UNIVERSITY OF CALIFORNIA,
IRVINE

Topics on Hessian type equations

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

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2024

TABLE OF CONTENTS

ACKNOWLEDGMENTS

Words cannot express my gratitude to my advisor, Professor Zhang. He has always patiently guided me, his profound insights into mathematics have benefited me greatly, and his keen sense of research is admirable. I am also very grateful to my defense committee. Professor Mooney's PDE courses are humorous and engaging; his intuitive and profound understanding of PDE, along with his warm teaching style, is so refreshing. Professor Lu's geometry courses are also fascinating; his passion for mathematics and meticulous study are inspiring. Furthermore, I would like to thank the professors in the department. Your wealth of knowledge and engaging lectures always make me passionate about mathematics.

I am also very thankful to my classmates and roommates. Without you, the past few years would have been much duller. Whether it is mathematical discussions or casual chats, you make my graduate school life colorful. Especially during the pandemic, I am grateful for the interactions with my roommates that kept me connected to the world.

Lastly, I must thank my family. Without your support and upbringing, I could not have come this far. I also want to thank my friends; your occasional conversations have been a great stress reliever. I also want to thank my pet; your emotional support has been priceless.

Along this journey, there have been too many people who directly or indirectly helped me. A few words cannot fully express my gratitude towards everything, and my reverence and admiration for this world.

Financial support is provided by the University of California, Irvine.

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ABSTRACT OF THE DISSERTATION

Topics on Hessian type equations

By

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Doctor of Philosophy in Mathematics

University of California, Irvine, 2024

Professor Xiangwen Zhang, Chair

We study some selected topics on Hessian type equations.

In the first chapter, our goal to generalize the quantitative version of the constant rank theorem by Székelyhidi-Weinkove onto Hermitian manifolds and the complex coordinate space. As an application, we also study some properties of Ricci tensors based on the theorem we get.

In the second chapter, we consider C^2 estimates for complex Hessian equations involving gradient terms. In particular, we study special cases when the eigenvalues are bounded below.

In the third chapter, we study the long-time existence and convergence of parabolic complex Hessian type equations whose second order operator is not necessarily convex or concave.

Chapter 1

Constant rank theorem on Hermitian manifolds

1.1 Introduction

Constant rank theorem (or microscopic convexity principle) was first studied by Caffarelli-Friedman [6] for convex solutions of the following semilinear elliptic equations,

$$\Delta u = f(\nabla u, u, x) \text{ on } \Omega \subset \mathbb{R}^2.$$

Yau [20] has a similar result at about the same time. Later, it was further studied by Korevaar-Lewis [16], Caffarelli-Guan-Ma [7], Bian-Guan [3, 4], and many other mathematicians under different settings. For related generalization to complex cases, it was studied by Li, Guan and Zhang [17, 13]. The theorem is a useful tool in the study of the existence of convex solutions of PDEs [14] and geometric properties of the solutions. In the paper [3], Bian-Guan consider the convex solutions of

$$F\left(D^2u, Du, u, x\right) = 0$$

under some structural conditions on F. The original proof of constant rank theorems consider test functions of the form $\sigma_{l+1} + \frac{\sigma_{l+2}}{\sigma_{l+1}}$, where σ_k is the elementary symmetric function on the eigenvalues of D^2u. Székelyhidi-Weinkove [21] later provided a new proof by considering the following test function.

$$\lambda_k + 2\lambda_{k-1} + ... + k\lambda_1,$$

where $0 \leq \lambda_1 \leq ... \leq \lambda_n$ are eigenvalues of D^2u. The key point is that $\lambda_k + 2\lambda_{k-1} + ... + k\lambda_1$ is semi-concave, so it is twice differentiable almost everywhere. Motivated by Brendle-Choi-Daskalopoulos [5], they compute the derivatives of eigenvalues directly and give a quantitative version of the constant rank theorem by the weak Harnack inequality in [22]. We would like to apply their methods to Hermitian manifolds and general cases on complex coordinate spaces. We will describe the setting below.

Let $W_{i\bar{j}}$ be a $(1,1)$ smooth Hermitian tensor on a Hermitian manifold (M, g). Define $A_a^b \equiv W_{a\bar{k}}g^{\bar{k}b}$. Suppose A is semi-positive definite, we consider the following C^2 real function on (M, g).

$$F(A, z) = f(\lambda(A), z) = 0, \text{ where } f \text{ is a symmetric function of eigenvalues.}$$

Fix a coordinate ball B, we require F to be elliptic with the elliptic constant $\Lambda(B)$.

$$\Lambda^{-1}(B)|\xi|^2 \leq F^{a\bar{b}}\xi_a\xi_{\bar{b}} \leq \Lambda(B)|\xi|^2.$$

Also, we require F to satisfy the following conjugate condition.

$$\overline{F^{a\bar{b}}} = F^{b\bar{a}},\ \overline{F^{i\bar{j},r\bar{s}}} = F^{s\bar{r},j\bar{i}},\ \text{and}\ \overline{F^{a\bar{b},\bar{z}_\alpha}} = F^{z_\alpha,b\bar{a}},\ \text{where}\ \frac{\partial F}{\partial A_a^k} g^{k\bar{b}} = F^{a\bar{b}}. \tag{1.1.1}$$

This condition is used when we do term-by-term calculations. The conjugate condition ensures that the terms we study are real. Next, we need certain positivity conditions and constraints on the tensor. We first recall the conditions that is required for the constant rank theorem on the Riemannian manifolds [7][3].

$$F\left(A^{-1}, x\right) \text{ is locally convex in } (A, x)\,. \tag{1.1.2}$$

W_{ijk} is symmetric in i, j and k.

Our main challenge is to find suitable conditions on Hermitian manifolds. Furthermore, our conditions should be consistent with the ones studied by Li, Guan and Zhang [17, 13] in the complex setting. We will discuss these two questions below.

Question 1: What is a suitable positivity condition on F?

In [2], Andrews uses inverse concavity to describe (1.1.2).

$$f \text{ is inverse-concave if } \widetilde{f}\left(\lambda_1, ..., \lambda_n\right) = f\left(\lambda_1^{-1}, ..., \lambda_n^{-1}\right) \text{ is a convex function.} \tag{1.1.3}$$

The inverse concavity of f is the same as $F\left(A^{-1}\right)$ being locally convex in A when f is defined on the positive cone Γ_n. Motivated by the definition of inverse-concave, we say a function h is exponential-convex if

$$h^*\left(\lambda_1, ..., \lambda_n\right) = h\left(e^{\lambda_1}, ..., e^{\lambda_n}\right) \text{ is a convex function.} \tag{1.1.4}$$

In fact, the exponential-convexity is our desired condition. We will discuss this condition in section 1.2.3 in details. We first state our condition precisely. Define $f^*(\lambda, z) = f\left(e^{\lambda}, z\right)$, where $e^{\lambda} = \left(e^{\lambda_1}, ..., e^{\lambda_n}\right)$. And μ is a $2n$ dimensional vector.

$$\mu_k = \lambda_k,\ 1 \leq k \leq n. \quad \mu_{n+k} = z_k,\ 1 \leq k \leq n.$$

When $\lambda_i > 0$, for all i, we require the following condition.

$$\frac{\partial^2 f^*}{\partial \mu_i \partial \overline{\mu_j}} \geq 0. \tag{1.1.5}$$

Note that since λ_k are real, $\mu_k = \overline{\mu_k}$ for $1 \leq k \leq n$.

Question 2: What is a suitable condition on W?

Similar to the symmetry condition on Riemannian manifolds, the condition people usually impose on the Kähler manifolds is the following symmetry condition.

$$W_{i\bar{j}k} = W_{k\bar{j}i}.$$

We can require the same symmetry condition on the Hermitian manifolds as well. But based on calculations, we believe that W being *closed* should be a more suitable condition. And on Kähler manifolds, the conditions that W being closed coincides with $W_{i\bar{j}k} = W_{k\bar{j}i}$. We have the following theorem.

Theorem 1.1.1. *Suppose $F(A, z) = f(\lambda(A), z)$ being C^2 and elliptic satisfies the conjugate condition (1.1.1) and the positivity condition (1.1.5). Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of A_a^b. If either one of the following condition holds,*

1. *W is closed and the orthogonal bisectional curvature $R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq 0$*
2. *$W_{i\bar{j}k} = W_{k\bar{j}i}$ and $R_{\alpha\bar{\beta}\beta\bar{\alpha}} \geq 0$,*

then for any given coordinate chart $B_1(p)$ *on* M*, there exist positive constants* C_0, q *depending on* n*,* p*,* W*,* F*,* M *and* g *such that for each* $l = 1, \ldots, n$*,*

$$\|\lambda_l\|_{L^q(B_{1/2})} \leq C_0 \inf_{B_{1/2}} \lambda_l.$$

In particular, A_a^b *has constant rank in* $B_1(p)$*. Suppose that furthermore the smallest eigenvalue of* A_a^b *is* 0 *at one point and the curvature tensor (*$R_{\alpha\bar{\alpha}\beta\bar{\beta}}$ *or* $R_{\alpha\bar{\beta}\beta\bar{\alpha}}$*) is strictly positive at some point in* B_1*, then* $A_a^b \equiv 0$*.*

Besides on the Hermitian manifold, we also have the following version for the complex coordinate space. We consider the following function on $\mathbb{C}^n$.

$$F\left(u_{i\bar{j}}, u_k, u_{\bar{k}}, u, z\right) = 0 \text{ on } B_1, \tag{1.1.6}$$

where $\left(u_{i\bar{j}}\right) \geq 0$, $u \in C^4$ and F is a C^2 uniformly elliptic functions. Also we require F to satisfy similar conjugate conditions and positivity conditions like before. For the exact conditions, please see section 1.4.3. We have the following theorem.

Theorem 1.1.2. *Suppose* $F\left(u_{i\bar{j}}, u_k, u_{\bar{k}}, u, z\right)$ *satisfies the conjugate condition (1.4.6) and the positivity condition (1.4.7) or (1.4.8). Let* $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ *be the eigenvalues of* $u_{i\bar{j}}$*. There exist positive constants* C_0, q *depending on* n*,* $\|u\|_{C^4}$*,* $\|F\|_{C^2}$ *and elliptic constants such that for each* $l = 1, \ldots, n$*,*

$$\|\lambda_l\|_{L^q(B_{1/2})} \leq C_0 \inf_{B_{1/2}} \lambda_l.$$

We now describe the outline of this chapter. In section 2, we first recall some definitions for Hermitian manifolds and semi-concavity. And we discuss the positivity conditions in details.

In section 3, our goal is to prove a key differential inequality similar to [22, Lemma 3.1].

$$F^{a\bar{a}}Q_{a\bar{a}} \leq C_1 Q + C_2 \sum_{a=1}^{n}\sum_{\alpha=1}^{l} |\lambda_{\alpha,a}| + C_0, \tag{1.1.7}$$

where $Q = \sum_{m=1}^{l}\sum_{\alpha=1}^{m} \lambda_\alpha$. The first half of this section is devoted to studying the second derivatives of eigenvalues on Hermitian manifolds, and the second part is to get above inequality from term-by-term calculations.

In section 4, we finish the proof for theorem 1.1.1, study the special case and give an outline for the proof of theorem 1.1.2.

In section 5, we consider the Ricci curvature tensor as an example. In the first half, we consider the case when the Ricci tensor is closed. We study the condition for it to be closed, and pick up a function satisfying the structural conditions and get corollary 1.5.2. In the second half, we consider the case when the Ricci tensor satisfies $R_{i\bar{j}k} = R_{k\bar{j}i}$. Motivated by Kähler-like [24] manifolds and CAS [18] manifolds, we consider a special type of Hermitian manifold and obtain corollary 1.5.5 by applying theorem 1.1.1.

1.2 Preliminary

1.2.1 Basic notations for Hermitian manifolds

We follow the notation of [11]. For a quick introduction to the background materials for Hermitian manifolds, we refer to the first chapter of [19]. Let (M^n, g) be a Hermitian manifold. ∇ will always denote the Chern connection in this note. In local coordinate

$z = (z^1, ..., z^n)$,

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) \text{ and } \left[g_{i\bar{j}}\right]^{-1} = \left[g^{i\bar{j}}\right].$$

The Christoffel symbols, torsion tensors and curvature tensors in local coordinates are defined as follows.

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^k} = \Gamma^l_{jk} \frac{\partial}{\partial z^l} = g^{l\bar{m}} \partial_j g_{k\bar{m}}.$$

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} = g^{k\bar{l}} \left(\partial_i g_{j\bar{l}} - \partial_j g_{i\bar{l}}\right).$$

$$R_{i\bar{j}k\bar{l}} = -g_{m\bar{l}} \partial_{\bar{j}} \Gamma^m_{ik} = -\partial_i \partial_{\bar{j}} g_{k\bar{l}} + g^{p\bar{q}} \partial_i g_{k\bar{q}} \partial_{\bar{j}} g_{p\bar{l}}.$$

In particular,

$$\overline{\Gamma^l_{jk}} = \Gamma^{\bar{l}}_{\bar{j}\bar{k}} = g^{m\bar{l}} \partial_{\bar{j}} g_{m\bar{k}} \text{ and } \overline{R_{i\bar{j}k\bar{l}}} = R_{j\bar{i}l\bar{k}}.$$

A Hermitian manifold is said to have *nonnegative orthogonal bisectional curvature* if for any orthonormal basis $\{e_\alpha\}$, $R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq 0$ for any $\alpha \neq \beta$. Also, we have Bianchi identities for curvature tensors.

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = g_{m\bar{l}} \nabla_{\bar{j}} T^m_{ki} = \nabla_{\bar{j}} T_{ki\bar{l}}. \tag{1.2.1}$$

$$\nabla_m R_{k\bar{j}i\bar{l}} - \nabla_k R_{m\bar{j}i\bar{l}} = T^r_{km} R_{r\bar{j}i\bar{l}}. \tag{1.2.2}$$

1.2.2 Concavity of eigenvalues

A real-valued function f on a bounded convex set B is *semi-concave* if there exists a constant M such that

$$\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\leq M\left|x-y\right|^{2}, \quad \text{for all } x,y\in B. \tag{1.2.3}$$

Observe that all C^2 functions are semi-concave. By Alexandrov's theorem, a semi-concave function is twice differentiable almost everywhere (it means that there is a second order Taylor expansion almost everywhere.) For more details, we refer to [8]. We have the following well-known proposition.

Proposition 1.2.1. *Let $\lambda_1(x)\leq\lambda_2(x)\leq ...\leq\lambda_n(x)$ be eigenvalues of $g^{-1}W$. On a coordinate ball $B_1(0)$, the map $\overline{B_1(0)}\to\mathbb{R}$ given by*

$$x\mapsto\lambda_1(x)+\lambda_2(x)+...+\lambda_k(x)$$

is semi-concave for each $k=1,...,n$. In particular, λ_i is twice differentiable almost everywhere for all i on the coordinate ball.

Proof. The proof is similar to the proof for real case [22, Proposition 2.1]. Let $\sigma(A)=\lambda_1(A)+\lambda_2(A)+...+\lambda_k(A)$ be a function defined on Hermitian matrices. Then σ is increasing, concave and Lipschitz continuous with Lipschitz constant depending only on n and k (see [22]). Write

$$A=g^{-1}W=X+iY,$$

where X and Y are real symmetric n by n matrices. Denote the (a,b) component of A as A_{ab}. Note that all the components of X and Y are C^2 functions, so X_{ab} and Y_{ab} are semi-concave.

