

Here, m is the speed measure, and it is defined as

$$m(dy) = \frac{2dy}{\sigma^2(y)s(y)}, \tag{2.4}$$

where s is the scale density, defined as

$$s(y) = \exp\left\{-\int_0^y 2\mu(\xi)/\sigma^2(\xi)d\xi\right\}. \tag{2.5}$$

Additionally, $l_t^{x_i}$ denotes the Markov local time at the point x_i up to time t.

It is important to point out that there is a difference between semi-martingale local time, and Markov local time. Here, and in what follows, we will use L (resp. l) for semi-martingale local time (resp. Markov local time). We will introduce the definition of semi-martingale local time first. More detail can be found in D. Revuz and M. Yor [11].

Given a continuous semi-martingale $\{Y_t\}_{t\geq 0}$, the local time process at a point a is defined to be positive increasing process appearing as the remainder term when expanding $|Y_t - a|$ by the Tanaka's formula. The precise statement is as follows:

Theorem 2. *For each real number a, there exists an increasing continuous adapted process L^a, called the local time of Y in a, such that*

$$|Y_t - a| = |Y_0 - a| + \int_0^t \operatorname{sgn}(Y_s - a)dY_s + L_t^a, \tag{2.6}$$

Notice that because $L^a_\cdot$ is an increasing process, we can associate to it a random measure dL^a_t on $\mathbb{R}_+$. It can be shown that this random measure is singular to Lebesgue's measure.

Proposition 1. *The measure dL^a_t is a.s. carried by the set $\{t : Y_t = a\}$*

Proof. First, we apply Itô's formula to the semimartingale $|Y_t - a|$, and get

$$(Y_t - a)^2 = (Y_0 - a)^2 + 2\int_0^t |Y_s - a| d(|Y - a|)_s + \langle |Y|, |Y| \rangle_t$$

and using Tanaka's formula, we can expand the term further

$$(Y_0 - a)^2 + \int_0^t |Y_s - a| \text{sgn}(Y_s - a) dY_s + 2\int_0^t |Y_s - a| dL^a_s + \langle Y, Y \rangle_t$$

However, by applying Itô's formula to $(Y_t - a)^2$ we get

$$(Y_t - a)^2 = (Y_0 - a)^2 + 2\int_0^t |Y_s - a| dY_s + \langle Y, Y \rangle_t$$

Thus, we reach the conclusion that

$$\int_0^t |Y_s - a| dL^a_s = 0$$

□

In fact, Tanaka's formula is part of a more general result:

Theorem 3. ***(Itô-Tanaka formula)*** *If f is the difference of two convex functions and if Y is a continuous semimartingale*

$$f(Y_t) = f(Y_0) + \int_0^t f'_-(Y_s) dY_s + \frac{1}{2}\int_{\mathbb{R}} L^a_t f''(da)$$

If we let f be a positive and twice differentiable function, comparing Itô-Tanaka formula and Itô's formula, gives us that

$$\int_0^t f(Y_s) \langle Y, Y \rangle_s = \int_{-\infty}^{\infty} f(a) L^a_t da \tag{2.7}$$

By monotone class argument, we can conclude the above result will hold for all positive Borel measurable f.

Heuristically, the local time L^a_t can be thought of as the amount of time the process Y spends around the point a up to time t. The next result illustrates this point.

Proposition 2.

$$L_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[a,a+\varepsilon)}(Y_s) d\langle Y, Y \rangle_s \quad a.s. \tag{2.8}$$

Remark 1. *L_t^a in equation (2.8) refers to a semi-martingale local time. Markov local time, denoted l_t^a, is defined as,*

$$l_t^a = \lim_{\varepsilon \downarrow 0} \frac{1}{m([a, a+\varepsilon))} \int_0^t 1_{[a,a+\varepsilon)}(Y_s) ds. \tag{2.9}$$

In the next section, we will show that $\mathbb{E}^x[l^y_{T_1}] = G(x, y)$, for the Green function G. Comparing equation (2.8) and (2.9), we have the relationship between the two as,

$$l_t^a = \frac{L_t^a}{\sigma^2(a) m(a)}. \tag{2.10}$$

The last result we list here points out that the local time L_t^a is cadlag with respect to the spacial variable, so it enables us to approximate (2.3) using Riemann sum.

Theorem 4. *For any continuous semimartingale $\{Y_t\}_{t \geq 0}$, there exists a modification of the process $\{L_t^a, a \in \mathbb{R}, t \in \mathbb{R}_+\}$ such that the map $(a, t) \to L_t^a$ is a.s. continuous in t and cadlag in a. Moreover, if $Y = M + V$, then*

$$L_t^a - L_t^{a-} = 2 \int_0^t 1_{\{Y_s = a\}} dV_s = 2 \int_0^t 1_{\{Y_s = a\}} dY_s.$$

Thus, in particular, if $\{Y_t\}_{t \geq 0}$ is a local martingale, there is a bicontinuous modification of the family L^a of local times.

Remark 2. *For a diffusion process $\{X_t\}_{t \geq 0}$ with continuous drift $\mu(x)$ and variance parameter $\sigma^2(x)$, the occupation time formula (2.7) implies,*

$$L_t^a - L_t^{a-} = 2 \int_0^t 1_{\{X_s = a\}} \mu(X_s) ds = 2 \int_0^1 1_{\{x = a\}} \frac{\mu(x)}{\sigma^2(x)} L_t^x dx = 0$$

The last equality is a consequence of the fact that $L_t^x dx$ is absolutely continuous with respect to Lebesgue measure. Thus, for the process we consider, L_t^a is jointly continuous in (t, a). From equation (2.10), we also have the joint continuity of l_t^a.

2.2 Green Function

In this section, we will show that with the appropriate normalization of the Green function G, we have

$$\mathbb{E}^x[l^y_{T_1}] = G(x, y).$$

We will derive the above equation with the corresponding boundary condition that is suitable for our purpose.

For the diffusion process $\{X_t\}_{t\geq 0}$ with drift $\mu(x)$ and variance parameter $\sigma^2(x) > 0$, its infinitesimal generator is

$$\mathcal{L} = \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}.$$

In the next chapter, we will show that for a positive integrable function f, the function φ defined as:

$$\varphi(x) = \mathbb{E}^x\left[\exp\left\{-\lambda\int_0^{T_1} f(X_s)ds\right\}\right], \tag{2.11}$$

satisfies

$$\mathcal{L}\varphi = \lambda f\varphi. \tag{2.12}$$

Differentiating both sides of (2.11) with respect to λ, due to the fact that

$$\left|\exp\left\{-\lambda\int_0^{T_1} f(X_s)ds\right\}\right| \leq 1,$$

and

$$\left|\int_0^{T_1} f(X_s)ds\exp\left\{-\lambda\int_0^{T_1} f(X_s)ds\right\}\right| \leq \int_0^{T_1} f(X_s)ds < \infty.$$

we can switch the order of $\frac{\partial}{\partial\lambda}$ and the expectation $\mathbb{E}^x$, obtaining

$$\frac{\partial\varphi}{\partial\lambda} = \mathbb{E}^x\left[-\int_0^{T_1} f(X_s)ds\exp\left\{-\lambda\int_0^{T_1} f(X_s)ds\right\}\right].$$

$$\frac{\partial}{\partial\lambda}\mathcal{L}\varphi = \mathcal{L}\frac{\partial\varphi}{\partial\lambda} = f\varphi + \lambda f\frac{\partial\varphi}{\partial\lambda}. \tag{2.13}$$

Let $\lambda \to 0$, with the help of dominated convergence theorem,

$$\varphi(\lambda) \to 1, \qquad \text{and} \quad \frac{\partial\varphi}{\partial\lambda} \to -\mathbb{E}^x\left[\int_0^{T_1} f(X_s)ds\right].$$

If we define w as:

$$w(x) = \mathbb{E}^x\left[\int_0^{T_1} f(X_s)ds\right], \qquad \text{for} \quad x \in (0,1), \tag{2.14}$$

equation (2.13) becomes

$$\mathcal{L}w = -f. \tag{2.15}$$

The solution to such an ordinary differential equation is well-known, and is given as

$$w(x) = \int_0^1 G(x,y)f(y)m(dy), \tag{2.16}$$

where G is the Green function.

Combining (2.14) and (2.16) we get:

$$\mathbb{E}^x\left[\int_0^{T_1} f(X_s)ds\right] = \int_0^1 G(x,y)f(y)m(dy).$$

By Proposition 2, if we set $f = \frac{1}{m[y,y+\epsilon)}1_{[y,y+\epsilon)}$ and let $\epsilon \downarrow 0$, we get the desired identity:

$$\mathbb{E}^x[l^y_{T_1}] = G(x,y).$$

Now we will construct the Green function. More detail can be found in Itô and McKean [7].

Here, we denote the scale derivative g^+ as:

$$g^+(y) = \lim_{y \downarrow x} \frac{g(y) - g(x)}{S(y) - S(x)}, \qquad S(x) = \int_0^x s(\eta)d\eta.$$

Let g_1 (resp. g_2) be the increasing (resp. decreasing) solution to

$$\mathcal{L}g = 0, \tag{2.17}$$

with

$$g_1'(0) = 0 \quad \text{and} \quad g_2(1) = 0.$$

Both g_1 and g_2 are uniquely determined up to a positive constant. Moreover, the Wronskian $W = g_1^+ g_2 - g_2^+ g_1$ is a constant, so g_1 and g_2 are linearly independent. The Green function is:

$$G(x,y) = G(y,x) = g_1(x)g_2(y)/W, \qquad x \le y. \tag{2.18}$$

By the strong Markov property and the terminal time property, for $x \leq y$, we can write: $T_1 = T_y + T_1 \circ \theta_{T_y}$, where $\theta_\cdot$ is the shift operator, and have:

$$G(x,y) = \mathbb{E}^x[l^y_{T_1}] = \mathbb{E}^x[l^y_{T_y} + l^y_{T_1 \circ \theta_{T_y}}] = \mathbb{E}^y[l^y_{T_1}] = G(y,y). \tag{2.19}$$

Equations (2.19) and (2.18) tells us that the g_1 is a constant, and without loss of generality, we set $g_1 = 1$.

Moreover, from the fact that $G(1,1) = \mathbb{E}^1[l^1_{T_1}] = 0$, consistent with the right boundary condition

$$g_2(1) = 0.$$

A simple calculation will show that we can take

$$g_2(x) = \int_x^1 s(\xi)d\xi \quad \text{and} \quad s(\xi) = \exp\{-\int_0^\xi 2\mu(\eta)/\sigma^2(\eta)d\eta\}. \tag{2.20}$$

so that the Wronskian is $W = 1$.

Summarizing:

$$G(x,y) = g_2(x \vee y),$$

where g_2 is given by (2.20).

We can now check that $w(x)$ defined as in (2.14), solves the boundary value problem:

$$\begin{cases} \mathcal{L}w = -f \\ w'(0) = 0 \\ w(1) = 0 \end{cases}.$$

First, we rewrite w as:

$$w(x) = g_1(x)\int_x^1 g_2(y)f(y)m(dy) + g_2(x)\int_0^x g_1(y)f(y)m(dy).$$

It is easy to see that $w(1) = g_2(1)\int_0^1 g_1(y)f(y)m(dy) = 0$.

Also,

$$w'(x) = g_2'(x)\int_0^x g_1(y)f(y)m(dy).$$

Clearly, $w'(0) = 0$.

Next,

$$w''(x) = g_2''(x) \int_0^x g_1(y) f(y) m(dy) - 2f(x)/\sigma^2(x).$$

Hence, $\mathcal{L}w = -f$.

Remark 3. *To see that w is the unique solution to the boundary value problem, suppose w_1 and w_2 solve the same boundary value problem. Then, for $g(x) = w_1(x) - w_2(x)$, g satisfies $g'(0) = 0$ and $g(1) = 0$.*

Notice $\mathcal{L}g = 0$ implies $g''(x) = -(2\mu(x)/\sigma^2(x))g'(x)$. Let $h(x) = g'(x)$, then solving the first degree differential equation will give $h(x) = C \cdot \exp\{-\int_0^x -2\mu(\eta)/\sigma^2(\eta)d\eta\}$. $h(0) = g'(0)$ implies $C = 0$. In other words, g must be a constant, and since $g(1) = 0$, g must be identically zero.

Remark 4. *For a Brownian motion on $[0, 1)$, reflected at 0 and absorbed at 1. We have $\mu(x) \equiv 0$ and $\sigma^2(x) \equiv 1$. Therefore, $g_2(x) = 1 - x$, and the Green function is $G(x, y) = 1 - (x \vee y)$.*

2.3 Marcus and Rosen Identity

The main tool we will be using to prove our main result is the identity:

Proposition 3.

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=1}^n \lambda_i \xi_i\right\}\right] = \frac{1}{\det(I + \Sigma\Lambda)}, \tag{2.21}$$

where $\xi_i = l_{T_1}^{x_i}$, for $0 = x_1 < \cdots < x_n \leq 1$. Σ is the $n \times n$ matrix with entries

$$\Sigma_{i,j} = G(x_i, x_j), \quad \textit{and} \quad \Sigma_{i,j} = \lambda_i \delta_{i,j}.$$

Here, $\lambda_1, \cdots, \lambda_n$ are all strictly positive.

To show (2.21), we will use the following result from M. Marcus and J. Rosen [9]:

Lemma 1. ***(Marcos and Rosen)*** *Let X be a Markov process with continuous $0-$potential density $u(x, y)$. Assume that a local time l_t^y exists for each y, normalized so that $\mathbb{E}^x[l_\infty^y] =$*

$u(x,y)$. Let Σ be the matrix with element $\Sigma_{i,j} = u(x,y)$, $i,j = 1,\cdots,n$. Let Λ be the matrix with elements $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$. For all $\lambda_1,\cdots,\lambda_n$ sufficiently small and $1 \le l \le n$,

$$\mathbb{E}^{x_l}\left[\exp\left\{\sum_{i=1}^{n}\lambda_i l_\infty^{x_i}\right\}\right] = \frac{\det(I-\widehat{\Sigma}\Lambda)}{\det(I-\Sigma\Lambda)},$$

$$\textit{where}\quad \widehat{\Sigma}_{j,k} = \Sigma_{j,k} - \Sigma_{l,k}, \qquad j,k = 1,\cdots,n.$$

From previous calculations, we have

$$\Sigma_{i,j} = G(x_i,x_j) = g_2(x_i \vee x_j).$$

For $l < j,k$

$$\begin{aligned}\widehat{\Sigma^1}_{j,k} &= \Sigma^1_{j,k} - \Sigma^1_{l,k}\\ &= g(x_j \vee x_k) - g(x_l \vee x_k)\\ &= \begin{cases} g(x_j) - g(x_k), & \text{if } j > k,\\ 0, & j \le k,\end{cases}\end{aligned}$$

so

$$(I-\widehat{\Sigma}\Lambda)_{i,j} = \begin{cases} g(x_k) - g(x_j), & j > k,\\ 1, & j = k,\\ 0, & j < k.\end{cases}$$

Therefore, we have $\det(I-\widehat{\Sigma}\Lambda) = 1$ in the case where $x_1 = 0$. Setting $\xi_i = l_{T_1}^{x_i}$, we have

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=1}^{n}\lambda_i\xi_i\right\}\right] = \frac{1}{\det(I+\Sigma\Lambda)}, \qquad \text{for } \lambda_i \text{ sufficiently small.}$$

Remark 5. *In the paper of M.Marcus and J. Rosen [9], there is a more general lemma than the one given above. The precise statement is given as follow:*

Lemma 2. *Let X be a Markov process with finite $0-$potential density $u(x,y)$. Assume that a local time l_t^y exists for each y, normalized so that $\mathbb{E}^x[l_\infty^y] = u(x,y)$. Let Θ be the*

matrix with elements $\Theta_{i,j} = u(x_i, x_j)$, $i, j = 1, \cdots, n$. *Let* Λ *be the matrix with elements* $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$. *For all* $\lambda_1, \cdots, \lambda_n$ *sufficiently small and* $1 \le l \le n$,

$$\mathbb{E}^{x_l}\left[\exp\left\{\sum_{i=1}^{n} \lambda_i l_\infty^{x_i}\right\}\right] = \frac{\det((I - \Theta\Lambda)^{(l)})}{\det(I - \Theta\Lambda)}$$

here $A^{(l)}$ *denotes the matrix obtained by replacing the* l*th column of the* $n \times n$ *matrix* A *by a column of* 1.

The following calculation shows that both lemmas lead to the same identity. We set $g_i = g_2(x_i)$.

$$\begin{aligned}
\det((I + \Sigma\Lambda)^{(1)}) &= \det\begin{pmatrix} 1 & g_2\lambda_2 & \cdots & g_n\lambda_n \\ 1 & 1 + g_2\lambda_2 & \cdots & g_n\lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 + g_n\lambda_n \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & g_2\lambda_2 & \cdots & g_n\lambda_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (g_n - g_2)\lambda_2 & \cdots & 1 \end{pmatrix} \\
&= 1.
\end{aligned}$$

The second to third equality is obtained by using row reduction.

Our next task is to remove the restriction on the size of λ_i. The constraint on the λ_i's is due to the fact that both $\mathbb{E}^{x_l}\left[\exp\left\{-\sum_{i=1}^{n} \lambda_i l_\infty^{x_i}\right\}\right]$ and $1/\det(I + \Sigma\Lambda)$ are represented as a formal power series (if the series converges absolutely):

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=1}^{n} \lambda_i l_{T_1}^{x_i}\right\}\right] = \sum_{k=0}^{\infty} \{(-1)^k (\Sigma\Lambda)^k 1^t\}_1 = \frac{1}{\det(I + \Sigma\Lambda)}, \tag{2.22}$$

as functions of $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, wherever such power series is well-defined.

Analytic continuation will enable us to extend the validity of the identity to a larger domain to where both $\mathbb{E}^{x_l}\left[\exp\left\{-\sum_{i=1}^{n} \lambda_i l_\infty^{x_i}\right\}\right]$ and $1/\det(I + \Sigma\Lambda)$ are defined.

Our first step is to find a modest bound on λ_i's, so that the formal power series will converge. For simplicity, we choose the open ball $B(0, q) \in \mathbb{C}^n$, with center at the

origin and radius $q < 1$.

Lemma 3. *Assuming $\mu(x) \geq 0$, for all $x \in [0,1]$, and $n \cdot \lambda_i < q < 1$ for $i = 1$ to n. Then,*

$$\sum_{k=0}^{\infty} \{(\Sigma\Lambda)^k 1^t\}_1 < \infty \tag{2.23}$$

Proof. First, notice that for $a_i \in (0,1)$, have

$$g_i = g(a_i) = \int_{a_i}^{1} \exp\left\{-\int_0^t 2\mu(s)/\sigma^2(s)ds\right\} dt < 1.$$

Next, we claim that

$$\{(\Sigma\Lambda)^k 1^t\}_1 < q^k.$$

We will proceed by induction on k.

For $k = 1$,

$$\{(\Sigma\Lambda)1^t\}_1 = \sum_{i=1}^{n} g_i\lambda_i < \sum_{i=1}^{n} \lambda_i \leq n \cdot (\max_i \lambda_i) < q < 1.$$

and

$$\begin{aligned} \{(\Sigma\Lambda)^{k+1}1^t\}_1 &= \left\{(\Sigma\Lambda)(\Sigma\Lambda)^k 1^t\right\}_1 \\ &< \{q^k(\Sigma\Lambda)1^t\}_1 \\ &= q^{k+1}. \end{aligned}$$

Finally, we have

$$\sum_{k=0}^{\infty} \{(\Sigma\Lambda)^k 1^t\}_1 < \sum_{k=0}^{\infty} q^k < \infty.$$

□

For each i, both $\mathbb{E}^0[\exp\{-\sum_{i=1}^{n} \lambda_i l_{T_1}^{x_i}\}]$ and $1/\det(I + \Sigma\Lambda)$ are functions of λ_i, and they are continuous and differentiable on $\Omega = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : \text{Real}(z_i) > 0, \forall\ i\}$. By Osgood's lemma, both function are holomorphic on Ω (see Appendix C). Both $\mathbb{E}^0[\exp\{-\sum_{i=1}^{n} \lambda_i l_{T_1}^{x_i}\}]$ and $1/\det(I + \Sigma\Lambda)$ can be represented by the formal power series

$\sum_{k=0}^{\infty}\{(\Sigma\Lambda)^k 1^t\}_1$ on the open set $B(0,q)\bigcap\Omega$. Thus, by the Identity Theorem (see Appendix C), the two functions are equal on Ω.

In Pruss' paper, he shows that if the probability of the random walk jumping up is increased, the probability of safe arrival at the end point will also increase. In the context of our set up, we have the following analogous inequality.

Theorem 5. *Let* $\{\lambda_0,\cdots,\lambda_{n-1}\}$ *be all positive numbers, and* $0 = x_0 < \cdots x_{n-1} \leq 1$. *Consider diffusions* $X^{(1)}$ *and* $X^{(2)}$*, such that their corresponding infinitesimal generators satisfying the inequality*

$$\frac{\mu_1(x)}{\sigma_1^2(x)} \geq \frac{\mu_2(x)}{\sigma_2^2(x)}, \quad \textit{for all} \quad x\in[0,1].$$

In this case,

$$\mathbb{E}_1^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i l_{T_1}^{x_i}\right\}\right] \geq \mathbb{E}_2^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i l_{T_1}^{x_i}\right\}\right].$$

Proof. Let $g_2^{(1)}$ and $g_2^{(2)}$ denote the decreasing right boundary solution for $X^{(1)}$ and $X^{(2)}$ discussed before respectively. Recall from our discussion of Green function, that

$$g_2^{(1)}(x) = \int_x^1 \exp\left\{-\int_0^\xi 2\frac{\mu_1(\eta)}{\sigma_1^2(\eta)}d\eta\right\}d\xi \leq g_2^{(2)}(x) = \int_x^1 \exp\left\{-\int_0^\xi 2\frac{\mu_2(\eta)}{\sigma_2^2(\eta)}d\eta\right\}d\xi.$$

Thus, we have

$$\Sigma_{i,j}^{(1)} = g_1^{(1)}(x_i\vee x_j) \leq \Sigma_{i,j}^{(2)} = g_2^{(2)}(x_i\vee x_j),$$

so

$$\det(I+\Sigma^{(1)}\Lambda) \leq \det(I+\Sigma^{(2)}\Lambda),$$

which gives the desire inequality. □

Chapter 2 is based on the paper "A Local Time Inequality for Reflecting Brownian Motion" written jointly with Patrick Fitzsimmons, which is currently in preparation. The dissertation author is the primary author of this work.

Chapter 3

Proof of Main Result

We will focus our effort on reflecting Brownian motion. In what follows, $\{B\}_{t\geq 0}$ denotes a Brownian motion on $[0, 1)$ with reflecting boundary at 0, and an absorbing boundary at 1. The interval $[0, 1)$ is partitioned into n equal space subintervals so that $x_i = \dfrac{i}{n}$ for $i = 0$ to $n-1$. Thus, the entries of the matrix Σ in our set up

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i\xi_i\right\}\right] = \frac{1}{\det(I+\Sigma\Lambda)}, \quad \xi_i = l_{T_1}^{x_i}, \tag{3.1}$$

are

$$\Sigma_{i,j} = g_i \wedge g_j = g(x_i \wedge x_j) = 1 - \frac{i\wedge j}{n}.$$

Let $\{\lambda_0^*, \cdots, \lambda_{n-1}^*\}$ be the increasing rearrangement of the sequence $\{\lambda_0, \cdots, \lambda_{n-1}\}$. With equation (3.1), to show

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i\xi_i\right\}\right] \leq \mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i^*\xi_i\right\}\right], \quad \text{where} \quad \xi_i = l_{T_1}^{x_i},$$

it is suffices to show the corresponding inequality

$$\det(I+\Sigma\Lambda) \geq \det(I+\Sigma\Lambda^*),$$

where

$$\Lambda_{i,j}^* = \lambda_i^*\delta_{i,j}, \quad \text{for} \quad i = 0, \cdots, n-1.$$

Instead of rearranging the order of λ_i's all at once, we will do it in steps. For a

positive sequence of distinct numbers $\{\lambda_0, \cdots, \lambda_{n-1}\}$. We pick two indices s and k by:

$$\begin{aligned} s &:= \max\{j : \quad \lambda_0 < \cdots < \lambda_j < \min_{i \geq j+1} \lambda_i\}, \\ k &:= \{l : \quad \lambda_l = \min_{i \geq s} \lambda_i\}. \end{aligned}$$

We set

$$\hat{\lambda}_j = \begin{cases} \lambda_j, & \text{for} \quad 0 \leq j \leq s, \quad \text{or} \quad j > k, \\ \lambda_{j-1}, & \text{for} \quad s < j \leq k \\ \lambda_k, & \text{for} \quad j = s+1. \end{cases}$$

Clearly, by repeating the above step in which $\{\lambda_i\}$ is replaced by $\{\hat{\lambda}_i\}$, we can rearrange λ_i's into increasing order.

We set $\mathcal{M}(s+1, k) = \det(I+\Sigma\Lambda)$, and $\widehat{\mathcal{M}}(s+1, k) = \det(I+\Sigma\widehat{\Lambda})$, where $\widehat{\Lambda}_{i,j} = \hat{\lambda}_i \delta_{i,j}$. Thus, to get the desired result, it is suffices to show

$$\mathcal{M}(s+1, k) \geq \widehat{\mathcal{M}}(s+1, k).$$

3.1 Preliminary and Notation

To simplify the caculation, we observe that by row reduction

$$\det(I + \Sigma\Lambda) = \det(I + N),$$

where

$$N_{i,j} = \begin{cases} \lambda_j/n, & j \leq i, \\ -1, & j = i+1, \\ 0, & j > i+1. \end{cases}$$

Next, notice that although the position of the entries from λ_{s+1} to λ_k have changed, the position from λ_0 to λ_s, and from λ_{k+1} to λ_{n-1} is the same for both $\mathcal{M}(s+1, k)$ and $\widehat{\mathcal{M}}(s+1, k)$. Thus, it makes sense to break those two determinant down into parts.

To simplify our discussion, we adopt the following notation:

$$\mathbb{A}_l = \begin{vmatrix} 1+\frac{\lambda_0}{n} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_0}{n} & \frac{\lambda_1}{n} & \cdots & 1+\frac{\lambda_l}{n} \end{vmatrix}, \quad \text{and} \quad \mathbb{A}_l^- = \begin{vmatrix} 1+\frac{\lambda_0}{n} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_0}{n} & \frac{\lambda_1}{n} & \cdots & \frac{\lambda_l}{n} \end{vmatrix}.$$

For $l \geq 0$, $\mathbb{A}_l$ is the determinant of the matrix with the indices of the entries starting at 0 and ending at l, and $\mathbb{A}_l^-$ is the determinant of the matrix that is the same as that of $\mathbb{A}_l$ except the entry$1 + \dfrac{\lambda_l}{n}$ in the position of (l, l) replaced by $\dfrac{\lambda_l}{n}$.

By expanding along the last coloumn, we have the relation:

Lemma 4.

$$\mathbb{A}_l = \mathbb{A}_{l-1} + \mathbb{A}_l^-.$$

In the subsequent calculations, we will use this lemma in the case when $l = 1$. In order to keep the notation consistent, we set $\mathbb{A}_{-1} = 1$ and $\mathbb{A}_{-1}^- = 0$.