For x and y in $B_1(0)$, we have

$$\begin{aligned}
&\left|\frac{A_{ab}(x)+A_{ab}(y)}{2}-A_{ab}\left(\frac{x+y}{2}\right)\right| \\
&\leq \left|\frac{X_{ab}(x)+X_{ab}(y)}{2}-X_{ab}\left(\frac{x+y}{2}\right)\right|+\left|\frac{Y_{ab}(x)+Y_{ab}(y)}{2}-Y_{ab}\left(\frac{x+y}{2}\right)\right| \\
&\leq M_1 |x-y|^2 .
\end{aligned}$$

Now the absolute value of every entry for the matrix $\frac{A(x)+A(y)}{2}-A\left(\frac{x+y}{2}\right)$ is bounded. By Gershgorin circle theorem, all the eigenvalues are hence bounded. Therefore, there exists a constant M_2 such that

$$\frac{A(x)+A(y)}{2}-A\left(\frac{x+y}{2}\right) \leq M_2 |x-y|^2 Id.$$

Since σ is increasing, concave and Lipschitz continuous, we have

$$\begin{aligned}
&\frac{\sigma(A(x))+\sigma(A(y))}{2}-\sigma\left(A\left(\frac{x+y}{2}\right)\right) \\
&\leq \sigma\left(\frac{A(x)+A(y)}{2}\right)-\sigma\left(A\left(\frac{x+y}{2}\right)\right) \\
&\leq \sigma\left(A\left(\frac{x+y}{2}\right)+M_2|x-y|^2 Id\right)-\sigma\left(A\left(\frac{x+y}{2}\right)\right) \\
&\leq C|x-y|^2 .
\end{aligned}$$

□

1.2.3 Positivity condition

To study the positivity condition, let's simplify our function to be $F(A)$. For the real case, when we study the constant rank theorem [7], we require the following condition.

$$A \in Sym_n^+(\mathbb{R}) \mapsto F\left(A^{-1}\right) \quad \text{is locally convex.} \tag{1.2.4}$$

Diagonalize A and write in terms of index, it becomes

$$F^{ab,rs}X_{ab}X_{rs} + F^{ab}A^{rr}X_{ar}X_{br} + F^{ab}A^{rr}X_{ra}X_{rb} \geq 0, \ X \in M_n(\mathbb{R}). \tag{1.2.5}$$

In the real case, we can actually require X_{ab} to be *symmetric* during our computation. For example,

$$if \ A = D^2u, \ X_{ab} = u_{ab\alpha}. \ \alpha \text{ is a given direction.}$$

And both terms $F^{ab}A^{rr}X_{ar}X_{br} + F^{ab}A^{rr}X_{ra}X_{rb}$ can be combined in the computation once X_{ab} is symmetric. However, in complex case, the above scenario doesn't hold. If we require a similar condition like (1.2.4), equation (1.2.5) becomes

$$F^{a\bar{b},r\bar{s}}X_{a\bar{b}}\overline{X_{s\bar{r}}} + F^{a\bar{b}}A^{r\bar{r}}X_{a\bar{r}}\overline{X_{b\bar{r}}} + F^{a\bar{b}}A^{r\bar{r}}X_{r\bar{b}}\overline{X_{r\bar{a}}} \geq 0, \ X \in M_n(\mathbb{C}).$$

In the complex case, we cannot require $X_{a\bar{b}}$ to be Hermitian. For example,

$$if \ A = \left(u_{i\bar{j}}\right), \ X_{a\bar{b}} = u_{a\bar{b}\alpha}. \ \alpha \text{ is a given direction.}$$

Therefore, these the two terms $F^{a\bar{b}}A^{r\bar{r}}X_{a\bar{r}}\overline{X_{b\bar{r}}} + F^{a\bar{b}}A^{r\bar{r}}X_{r\bar{b}}\overline{X_{r\bar{a}}}$ cannot be combined since $X_{a\bar{b}} = u_{a\bar{b}\alpha}$ is not Hermitian, which will cause troubles during the computation. We can only deal with one term, and therefore, the possible positivity condition we can impose becomes (see also [13])

$$F^{a\bar{b},r\bar{s}}X_{a\bar{b}}\overline{X_{s\bar{r}}}+F^{a\bar{b}}A^{r\bar{r}}X_{a\bar{r}}\overline{X_{b\bar{r}}}\geq 0,\ X\in M_n\left(\mathbb{C}\right). \tag{1.2.6}$$

If $F\left(A\right)=f\left(\lambda\left(A\right)\right)$ is in fact a symmetric function of the eigenvalues, we have an alternative viewpoint to understand the condition (1.2.6). Motivated by Andrews' work [2], in which he describes (1.2.5) by using inverse concavity (1.1.3), we have the following proposition.

Proposition 1.2.2. *Let $f\left(\lambda\left(A\right)\right)$ be a symmetric function of eigenvalues defining on the positive cone $\Gamma_n=\{\lambda\in\mathbb{R}^n|\lambda_i>0 \text{ for all } i\}$. The following are equivalent.*

1. *f is exponential-convex (1.1.4).*
2. *$f_{ij}+\frac{f_i}{\lambda_j}\delta_{ij}$ is positive semidefinite.*
3. *$f_{ij}+\frac{f_i}{\lambda_j}\delta_{ij}$ is positive semidefinite and $\frac{f_a}{\lambda_q}+\frac{f_a-f_q}{\lambda_a-\lambda_q}\geq 0$ for $a\neq q$.*
4. *$F^{a\bar{b},r\bar{s}}X_{a\bar{b}}\overline{X_{s\bar{r}}}+F^{a\bar{b}}A^{r\bar{r}}X_{a\bar{r}}\overline{X_{b\bar{r}}}\geq 0$, $X\in M_n\left(\mathbb{C}\right)$ and $A^{r\bar{r}}$ is the inverse matrix of A.*

Remark 1.2.3. It is well-known that quite a few functions satisfy condition 2 in proposition 1.2.2. For example,

$$f=\log\sigma_k\left(\lambda\right)\ \ or\ \ f=\sigma_k^p\left(\lambda\right)\ \ for\ any\ p>0, \tag{1.2.7}$$

where

$$\sigma_k\left(\lambda_1,...,\lambda_n\right)=\sum_{1\leq i_1<i_2<...<i_k\leq n}\lambda_{i_1}\lambda_{i_2}...\lambda_{i_k},\ \ k\geq 1.$$

Proof. We first show (3) and (4) are equivalent. Let A be diagonalized. Since F is a symmetric function of the eigenvalues, it is known that (see Andrews [1], Gerhardt [10])

$$F^{i\bar{j}}=\frac{\partial f}{\partial\lambda_i}\delta_{ij}=f_i\delta_{ij}. \tag{1.2.8}$$

$$F^{a\bar{b},r\bar{s}}X_{a\bar{b}}\overline{X_{s\bar{r}}} = f_{ij}X_{i\bar{i}}\overline{X_{j\bar{j}}} + \sum_{a\neq q}\frac{f_a - f_q}{\lambda_a - \lambda_q}\left|X_{a\bar{q}}\right|^2. \tag{1.2.9}$$

Therefore,

$$\begin{aligned} &F^{a\bar{b},r\bar{s}}X_{a\bar{b}}\overline{X_{s\bar{r}}} + F^{a\bar{b}}A^{r\bar{r}}X_{a\bar{r}}\overline{X_{b\bar{r}}} \\ &= f_{ij}X_{i\bar{i}}\overline{X_{j\bar{j}}} + \frac{f_a}{\lambda_a}\left|X_{a\bar{a}}\right|^2 + \sum_{a\neq q}\left(\frac{f_a - f_q}{\lambda_a - \lambda_q} + \frac{f_a}{\lambda_q}\right)\left|X_{a\bar{q}}\right|^2. \end{aligned} \tag{1.2.10}$$

By choosing suitable $X_{a\bar{b}}$, it is clear that (3) and (4) are equivalent from (1.2.10).

Define $f^*(\lambda) = f\left(e^{\lambda}\right)$, where $e^{\lambda} \equiv \left(e^{\lambda_1}, ..., e^{\lambda_n}\right)$. We next show that (1) and (2) are equivalent.

$$\frac{\partial^2 f^*}{\partial\lambda_i\partial\lambda_j} \geq 0 \text{ on } \mathbb{R}^n \text{ if and only if } \left(f_{ij} + \frac{f_i}{\lambda_j}\delta_{ij}\right) \geq 0 \text{ on } \Gamma_n. \tag{1.2.11}$$

Indeed, suppose f is exponential convex,

$$\begin{aligned} \frac{\partial^2 f^*}{\partial\lambda_i\partial\lambda_j} &= \frac{\partial}{\partial\lambda_i}\left(\frac{\partial f}{\partial e^{\lambda_j}}\frac{\partial e^{\lambda_j}}{\partial\lambda_j}\right) = \frac{\partial^2 f}{\partial e^{\lambda_i}\partial e^{\lambda_j}}\frac{\partial e^{\lambda_j}}{\partial\lambda_j}\frac{\partial e^{\lambda_i}}{\partial\lambda_i} + \frac{\partial f}{\partial e^{\lambda_j}}\frac{\partial^2 e^{\lambda_j}}{\partial\lambda_i\partial\lambda_j} \\ &= \frac{\partial^2 f}{\partial e^{\lambda_i}\partial e^{\lambda_j}}e^{\lambda_j}e^{\lambda_i} + \frac{\partial f}{\partial e^{\lambda_j}}e^{\lambda_i}\delta_{ij} \geq 0 \end{aligned}$$

Substituting $\lambda_j = \log\mu_j$ into above equation and multiplying the equation with $\mu_j^{-1}\mu_i^{-1}$ give the result. Reversing above procedures proves the other direction.

Our last goal is to prove (1) implies (3). First we observe that the following two conditions are equivalent.

$$\frac{f_a}{\lambda_q} + \frac{f_a - f_q}{\lambda_a - \lambda_q} \geq 0 \text{ if and only if } \frac{\partial f}{\partial\lambda_i}\lambda_i \geq \frac{\partial f}{\partial\lambda_j}\lambda_j \text{ if } \lambda_i \geq \lambda_j. \tag{1.2.12}$$

By restriction to the variables (λ_i, λ_j), it suffices to prove (1.2.12) when there are only two variables (see also [9, Lemma 2]). In other words, we will show

$$x_0 \frac{\partial f}{\partial x}(x_0, y_0) \geq y_0 \frac{\partial f}{\partial y}(x_0, y_0) \text{ if } x_0 \geq y_0. \tag{1.2.13}$$

Consider the following staight line $\gamma(t)$ orthogonal to the diagonal $\{x = y\}$ through (α_0, β_0).

$$\gamma(t) = (\alpha_0 - t, \beta_0 + t) \text{ and } \alpha_0 \geq \beta_0.$$

Since f^* is convex and symmetric, $f^*|_\gamma$ attains minimum at $t = \frac{1}{2}(\alpha_0 - \beta_0)$. At $t = 0$, $\gamma(t)$ moves toward the diagonal when t increases. Therefore, $\frac{d}{dt} f^* \leq 0$ when $t = 0$.

$$\begin{aligned} \frac{df^*}{dt}|_{t=0} &= \frac{\partial f}{\partial e^x}\frac{\partial e^x}{\partial t} + \frac{\partial f}{\partial e^y}\frac{\partial e^y}{\partial t} \\ &= -f_x\left(e^{\alpha_0}, e^{\beta_0}\right)e^{\alpha_0} + f_y\left(e^{\alpha_0}, e^{\beta_0}\right)e^{\beta_0}. \end{aligned}$$

Plug in $\alpha_0 = \log x_0$ and $\beta_0 = \log y_0$. (1.2.13) is proved. Therefore, (1) implies (3), and the proof is complete. □

For $F(A, x) = f(\lambda(A), x)$, we have similar results as above proposition. One can show that the following are equivalent.

Corollary 1.2.4. *Let $f(\lambda(A), x) \in C^2(\Gamma_n, M)$. The following positivity conditions are equivalent.*

1. *f is exponential-convex (1.1.5).*

2.

$$f_{ij}\zeta_i\overline{\zeta_j} + f_i\lambda_i^{-1}|\zeta_i|^2 + f_i^{\bar{z}_\beta}\zeta_i\overline{\eta_\beta} + f_i^{z_\alpha}\eta_\alpha\overline{\zeta_i} + +f^{z_\alpha,\bar{z}_\beta}\eta_\alpha\overline{\eta_\beta} \geq 0, \tag{1.2.14}$$

where η and $\zeta \in \mathbb{C}^n$.

3. *f satisfies equation (1.2.14) and $\frac{f_a}{\lambda_q} + \frac{f_a - f_q}{\lambda_a - \lambda_q} \geq 0$ for $a \neq q$.*

4.

$$F^{a\bar{b},r\bar{s}} X_{a\bar{b}} \overline{X_{s\bar{r}}} + F^{a\bar{b}} A^{r\bar{s}} X_{a\bar{s}} \overline{X_{b\bar{r}}} + F^{a\bar{b},\bar{z}_\beta} X_{a\bar{b}} \overline{\eta_\beta} + F^{z_\alpha,b\bar{a}} \eta_\alpha \overline{X_{a\bar{b}}} + F^{z_\alpha,\bar{z}_\beta} \eta_\alpha \overline{\eta_\beta} \geq 0,$$

where $X \in M_n(\mathbb{C})$ and $\eta \in \mathbb{C}^n$.

1.3 A differential inequality

1.3.1 Derivatives of eigenvalues

Based on the approach of Székelyhidi-Weinkove [22] and Brendle-Choi-Daskalopoulos [5], we try to compute the derivatives of eigenvalue on Hermitian manifolds.

Fix an x_0 at which the λ_i are **twice differentiable**. We choose a local coordinate so that at the point x_0, $g_{i\bar{j}} = \delta_{ij}$, and W is diagonal. In particular, we may assume $\lambda_i = W_{i\bar{i}}$ at x_0.

$$\lambda_1 = \cdots = \lambda_{\mu_1} < \lambda_{1+\mu_1} = \cdots = \lambda_{\mu_2} < \lambda_{1+\mu_2} = \cdots\cdots = \lambda_{\mu_N} = \lambda_n.$$

Define $\mu_0 = 0$ so that the multiplicities of the eigenvalues of A at x_0 are $\mu_1 - \mu_0, \mu_2 - \mu_1, \ldots, \mu_N - \mu_{N-1}$. We are using covariant derivatives with respect to Chern connection below.

Lemma 1.3.1. *For each $j = 1, 2, \ldots, N$ we have at x_0,*

$$\nabla_i (W_{k\bar{l}}) = \nabla_i(\lambda_{1+\mu_{j-1}})\delta_{kl} \tag{1.3.1}$$

$$\nabla_{\bar{i}} (W_{k\bar{l}}) = \nabla_{\bar{i}}(\lambda_{1+\mu_{j-1}})\delta_{kl} \tag{1.3.2}$$

for $1 + \mu_{j-1} \leq k, l \leq \mu_j$ and $i = 1, 2, \ldots, n$.

Proof. We will prove by induction. First show $j = 1$

$$\nabla_i (W_{k\bar{l}}) = \nabla_i(\lambda_{1+\mu_{j-1}})\delta_{kl}, \text{ for } 1 \leq k, l \leq \mu_1, \quad 1 \leq i \leq n. \tag{1.3.3}$$

Consider the following function in a neighborhood U of x_0.

$$\begin{aligned} h &:= A_k^l V^k V_l - \lambda_1 g_{k\bar{l}} V^k \overline{V^l} \\ &= W_{k\bar{l}} V^k \overline{V^l} - \lambda_1 g_{k\bar{l}} V^k \overline{V^l}, \end{aligned}$$

where $V = V^i \frac{\partial}{\partial z^i}$ is a vector field defined on U, and λ_1 is the first eigenvalue of A. Since h is a scalar function, it is invariant under coordinate transformation. At each point $y \in U$, we may pick a coordinate such that $g_{i\bar{j}}(y) = \delta_{ij}$, and λ_1 will become the first eigenvalue of $W_{k\bar{l}}(y)$ in this particular coordinate. Hence $h \geq 0$ in U. We set $V^k = 0$ for $k > \mu_1$. It follows that at x_0, $h(x_0) = 0$, and $\nabla_i h = \nabla_{\bar{i}} h = 0$. Observe that at x_0,

$$\begin{aligned} &W_{k\bar{l}} \left(\nabla_i V^k\right) \overline{V^l} - \lambda_1 g_{k\bar{l}} \left(\nabla_i V^k\right) \overline{V^l} \\ &= \sum_{k \leq \mu_1} W_{k\bar{k}} \left(\nabla_i V^k\right) \overline{V^k} - \lambda_1 \left(\nabla_i V^k\right) \overline{V^k} = 0. \end{aligned}$$

Similarly, $W_{k\bar{l}}V^k\left(\nabla_i\overline{V^l}\right)-\lambda_1 g_{k\bar{l}}V^k\left(\nabla_i\overline{V^l}\right)=0$. Therefore,

$$\begin{aligned}0&=\nabla_i h+\nabla_{\bar{i}}h\\&=\left(\nabla_i W_{k\bar{l}}+\nabla_{\bar{i}}W_{k\bar{l}}\right)V^k\overline{V^l}-\left(\nabla_i\lambda_1+\nabla_{\bar{i}}\lambda_1\right)g_{k\bar{l}}V^k\overline{V^l}.\end{aligned}$$

Observe that $\nabla_i W_{k\bar{l}}+\nabla_{\bar{i}}W_{k\bar{l}}-\left(\nabla_i\lambda_1+\nabla_{\bar{i}}\lambda_1\right)g_{k\bar{l}}$ is a $(\mu_1\times\mu_1)$ Hermitian matrix. Since $V^k(x_0)$ can be arbitrary for $k\leq\mu_1$, we have

$$\nabla_i W_{k\bar{l}}+\nabla_{\bar{i}}W_{k\bar{l}}=\left(\nabla_i\lambda_1+\nabla_{\bar{i}}\lambda_1\right)g_{k\bar{l}},\ 1\leq k,l\leq\mu_1 \text{ and } 1\leq i\leq n.$$

Similarly, $\sqrt{-1}\left(\nabla_{\bar{i}}h-\nabla_i h\right)=0$ and $\sqrt{-1}\left(\nabla_{\bar{i}}W_{k\bar{l}}-\nabla_i W_{k\bar{l}}\right)-\sqrt{-1}\left(\nabla_{\bar{i}}\lambda_1-\nabla_i\lambda_1\right)g_{k\bar{l}}$ is a $(\mu_1\times\mu_1)$ Hermitian matrix. We have

$$\left(\nabla_{\bar{i}}W_{k\bar{l}}-\nabla_i W_{k\bar{l}}\right)=\left(\nabla_{\bar{i}}\lambda_1-\nabla_i\lambda_1\right)g_{k\bar{l}},\ 1\leq k,l\leq\mu_1 \text{ and } 1\leq i\leq n.$$

Therefore, (1.3.3) holds.

Assume (1.3.1) and (1.3.2) holds for $1\leq j\leq p$. Let $V_1,\ldots,V_{1+\mu_p}$ be orthonormal vector fields in U. Also let $V_1(x_0),\ldots,V_{\mu_p}(x_0)$ be the constant vectors in the $\partial/\partial z_1,\ldots,\partial/\partial z_{\mu_p}$ directions. In other words,

$$V_i(x_0)=(0,...1,...,0)\,,\text{ only nonzero in } i^{th} \text{ component. } 1\leq i\leq\mu_p$$

Let $V_{1+\mu_p}(x_0)$ be the unit vector in the span of the directions $\partial/\partial z_{1+\mu_p},\ldots,\partial/\partial z_{\mu_{p+1}}$. Denote $X=V_{1+\mu_p}$.

Recall that $\lambda_{1+\mu_{j-1}}$ has multiplicity $(\mu_j-\mu_{j-1})$

$$
\begin{aligned}
h &= \sum_{\alpha=1}^{1+\mu_p} A_k^l V_\alpha^k V_l^\alpha - \left(\sum_{j=1}^{p} (\mu_j - \mu_{j-1}) \lambda_{1+\mu_{j-1}} \right) - \lambda_{1+\mu_p} \\
&= \sum_{\alpha=1}^{1+\mu_p} W_{k\bar{l}} V_\alpha^k \overline{V_\alpha^l} - \sum_{j=1}^{p} \sum_{1+\mu_{j-1} \le \alpha \le \mu_j} \lambda_{1+\mu_{j-1}} g_{k\bar{l}} V_\alpha^k \overline{V_\alpha^l} - \lambda_{1+\mu_p} \\
&= \sum_{j=1}^{p} \sum_{1+\mu_{j-1} \le \alpha \le \mu_j} (W_{k\bar{l}} - \lambda_{1+\mu_{j-1}} g_{k\bar{l}}) V_\alpha^k \overline{V_\alpha^l} + W_{k\bar{l}} X^k \overline{X^l} - \lambda_{1+\mu_p}.
\end{aligned}
$$

$h(x_0) = 0$ and $h(x) \ge 0$ for x near x_0. By induction hypothesis and the same reasoning as above, we have

$$
\begin{aligned}
0 &= (\nabla_i + \nabla_{\bar{i}})\, h(x_0) \\
&= \sum_{1+\mu_p \le k,l \le \mu_{p+1}} (\nabla_i W_{k\bar{l}} + \nabla_{\bar{i}} W_{k\bar{l}})\, X^k \overline{X^l} - \left(\nabla_i \lambda_{1+\mu_p} + \nabla_{\bar{i}} \lambda_{1+\mu_p} \right) g_{k\bar{l}} X^k \overline{X^l}.
\end{aligned}
$$

Observe that $(\nabla_i W_{k\bar{l}} + \nabla_{\bar{i}} W_{k\bar{l}}) - \left(\nabla_i \lambda_{1+\mu_p} + \nabla_{\bar{i}} \lambda_{1+\mu_p} \right) g_{k\bar{l}}$ can be viewed as a $(\mu_{p+1} - \mu_p) \times (\mu_{p+1} - \mu_p)$ Hermitian matrix. Since $X^k(x_0)$ can be arbitrary for $1 + \mu_p \le k \le \mu_{p+1}$, we obtain

$$
(\nabla_i + \nabla_{\bar{i}})\, W_{k\bar{l}} = (\nabla_i + \nabla_{\bar{i}})\, \lambda_{1+\mu_p} g_{k\bar{l}},\ 1 + \mu_p \le k, l \le \mu_{p+1} \text{ and } 1 \le i \le n.
$$

Same as the case for $j = 1$, by considering $(\nabla_{\bar{i}} - \nabla_i)\, h$ as well, we proved (1.3.1) and (1.3.2) for $j = p + 1$. By induction, equation (1.3.1) and (1.3.2) are true.

■

At the same x_0, we fix m between 1 and n. Define $\rho \in \{m, m+1, \ldots, n\}$ to be the largest integer such that $\lambda_\rho = \lambda_m$ at x_0, so that

$$
0 \le \lambda_1 \le \cdots \le \lambda_m = \lambda_{m+1} = \cdots = \lambda_\rho < \lambda_{\rho+1} \le \cdots \le \lambda_n.
$$

Lemma 1.3.2. *As a Hermitian $n \times n$ matrices we have at x_0,*

$$\sum_{\alpha=1}^{m} (\lambda_\alpha)_{a\bar{b}} \leq \sum_{\alpha=1}^{m} W_{\alpha\bar{\alpha}a\bar{b}} + \sum_{\alpha=1}^{m} \sum_{q>\rho} \left(\frac{W_{q\bar{\alpha}a} W_{\alpha\bar{q}\bar{b}}}{\lambda_\alpha - \lambda_q} + \frac{W_{\alpha\bar{q}a} W_{q\bar{\alpha}\bar{b}}}{\lambda_\alpha - \lambda_q} \right) \tag{1.3.4}$$

Proof. Let $V_1, \ldots, V_m$ be smooth holomorphic vector fields defined in a neighborhood of x_0, and these vector fields are mutually orthonormal to each other. We assume $V_\alpha(x_0)$ is the unit vector in the $\partial/\partial z_\alpha$ direction. In particular, writing $V_\alpha = V_\alpha^i \frac{\partial}{\partial z^i}$, we have $V_\alpha^q = \delta_{q\alpha}$ at x_0.

We consider the quantity

$$\begin{aligned} h(x) &= \sum_{\alpha=1}^{m} A_k^l V_\alpha^k V_l^\alpha - \sum_{\alpha=1}^{m} \lambda_\alpha g_{k\bar{l}} V_\alpha^k \overline{V_\alpha^l} \\ &= \sum_{\alpha=1}^{m} W_{k\bar{l}} V_\alpha^k \overline{V_\alpha^l} - \sum_{\alpha=1}^{m} \lambda_\alpha g_{k\bar{l}} V_\alpha^k \overline{V_\alpha^l}, \end{aligned}$$

which has $h(x_0) = 0$ and $h(x) \geq 0$ for x near x_0. In particular, h achieves its local minimum at x_0, and moreover, h is twice differentiable at x_0. We prescribe the first and second derivatives of the V_α at x_0 as follows (see lemma 1.6.1 for the existence). For $1 \leq \alpha \leq m$, and $1 \leq a \leq n$,

$$\nabla_a V_\alpha^q(x_0) = \begin{cases} 0, & q \leq \rho. \\ \frac{W_{\alpha\bar{q}a}}{\lambda_\alpha - \lambda_q}, & q > \rho. \end{cases} \quad \nabla_{\bar{a}} \overline{V_\alpha^q}(x_0) = \begin{cases} 0, & q \leq \rho. \\ \frac{W_{q\bar{\alpha}\bar{a}}}{\lambda_\alpha - \lambda_q}, & q > \rho. \end{cases}$$

$$\nabla_a \overline{V_\alpha^q}(x_0) = \begin{cases} 0, & q \leq \rho. \\ \frac{W_{q\bar{\alpha}a}}{\lambda_\alpha - \lambda_q}, & q > \rho. \end{cases} \quad \nabla_{\bar{a}} V_\alpha^q(x_0) = \begin{cases} 0, & q \leq \rho. \\ \frac{W_{\alpha\bar{q}\bar{a}}}{\lambda_\alpha - \lambda_q}, & q > \rho. \end{cases}$$

For $1 \leq \alpha, \beta \leq m$ and $1 \leq a, b \leq n$, we define

$$\nabla_{\bar{b}}\nabla_a\overline{V_\beta^\alpha}\,(x_0) = -\frac{1}{2}\sum_{q>\rho}\frac{W_{\alpha\bar{q}a}W_{q\bar{\beta}\bar{b}}}{(\lambda_\alpha-\lambda_q)(\lambda_\beta-\lambda_q)} - \frac{1}{2}\sum_{q>\rho}\frac{W_{\alpha\bar{q}\bar{b}}W_{q\bar{\beta}a}}{(\lambda_\alpha-\lambda_q)(\lambda_\beta-\lambda_q)}$$
$$-\frac{1}{2}\partial_{\bar{b}}\partial_a g_{\alpha\bar{\beta}} + \frac{1}{2}\partial_{\bar{b}}g_{m\bar{\beta}}\partial_a g_{\alpha\bar{m}}.$$

Similarly, we can define $\nabla_{\bar{b}}\nabla_a V_\alpha^\beta$ by the following relation.

$$\begin{aligned}\nabla_{\bar{b}}\nabla_a V_\alpha^\beta\,(x_0) &= \overline{\nabla_{\bar{a}}\nabla_b\overline{V_\alpha^\beta}} + \partial_{\bar{b}}\Gamma_{a\alpha}^\beta \\ &= \overline{\nabla_{\bar{a}}\nabla_b\overline{V_\alpha^\beta}} + \partial_{\bar{b}}\partial_a g_{\alpha\bar{\beta}} - \partial_{\bar{b}}g_{m\bar{\beta}}\partial_a g_{\alpha\bar{m}}.\end{aligned}$$

We check that these prescribed values are consistent with the V_α being orthonormal vectors. At x_0, for $\alpha, \beta = 1, \ldots, m$,

$$\nabla_a\left(g_{k\bar{l}}V_\alpha^k\overline{V_\beta^l}\right) = \nabla_a\left(\sum_q V_\alpha^q\overline{V_\beta^q}\right) = \sum_q(\nabla_a V_\alpha^q)\overline{V_\beta^q} + \sum_q V_\alpha^q\nabla_a\overline{V_\beta^q} = 0.$$

Note that $V_\alpha^q = 0$ if $q > m$ and $\nabla_a V_\alpha^q$ and $\nabla_a\overline{V_\beta^q}$ vanish when $q \leq m$. And

$$\begin{aligned}&\nabla_{\bar{b}}\nabla_a\left(g_{k\bar{l}}V_\alpha^k\overline{V_\beta^l}\right)\\ &= \sum_{q>\rho}(\nabla_a V_\alpha^q)(\nabla_{\bar{b}}\overline{V_\beta^q}) + \sum_{q>\rho}(\nabla_{\bar{b}}V_\alpha^q)(\nabla_a\overline{V_\beta^q}) + \nabla_{\bar{b}}\nabla_a V_\alpha^\beta + \nabla_{\bar{b}}\nabla_a\overline{V_\beta^\alpha}\\ &= \sum_{q>\rho}\frac{W_{\alpha\bar{q}a}}{\lambda_\alpha-\lambda_q}\frac{W_{q\bar{\beta}\bar{b}}}{\lambda_\beta-\lambda_q} + \sum_{q>\rho}\frac{W_{\alpha\bar{q}\bar{b}}}{\lambda_\alpha-\lambda_q}\frac{W_{q\bar{\beta}a}}{\lambda_\beta-\lambda_q}\\ &-\sum_{q>\rho}\frac{W_{\alpha\bar{q}a}W_{q\bar{\beta}\bar{b}}}{(\lambda_\alpha-\lambda_q)(\lambda_\beta-\lambda_q)} - \sum_{q>\rho}\frac{W_{\alpha\bar{q}\bar{b}}W_{q\bar{\beta}a}}{(\lambda_\alpha-\lambda_q)(\lambda_\beta-\lambda_q)}\\ &= 0\end{aligned}$$

as required.

Since h has a minimum at x_0, the complex Hessian is positive definite at x_0.

$$
\begin{aligned}
0 \leq h_{a\bar{b}} = &\sum_{\alpha=1}^{m} W_{\alpha\bar{\alpha}a\bar{b}} - (\lambda_a)_{a\bar{b}} \\
&+ W_{k\bar{l}a}\left(\nabla_{\bar{b}} V_\alpha^k \overline{V_\alpha^l} + V_\alpha^k \nabla_{\bar{b}} \overline{V_\alpha^l}\right) + W_{k\bar{l}\bar{b}}\left(\nabla_a V_\alpha^k \overline{V_\alpha^l} + V_\alpha^k \nabla_a \overline{V_\alpha^l}\right) \\
&+ W_{k\bar{l}}\left(\nabla_{\bar{b}}\nabla_a V_\alpha^k \overline{V_\alpha^l} + \nabla_a V_\alpha^k \nabla_{\bar{b}} \overline{V_\alpha^l} + \nabla_{\bar{b}} V_\alpha^k \nabla_a \overline{V_\alpha^l} + V_\alpha^k \nabla_{\bar{b}}\nabla_a \overline{V_\alpha^l}\right).
\end{aligned}
\tag{1.3.5}
$$

For α fixed

$$
\begin{aligned}
W_{k\bar{l}a}\left(\nabla_{\bar{b}} V_\alpha^k \overline{V_\alpha^l} + V_\alpha^k \nabla_{\bar{b}} \overline{V_\alpha^l}\right) &= \sum_{q>\rho} W_{q\bar{\alpha}a} \frac{W_{\alpha\bar{q}\bar{b}}}{\lambda_\alpha - \lambda_q} + W_{\alpha\bar{q}a} \frac{W_{q\bar{\alpha}\bar{b}}}{\lambda_\alpha - \lambda_q} \\
W_{k\bar{l}\bar{b}}\left(\nabla_a V_\alpha^k \overline{V_\alpha^l} + V_\alpha^k \nabla_a \overline{V_\alpha^l}\right) &= \sum_{q>\rho} W_{q\bar{\alpha}\bar{b}} \frac{W_{\alpha\bar{q}a}}{\lambda_\alpha - \lambda_q} + W_{\alpha\bar{q}\bar{b}} \frac{W_{q\bar{\alpha}a}}{\lambda_\alpha - \lambda_q}
\end{aligned}
\tag{1.3.6}
$$

$$
W_{k\bar{l}}\left(\nabla_{\bar{b}}\nabla_a V_\alpha^k \overline{V_\alpha^l} + \nabla_a V_\alpha^k \nabla_{\bar{b}} \overline{V_\alpha^l} + \nabla_{\bar{b}} V_\alpha^k \nabla_a \overline{V_\alpha^l} + V_\alpha^k \nabla_{\bar{b}}\nabla_a \overline{V_\alpha^l}\right) \tag{1.3.7}
$$

$$
\begin{aligned}
&= W_{\alpha\bar{\alpha}}\left(\nabla_{\bar{b}}\nabla_a V_\alpha^\alpha + \nabla_{\bar{b}}\nabla_a \overline{V_\alpha^\alpha}\right) + W_{q\bar{q}}\left(\nabla_a V_\alpha^q \nabla_{\bar{b}} \overline{V_\alpha^q} + \nabla_{\bar{b}} V_\alpha^q \nabla_a \overline{V_\alpha^q}\right) \\
&= \lambda_\alpha \left(-\sum_{q>\rho} \frac{W_{\alpha\bar{q}a} W_{q\bar{\alpha}\bar{b}}}{(\lambda_\alpha - \lambda_q)(\lambda_\alpha - \lambda_q)} - \sum_{q>\rho} \frac{W_{\alpha\bar{q}\bar{b}} W_{q\bar{\alpha}a}}{(\lambda_\alpha - \lambda_q)(\lambda_\alpha - \lambda_q)}\right) \\
&+ \lambda_q \sum_{q>\rho} \frac{W_{\alpha\bar{q}a}}{\lambda_\alpha - \lambda_q} \frac{W_{q\bar{\alpha}\bar{b}}}{\lambda_\alpha - \lambda_q} + \frac{W_{\alpha\bar{q}\bar{b}}}{\lambda_\alpha - \lambda_q} \frac{W_{q\bar{\alpha}a}}{\lambda_\alpha - \lambda_q} \qquad (1.3.8) \\
&= -\left(\sum_{q>\rho} W_{\alpha\bar{q}a} \frac{W_{q\bar{\alpha}\bar{b}}}{\lambda_\alpha - \lambda_q} + W_{\alpha\bar{q}\bar{b}} \frac{W_{q\bar{\alpha}a}}{\lambda_\alpha - \lambda_q}\right).
\end{aligned}
$$

Substituting (1.3.6) and (1.3.7) into (1.3.5) gives (1.3.4).