In the more general context, if the entries of the matrix start at λ_s and end at λ_l, we use the notation

$$\mathbb{A}_l^s = \begin{vmatrix} 1 + \frac{\lambda_s}{n} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_s}{n} & \frac{\lambda_{s+1}}{n} & \cdots & 1 + \frac{\lambda_l}{n} \end{vmatrix}, \quad \text{and} \quad {}^-\mathbb{A}_l^s = \begin{vmatrix} 1 + \frac{\lambda_s}{n} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_s}{n} & \frac{\lambda_{s+1}}{n} & \cdots & \frac{\lambda_l}{n} \end{vmatrix}.$$

We have the relation

Lemma 5.

$$\mathbb{A}_l^s = \mathbb{A}_{l-1}^s + {}^-\mathbb{A}_l^s.$$

Remark 6. *Similar to the previous remark, we will be using the above lemma in the case when $s = l$. In this case, ${}^-\mathbb{A}_s^s = \dfrac{\lambda_s}{n}$ and $\mathbb{A}_s^s = 1 + {}^-\mathbb{A}_s^s$, so we set $\mathbb{A}_{s-1}^s = 1$.*

Another special case is when the indices of the entries start at l and end at $n - 1$. We use the notation

$$\mathbb{A}_l^n = \begin{vmatrix} 1 + \frac{\lambda_l}{n} & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_l}{n} & \frac{\lambda_{l+1}}{n} & \cdots & 1 + \frac{\lambda_{n-1}}{n} \end{vmatrix}.$$

We can also evaluate the value of $\mathbb{A}_l^n$ and $\mathbb{A}_l^s$ by expanding along the first row. To keep the calculation tidy, we adopt the following notation:

$$\mathbb{B}_l^s = \begin{vmatrix} \frac{1}{n} & -1 & \cdots & 0 \\ \frac{1}{n} & 1 + \frac{\lambda_s}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{\lambda_s}{n} & \cdots & 1 + \frac{\lambda_l}{n} \end{vmatrix} \quad \text{and} \quad {}^1\mathbb{B}_l^s = \begin{vmatrix} 1 & -1 & \cdots & 0 \\ 1 & 1 + \frac{\lambda_s}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\lambda_s}{n} & \cdots & 1 + \frac{\lambda_l}{n} \end{vmatrix}.$$

Obviously, $\mathbb{B}_l^s = \frac{1}{n}{}^1\mathbb{B}_l^s$.

By expanding along the first row we have

Lemma 6.

$$\begin{aligned}\mathbb{A}_l^s &= \lambda_s \cdot \mathbb{B}_l^{s+1} + \mathbb{A}_l^{s+1}, \\ &= \frac{\lambda_s}{n} \cdot {}^1\mathbb{B}_l^{s+1} + \mathbb{A}_l^{s+1}.\end{aligned}$$

For the special case where the indices of the entries start at l and end at $n-1$, we use the notation

$$\mathbb{B}_l^1 = \begin{vmatrix} 1 & -1 & \cdots & 0 \\ 1 & 1+\frac{\lambda_l}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\lambda_l}{n} & \cdots & 1+\frac{\lambda_{n-1}}{n} \end{vmatrix}.$$

Additionally, we have

$${}^-\mathbb{B}_l^s = \begin{vmatrix} \frac{1}{n} & -1 & \cdots & 0 \\ \frac{1}{n} & 1+\frac{\lambda_s}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{\lambda_s}{n} & \cdots & \frac{\lambda_l}{n} \end{vmatrix} \quad \text{and} \quad {}^{1,-}\mathbb{B}_l^s = \begin{vmatrix} 1 & -1 & \cdots & 0 \\ 1 & 1+\frac{\lambda_s}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{\lambda_s}{n} & \cdots & \frac{\lambda_l}{n} \end{vmatrix}.$$

Here are some additional properties of these determinants

Lemma 7.

$$\mathbb{A}_l^s = (1+\frac{\lambda_s}{n}) \cdot \mathbb{A}_l^{s+1} + \frac{\lambda_s}{n} \cdot {}^1\mathbb{B}_l^{s+2}. \tag{3.2}$$

$$\mathbb{A}_l^s = \frac{\lambda_s}{n} \cdot {}^1\mathbb{B}_l^{s+1} + \mathbb{A}_l^{s+1} \tag{3.3}$$

$$\mathbb{A}_l^s = (1+\frac{\lambda_l}{n}) \cdot \mathbb{A}_{l-1}^s + {}^-\mathbb{A}_{l-1}^s = \mathbb{A}_{l-1}^s + {}^-\mathbb{A}_l^s. \tag{3.4}$$

$${}^1\mathbb{B}_l^s = \mathbb{A}_l^s + {}^1\mathbb{B}_l^{s+1}. \tag{3.5}$$

$$\mathbb{B}_l^s = \frac{1}{n} \cdot \mathbb{A}_l^s + \mathbb{B}_l^{s+1} \tag{3.6}$$

$$\mathbb{B}_l^s = {}^-\mathbb{B}_l^s + \mathbb{B}_{l-1}^s. \tag{3.7}$$

Remark 7. *Mindful of the fact that the above identities also works for the end points, we adopt the following conventions:*

- *First, notice that* $\mathbb{A}^n_{n-1} = (1 + \frac{\lambda_{n-1}}{n}) \cdot \mathbb{A}^n_n + \frac{\lambda_{n-1}}{n} \cdot \mathbb{B}^1_{n+1}$. *Therefore, we set* $\mathbb{A}^n_n = 1$ *and* $\mathbb{B}^1_{n+1} = 0$. *By the same token, we set* $\mathbb{A}^{s+1}_s = 1$ *and* ${}^1\mathbb{B}^{s+2}_s = 0$.

- *Second,* $\mathbb{B}^1_{n-1} = \mathbb{A}^n_{n-1} + 1 = \mathbb{A}^n_{n-1} + \mathbb{B}^1_n$, *so we set* $\mathbb{B}^1_n = 1$. *By the same token, we set* ${}^1\mathbb{B}^{s+1}_s = 1$, $\mathbb{B}^{s+1}_s = \frac{1}{n}$, *and* $\mathbb{B}^1_n = \frac{1}{n}$.

- *Third,* $\mathbb{B}^{s+1}_s = \frac{1}{n} = {}^-\mathbb{B}^{s+1}_s + \mathbb{B}^{s+1}_{s-1} = {}^-\mathbb{B}^{s+1}_s$. *Likewise,* ${}^{1,-}\mathbb{B}^{s+1}_s = 1$.

- *Lastly,* $\frac{\lambda_s}{n} = {}^-\mathbb{A}^s_s = \frac{\lambda_s}{n}\mathbb{A}^s_{s-1} + {}^-\mathbb{A}^s_{s-1}$, *so we have* ${}^-\mathbb{A}^s_{s-1} = 0$.

The next lemma allows us the break down the determinant $\mathbb{A}^s_l$ at points other than the end points.

Lemma 8. *For* $s \le r \le l$,

$$\mathbb{A}^s_l = \mathbb{A}^s_r \cdot \mathbb{A}^{r+1}_l + {}^-\mathbb{A}^s_r \cdot {}^1\mathbb{B}^{r+2}_l.$$

Proof. We will use induction. First, let $k = s + 1$, and evaluate the determinant $\mathbb{A}^s_l$ by expanding along first row.

$$\begin{aligned} \mathbb{A}^s_l &= (1 + \frac{\lambda_s}{n}) \cdot \mathbb{A}^{s+1}_l + \frac{\lambda_s}{n} \cdot {}^1\mathbb{B}^{s+2}_l \\ &= \mathbb{A}^s_s \cdot \mathbb{A}^{s+1}_l + {}^-\mathbb{A}^s_s \cdot {}^1\mathbb{B}^{s+2}_l. \end{aligned}$$

Using the inductive hypothesis, for general entry $s \le k \le l$,

$$\begin{aligned} \mathbb{A}^s_l &= \mathbb{A}^s_k \cdot \mathbb{A}^{k+1}_l + {}^-\mathbb{A}^s_k \cdot {}^1\mathbb{B}^{k+2}_l \\ &= \mathbb{A}^s_k \cdot \left((1 + \frac{\lambda_{k+1}}{n}) \cdot \mathbb{A}^{k+2}_l + \frac{\lambda_{k+1}}{n} \cdot {}^1\mathbb{B}^{k+3}_l\right) + {}^-\mathbb{A}^s_k \cdot \left(\mathbb{A}^{k+2}_l + {}^1\mathbb{B}^{k+3}_l\right) \\ &= \left((1 + \frac{\lambda_{k+1}}{n}) \cdot \mathbb{A}^s_k + {}^-\mathbb{A}^s_k\right) \cdot \mathbb{A}^{k+2}_l + \left(\frac{\lambda_{k+1}}{n}\mathbb{A}^s_k + {}^-\mathbb{A}^s_k\right) \cdot {}^1\mathbb{B}^{k+3}_l \\ &= \mathbb{A}^s_{k+1} \cdot \mathbb{A}^{k+2}_l + {}^-\mathbb{A}^s_{k+1} \cdot {}^1\mathbb{B}^{k+3}_l. \end{aligned}$$

□

There are times when, in order to show recursive relation, we need to expand ${}^1\mathbb{B}^s_l$ other than at the diagonal end points. In those instance, the next lemma will be useful.

Lemma 9.

$${}^1\mathbb{B}^s_l = (1 + \frac{\lambda_s}{n}) \cdot {}^1\mathbb{B}^{s+1}_l + \mathbb{A}^{s+1}_l.$$

Proof.

$$\begin{aligned} {}^{1}\mathbb{B}_{l}^{s} &= \mathbb{A}_{l}^{s} + {}^{1}\mathbb{B}_{l}^{s+1} \\ &= \frac{\lambda_s}{n} \cdot {}^{1}\mathbb{B}_{l}^{s+1} + \mathbb{A}_{l}^{s+1} + {}^{1}\mathbb{B}_{l}^{s+1} \\ &= (1 + \frac{\lambda_s}{n}) \cdot {}^{1}\mathbb{B}_{l}^{s+1} + \mathbb{A}_{l}^{s+1}. \end{aligned}$$

□

3.2 Main Steps

We are now ready for the main steps. Using Lemma 8, we have

$$\begin{aligned} \mathcal{M}(s+1,k) &= \mathbb{A}_s \cdot \mathbb{A}_{s+1}^{n} + \mathbb{A}_{s}^{-} \cdot \mathbb{B}_{s+2}^{1} \\ &= \mathbb{A}_s \cdot \left(\mathbb{A}_{k}^{s+1} \cdot \mathbb{A}_{k+1}^{n} + {}^{-}\mathbb{A}_{k}^{s+1} \cdot \mathbb{B}_{k+2}^{1} \right) + \mathbb{A}_{s}^{-} \cdot \left({}^{1}\mathbb{B}_{k}^{s+2} \cdot \mathbb{A}_{k+1}^{n} + {}^{1,-}\mathbb{B}_{k}^{s+2} \cdot \mathbb{B}_{k+2}^{1} \right). \end{aligned}$$

We break down $\widehat{\mathcal{M}}(s+1,k)$ the same way:

$$\widehat{\mathcal{M}}(s+1,k) = \mathbb{A}_s \cdot \left({}^{*}\mathbb{A}_{k}^{s+1} \cdot \mathbb{A}_{k+1}^{n} + {}^{-}\tilde{\mathbb{A}}_{k}^{s+1} \cdot \mathbb{B}_{k+2}^{1} \right) + \mathbb{A}_{s}^{-} \cdot \left({}^{1}\mathbb{B}_{k-1}^{s+1} \cdot \mathbb{A}_{k+1}^{n} + {}^{1,-}\mathbb{B}_{k-1}^{s+1} \cdot \mathbb{B}_{k+2}^{1} \right).$$

To be more clear,

$${}^{*}\mathbb{A}_{k}^{s+1} = \begin{vmatrix} 1+\frac{\lambda_k}{n} & -1 & \cdots & 0 \\ \frac{\lambda_k}{n} & 1+\frac{\lambda_{s+1}}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_k}{n} & \frac{\lambda_{s+1}}{n} & \cdots & 1+\frac{\lambda_{k-1}}{n} \end{vmatrix}, \quad {}^{-}\tilde{\mathbb{A}}_{k}^{s+1} = \begin{vmatrix} 1+\frac{\lambda_k}{n} & -1 & \cdots & 0 \\ \frac{\lambda_k}{n} & 1+\frac{\lambda_{s+1}}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_k}{n} & \frac{\lambda_{s+1}}{n} & \cdots & \frac{\lambda_{k-1}}{n} \end{vmatrix}.$$

We let

1. $(I) = \mathbb{A}_{k}^{s+1} - {}^{*}\mathbb{A}_{k}^{s+1}$,
2. $(II) = {}^{-}\mathbb{A}_{k}^{s+1} - {}^{-}\tilde{\mathbb{A}}_{k}^{s+1}$,
3. $(III) = {}^{1}\mathbb{B}_{k}^{s+2} - {}^{1}\mathbb{B}_{k-1}^{k+1}$,
4. $(IV) = {}^{1,-}\mathbb{B}_{k}^{s+2} - {}^{1,-}\mathbb{B}_{k-1}^{s+1}$.

Our next result shows that

Lemma 10.

$$
\begin{aligned}
& \mathcal{M}(s+1,k) - \widehat{M}(s+1,k) \\
= \;& \mathbb{A}_s \cdot \left((I) \cdot \mathbb{A}^n_{k+1} + (II) \cdot \mathbb{B}^1_{k+2} \right) + \mathbb{A}^-_s \cdot \left((III) \cdot \mathbb{A}^n_{k+1} + (IV) \cdot \mathbb{B}^1_{k+2} \right) \\
= \;& \mathbb{A}_s \cdot \left\{ \begin{array}{l} \sum_{j=0}^{\nu} \left((\lambda_{s+1+j} - \lambda_k) \cdot {}^-\mathbb{B}^{s+2+j}_{k-1-j} + \frac{1}{n} \cdot (\lambda_{k-1-j} - \lambda_k) \cdot \mathbb{A}^{s+2+j}_{k+2+j} \right) \cdot \mathbb{A}^n_{k+1} \\ \\ + \frac{\lambda_k}{n} \cdot \left(\sum_{j=0}^{\nu} (\lambda_{s+1+j} - \lambda_{k-1-j}) \cdot \mathbb{B}^{s+2+j}_{k-2-j} \right) \cdot \mathbb{B}^1_{k+2} \end{array} \right\} \\
& + \mathbb{A}^-_s \cdot \left\{ \begin{array}{l} \left(\sum_{j=0}^{\nu} (\lambda_{k-j} - \lambda_{s+1+j}) \cdot \mathbb{B}^{s+2+j}_{k-2-j} \right) \cdot \mathbb{A}^n_{k+1} \\ \\ - \sum_{j=0}^{\nu} \left((\lambda_{s+1+j} - \lambda_k) {}^-\mathbb{B}^{s+2+j}_{k-1-j} + \frac{1}{n} \cdot (\lambda_{k-1-j} - \lambda_k) \mathbb{A}^{s+2+j}_{k-2-j} \right) \cdot \mathbb{B}^1_{k+2} \end{array} \right\},
\end{aligned}
$$

where

$$
\nu = \left\{ \begin{array}{ll} \frac{k-s-3}{2}, & k-s \quad \textit{is odd}, \\ \\ \frac{k-s-2}{2}, & k-s \quad \textit{is even}. \end{array} \right.
$$

Proof.

$$
\begin{aligned}
(I) &= \mathbb{A}^{s+1}_k - {}^*\mathbb{A}^{s+1}_k \\
&= (1 + \frac{\lambda_k}{n}) \cdot \mathbb{A}^{s+1}_{k-1} + {}^-\mathbb{A}^{s+1}_{k-1} - (1 + \frac{\lambda_k}{n}) \cdot \mathbb{A}^{s+1}_{k-1} - \lambda_k \cdot \mathbb{B}^{s+2}_{k-1} \\
&= \lambda_{s+1} \cdot {}^-\mathbb{B}^{s+2}_{k-1} + {}^-\mathbb{A}^{s+2}_{k-1} - \lambda_k \cdot {}^-\mathbb{B}^{s+2}_{k-1} - \lambda_k \cdot \mathbb{B}^{s+2}_{k-2} \\
&= (\lambda_{s+1} - \lambda_k) \cdot {}^-\mathbb{B}^{s+2}_{k-1} + \frac{\lambda_{k-1}}{n} \cdot \mathbb{A}^{s+2}_{k-2} + {}^-\mathbb{A}^{s+2}_{k-2} - \frac{\lambda_k}{n} \cdot \mathbb{A}^{s+2}_{k-2} - \lambda_k \cdot \mathbb{B}^{s+3}_{k-2} \\
&= (\lambda_{s+1} - \lambda_k) \cdot {}^-\mathbb{B}^{s+2}_{k-1} + \frac{\lambda_{k-1} - \lambda_k}{n} \cdot \mathbb{A}^{s+2}_{k-2} + {}^-\mathbb{A}^{s+2}_{k-2} - \lambda_k \cdot \mathbb{B}^{s+3}_{k-2}.
\end{aligned}
$$

Notice the recursive relation

$$
{}^-\mathbb{A}^{s+1+j}_{k-1-j} - \lambda_k \mathbb{B}^{s+2+j}_{k-1-j} = (\lambda_{s+1+j} - \lambda_k) \cdot {}^-\mathbb{B}^{s+2+j}_{k-1-j} + \frac{\lambda_{k-1-j} - \lambda_k}{n} \cdot \mathbb{A}^{s+2+j}_{k-2-j} + {}^-\mathbb{A}^{s+2+j}_{k-2-j} - \lambda_k \cdot \mathbb{B}^{s+3+j}_{k-2-j}.
$$

By repeating the steps, we have

$$
\begin{aligned}
(I) &= {}^-\mathbb{A}^{s+1}_{k-1} - \lambda_k \cdot \mathbb{B}^{s+2}_{k-1} \\
&= \sum_{j=0}^{\nu-1} \left((\lambda_{s+1+j} - \lambda_k) \cdot {}^-\mathbb{B}^{s+2+j}_{k-1-j} + \frac{\lambda_{k-1-j} - \lambda_k}{n} \cdot \mathbb{A}^{s+2+j}_{k-2-j} \right) + I^1_\nu,
\end{aligned}
$$

where ν and I_ν^1 are to be determined next.

Case 1: $k-s$ is odd. In this case, we have $s+\nu+2 = k-\nu-1$, or $\nu = \dfrac{k-s-3}{2}$.

$$\begin{aligned} I_\nu^1 &= {}^{-}\mathbb{A}_{k-\nu-1}^{s+\nu+1} - \lambda_k \cdot \mathbb{B}_{k-\nu-1}^{s+\nu+2} \\ &= (\lambda_{s+\nu+1} - \lambda_k) \cdot {}^{-}\mathbb{B}_{k-\nu-1}^{s+\nu+2} + \frac{\lambda_{k-\nu-1} - \lambda_k}{n}. \end{aligned}$$

Case 2: $k-s$ is even. $s+\nu+1 = k-\nu-1$, or $\nu = \dfrac{k-s-2}{2}$.

$$\begin{aligned} I_\nu^1 &= {}^{-}\mathbb{A}_{k-\nu-1}^{s+\nu+1} - \lambda_k \cdot \mathbb{B}_{k-\nu-1}^{s+\nu+2} \\ &= \frac{\lambda_{k-\nu-1} - \lambda_k}{n}. \end{aligned}$$

Thus, we set

$$\nu = \begin{cases} \frac{k-s-3}{2}, & k-s \quad \text{is odd,} \\ \\ \frac{k-s-2}{2}, & k-s \quad \text{is even.} \end{cases}$$

and

$$I_\nu^1 = \begin{cases} (\lambda_{s+\nu+1} - \lambda_k) \cdot {}^{-}\mathbb{B}_{k-\nu-1}^{s+\nu+2} + \frac{\lambda_{k-\nu-1}-\lambda_k}{n}, & \nu = \frac{k-s-3}{2}, \\ \\ \frac{\lambda_{k-\nu-1}-\lambda_k}{n}, & \nu = \frac{k-s-2}{2}. \end{cases}$$

$$\begin{aligned} (II) &= {}^{-}\mathbb{A}_k^{s+1} - {}^{-}\tilde{A}_k^{s+1} \\ &= \frac{\lambda_k}{n} \cdot \mathbb{A}_{k-1}^{s+1} + {}^{-}\mathbb{A}_{k-1}^{s+1} - (1 + \frac{\lambda_k}{n}) \cdot {}^{-}\mathbb{A}_{k-1}^{s+1} - \lambda_k \cdot {}^{-}\mathbb{B}_{k-1}^{s+2} \\ &= \frac{\lambda_k}{n} \cdot \left({}^{-}\mathbb{A}_{k-1}^{s+1} + \mathbb{A}_{k-2}^{s+1}\right) + {}^{-}\mathbb{A}_{k-1}^{s+1} - (1 + \frac{\lambda_k}{n}) \cdot {}^{-}\mathbb{A}_{k-1}^{s+1} - \lambda_k \cdot {}^{-}\mathbb{B}_{k-1}^{s+2} \\ &= \frac{\lambda_k}{n} \cdot \mathbb{A}_{k-2}^{s+1} - \lambda_k \cdot {}^{-}\mathbb{B}_{k-1}^{s+2} \\ &= \frac{\lambda_k}{n} \cdot \left(\lambda_{s+1} \cdot \mathbb{B}_{k-2}^{s+2} + \mathbb{A}_{k-2}^{s+2}\right) - \frac{\lambda_{k-1}}{n} \cdot \lambda_k \cdot \mathbb{B}_{k-2}^{s+2} - \lambda_k \cdot {}^{-}\mathbb{B}_{k-2}^{s+2} \\ &= \frac{\lambda_k}{n} \cdot (\lambda_{s+1} - \lambda_{k-1}) \cdot \mathbb{B}_{k-2}^{s+2} + \frac{\lambda_k}{n} \cdot \mathbb{A}_{k-2}^{s+2} - \lambda_k \cdot {}^{-}\mathbb{B}_{k-2}^{s+2} \\ &= \frac{\lambda_k}{n} \cdot (\lambda_{s+1} - \lambda_{k-1}) \cdot \mathbb{B}_{k-2}^{s+2} + \frac{\lambda_k}{n} \cdot \left({}^{-}\mathbb{A}_{k-2}^{s+2} + \mathbb{A}_{k-3}^{s+2}\right) - \left(\frac{\lambda_k}{n} \cdot {}^{-}\mathbb{A}_{k-2}^{s+2} + \lambda_k \cdot {}^{-}\mathbb{B}_{k-2}^{s+3}\right) \\ &= \frac{\lambda_k}{n} \cdot (\lambda_{s+1} - \lambda_{k-1}) \cdot \mathbb{B}_{k-2}^{s+2} + \frac{\lambda_k}{n} \cdot \mathbb{A}_{k-3}^{s+2} - \lambda_k \cdot {}^{-}\mathbb{B}_{k-2}^{s+3}. \end{aligned}$$

Notice the recursive relation

$$\frac{\lambda_k}{n}\cdot \mathbb{A}^{s+1+j}_{k-2-j} - \lambda_k \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-1-j} = \frac{\lambda_k}{n}\cdot(\lambda_{s+1+j}-\lambda_{k-1-j})\cdot \mathbb{B}^{s+2+j}_{k-2-j} + \frac{\lambda_k}{n}\cdot \mathbb{A}^{s+2+j}_{k-3-j} - \lambda_k \cdot {}^{-}\mathbb{B}^{s+3+j}_{k-2-j}.$$

By keeping the notation consistent with the previous calculation, we denote

$$(II) = \frac{\lambda_k}{n}\cdot\left(\sum_{j=0}^{\nu-1}(\lambda_{s+1+j}-\lambda_{k-1-j})\cdot \mathbb{B}^{s+2+j}_{k-2-j} + I^2_\nu\right).$$

Case 1: $s+\nu+2 = k-\nu-1$ or $s+\nu+1 = k-\nu-2$.

$$\frac{\lambda_k}{n}\cdot \mathbb{A}^{s+\nu+1}_{k-\nu-2} - \lambda_k \cdot {}^{-}\mathbb{B}^{s+\nu+2}_{k-\nu-1} = \frac{\lambda_k}{n}\cdot\frac{\lambda_{s+\nu+1}-\lambda_{k-\nu-1}}{n}.$$

Case 2: $s+\nu+1 = k-\nu-1$.

Recall from our remark that $\mathbb{A}^{s+\nu+1}_{k-\nu-2} = 1$ and ${}^{-}\mathbb{B}^{s+\nu+2}_{k-\nu-1} = \dfrac{1}{n}$.

$$\frac{\lambda_k}{n}\cdot \mathbb{A}^{s+\nu+1}_{s+\nu+2} - \lambda_k \cdot {}^{-}\mathbb{B}^{s+\nu+2}_{k-\nu-1} = 0.$$

Keeping ν the same as the previous calculation, and we set

$$I^2_{\nu+1} = \begin{cases} \frac{\lambda_{s+\nu+1}-\lambda_{k-\nu-1}}{n}, & \nu = \frac{k-s-3}{2}, \\ \\ 0, & \nu = \frac{k-s-2}{2}. \end{cases}$$

By Lemma 9

$$\begin{aligned}
(III) &= {}^{1}\mathbb{B}^{s+2}_{k} - {}^{1}\mathbb{B}^{s+1}_{k-1} \\
&= (1+\frac{\lambda_k}{n})\cdot {}^{1}\mathbb{B}^{s+2}_{k-1} + {}^{1,-}\mathbb{B}^{s+2}_{k-1} - \left(1+\frac{\lambda_{s+1}}{n}\right)\cdot {}^{1}\mathbb{B}^{s+2}_{k-1} - \mathbb{A}^{s+2}_{k-1} \\
&= (\lambda_k-\lambda_{s+1})\cdot \mathbb{B}^{s+2}_{k-1} + \left({}^{-}\mathbb{A}^{s+2}_{k-1} + {}^{1,-}\mathbb{B}^{s+3}_{k-1}\right) - \left({}^{-}\mathbb{A}^{s+2}_{k-1} - \mathbb{A}^{s+2}_{k-2}\right) \\
&= (\lambda_k-\lambda_{s+1})\cdot \mathbb{B}^{s+2}_{k-1} + \left(\frac{\lambda_{k-1}}{n}\cdot {}^{1}\mathbb{B}^{s+3}_{k-2} + {}^{1,-}\mathbb{B}^{s+3}_{k-2}\right) - \left(\frac{\lambda_{s+2}}{n}\cdot {}^{1}\mathbb{B}^{s+3}_{k-2} + \mathbb{A}^{s+3}_{k-2}\right) \\
&= (\lambda_k-\lambda_{s+1})\cdot \mathbb{B}^{s+2}_{k-1} + (\lambda_{k-1}-\lambda_{s+2})\cdot \mathbb{B}^{s+3}_{k-2} + {}^{1,-}\mathbb{B}^{s+3}_{k-2} - \mathbb{A}^{s+3}_{k-2}.
\end{aligned}$$

We use the recursive relation, which can be deduced from the second and fifth equality,

$${}^{1,-}\mathbb{B}^{s+1+j}_{k-j} - \mathbb{A}^{s+1+j}_{k-j} = (\lambda_{k-j}-\lambda_{s+j+1})\cdot \mathbb{B}^{s+2+j}_{k-j-1} + {}^{1,-}\mathbb{B}^{s+2+j}_{k-j-1} - \mathbb{A}^{s+2+j}_{k-1-j}.$$

We can write

$$(III) = \sum_{j=0}^{\nu-1} (\lambda_{k-j} - \lambda_{s+j+1}) \cdot \mathbb{B}_{k-j-1}^{s+2+j} + I_\nu^3.$$

Case 1: $s + \nu + 2 = k - \nu - 1$.

$$\begin{aligned} {}^{1,-}\mathbb{B}_{k-\nu}^{s+1+\nu} - \mathbb{A}_{k-\nu}^{s+1+\nu} &= (\lambda_{k-\nu} - \lambda_{s+\nu+1}) \cdot \mathbb{B}_{k-\nu-1}^{s+2+\nu} + {}^{1,-}\mathbb{B}_{k-\nu-1}^{k-\nu-1} - \mathbb{A}_{k-\nu-1}^{k-\nu-1} \\ &= (\lambda_{k-\nu} - \lambda_{s+\nu+1}) \cdot \mathbb{B}_{k-\nu-1}^{s+2+\nu}. \end{aligned}$$

Case 2: $s + \nu + 1 = k - \nu - 1$

$${}^{1,-}\mathbb{B}_{k-\nu}^{s+\nu+1} - \mathbb{A}_{k-\nu}^{s+\nu+1} = 0.$$

Thus, we set

$$I_\nu^3 = \begin{cases} (\lambda_{k-\nu} - \lambda_{s+\nu+1}) \cdot \mathbb{B}_{k-\nu-1}^{s+2+\nu}, & \nu = \frac{k-s-3}{2}, \\ \\ 0, & \nu = \frac{k-s-2}{2}. \end{cases}$$

Lastly, we show that $(IV) = -(I)$.

$$\begin{aligned} (IV) &= {}^{1,-}\mathbb{B}_k^{s+2} - {}^{1,-}\mathbb{B}_{k-1}^{s+1} \\ &= \left(\frac{\lambda_k}{n} \cdot \left({}^{1,-}\mathbb{B}_{k-1}^{s+2} + \mathbb{B}_{k-2}^{s+2}\right) + {}^{1,-}\mathbb{B}_{k-1}^{s+2}\right) - \left(\left(\frac{\lambda_{s+1}}{n} \cdot {}^{1,-}\mathbb{B}_{k-1}^{s+2} + {}^{-}\mathbb{A}_{k-1}^{s+2}\right) + {}^{1,-}\mathbb{B}_{k-1}^{s+2}\right) \\ &= (\lambda_k - \lambda_{s+1}) \cdot {}^{-}\mathbb{B}_{k-1}^{s+2} + \lambda_k \cdot \mathbb{B}_{k-2}^{s+2} - {}^{-}\mathbb{A}_{k-1}^{s+2} \end{aligned}$$

Recall from the remark 6 and 7, we can simplify the notation of I_ν^1, I_ν^2 and I_ν^3, and the result is the identity as stated in the lemma. □

Lemma 11.

$$
\begin{aligned}
& \mathcal{M}(s+1,k) - \widehat{\mathcal{M}}(s+1,k) \\
= \quad & \mathbb{A}_s \left\{ \begin{array}{l} \left(\sum_{j=0}^{\nu} (\lambda_{s+1+j} - \lambda_k) \cdot {}^{-}\mathbb{B}_{k-1-j}^{s+2+j} + \frac{1}{n} \cdot (\lambda_{k-1-j} - \lambda_k) \cdot \mathbb{A}_{k-2-j}^{s+2+j}\right) \cdot \mathbb{A}_{k+1}^{n} \\ \\ + \frac{\lambda_k}{n} \cdot \left(\sum_{j=0}^{\nu} (\lambda_{s+1+j} - \lambda_{k-1-j}) \cdot \mathbb{B}_{k-2-j}^{s+2+j}\right) \cdot \mathbb{B}_{k+2}^{1} \end{array} \right\} \\
& + \mathbb{A}_s^{-} \left\{ \begin{array}{l} \left(\sum_{j=0}^{\nu} (\lambda_{k-j} - \lambda_{s+1+j}) \cdot \mathbb{B}_{k-1-j}^{s+2+j}\right) \cdot \mathbb{A}_{k+1}^{n} \\ \\ - \left(\sum_{j=0}^{\nu} (\lambda_{s+1+j} - \lambda_k) \cdot {}^{-}\mathbb{B}_{k-1-j}^{s+2+j} + \frac{1}{n} \cdot (\lambda_{k-1-j} - \lambda_k) \cdot \mathbb{A}_{k-2-j}^{s+2+j}\right) \cdot \mathbb{B}_{k+2}^{1} \end{array} \right\} \\
= \quad & \sum_{j=0}^{\nu} \left(\begin{array}{l} (\lambda_{s+1+j} - \lambda_k) \cdot \{\mathcal{L}_1 \cdot {}^{-}\mathbb{B}_{k-1-j}^{s+2+j} + \mathcal{L}_2 \cdot \mathbb{B}_{k-2-j}^{s+2+j}\} \\ \\ \qquad + (\lambda_{k-1-j} - \lambda_k)\{\frac{1}{n} \cdot \mathcal{L}_1 \cdot \mathbb{A}_{k-2-j}^{s+2+j} - \mathcal{L}_2 \cdot \mathbb{B}_{k-2-j}^{s+2+j}\} \end{array} \right)
\end{aligned}
$$

where

$$
\mathcal{L}_1 = \mathbb{A}_{s-1} \cdot \mathbb{A}_{k+1}^{n} - \mathbb{A}_s^{-} \cdot \mathbb{B}_{k+2}^{1}, \quad \textit{and} \quad \mathcal{L}_2 = \frac{\lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{B}_{k+2}^{1} - \mathbb{A}_s^{-} \cdot \mathbb{A}_{k+1}^{n}.
$$

Proof. For $j = 0, \cdots, \nu - 1$

$$
\begin{aligned}
& (\lambda_{s+1+j} - \lambda_k) \cdot \mathbb{A}_s \cdot {}^{-}\mathbb{B}_{k-1-j}^{s+2+j} \cdot \mathbb{A}_{k+1}^{n} + (\lambda_{k-j} - \lambda_{s+1+j}) \cdot \mathbb{A}_s^{-} \cdot \mathbb{B}_{k-1-j}^{s+2+j} \cdot \mathbb{A}_{k+1}^{n} \\
& -(\lambda_{s+1+j} - \lambda_k) \cdot \mathbb{A}_s^{-} \cdot {}^{-}\mathbb{B}_{k-1-j}^{s+2+j} \cdot \mathbb{B}_{k+2}^{1} + \frac{\lambda_k}{n} \cdot (\lambda_{s+1+j} - \lambda_{k-1-j}) \cdot \mathbb{A}_s \cdot \mathbb{B}_{k-2-j}^{s+2+j} \cdot \mathbb{B}_{k+2}^{1} \\
& + \frac{\lambda_{k-1-j} - \lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{A}_{k-2-j}^{s+2+j} \cdot \mathbb{A}_{k+1}^{n} - \frac{\lambda_{k-1-j} - \lambda_k}{n} \cdot \mathbb{A}_s^{-} \cdot \mathbb{A}_{k-2-j}^{s+2+j} \cdot \mathbb{B}_{k+2}^{1} \\
= \quad & (\lambda_{s+1+j} - \lambda_k) \cdot (\mathbb{A}_{s-1} - \mathbb{A}_s^{-}) \cdot {}^{-}\mathbb{B}_{k-1-j}^{s+2+j} \cdot \mathbb{A}_{k+1}^{n} \\
& + (\lambda_{k-j} - \lambda_k) \cdot \mathbb{A}_s^{-} \cdot \mathbb{B}_{k-1-j}^{s+2+j} \cdot \mathbb{A}_{k+1}^{n} + (\lambda_k - \lambda_{s+1+j}) \cdot \mathbb{A}_s^{-} \cdot \left({}^{-}\mathbb{B}_{k-1-j}^{s+2+j} + \mathbb{B}_{k-2-j}^{s+2+j}\right) \cdot \mathbb{A}_{k+1}^{n} \\
& \frac{\lambda_k}{n} \cdot (\lambda_{s+1+j} - \lambda_k + \lambda_k - \lambda_{k-1-j}) \cdot \mathbb{A}_s \cdot \mathbb{B}_{k-2-j}^{s+2+j} \cdot \mathbb{B}_{k+2}^{1}
\end{aligned}
$$

$$+\frac{\lambda_{k-1-j}-\lambda_k}{n}\cdot(\mathbb{A}_{s-1}+\mathbb{A}_s^-)\cdot\mathbb{A}_{k-2-j}^{s+2+j}\cdot\mathbb{A}_{k+1}^n$$

$$-\frac{\lambda_{k-1-j}-\lambda_k}{n}\cdot\mathbb{A}_s^-\cdot\mathbb{A}_{k-2-j}^{s+2+j}\cdot\mathbb{B}_{k+2}^1\pm(\lambda_{k-1-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-2-j}^{s+3+j}\cdot\mathbb{A}_{k+1}^n$$

$$\begin{aligned}
= \quad & (\lambda_{k-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-1-j}^{s+2+j}\cdot\mathbb{A}_{k+1}^n+(\lambda_{s+1+j}-\lambda_k)\cdot\left\{\mathcal{L}_1\cdot{}^-\mathbb{B}_{k-1-j}^{s+2+j}+\mathcal{L}_2\cdot\mathbb{B}_{k-2-j}^{s+2+j}\right\}\\
& +\frac{\lambda_k}{n}(\lambda_k-\lambda_{k-1-j})\cdot\mathbb{A}_s\cdot\mathbb{B}_{k-2-j}^{s+2+j}\cdot\mathbb{B}_{k+2}^1\\
& \qquad +(\lambda_{k-1-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\left(\frac{1}{n}\cdot\mathbb{A}_{k-2-j}^{s+2+j}+\mathbb{B}_{k-2-j}^{s+3+j}\right)\cdot\mathbb{A}_{k+1}^n\\
& +\frac{\lambda_{k-1-j}-\lambda_k}{n}\cdot\left(\mathbb{A}_{s-1}\cdot\mathbb{A}_{k+1}^n-\mathbb{A}_s^-\cdot\mathbb{B}_{k+2}^1\right)\cdot\mathbb{A}_{k-2-j}^{s+2+j}-(\lambda_{k-1-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-2-j}^{s+3+j}\cdot\mathbb{A}_{k+1}^n
\end{aligned}$$

$$\begin{aligned}
= \quad & (\lambda_{k-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-1-j}^{s+2+j}\cdot\mathbb{A}_{k+1}^n-(\lambda_{k-1-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-2-j}^{s+3+j}\cdot\mathbb{A}_{k+1}^n\\
& +(\lambda_{s+1+j}-\lambda_k)\cdot\left\{\mathcal{L}_1\cdot{}^-\mathbb{B}_{k-1-j}^{s+2+j}+\mathcal{L}_2\cdot\mathbb{B}_{k-2-j}^{s+2+j}\right\}\\
& \qquad +(\lambda_{k-1-j}-\lambda_k)\cdot\left\{\frac{1}{n}\cdot\mathcal{L}_1\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\mathcal{L}_2\cdot\mathbb{B}_{k-2-j}^{s+2+j}\right\}.
\end{aligned}$$

Notice for this step, we have an extra term $(\lambda_{k-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-1-j}^{s+2+j}\cdot\mathbb{A}_{k+1}^n$ and we need to borrow a term $(\lambda_{k-1-j}-\lambda_k)\cdot\mathbb{A}_s^-\cdot\mathbb{B}_{k-2-j}^{s+3+j}\cdot\mathbb{A}_{k+1}^n$ from the next step of calculation. The borrowed term is needed for the identity

$$\frac{1}{n}\cdot\mathbb{A}_{k-2-j}^{s+2+j}+\mathbb{B}_{k-2-j}^{s+3+j}=\mathbb{B}_{k-2-j}^{s+2+j}.$$

Since there are no more terms to borrow from for the last step, we need to treat those two cases separately.

Before further calculation, recall that for the first case when $k-s$ is odd, and we

set $\nu = \frac{k-s-3}{2}$

$$\mathbb{A}_s \cdot \left\{ \begin{array}{l} \left(\begin{array}{ccc} (\lambda_{s+1} - \lambda_k) \cdot {}^{-}\mathbb{B}^{s+2}_{k-1} & + & \frac{1}{n} \cdot (\lambda_{k-1} - \lambda_k) \cdot \mathbb{A}^{s+2}_{k-2} \\ + \quad \vdots & + & \vdots \\ + \quad (\lambda_{s+1+\nu} - \lambda_k) \cdot {}^{-}\mathbb{B}^{k-1-\nu}_{k-1-\nu} & + & \frac{1}{n} \cdot (\lambda_{k-1-\nu} - \lambda_k) \end{array} \right) \cdot \mathbb{A}^n_{k+1} \\ \\ + \frac{\lambda_k}{n} \cdot \left((\lambda_{s+1} - \lambda_{k-1}) \cdot \mathbb{B}^{s+2}_{k-2} + \cdots + \frac{1}{n} \cdot (\lambda_{s+1+\nu} - \lambda_{k-1-\nu}) \right) \cdot \mathbb{B}^1_{k+2} \end{array} \right\}$$

$$+ \quad \mathbb{A}^-_s \cdot \left\{ \begin{array}{l} \left((\lambda_k - \lambda_{s+1}) \cdot \mathbb{B}^{s+2}_{k-1} + \cdots + (\lambda_{k-\nu} - \lambda_{s+1+\nu}) \cdot \mathbb{B}^{k-1-\nu}_{k-1-\nu} \right) \cdot \mathbb{A}^n_{k+1} \\ \\ - \left(\begin{array}{ccc} (\lambda_{s+1} - \lambda_k) \cdot {}^{-}\mathbb{B}^{s+2}_{k-1} & + & \frac{1}{n} \cdot (\lambda_{k-1} - \lambda_k) \cdot \mathbb{A}^{s+2}_{k-2} \\ + \quad \vdots & + & \vdots \\ + \quad (\lambda_{s+1+\nu} - \lambda_k) \cdot {}^{-}\mathbb{B}^{k-1-\nu}_{k-1-\nu} & + & \frac{1}{n} \cdot (\lambda_{k-1-\nu} - \lambda_k) \end{array} \right) \cdot \mathbb{B}^1_{k+2} \end{array} \right\}.$$

For the end point,

$$(\lambda_{s+1+\nu} - \lambda_k) \cdot \mathbb{A}_s \cdot {}^{-}\mathbb{B}^{k-1-\nu}_{k-1-\nu} \cdot \mathbb{A}^n_{k+1} + (\lambda_{k-\nu} - \lambda_{s+\nu+1}) \cdot \mathbb{A}^-_s \cdot \mathbb{B}^{k-1-\nu}_{k-1-\nu} \cdot \mathbb{A}^n_{k+1}$$

$$-(\lambda_{s+1+\nu} - \lambda_k) \cdot \mathbb{A}^-_s \cdot {}^{-}\mathbb{B}^{k-1-\nu}_{k-1-\nu} \cdot \mathbb{B}^1_{k+2} + \frac{\lambda_k}{n} \cdot \frac{\lambda_{s+1+\nu} - \lambda_{k-1-\nu}}{n} \cdot \mathbb{B}^1_{k+2}$$

$$\begin{aligned} & \frac{\lambda_{k-1-\nu} - \lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{A}^n_{k+1} - \frac{\lambda_{k-1-\nu} - \lambda_k}{n} \cdot \mathbb{A}^-_s \cdot \mathbb{B}^1_{k+2} \\ = \quad & (\lambda_{k-\nu} - \lambda_k) \cdot \mathbb{A}^-_s \cdot \mathbb{B}^{k-1-\nu}_{k-1-\nu} \cdot \mathbb{A}^n_{k+1} \end{aligned}$$

$$+(\lambda_{s+1+\nu} - \lambda_k) \cdot \left(\mathbb{A}_{s-1} \cdot \mathbb{A}^n_{k+1} - \mathbb{A}^-_s \cdot \mathbb{B}^1_{k+2} \right) \cdot {}^{-}\mathbb{B}^{k-1-\nu}_{k-1-\nu}$$

$$+\frac{\lambda_{s+1+\nu} - \lambda_k}{n} \cdot \left(\frac{\lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{B}^1_{k+2} - \mathbb{A}^-_s \cdot \mathbb{A}^n_{k+1} \right)$$

$$+\frac{\lambda_{k-1-\nu} - \lambda_k}{n} \cdot \left(\left(\mathbb{A}_{s-1} \cdot \mathbb{A}^n_{k+1} - \mathbb{A}^-_s \cdot \mathbb{B}^1_{k+2} \right) + \left(\mathbb{A}^-_s \cdot \mathbb{A}^n_{k+1} - \frac{\lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{B}^1_{k+2} \right) \right)$$

$$= \quad (\lambda_{k-\nu} - \lambda_k) \cdot \mathbb{A}^-_s \cdot \mathbb{B}^{k-1-\nu}_{k-1-\nu} \cdot \mathbb{A}^n_{k+1}$$

$$+(\lambda_{s+1+\nu}-\lambda_k)\cdot\left\{\mathcal{L}_1\cdot{}^{-}\mathbb{B}^{k-1-\nu}_{k-1-\nu}+\frac{1}{n}\cdot\mathcal{L}_2\right\}+\frac{\lambda_{k-1-\nu}-\lambda_k}{n}\cdot\{\mathcal{L}_1-\mathcal{L}_2\}\,.$$

For the second case where $k-s$ is even, and $\nu=\frac{k-s-2}{2}$, we have

$$\mathbb{A}_s\cdot\left\{\begin{array}{l}\left(\begin{array}{lcl}(\lambda_{s+1}-\lambda_k)\cdot{}^{-}\mathbb{B}^{s+2}_{k-1} & + & \frac{1}{n}\cdot(\lambda_{k-1}-\lambda_k)\cdot\mathbb{A}^{s+2}_{k-2}\\ \vdots & & \vdots\\ (\lambda_{s+\nu}-\lambda_k)\cdot{}^{-}\mathbb{B}^{s+1+\nu}_{k-\nu} & + & \frac{1}{n}\cdot(\lambda_{k-\nu}-\lambda_k)\cdot\mathbb{A}^{s+1+\nu}_{k-1-\nu}\\ \\ \frac{1}{n}\cdot(\lambda_{k-1-\nu}-\lambda_k) & & \end{array}\right)\cdot\mathbb{A}^{n}_{k+1}\\ \\ +\frac{\lambda_k}{n}\cdot\left((\lambda_{s+1}-\lambda_{k-1})\cdot\mathbb{B}^{s+2}_{k-2}+\cdots+(\lambda_{s+\nu}-\lambda_{k-\nu})\cdot\mathbb{B}^{s+1+\nu}_{k-1-\nu}\right)\cdot\mathbb{B}^{1}_{k+2}\end{array}\right\}$$

$$+\quad\mathbb{A}^{-}_s\cdot\left\{\begin{array}{l}\left((\lambda_k-\lambda_{s+1})\cdot\mathbb{B}^{s+2}_{k-1}+\cdots+\frac{1}{n}\cdot(\lambda_{k-\nu}-\lambda_{s+1+\nu})\right)\cdot\mathbb{A}^{n}_{k+1}\\ \\ -\left(\begin{array}{lcl}(\lambda_{s+1}-\lambda_k)\cdot{}^{-}\mathbb{B}^{s+2}_{k-1} & + & \frac{1}{n}\cdot(\lambda_{k-1}-\lambda_k)\cdot\mathbb{A}^{s+2}_{k-2}\\ \vdots & & \vdots\\ (\lambda_{s+\nu}-\lambda_k)\cdot{}^{-}\mathbb{B}^{s+1+\nu}_{k-\nu} & + & \frac{1}{n}\cdot(\lambda_{k-\nu}-\lambda_k)\cdot\mathbb{A}^{s+1+\nu}_{k-1-\nu}\\ \\ \frac{1}{n}\cdot(\lambda_{s+1+\nu}-\lambda_k) & & \end{array}\right)\cdot\mathbb{B}^{1}_{k+2}\end{array}\right\}.$$

Bear in mind that $\lambda_{s+1+\nu}=\lambda_{k-1-\nu}$. The last term becomes

$$\begin{aligned}&\frac{\lambda_{k-1-\nu}-\lambda_k}{n}\cdot\mathbb{A}_s\cdot\mathbb{A}^{n}_{k+1}+\frac{\lambda_{k-\nu}-\lambda_{s+1+\nu}}{n}\cdot\mathbb{A}^{-}_s\cdot\mathbb{A}^{n}_{k+1}-\frac{\lambda_{k-1-\nu}-\lambda_k}{n}\cdot\mathbb{A}^{-}_s\cdot\mathbb{B}^{1}_{k+2}\\ =\quad&\frac{\lambda_{k-\nu}-\lambda_k}{n}\cdot\mathbb{A}^{-}_s\cdot\mathbb{A}^{n}_{k+1}+\frac{\lambda_{k-1-\nu}-\lambda_k}{n}\cdot\mathcal{L}_1.\end{aligned}$$

With our previous adopted notation, the calculation results in the identity of the lemma. □

From the previous lemma, we can see that to show $\mathcal{M}(s+1,k)-\widehat{\mathcal{M}}(s+1,k)\geq 0$, it is enough to show that

$$\frac{1}{n}\cdot\mathcal{L}_1\cdot\mathbb{A}^{s+2+j}_{k-2-j}-\mathcal{L}_2\cdot\mathbb{B}^{s+2+j}_{k-2-j}\geq 0,\quad\text{and}\quad\mathcal{L}_1\cdot{}^{-}\mathbb{B}^{s+2+j}_{k-1-j}+\mathcal{L}_2\cdot\mathbb{B}^{s+2+j}_{k-2-j}\geq 0.$$

It is worth pointing out that the latter is considerably easier, so we will proceed with the first inequality. To that end, we will first need to break down the expression of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Lemma 12.

$$\begin{aligned}\mathcal{L}_1 &= \mathbb{A}_{s-1}\cdot\mathbb{A}^n_{k+1} - \mathbb{A}^-_s\cdot\mathbb{B}^1_{k+2}\\ &= \frac{\lambda_{k+1}-\lambda_s}{n}\cdot\mathbb{A}_{s-1}\cdot\mathbb{B}^1_{k+2} + \mathbb{A}_{s-2}\cdot\mathbb{A}^n_{k+2} - \mathbb{A}^-_{s-1}\cdot\mathbb{B}^1_{k+3}\\ &= \sum_{l=0}^{\mu}\frac{\lambda_{k+1+l}-\lambda_{s-l}}{n}\cdot\mathbb{A}_{s-1-l}\cdot\mathbb{B}^1_{k+2+l} + \mathcal{L}^1_{\mu+1}\end{aligned}$$

where $\mu = \min\{s-1, n-k-2\}$*, and*

$$\mathcal{L}^1_{\mu+1} = \begin{cases}\mathbb{A}^n_{k+s+1}, & \mu = s-1,\\ \mathbb{A}_{s-n+k}, & \mu = n-k-2.\end{cases}$$

Proof. We have the following recursive relation.

$$\begin{aligned}&\mathbb{A}_{s-1}\cdot\mathbb{A}^n_{k+1} - \mathbb{A}^-_s\cdot\mathbb{B}^1_{k+2}\\ =\ &\mathbb{A}_{s-1}\cdot\left(\frac{\lambda_{k+1}}{n}\cdot\mathbb{B}^1_{k+2} + \mathbb{A}^n_{k+2}\right) - \left(\frac{\lambda_s}{n}\cdot\mathbb{A}_{s-1} + \mathbb{A}^-_{s-1}\right)\cdot\mathbb{B}^1_{k+2}\\ =\ &\frac{\lambda_{k+1}-\lambda_s}{n}\cdot\mathbb{A}_{s-1}\cdot\mathbb{B}^1_{k+2} + \left(\mathbb{A}^-_{s-1} + \mathbb{A}_{s-2}\right)\cdot\mathbb{A}^n_{k+2} - \mathbb{A}^-_{s-1}\cdot\left(\mathbb{A}^n_{k+2} + \mathbb{B}^1_{k+3}\right)\\ =\ &\frac{\lambda_{k+1}-\lambda_s}{n}\cdot\mathbb{A}_{s-1}\cdot\mathbb{B}^1_{k+2} + \mathbb{A}_{s-2}\cdot\mathbb{A}^n_{k+2} - \mathbb{A}^-_{s-1}\cdot\mathbb{B}^1_{k+3}.\end{aligned}$$

For the end points, first set $\mu = s-1$,

$$\begin{aligned}&\mathbb{A}_0\cdot\mathbb{A}^n_{k+s} - \mathbb{A}^-_1\cdot\mathbb{B}^1_{k+s+1}\\ =\ &\frac{\lambda_{k+s}-\lambda_1}{n}\cdot\mathbb{B}^1_{k+s+1} + \mathbb{A}^n_{k+s+1}.\end{aligned}$$

For the second case, set $k+\mu+1 = n-1$,

$$\begin{aligned}&\mathbb{A}_{s-\mu-1}\cdot\mathbb{A}^n_{n-1} - \mathbb{A}^-_{s-\mu}\cdot\mathbb{B}^1_n\\ =\ &\frac{\lambda_{n-1}-\lambda_{s-\mu}}{n}\cdot\mathbb{A}_{s-1-\mu} + \mathbb{A}_{s-2-\mu}\\ =\ &\frac{\lambda_{n-1}-\lambda_{s-n+k+2}}{n}\cdot\mathbb{A}_{s-n+k+1} + \mathbb{A}_{s-n+k}.\end{aligned}$$

Note: With our assumption, $\mathcal{L}_1 \geq 0$. □

Lemma 13.

$$\begin{aligned}
-\mathcal{L}_2 &= \mathbb{A}_s^- \cdot \mathbb{A}_{k+1}^n - \frac{\lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{B}_{k+2}^1 \\
&= \frac{\lambda_{k+1} - \lambda_k}{n} \cdot \mathbb{A}_s^- \cdot \mathbb{B}_{k+2}^1 + \frac{\lambda_s - \lambda_k}{n} \cdot \mathbb{A}_{s-1} \cdot \mathbb{A}_{k+2}^n \\
&\quad + \mathbb{A}_{s-1}^- \cdot \mathbb{A}_{k+2}^n - \frac{\lambda_k}{n} \cdot \mathbb{A}_{s-1} \cdot \mathbb{B}_{k+3}^1 \\
&= \sum_{l=0}^{\mu} \left(\frac{\lambda_{k+1+l} - \lambda_k}{n} \cdot \mathbb{A}_{s-l}^- \cdot \mathbb{B}_{k+2+l}^1 + \frac{\lambda_{s-l} - \lambda_k}{n} \cdot \mathbb{A}_{s-1-l} \cdot \mathbb{A}_{k+2+l}^n \right) + \mathcal{L}_{\mu+1}^2
\end{aligned}$$

where $\mu = \min\{s-1, n-k-2\}$*, and*

$$\mathcal{L}_{\mu+1}^2 = \begin{cases} -\frac{\lambda_k}{n} \cdot \mathbb{B}_{k+s+2}^1, & \mu = s-1, \\ \\ \mathbb{A}_{s-n+k}^-, & \mu = n-k-2. \end{cases}$$

Proof.

$$\begin{aligned}
&\mathbb{A}_s^- \cdot \mathbb{A}_{k+1}^n - \frac{\lambda_k}{n} \cdot \mathbb{A}_s \cdot \mathbb{B}_{k+2}^1 \\
= \;& \mathbb{A}_s^- \cdot \left(\frac{\lambda_{k+1}}{n} \cdot \mathbb{B}_{k+2}^1 + \mathbb{A}_{k+2}^n \right) - \frac{\lambda_k}{n} \cdot (\mathbb{A}_s^- + \mathbb{A}_{s-1}) \cdot \mathbb{B}_{k+2}^1 \\
= \;& \frac{\lambda_{k+1} - \lambda_k}{n} \cdot \mathbb{A}_s^- \cdot \mathbb{B}_{k+2}^1 + \left(\frac{\lambda_s}{n} \cdot \mathbb{A}_{s-1} + \mathbb{A}_{s-1}^- \right) \cdot \mathbb{A}_{k+2}^n - \frac{\lambda_k}{n} \cdot \mathbb{A}_{s-1} \left(\mathbb{A}_{k+2}^n + \mathbb{B}_{k+3}^1 \right) \\
= \;& \frac{\lambda_{k+1} - \lambda_k}{n} \cdot \mathbb{A}_s^- \cdot \mathbb{B}_{k+2}^1 + \frac{\lambda_s - \lambda_k}{n} \cdot \mathbb{A}_{s-1} \cdot \mathbb{A}_{k+2}^n \\
&+ \mathbb{A}_{s-1}^- \cdot \mathbb{A}_{k+2}^n - \frac{\lambda_k}{n} \cdot \mathbb{A}_{s-1} \cdot \mathbb{B}_{k+3}^1.
\end{aligned}$$

For the end points, the first case is clear because $\mathbb{A}_0^- = 0$.