■

1.3.2 The key differential inequality.

Let B_1 be a coordinate chart. At a point $x_0 \in B_1$ where the λ_i are **twice differentiable,** we pick a coordinate such that $g_{i\bar{j}} = \delta_{ij}$, and $W_{i\bar{j}}$ is diagonal with entries $W_{i\bar{i}} = W_{i\bar{i}} g^{i\bar{i}} = \lambda_i$. Let

$$Q = \sum_{m=1}^{l} \sum_{\alpha=1}^{m} \lambda_\alpha.$$

Lemma 1.3.3. *The following inequality holds at* x_0*.*

$$F^{a\bar{a}} Q_{a\bar{a}} \leq C_1 Q + C_2 \sum_{a=1}^{n} \sum_{\alpha=1}^{l} |\lambda_{\alpha,a}| + C_0. \tag{1.3.9}$$

When W *is closed, the constant* $C_0 \leq 0$ *if* $R_{i\bar{i}j\bar{j}} \geq 0$*. When* $W_{i\bar{j}k} = W_{k\bar{j}i}$*, the constant* $C_0 \leq 0$ *if* $R_{i\bar{j}j\bar{i}} > 0$*. And all the constants above are independent of any choice of orthonormal basis.*

Note that we have the following constants.

$$C_\Lambda = \sup_{z \in \overline{B_1}} \sup \left\{ \Lambda : \Lambda^{-1} |\xi|^2 \leq F^{a\bar{b}} \xi_a \xi_{\bar{b}} \leq \Lambda |\xi|^2 \right\}.$$

$$C_W = \sup_{z \in \overline{B_1}} \sup \left\{ \left| W_{i\bar{j}k} \right| + \left| W_{i\bar{j}\bar{k}} \right| : \text{the basis } z_1, ...z_n, \bar{z}_1, ..., \bar{z}_n \text{ is orthonormal w.r.t. } g \right\}.$$

$$C_R = \sup_{z \in \overline{B_1}} \sup \left\{ R_{\alpha\bar{\alpha}a\bar{a}} : \text{the basis } z_1, ...z_n, \bar{z}_1, ..., \bar{z}_n \text{ is orthonormal w.r.t. } g \right\}.$$

$$C_f = \sup_{z \in \overline{B_1}} \sup \left\{ \|f\|_{C^2} : \text{the basis } z_1, ...z_n, \bar{z}_1, ..., \bar{z}_n \text{ is orthonormal w.r.t. } g \right\}$$

C_Λ, C_W, C_R and C_f are well-defined constants since orthonormal frame at each point is homeomorphic to the orthogonal group which is compact, and $\overline{B_1}$ is compact as well.

We first consider the case that W is closed. By taking two covariant derivative, at x_0, we have

$$0 = F^{a\bar{a}} W_{a\bar{a}\alpha\bar{\alpha}} + F^{a\bar{b},r\bar{s}} W_{a\bar{b}\alpha} W_{r\bar{s}\bar{\alpha}} + F^{a\bar{b},\bar{z}_\alpha} W_{a\bar{b}\alpha} + F^{z_\alpha,a\bar{b}} W_{a\bar{b}\bar{\alpha}} + F^{z_\alpha,\bar{z}_\alpha}. \tag{1.3.10}$$

Since W is closed, $\partial_k W_{i\bar{j}} = \partial_i W_{k\bar{j}}$. We have

$$W_{i\bar{j}k} - W_{k\bar{j}i} = T^l_{ik} W_{l\bar{j}}. \tag{1.3.11}$$

Since W is closed and Hermitian, by direct computation, we have the following formula for commuting derivatives at x_0. (See Lemma 1.6.2.)

$$W_{s\bar{s}k\bar{k}} - W_{k\bar{k}s\bar{s}} = R_{k\bar{k}s\bar{s}} W_{s\bar{s}} - R_{s\bar{s}k\bar{k}} W_{k\bar{k}} + 2\,\mathrm{Re}\left\{\overline{T^j_{sk}} W_{s\bar{j}k}\right\} - \left|T^j_{sk}\right|^2 W_{j\bar{j}}.$$

By above formula, we have

$$\begin{aligned} &F^{a\bar{a}} W_{\alpha\bar{\alpha}a\bar{a}} \\ &\leq F^{a\bar{a}} W_{a\bar{a}\alpha\bar{\alpha}} + C\lambda_\alpha + F^{a\bar{a}} \left(-R_{\alpha\bar{\alpha}a\bar{a}} W_{a\bar{a}} + T^q_{\alpha a} W_{q\bar{\alpha}\bar{a}} + \overline{T^q_{\alpha a}} W_{\alpha\bar{q}a} - |T^q_{\alpha a}|^2 W_{q\bar{q}}\right). \end{aligned} \tag{1.3.12}$$

Note that

$$\begin{aligned} &2\mathfrak{Re}\left\{\overline{T^q_{\alpha a}} W_{\alpha\bar{q}a}\right\} - |T^q_{\alpha a}|^2 W_{q\bar{q}} \\ &\leq C \sum_{b\leq\rho_l, b\neq\alpha} |W_{\alpha\bar{b}a}| + C\lambda_{\alpha,a} + \sum_{q>\rho_l} \left(T^q_{\alpha a} W_{q\bar{\alpha}\bar{a}} + \overline{T^q_{\alpha a}} W_{\alpha\bar{q}a} - |T^q_{\alpha a}|^2 W_{q\bar{q}}\right), \end{aligned} \tag{1.3.13}$$

where $\rho_l \in \{l, l+1, \ldots, n\}$ is the largest integer such that $\lambda_{\rho_l} = \lambda_l$ at x_0. Using equation

(1.3.4), we get

$$
\begin{aligned}
F^{a\bar{a}}Q_{a\bar{a}} &= \sum_{m=1}^{l}\sum_{\alpha=1}^{m} F^{a\bar{a}}(\lambda_\alpha)_{a\bar{a}} \\
&\le \sum_{m=1}^{l}\sum_{\alpha=1}^{m} F^{a\bar{a}}W_{\alpha\bar{\alpha}a\bar{a}} + \sum_{m=1}^{l}\sum_{\alpha=1}^{m}\sum_{q>\rho_m} F^{a\bar{a}}\left(\frac{W_{q\bar{\alpha}a}W_{\alpha\bar{q}\bar{a}}}{\lambda_\alpha-\lambda_q} + \frac{W_{\alpha\bar{q}a}W_{q\bar{\alpha}\bar{a}}}{\lambda_\alpha-\lambda_q}\right).
\end{aligned}
\tag{1.3.14}
$$

By equation (1.3.10) and (1.3.12), we have

$$
\begin{aligned}
F^{a\bar{a}}W_{\alpha\bar{\alpha}a\bar{a}} \le{}& -F^{a\bar{b},r\bar{s}}W_{a\bar{b}\alpha}W_{r\bar{s}\bar{\alpha}} - F^{a\bar{b},\bar{z}_\alpha}W_{a\bar{b}\alpha} - F^{z_\alpha,a\bar{b}}W_{a\bar{b}\bar{\alpha}} - F^{z_\alpha,\bar{z}_\alpha} \\
&+ F^{a\bar{a}}\left(-R_{\alpha\bar{\alpha}a\bar{a}}W_{a\bar{a}} + T^q_{\alpha a}W_{q\bar{\alpha}\bar{a}} + \overline{T^q_{\alpha a}}W_{\alpha\bar{q}a} - \left|T^q_{\alpha a}\right|^2 W_{q\bar{q}}\right) + C\lambda_\alpha.
\end{aligned}
\tag{1.3.15}
$$

Let's analyze above equation term by term. First of all,

$$
\begin{aligned}
&- F^{a\bar{b},\bar{z}_\alpha}W_{a\bar{b}\alpha} - F^{z_\alpha,b\bar{a}}W_{b\bar{a}\bar{\alpha}} \\
&= -2\,\mathrm{Re}\left\{F^{a\bar{b},\bar{z}_\alpha}W_{a\bar{b}\alpha}\right\} = -2\,\mathrm{Re}\left\{F^{a\bar{b},\bar{z}_\alpha}\left(W_{\alpha\bar{b}a} + T^l_{a\alpha}W_{l\bar{b}}\right)\right\} \\
&\le -\sum_{b>\rho_l}\left(F^{a\bar{b},\bar{z}_\alpha}W_{a\bar{b}\alpha} + F^{z_\alpha,b\bar{a}}W_{b\bar{a}\bar{\alpha}}\right) + C\left|\lambda_{\alpha,a}\right| + \sum_{\substack{1\le a\le n\\ b\le\rho_l,b\ne\alpha}} C\left|W_{\alpha\bar{b}a}\right| + CQ
\end{aligned}
\tag{1.3.16}
$$

For the following term,

$$
-F^{a\bar{b},r\bar{s}}W_{a\bar{b}\alpha}W_{r\bar{s}\bar{\alpha}} = -\sum_{b>\rho_l,r>\rho_l} F^{a\bar{b},r\bar{s}}W_{a\bar{b}\alpha}W_{r\bar{s}\bar{\alpha}} + \text{remaining terms.}
$$

There are three cases in the remaining terms. $(i)\, b\le\rho_l$ and $r\le\rho_l$. $(ii)\, b>\rho_l$ and $r\le\rho_l$. $(iii)\, b\le\rho_l$ and $r>\rho_l$. Let $b=\alpha$ and $1\le r\le n$ be an arbitrary fixed integer. Consider the following expression. If $(s,r)\ne(a,\alpha)$, we have

$$
\begin{aligned}
&- F^{a\bar{\alpha},r\bar{s}}W_{a\bar{\alpha}\alpha}W_{r\bar{s}\bar{\alpha}} - F^{s\bar{r},\alpha\bar{a}}W_{s\bar{r}\alpha}W_{\alpha\bar{a}\bar{\alpha}} \\
&= -2\,\mathrm{Re}\left\{F^{a\bar{\alpha},r\bar{s}}\left(W_{\alpha\bar{\alpha}a} + T^l_{a\alpha}W_{l\bar{\alpha}}\right)W_{r\bar{s}\bar{\alpha}}\right\} \le C\left|\lambda_{\alpha,a}\right| + C\lambda_\alpha.
\end{aligned}
$$

If $(s, r) = (a, \alpha)$, we have

$$-F^{a\bar{\alpha},\alpha\bar{a}} W_{a\bar{\alpha}\alpha} W_{\alpha\bar{a}\bar{\alpha}} \leq C |\lambda_{\alpha,a}| + C\lambda_\alpha.$$

Let $b \neq \alpha$, $r \neq \alpha$, $b \leq \rho_l$ and $r \leq \rho_l$. If $(s, r) \neq (a, b)$, we have

$$-F^{a\bar{b},r\bar{s}} W_{a\bar{b}\alpha} W_{r\bar{s}\bar{\alpha}} - F^{s\bar{r},b\bar{a}} W_{s\bar{r}\alpha} W_{b\bar{a}\bar{\alpha}} \leq C |W_{\alpha\bar{b}a}| + CQ.$$

If $(s, r) = (a, b)$, we have

$$-F^{a\bar{b},b\bar{a}} |W_{a\bar{b}\alpha}|^2 \leq C |W_{\alpha\bar{b}a}| + CQ.$$

Let $b \neq \alpha$, $r \neq \alpha$, $b > \rho_l$ and $r \leq \rho_l$. We have

$$-F^{a\bar{b},r\bar{s}} W_{a\bar{b}\alpha} W_{r\bar{s}\bar{\alpha}} - F^{s\bar{r},b\bar{a}} W_{s\bar{r}\alpha} W_{b\bar{a}\bar{\alpha}} \leq C |W_{\alpha\bar{r}s}| + CQ.$$

Combine all the cases, we get

$$-F^{a\bar{b},r\bar{s}} W_{a\bar{b}\alpha} W_{r\bar{s}\bar{\alpha}} \leq -\sum_{b>\rho_l, r>\rho_l} F^{a\bar{b},r\bar{s}} W_{a\bar{b}\alpha} W_{r\bar{s}\bar{\alpha}} + \sum_a C |\lambda_{\alpha,a}| + \sum_{\substack{1 \leq a \leq n \\ b \leq \rho_l, b \neq \alpha}} C |W_{\alpha\bar{b}a}| + CQ. \quad (1.3.17)$$

Next, we show that following inequality is true.

$$\sum_{\substack{1 \leq a \leq n \\ b \leq \rho_l, b \neq \alpha}} C |W_{\alpha\bar{b}a}| \leq CQ + \sum_{m=1}^{l-1} \sum_{\alpha=1}^{m} \sum_{\rho_m < q \leq \rho_l} F^{a\bar{a}} \left(\frac{W_{q\bar{\alpha}a} W_{\alpha\bar{q}\bar{a}}}{\lambda_q - \lambda_\alpha} + \frac{W_{\alpha\bar{q}a} W_{q\bar{\alpha}\bar{a}}}{\lambda_q - \lambda_\alpha} \right). \quad (1.3.18)$$

1.3.2.1 Estimate $|W_{\alpha\bar{b}a}|$

For the term $\sum_{\substack{1 \leq a \leq n \\ b \leq \rho_l, b \neq \alpha}} C |W_{\alpha\bar{b}a}|$. We separate it into two cases.

1.3.2.1.1 Case 1. $b > \alpha$ Rewrite b as q. By Lemma 1.3.1, it follows that $|W_{\alpha\bar{q}a}| = 0$ if $\lambda_q = \lambda_\alpha$ since $q \neq \alpha$. Therefore, $q > \rho_\alpha$.

$$|W_{\alpha\bar{q}a}| \leq C_0 \left(\lambda_q - \lambda_\alpha\right) + \frac{|W_{\alpha\bar{q}a}|^2}{C_0 \left(\lambda_q - \lambda_\alpha\right)}. \; C_0 \text{ to be fixed.}$$

$$\begin{aligned}
\sum_{m=1}^{l} \sum_{\alpha=1}^{m} \sum_{\substack{1 \leq a \leq n \\ \alpha < q \leq \rho_l}} C |W_{\alpha\bar{q}a}| &\leq C \sum_{\substack{1 \leq a \leq n \\ 1 \leq \alpha < q \leq \rho_l}} |W_{\alpha\bar{q}a}| \leq C \sum_{\alpha=1}^{l-1} \sum_{\rho_\alpha < q \leq \rho_l} \sum_{a=1}^{n} |W_{\alpha\bar{q}a}| \\
&\leq C \sum_{\alpha=1}^{l-1} \sum_{\rho_\alpha < q \leq \rho_l} \left(\lambda_q - \lambda_\alpha\right) + \sum_{\alpha=1}^{l-1} \sum_{\rho_\alpha < q \leq \rho_l} \frac{F^{a\bar{a}} W_{\alpha\bar{q}a} W_{q\bar{\alpha}\bar{a}}}{\left(\lambda_q - \lambda_\alpha\right)} \\
&\leq CQ + \sum_{m=1}^{l-1} \sum_{\alpha=1}^{m} \sum_{\rho_m < q \leq \rho_l} \frac{F^{a\bar{a}} W_{\alpha\bar{q}a} W_{q\bar{\alpha}\bar{a}}}{\left(\lambda_q - \lambda_\alpha\right)}, \qquad (1.3.19)
\end{aligned}$$

where in the second line we have to pick a large constant C_0 depending on the elliptic constant of $\left(F^{a\bar{b}}\right)$.

1.3.2.1.2 Case 2. $b < \alpha$ $|W_{\alpha\bar{b}a}| = 0$ if $\lambda_b = \lambda_\alpha$ since $b \neq \alpha$. Therefore, $\alpha > \rho_b$.

$$|W_{\alpha\bar{b}a}| \leq C_0 \left(\lambda_\alpha - \lambda_b\right) + \frac{|W_{\alpha\bar{b}a}|^2}{C_0 \left(\lambda_\alpha - \lambda_b\right)}. \; C_0 \text{ to be fixed.}$$

$$\begin{aligned}
\sum_{m=1}^{l} \sum_{\alpha=1}^{m} \sum_{\substack{1 \leq a \leq n \\ b < \alpha}} C |W_{\alpha\bar{b}a}| &\leq C \sum_{\substack{1 \leq a \leq n \\ 1 \leq b < \alpha \leq \rho_l}} |W_{\alpha\bar{b}a}| \leq C \sum_{b=1}^{l-1} \sum_{\rho_b < \alpha \leq \rho_l} \sum_{a=1}^{n} |W_{\alpha\bar{b}a}| \\
&\leq C \sum_{b=1}^{l-1} \sum_{\rho_b < \alpha \leq \rho_l} \left(\lambda_\alpha - \lambda_b\right) + \sum_{b=1}^{l-1} \sum_{\rho_b < \alpha \leq \rho_l} \frac{F^{k\bar{k}} W_{\alpha\bar{b}k} W_{b\bar{\alpha}\bar{k}}}{\left(\lambda_\alpha - \lambda_b\right)} \\
&\leq CQ + \sum_{m=1}^{l-1} \sum_{\alpha=1}^{m} \sum_{\rho_m < q \leq \rho_l} \frac{F^{k\bar{k}} W_{q\bar{\alpha}k} W_{\alpha\bar{q}\bar{k}}}{\left(\lambda_q - \lambda_\alpha\right)}. \qquad (1.3.20)
\end{aligned}$$

Althogh W is only semi-positive definite, we can still apply the positivity condition (1.2.4) (see [3, Lemma 3.1]). At x_0, our choice of coordinate implies

$$F^{a\bar{b},r\bar{s}}X_{a\bar{b}}\overline{X_{s\bar{r}}}+F^{a\bar{a}}\frac{1}{W_{q\bar{q}}}X_{a\bar{q}}\overline{X_{a\bar{q}}}+F^{a\bar{b},\bar{z}_\beta}X_{a\bar{b}}\overline{\eta_\beta}+F^{z_\alpha,b\bar{a}}\eta_\alpha\overline{X_{a\bar{b}}}+F^{z_\alpha,\bar{z}_\beta}\eta_\alpha\overline{\eta_\beta}\geq 0,$$

where $q>\rho_l$ and $X_{a\bar{b}}=0$ if $b\leq\rho_l$. For each fixed α, we set

$$X_{a\bar{b}}=\begin{cases}W_{a\bar{b}\alpha} & \text{if } b>\rho_l\\ 0 & \text{otherwise,}\end{cases}\qquad \eta_a=\begin{cases}1 & \text{if } a=\alpha\\ 0 & \text{otherwise,}\end{cases}\quad. \tag{1.3.21}$$

Therefore, we have

$$-\sum_{b>\rho_l,r>\rho_l}F^{a\bar{b},r\bar{s}}W_{a\bar{b}\alpha}W_{r\bar{s}\bar{\alpha}}-\sum_{b>\rho_l}\left(F^{a\bar{b},\bar{z}_\alpha}W_{a\bar{b}\alpha}+F^{z_\alpha,b\bar{a}}W_{b\bar{a}\bar{\alpha}}\right)-F^{z_\alpha,\bar{z}_\alpha}\leq\sum_{q>\rho_l}F^{a\bar{a}}\frac{W_{a\bar{q}\alpha}W_{q\bar{a}\bar{\alpha}}}{\lambda_q}. \tag{1.3.22}$$

Note that

$$\begin{aligned}
&\sum_{q>\rho_l}F^{a\bar{a}}\frac{W_{a\bar{q}\alpha}W_{q\bar{a}\bar{\alpha}}}{\lambda_q}\\
&=\sum_{q>\rho_l}F^{a\bar{a}}\frac{\left(W_{\alpha\bar{q}a}+T^l_{a\alpha}W_{l\bar{q}}\right)\left(W_{q\bar{\alpha}\bar{a}}+\overline{T^l_{a\alpha}}W_{q\bar{l}}\right)}{\lambda_q}\\
&\leq\sum_{q>\rho_l}F^{a\bar{a}}\frac{W_{\alpha\bar{q}a}W_{q\bar{\alpha}\bar{a}}}{\lambda_q-\lambda_\alpha}+F^{a\bar{a}}\left(W_{\alpha\bar{q}a}\overline{T^q_{a\alpha}}+T^q_{a\alpha}W_{q\bar{\alpha}\bar{a}}+T^q_{a\alpha}\overline{T^q_{a\alpha}}W_{q\bar{q}}\right).
\end{aligned} \tag{1.3.23}$$

Combining equation (1.3.13), (1.3.14), (1.3.15), (1.3.16), (1.3.17), (1.3.18), (1.3.22), and (1.3.23), we get

$$\begin{aligned}
F^{a\bar{a}}Q_{a\bar{a}}&=\sum_{m=1}^{l}\sum_{\alpha=1}^{m}F^{a\bar{a}}(\lambda_\alpha)_{a\bar{a}}\\
&\leq\sum_{m=1}^{l}\sum_{\alpha=1}^{m}F^{a\bar{a}}\left(-R_{\alpha\bar{\alpha}a\bar{a}}W_{a\bar{a}}\right)+C\left|\lambda_{\alpha,a}\right|+CQ.
\end{aligned} \tag{1.3.24}$$

The constants C depending on C_Λ, C_W, C_R and C_f are independent of any orthonormal basis.

Next, we consider the case $W_{i\bar{j}k} = W_{k\bar{j}i}$. The only difference is how we commute the derivatives. By direct computation (see Lemma 1.6.2), it follows that

$$W_{l\bar{l}k\bar{k}} - W_{k\bar{k}l\bar{l}} = R_{l\bar{k}k\bar{l}} W_{l\bar{l}} - R_{l\bar{k}k\bar{l}} W_{k\bar{k}}.$$

By above formula, we have

$$\begin{aligned} F^{a\bar{a}} W_{\alpha\bar{\alpha}a\bar{a}} &= F^{a\bar{a}} W_{a\bar{a}\alpha\bar{\alpha}} - F^{a\bar{a}} \left(R_{\alpha\bar{a}a\bar{\alpha}} W_{a\bar{a}} - R_{\alpha\bar{a}a\bar{\alpha}} W_{\alpha\bar{\alpha}} \right) \\ &\leq F^{a\bar{a}} W_{a\bar{a}\alpha\bar{\alpha}} - F^{a\bar{a}} R_{\alpha\bar{a}a\bar{\alpha}} W_{a\bar{a}} + C\lambda_\alpha. \end{aligned}$$

By similar reasoning above, we have

$$F^{a\bar{a}} Q_{a\bar{a}} \leq -\sum_{m=1}^{l} \sum_{\alpha=1}^{m} F^{a\bar{a}} R_{\alpha\bar{a}a\bar{\alpha}} W_{a\bar{a}} + C \left| \lambda_{\alpha,a} \right| + CQ.$$

1.4 Proof of the theorem

1.4.1 Eigenvalue inequality

With the key differential inequality, the proof of eigenvalue inequality is the same as the proof in Székelyhidi-Weinkove [22]. We only outline the key steps below. Let $Q_k = \lambda_k + 2\lambda_{k-1} + \cdots + k\lambda_1$, which is semi-concave. From previous section, once the curvature condition is satisfied, we have almost everywhere

$$F^{a\bar{b}} (Q_k)_{a\bar{b}} \leq CQ_k + C \sum_{a=1}^{n} \sum_{\alpha=1}^{k} \left| \lambda_{\alpha,a} \right| \text{ for } C \text{ uniformly bounded.} \tag{1.4.1}$$

For $\varepsilon > 0$ and a fixed $l \in \{1, 2, \ldots, n\}$, we consider

$$R = \sum_{k=1}^{l} (Q_k + \varepsilon)^{1/2}.$$

Following [22], R is semi-concave with a constant of semi-concavity depending on ε. At a twice differentiable point x_0, we pick a coordinate such that $g_{i\bar{j}} = \delta_{ij}$ and $W_{i\bar{j}}$ is diagonal. We have

$$\begin{aligned} F^{a\bar{a}} R_{a\bar{a}} &= \frac{1}{2} \sum_{k=1}^{l} (Q_k + \varepsilon)^{-\frac{1}{2}} F^{a\bar{a}} (Q_k)_{a\bar{a}} - \frac{1}{4} \sum_{k=1}^{l} (Q_k + \varepsilon)^{-\frac{3}{2}} F^{a\bar{a}} (Q_k)_a (Q_k)_{\bar{a}} \\ &\leq \frac{1}{2} \sum_{k=1}^{l} (Q_k + \varepsilon)^{-\frac{1}{2}} \left(C (Q_k + \varepsilon) + C \sum_{a,\alpha} |\lambda_{\alpha,a}| \right) - c \sum_{k=1}^{l} (Q_k + \varepsilon)^{-\frac{3}{2}} |DQ_k|^2 \\ &\leq CR + C \sum_{k=1}^{l} (Q_k + \varepsilon)^{-\frac{1}{2}} \sum_{\alpha=1}^{l} |D\lambda_\alpha| - c \sum_{k=1}^{l} (Q_k + \varepsilon)^{-\frac{3}{2}} |DQ_k|^2 . \end{aligned}$$

for uniform constants $C, c > 0$. The rest computation is the same as [22, (4.1)]. We get at x_0,

$$F^{a\bar{b}} R_{a\bar{b}} = F^{a\bar{a}} R_{a\bar{a}} \leq CR. \tag{1.4.2}$$

Note that the constant (1.4.2) depends only on n, C_Λ, C_W, C_R and C_f. Furthermore, since equation (1.4.2) is a scalar equation, it is independent of choice of the basis. In particular, we have $F^{a\bar{b}} R_{a\bar{b}} \leq CR$ almost everywhere in a coordinate ball. Since R is semi-concave, the weak Harnack inequality [22, Proposition 2.3] implies that for a uniform $q > 0$ and C,

$$\|R\|_{L^q(B_{1/2})} \leq C \inf_{B_{1/2}} R, \tag{1.4.3}$$

where the constant C is independent of ε. Hence we can let $\varepsilon \to 0$ and get

$$\|\lambda_l\|_{L^{q/2}(B_{1/2})} \leq C \inf_{B_{1/2}} \lambda_l. \tag{1.4.4}$$

This completes the proof of the eigenvalue estimate.

1.4.2 Special case in theorem 1.1.1

With the eigenvalue estimate (1.4.4) in, it follows that $A_a^b \equiv W_{a\bar{k}} g^{\bar{k}b}$ has constant rank in a coordinate ball. Indeed, since eigenvalues are continuous,

$$\{x \in B_1 | rank\,(A) \geq k\} \text{ is open in } B_1.$$

For any $x \in B_1$, we apply the eigenvalue estimate (1.4.4) on $B_\varepsilon(x) \subset B_1$ for some small ε.

$$\{x \in B_1 | rank\,(A) \leq k\} \text{ is open in } B_1.$$

Therefore $\{x \in B_1 | rank\,(A) = k\}$ is open and closed in B_1. A has constant rank. Let's consider the special case when the smallest eigenvalue is zero at some point, and $R_{\alpha\bar{\alpha}a\bar{a}} > 0$ at some point. Since A has constant rank, it follows that $\lambda_1 \equiv 0$ in B_1. Substitute $Q = \lambda_1$ into equation (1.3.24), we have

$$\begin{aligned} F^{a\bar{a}} (\lambda_1)_{a\bar{a}} &\leq \sum_{a>1} F^{a\bar{a}} \left(-R_{1\bar{1}a\bar{a}} W_{a\bar{a}}\right) + C\,|\lambda_{1,a}| + C\lambda_1 \\ &\leq \sum_{a>1} F^{a\bar{a}} \left(-R_{1\bar{1}a\bar{a}} W_{a\bar{a}}\right). \end{aligned}$$

At the point $R_{1\bar{1}a\bar{a}} > 0$, for above inequality to hold, $W_{a\bar{a}}$ has to be 0 for all $a > 1$. And once $W_{a\bar{a}}$ is zero at one point, by the property of constant rank, it is zero everywhere. Therefore, all the eigenvalues are zero in B_1.

1.4.3 C^n case

We consider

$$F(A, p, q, u, z) \in C^2 \left(Hermitian_n(\mathbb{C}) \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R} \times B_1(0)\right).$$

F is uniformly elliptic. For simplicity, we assume $u \in C^4(B_1)$ and $\left(u_{i\bar{j}}\right) \geq 0$ is a solution to

$$F\left(u_{i\bar{j}}, u_k, u_{\bar{k}}, u, z\right) = 0. \tag{1.4.5}$$

Similar to case on Hermitian manifolds, we require a conjugate condition and positivity condition.

1. *Conjugate condition.*

$$\overline{F^{A_\theta}} = F^{\overline{A_\theta}} = F^{A_{\bar{\theta}}},\ F^{\overline{A_\theta} B_\tau} = F^{A_{\bar{\theta}} B_\tau},\ \text{and } \overline{F^{A_\theta B_\tau}} = F^{\overline{A_\theta B_\tau}} = F^{A_{\bar{\theta}} B_{\bar{\tau}}}. \tag{1.4.6}$$

   Here $A_\theta,\ B_\tau \in \left\{u_{i\bar{j}}, \overline{u_{i\bar{j}}}, u_j, \overline{u_j}, u_{\bar{k}}, \overline{u_{\bar{k}}}, z_\beta, \bar{z}_\beta\right\}$. For example, if $A_\theta = u_{a\bar{b}}$ and $B_\tau = \overline{u_{s\bar{r}}}$, above condition means

$$\overline{F^{a\bar{b}}} = \overline{F^{u_{a\bar{b}}}} = F^{\overline{u_{a\bar{b}}}} = F^{u_{b\bar{a}}} = F^{b\bar{a}} \Rightarrow \left(F^{a\bar{b}}\right) \text{ is Hermitian.}$$

$$\overline{F^{i\bar{j},\overline{s\bar{r}}}} = \overline{F^{u_{i\bar{j}},\overline{u_{s\bar{r}}}}} = F^{\overline{u_{i\bar{j}}},u_{s\bar{r}}} = F^{s\bar{r},\overline{i\bar{j}}} \Rightarrow \left(F^{i\bar{j},\overline{s\bar{r}}}\right) \text{ is Hermitian.}$$

In particular, for any complex-valued vector $X_{a\bar{b}}$, we have

$$F^{a\bar{b},\overline{s\bar{r}}} X_{a\bar{b}} \overline{X_{s\bar{r}}} = F^{a\bar{b},r\bar{s}} X_{a\bar{b}} \overline{X_{s\bar{r}}} \in \mathbb{R}.$$

2. *Positivity condition.* We will consider two cases.

   (a) F is in fact a function of eigenvalues. $F(A, p, q, u, z) = f(\lambda(A), p, q, u, z)$. Let μ be a complex vector in $\mathbb{C}^{3n+1}$. We define μ as follows.

$$\mu_i = \lambda_i,\ 1 \leq i \leq n. \qquad \mu_{n+k} = p_k,\ 1 \leq k \leq n.$$
$$\mu_{2n+\beta} = z_\beta,\ 1 \leq \beta \leq n. \quad \mu_{3n+1} = u.$$

   Define $f^*(\lambda, p, q, u, z) = f\left(e^\lambda, p, q, u, z\right)$, where $e^\lambda \equiv \left(e^{\lambda_1}, ..., e^{\lambda_n}\right)$. When $\lambda_i > 0$, $\forall i$, for each $q \in \mathbb{C}^n$, we require

$$\frac{\partial^2 f^*}{\partial \mu_i \partial \overline{\mu_j}} \geq 0. \tag{1.4.7}$$

   Note that since λ_k are real, $\mu_k = \overline{\mu_k}$ for $1 \leq k \leq n$.

   (b) The general case. Let w be a complex vector in $\mathbb{C}^{n^2+2n+1}$. We define w as follows.

$$w_{(n-1)i+j} = A_{i\bar{j}},\ 1 \leq i, j \leq n. \quad w_{n^2+k} = p_k,\ 1 \leq k \leq n.$$
$$w_{n^2+n+\beta} = z_\beta,\ 1 \leq \beta \leq n. \qquad w_{(n+1)^2} = u.$$

   When A is positive definite, for each $q \in \mathbb{C}^n$, we require

$$\frac{\partial^2 F}{\partial w_i \partial \overline{w_j}} V_i \overline{V_j} + F^{a\bar{b}} A^{r\bar{s}} X_{a\bar{s}} \overline{X_{b\bar{r}}} \geq 0, \tag{1.4.8}$$

   where V is an arbitrary vector in $\mathbb{C}^{(n+1)^2}$ with $V_{(n-1)i+j} = X_{i\bar{j}}$. In our current setting, $A_{i\bar{j}} = u_{i\bar{j}}$, $p_k = u_k$ and $q_k = u_{\bar{k}}$. Note that when F is a function of eigenvalues, these two conditions are equivalent by the same reasoning as proposition

1.2.2.

Observe that in condition (1.4.8), we only need to consider derivatives with respect to u_k, and don't have to consider derivatives with respect to $u_{\bar{k}}$ since $u_{\bar{k}\alpha} = \lambda_\alpha \delta_{k\alpha}$. With these two conditions, the rest of computation is essentially the same as the Hermitian manifold case once we set $W_{i\bar{j}} = u_{i\bar{j}}$. We outline the steps below, and leave the details to the reader.