For the second case, set $k + 1 + \mu = n - 1$, or $\mu = n - k - 2$

$$\begin{aligned}
&\mathbb{A}_{s-\mu}^- \cdot \mathbb{A}_{n-1}^n - \frac{\lambda_k}{n} \cdot \mathbb{A}_{s-\mu} \\
= \;& \frac{\lambda_{n-1} - \lambda_k}{n} \cdot \mathbb{A}_{s-\mu}^- + \frac{\lambda_{s-\mu} - \lambda_k}{n} \cdot \mathbb{A}_{s-1-\mu} + \mathbb{A}_{s-2-\mu}^- \\
= \;& \frac{\lambda_{n-1} - \lambda_k}{n} \cdot \mathbb{A}_{s-n+k+2}^- + \frac{\lambda_{s-n+k+2}}{n} \cdot \mathbb{A}_{s-n+k+1} + \mathbb{A}_{s-n+k}^-.
\end{aligned}$$

□

By combining the two previous lemmas, we obtain

Lemma 14.

$$\begin{aligned}
&\frac{1}{n}\cdot\mathcal{L}_1\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\mathcal{L}_2\cdot\mathbb{B}_{k-2-j}^{s+2+j}\\
=\;&\frac{1}{n}\cdot\left\{\sum_{l=0}^{\mu}\frac{\lambda_{k+1+l}-\lambda_k}{n}\cdot\mathbb{A}_{s-l}\cdot\mathbb{B}_{k+2+l}^{1}\right\}\cdot\mathbb{A}_{k-2-j}^{s+2+j}\\
&+\left\{\sum_{l=0}^{\mu}\frac{\lambda_{k+1+l}-\lambda_k}{n}\cdot\mathbb{A}_{s-l}^{-}\cdot\mathbb{B}_{k+2+l}^{1}\right\}\cdot\mathbb{B}_{k-2-j}^{s+3+j}\\
&+\sum_{l=0}^{\mu}\frac{\lambda_k-\lambda_{s-l}}{n}\cdot\mathbb{A}_{s-1-l}\left(\frac{1}{n}\cdot\mathbb{B}_{k+2+l}^{1}\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\mathbb{A}_{k+2+l}^{n}\cdot\mathbb{B}_{k-2-j}^{s+2+j}\right)\\
&+\frac{1}{n}\cdot\mathcal{L}_{\mu+1}^{1}\cdot\mathbb{A}_{k-2-j}^{s+2+j}+\mathcal{L}_{\mu+1}^{2}\cdot\mathbb{B}_{k+2+j}^{s+2+j}.
\end{aligned}$$

Lemma 15.

$$\begin{aligned}
&\frac{1}{n}\cdot\mathbb{B}_{k+2+l}^{1}\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\mathbb{A}_{k+2+l}^{n}\cdot\mathbb{B}_{k-2-j}^{s+2+j}\\
=\;&\frac{\lambda_{s+2+j}-\lambda_{k+2+l}}{n}\cdot\mathbb{B}_{k+3+l}^{1}\cdot\mathbb{B}_{k-2-j}^{s+3+j}+\frac{1}{n}\cdot\mathbb{B}_{k+3+l}^{1}\cdot\mathbb{A}_{k-2-j}^{s+3+j}-\mathbb{A}_{k+3+l}^{n}\cdot\mathbb{B}_{k-2-j}^{s+3+j}\\
=\;&\sum_{t=0}^{m_j(l)}\frac{\lambda_{s+2+j+t}-\lambda_{k+2+l+t}}{n}\cdot\mathbb{B}_{k+2+l+t}^{1}\cdot\mathbb{B}_{k-2-j}^{s+3+j+t}+\mathcal{M}_{m_j(l)+1},
\end{aligned}$$

where $m_j(l)=\min\{k-s-4-2j,n-k-2-l\}$*, and*

$$\mathcal{M}_{m_j(l)+1}=\begin{cases}\frac{1}{n}\mathbb{B}_{k+4+l+m_j(l)}^{1}, & m_j(l)=k-s-4-2j,\\ \\ -\mathbb{B}_{k-2-j}^{s+3+j+m_j(l)}, & m_j(l)=n-k-2-l.\end{cases}$$

Proof. First, we prove the following recursive relation:

$$\begin{aligned}
&\frac{1}{n}\cdot\mathbb{B}_{k+2+l}^{1}\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\mathbb{A}_{k+2+l}^{n}\cdot\mathbb{B}_{k-2-j}^{s+2+j}\\
=\;&\frac{1}{n}\cdot\mathbb{A}_{k+2+l}^{n}\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\frac{1}{n}\cdot\mathbb{A}_{k+2+l}^{n}\cdot\mathbb{A}_{k-2-j}^{s+2+j}\\
&+\frac{1}{n}\cdot\mathbb{B}_{k+3+l}^{1}\cdot\mathbb{A}_{k-2-j}^{s+2+j}-\mathbb{A}_{k+2+l}^{n}\cdot\mathbb{B}_{k-2-j}^{s+3+j}\\
=\;&\frac{\lambda_{s+2+j}-\lambda_{k+2+l}}{n}\cdot\mathbb{B}_{k+3+l}^{1}\cdot\mathbb{B}_{k-2-j}^{s+3+j}+\frac{1}{n}\cdot\mathbb{B}_{k+3+l}^{1}\cdot\mathbb{A}_{k-2-j}^{s+3+j}-\mathbb{A}_{k+3+l}^{n}\cdot\mathbb{B}_{k-2-j}^{s+3+j}
\end{aligned}$$

In other words, we have

$$\begin{aligned}
&\frac{1}{n}\cdot\mathbb{B}^{1}_{k+2+l+t}\cdot\mathbb{A}^{s+2+j+t}_{k-2-j}-\mathbb{A}^{n}_{k+2+l+t}\cdot\mathbb{B}^{s+2+j+t}_{k-2-j}\\
=\;&\frac{\lambda_{s+2+j+t}-\lambda_{k+2+l+t}}{n}\cdot\mathbb{B}^{1}_{k+3+l+t}\cdot\mathbb{B}^{s+3+j+t}_{k-2-j}\\
&\qquad+\frac{1}{n}\cdot\mathbb{B}^{1}_{k+3+l+t}\cdot\mathbb{A}^{s+3+j+t}_{k-2-j}-\mathbb{A}^{n}_{k+3+l+t}\cdot\mathbb{B}^{s+3+j+t}_{k-2-j}.
\end{aligned}$$

Next, we will deal with end points where the recursion ends.

The first case is when $s+2+j+m_j(l)=k-2-j$ or $m_j(l)=k-s-4-2j$. It is important to note that we can safely assume $k-s-4-2j\geq 0$, otherwise both the term $\mathbb{A}^{s+2+j+t}_{k-2-j}$ and $\mathbb{B}^{s+2+j+t}_{k-2-j}$ will vanish. Recall that in our notation

$$\mathbb{A}^{k-1-j}_{k-2-j}=1,\quad\text{and}\quad\mathbb{B}^{k-1-j}_{k-2-j}=\frac{1}{n}.$$

We have

$$\frac{1}{n}\cdot\mathbb{B}^{1}_{k+3+l+m_j(l)}-\frac{1}{n}\cdot\mathbb{A}^{n}_{k+3+l+m_j(l)}=\frac{1}{n}\cdot\mathbb{B}^{1}_{k+4+l+m_j(l)}.$$

Otherwise, have $k+2+l+m_j(l)=n$ or $m_j(l)=n-k-l-2$. In this case, we have $\mathbb{B}^{1}_{n}=\mathbb{A}^{n}_{n}=1$, and the remainder term is

$$\mathbb{A}^{s+2+j+m_j(l)}_{k-2-j}-\mathbb{B}^{s+2+j+m_j(l)}_{k-2-j}=-\mathbb{B}^{s+3+j+m_j(l)}_{k-2-j}.$$

□

Remark 8. *Here, we use the notation $m_j(l)$ to emphasize the fact that it depends on both l and j. For each step of calculation, j is fixed, but l varies from 0 to μ. Moreover, it is obvious that $m_j(\mu)\leq\cdots\leq m_j(0)$.*

Again, we will combine the two previous lemmas, to express

$$\begin{aligned}
&\frac{1}{n}\cdot\mathcal{L}_1\cdot\mathbb{A}^{s+2+j}_{k-2-j}-\mathcal{L}_2\cdot\mathbb{B}^{s+2+j}_{k-2-j}\\
&=\frac{1}{n}\cdot\left\{\sum_{l=0}^{\mu}\frac{\lambda_{k+1+l}-\lambda_k}{n}\cdot\mathbb{A}_{s-l}\cdot\mathbb{B}^{1}_{k+2+l}\right\}\cdot\mathbb{A}^{s+2+j}_{k-2-j}+\left\{\sum_{l=0}^{\mu}\frac{\lambda_{k+1+l}-\lambda_k}{n}\cdot\mathbb{A}^{-}_{s-l}\cdot\mathbb{B}^{1}_{k+2+l}\right\}\cdot\mathbb{B}^{s+3+j}_{k-2-j}\\
&+\sum_{l=0}^{\mu}\frac{\lambda_k-\lambda_{s-l}}{n}\cdot\mathbb{A}_{s-1-l}\cdot\left(\sum_{t=0}^{m_j(l)}\frac{\lambda_{s+2+j+t}-\lambda_{k+2+l+t}}{n}\cdot\mathbb{B}^{1}_{k+3+l+t}\cdot\mathbb{B}^{s+3+j+t}_{k-2-j}+\mathcal{M}_{m_j(l)+1}\right)\\
&+\frac{1}{n}\cdot\mathcal{L}^{1}_{\mu+1}\cdot\mathbb{A}^{s+2+j}_{k-2-j}+\mathcal{L}^{2}_{\mu+1}\cdot\mathbb{B}^{s+2+j}_{k-2-j}.
\end{aligned}\tag{3.8}$$

Before we move on, we will deal with $\frac{1}{n}\mathcal{L}^1_{\mu+1}\cdot\mathbb{A}^{s+2+j}_{k-2-j}+\mathcal{L}^2_{\mu+1}\cdot\mathbb{B}^{s+2+j}_{k-2-j}$ in the case when $\mu=s-1$.

Lemma 16.

$$\begin{aligned}&\frac{1}{n}\cdot\mathbb{A}^n_{k+s+1}\cdot\mathbb{A}^{s+2+j}_{k-2-j}-\frac{\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+2}\cdot\mathbb{B}^{s+2+j}_{k-2-j}\\ =&\sum_{t=0}^{m_j(s-1)}\left(\frac{1}{n}\cdot\frac{\lambda_{k+s+1+t}-\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+2+t}\cdot\mathbb{A}^{s+2+j+t}_{k-2-j}+\frac{\lambda_{s+2+j+t}-\lambda_k}{n}\cdot\mathbb{A}^n_{k+s+2+t}\cdot\mathbb{B}^{s+3+j+t}_{k-2-j}\right)\\ &+\mathcal{J}_{m_j(s)},\end{aligned}$$

where

$$\mathcal{J}_{m_j(s)}=\begin{cases}\frac{1}{n}\cdot\mathbb{A}^{s+3+j+m_j(s)}_{k-2-j}, & m_j(s)=n-k-s-2,\\ \frac{1}{n}\cdot\frac{\lambda_{k+s+2+m_j(s)}-\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+3+m_j(s)} & \\ \quad+\frac{1}{n}\cdot\mathbb{A}^n_{k+s+3+m_j(s)}, & m_j(s)=k-s-4-2j.\end{cases}$$

Proof. As usual, we begin with a recursive relation,

$$\begin{aligned}&\frac{1}{n}\cdot\mathbb{A}^n_{k+s+1}\cdot\mathbb{A}^{s+2+j}_{k-2-j}-\frac{\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+2}\cdot\mathbb{B}^{s+2+j}_{k-2-j}\\ =&\frac{1}{n}\cdot\left(\frac{\lambda_{k+s+1}}{n}\cdot\mathbb{B}^1_{k+s+2}+\mathbb{A}^n_{k+s+2}\right)\cdot\mathbb{A}^{s+2+j}_{k-2-j}-\frac{\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+2}\cdot\left(\frac{1}{n}\cdot\mathbb{A}^{s+2+j}_{k-2-j}+\mathbb{B}^{s+3+j}_{k-2-j}\right)\\ =&\frac{1}{n}\cdot\frac{\lambda_{k+s+1}-\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+2}\cdot\mathbb{A}^{s+2+j}_{k-2-j}\\ &+\mathbb{A}^n_{k+s+2}\cdot\left(\frac{\lambda_{s+2+j}}{n}\cdot\mathbb{B}^{s+3+j}_{k-2-j}+\frac{1}{n}\cdot\mathbb{A}^{s+3+j}_{k-2-j}\right)-\frac{\lambda_k}{n}\cdot\left(\mathbb{A}^n_{k+s+2}+\mathbb{B}^1_{k+s+3}\right)\cdot\mathbb{B}^{s+3+j}_{k-2-j}\\ =&\frac{1}{n}\cdot\frac{\lambda_{k+s+1}-\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+2}\cdot\mathbb{A}^{s+2+j}_{k-2-j}+\frac{\lambda_{s+2+j}-\lambda_k}{n}\cdot\mathbb{A}^n_{k+s+2}\cdot\mathbb{B}^{s+3+j}_{k-2-j}\\ &+\frac{1}{n}\cdot\mathbb{A}^n_{k+s+2}\cdot\mathbb{A}^{s+3+j}_{k-2-j}-\frac{\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+3}\cdot\mathbb{B}^{s+3+j}_{k-2-j}.\end{aligned}$$

For the end points, we have either $k+s+1+t=n-1$ or $s+2+j+t=k-2-j$. Thus, we have either $t=n-k-s-2$ or $k-s-4-2j$. Notice, this is $m_j(l)$ when $l=s$.

If $t=n-k-s-2$, the result is clear since we have $\mathbb{A}^n_n=1$ and $\mathbb{B}^1_{n+1}=0$.

For the second case, $\mathbb{A}^{k-1-j}_{k-2-j}=1$ and $\mathbb{B}^{k-1-j}_{k-2-j}=\frac{1}{n}$. We have

$$\begin{aligned}&\frac{1}{n}\cdot\mathbb{A}^n_{k+s+2+m_j(s)}-\frac{1}{n}\cdot\frac{\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+3+m_j(s)}\\ =&\frac{1}{n}\cdot\frac{\lambda_{k+s+2+m_j(s)}-\lambda_k}{n}\cdot\mathbb{B}^1_{k+s+3+m_j(s)}+\frac{1}{n}\cdot\mathbb{A}^n_{k+s+3+m_j(s)}.\end{aligned}$$

□

By writing $\lambda_{s+2+j+t} - \lambda_{k+2+l+t} = \lambda_{s+2+j+t} - \lambda_k + \lambda_k - \lambda_{k+2+l+r}$, we can split the last sum of equation (3.8) into two parts.

$$\begin{aligned}
&\sum_{l=0}^{\mu} \frac{\lambda_k - \lambda_{s-l}}{n} \cdot \mathbb{A}_{s-1-l} \left(\sum_{t=0}^{m_j(l)} \frac{\lambda_{s+2+j+t} - \lambda_{k+2+l+t}}{n} \cdot \mathbb{B}^1_{k+3+l+r} \cdot \mathbb{B}^{s+3+j+t}_{k-2-j} + \mathcal{M}_{m_j(l)+1} \right) \\
= &\sum_{l=0}^{\mu} \sum_{t=0}^{m_j(l)} \frac{\lambda_k - \lambda_{s-l}}{n} \cdot \frac{\lambda_{s+2+j+t} - \lambda_k}{n} \cdot \mathbb{A}_{s-1-l} \cdot \mathbb{B}^1_{k+3+l+t} \cdot \mathbb{B}^{s+3+j+t}_{k-2-j} \\
&+ \sum_{l=0}^{\mu} \frac{\lambda_k - \lambda_{s-l}}{n} \cdot \mathbb{A}_{s-1-l} \cdot \left(\sum_{t=0}^{m_j(l)} \frac{\lambda_k - \lambda_{k+2+l+t}}{n} \cdot \mathbb{B}^1_{k+3+l+r} \cdot \mathbb{B}^{s+3+j+t}_{k-2-j} + \mathcal{M}_{m_j(l)+1} \right).
\end{aligned}$$

With our assumption, the first sum is positive and the second sum is negative, so our remaining job is to show that it is smaller than the first two sums of equation (3.8).

First, observe that for various l and t such that $l + t = w_0$, for some constant w_0, their corresponding terms $\frac{\lambda_k - \lambda_{s-l}}{n} \cdot \frac{\lambda_k - \lambda_{k+2+l+t}}{n} \cdot \mathbb{A}_{s-1-l} \cdot \mathbb{B}^1_{k+3+l+t} \cdot \mathbb{B}^{s+3+j+t}_{k-2-j}$ have the common term $\frac{\lambda_k - \lambda_{k+2+w_0}}{n} \cdot \mathbb{B}^1_{k+3+w_0}$. Thus, we set $r = t$, and let r varies from 0 to w_0, and let w_0 varies from 0 to $\mu + m_j(0)$. We can relabel

$$\begin{aligned}
&\sum_{l=0}^{\mu} \frac{\lambda_k - \lambda_{s-l}}{n} \cdot \mathbb{A}_{s-1-l} \cdot \left(\sum_{t=0}^{m_j(l)} \frac{\lambda_k - \lambda_{k+2+l+t}}{n} \cdot \mathbb{B}^1_{k+3+l+r} \cdot \mathbb{B}^{s+3+j+t}_{k-2-j} + \mathcal{M}_{m_j(l)+1} \right) \\
= &\sum_{w_0=0}^{\mu+m_j(0)+1} \sum_{r=0}^{w_0} \frac{\lambda_k - \lambda_{s-w_0+r}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathcal{M}_r(w_0),
\end{aligned}$$

where

$$\mathcal{M}_r(w_0) = \begin{cases} \frac{\lambda_k - \lambda_{k+2+w_0}}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{B}^{s+3+j+r}_{k-2-j}, & r \le m_j(w_0 - r) \\ & \text{and} \quad w_0 - r \le \mu, \\ \mathcal{M}_{m_j(l)+1}, & r = m_j(w_0 - r) + 1 \\ & \text{and} \quad w_0 - r \le \mu, \\ 0, & r > m_j(w_0 - r) + 1 \\ & \text{or} \quad w_0 - r > \mu. \end{cases}$$

For each w_0 and $\sum_{r=0}^{w_0} \frac{\lambda_k - \lambda_{s-w_0+r}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathcal{M}_r(w_0)$, we pick two terms, from the first and second sum of equation (3.8), $\frac{1}{n} \cdot \frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{A}_{s-w_0+1} \cdot \mathbb{A}^{s+2+j}_{k-2-j}$ and $\frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{A}^-_{s-w_0+1} \cdot \mathbb{B}^{s+3+j}_{k-2-j}$ to show that

$$\frac{\lambda_{k+2+w_0}-\lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \left(\frac{1}{n} \cdot \mathbb{A}_{s-1-w_0} \cdot \mathbb{A}^{s+2+j}_{k-2-j} + \mathbb{A}^-_{s-1-w_0} \cdot \mathbb{B}^{s+3+j}_{k-2-j}\right)$$

$$+ \textstyle\sum_{r=0}^{w_0} \frac{\lambda_k-\lambda_{s-w_0+r}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathcal{M}_r(w_0) \geq 0.$$

To do that, we need the next lemma.

Lemma 17.

$$\begin{aligned}
&\frac{1}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{A}^{s+2+j}_{k-2-j} + \mathbb{A}^-_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} \\
= \;& \frac{\lambda_{s+2+j} - \lambda_{s-w_0}}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} + \frac{1}{n} \cdot \mathbb{A}_{s-w_0} \cdot \mathbb{A}^{s+3+j}_{k-2-j} + \mathbb{A}^-_{s-w_0} \cdot \mathbb{B}^{s+4+j}_{k-2-j} \\
= \;& \sum_{r=0}^{\zeta} \frac{\lambda_{s+2+j+r} - \lambda_{s-w_0+r}}{n} \cdot \mathbb{A}_{s-w_0-1+r} \cdot \mathbb{B}^{s+3+j+r}_{k-2-j} + \mathcal{W}_{\zeta+1},
\end{aligned}$$

where $\zeta = \min\{w_0, k-4-s-2j\}$, *and*

$$\mathcal{W}_{\zeta+1} = \begin{cases} \frac{1}{n} \cdot \mathbb{A}_s \cdot \mathbb{A}^{s+3+j+w_0}_{k-2-j} + \mathbb{A}^-_s \cdot \mathbb{B}^{s+4+j+w_0}_{k-2-j}, & \zeta = w_0, \\ \frac{1}{n} \cdot \mathbb{A}_{s-w_0+r}, & \zeta = k-s-4-2j. \end{cases}$$

Proof.

$$\begin{aligned}
&\frac{1}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{A}^{s+2+j}_{k-2-j} + \mathbb{A}^-_{s-w_0-1} \cdot \mathbb{B}^{s+2+j}_{k-2-j} \\
= \;& \frac{\lambda_{s+2+j} - \lambda_{s-w_0}}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} + \frac{\lambda_{s-w_0}}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} \\
& + \frac{1}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{A}^{s+3+j}_{k-2-j} + \mathbb{A}^-_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} \\
= \;& \frac{\lambda_{s+2+j} - \lambda_{s-w_0}}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} + \frac{1}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{A}^{s+3+j}_{k-2-j} + \mathbb{A}^-_{s-w_0} \cdot \mathbb{B}^{s+3+j}_{k-2-j} \\
= \;& \frac{\lambda_{s+2+j} - \lambda_{s-w_0}}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} + \frac{1}{n} \mathbb{A}_{s-w_0-1} \cdot \mathbb{A}^{s+3+j}_{k-2-j} \\
& + \mathbb{A}^-_{s-w_0} \cdot \left(\frac{1}{n} \mathbb{A}^{s+3+j}_{k-2-j} + \mathbb{B}^{s+4+j}_{k-2-j}\right) \\
= \;& \frac{\lambda_{s+2+j} - \lambda_{s-w_0}}{n} \cdot \mathbb{A}_{s-w_0-1} \cdot \mathbb{B}^{s+3+j}_{k-2-j} + \frac{1}{n} \cdot \mathbb{A}_{s-w_0} \cdot \mathbb{A}^{s+3+j}_{k-2-j} + \mathbb{A}^-_{s-w_0} \cdot \mathbb{B}^{s+4+j}_{k-2-j}.
\end{aligned}$$

Here, the recursion ends either when $s - w_0 + \zeta = s$ or $s + 2 + j + \zeta = k - 2 - j$. In the first case, we have the tail term being

$$\frac{\lambda_{s+2+j+w_0} - \lambda_s}{n} \cdot \mathbb{A}_{s-1} \cdot \mathbb{B}^{s+3+j+w_0}_{k-2-j} + \frac{1}{n} \cdot \mathbb{A}_s \cdot \mathbb{A}^{s+3+j+w_0}_{k-2-j} + \mathbb{A}^-_s \cdot \mathbb{B}^{s+4+j+w_0}_{k-2-j}.$$

In the latter case, we can use the fact that $\mathbb{B}^{k-1-j}_{k-2-j} = \frac{1}{n}$ and $\mathbb{B}^{k-j}_{k-2-j} = 0$ to conclude the tail term is $\frac{1}{n} \cdot \mathbb{A}_{s-w_0+\zeta}$. □

We are now ready to show

Lemma 18.

$$\frac{1}{n} \cdot \mathcal{L}_1 \cdot \mathbb{A}^{s+2+j}_{k-2-j} - \mathcal{L}_2 \cdot \mathbb{B}^{s+2+j}_{k-2-j} \geq 0. \tag{3.9}$$

Proof. First, notice that $\zeta \geq w_0$. Second, with our assumption, $\lambda_{s+2+j+r} > \lambda_k$ for all $r = 0$ to w_0. Thus, in the case when $w_0 \leq \mu$ and $r \leq m_j(w_0 - r)$, Lemma 17 gives us

$$\begin{aligned}
&\frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \left(\frac{1}{n} \cdot \mathbb{A}_{s-1-w_0} \cdot \mathbb{A}^{s+2+j}_{k-2-j} + \mathbb{A}^-_{s-1-w_0} \cdot \mathbb{B}^{s+3+j}_{k-2-j} \right) \\
&\qquad + \sum_{r=0}^{w_0} \frac{\lambda_k - \lambda_{s-w_0+r}}{n} \cdot \frac{\lambda_k - \lambda_{k+2+w_0}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{B}^{s+3+j+r}_{k-2-j} \\
= \quad &\frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \left(\sum_{r=0}^{\zeta} \frac{\lambda_{s+2+j+r} - \lambda_{s-w_0-r}}{n} \cdot \mathbb{A}_{s-w_0-1+r} \cdot \mathbb{B}^{s+3+j+r}_{k-2-j} + \mathcal{W}_{\zeta+1} \right) \\
&\qquad + \sum_{r=0}^{w_0} \frac{\lambda_k - \lambda_{s-w_0+r}}{n} \cdot \frac{\lambda_k - \lambda_{k+2+w_0}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{B}^{s+3+j+r}_{k-2-j} \geq 0.
\end{aligned}$$

We need to pay special attention to the case when $\mu = s - 1$, and $w_0 > \mu$. In such case, we have $s - w_0 - 1 + r = 0$, and $A_0 = 1$, so the term

$$\sum_{r=w_0-s+1}^{w_0} \frac{\lambda_k - \lambda_{s-w_0+r}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathcal{M}_r(w_0),$$

cannot be dealt with as before. Instead, we will need to borrow the term

$$\frac{1}{n} \cdot \frac{\lambda_{k+s+1+t} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+s+2+t} \cdot \mathbb{A}^{s+2+j+t}_{k-s-j},$$

from the decomposition of

$$\frac{1}{n} \cdot \mathbb{A}^n_{k+s+1} \cdot \mathbb{A}^{s+2+j}_{k-2-j} - \frac{\lambda_k}{n} \cdot \mathbb{B}^1_{k+s+2} \cdot \mathbb{B}^{s+2+j}_{k-2-j},$$

as stated in Lemma 16. By picking a t such that $t = w_0 - s + 1$, which implies $k+s+1+t = k+2+w_0$, the above term becomes

$$\frac{1}{n} \cdot \frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{A}^{w_0+3+j}_{k-2-j}.$$

We can modify the previous lemma by relabel the recursive relation as:

$$\begin{aligned} & \frac{1}{n} \cdot \mathbb{A}_q \cdot \mathbb{A}^i_h + \mathbb{A}^-_q \cdot \mathbb{B}^{i+1}_h \\ = \quad & \frac{\lambda_i - \lambda_{q+1}}{n} \cdot \mathbb{A}_q \cdot \mathbb{B}^{i+1}_h + \frac{1}{n} \cdot \mathbb{A}_{q+1} \cdot \mathbb{A}^{i+1}_h + \mathbb{A}^-_q \cdot \mathbb{B}^{i+2}_h. \end{aligned}$$

By setting $q = 0$ and $i = s + 2 + j + t$, we have

$$\frac{1}{n} \cdot \mathbb{A}^{w_0+3+j}_{k-2-j} = \sum_{a=0}^{\beta} \frac{\lambda_{w_0+3+j+a} - \lambda_{a+1}}{n} \cdot \mathbb{A}_a \cdot \mathbb{B}^{w_0+4+j+a}_{k-2-j} + \mathcal{W}_{\beta+1}.$$

In this setting, $\beta = \min\{s-1, k-w_0-6-2j\}$, and $\mathcal{W}_{\beta+1}$ remains the same except for the change on the index.

For the case $r \le m_j(w_0 - r)$ and $w_o > \mu$, we have

$$\begin{aligned} & \frac{1}{n} \cdot \frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{A}^{w_0+3+j}_{k-2-j} + \sum_{r=w_0-s+1}^{w_0} \frac{\lambda_k - \lambda_{s-w_0+r}}{n} \cdot \mathbb{A}_{s-1-w_0+r} \cdot \mathcal{M}_r(w_0) \\ = \quad & \frac{1}{n} \cdot \frac{\lambda_{k+2+w_0} - \lambda_k}{n} \cdot \mathbb{B}^1_{k+3+w_0} \cdot \left(\sum_{a=0}^{\beta} \frac{\lambda_{w_0+3+j+a} - \lambda_{a+1}}{n} \cdot \mathbb{A}_a \cdot \mathbb{B}^{w_0+4+j+a}_{k-2-j} + \mathcal{W}_{\beta+1} \right) \\ & + \sum_{a=0}^{2w_0-s+1} \frac{\lambda_k - \lambda_{a+1}}{n} \cdot \frac{\lambda_k - \lambda_{k+2+w_0}}{n} \cdot \mathbb{A}_a \cdot \mathbb{B}^1_{k+3+w_0} \cdot \mathbb{B}^{w_0+4+j+a}_{k-2-j} \ge 0. \end{aligned}$$

The rest of the cases are clear, and we have the desire result. □

Next, we will show that

$$\mathcal{L}_1 \cdot {}^-\mathbb{B}^{s+2+j}_{k-1-j} + \mathcal{L}_2 \cdot \mathbb{B}^{s+2+j}_{k-2-j} \ge 0.$$

We will need the following Lemma.

Lemma 19.

$$\mathbb{A}_{q-1} \cdot {}^-\mathbb{B}^{s+2+j}_{k-1-j} - \mathbb{A}^-_q \cdot \mathbb{B}^{s+2+j}_{k-2-j} \ge 0, \quad \textit{for} \quad 1 \le q \le s.$$

Proof. First, we have the recursive relation:

$$\begin{aligned}
& \mathbb{A}_{q-1} \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-1-j} - \mathbb{A}^{-}_{q} \cdot \mathbb{B}^{s+2+j}_{k-2-j} \\
= \; & \mathbb{A}_{q-1} \cdot \left(\frac{\lambda_{k-1-j}}{n} \cdot \mathbb{B}^{s+2+j}_{k-2-j} + {}^{-}\mathbb{B}^{s+2+j}_{k-2-j} \right) - \left(\frac{\lambda_q}{n} \cdot \mathbb{A}_{q-1} + \mathbb{A}^{-}_{q-1} \right) \cdot \mathbb{B}^{s+2+j}_{k-2-j} \\
= \; & \frac{\lambda_{k-1-j} - \lambda_q}{n} \cdot \mathbb{A}_{q-1} \cdot \mathbb{B}^{s+2+j}_{k-2-j} + \left(\mathbb{A}^{-}_{q-1} + \mathbb{A}_{q-2} \right) \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-2-j} \\
& \qquad -\mathbb{A}^{-}_{q-1} \cdot \left({}^{-}\mathbb{B}^{s+2+j}_{k-2-j} + \mathbb{B}^{s+2+j}_{k-3-j} \right) \\
= \; & \frac{\lambda_{k-1-j} - \lambda_q}{n} \cdot \mathbb{A}_{q-1} \cdot \mathbb{B}^{s+2+j}_{k-2-j} + \mathbb{A}_{q-2} \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-2-j} - \mathbb{A}^{-}_{q-1} \cdot \mathbb{B}^{s+2+j}_{k-3-j}.
\end{aligned}$$

For the end point, if $q = 1$, we have

$${}^{-}\mathbb{B}^{s+2+j}_{k-1-j-t} - \frac{\lambda_1}{n} \cdot \mathbb{B}^{s+2+j}_{k-2-j-t} = \frac{\lambda_{k-1-j-t} - \lambda_1}{n} \cdot \mathbb{B}^{s+2+j}_{k-2-j-t} + {}^{-}\mathbb{B}^{s+2+j}_{k-2-j-t}.$$

If $k - 1 - j - t = s + 2 + j$, we have

$$\mathbb{A}_{q-1} \cdot {}^{-}\mathbb{B}^{s+2+j}_{s+2+j} - \frac{1}{n} \cdot \mathbb{A}^{-}_{q} = \frac{1}{n} \cdot \frac{\lambda_{s+2+j} - \lambda_q}{n} \cdot \mathbb{A}_{q-1} + \frac{1}{n} \cdot \mathbb{A}_{q-2}.$$

For $1 \le q \le s$, and t from 0 to $k - s - 3 - 2j$, we have $\lambda_q \le \lambda_{k-1-j-t}$. Thus we have the result of the lemma. □

Lemma 20.

$$\mathcal{L}_1 \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-1-j} + \mathcal{L}_2 \cdot \mathbb{B}^{s+2+j}_{k-2-j} \ge 0.$$

Proof. Recall that for $\mu = \min\{s - 1, n - k - 2\}$,

$$\mathcal{L}^1_{\mu+1} = \begin{cases} \mathbb{A}^n_{k+s+1}, & \mu = s - 1 \\ \mathbb{A}_{s-n+k}, & \mu = n - k - 2 \end{cases}, \quad \text{and} \quad -\mathcal{L}^2_{\mu+1} = \begin{cases} \frac{\lambda_k}{n}\mathbb{B}^1_{k+s+2}, & \mu = s - 1 \\ -\mathbb{A}^{-}_{s-n+k}, & \mu = n - k - 2 \end{cases},$$

we have

$$\begin{aligned}
& \mathcal{L}_1 \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-1-j} + \mathcal{L}_2 \cdot \mathbb{B}^{s+2+j}_{k-2-j} \\
= \; & \left(\sum_{l=0}^{\mu} \frac{\lambda_{k+1+l} - \lambda_{s-l}}{n} \cdot \mathbb{A}_{s-1-l} \cdot \mathbb{B}^1_{k+2+l} + \mathcal{L}^1_{\mu+1} \right) \cdot {}^{-}\mathbb{B}^{s+2+j}_{k-1-j} \\
& + \left(\sum_{l=0}^{\mu} \left(\frac{\lambda_k - \lambda_{k+1+l}}{n} \cdot \mathbb{A}^{-}_{s-1-l} \cdot \mathbb{B}^1_{k+2+l} + \frac{\lambda_k - \lambda_{s-l}}{n} \cdot \mathbb{A}_{s-1-l} \cdot \mathbb{A}^n_{k+2+l} \right) - \mathcal{L}^2_{\mu+1} \right) \cdot \mathbb{B}^{s+2+j}_{k-2-j}
\end{aligned}$$

Now, we match the term

$$\begin{aligned}
&\frac{\lambda_{k+1+l}-\lambda_{s-l}}{n}\cdot\mathbb{A}_{s-1-l}\cdot\mathbb{B}^{1}_{k+2+l}\cdot{}^{-}\mathbb{B}^{s+2+j}_{k-1-j}+\frac{\lambda_k-\lambda_{k+1+l}}{n}\cdot\mathbb{A}^{-}_{s-l}\cdot\mathbb{B}^{1}_{k+s+l}\cdot\mathbb{B}^{s+2+j}_{k-2-j}\\
=\;&\frac{\lambda_k-\lambda_{s-l}}{n}\mathbb{A}_{s-1-l}\cdot\mathbb{B}^{1}_{k+2+l}\cdot{}^{-}\mathbb{B}^{s+2+l}_{k-1-l}\\
&+\frac{\lambda_{k+1+l}-\lambda_k}{n}\cdot\mathbb{B}^{1}_{k+2+l}\cdot\left(\mathbb{A}_{s-1-l}\cdot{}^{-}\mathbb{B}^{s+2+j}_{k-1-j}-\mathbb{A}^{-}_{s-l}\cdot\mathbb{B}^{s+2+j}_{k-2-j}\right).
\end{aligned}$$

By setting $q = s - l$ for the previous Lemma, we have

$$\mathbb{A}_{s-1-l}\cdot{}^{-}\mathbb{B}^{s+2+j}_{k-1-j}-\mathbb{A}^{-}_{s-l}\cdot\mathbb{B}^{s+2+j}_{k-2-j}\geq 0.$$

For $\mathcal{L}^{1}_{\mu+1}\cdot{}^{-}\mathbb{B}^{s+2+j}_{k-1-j}-\mathcal{L}^{2}_{\mu+1}\cdot\mathbb{B}^{s+2+j}_{k-2-j}$, in the case when $\mu = s-1$, there is nothing to prove. For the case when $\mu = n-k-2$, we set $\mathcal{L}^{1}_{\mu+1}=\mathbb{A}^{-}_{s-n+k}+\mathbb{A}_{s-n+k-1}$. By setting $q = s-n+k-1$, (recall from our calculation in Lemma 12, in such case, $s-n+k+1\leq s$) we have

$$\mathbb{A}_{s-n+k-1}\cdot{}^{-}\mathbb{B}^{s+2+j}_{k-1-j}-\mathbb{A}^{-}_{s-n+k}\cdot\mathbb{B}^{s+2+j}_{k-2-j}\geq 0.$$

□

With the above step, we conclude our proof that

$$\mathcal{M}(s+1,k)\geq\widehat{\mathcal{M}}(s+1,k).$$

Clearly, by iterating such rearrangements, we can conclude

Theorem 6. *For $\lambda_1,\cdots,\lambda_{n-1}$ strictly postive, and $\lambda^*_1,\cdots,\lambda^*_{n-1}$ its increasing rearrangement, and $x_i = i/n$, a Brownian motion on $[0,1)$, reflected at 0 and absorbed at 1, satisfies*

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda_i\xi_i\right\}\right]\leq\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1}\lambda^*_i\xi_i\right\}\right],\quad \textit{where,}\quad \xi_i = l^{x_i}_{T_1}. \tag{3.10}$$

We can now conclude our main result by applying the occupation time formula.

Theorem 7. *Given $k(x)\geq 0$ is continuous on the interval $[0,1]$, let k^* be its unique right continuous non-decreasing rearrangement. Let B be a Brownian motion on $[0,1)$ reflected at 0 and absorbed at 1. Then*

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}k(B_t)dt\right\}\right]\leq\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}k^*(B_t)dt\right\}\right]. \tag{3.11}$$

Proof. Recall the occupation time formula for Brownian motion:

$$\int_0^{T_1} k(B_t)dt = \int_0^1 k(x) l^x_{T_1} dx.$$

We have also shown earlier that $l^x_{T_1}$ is jointly continuous. Thus, by a Reimann sum approximation we have:

$$\int_0^{T_1} k(B_t)dt = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} k(x_i) l^{x_i}_{T_1} \quad \text{a.s.}$$

where $x_i = i/n$.

Likewise, we have,

$$\int_0^{T_1} k^*(B_t)dt = \lim_{n\to 0} \frac{1}{n} \sum_{i=0}^{n-1} k^*(x_i) l^{x_i}_{T_1} \quad \text{a.s.}$$

If we set $\lambda_i = k(x_i)/n$, and apply inequality (3.10) to get

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1} \lambda_i l^{x_i}_{T_1}\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1} \lambda^*_i l^{x_i}_{T_1}\right\}\right].$$

Clearly,

$$\exp\left\{-\sum_{i=0}^{n-1} \lambda_i l^{x_i}_{T_1}\right\} \le 1 \quad \text{a.s.} \tag{3.12}$$

Dominated convergence theorem will now allows us to pass to the limit to obtain inequality (3.11). □

Remark 9. *One might ask whether our rearrangement result will hold for a broader set of diffusion processes. Here, we propose a simple extension to our result.*

For a diffusion process $\{X_t\}_{t\ge 0}$ *with drift* $\mu(x) \equiv 0$ *and* $\sigma^2(x) \ge \epsilon > 0$ *for all* $x \in [0,1]$ *, the decreasing right boundary condition of the Green function is:*

$$\begin{aligned} g_2(x) &= \int_x^1 \exp\left\{-\int_0^\xi 2\frac{\mu(\eta)}{\sigma^2(\eta)}\right\} d\xi \\ &= 1 - x, \end{aligned} \tag{3.13}$$

Thus, without any modification to our proof, we retain the same rearrangement inequality for such diffusion process.

However, recall from the previous chapter, the speed measure is no longer identically 1*, but instead becomes*

$$m(dx) = \frac{2}{\sigma^2(x)}dx. \tag{3.14}$$

The occupation time formula says

$$\int_0^{T_1} k(X_t)dt = \int_0^1 k(x) l^x_{T_1} m(dx). \tag{3.15}$$

The Riemann-sum approximation is:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \frac{2 \cdot k(x_i)}{\sigma^2(x_i)} l^{x_i}_{T_1} = \int_0^{T_1} k(X_t)dt \quad a.s. \tag{3.16}$$

If we keep the same set up, we will have $\lambda_i = \dfrac{2k(x_i)}{\sigma^2(x_i)}$*. However, we cannot have* $\lambda_i^* = \left(\dfrac{2k(x_i)}{\sigma^2(x_i)}\right)^*$ *because* $2/\sigma^2(x_i)$ *is part of the speed measure, and its position is fixed.*

Thus, for a diffusion process X *with infinitesimal generator* $\sigma(x) > 0$ *for all* $x \in [0, 1]$ *and* $\mu \equiv 1$*, we still have the local time rearrangement inequality*

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1} \lambda_i l^{x_i}_{T_1}\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{n-1} \lambda_i^* l^{x_i}_{T_1}\right\}\right],$$

but it will not translate into the rearrangement inequality

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(X_t)dt\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(X_t)dt\right\}\right],$$

without some restriction on $\sigma(x)$*. We have not explored such possibilities further.*

Remark 10. *Our method of rearranging the entries is motivated by that of Essén. It is worth pointing out that some of the simpler rearrangement schemes will not work in our context. For example, the "bubble sort" method of switching the order of two adjacent entries by moving the smaller entry to the front will not work.*

To find a counter example, we can take advantage of the identity we have from Lemma 11. Let $k = s + 2$ *and* $\mathcal{M}^*(s + 1, s + 2)$ *denotes the determinant resulting from the rearrangement. We have*

$$\mathcal{M}(s + 1, s + 2) - \mathcal{M}^*(s + 1, s + 2) = \frac{\lambda_{s+1} - \lambda_{s+2}}{n} \cdot \mathcal{L}_1. \tag{3.17}$$

Recall from the expression of $\mathcal{L}_1$*, the sufficient condition for* $\mathcal{L}_1 \geq 0$ *is* $\lambda_i \leq \lambda_j$ *for all* $i < s$ *and* $j > s+1$*. Thus, it is easy to come up with a counter example. The reader can check that*

$$\lambda_1 = 10, \quad \lambda_2 = 4, \quad \lambda_3 = 2, \quad \lambda_4 = 1, \quad \textit{and} \quad \lambda_5 = 0.$$

Then,

$$\mathcal{M}(2,3) < \mathcal{M}^*(2,3).$$

Even if we keep the set up same as before, but swap just the position of λ_{s+1} *and* λ_k*, and keep the position of* λ_{s+2} *through* λ_{k-1} *unchanged, such method won't work either. We let*

$$\lambda_1 = 1, \quad \lambda_2 = 10, \quad \lambda_3 = 1000, \quad \lambda_4 = 2, \quad \lambda_5 = 1, \quad \textit{and} \quad \lambda_6 = 1.$$

Here, we switch the position of λ_2 *and* λ_5*, and after the rearrangement, we have*

$$\lambda_1^* = 1, \quad \lambda_2^* = 1, \quad \lambda_3^* = 1000, \quad \lambda_4^* = 2, \quad \lambda_5^* = 10, \quad \textit{and} \quad \lambda_6^* = 1.$$

Let $\mathcal{M}(2,5)$ *be the determinant of the original matrix, and* $\mathcal{M}^*(2,5)$ *be the determinant of the matrix after the rearrangement. The reader can check*

$$\mathcal{M}(2,5) < \mathcal{M}^*(2,5).$$

Chapter 3 is based on the paper "A Local Time Inequality for Reflecting Brownian Motion" written jointly with Patrick Fitzsimmons, which is currently in preparation. The dissertation author is the primary author of this work.

Chapter 4

Alternative Proof of the Rearrangement Inequality and its Probabilistic Interpretation

In this chapter, we will present two Feynman-Kac identities, and use them to give an alternative proof of the rearrangement inequality of last chapter

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(B_s)ds\right\}\right] \leq \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(B_s)ds\right\}\right]. \tag{4.1}$$

Many of our calculations came from Essén's proof, and we will keep our notation conistent with his paper [4]. However, such approach allows us to draw several interesting conclusions from the rearrangement result.

We begin with the statement of Essén's result:

Theorem 8. *Let $p : [-\infty, 0] \to [0, \infty)$ be a lower semi-continuous, piece-wise constant function such that the range of p is finite. Assume that there exists a solution Φ of the inequality*

$$\Phi''(t) - p(t)^2\Phi(t) \geq 0, \qquad -\infty < t \leq 0,$$

such that $\Phi(0) = 1$, and $\lim_{t\to-\infty} \Phi(t)$ exists.

Let t_0 be given, $t_0 < 0$. If $\inf_t p(t) > 0$, there exists a non-negative solution Φ^* of the equation

$$\Phi^{*''}(t) - (p^*)^2(t)\Phi^*(t) = 0,$$

such that $\Phi^*(0) = 1$, $\Phi^*(-\infty) = 0$ *and*

$$\Phi(t_0) \leq \Phi^*(t_0).$$

Here, p^* *is the measure preserving, non-decreasing rearrangement of* p *on* $[t_0, 0]$ *and* $p^*(t) = \inf_s p(s)$ *for* $t < t_0$.

Remark 11. *Since* p *is a piecewise function,* Φ'' *needs not be continuous. In the case when* p *is discontinuous at* $x = a$*, we define*

$$\Phi''(a) = \lim_{x \to a^+} \Phi''(x).$$

4.1 Feynman-Kac Equation

The process $\{B_t\}_{t\geq 0}$ is a Brownian motion on $[0, 1)$ with a reflecting boundary at 0, and an absorbing boundary at 1. Proposition 5 of Appendix C says that

$$\varphi(x) = \mathbb{E}^x \left[\exp\left\{ -\int_0^{T_1} k(B_s) ds \right\} \right],$$

satisfies

$$\frac{1}{2}\varphi'' = k\varphi, \quad \text{and} \quad \varphi(1) = 1, \quad \varphi'(0) = 0.$$

By setting k to be a positive piecewise constant function, φ satisfies the condition of Essén's result after appropriate shifting. Thus, Essén's result gives a probabilistic interpretation as:

Theorem 9. *Let* $\{B_t\}_{t\geq 0}$ *be a Brownian motion on* $[0, 1)$ *with a reflecting boundary at* 0*, and a absorbing boundary at* 1*.* k *is a positive piecewise constant function and* k^* *is its non-decreasing rearrangement. We have*

$$\mathbb{E}^0 \left[\exp\left\{ -\int_0^{T_1} k(B_s) ds \right\} \right] \leq \mathbb{E}^0 \left[\exp\left\{ -\int_0^{T_1} k^*(B_s) ds \right\} \right]. \tag{4.2}$$

Remark 12. *Inequality (4.2) does not hold if the starting point* 0 *is replaced by an arbitrary* $x \in [0, 1]$*. For a simple-counter example, we can set* $k(x) = 1_{[0,\frac{1}{2}]}$*, so* $k^*(x) = 1_{[\frac{1}{2},1]}$*. We can see from the graph that* $\varphi(0.8) > \varphi^*(0.8)$.

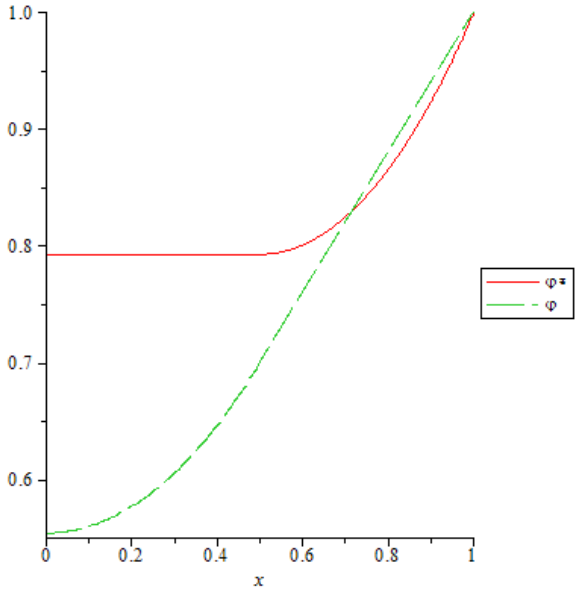

Figure 4.1: Example of $\varphi(x)$ vs $\varphi^*(x)$

To examine Essén's proof more closely, we need a scheme that will allow us to break the process $\{B_t\}_{t\geq 0}$ on the interval $[0,1]$ into four conditionally independ processes on segments: $[0, \triangle_1)$, $[\triangle_1, \triangle_1 + \triangle)$, $[\triangle_1 + \triangle, \widetilde{\triangle})$, and $[\widetilde{\triangle}, 1]$, where $\widetilde{\triangle} = \triangle_1 + \triangle + \triangle_2$. We use the fact that a Brownian motion can be split into two independent reflecting processes via the method of time change. It results in the theorem given next. We will give a short proof using Proposition 5 and Proposition 6. In the next section, we will discuss the method of time change in more detail, and give an alternative proof of the theorem.

Theorem 10. *Let k be a positive integrable function on $[0,1]$, and for $0 < a < 1$, define $k_1 = k1_{[0,a)}$ and $k_2 = k1_{[a,1]}$. Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on $[0,1)$ reflected at 0 and absorbed at 1. $\{B_t^{(1)}\}_{t\geq 0}$ is a Brownian motion on $[a,1)$ reflected at a and absorbed at 1. Define*

$$\begin{aligned}
\varphi(x) &= \mathbb{E}^x\left[\exp\left\{-\int_0^{T_1}(k_1(B_s)+k_2(B_s))ds\right\}\right], \quad \textit{for} \quad x\in[0,1].\\
\varphi_1(x) &= \mathbb{E}^x\left[\exp\left\{-\int_0^{T_a}k_1(B_s)ds\right\}\right], \quad \textit{for} \quad x\in[0,a].\\
\varphi_2(x) &= \mathbb{E}^x\left[\exp\left\{-\varphi_1'(a)l_{T_1}^a-\int_0^{T_1}k_2(B_s^{(1)})ds\right\}\right], \quad \textit{for} \quad x\in[a,1].
\end{aligned}$$

Then for $x \in [0, a)$

$$\begin{aligned}&\mathbb{E}^x\left[\exp\left\{-\int_0^{T_1}(k_1(B_s)+k_2(B_s))ds\right\}\right]\\ =&\mathbb{E}^x\left[\exp\left\{-\int_0^{T_a}k_1(B_s)ds\right\}\right]\cdot\mathbb{E}^a\left[\exp\left\{-\varphi_1'(a)l^a_{T_1}-\int_0^{T_1}k_2(B^{(1)}_s)ds\right\}\right].\end{aligned} \tag{4.3}$$

Proof. Let

$$\psi_1(x)=\frac{\varphi_1(x)}{\varphi_1(0)}.$$

Then, ψ_1 satisfies the initial condition

$$\psi_1(0)=1,\quad \psi_1'(0)=0.$$

Since

$$\varphi_1(a)=1,\quad \psi_1(a)=\frac{1}{\varphi_1(0)}\quad\text{and}\quad \psi_1'(a)=\frac{\varphi_1'(a)}{\varphi_1(0)}.$$

Define

$$\psi_2(x)=\frac{\varphi_2(x)}{\varphi_1(0)\varphi_2(a)}.$$

Then, ψ_2 satisfies the initial condition:

$$\psi_2(a)=\frac{1}{\varphi_1(0)}=\psi_1(a),$$

Proposition 6 says that $\varphi_2'(a)=\varphi_1'(a)\varphi_2(a)$. Thus

$$\psi_2'(a)=\frac{\varphi_2'(a)}{\varphi_1(0)\varphi_2(a)}=\frac{\varphi_1'(a)}{\varphi_1(0)}=\psi_1'(a).$$

Note that $\psi_1(a)=\psi_2(a)$, and $\psi_1'(a)=\psi_2'(a)$. Thus, $\psi(x)=\psi_1 1_{[0,a)}+\psi_2 1_{[a,1]}$, is continuously differentiable, and ψ satisfies

$$\frac{1}{2}\psi''=\begin{cases}k_1\psi, & \text{for}\quad x\in[0,a)\\ k_2\psi & \text{for}\quad x\in[a,1]\end{cases},\quad \psi(0)=1,\quad \psi'(0)=0.$$

The uniqueness part of Proposition 6 implies that $\varphi(x)=\dfrac{\psi(x)}{\psi(1)}=\dfrac{\psi(x)}{\psi_2(1)}$, and for $x\in[0,a)$

$$\begin{aligned}\varphi(x) &= \frac{\psi(x)}{\psi_2(1)}=\frac{\psi_1(x)}{\psi_2(1)}\\ &= \frac{\psi_1(x)}{\varphi_1(0)\psi_2(1)}=\frac{\varphi_1(x)}{\varphi_1(0)}\cdot\frac{\varphi_1(0)\varphi_2(a)}{\varphi_2(1)}\\ &= \varphi_1(x)\varphi_2(a).\end{aligned}$$

□

4.2 Method of Time Change and Knight's Theorem

We can split a Brownian motion B on $[0,1]$ into two process B^1 (resp. B^2) on $[0,a]$ (resp. $[a,1]$) for some $0 < a < 1$ via the method of time change:

$$\begin{aligned} \Gamma_1(t) &= \text{meas}\{0 \le \text{s} \le \text{t}: \quad \text{B}_\text{s} \in [0,\text{a}]\}, \\ s_1(\tau) &= \inf\{s \ge 0: \quad \Gamma_1(s) > \tau\}, \\ B^1_\tau &= B_{s_1(\tau)}. \end{aligned}$$

Likewise, we define B^2 the same way.