Step 1. Differentiate equation (1.4.5) twice, pair conjugate terms together, and analyze each term separately like how we do for $W_{i\bar{j}}$. We have the following inequality.

$$\begin{aligned} F^{a\bar{b}} u_{a\bar{b}\alpha\bar{\alpha}} \leq {} & C\lambda_\alpha + C\,|\lambda_{\alpha,a}| + \sum_{\substack{1\leq a\leq n \\ b\leq \rho_l, b\neq\alpha}} C\,|u_{\alpha\bar{b}a}| - \sum_{b>\rho_l, r>\rho_l} F^{a\bar{b},r\bar{s}} u_{a\bar{b}\alpha} u_{r\bar{s}\bar{\alpha}} \\ & - \sum_{b>\rho_l} (*) - F^{u_k,u_{\bar{j}}} u_{k\alpha} u_{\bar{j}\bar{\alpha}} - F^{u_k,u} u_{k\alpha} u_{\bar{\alpha}} - F^{u,u_{\bar{k}}} u_\alpha u_{\bar{k}\bar{\alpha}} - F^{u_k,\bar{z}_\alpha} u_{k\alpha} \\ & - F^{z_\alpha,u_{\bar{k}}} u_{\bar{k}\bar{\alpha}} - F^{u,u} u_\alpha u_{\bar{\alpha}} - F^{u,\bar{z}_\alpha} u_\alpha - F^{z_\alpha,u} u_{\bar{\alpha}} - F^{z_\alpha,\bar{z}_\alpha}, \end{aligned} \tag{1.4.9}$$

where

$$(*) = 2\,\mathrm{Re}\left\{ F^{a\bar{b},u_{\bar{k}}} u_{a\bar{b}\alpha} u_{\bar{k}\bar{\alpha}} + F^{a\bar{b},u} u_{a\bar{b}\alpha} u_{\bar{\alpha}} + F^{a\bar{b},\bar{z}_\alpha} u_{a\bar{b}\alpha} \right\}.$$

Step 2. Apply the positivity condition (1.4.8) to cancel the extra terms in equation (1.4.9). We have

$$F^{a\bar{b}} u_{a\bar{b}\alpha\bar{\alpha}} \leq C\lambda_\alpha + C\,|\lambda_{\alpha,a}| + \sum_{\substack{1\leq a\leq n \\ b\leq \rho_l, b\neq\alpha}} C\,|u_{\alpha\bar{b}a}| + \sum_{q>\rho_l} F^{a\bar{b}} u^{q\bar{q}} u_{a\bar{q}\alpha} u_{q\bar{b}\bar{\alpha}}. \tag{1.4.10}$$

Step 3. Eliminate $|u_{\alpha\bar{b}a}|$ in equation (1.4.9) by the following negative terms from equation

(1.3.4).

$$\sum_{m=1}^{l-1}\sum_{\alpha=1}^{m}\sum_{\rho_m<q\leq\rho_l}\left(\frac{u_{q\bar{\alpha}a}u_{\alpha\bar{q}\bar{b}}}{\lambda_\alpha-\lambda_q}+\frac{u_{\alpha\bar{q}a}u_{q\bar{\alpha}\bar{b}}}{\lambda_\alpha-\lambda_q}\right).$$

Step 4. Eliminate $F^{a\bar{b}}u^{q\bar{q}}u_{a\bar{q}\alpha}u_{q\bar{b}\bar{\alpha}}$ in equation (1.4.9) by the following negative terms from equation (1.3.4).

$$\sum_{m=1}^{l}\sum_{\alpha=1}^{m}\sum_{q>\rho_l}F^{a\bar{b}}\frac{u_{a\bar{q}\alpha}u_{q\bar{b}\bar{\alpha}}}{\lambda_\alpha-\lambda_q}.$$

Therefore, we obtain the same inequality as equation (1.4.1). Once we obtain this inequality, the rest of the proof is the same.

1.5 Application and Discussion

In this section, we will discuss some examples of W. Motivated by Guan-Li-Zhang [13], we would like to consider the curvature tensor $R_{i\bar{j}k\bar{l}}$ on M in the local coordinate. By taking the trace, we can define the following Ricci curvatures.

$$R_{k\bar{l}}=g^{i\bar{j}}R_{i\bar{j}k\bar{l}} \text{ and } R^{(2)}_{i\bar{j}}=g^{k\bar{l}}R_{i\bar{j}k\bar{l}}.$$

Also, we can define the scalar curvature.

$$R=g^{i\bar{j}}g^{k\bar{l}}R_{i\bar{j}k\bar{l}}.$$

We say a tensor is *quasi-positive* [15][23] if it is nonnegative everywhere and strictly positive at some point.

1.5.1 $W_{i\bar{j}}$ is closed

Lemma 1.5.1.

1. $R_{k\bar{l}}$ and $R^{(2)}_{i\bar{j}}$ are Hermitian.

2. $R^{(2)}_{i\bar{j}}$ is closed.

3. $R_{k\bar{l}}$ is closed when $(\operatorname{div} T)_{j\bar{k}} \equiv \nabla^i T_{ij\bar{k}} = 0$.

Proof. (1) is straightforward, and (2) follows from the fact that $R^{(2)}_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det (g_{k\bar{l}})$.

For (3), by Bianchi identity $R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}} + \nabla_{\bar{j}} T_{ki\bar{l}}$ and $\nabla_m R_{k\bar{j}i\bar{l}} - \nabla_k R_{m\bar{j}i\bar{l}} = T^r_{km} R_{r\bar{j}i\bar{l}}$, we have

$$
\begin{aligned}
\nabla_m R_{i\bar{j}k\bar{l}} &= \nabla_m R_{k\bar{j}i\bar{l}} + \nabla_m \nabla_{\bar{j}} T_{ki\bar{l}}. \\
\nabla_k R_{i\bar{j}m\bar{l}} &= \nabla_k R_{m\bar{j}i\bar{l}} + \nabla_k \nabla_{\bar{j}} T_{mi\bar{l}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\nabla_m R_{i\bar{j}k\bar{l}} - \nabla_k R_{i\bar{j}m\bar{l}} &= T^r_{km} R_{r\bar{j}i\bar{l}} + \nabla_m \nabla_{\bar{j}} T_{ki\bar{l}} - \nabla_k \nabla_{\bar{j}} T_{mi\bar{l}} \\
\nabla_m R_{k\bar{l}} - \nabla_k R_{m\bar{l}} &= g^{i\bar{j}} T^r_{km} \left(R_{i\bar{j}r\bar{l}} + \nabla_{\bar{j}} T_{ir\bar{l}} \right) - g^{i\bar{j}} \nabla_m \nabla_{\bar{j}} T_{ik\bar{l}} + g^{i\bar{j}} \nabla_k \nabla_{\bar{j}} T_{im\bar{l}} \\
&= T^r_{km} \left(R_{r\bar{l}} + \nabla^i T_{ir\bar{l}} \right) - \nabla_m \nabla^i T_{ik\bar{l}} + \nabla_k \nabla^i T_{im\bar{l}}
\end{aligned}
$$

It is clear that $R_{k\bar{l}}$ is closed from above equation when $\operatorname{div} T = 0$. □

From above lemma, by finding functions $F\left(g^{-1}W, z\right)$ satisfying the structural conditions, we have the following corollaries.

Corollary 1.5.2. *Let (M, g) be a connected Hermitian manifold with nonnegative orthogonal bisectional curvature and Ricci curvature. Then we have*

1. If $\operatorname{div} T = 0$ and $-\log(R+1)$ is plurisubharmonic, $R_{k\bar{l}}$ has constant rank.

2. If $-\log(R+1)$ is plurisubharmonic, $R^{(2)}_{i\bar{j}}$ has constant rank.

Remark 1.5.3. We add 1 to avoid $R(z) = 0$ at some point. It can be any positive constant.

Proof. The proof is immediate. Let's prove (2). Let $W_{i\bar{j}} = R^{(2)}_{i\bar{j}}$ and λ be the eigenvalues of W with respect to g. We have the following equation.

$$f(\lambda, z) = \log(\sigma_1(\lambda) + 1) - \log(R+1) = 0. \tag{1.5.1}$$

To apply the theorem, we require the following condition.

$$f_{ij}\zeta_i\overline{\zeta_j} + f_i\lambda_i^{-1}|\zeta_i|^2 + f_i^{\bar{z}_\beta}\zeta_i\overline{\eta_\beta} + f_i^{z_\alpha}\eta_\alpha\overline{\zeta_i} + f^{z_\alpha,\bar{z}_\beta}\eta_\alpha\overline{\eta_\beta} \geq 0, \tag{1.5.2}$$

where ζ and η are arbitrary complex-valued vector. From (1.2.7), we know that $g(\lambda, z) = \log\sigma_1(\lambda)$ satisfies these structure conditions. By direct computation, it follows that $h(\lambda, z) = \log(\sigma_1(\lambda) + 1)$ also satisfies

$$h_{ij}\zeta_i\overline{\zeta_j} + h_i\lambda_i^{-1}|\zeta_i|^2 \geq 0 \text{ for any } \zeta \in \mathbb{C}^n. \tag{1.5.3}$$

Therefore, if $-\log\left(R^{(2)} + 1\right)$ is plurisubharmonic, (1.5.2) is satisfied. Then we can apply theorem 1.1.1. The proof for (1) is the same. □

1.5.2 $W_{i\bar{j}k} = W_{k\bar{j}i}$

It is also natural to consider the condition $W_{i\bar{j}k} = W_{k\bar{j}i}$ on Hermitian manifolds. It is an interesting question to find a manifold such that the Ricci curvature tensors satisfy these conditions. Let's first consider the following two special types of Hermitian manifolds.

1. A Hermitian manifold is *Chern-Kähler-like* [24] if $R_{i\bar{j}k\bar{l}} = R_{k\bar{j}i\bar{l}}$.

2. A Hermitian manifold is a *CAS* manifold [18] if the Chern connection has parallel torsion and curvature.

We remark that a CAS manifold must be Kähler-like, and for a Kähler-like manifold,

$$R_{i\bar{j}} = R^{(2)}_{i\bar{j}}.$$

Obviously a CAS manifold will have $R_{i\bar{j}k} = R_{k\bar{j}i}$ since the curvature tensor is parallel. However, the condition seems to be very strong. Observe that the Kähler-like condition is the same as the first Bianchi identity (1.2.1) being zero.

$$R_{i\bar{j}k\bar{l}} - R_{k\bar{j}i\bar{l}} = g_{m\bar{l}} \nabla_{\bar{j}} T^m_{ki} = 0.$$

Motivated by the definition of being Kähler-like, we say a manifold is *special Kähler-like* if the first Bianchi identity and second Bianchi identity (1.2.2) are both zero. Then the special Kähler-like manifold will have $R_{i\bar{j}k} = R_{k\bar{j}i}$. Note that we have the following relations.

$$\text{CAS} \subseteq \text{special Kähler-like} \subseteq \text{Kähler-like}.$$

Remark 1.5.4. We can also consider the case that only the second Bianchi identity is zero. The second Bianchi identity being zero already implies $R^{(2)}_{i\bar{j}k} = R^{(2)}_{k\bar{j}i}$. But it feels more natural to have both Bianchi identities equal to zero.

Motivated by Guan-Li-Zhang's result [13], we have the following corollary on compact special Kähler-like manifold.

Corollary 1.5.5. *Let (M, g) be a compact special Kähler-like manifold with quasi-positive orthogonal bisectional curvature and nonnegative Ricci curvature. Suppose that there exists*

F satisfying the structural conditions in theorem 1.1.1 such that $F\left(g^{-1}Ric, z\right) = 0$. Then it is either Kähler-Einstein or Ricci-flat.

Proof. First of all, observe that the condition in theorem 1.1.1 has reduced to $R_{\alpha\bar{\alpha}\beta\bar{\beta}}$ being quasi-positive due to the Kähler-like condition.

Suppose it is not *Kähler*. We diagonalize $R_{i\bar{j}}$ at a fixed point. Since the manifold is Kähler-like, we have

$$\nabla_m R_{k\bar{l}} - \nabla_k R_{m\bar{l}} = T^l_{km} R_{l\bar{l}} = 0.$$

Since some torsion component is not zero, it follows that $R_{l\bar{l}}$ must be zero for some component. Since the orthogonal bisectional curvature is quasi-positive and the smallest eigenvalue of $\left(R_{i\bar{j}}\right)$ is zero, by theorem 1.1.1, the Ricci curvature must be zero.

Now suppose it is *Kähler* and a is the smallest eigenvalue of $R_{i\bar{j}}$ with respect to the metric. Let's consider $W_{i\bar{j}} = R_{i\bar{j}} - a g_{i\bar{j}}$, and λ be the eigenvalues of W with respect to g. Then

$$F\left(g^{-1}Ric, z\right) = F\left(g^{-1}W + aI, z\right) = G\left(g^{-1}W, z\right) = 0,$$

where $G(A) = F(A + aI, z)$. Then G still satisfies the structural condition in theorem 1.1.1 since a is nonnegative. Since the orthogonal bisectional curvature is quasi-positive and the smallest eigenvalue of $\left(W_{i\bar{j}}\right)$ is zero, by theorem 1.1.1, $\left(W_{i\bar{j}}\right)$ must be zero. Therefore, it's Kähler-Einstein. □

1.6 Miscellaneous lemmas

Lemma 1.6.1. *Let (M^n, g) be a Hermitian manifold. Fix a coordinate chart, for $m \leq n$, there exist mutually orthonormal vector fields $V_1, ..., V_m$ around 0 with prescribed $V_i(0)$, $\nabla V_i(0)$ and $\nabla\nabla V_i(0)$ as long as the prescribed values are consistent with the orthonormal conditions. In other words,*

$$V_\alpha^* g V_\beta = \delta_{\alpha\beta},\ \nabla\left(V_\alpha^* g V_\beta\right) = \nabla\nabla\left(V_\alpha^* g V_\beta\right) = 0 \text{ at the origin,} \tag{1.6.1}$$

where we write V_α as a column vector and g as a $n \times n$ matrix.

Proof. We will assume the lemma is true on $\mathbb{C}^n$ with the standard metric. For the case of $\mathbb{C}^n$, above lemma can be proved by using Taylor series expansions. We leave the details to the reader.

First observe that prescribe covariant derivatives at the origin is the same as prescribed partial derivatives in the coordinate chart since the metric is given. We will assume we have a set of prescribed values $V_i(0)$, $\partial V_i(0)$ and $\partial\partial V_i(0)$ that satisfy the orthonormal condition (1.6.1).

Since g is positive definite, we have a positive definite matrix B such that $g = B^* B$. We claim that there exist mutually orthonormal vector fields $U_1, ..., U_m$ around 0 in $\mathbb{C}^n$ with the following prescribed values at 0.

$$U_i(0) = B(0) V_i(0),\ \partial U_i(0) = (\partial B) V_i(0) + B(\partial V_i(0)).$$

$$\partial\partial U_i(0) = (\partial\partial B) V_i(0) + (\partial B) \partial V_i(0) + \partial B(\partial V_i(0)) + B(\partial\partial V_i(0)).$$

Since we assume the lemma is true on $\mathbb{C}^n$, it suffices to check that above prescribed values are consistent with being orthonormal in $\mathbb{C}^n$.

$$U_\beta^* U_\alpha(0) = V_\beta^* B^* B V_\alpha(0) = V_\alpha^* g V_\beta(0) = \delta_{\alpha\beta}.$$

$$\begin{aligned}\partial\left(U_\beta^* U_\alpha\right) &= V_\beta^* B^* \left[(\partial B) V_\alpha(0) + B(\partial V_\alpha(0))\right] \\ &\quad + \left[\partial V_\beta^*(0) B^* + V_\beta^*(0)\partial B^*\right] B V_\alpha \\ &= \partial\left(V_\beta^* B^* B V_\alpha\right) = 0.\end{aligned}$$

Similarly, $\partial\partial\left(U_\beta^* U_\alpha\right) = \partial\partial\left(V_\beta^* g V_\beta\right) = 0$. Therefore, $U_1, ..., U_m$ exists, and $U_\beta^* U_\alpha(z) = \delta_{\alpha\beta}$ in a neighborhood of 0.

We define $\widetilde{V}_\alpha(z) = B^{-1}(z) U_\alpha(z)$. Then $\widetilde{V}_\alpha^* g \widetilde{V}_\beta(z) = U_\beta^* U_\alpha(z) = \delta_{\alpha\beta}$ around 0. Therefore $\widetilde{V}_1, ..., \widetilde{V}_m$ are mutually orthonormal vector fields w.r.t. g. Furthurmore, $\widetilde{V}_\alpha(z)$ has the desired prescribed values at $z = 0$.

$$\begin{aligned}\widetilde{V}_\alpha(0) &= B^{-1} U_\alpha(0) = B^{-1} B V_\alpha(0) = V_\alpha(0). \\ \partial\widetilde{V}_\alpha(0) &= \partial\left(B^{-1} U_\alpha\right)(0) \\ &= \left[\left(\partial B^{-1}\right) B + B^{-1}(\partial B)\right] V_\alpha(0) + B^{-1} B(\partial V_\alpha(0)) \\ &= \partial V_\alpha(0).\end{aligned}$$

$\partial\partial\widetilde{V}_\alpha(0)$ satisfies the prescribed values as well by similar calculation. Therefore, we construct a set of mutually orthonormal vector fields $\widetilde{V}_\alpha$ with desired prescribed values at 0. Lastly, we remark that taking the square root of g and finding the inverse of B are both smooth. □

Lemma 1.6.2. *Fix a coordinate such that $g_{i\bar{j}} = \delta_{ij}$ and $W_{i\bar{j}}$ is diagonal at x_0. Then at x_0, the following holds.*

1. If W is closed, then

$$W_{s\bar{s}k\bar{k}} - W_{k\bar{k}s\bar{s}} = R_{k\bar{k}s\bar{s}}W_{s\bar{s}} - R_{s\bar{s}k\bar{k}}W_{k\bar{k}} + 2\,\mathrm{Re}\left\{\overline{T^j_{sk}}W_{s\bar{j}k}\right\} - \left|T^j_{sk}\right|^2 W_{j\bar{j}}.$$

2. If $W_{i\bar{j}k} = W_{k\bar{j}i}$, then

$$W_{l\bar{l}k\bar{k}} - W_{k\bar{k}l\bar{l}} = R_{l\bar{k}k\bar{l}}W_{l\bar{l}} - R_{l\bar{k}k\bar{l}}W_{k\bar{k}}.$$

proof for 1. First consider the case that W is closed. Then we have

$$W_{i\bar{j}k} - W_{k\bar{j}i} = T^l_{ik}W_{l\bar{j}}.$$

To exchange derivatives, we consider

$$W_{i\bar{i}k\bar{k}} - W_{k\bar{k}i\bar{i}} = (W_{i\bar{i}k\bar{k}} - W_{k\bar{i}i\bar{k}}) + (W_{k\bar{i}i\bar{k}} - W_{k\bar{i}\bar{k}i}) + (W_{k\bar{i}\bar{k}i} - W_{k\bar{k}\bar{i}i}) + (W_{k\bar{k}\bar{i}i} - W_{k\bar{k}i\bar{i}}). \quad (1.6.2)$$

First we will show that

$$W_{i\bar{j}k\bar{l}} - W_{i\bar{j}\bar{l}k} = R_{k\bar{l}i\bar{j}}\left(W_{j\bar{j}} - W_{i\bar{i}}\right). \quad (1.6.3)$$

Since $W_{j\bar{i}\bar{k}l} = \overline{W_{i\bar{j}k\bar{l}}}$, it suffices to compute $W_{i\bar{j}k\bar{l}}$ and take conjugates.

$$\begin{aligned}
W_{i\bar{j}k\bar{l}} =& \ \partial_{\bar{l}}W_{i\bar{j}k} - \overline{\Gamma^q_{lj}}W_{i\bar{q}k} \\
=& \ \partial_{\bar{l}}(\partial_k W_{i\bar{j}} - \Gamma^p_{ki}W_{p\bar{j}}) - \overline{\Gamma^q_{lj}}W_{i\bar{q}k} \\
=& \ \partial_{\bar{l}}\partial_k W_{i\bar{j}} - \partial_{\bar{l}}\Gamma^j_{ki}W_{j\bar{j}} - \Gamma^p_{ki}(W_{p\bar{j}\bar{l}} + \overline{\Gamma^p_{lj}}W_{p\bar{p}}) - \overline{\Gamma^q_{lj}}W_{i\bar{q}k} \\
=& \ \partial_{\bar{l}}\partial_k W_{i\bar{j}} + R_{k\bar{l}i\bar{j}}W_{j\bar{j}} - \Gamma^p_{ki}W_{p\bar{j}\bar{l}} - \overline{\Gamma^q_{lj}}W_{i\bar{q}k} - \Gamma^p_{ki}\overline{\Gamma^p_{lj}}W_{p\bar{p}}. \quad (1.6.4)
\end{aligned}$$

By equation (1.6.4), (1.6.3) is true. Substitute (1.6.3) into (1.6.2), we have

$$
\begin{aligned}
& W_{i\bar{i}k\bar{k}} - W_{k\bar{k}i\bar{i}} \\
&= \nabla_{\bar{k}}(T^l_{ik}W_{l\bar{i}}) + R_{i\bar{k}k\bar{i}}\left(W_{i\bar{i}} - W_{k\bar{k}}\right) + \nabla_i(\overline{T^l_{ik}}W_{k\bar{l}}) \\
&= \nabla_{\bar{k}}T^i_{ik}W_{i\bar{i}} + T^l_{ik}W_{l\bar{i}\bar{k}} + R_{i\bar{k}k\bar{i}}\left(W_{i\bar{i}} - W_{k\bar{k}}\right) + \nabla_i\overline{T^k_{ik}}W_{k\bar{k}} + \overline{T^l_{ik}}W_{k\bar{l}i} \\
&= \partial_{\bar{k}}\left(\Gamma^i_{ik} - \Gamma^i_{ki}\right)W_{i\bar{i}} + T^l_{ik}W_{l\bar{i}\bar{k}} - \partial_{\bar{k}}\Gamma^i_{ik}\left(W_{i\bar{i}} - W_{k\bar{k}}\right) + \partial_i\left(\overline{\Gamma^k_{ik}} - \overline{\Gamma^k_{ki}}\right)W_{k\bar{k}} + \overline{T^l_{ik}}W_{k\bar{l}i}
\end{aligned}
$$

Note that $R_{j\bar{\imath}l\bar{k}} = -g_{l\bar{l}}\partial_{\bar{\imath}}\Gamma^k_{jl}$ and $R_{j\bar{\imath}l\bar{k}} = \overline{R_{i\bar{j}k\bar{l}}} = -g_{l\bar{l}}\partial_j\overline{\Gamma^l_{ik}}$. So we have

$$
\begin{aligned}
& W_{i\bar{i}k\bar{k}} - W_{k\bar{k}i\bar{i}} \\
&= -\partial_{\bar{k}}\Gamma^i_{ki}W_{i\bar{i}} + T^l_{ik}W_{l\bar{i}\bar{k}} + \partial_{\bar{i}}\Gamma^k_{ik}W_{k\bar{k}} + \overline{T^l_{ik}}W_{k\bar{l}i} \\
&= R_{k\bar{k}i\bar{i}}W_{i\bar{i}} - R_{i\bar{i}k\bar{k}}W_{k\bar{k}} + T^l_{ik}W_{l\bar{i}\bar{k}} + \overline{T^l_{ik}}\left(W_{i\bar{l}k} - T^l_{ik}W_{l\bar{l}}\right).
\end{aligned}
$$

□

proof for 2. From equation (1.6.2), we have

$$
W_{l\bar{l}k\bar{k}} - W_{k\bar{k}l\bar{l}} = \left(W_{k\bar{l}l\bar{k}} - W_{k\bar{l}\bar{k}l}\right) + \left(W_{k\bar{k}\bar{l}l} - W_{k\bar{k}l\bar{l}}\right)
$$

First compute $W_{k\bar{l}l\bar{k}} - W_{k\bar{l}\bar{k}l}$.

$$
\begin{aligned}
W_{k\bar{l}l\bar{k}} &= \partial_{\bar{k}}W_{k\bar{l}l} - \overline{\Gamma^p_{kl}}W_{k\bar{p}l} = \partial_{\bar{k}}\left(\partial_l W_{k\bar{l}} - \Gamma^p_{lk}W_{p\bar{l}}\right) - \overline{\Gamma^p_{kl}}W_{k\bar{p}l}. \\
W_{k\bar{l}\bar{k}l} &= \partial_l W_{k\bar{l}\bar{k}} - \Gamma^p_{lk}W_{p\bar{l}\bar{k}} = \partial_l\left(\partial_{\bar{k}}W_{k\bar{l}} - \overline{\Gamma^p_{kl}}W_{k\bar{p}}\right) - \Gamma^p_{lk}W_{p\bar{l}\bar{k}}
\end{aligned}
$$

Therefore, we obtain

$$\begin{aligned}
&W_{k\bar{l}l\bar{k}} - W_{k\bar{l}\bar{k}l} \\
&= -\partial_{\bar{k}}\Gamma^{l}_{lk}W_{l\bar{l}} - \Gamma^{p}_{lk}\partial_{\bar{k}}W_{p\bar{l}} - \overline{\Gamma^{p}_{kl}}W_{k\bar{p}l} + \partial_{l}\overline{\Gamma^{k}_{kl}}W_{k\bar{k}} + \overline{\Gamma^{p}_{kl}}\partial_{l}W_{k\bar{p}} + \Gamma^{p}_{lk}W_{p\bar{l}\bar{k}}. \\
&= -\partial_{\bar{k}}\Gamma^{l}_{lk}W_{l\bar{l}} - \Gamma^{p}_{lk}\overline{\Gamma^{p}_{kl}}W_{p\bar{p}} + \overline{\Gamma^{p}_{kl}}\Gamma^{p}_{lk}W_{p\bar{p}} + \partial_{l}\overline{\Gamma^{k}_{kl}}W_{k\bar{k}} \\
&= -\partial_{\bar{k}}\Gamma^{l}_{lk}W_{l\bar{l}} + \partial_{l}\overline{\Gamma^{k}_{kl}}W_{k\bar{k}}.
\end{aligned}$$

Next compute $W_{k\bar{k}\bar{l}l} - W_{k\bar{k}l\bar{l}}$.

$$\begin{aligned}
W_{k\bar{k}\bar{l}l} &= \partial_{l}W_{k\bar{k}\bar{l}} - \Gamma^{p}_{lk}W_{p\bar{k}\bar{l}} = \partial_{l}\left(\partial_{\bar{l}}W_{k\bar{k}} - \overline{\Gamma^{p}_{lk}}W_{k\bar{p}}\right) - \Gamma^{p}_{lk}W_{p\bar{k}\bar{l}}. \\
W_{k\bar{k}l\bar{l}} &= \partial_{\bar{l}}W_{k\bar{k}l} - \overline{\Gamma^{p}_{lk}}W_{k\bar{p}l} = \partial_{\bar{l}}\left(\partial_{l}W_{k\bar{k}} - \Gamma^{p}_{lk}W_{p\bar{k}}\right) - \overline{\Gamma^{p}_{lk}}W_{k\bar{p}l}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&W_{k\bar{k}\bar{l}l} - W_{k\bar{k}l\bar{l}} \\
&= -\partial_{l}\overline{\Gamma^{k}_{lk}}W_{k\bar{k}} - \overline{\Gamma^{p}_{lk}}\partial_{l}W_{k\bar{p}} - \Gamma^{p}_{lk}W_{p\bar{k}\bar{l}} + \partial_{\bar{l}}\Gamma^{k}_{lk}W_{k\bar{k}} + \Gamma^{p}_{lk}\partial_{\bar{l}}W_{p\bar{k}} + \overline{\Gamma^{p}_{lk}}W_{k\bar{p}l} \\
&= -\partial_{l}\overline{\Gamma^{k}_{lk}}W_{k\bar{k}} + \partial_{\bar{l}}\Gamma^{k}_{lk}W_{k\bar{k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&W_{l\bar{l}k\bar{k}} - W_{k\bar{k}l\bar{l}} \\
&= -\partial_{\bar{k}}\Gamma^{l}_{lk}W_{l\bar{l}} + \partial_{l}\overline{\Gamma^{k}_{kl}}W_{k\bar{k}} - \partial_{l}\overline{\Gamma^{k}_{lk}}W_{k\bar{k}} + \partial_{\bar{l}}\Gamma^{k}_{lk}W_{k\bar{k}} \\
&= R_{l\bar{k}k\bar{l}}W_{l\bar{l}} - R_{l\bar{k}k\bar{l}}W_{k\bar{k}} + R_{l\bar{l}k\bar{k}}W_{k\bar{k}} - R_{l\bar{l}k\bar{k}}W_{k\bar{k}} \\
&= R_{l\bar{k}k\bar{l}}W_{l\bar{l}} - R_{l\bar{k}k\bar{l}}W_{k\bar{k}}.
\end{aligned}$$

□

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Chapter 2

C^2 estimates on complex Hessian equations

2.1 Introduction

Let (M, ω) be a compact Kähler manifold of complex dimension n. In any local coordinate chart, we write $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. In this chapter, we study the C^2 estimates of the following form of elliptic equations, for $n \geq k > 1$

$$\sigma_k\left(g^{i\bar{j}}\left(g_{k\bar{j}} + u_{k\bar{j}}\right)\right) = \psi\left(u, Du, z\right). \tag{2.1.1}$$

where and $\sigma_k\left(\chi\right) = \binom{n}{k}\frac{\chi^k \wedge \omega^{n-k}}{\omega^n}$ is the k-th elementary symmetric function for the eigenvalues of χ with respect to ω.

When ψ in equation 2.1.1 is independent of Du, it has been studied extensively. The most well-known complex Hessian equations should be the complex Monge-Ampère equations $\sigma_n\left(g\right) = a_0$ solved by Yau [19] on compact Kähler manifolds for the Calabi conjecture.

Later, it is also solved on compact Hermitian manifolds by Tosatti and Weinkove [16]. For general complex Hessian equation $\sigma_k = a_0$, it is studied by Dinew and Kolodziej [4] and Hou-Ma-Wu [9] on compact Kähler manifolds.

When ψ in equation 2.1.1 depends on Du, the C^2 estimates are much harder and less stuided. For the real Hessian equation, Guan-Ren-Wang [8] solves it completely when $k = 2$. For general Hessian equations, they solve it when the admissible solution is in the Γ_{k+1} cone, and Lu [10] extends their results to the semi-convex setting. When $k = n-1$ and $n-2$, Ren-Wang [13, 14] solve it completely by extremely complicated computation, and Spruck-Xiao [15] provide a simple proof for k = 2.

As for the complex Hessian equation, an important example with ψ depending on Du is studied by Fu-Yau [5, 6]. They study a Monge-Ampère type equation in two dimensions related to a Strominger system. Later Phong-Picard-Zhang [11] study the Fu-Yau (σ_2) equation in higher dimensions. For complex Hessian equations, Phong-Picard-Zhang [12] solve it in the Γ_{k+1} cone, and Chu-Huang-Zhu [3] solve the complex σ_2 equation by really involved calculation, which is hard to verify.

Motivated by Lu's [10] semi-convex assumption, we want to solve the complex Hessian equations in the semi-convex setting first. Here is our main result.

Theorem 2.1.1. *Let (M, ω) be a compact Kähler manifold. $k = 2$ or $n - 1$. Suppose u is a solution to the following equation*

$$\sigma_k \left(g^{i\bar{k}} \left(g_{j\bar{k}} + u_{j\bar{k}} \right) \right) = f \left(z, u, Du \right)$$

with eigenvalues in the Γ_k cone and bounded below by a constant K, then we have second order derivative estimates.

$$\left| D\overline{D}u \right|_\omega \leq C,$$

where C is a constant depending on n, k, K, ω, M, $\|u\|_{C^1}$, and $\|f\|_{C^2}$.

2.2 Preliminary

Let $\sigma_k(\boldsymbol{\lambda})$ denote the k-th elementary symmetric polynomial of $\boldsymbol{\lambda} \in \mathbb{R}^n$,

$$\sigma_k(\boldsymbol{\lambda}) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} . \tag{2.2.1}$$

Also we denote

$$\sigma_{k-1;i} = \sigma_{k-1}|_{\lambda_i = 0.}$$

$$\sigma_{k-2;pq} = \sigma_{k-2}|_{\lambda_p = \lambda_q = 0.}$$

The $k-1$-positive cone in $\mathbb{R}^n$ is defined as

$$\Gamma_{k-1} = \{\lambda \in \mathbb{R}^n | \sigma_1 > 0,\ \sigma_2 > 0, ...,\ \sigma_{k-1} > 0\}$$

In particular, for $\boldsymbol{\lambda} \in \Gamma_k$, suppose $\lambda_1 \ge \lambda_2 \ge ... \ge \lambda_n$, we have

$$\lambda_1 \sigma_{k-1;1} \ge C(n,k)\, \sigma_k, \text{ and}$$

$$\sigma_{k-1;n} \ge ... \ge \sigma_{k-1;1} > 0.$$

For more properties for σ_k, we refer the reader to Wang [18]. $A\left(u_{i\bar{j}}\right)$ denotes the Hermitian matrix with entries $A^i_j = g^{i\bar{k}}\left(g_{j\bar{k}} + u_{j\bar{k}}\right)$, and $\lambda\left(u_{i\bar{j}}\right)$ are the eigenvalues of A. We write

$$F\left(\lambda\left(D^2 u\right)\right) = f(\lambda) = \log \sigma_k(\lambda) = \psi(u, Du, x).$$

If $F(A) = f(\lambda_1, \ldots, \lambda_n)$ is a symmetric function of the eigenvalues, then at a diagonal matrix A, we have the following well-known results (e.g. see Gerhardt [7]).

$$\begin{aligned} F^{ij} &= \delta_{ij} f_i, \text{ and} \\ F^{ij,rs} &= f_{ir}\delta_{ij}\delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j}(1-\delta_{ij})\delta_{is}\delta_{jr}. \end{aligned} \tag{2.2.2}$$

Next, we state the concavity lemma due to Guan-Ren-Wang [8]. Note that the format of the following version is similar to the version of Chu [2]. We can obtain the following inequality by modifying [2, (3.26)].