Then $\{B^1_\tau\}_{\tau\ge 0} \subset [0,a]$ and $\{B^2_\tau\}_{\tau\ge 0} \subset [a,1]$. Next, with the theorem by F. B. Knight, we can show that on the shifted filtrations, $(B^1_\tau, \mathcal{F}_{s_1(\tau)})$ and $(B^2_\tau, \mathcal{F}_{s_2(\tau)})$ are independent.

Theorem 11. ***(F.B. Knight)*** *Let $M = \{M_t = (M^1_t, M^2_t, \cdots, M^d_t),\ \mathcal{F}_t, 0 \le t < \infty)\}$ be a continuous, adapted process such that the M^i's are continuous local martingales with $\lim_{t\to\infty}\langle M^i\rangle_t = \infty$ P-a.s., and*

$$\langle M^i, M^j\rangle_t = 0; \quad 1 \le i \ne j \le d, \quad 0 \le t < \infty.$$

Define

$$T_i(s) \triangleq \inf\{t \ge 0: \ \langle M^i\rangle_t \ge s\}; \quad 0 \le s < \infty, \quad 1 \le i \le d,$$

so that for each i and s, the random time $T_i(s)$ is a stopping time for the (right-continuous) filtration $\{\mathcal{F}_t\}$. Then the processes

$$B^i_s = M^i_{T_i(s)}; \quad 0 \le s < \infty, \quad 1 \le i \le d,$$

are independent, standard, one-dimensional Brownian motions.

Lemma 21. *$(B^1_\tau, \mathcal{F}_{s_1(\tau)})$ and $(B^2_\tau, \mathcal{F}_{s_2(\tau)})$ are independent P^x a.s.*

Proof. Let $l_1(t) = l^0_t - l^a_t$ and $l_2(t) = l^1_t - l^a_t$.

Let s_1 and s_2 be the right continuous inverses of Γ_1 and Γ_2 respectively.

We define l^1 and l^2 as follow:

$$\begin{aligned} l^1(\tau) &:= l_1(s_1(\tau)) \\ &= \max_{0\le t\le s_1(\tau)} \int_0^t 1_{[0,a]}(B_v)dB_v - \min_{0\le t\le s_1(\tau)} \int_0^t 1_{[0,a]}(B_v)dB_v \\ &= \max_{0\le \Gamma_1(t)\le \tau} \int_0^{s_1(t)} 1_{[0,a]}(B_v)dB_v - \min_{0\le \Gamma_1(t)\le \tau} \int_0^{s_1(t)} 1_{[0,a]}(B_v)dB_v \\ &= \max_{0\le l\le \tau} \int_0^{s_1(l)} 1_{[0,a]}(B_v)dB_v - \min_{0\le l\le \tau} \int_0^{s_1(l)} 1_{[0,a]}(B_v)dB_v. \end{aligned}$$

Likewise,

$$\begin{aligned} l^2(\tau) &:= l_2(s_2(\tau)) \\ &= \max_{0\le l\le \mu} \int_0^{s_2(l)} 1_{[a,1]}(B_v)dB_v - \min_{0\le l\le \tau} \int_0^{s_2(l)} 1_{[a,1]}(B_v)dB_v. \end{aligned}$$

Then P^x almost surely, we have:

$$\begin{aligned} \left\langle \int_0^t 1_{[0,a]}(B_s)dB_s, \int_0^t 1_{[a,1]}(B_s)dB_s \right\rangle &= \int_0^t 1_{\{a\}}(B_s)ds \\ &= \int_0^1 1_{\{a\}}(x) l_t^x dx \\ &= 0. \end{aligned}$$

By the Knight's theorem, we have $\left\{ \int_0^{s_1(\tau)} 1_{[0,a]}(B_s)dB_s \right\}_{\tau\ge 0}$ independent of $\left\{ \int_0^{s_2(\tau)} 1_{[a,1]}(B_s)dB_s \right\}_{\tau\ge 0}$ P^x a.s.

Consequently, we have l^1 independent of l^2 P^x -a.s. as well.

By the Skorohod representation, we can rewrite B^1 and B^2 as follow:

$$\begin{aligned} B^1(\tau) &= \int_0^{s_1(\tau)} 1_{[0,a]}(B_s)dB_s + \max_{0\le l\le \tau} \int_0^{s_1(l)} 1_{[0,a]}(B_v)dB_v \\ &\quad - \min_{0\le l\le \tau} \int_0^{s_1(l)} 1_{[0,a]}(B_v)dB_v. \end{aligned}$$

and

$$\begin{aligned} B^2(\tau) &= \int_0^{s_2(l)} 1_{[a,1]}(B_s)dB_s + \max_{0\le l\le \tau} \int_0^{s_2(l)} 1_{[a,1]}(B_v)dB_v \\ &\quad - \min_{0\le l\le \tau} \int_0^{s_2(l)} 1_{[a,1]}(B_v)dB_v. \end{aligned}$$

Hence, B^1 and B^2 are independent P^x a.s. □

For the next lemma, we will use the same notation for B^1 and B^2.

Lemma 22. *Let T be a stopping time of $\mathcal{G}_2(\tau) = \mathcal{F}_{s_2(\tau)}$. Define $\xi := l^a_{T_1}$, where $T_1 = \inf\{t \geq 0 : \quad B^2(t) = 1\}$ and $T^1_\xi = \inf\{\mu \geq 0 : l^a_{s_1(\tau)} > \xi\}$. Then given ξ, the processes $\widehat{B^1}(\tau) = B^1(\tau \wedge T^1_\xi)$ and $\widehat{B^2}(\tau) = B^2(\tau \wedge T_1)$ are independent.*

Proof. When ξ is given, T^1_ξ is a stopping time of the filtration of $\mathcal{G}_1(\tau) = \mathcal{F}_{s_1(\tau)}$. Since B_1 and B_2 are independent, $\widehat{B^1}$ and $\widehat{B^2}$ are as well. □

Lemma 23. *Let $T_\xi = \inf\{s \geq 0 : l^b_s > \xi\}$, where ξ is a positive constant. Then*

$$\mathbb{E}^x[e^{-K_{T_\xi}}] = \varphi(x)\exp\{-\xi \cdot \varphi'(b)\}.$$

Proof. By equation (B.6), and taking expectations on both side, we have:

$$\mathbb{E}^x[\varphi(B_{T_\xi})e^{-K_{T_\xi}}] = \varphi(x) - \mathbb{E}^x\left[\int_0^{T_\xi} e^{-K_s} dl^b_s\right]\varphi'(b).$$

Notice that $\varphi(B_{T_\xi}) = \varphi(b) = 1$. Thus the left hand side becomes $\mathbb{E}^x[e^{-K_{T_\xi}}]$.

By making the substitution $\mu(s) = l^b_s$, we get $s = \inf\{t \geq 0 : l^b_t > \mu\} = T_\mu$. With Fubini, the expectation on the right becomes,

$$\mathbb{E}^x\left[\int_0^{T_\xi} e^{-K_s} dl^b_s\right] = \int_0^\xi \mathbb{E}^x[e^{-K_{T_\mu}}]d\mu.$$

If we define a function of ξ as $f(\xi) = \mathbb{E}^x[e^{-K_{T_\xi}}]$, we have the following equation:

$$f(\xi) = \varphi(x) - \varphi'(b)\int_0^\xi f(\mu)d\mu.$$

Clearly, $f(0) = \varphi(x)$, and we also have the differential equation

$$f'(\xi) = -\varphi'(b)f(\xi).$$

By solving the differential equation with initial condition, we get

$$f(\xi) = \mathbb{E}^x[e^{-K_{T_\xi}}] = \varphi(x)e^{-\xi\varphi'(b)}.$$

□

We are now ready to give an alternative proof for Theorem 10.

Proof. By Knight's Theorem, $(B^1_\tau, \mathcal{F}_{s_1(\tau)})$ and $(B^2_\tau, \mathcal{F}_{s_2(\tau)})$ are independent.

We set $\eta = \frac{1}{2} l^a_2(T_1)$, where $T_1 = \inf\{\tau \geq 0 : \quad B^2_\tau = 1\} \subset \mathcal{F}_{s_2(\tau)}$, and $l^a_2(t)$ refers to the local time of B^2 at point a up to time t. $T_\eta = \inf\{t \geq 0 : \quad l^a_1(t) > \eta\}$, where $l^a_1(t)$ is the local time of B^1 at point a up to time t. Then, first by Lemma 22 and next by Lemma 23, we get

$$\begin{aligned}
& \mathbb{E}^x\left[\exp\left\{-\int_0^{T_1}(k_1(B_s)+k_2(B_s))ds\right\}\right] \\
= \; & \mathbb{E}^x\left[\mathbb{E}^x\left[\exp\left\{-\int_0^{T_\eta} k_1(B_{s_1})ds_1\right\}\middle|\eta\right]\cdot\mathbb{E}^x\left[\exp\left\{-\int_0^{T_1} k_2(B_{s_2})ds_2\right\}\middle|\eta\right]\right] \\
= \; & \mathbb{E}^x\left[\varphi_1(x)\mathbb{E}^a\left[\exp\left\{-\varphi_1'(x)l^a_{T_\eta} - \int_0^{T_1} k_2(B_{s_2})ds_2\right\}\middle|\eta\right]\right] \\
= \; & \varphi_1(x)\mathbb{E}^a\left[\exp\left\{-\varphi_1'(x)l^a_{T_1} - \int_0^{T_1} k_2(B_s)ds\right\}\right].
\end{aligned}$$

□

4.3 Set-Up and Preliminary

We will concentrate on the inductive step of the proof. To keep our notation consistent with in Essén's paper [4], we let p be a piecewise constant function with values be $\sigma_0 < \sigma_1 < \cdots$, such that:

$$p^2 = \begin{cases} < \sigma_1, & \text{for} \quad x \in [0, \triangle_1) \\ > \sigma_1, & \text{for} \quad x \in [\triangle_1, \triangle_1 + \triangle) \\ \sigma_1, & \text{for} \quad x \in [\triangle_1 + \triangle, \widetilde{\triangle}) \\ > \sigma_1, & \text{for} \quad x \in [\widetilde{\triangle}, 1] \end{cases} \tag{4.4}$$

where $\widetilde{\triangle} = \triangle_1 + \triangle_2 + \triangle$.

$(p^*)^2$ is the function resulting from taking the segment whose level is σ_1 and pushing it to the left in the following way.

$$(p^*)^2 = \begin{cases} < \sigma_1, & \text{for} \quad x \in [0, \triangle_1) \\ \sigma_1, & \text{for} \quad x \in [\triangle_1, \triangle_1 + \triangle_2) \\ > \sigma_1, & \text{for} \quad x \in [\triangle_1 + \triangle_2, \widetilde{\triangle}) \\ > \sigma_1, & \text{for} \quad x \in [\widetilde{\triangle}, 1] \end{cases} \tag{4.5}$$

Example:

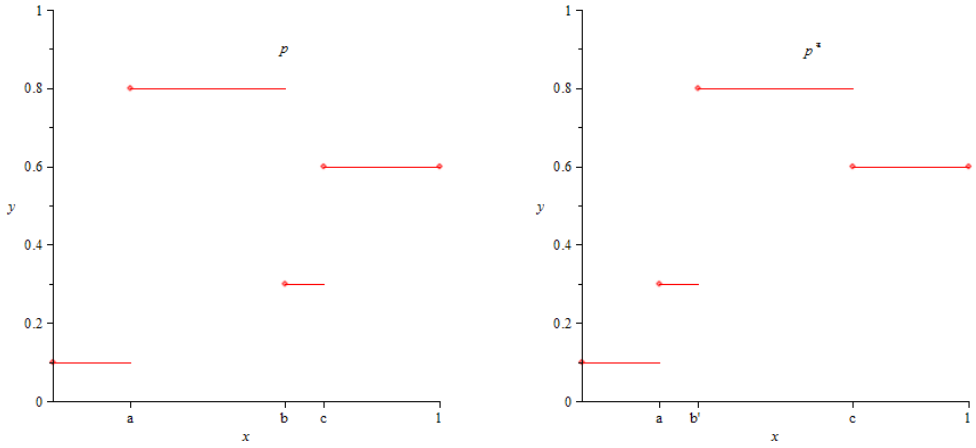

Figure 4.2: Example of $p(x)$ vs $p^*(x)$.

Remark 13. *It is clear that for a simple function p, we have* $(p^2)^* = (p^*)^2$. *For a Lebesgue measurable function p, we can pick a sequence of simple functions* $\{p_n\}$ *such that* $p_n \to p$ *almost everywhere. A result from next chapter will show that* $p_n^* \to p^*$ *almost everywhere. Thus, we have* $(p^2)^* = (p^*)^2$ *for Lebesgue measurable function as well.*

To simplify our calculation, we consider φ (resp. φ^*) satisfying

$$\varphi'' = p^2\varphi. \quad (\text{ resp. } \quad \varphi^*) \tag{4.6}$$

The next few lemmas give us the tools to prove our result. We will translate equation (4.6) into a Ricatti equation, which simplifies many of the calculations.

If φ is a solution to (4.6), for $x \geq 0$, and $\varphi > 0$, then define

$$g(x) = \frac{\varphi'(x)}{\varphi(x)}, \quad \text{for} \quad x \geq 0.$$

Then g satisfies the Riccati equation

$$g'(x) = p^2(x) - g^2(x). \tag{4.7}$$

For what follows, p satisfies (4.4).

Lemma 24. *Let σ be a strictly positive constant such that $p^2(x) \geq \sigma$ for all $x \geq 0$. Let z solved equation (4.6) with*

$$z(0) = A > 0, \quad z'(0) = -B < 0, \tag{4.8}$$

such that

$$\sigma \cdot z(0) \geq -z'(0).$$

Then, $z(x) > 0$ for all $x \geq 0$.

Proof. Let g be the solution to (4.7) with initial condition $g(0) = z'(0)/z(0)$. Then consider

$$\varphi(x) = z(0) \exp\left\{\int_0^x g(t)dt\right\}.$$

Notice,

$$\begin{aligned}
\varphi'(x) &= g(x)\varphi(x) \\
\varphi''(x) &= g'(x)\varphi(x) + g(x)\varphi'(x) \\
&= (p^2(x) - g^2(x))\varphi(x) + g^2(x)\varphi(x) \\
&= p^2(x)\varphi(x).
\end{aligned}$$

Since $\varphi(0) = z(0)$ and

$$\varphi'(0) = g(0) \cdot \varphi(0) = \frac{z'(0)}{z(0)} \cdot z(0) = z'(0).$$

we have $\varphi(x) = z(x)$.

Because, $B/A \geq -\sigma$, we have g is bounded. Hence, $z(x) > 0$. □

Remark 14. *The conclusion of Lemma 24 could fail if the condition (4.8) is not satisfied. One counter-example is as follows.*

Let $z_1(x) = \cosh(\sigma x)$, and $z_2(x) = \sigma^{-1}\sinh(\sigma x)$ where $0 < \sigma < 1$.

Then, $\sigma(z_1(0) - z_2(0)) < (z_2'(0) - z_1'(0))$, but $z_1(x) < z_2(x)$ when $x > \sigma^{-1}\tanh^{-1}(\sigma)$.

Lemma 25. *Suppose both φ_1 and φ_2 solve equation (4.6), with*

$$\begin{aligned}
\varphi_1(0) = a_1, &\quad \varphi_2(0) = a_2, \\
\varphi_1'(0) = b_1, &\quad \varphi_2'(0) = b_2,
\end{aligned}$$

such that $a_1 > a_2$ *and* $b_1 < b_2$.

If $\min_{x\geq 0} p^2(x) > \sigma_1$ *and*

$$\sigma_1(a_1 - a_2) - (b_2 - b_1) \geq 0, \tag{4.9}$$

then $\varphi_1(x) > \varphi_2(x)$ *for all* $x \geq 0$.

Lemma 26. *Suppose* φ *satisfies equation (4.6) on the interval* $[0, \triangle]$ *with*

$$\varphi(1) = 1, \quad \textit{and} \quad \varphi'(0) = 0.$$

Define

$$\varphi(\triangle) \equiv C_1, \quad \textit{and} \quad \varphi'(\triangle) \equiv D_1.$$

If $p^2 \leq \sigma_1$ *on* $[0, \triangle]$*, then*

$$C_1 - \sigma_1^{-1} D_1 \geq 0.$$

Proof. We set $g(x) = \dfrac{\varphi'(x)}{\varphi(x)}$. Then $p^2 \leq \sigma_1$ implies than $g(x) \leq \sigma_1$ for all $x \in [0, \triangle]$.

Thus,

$$g(\triangle) = \frac{C_1}{D_1} \leq \sigma_1.$$

□

Lemma 27. *Let* φ *satisfy equation (4.6) on the interval* $(\triangle_1, \triangle_1 + \triangle) = (a, b)$ *with boundary conditions*

$$\varphi(b) = 1, \quad \varphi'(a) = \sigma_1 \varphi(a).$$

Then,

$$\varphi'(b) \geq \sigma_1.$$

Proof. Let

$$g(x) = \frac{\varphi'(x)}{\varphi(x)} \quad \text{for} \quad x \in (a, b).$$

Then, g satisfies the Riccati equation with initial condition

$$g' = p^2 - g^2, \quad g(a) = \sigma_1.$$

Because $p^2(x) \geq \sigma_1$ for all $x \in (a, b)$, we have $g'(a) \geq 0$. Hence, $g(b) \geq \sigma_1$, which is the desired inequality. □

Lemma 28. *Let φ be the solution of (4.6) on (a, b) with the boundary conditions*

$$\varphi(b) = 1, \quad \varphi'(a) = -\sigma_1\varphi(a).$$

Then

$$\varphi'(b) \geq -\sigma_1.$$

Proof. Again, we let

$$g(x) = \frac{\varphi'(x)}{\varphi(x)}.$$

Then $g'(x) \geq 0$ for all $x \in (a, b)$.

Hence $g(b) \geq -\sigma_1$, which is the same as the inequality we want to prove. □

Lemma 29. *Given that $p^2(x) > \sigma_1$ on the interval $(\triangle_1, \triangle_1 + \triangle) = (a, b)$. Let v_1 and v_2 be the solutions of (4.6) on the same interval such that*

$$\begin{aligned} v_1(a) = 1, \quad v_1'(a) = 0, \quad v_2(a) = 0, \quad v_2'(a) = 1, \\ v_1(b) = A_1, \quad v_1'(b) = B_1, \quad v_2(b) = A_2, \quad v_2'(b) = B_2. \end{aligned}$$

Then,

$$B_1 - \sigma_1^2 A_2 \geq \sigma_1 |B_2 - A_1|. \tag{4.10}$$

Proof. Let $v = v_1 + \sigma_1 v_2$. Then $v(a) = 1$ and $v'(a) = \sigma_1$. Notice, if we set

$$\varphi(x) = \frac{v(x)}{v(b)},$$

then φ solves equation (4.6) on the interval (a, b) with the boundary condition

$$\varphi(b) = 1, \quad \varphi'(a) = \sigma_1\varphi(a).$$

By Lemma 27, we have

$$\varphi'(b) = \frac{v'(b)}{v(b)} \geq \sigma_1.$$

Hence, we have

$$v'(b) \geq \sigma_1 v(b),$$

which implies

$$B_1 + \sigma_1 B_2 \geq \sigma_1 A_1 + \sigma_1^2 A_2. \tag{4.11}$$

Similarly, setting $v = v_1 - \sigma_1 v_2$ and

$$\psi(x) = \frac{v(x)}{v(b)},$$

which implies

$$\psi(b) = 1, \quad \psi'(a) = -\sigma_1 \psi(a).$$

By Lemma 28, we get

$$v'(b) \geq -\sigma_1 v(b),$$

or

$$B_1 - \sigma_1 B_2 \geq -\sigma_1 A_1 + \sigma_1^2 A_2. \tag{4.12}$$

Combining (4.11) and (4.12) gives the desire inequality. □

4.4 Alternative Proof of the Rearrangement Inequality

To keep our notation consistent, we set

$$\varphi(x) = \mathbb{E}^x \left[\exp\left\{ -\int_0^{T_1} \frac{1}{2} p^2(B_s) ds \right\} \right],$$

where p is defined as in the previous section. Therefore

$$\varphi'' = p^2 \varphi, \quad \varphi(1) = 1, \quad \text{and} \quad \varphi'(0) = 0.$$

We can restate our rearrangement result as

Theorem 12. *Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on $[0, 1)$ reflected at 0 and absorbed at 1. Suppose p and p^* are piece-wise positive function on $[0, 1]$ as defined in (4.15) and (4.5), respectively. Then*

$$\mathbb{E}^0 \left[\exp\left\{ -\int_0^{T_1} \frac{1}{2} p^2(B_s) ds \right\} \right] \leq \mathbb{E}^0 \left[\exp\left\{ -\int_0^{T_1} \frac{1}{2} (p^*)^2(B_s) ds \right\} \right]. \tag{4.13}$$

Proof. Let $B^{(1)}$, $B^{(2)}$, $B^{(3)}$, and $B^{(4)}$ be Brownian motion on the interval $[0, \triangle)$, $[\triangle, b)$, $[b, \widetilde{\triangle})$ and $[\widetilde{\triangle}, 1)$ respectively, with lower reflecting boundary and upper absorbing boundary.

Define,

$$\begin{aligned}
\varphi_1(x) &= \mathbb{E}^x\left[\exp\left\{-\int_0^{T_\triangle} p_1(B_s^{(1)})ds\right\}\right],\\
\varphi_2(x) &= \mathbb{E}^x\left[\exp\left\{-\varphi_1'(\triangle_1)l_{T_b}^{\triangle_1} - \int_0^{T_b} p_2(B_s^{(2)})ds\right\}\right],\\
\varphi_3(x) &= \mathbb{E}^x\left[\exp\left\{-\varphi_2'(b)l_{T_{\widetilde{\triangle}}}^{b} - \int_0^{T_{\widetilde{\triangle}}} p_3(B_s^{(3)})ds\right\}\right],\\
\varphi_4(x) &= \mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\varphi_3'(\widetilde{\triangle})l_{T_1}^{\widetilde{\triangle}} - \int_0^{T_1} p_4(B_s^{(4)})ds\right\}\right],
\end{aligned} \tag{4.14}$$

where

$$p_1 = \frac{1}{2}p^2 1_{[0,\triangle_1)}, \quad p_2 = \frac{1}{2}p^2 1_{[\triangle_1,b)}, \quad p_3 = \frac{1}{2}p^2 1_{[b,\widetilde{\triangle})}, \quad p_4 = \frac{1}{2}p^2 1_{[\widetilde{\triangle},1]}. \tag{4.15}$$

By repeatedly applying Theorem 10, we have

$$\begin{aligned}
\varphi(0) &= \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}p^2(B_s)ds\right\}\right]\\
&= \varphi_1(0)\cdot\varphi_2(\triangle_1)\cdot\varphi_3(b)\cdot\varphi_3(\widetilde{\triangle}).
\end{aligned} \tag{4.16}$$

We calculate $\varphi_1,\cdots,\varphi_4$ (resp. $\psi_1,\cdots,\psi_4$), by calculating their corresponding initial condition solutions $\xi_1,\cdots,\xi_4$ (resp. $\zeta_1,\cdots,\zeta_4$.), as

$$\begin{aligned}
&\xi_1(0) = 1, \quad \xi_2(\triangle_1+\triangle) = \xi_1(\triangle_1+\triangle), \quad \xi_3(b) = \xi_2(b), \quad \xi_4(\widetilde{\triangle}) = \xi_3(\widetilde{\triangle}),\\
&\xi_1'(0) = 0, \quad \xi_2'(\triangle_1+\triangle) = \xi_1'(\triangle_1+\triangle), \quad \xi_3'(b) = \xi_2'(b), \quad \xi_4'(\widetilde{\triangle}) = \xi_3'(\widetilde{\triangle}).
\end{aligned}$$

and set:

$$\varphi_1(x) = \frac{\xi_1(x)}{\xi_1(\triangle_1)}, \quad \varphi_2(x) = \frac{\xi_2(x)}{\xi_2(b)}, \quad \varphi_3(x) = \frac{\xi_3(x)}{\xi_3(\widetilde{\triangle})}, \quad \text{and} \quad \varphi_4 = \frac{\xi_4(x)}{\xi_4(1)}.$$

We denote

$$\xi_1(\triangle_1) = C_1 \quad \text{and} \quad \xi_1'(\triangle_1) = D_1.$$

Let v_1 and v_2 be the solution of (4.6) on $[\triangle_1, b)$ with the initial condition

$$v_1(\triangle_1) = 1, \quad v_1'(\triangle_1) = 0; \quad v_2(\triangle_1) = 0, \quad v_2'(\triangle_1) = 1.$$

and we denote

$$v_1(b) = A_1, \quad v_1'(b) = B_1; \quad v_2(b) = A_2, \quad v_2'(b) = B_2.$$

Setting ξ_2 be the solution of (4.6) on $[\triangle_1, b)$ with initial condition

$$\xi_2(\triangle_1) = C_1, \quad \xi_2'(\triangle_1) = D_1.$$

Then

$$\xi_2(x) = C_1 v_1(x) + D_1 v_2(x).$$

Hence,

$$\xi_2(b) = C_1 A_1 + D_1 A_2; \quad \xi_2'(b) = C_1 B_1 + D_1 B_2.$$

Setting ξ_3 be the solution of (4.6) on $[b, \widetilde{\triangle})$ with initial condition

$$\xi_3(b) = \xi_2(b), \quad \xi_3'(b) = \xi_2'(b).$$

Then,

$$\xi_3(x) = \xi_2(b)\cosh(\sigma_1(x-b)) + \frac{\xi_2'(b)}{\sigma_1}\sinh(\sigma_1(x-b)).$$

Now consider p^*, and

$$\psi'' = (p^*)^2\psi. \tag{4.17}$$

Recall that p is a piecewise function, so $2(p^2)^*$ is the increasing rearrangement of $2p^2$. Setting $c = \triangle_1 + \triangle_2$, we denote

$$p_1 = \frac{1}{2}(p^*)^2 1_{[0,\triangle_1)}, \quad p_2^* = \frac{1}{2}(p^*)^2 1_{[\triangle_1, c)}, \quad p_3^* = \frac{1}{2}(p^*)^2 1_{[c,\widetilde{\triangle})}, \quad \text{and} p_4 = \frac{1}{2}(p^*)^2 1_{[\widetilde{\triangle},1]}. \tag{4.18}$$

We have

$$\begin{aligned}
\psi_1(x) &= \mathbb{E}^x\left[\exp\left\{-\int_0^{T_{\triangle_1}} p_1(W_s^{(1)})ds\right\}\right],\\
\psi_2(x) &= \mathbb{E}^x\left[\exp\left\{-\psi_1'(\triangle_1)l_{T_c}^{\triangle_1} - \int_0^{T_c} p_2^*(W_s^{(1)})ds\right\}\right],\\
\psi_3(x) &= \mathbb{E}^x\left[\exp\left\{-\psi_2'(c)l_{T_{\widetilde{\triangle}}}^{c} - \int_0^{T_{\widetilde{\triangle}}} p_3^*(W_s^{(3)})ds\right\}\right],\\
\psi_4(x) &= \mathbb{E}^x\left[\exp\left\{-\psi_3'(\widetilde{\triangle})l_{T_1}^{\widetilde{\triangle}} - \int_0^{T_1} p_4(W_s^{(4)})ds\right\}\right],
\end{aligned} \tag{4.19}$$

where $W^{(1)}$, $W^{(2)}$, $W^{(3)}$ and $W^{(4)}$ are Brownian motions on the intervals $[0, \triangle_1)$, $[\triangle_1, c)$, $[c, \widetilde{\triangle})$ and $[\widetilde{\triangle}, 1)$ respectively, with lower reflecting boundary and upper absorbing boundary.

Thus, we have

$$\begin{aligned}\psi(0) &= \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}(p^2)^*(B_s)ds\right\}\right] \\ &= \psi_1(0)\cdot\psi_2(\triangle_1)\cdot\psi_3(c)\cdot\psi_4(\widetilde{\triangle}).\end{aligned} \tag{4.20}$$

Likewise, we set

$$\psi_1(x)=\frac{\zeta_1(x)}{\zeta_1(\triangle_1)},\quad \psi_2(x)=\frac{\zeta_2(x)}{\zeta_2(b)},\quad \psi_3(x)=\frac{\zeta_3(x)}{\zeta_3(\widetilde{\triangle})},\quad \text{and}\quad \psi_4=\frac{\zeta_4(x)}{\zeta_4(1)}.$$

Note that $\xi_1(x)=\zeta_1(x)$. For the next two intervals, we have

$$\begin{aligned}\zeta_2(x) &= C_1\cosh(\sigma_1(x-\triangle_1))+\frac{D_1}{\sigma_1}\sinh(\sigma_1(x-\triangle_1)),\\ \zeta_2(b) &= C_1A_1+D_1A_2,\\ \zeta_2'(b) &= C_1B_1+D_1B_2.\end{aligned}$$

$$\begin{aligned}\zeta_3(x) &= \zeta_2(\triangle_1+\triangle_2)v_1(x-\triangle_2)+\zeta_2'(\triangle_1+\triangle_2)v_2(x-\triangle_2),\\ \zeta_3(\widetilde{\triangle}) &= \zeta_2(\triangle_1+\triangle_2)A_1+\zeta_2'(\triangle_1+\triangle_2)A_2\\ &= (C_1\cosh(\sigma_1\triangle_2)+\sigma_1^{-1}D_1\sinh(\sigma_1\triangle_2))A_1+(\sigma_1C_1\sinh(\sigma_1\triangle_2)\\ &\quad +D_1\cosh(\sigma_1\triangle_2))A_2\\ &= (C_1A_1+D_1A_1)\cosh(\sigma_1\triangle_2)+(\sigma_1^{-1}D_1A_1+\sigma_1C_1A_2)\sinh(\sigma_1\triangle_2),\\ \zeta_3'(\widetilde{\triangle}) &= (C_1B_1+D_1B_2)\cosh(\sigma_1\triangle_2)+(\sigma_1^{-1}D_1B_1+\sigma_1C_1B_2)\sinh(\sigma_1\triangle_2).\end{aligned}$$

We have

$$\begin{aligned}\sigma_1(\xi_3(\widetilde{\triangle})-\zeta_3(\widetilde{\triangle})) &= [C_1(B_1-\sigma_1^2A_2)+D_1(B_2-A_1)]\cdot\sinh(\sigma_1\triangle_2),\\ \zeta_3'(\widetilde{\triangle})-\xi_3'(\widetilde{\triangle}) &= [\sigma_1C_1(A_1-B_2)+D_1(\sigma_1A_2-\sigma_1^{-1}B_1)]\cdot\sinh(\sigma_1\triangle_2).\end{aligned}$$

Thus,

$$\begin{aligned}&\sigma_1(\xi_3(\widetilde{\triangle})-\zeta_3(\widetilde{\triangle}))-(\zeta_3'(\widetilde{\triangle})-\xi_3'(\widetilde{\triangle}))\\ =\ &[B_1-A_2\sigma_1^2-\sigma_1(B_2-A_1)]\cdot(C_1-\sigma_1^{-1}D_1)\cdot\sinh(\sigma_1\triangle_2)\end{aligned}$$

By Lemma 26, we have

$$C_1-\sigma_1^{-1}D_1\geq 0. \tag{4.21}$$

Also, by Lemma 29

$$B_1 - A_2\sigma_1^2 - \sigma_1(B_2 - A_1) \geq 0. \tag{4.22}$$

Thus, condition (4.9) of Lemma 25 is satisfied, and we have

$$\xi_4(1) > \zeta_4(1).$$

Recall that

$$\varphi(0) = \frac{\xi_1(0)}{\xi_1(\triangle_1)} \cdot \frac{\xi_2(\triangle_1)}{\xi_2(b)} \cdot \frac{\xi_3(b)}{\xi_3(\widetilde{\triangle})} \cdot \frac{\xi_4(\widetilde{\triangle})}{\xi_4(1)} = \frac{1}{\xi_4(1)}.$$

Likewise, $\psi(0) = \dfrac{1}{\zeta_4(1)}$. Hence

$$\varphi(0) < \psi(0).$$

□

4.5 Some Observations of the Proof

As we can see, Essén's rearrangement scheme is a very important part of his proof. To get equation (4.21), we need $\max\limits_{x\in[0,\triangle_1)} p^2(x) \leq \sigma_1$; equation (4.22) requires $\min\limits_{x\in[\triangle_1,\triangle_1+\triangle)} p^2(x) \geq \sigma_1$; and the condition for

$$\sigma_1(\xi_3(\widetilde{\triangle}) - \zeta_3(\widetilde{\triangle})) - (\zeta_3'(\widetilde{\triangle}) - \xi_3'(\widetilde{\triangle})) \geq 0, \tag{4.23}$$

to imply $\varphi(0) \leq \psi(0)$ is $\min\limits_{x\in[\widetilde{\triangle},1]} p^2(x) \geq \sigma_1$. As a counter example exhibited in Chapter 3 has shown, this type of rearrangement is necessary.

In fact, those restrictions on the value of $p(x)$ have interesting probabilitic meanings. To examine them more closely, we first need to rewrite Lemma 25 as:

Theorem 13. *Let $\{B_t\}_{t\geq 0}$ be a Brownian motion on $[b, 1)$ for some $0 \leq b < 1$, with a reflecting boundary at b and an aborbing boundary at 1. Suppose p is a positive piece-wise function on $[b, 1]$, such that $\min\limits_{x\in[b,1]} p^2(x) > \sigma_1 > 0$.*

Given $\alpha_2 > \alpha_1$ and $\beta_2 > \beta_1$, define

$$\begin{aligned}\varphi(x) &= \alpha_1 \cdot \mathbb{E}^x\left[\exp\left\{-\beta_1 l^b_{T_1} - \int_0^{T_1} \frac{1}{2}p^2 1_{[b,1)}(B_s)ds\right\}\right],\\ \psi(x) &= \alpha_2 \cdot \mathbb{E}^x\left[\exp\left\{-\beta_2 l^b_{T_1} - \int_0^{T_1} \frac{1}{2}p^2 1_{[b,1)}(B_s)ds\right\}\right].\end{aligned}$$

If

$$\frac{\beta_2 \cdot \alpha_1 - \beta_1 \cdot \alpha_2}{\alpha_2 - \alpha_1} < \sigma_1, \tag{4.24}$$

then $\psi(b) > \varphi(b)$.

Let p_1, p_2, p_3 be defined as in (4.15). Define

$$\begin{aligned}\hat{\varphi}(x) &= \mathbb{E}^x\left[\exp\left\{-\int_0^{T_{\widetilde{\triangle}}}(p_1(B_s)+p_2(B_s)+p_3(B_s))ds\right\}\right]\\ \varphi(0) &= \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}p^2(B_s)ds\right\}\right]\\ &= \hat{\varphi}(0)\cdot\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\hat{\varphi}'(\widetilde{\triangle})l^{\widetilde{\triangle}}_{T_1} - \int_0^{T_1} p_4(B^{(4)}_s)ds\right\}\right].\end{aligned}$$

Likewise,

$$\begin{aligned}\hat{\psi}(x) &= \mathbb{E}^x\left[\exp\left\{-\int_0^{T_{\widetilde{\triangle}}}(p_1(B_s)+p^*_2(B_s)+p^*_3(B_s))ds\right\}\right]\\ \psi(0) &= \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}(p^*)^2(B_s)ds\right\}\right]\\ &= \hat{\psi}(0)\cdot\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\hat{\psi}'(\widetilde{\triangle})l^{\widetilde{\triangle}}_{T_1} - \int_0^{T_1} p_4(B^{(4)}_s)ds\right\}\right].\end{aligned}$$

From the previous calculation, we have

$$\begin{aligned}\alpha_1 = \hat{\varphi}(0) = 1/\xi_3(\widetilde{\triangle}), \quad \beta_1 = \hat{\varphi}'(\widetilde{\triangle}) = \xi_3'(\widetilde{\triangle})/\xi_3(\widetilde{\triangle}),\\ \alpha_2 = \hat{\psi}(0) = 1/\zeta_3(\widetilde{\triangle}), \quad \beta_2 = \hat{\psi}'(\widetilde{\triangle}) = \zeta_3'(\widetilde{\triangle})/\psi_3(\widetilde{\triangle}).\end{aligned}$$

The condition (4.24) can be written explicitly as

$$\frac{\hat{\psi}'(\widetilde{\triangle})\cdot\hat{\varphi}(0) - \hat{\varphi}'(\widetilde{\triangle})\cdot\hat{\psi}(0)}{\hat{\psi}(0)-\hat{\varphi}(0)} < \sigma_1.$$

Next, we break the event into two parts: the process starts from 0 and reaches $\widetilde{\triangle}$ for the first time; and the process starts from $\widetilde{\triangle}$ and reaches 1. Using the strong Markov

property, we have

$$\begin{aligned}&\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}p^2(B_s)ds\right\}\right]\\ =&\mathbb{E}^0\left[\exp\left\{-\int_0^{T_{\widetilde{\triangle}}}\frac{1}{2}p^2(B_s)ds\right\}\right]\cdot\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}p^2(B_s)ds\right\}\right].\end{aligned}\tag{4.25}$$

Since $\xi_3(\widetilde{\triangle}) > \zeta_3(\widetilde{\triangle})$, we have $\alpha_2 > \alpha_1$. In the context of (4.25), it means that by moving the "safer" interval $[\triangle_1 + \triangle, \widetilde{\triangle})$ closer to the entrance, the probability of "safe arrival" to the level $\widetilde{\triangle}$ will improve.

With $\zeta_3'(\widetilde{\triangle}) > \xi_3'(\widetilde{\triangle})$, we have

$$\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\hat{\varphi}'(\widetilde{\triangle})l_{T_1}^{\widetilde{\triangle}}-\int_0^{T_1}p_4(B_s^{(4)})ds\right\}\right]\geq\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\hat{\psi}'(\widetilde{\triangle})l_{T_1}^{\widetilde{\triangle}}-\int_0^{T_1}p_4(B_s^{(4)})ds\right\}\right].\tag{4.26}$$

Recall from Theorem 10, if we set $x \to \widetilde{\triangle}$, (or by matching the terms in (4.25)) we have

$$\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}p^2(B_s)ds\right\}\right]=\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\hat{\varphi}'(\widetilde{\triangle})l_{T_1}^{\widetilde{\triangle}}-\int_0^{T_1}p_4(B_s^{(4)})ds\right\}\right].\tag{4.27}$$

Hence,

$$\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}p^2(B_s)ds\right\}\right]\geq\mathbb{E}^{\widetilde{\triangle}}\left[\exp\left\{-\int_0^{T_1}\frac{1}{2}(p^2)^*(B_s)ds\right\}\right].\tag{4.28}$$

Such an inequality makes sense, since for a process that starts at $\widetilde{\triangle}$, it spend more time on average near $\widetilde{\triangle}$, making the traversal on the interval $[0, \widetilde{\triangle})$ more "dangerous" after the rearrangement. Note that $\hat{\varphi}'(\widetilde{\triangle})$ (resp. $\hat{\psi}'(\widetilde{\triangle})$) is the "risk" the process accumulated from traversing the interval $[0, \widetilde{\triangle})$ starting at $\widetilde{\triangle}$.

At last, we should keep in mind that in order to get $\varphi(0) \leq \psi(0)$, we need $\min_{x\in[\widetilde{\triangle},1]} p^2(x) > \sigma_1$. Meaning, if there is any sub-interval $I \subset [\widetilde{\triangle}, 1]$ that is "safer" than that in $[\triangle_1 + \triangle, \widetilde{\triangle})$, we should move the interval I first.

Chapter 5

Extension of Main Result

Some conditions of our main result can be relaxed. In this chapter, we show two of such refinements. First, we can weaken the requirement for k from being positive continuous to positive L^1 integrable. We will begin this discussion by introducing the concept of a general measure preserving rearrangement function.

Second, since our main result holds for diffusion process with zero drift, it is natural to ask whether the same result will hold if we introduce drift to the process. In the case when the process is a Brownian motion, a Girsonov's argument will allows us to show the same rearrangement result for a Brownian motion with a constant upward drift.

5.1 General Measure Preserving Rearrangement

For a general Lebesgue measurable function f on the interval $[0, 1]$, we define its equi-measurable non-decreasing rearrangement by first defining $\tau(s)$ as:

$$\tau(s) = \text{meas}\{0 \leq \text{t} \leq 1 : \quad \text{f(t)} \leq \text{s}\}, \tag{5.1}$$

where "meas" is the Lebesgue measure.

Then the equi-measurable non-decreasing rearrangement f^* is defined as:

$$f^*(t) = \inf\{s \geq 0 : \quad \tau(s) > t\} \tag{5.2}$$

Remark 15. *Clearly, $\tau(s)$ is non-decreasing. Moreover, given a sequence $s_n \downarrow s$, we have $\{f(t) \leq s\} = \bigcap_n \{f(t) \leq s_n\}$, and by continuity from above of the Lebesgue measure, we conclucde that $\tau(t)$ is right-continuous.*

The following result can be found in the paper by K.M. Chong. ([1], [2])

Proposition 4. *1. f^* is non-decreasing, and right-continuous.*

2. *For any $s > 0$,*

$$\text{meas}\{0 \leq t \leq 1 : \quad f(t) \leq s\} = \text{meas}\{0 \leq t \leq 1 : \quad f^*(t) \leq s\}.$$

3. *If $\{f_n\}_{n\geq 0}$ is uniformly integrable, so is $\{f_n^*\}_{n\geq 0}$.*

4. *If $f_n \to f$ pointwise, a.e, or in measure, $f_n^* \to f^*$ in like manner.*

5. *If there exists a sequence of $L^1[0,1]$ functions f_n and f such that $f_n \to f$ in L^1, then, f_n^* and f^* are also in L^1 and $f_n^* \to f^*$ in L^1.*

Proof. 1. For $0 \leq t_1 < t_2 \leq 1$, let $s_1 = f^*(t_1)$, then, $\tau(s_1) > t_1$. Since m is non-decreasing and right-continuous, we have either $\tau(s_1) > t_2$ or there exists $s_2 > s_1$ such that $\tau(s_2) > t_2$. Taking the infimum on all the s_2, we have $f^*(t_2) \geq f^*(t_1)$.

To show that f^* is right continuous, let $f^*(t_0) = s_0$ where t_0 and s_0 are given. For any $\delta > 0$, we set $\tau(s + \frac{\delta}{2}) = t_1$. Then, $f^*(t_1) - f^*(t_0) \leq \frac{\delta}{2}$. Since f is non-decreasing, for all $t \in (t_0, t_1)$, $f^*(t) - f^*(t_0) < \delta$.

Hence, f^* is right-continuous.

2. Suppose $s > 0$ is given, and $f^*(\tau(s)) = s'$. Since f^* is right-continuous, we have

$$\text{meas}\{0 \leq t \leq 1 : \qquad s \leq f(t) \leq s'\} = 0$$

which implies

$$\text{meas}\{0 \le t \le 1 : f^*(t) \le s\} \quad = \quad \text{meas}\{0 \le t \le 1 : f(t) \le s\}.$$

3. From the previous item, we also have

$$\int_0^1 f_n(t) 1_{\{f_n(t) \ge s\}} dt = \int_0^1 f_n^*(t) 1_{\{f_n^*(t) \ge s\}} dt.$$

Thus, the rearrangement preserve uniform integrability.

4. The statement for pointwise and a.e convergence are obvious. If $f_n \to f$ in measure, we can pick a subsequence $\{f_{n_i}\}$ converges to f a.e, and we can pick another subsequence $\{f_{n_{i_j}}\}$ that converges to f pointwise. Thus, we have $\{f^*_{n_{i_j}}\}$ converges to f^* pointwise, which implies $f_n^* \to f^*$ in measure.

5. Since the space has finite measure, $f_n \to f$ in L^1 will imply convergence in measure, whence $f_n^* \to f^*$ in measure. Moreover, the rearrangement preserve uniform integrability. Thus, we have $f_n^* \to f^*$ in L^1.

□

Theorem 14. *Let $k \in L^1[0,1]$ be non-negative, and let $\{B_t\}_{t \ge 0}$ be a Brownian motion on $[0,1)$ reflected at 0 and absorbed at 1. Then*

$$\mathbb{E}^0 \left[\exp\left\{ - \int_0^{T_1} k(B_s) ds \right\} \right] \le \mathbb{E}^0 \left[\exp\left\{ - \int_0^{T_1} k^*(B_s) ds \right\} \right]. \tag{5.3}$$

Proof. Since step functions are dense in $L^1[0,1]$, we can pick a sequence $k_n \to k$ in L^1. By the occupation time formula:

$$\int_0^{T_1} k(B_s) ds = \int_0^1 k(x) l^x_{T_1} dx, \tag{5.4}$$

$l^x_{T_1}$ is continuous with respect to $x \in [0,1]$ almost surely. Therefore, $l^x_{T_1} dx$ is absolutely continuous with respect the Lebesgue measure. Consequently, $k_n \to k$ in L^1 will imply

$$\int_0^1 k_n(x) l^x_{T_1} dx \to \int_0^1 k(x) l^x_{T_1} dx, \qquad \text{a.s.}$$

Hence,

$$\exp\left\{ - \int_0^{T_1} k_n(B_s) ds \right\} \to \exp\left\{ - \int_0^{T_1} k(B_s) ds \right\} \qquad \text{a.s..}$$

Both $\exp\left\{-\int_0^{T_1} k_n(B_s)ds\right\}$ and $\exp\left\{-\int_0^{T_1} k(B_s)ds\right\}$ are bounded by 1. By Dominated convergence theorem, we get

$$\lim_{n\to\infty} \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k_n(B_s)ds\right\}\right] = \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(B_s)ds\right\}\right].$$

By item three of the previous theorem, we have $k_n^* \to k^*$ in L^1. Thus we have

$$\lim_{n\to\infty} \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k_n^*(B_s)ds\right\}\right] = \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(B_s)ds\right\}\right].$$

By theorem 1.2, we have

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k_n(B_s)ds\right\}\right] \le \mathbb{E}^0\left[\exp\left\{\int_0^{T_1} k_n^*(B_s)ds\right\}\right].$$

Taking limit on both sides of the above inequality gives us:

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(B_s)ds\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(B_s)ds\right\}\right].$$

□

5.2 Reflecting Brownian Motion with Constant Drift

Given the rearrangement inequality for reflecting Brownian motion, with the help of Girsonov's transform, our next result gives a simple proof that the rearrangement inequality holds true for all reflecting Brownian motion with a constant drift μ.

Theorem 15. *Let $Y_t = B_t + \mu t$ where $\{B_t\}_{t\ge 0}$ is a Brownian Motion on $[0,1)$ reflected at 0 and absorbed at 1, and μ is a constant. Then, for a positive integrable function k on $[0,1]$ and k^* its measure preserving increasing rearrangement function, $\{Y_t\}_{t\ge 0}$ also satisfies the rearrangement inequality:*

$$\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(Y_s)ds\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(Y_s)ds\right\}\right],$$

where $T_1 = \inf\{t \ge 0, Y_t = 1\}$.

Proof. By dominating convergence theorem, it is suffice to prove the result for the case when k is a piecewise constant function. Given a partition $0 = x_0 < \cdots < x_n = 1$, and positive sequence $\{\lambda_1, \cdots, \lambda_n\}$, we define

$$k(x) = \sum_{i=1}^{n} \lambda_i I^i(x), \quad \text{where} \quad I^i(x) = 1_{[x_{i-1}, x_i)}(x).$$

Define $K_t = \int_0^t k(B_s)ds$. By Girsonov's theorem we have

$$\begin{aligned}
\mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k(Y_s)ds\right\}\right] &= \mathbb{E}^0[\exp\{-K_{T_1} + \mu B_{T_1} - \frac{1}{2}\mu^2 T_1\}] \\
&= e^{\mu}\mathbb{E}^0\left[\exp\left\{-(\sum_{i=1}^{n}(\lambda_i + \frac{1}{2}\mu^2)\int_0^{T_1} I^i(B_s)ds)\right\}\right] \\
&\leq e^{\mu}\mathbb{E}^0\left[\exp\left\{-(\sum_{i=1}^{n}(\lambda_i + \frac{1}{2}\mu^2)^*\int_0^{T_1} I^i(B_s)ds)\right\}\right] \\
&= e^{\mu}\mathbb{E}^0\left[\exp\left\{-(\sum_{i=1}^{n}(\lambda_i^* + \frac{1}{2}\mu^2)\int_0^{T_1} I^i(B_s)ds)\right\}\right] \\
&= \mathbb{E}^0\left[\exp\left\{-K^*_{T_1} + \mu B_{T_1} - \frac{1}{2}\mu^2 T_1\right\}\right] \\
&= \mathbb{E}^0\left[\exp\left\{-\int_0^{T_1} k^*(Y_s)ds\right\}\right]
\end{aligned}$$

□

Chapter 6

An Application of the Main Result

In this chapter, we show that by picking an appropriate continuous additive functional, we can construct a birth-death process on the state space of $\{0, \cdots, N\}$ from a Brownian motion B on $[0, 1)$, reflected at 0 and absorbed at 1. More detail of such construction can be found in Sharpe [12]. We can then apply our main result to the constructed birth-death process to give a holding rate rearrangement inequality.

Recall from our main result (1.2) that the set of points we pick for the local time inequality are equally spaced. In our construction, we set $0 = x_0, \cdots, x_{N-1}, x_N = 1$, such that $x_i = i/N$. $\beta_0, \cdots, \beta_{N-1}$ are all strictly positive. Define a continuous additive functional A by

$$A_t = \sum_{i=0}^{N-1} \beta_i l_t^{x_i}.$$

τ is the right continuous inverse of A

$$\tau(a) = \inf\{s \geq 0 : A_s > a\}.$$

We define a process Y as

$$Y_t := S(B_{\tau(t)}),$$

where $S(x_i) = i$ for $i = 0, 1, \cdots, N$.

Y is reflected at 0 and absorbed at N. With respect to the time changed filtration, Y is a strong Markov process (see Sharpe [12] Lemma (65.8) and Theorem (65.9)). It is clear that Y is a birth-death process. Moreover, starting at state i, for $i = 1, \cdots, N-1$,

the probability for Y to reach $i+1$ and $i-1$ are the same, since the probability of the underlying Brownian motion to reach x_{i-1} and x_{i+1} are the same when it starts at x_i. The birth rate and the death rate are the same for each of those states. To determine the holding rate for state 0 to $N-1$, conisider the process B that starts at x_i, the holding time for Y before it reaches the next state (x_{i+1} or x_{i-1}) is $\beta_i l_T^{x_i}$, where $T = T_{x_{i+1}} \wedge T_{x_{i-1}}$, and $T_{x_{i+1}}$ and $T_{x_{i-1}}$ are the hitting time to x_{i+1} and x_{i-1} respectively.

Using Tanaka's formula,

$$|B_t - x_i| = \int_0^t \operatorname{sgn}(B_s - x_i) dB_s + l_t^{x_i}. \tag{6.1}$$

Taking expectation of both sides and setting $T = T_{x_{i+1}} \wedge T_{x_{i-1}}$ yields

$$\mathbb{E}^{x_i}\left[|B_T - x_i|\right] = \mathbb{E}^{x_i}\left[l_T^{x_i}\right]. \tag{6.2}$$

We know

$$\mathbb{E}^{x_i}\left[|B_T - x_i|\right] = \frac{1}{2}(x_{i+1} - x_i) + \frac{1}{2}(x_i - x_{i-1}) = \frac{1}{N} = x_1.$$

Thus, the holding rate $\zeta_i = 1/x_1\beta_i$.

Let T_1 be the stopping time of the process B reaching level 1. The lifetime of Y is $\sum_{i=0}^{N-1} \beta_k l_{T_1}^{x_i}$.

Recall our main result says:

$$\mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{N-1}\beta_i l_{T_1}^{x_i}\right\}\right] \le \mathbb{E}^0\left[\exp\left\{-\sum_{i=0}^{N-1}\beta_i^* l_{T_1}^{x_i}\right\}\right],$$

with $\{\beta_i^*\}_{i=0}^{N-1}$ being the non-decreasing rearrangement of $\{\beta_i\}_{i=0}^{N-1}$.

Setting $\zeta_i^\# = 1/x_1\beta_i^*$, $\zeta_i^\#$ are in non-increasing order. In the context of birth-death process, we have

Theorem 16. *Let Y be a birth-death process on $\{0, 1, \cdots, N\}$ that has strictly positive holding rate $\{\zeta_i\}_{i=0}^N$ (the death rate at 0 is zero), and its birth rate and death rate are equal on the states of $1, \cdots, N-1$. The process starts at 0 and stop the first time it reaches state N, and T is its lifetime.*

Denote $\{\zeta_i^\#\}_{i=0}^{N-1}$ as the non-increasing rearrangement of $\{\zeta_i\}_{i=0}^{N-1}$. $T^\#$ is the lifetime of the process $Y^\#$ results from replacing the holding rate from ζ_i to $\zeta_i^\#$. Then

$$\mathbb{E}[\exp\{-T\}] \le \mathbb{E}[\exp\{-T^\#\}]. \tag{6.3}$$

Appendix A

Statement of Pruss' Result

Let $\mathbb{Z}_0^+ = \{0\} \cup \mathbb{Z}^+$. Fix $p \in [0,1]$. Let $\{r_i^p : i \in Z_0^+\}$ be a random walk on $\{1, 2, \cdots, N+1\}$, with $r_0^p = 1$,

$$P(r_{i+1}^p = r_i^p + 1 | r_i^p) = p,$$

$$P(r_{i+1}^p = n - 1 | r_i^p = n) = 1 - p, \quad \text{if} \quad n > 1$$

and

$$P(r_{i+1}^p = 1 | r_i^p = 1) = 1 - p.$$

Thus, r^p is a random walk on a "blind alley".

Let $s_1, s_2, \cdots, s_N \in [0,1]$ be given as the probability of survival at the site n. $P_N^p(s_1, \cdots, s_N)$ be the probability that the random walk has survived all the time up to its arrival at the point $N+1$. The precise statement of Pruss' result is:

Theorem 17. *Let $s_1, s_2, \cdots, s_N \in [0,1]$, and $s_1^*, s_2^*, \cdots, s_N^*$ be the non-decreasing rearrangement. Then for $p \in [0,1]$, we have*

$$P_N^p(s_1, \cdots, s_N) \leq P_N^p(s_1^*, \cdots, s_N^*).$$

The intuition behind this theorem is that the random walk spends more time further away from the site $N+1$ than near it. Therefore, we improve safety by rearranging the order of the danger so that it is more concentrated toward the exit.

The next result implies that if a random walk has more of a tendency to move up then, its chance of safe arrival to site $N+1$ increases.

Theorem 18. *Let* $0 \leq p < r \leq 1$*, and let* $s_1, \cdots, s_N \in [0, 1]$*. Then,*

$$P_N^p(s_1, \cdots, s_N) \leq P_N^r(s_1, \cdots, s_N).$$

with equality if and only if one of the following conditions holds:

1. $s_k = 0$ *for some* $k \in \{1, \cdots, N\}$*,*

2. $s_1 = \cdots = s_N = 1$ *and* $p > 0$*.*

Appendix B

Feynman-Kac Equation

Let $\{X_t\}_{t\geq 0}$ be a diffusion process on the interval $[0, 1]$ with a reflecting boundary at 0, an absorbing boundary at 1, and its infinitesimal generator be:

$$\frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx}, \tag{B.1}$$

$\sigma^2(x) \geq \epsilon > 0$ for all $x \in [0, 1]$.

We then have the following result:

Proposition 5. *Let k be a positive continuous function on $[0, 1]$. Define $K_t = \int_0^t k(X_s)ds$, and $T_1 = \inf\{t > 0 : X_t = 1\}$. Then*

1. *If φ is the unique solution to the boundary value problem*

$$\frac{1}{2}\sigma^2\varphi'' + \mu\varphi' = k\varphi, \qquad \varphi'(0) = 0, \quad \varphi(1) = 1 \tag{B.2}$$

then,

$$\varphi(x) = \mathbb{E}^x[\exp\{-K_{T_1}\}]. \tag{B.3}$$

2. *Conversely, if φ is defined as in (B.3), then φ satisfies (B.2).*

Proof. By the Itô formula,

$$\begin{aligned}\varphi(X_t)e^{-K_t} &= \varphi(X_0) - \int_0^t e^{-K_s}\varphi(X_s)K_s ds + \int_0^t e^{-K_s}\varphi'(X_s)dX_s \\ &\quad + \frac{1}{2}\int_0^t e^{-K_s}\varphi''(X_s)\sigma^2(s)ds.\end{aligned} \tag{B.4}$$

The diffusion $\{X_t\}_{t\geq 0}$ satisfies the SDE:

$$dX_t = \sigma(X_t)dB_t + \mu(X_t)dt + dl_t^0, \qquad X_0 = x, \tag{B.5}$$

where $\{B_t\}_{t\geq 0}$ is a standard Brownian motion, and l_t^0 is its local time at 0.

Thus, the equation (B.4) becomes

$$\begin{aligned} &= \varphi(X_0) + \int_0^t e^{-K_s}(-\varphi(X_s)K_s + \varphi'(X_s)\mu(X_s) + \frac{1}{2}\sigma^2(X_s)\varphi''(X_s))ds \\ &\quad + \int_0^t e^{-K_s}\varphi'(X_s)\sigma(X_s)dB_s + \varphi'(0)\int_0^t e^{-K_s}dl_s^0 \\ &= \varphi(X_0) + \int_0^t e^{-K_s}\varphi'(X_s)\sigma(X_s)dB_s \end{aligned} \tag{B.6}$$

Now, taking expectation on both sides, and taking t to be the stopping time T_1:

$$\mathbb{E}^x[\exp\{-K_{T_1}\}] = \varphi(x), \qquad 0 \leq x < 1.$$

Conversely, the function

$$\varphi(x) := \mathbb{E}^x[\exp\{-K_{T_1}\}]$$

solves the integral equation:

$$1 - e^{-K_{T_1}} = \int_0^{T_1} \exp(-K_{T_1\circ\theta_s})dK_s.$$

Therefore,

$$\begin{aligned} 1 - \varphi(x) &= \mathbb{E}^x\left[\int_0^{T_1} \varphi(x)k(X_s)ds\right] \\ &= \int_0^1 G(x,y)\varphi(y)k(y)m(dy). \end{aligned}$$

Hence,

$$\varphi(x) = 1 - \int_0^1 G(x,y)\varphi(y)k(y)m(dy).$$

By inspection, we have:

$$\begin{aligned} \varphi'(0) &= 0 \\ \varphi(1) &= 1 \\ \frac{1}{2}\varphi''(x) + \mu(x)\varphi'(x) &= k(x)\varphi(x) \end{aligned}$$

□

Remark 16. *If we replace* $K_t = \int_0^t k(X_s)ds$ *by* $K_t = \lambda l_t^a + \int_0^t k(X_s)ds$, *we have*

$$dK_t = \lambda dl_t^a + k(X_s)ds.$$

The same argument of the proposition will yield

Proposition 6. *Let X be a diffusion process with infinitesimal generator satisfying (B.1), and a reflecting boundary* $0 < a < 1$, *and* φ *be the unique solution to the boundary value problem*

$$\frac{1}{2}\sigma^2(x)\varphi'' + \mu(x)\varphi' = k\varphi, \quad \varphi(1) = 1, \quad \text{and} \quad \varphi'(a) = \lambda\varphi(a) \tag{B.7}$$

for $x \in [a, 1]$

$$\varphi(x) = \mathbb{E}^x\left[\exp\left\{-\lambda l_{T_1}^a - \int_0^{T_1} k(X_s)ds\right\}\right],$$

where k is a positive integrable function in $[0, 1]$, *and* $T_1 = \inf\{t > 0 : \quad X_t = 1\}$.

Appendix C

Proof of Marcus and Rosen

Let 1^t denote the transpose of the n-dimensional vector $(1, \cdots, 1)$. $A^{(l)}$ denote the matrix obtained by replacing the lth column of the $n \times n$ matrix A by 1^t. Also, $\{Y\}_l$ denote the lth element of the vector Y.

Lemma 30. *Let X be a Markov process with finite $0-$ potential density $u(x, y)$. Assume that a local time l_t^y exists for each y, normalized so that $\mathbb{E}^x[l_\infty^y] = u(x, y)$. Let Θ be the matrix with elements $\Theta_{i,j} = u(x_i, x_j)$, $i, j = 1, \cdots, n$. Let Σ be the matrix with elements $\Sigma_{i,j} = \lambda_i \delta_{i,j}$. For all $\lambda_1, \cdots, \lambda_n$ sufficiently small and $1 \leq l \leq n$,*

$$\mathbb{E}^{x_l}\left[\exp\left\{\sum_{i=1}^{n} \lambda_i L_\infty^{x_i}\right\}\right] = \frac{\det(I - \Theta\Sigma)^{(l)}}{\det(I - \Theta\Sigma)}. \tag{C.1}$$

Proof. By Kac's moment formula ([5]),

$$\mathbb{E}^{x_l}\left(\prod_{i=1}^{n} L_\infty^{y_i}\right) = \sum_{\pi} u(x, y_{\pi(1)}) u(y_{\pi(1)}, y_{\pi(2)}) \cdots u(y_{\pi(n-1)}, y_{\pi(n)})$$

where the sum goes over all permutation π of $\{1, \cdots, n\}$. Hence

$$\begin{aligned}
&\mathbb{E}^{x_l}\left(\left(\sum_{i=1}^{n}\lambda_i L_\infty^{x_i}\right)^k\right) \\
&= k!\sum_{j_1,\cdots,j_k=1}^{n} u(x_l, x_{j_1})\lambda_{j_1}u(x_{j_2}, u_{j_2})\lambda_{j_2}u(x_{j_2}, x_{j_3}) \\
&\qquad \cdots u(x_{j_{k-2}}, x_{j_{k-1}})\lambda_{j_{k-1}}u(x_{j_{k-1}}, x_{j_k})\lambda_{j_k} \\
&= k!\sum_{j_k=1}^{n}\{(\Theta\Sigma)^k\}_{l,j_k} \\
&= k!\{(\Theta\Sigma)^k 1^t\}_l
\end{aligned} \tag{C.2}$$

for all k.

It follows from this that

$$\mathbb{E}^{x_l}\left[\exp\left(\sum_{i=1}^{n}\lambda_i L_\infty^{x_i}\right)\right] = \sum_{i=0}^{\infty}\left\{(\Theta\Sigma)^k\right\}_l = \{(I-\Theta\Sigma)^{-1}1^t\}_l.$$

Consequently,

$$(I-\Theta\Sigma)Y = 1^t$$

where Y is an $n-$dimensional vector with components $\mathbb{E}^{x_l}\left[\exp\left\{\sum_{i=1}^{n}\lambda_i L_\infty^{x_i}\right\}\right]$, $l = 1, \cdots, n$. By Cramér's theorem, we have the desired result. □

Appendix D

Analytic Continuation

More detail can be found in R.C. Cunning and H. Rossi [3].

Definition 1. *A complex-valued function f defined on an open subset $D \subset \mathbb{C}^n$ is called holomorphic in D if each point $w \in D$ has an open neighborhood U, $w \in U \subset D$, such that the function f has a power series expansion*

$$f(z) = \sum_{v_1,\cdots,v_n}^{\infty} a_{v_1\cdots v_n}(z_1 - w_1)^{v_1} \cdots (z_n - w_n)^{v_n},$$

which converges for all $z \in U$.

Theorem 19. *(Osgood's Lemma) If a complex-valued function f is continuous in an open set $D \subset \mathbb{C}^n$, and is holomorphic in each variable separately, then it is holomorphic in D.*

Theorem 20. *(Identity Theorem) If $f(z)$ and $g(z)$ are holomorphic functions in a connected open set $D \subset \mathbb{C}^n$, and if $f(z) = g(z)$ for all point z in a non-empty open subset $U \subset D$, then $f(z) = g(z)$ for all points $z \in D$.*

Appendix E

A Short Example of the Main Steps

In this example, we let $n = 16$. Suppose we have $\max\{\lambda_1, \cdots, \lambda_4\} \leq \lambda_{12}$, $\min\{\lambda_5, \cdots, \lambda_{11}\} > \lambda_{12}$, and $\min\{\lambda_{13}, \cdots, \lambda_{15}\} \geq \lambda_{12}$. Thus, we move λ_{12} to the position of λ_5, and shift each of the $\lambda_5, \cdots, \lambda_{11}$ one position to the right. We denote $\mathcal{M}(5, 12)$ the determinant before the rearrangment, and $\widehat{\mathcal{M}}(5, 12)$ the determinant after.

$$
\begin{aligned}
& \mathcal{M}(5,12) - \widehat{\mathcal{M}}(5,12) \\
= \quad & \mathbb{A}_4 \cdot \left\{ \begin{array}{l} \left(\begin{array}{l} (\lambda_5 - \lambda_{12}) \cdot {}^{-}\mathbb{B}^6_{11} + \dfrac{\lambda_{11} - \lambda_{12}}{n} \cdot \mathbb{A}^6_{10} \\ + (\lambda_6 - \lambda_{12}) \cdot {}^{-}\mathbb{B}^7_{10} + \dfrac{\lambda_{10} - \lambda_{12}}{n} \cdot \mathbb{A}^7_9 \\ + (\lambda_7 - \lambda_{12}) \cdot {}^{-}\mathbb{B}^8_9 + \dfrac{\lambda_9 - \lambda_{12}}{n} \cdot (1 + \dfrac{\lambda_8}{n}) \\ + \dfrac{\lambda_8 - \lambda_{12}}{n} \end{array} \right) \cdot \mathbb{A}^n_{13} \\ \quad \frac{\lambda_{12}}{n} \cdot \left(\begin{array}{l} \dfrac{\lambda_5 - \lambda_{11}}{n} \cdot \mathbb{B}^6_{10} + \dfrac{\lambda_6 - \lambda_{10}}{n} \cdot \mathbb{B}^7_9 \\ + \dfrac{\lambda_7 - \lambda_9}{n} \cdot \mathbb{B}^8_8 \end{array} \right) \cdot \mathbb{B}^1_{14} \end{array} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \mathbb{A}_4^- \cdot \left\{ \begin{array}{l} \left(\begin{array}{l} (\lambda_{12}-\lambda_5)\cdot \mathbb{B}_{11}^6 + (\lambda_{11}-\lambda_6)\cdot \mathbb{B}_{10}^7 \\ +(\lambda_{10}-\lambda_7)\cdot \mathbb{B}_9^8 + \dfrac{\lambda_9-\lambda_8}{n} \end{array} \right)\cdot \mathbb{A}_{13}^n \\ -\left(\begin{array}{l} (\lambda_5-\lambda_{12})\cdot {}^-\mathbb{B}_{11}^6 + \dfrac{\lambda_{11}-\lambda_{12}}{n}\cdot \mathbb{A}_{10}^6 \\ +(\lambda_6-\lambda_{12})\cdot {}^-\mathbb{B}_{10}^7 + \dfrac{\lambda_{10}-\lambda_{12}}{n}\cdot \mathbb{A}_9^7 \\ +(\lambda_7-\lambda_{12})\cdot {}^-\mathbb{B}_9^8 + \dfrac{\lambda_9-\lambda_{12}}{n}\cdot (1+\dfrac{\lambda_8}{n}) \\ +\dfrac{\lambda_8-\lambda_{12}}{n} \end{array} \right)\cdot \mathbb{B}_{14}^1 \end{array} \right\} \\
&= \quad (\lambda_5-\lambda_{12})\cdot \left\{ \mathcal{L}_1 \cdot {}^-\mathbb{B}_{11}^6 + \mathcal{L}_2 \cdot \mathbb{B}_{10}^6 \right\} + (\lambda_{11}-\lambda_{12})\cdot \left\{ \frac{1}{n}\cdot \mathcal{L}_1 \cdot \mathbb{A}_{10}^6 - \mathcal{L}_2 \cdot \mathbb{B}_{10}^6 \right\} \\
&\quad +(\lambda_6-\lambda_{12})\cdot \left\{ \mathcal{L}_1 \cdot {}^-\mathbb{B}_{10}^7 + \mathcal{L}_2 \cdot \mathbb{B}_9^7 \right\} + (\lambda_{10}-\lambda_{12})\cdot \left\{ \frac{1}{n}\cdot \mathcal{L}_1 \cdot \mathbb{A}_9^7 - \mathcal{L}_2 \cdot \mathbb{B}_9^7 \right\} \\
&\quad +(\lambda_7-\lambda_{12})\cdot \left\{ \mathcal{L}_1 \cdot {}^-\mathbb{B}_9^8 + \mathcal{L}_2 \cdot \mathbb{B}_8^8 \right\} + (\lambda_9-\lambda_{12})\cdot \left\{ \frac{1}{n}\cdot \mathcal{L}_1 \cdot (1+\frac{\lambda_8}{n}) - \mathcal{L}_2 \cdot \mathbb{B}_8^8 \right\} \\
&\quad +(\lambda_8-\lambda_{12})\cdot \left\{ \frac{1}{n}\cdot \mathcal{L}_1 \right\}.
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{L}_1 &= \mathbb{A}_3 \cdot \mathbb{A}_{13}^n - \mathbb{A}_4^- \cdot \mathbb{B}_{14}^1 \\
&= \frac{\lambda_{13}-\lambda_4}{n}\cdot \mathbb{A}_3 \cdot \mathbb{B}_{14}^1 + \frac{\lambda_{14}-\lambda_3}{n}\cdot \mathbb{A}_2 \cdot \mathbb{B}_{15}^1 + \frac{\lambda_{15}-\lambda_2}{n}\cdot \mathbb{A}_1 + 1
\end{aligned}
$$

and

$$
\begin{aligned}
-\mathcal{L}_2 &= \mathbb{A}_4^- \cdot \mathbb{A}_{13}^n - \frac{\lambda_{12}}{n}\cdot \mathbb{A}_4 \cdot \mathbb{B}_{14}^1 \\
&= \frac{\lambda_{13}-\lambda_{12}}{n}\cdot \mathbb{A}_4^- \cdot \mathbb{B}_{14}^1 + \frac{\lambda_4-\lambda_{12}}{n}\cdot \mathbb{A}_3 \cdot \mathbb{A}_{14}^n \\
&\quad + \frac{\lambda_{14}-\lambda_{12}}{n}\cdot \mathbb{A}_3^- \cdot \mathbb{B}_{15}^1 + \frac{\lambda_3-\lambda_{12}}{n}\cdot \mathbb{A}_2 \cdot \mathbb{A}_{15}^n \\
&\quad + \frac{\lambda_{15}-\lambda_{12}}{n}\cdot \mathbb{A}_2^- + \frac{\lambda_2-\lambda_{12}}{n}\cdot \mathbb{A}_1 \\
&\quad + \mathbb{A}_1^-
\end{aligned}
$$

To illustrate our steps, we will show

$$\frac{1}{n}\cdot\mathcal{L}_1\cdot\mathbb{A}^6_{10}-\mathcal{L}_2\cdot\mathbb{B}^6_{10}\geq 0,$$

and

$$\mathcal{L}_1\cdot{}^{-}\mathbb{B}^6_{11}+\mathcal{L}_2\cdot\mathbb{B}^6_{10}\geq 0.$$

We ommit the rest of the calculation because they are identical to the one we have shown.

$$\begin{aligned}
&\frac{1}{n}\cdot\mathcal{L}_1\cdot\mathbb{A}^6_{10}-\mathcal{L}_2\cdot\mathbb{B}^6_{10}\\
=\;&\frac{1}{n}\cdot\left\{\frac{\lambda_{13}-\lambda_{12}}{n}\cdot\mathbb{A}_4\cdot\mathbb{B}^1_{14}+\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}^1_{15}+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\mathbb{A}_2+\mathbb{A}_1\right\}\cdot\mathbb{A}^6_{10}\\
&+\left\{\frac{\lambda_{13}-\lambda_{12}}{n}\cdot\mathbb{A}^-_4\cdot\mathbb{B}^1_{14}+\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\mathbb{A}^-_3\cdot\mathbb{B}^1_{15}+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\mathbb{A}^-_2+\mathbb{A}^-_1\right\}\cdot\mathbb{B}^7_{10}\\
&+\left\{\begin{array}{l}\frac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\left\{\frac{1}{n}\cdot\mathbb{B}^1_{14}\cdot\mathbb{A}^6_{10}-\mathbb{A}^n_{14}\cdot\mathbb{B}^6_{10}\right\}\\ +\frac{\lambda_{12}-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\left\{\frac{1}{n}\cdot\mathbb{B}^1_{15}\cdot\mathbb{A}^6_{10}-\mathbb{A}^n_{15}\cdot\mathbb{B}^6_{10}\right\}\\ +\frac{\lambda_{12}-\lambda_2}{n}\cdot\mathbb{A}_1\cdot\left\{\frac{1}{n}\cdot\mathbb{A}^6_{10}-\mathbb{B}^6_{10}\right\}\end{array}\right\}\\
=\;&\frac{1}{n}\cdot\left\{\frac{\lambda_{13}-\lambda_{12}}{n}\cdot\mathbb{A}_4\cdot\mathbb{B}^1_{14}+\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}^1_{15}+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\mathbb{A}_2+\mathbb{A}_1\right\}\cdot\mathbb{A}^6_{10}\\
&+\left\{\frac{\lambda_{13}-\lambda_{12}}{n}\cdot\mathbb{A}^-_4\cdot\mathbb{B}^1_{14}+\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\mathbb{A}^-_3\cdot\mathbb{B}^1_{15}+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\mathbb{A}^-_2+\mathbb{A}^-_1\right\}\cdot\mathbb{B}^7_{10}\\
&+\left\{\begin{array}{l}\frac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\left\{\frac{\lambda_6-\lambda_{14}}{n}\cdot\mathbb{B}^1_{15}\cdot\mathbb{B}^7_{10}+\frac{\lambda_7-\lambda_{15}}{n}\cdot\mathbb{B}^8_{10}-\mathbb{B}^9_{10}\right\}\\ +\frac{\lambda_{12}-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\left\{\frac{\lambda_6-\lambda_{15}}{n}\cdot\mathbb{B}^7_{10}-\mathbb{B}^8_{10}\right\}\\ +\frac{\lambda_{12}-\lambda_2}{n}\cdot\mathbb{A}_1\cdot\left\{-\mathbb{B}^7_{10}\right\}\end{array}\right\}
\end{aligned}$$

First, we look at

$$\begin{aligned}
&\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\mathbb{B}^1_{15}\cdot\left\{\frac{1}{n}\cdot\mathbb{A}_3\cdot\mathbb{A}^6_{10}+\mathbb{A}^-_3\cdot\mathbb{B}^7_{10}\right\}+\frac{\lambda_{12}-\lambda_4}{n}\cdot\frac{\lambda_6-\lambda_{14}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}^1_{15}\cdot\mathbb{B}^7_{10}\\
=\;&\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\mathbb{B}^1_{15}\cdot\left\{\frac{\lambda_6-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}^7_{10}+\frac{1}{n}\cdot\mathbb{A}_4\cdot\mathbb{A}^7_{10}+\mathbb{A}^-_4\cdot\mathbb{B}^8_{10}\right\}\\
&+\frac{\lambda_{12}-\lambda_4}{n}\cdot\left(\frac{\lambda_6-\lambda_{12}}{n}+\frac{\lambda_{12}-\lambda_{14}}{n}\right)\cdot\mathbb{A}_3\cdot\mathbb{B}^1_{15}\cdot\mathbb{B}^7_{10}\\
=\;&2\cdot\frac{\lambda_{14}-\lambda_{12}}{n}\cdot\frac{\lambda_6-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}^1_{15}\cdot\mathbb{B}^7_{10}+\frac{1}{n}\cdot\mathbb{A}_4\cdot\mathbb{A}^7_{10}+\mathbb{A}^-_4\cdot\mathbb{B}^8_{10}\geq 0.
\end{aligned}$$

Second,

$$\begin{aligned}
&\frac{\lambda_{12}-\lambda_3}{n}\cdot\frac{\lambda_6-\lambda_{15}}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^7+\frac{\lambda_{12}-\lambda_4}{n}\cdot\frac{\lambda_7-\lambda_{15}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^8\\
&+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\left(\frac{1}{n}\cdot\mathbb{A}_2\cdot\mathbb{A}_{10}^6+\mathbb{A}_2^-\cdot\mathbb{B}_{10}^7\right)\\
=\;&\frac{\lambda_{12}-\lambda_3}{n}\cdot\frac{\lambda_6-\lambda_{12}}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^7+\frac{\lambda_{12}-\lambda_4}{n}\cdot\frac{\lambda_7-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^8\\
&+\frac{\lambda_{12}-\lambda_3}{n}\cdot\frac{\lambda_{12}-\lambda_{15}}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^7+\frac{\lambda_{12}-\lambda_4}{n}\cdot\frac{\lambda_{12}-\lambda_{15}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^8\\
&+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\left\{\frac{\lambda_6-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^7+\frac{\lambda_7-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^8+\frac{1}{n}\cdot\mathbb{A}_4\cdot\mathbb{A}_{10}^8+\mathbb{A}_4^-\cdot\mathbb{B}_{10}^9\right\}\\
=\;&\frac{\lambda_{12}-\lambda_3}{n}\cdot\frac{\lambda_6-\lambda_{12}}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^7+\frac{\lambda_{12}-\lambda_4}{n}\cdot\frac{\lambda_7-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^8\\
&+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\frac{\lambda_6-\lambda_{12}}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^7+\frac{\lambda_{15}-\lambda_{12}}{n}\cdot\frac{\lambda_7-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^8\\
&+\frac{1}{n}\cdot\mathbb{A}_4\cdot\mathbb{A}_{10}^8+\mathbb{A}_4^-\cdot\mathbb{B}_{10}^9\geq 0.
\end{aligned}$$

Third,

$$\begin{aligned}
&\frac{1}{n}\cdot\mathbb{A}_1\cdot\mathbb{A}_{10}^6+\mathbb{A}_1^-\cdot\mathbb{B}_{10}^7\\
&-\left(\frac{\lambda_{12}-\lambda_2}{n}\cdot\mathbb{A}_1\cdot\mathbb{B}_{10}^7+\frac{\lambda_{12}-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^8+\frac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^9\right)\\
=\;&\frac{\lambda_6-\lambda_2}{n}\cdot\mathbb{A}_1\cdot\mathbb{B}_{10}^7+\frac{\lambda_7-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^8+\frac{\lambda_8-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^9+\frac{1}{n}\cdot\mathbb{A}_4\cdot\mathbb{A}_{10}^9\\
&+\mathbb{A}_4^-\cdot\mathbb{B}_{10}^{10}-\left(\frac{\lambda_{12}-\lambda_2}{n}\cdot\mathbb{A}_1\cdot\mathbb{B}_{10}^7+\frac{\lambda_{12}-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^8+\frac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^9\right)\\
=\;&\frac{\lambda_6-\lambda_{12}}{n}\cdot\mathbb{A}_1\cdot\mathbb{B}_{10}^7+\frac{\lambda_7-\lambda_{12}}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_{10}^8+\frac{\lambda_8-\lambda_{12}}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^9\\
&+\frac{1}{n}\cdot\mathbb{A}_4\cdot\mathbb{A}_{10}^9+\mathbb{A}_4^-\cdot\mathbb{B}_{10}^{10}\geq 0.
\end{aligned}$$

Next, we check

$$\begin{aligned}
&\mathcal{L}_1\cdot{}^-\mathbb{B}_{11}^6+\mathcal{L}_2\cdot{}^-\mathbb{B}_{10}^6\\
=\;&\left(\frac{\lambda_{13}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{14}^1+\frac{\lambda_{14}-\lambda_3}{n}\cdot\mathbb{A}_2\mathbb{B}_{15}^1+\frac{\lambda_{15}-\lambda_2}{n}\cdot\mathbb{A}_1+1\right)\cdot{}^-\mathbb{B}_{11}^6
\end{aligned}$$

$$+\left\{\begin{array}{l}\dfrac{\lambda_{12}-\lambda_{13}}{n}\cdot\mathbb{A}_4^-\cdot\mathbb{B}_{14}^1+\dfrac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{A}_{14}^n\\ +\dfrac{\lambda_{12}-\lambda_{14}}{n}\cdot\mathbb{A}_3^-\cdot\mathbb{B}_{15}^1+\dfrac{\lambda_{12}-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\mathbb{A}_{15}^n\\ +\dfrac{\lambda_{12}-\lambda_{15}}{n}\cdot\mathbb{A}_2^-+\dfrac{\lambda_{12}-\lambda_2}{n}\cdot\mathbb{A}_1\\ +\mathbb{A}_1^-\end{array}\right\}\cdot\mathbb{B}_{10}^6.$$

For the pair

$$\begin{aligned}&\frac{\lambda_{13}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{14}^1\cdot{}^-\mathbb{B}_{11}^6+\frac{\lambda_{12}-\lambda_{13}}{n}\cdot\mathbb{A}_4^-\cdot\mathbb{B}_{14}^1\cdot\mathbb{B}_{10}^6\\ =\;&\frac{\lambda_{13}-\lambda_{12}}{n}\cdot\left(\mathbb{A}_3\cdot{}^-\mathbb{B}_{11}^6-\mathbb{A}_4^-\cdot\mathbb{B}_{10}^6\right)\\ &+\frac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{14}^1\cdot{}^-\mathbb{B}_{11}^6\\ =\;&\frac{\lambda_{13}-\lambda_{12}}{n}\cdot\left\{\begin{array}{l}\dfrac{\lambda_{11}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{10}^6+\dfrac{\lambda_{10}-\lambda_3}{n}\cdot\mathbb{A}_2\cdot\mathbb{B}_9^6+\dfrac{\lambda_9-\lambda_2}{n}\cdot\mathbb{A}_1\cdot\mathbb{B}_8^6\\ +\dfrac{\lambda_8-\lambda_1}{n}\cdot\mathbb{B}_7^6+{}^-\mathbb{B}_7^6\end{array}\right\}\\ &\frac{\lambda_{12}-\lambda_4}{n}\cdot\mathbb{A}_3\cdot\mathbb{B}_{14}^1\cdot{}^-\mathbb{B}_{11}^6\geq 0.\end{aligned}$$

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