Lemma 2.2.1. *Suppose $\boldsymbol{\lambda} \in \Gamma_k$ and $\sigma_{k;1} > -\varepsilon\lambda_1\sigma_{k-1;1}$. For ε, $\delta \in \left(0, \frac{1}{3}\right)$ and $1 \le l \le k-1$, there exists a constant δ' depending on ε, δ, n, k, $\inf f$ and $\|f\|_{C^1}$ such that if $\lambda_l \ge \delta\lambda_1$ and $\lambda_{l+1} \le \delta'\lambda_1$, then*

$$(1-3\varepsilon)\frac{\sigma_{k-1;1}}{\lambda_1}\left|u_{1\bar{1}1}\right|^2 \le -\sigma_{k-2;pq}u_{p\bar{p}1}u_{q\bar{q}\bar{1}} + \sum_{p>l}\frac{\sigma_{k-1;p}\left|u_{p\bar{p}1}\right|^2}{\lambda_1} + \frac{\left|\sum_p \sigma_{k-1;p}u_{p\bar{p}1}\right|^2}{\sigma_k}.$$

2.3 Key Lemma for k=2

2.3.1 Inequality for σ_2 equations.

Let $\lambda_1 = ... = \lambda_m > \lambda_{m+1} \ge ... \ge \lambda_n$.

Lemma 2.3.1. *Suppose that $k = 2$ and $\xi_p = 0$ for $1 < p \le m$, then there exists δ depending on ε and n such that when $\lambda_2 < \delta\lambda_1$, the following inequality holds.*

$$\sigma_{k-1;p}\sigma_{k-1;q}\xi_p\overline{\xi_q} - \sigma_k\sigma_{k-2;pq}\xi_p\overline{\xi_q} + \sigma_k\sum_{p>m}\frac{\sigma_{k-1;p}}{\lambda_1-\lambda_p}\left|\xi_p\right|^2 - (1-\varepsilon)\,\sigma_k\frac{\sigma_{k-1;1}}{\lambda_1}\left|\xi_1\right|^2 \ge 0. \tag{2.3.1}$$

Note that

$$\frac{\sigma_{1;p}}{\lambda_1 - \lambda_p} \geq 1 \text{ and } \sigma_{k-2;pq} = 1 \text{ when } p \neq q.$$

We define three matrices $A = (a_{pq})$, $B = (b_{pq})$, and $C = (c_{pq})$ as follows.

$$a_{pq} = \sigma_{k-1;p}\sigma_{k-1;q}.$$

$$b_{pp} = \sigma_k \text{ for all } p \text{ and } b_{pq} = -\sigma_k \text{ if } p \neq q.$$

$$c_{11} = \ -(1-\varepsilon)\,\sigma_k \frac{\sigma_{k-1;1}}{\lambda_1} - \sigma_k \text{ and } c_{pq} = 0 \text{ for all } \ (p,q) \neq (1,1)\,.$$

To prove inequality (2.3.1), it suffices to prove $A+B+C$ is positive definite. Observe that $A+B$ is a positive definite matrix and $-C$ is a rank 1 matrix. Therefore, there is at most one negative eigenvalue. Therefore, it suffices to show that $\det(A+B+C) > 0$. We have the following lemma.

Lemma 2.3.2. *Denote* $a = \frac{1}{1-\varepsilon}$.

$$\det(A+B+C) = \frac{2^{n-3}\sigma_2^{n-1}\sigma_{1;1}\left[(2n-2)(a-1)\lambda_1^2 + (2-2n+an)\lambda_1\sigma_{1;1} + (2-n)(\sigma_{1;1})^2\right]}{a\lambda_1}.$$

The proof of Lemma 2.3.2 is quite lengthy and tedious. We outline some key steps below and leave the details to the readers.

Step 1 We first compute the determinant of the matrix $D = (d_{pq})$.

$$d_{p1} = a_1 a_p \text{ for all } p, \qquad d_{pp} = x \text{ for all } p > 1$$

$$d_{1p} = -x \text{ for all } p > 1. \quad d_{pq} = -x \text{ if } p \neq q,\ p > 1 \text{ and } q > 1.$$

By row operations, we obtain

$$\det(D) = 2^{n-2} a_1 x^{n-1} \left[(3-n) a_1 + a_2 + ... + a_n \right]. \tag{2.3.2}$$

In particular, equation (2.3.2) implies

$$\det(B) = -(n-2) 2^{n-1} x^n. \tag{2.3.3}$$

Step 2 Compute the $\det(A+B)$. We set $a_p = \sigma_{k-1;p}$ and $x = \sigma_k$. Apply row operations, equation (2.3.2) and (2.3.3), we have

$$\det(A+B) = 2^{n-2} x^{n-1} \left[\sum_{k=1}^{n} (3-n) a_k^2 + 2 \sum_{i<j} a_i a_j \right] - (n-2) 2^{n-1} x^n. \tag{2.3.4}$$

Next, by computing every term carefully, we obtain

$$\sum_{k=1}^{n} (3-n) a_k^2 = (3-n)(n-1)^2 \sigma_1^2 - 2(3-n) \sum_{i<j} a_i a_j. \tag{2.3.5}$$

$$\sum_{i<j} a_i a_j = (n-2)(n-1) \sum_{k=1}^{n} \lambda_k^2 + (n-2)(n-3) \sigma_2 + n(n-1) \sigma_2. \tag{2.3.6}$$

Combining equation (2.3.4), (2.3.5) and (2.3.6), we obtain

$$\det(A+B) = 2^{n-2} \sigma_2^{n-1} (n-1) \sigma_1^2. \tag{2.3.7}$$

Step 3 Let M be the $(n-1) \times (n-1)$ submatrix obtained by deleting the first row and first column of $A+B$. Applying equation (2.3.4) once again, we compute each term separately. It follows that

$$\sum_{k=2}^{n} (4-n)\, a_k^2 = (4-n)\left[(n-2)\,\sigma_1 + \lambda_1\right]^2 + (2n-8) \sum_{\substack{i<j \\ i,j>1}} a_i a_j. \tag{2.3.8}$$

$$\sum_{\substack{i<j \\ i,j>1}} a_i a_j - x = \lambda_1 \sigma_{1;1} \left(n^2 - 4n + 3\right) + \binom{n-1}{2} \lambda_1^2 + \binom{n-2}{2} \left(\sum_{k>1} \lambda_k\right)^2. \tag{2.3.9}$$

Denote $\lambda_1 = u$ and $\sigma_{1;1} = v$. Combining equation (2.3.4), (2.3.8) and (2.3.9), we obtain

$$\det(M) = 2^{n-3} \sigma_2^{n-2} \left\{ \begin{array}{c} (4-n)\left[(n-1)\,u + (n-2)\,v\right]^2 + (n-1)(n-2)(n-3)\,u^2 \\ +2uv\,(n-3)^2 (n-1) + v^2 (n-3)^2 (n-2) \end{array} \right\}. \tag{2.3.10}$$

Step 4 By properties of determinant, it follows that

$$\det(A+B+C) = \det(A+B) - \sigma_k \left(1 + (1-\varepsilon)\frac{\sigma_{1;1}}{\lambda_1}\right) \det(M).$$

By equation (2.3.7) and (2.3.10), we finish the proof for Lemma 2.3.2. Lastly, observe that $\det(A+B+C) > 0$ when $\sigma_{1;1} < \delta_0 \lambda_1$ for some small δ_0 depending on ε. In particular, when $\lambda_2 < \frac{\delta_0}{n}\lambda_1$, $\det(A+B+C) > 0$. We finish the proof for Lemma 2.3.1.

2.4 Key Lemma for $k = n-1$

Lemma 2.4.1. *Suppose that λ_p and λ_q are both nonzero,*

$$\sigma_{k-1;p}\sigma_{k-1;q} - \sigma_k \sigma_{k-2;pq} = \frac{1}{\lambda_p \lambda_q}\left(\sigma_{k;p}\sigma_{k;q} - \sigma_{k;pq}\right) \ \textit{when } p \neq q.$$

Proof.

$$\begin{aligned}\sigma_k &= \lambda_p \sigma_{k-1;p} + \sigma_{k;p} \\ &= \lambda_p \lambda_q \sigma_{k-2;pq} + \lambda_p \sigma_{k-1;pq} + \sigma_{k;p} \\ &= \lambda_p \lambda_q \sigma_{k-2;pq} + \sigma_{k;q} - \sigma_{k;pq} + \sigma_{k;p}.\end{aligned}$$

Therefore,

$$\sigma_{k-1;p} = \frac{1}{\lambda_p}\left(\sigma_k - \sigma_{k;p}\right), \text{ and} \tag{2.4.1}$$

$$\sigma_{k-2;pq} = \frac{1}{\lambda_p \lambda_q}\left(\sigma_k - \sigma_{k;p} - \sigma_{k;q} + \sigma_{k;pq}\right). \tag{2.4.2}$$

Combining equation (2.4.1) and (2.4.2) gives the desired result. □

Lemma 2.4.2. *Let $\lambda \in \Gamma_{n-1}$, $n \geq 4$. Suppose the multiplicity of λ_1 is m, and $\xi_p = 0$ for $1 < p \leq m$. There exists δ depending on ε_0 such that when $\lambda_{n-1} \leq \delta \lambda_1$ and $-\sigma_{n-1;1} > \varepsilon_0 \lambda_1 \sigma_{n-2;1}$, we have*

$$\sigma_{n-2;p}\sigma_{n-2;q}\xi_p\overline{\xi_q} - \sigma_{n-1}\sigma_{n-3;pq}\xi_p\overline{\xi_q} + \sum_{p>m} \frac{\sigma_{n-1}\sigma_{n-2;p}}{\lambda_1 - \lambda_p}\left|\xi_p\right|^2 - (1-\varepsilon)\frac{\sigma_{n-1}\sigma_{n-2;1}}{\lambda_1}\left|\xi_1\right|^2 \geq 0. \tag{2.4.3}$$

Proof. Denote the LHS of equation (2.4.3) as $\frac{1}{\lambda_p \lambda_q} a_{pq}\xi_p\overline{\xi_q}$. When $\lambda_k = \lambda_1$, $\xi_k = 0$. By deleting the rows and columns where $\xi_k = 0$, we obtain an $(n-m+1) \times (n-m+1)$ submatrix of (a_{pq}). We want to show this submatrix of (a_{pq}) is positive definite. By Lemma 2.4.1, we can rewrite (a_{pq}) as the sum of a rank 1 matrix $s^T s$ and a diagonal matrix $D = diag(d_1, d_2, ..., d_{n-m+1})$, where

$$\begin{aligned} s &= \left[\sigma_{n-1;1}, \sigma_{n-1;2}, ..., \sigma_{n-1;n}\right], \\ d_1 &= \lambda_1^2\left(\sigma_{n-2;1}\right)^2 - \left(\sigma_{n-1;1}\right)^2 - (1-\varepsilon)\lambda_1\sigma_{n-2;1}\sigma_{n-1}, \text{ and} \\ d_p &= \lambda_p^2\left(\sigma_{n-2;p}\right)^2 - \left(\sigma_{n-1;p}\right)^2 + \frac{\lambda_p^2}{\lambda_1 - \lambda_p}\sigma_{n-1}\sigma_{n-2;p} \text{ for all } p > 1.\end{aligned}$$

Suppose $-\sigma_n = a\lambda_1\sigma_{n-1}$, we have

$$\sigma_{n-1;p} = -a\frac{\lambda_1}{\lambda_p}\sigma_{n-1} \text{ and } \lambda_p\sigma_{n-2;p} = \sigma_{n-1}\left(1 + a\frac{\lambda_1}{\lambda_p}\right).$$

Note that a is bounded below by ε_0 since

$$-\sigma_n = -\lambda_1\sigma_{n-1;1} \geq \lambda_1\left(\varepsilon_0\lambda_1\sigma_{n-2;1}\right) > \varepsilon_0\lambda_1\sigma_{n-1}.$$

Therefore,

$$d_1 = \sigma_{n-1}\left(-\sigma_{n-1;1} + \varepsilon\lambda_1\sigma_{n-2;1}\right) = \sigma_{n-1}^2\left(a + \varepsilon\left(1 + a\right)\right).$$
$$d_p = \sigma_{n-1}\left(-\sigma_{n-1;p} + \frac{\lambda_1\lambda_p}{\lambda_1 - \lambda_p}\sigma_{n-2;p}\right) = \sigma_{n-1}^2\left(a\frac{\lambda_1}{\lambda_p} + \frac{\lambda_1}{\lambda_1 - \lambda_p}\left(1 + a\frac{\lambda_1}{\lambda_p}\right)\right).$$

Since $\sigma_{n-1;p} < 0$ for all $p \neq n$ and $\sigma_{n-1;n} > 0$, it follows that the only negative entry in the diagonal matrix D is d_n. Note that

$$\begin{aligned}\det\left(D + s^Ts\right) &= \det\left(D\right)\det\left(I_{n-m+1} + D^{-1}s^Ts\right) = \det\left(D\right)\left(1 + sD^{-1}s^T\right)\\ &= \det\left(D\right)\left(1 + \sum_p \frac{s_p^2}{d_p}\right).\end{aligned} \tag{2.4.4}$$

Therefore,

$$\begin{aligned}1 + \sum_p \frac{s_p^2}{d_p} &= 1 + a\left[\frac{a}{a + \varepsilon\left(1 + a\right)} + \sum_{p>m}\frac{2a\lambda_1\left(\lambda_1 - \lambda_p\right)}{2\lambda_p\left(2a\lambda_1 + \left(1 - a\right)\lambda_p\right)}\right]\\ &\leq 1 + a\left[\frac{1}{1 + \varepsilon} + \sum_{p=1}^n\left(\frac{\lambda_1\left(2a\lambda_1 + \left(1 - a\right)\lambda_p\right)}{2\lambda_p\left(2a\lambda_1 + \left(1 - a\right)\lambda_p\right)} - \frac{\left(1 + a\right)\lambda_1}{2\left(2a\lambda_1 + \left(1 - a\right)\lambda_p\right)}\right)\right]\\ &= 1 + a\left[\frac{1}{1 + \varepsilon} + \left(\frac{\lambda_1\sigma_{n-1}}{2\sigma_n} - \sum_p\frac{\left(1 + a\right)\lambda_1}{2\left(2a\lambda_1 + \left(1 - a\right)\lambda_p\right)}\right)\right]\\ &= \frac{1}{2} + a\left[\frac{1}{1 + \varepsilon} - \sum_p\frac{\left(1 + a\right)\lambda_1}{2\left(2a\lambda_1 + \left(1 - a\right)\lambda_p\right)}\right].\end{aligned}$$

Since $\lambda_{n-1} < \delta\lambda_1$, it follows that $0 > \lambda_n > -\delta\lambda_1$. Picking $\delta < \varepsilon_0$, we have

$$-\frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_n)} \le -\,(1-\varepsilon_1)\,\frac{(1+a)}{4a} \text{ and} \tag{2.4.5}$$

$$-\frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_{n-1})} \le -\,(1-\varepsilon_1)\,\frac{(1+a)}{4a}, \tag{2.4.6}$$

where $\varepsilon_1 = C\,(\varepsilon_0)\,\delta$ since a is bounded below by ε_0. Also

$$-\frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_1)} = -\frac{1}{2}. \tag{2.4.7}$$

(Case 1) When $a < 1$.

$$-\frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_2)} \le -\frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_1)} = -\frac{1}{2}. \tag{2.4.8}$$

By equation (2.4.5), (2.4.6), (2.4.7) and (2.4.8), we have

$$\begin{aligned}
&1 + \sum_p \frac{s_p^2}{d_p} \\
&\le \frac{1}{2} + a\left[\frac{1}{1+\varepsilon} - \sum_p \frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_p)}\right] \\
&\le \frac{1}{2} + a\left[\frac{1}{1+\varepsilon} - 1 - \frac{1}{2a} - \frac{1}{2} + \varepsilon_1 \frac{(1+a)}{2a}\right] \\
&\le a\left[\frac{1}{1+\varepsilon} - 1 - \frac{1}{2} + \frac{\varepsilon_1}{\varepsilon_0}\right] < 0 \text{ when } \varepsilon_1 < \frac{\varepsilon_0}{2}.
\end{aligned}$$

(Case 2) When $a \ge 1$.

$$-\frac{(1+a)\,\lambda_1}{2\,(2a\lambda_1 + (1-a)\,\lambda_2)} \le -\frac{(1+a)}{4a}. \tag{2.4.9}$$

By equation (2.4.5), (2.4.6), (2.4.7) and (2.4.9), we have

$$
\begin{aligned}
& 1+\sum_p \frac{s_p^2}{d_p} \\
& \leq \frac{1}{2}+a\left[\frac{1}{1+\varepsilon}-\sum_p \frac{(1+a)\,\lambda_1}{2\left(2a\lambda_1+(1-a)\,\lambda_p\right)}\right] \\
& \leq \frac{1}{2}+a\left[\frac{1}{1+\varepsilon}-\frac{1}{2}-\frac{(1+a)}{4a}-\frac{1}{2a}-\frac{1}{2}+\varepsilon_1\frac{(1+a)}{2a}\right] \\
& \leq a\left[\frac{1}{1+\varepsilon}-\frac{1}{2}-\frac{1}{2}-\left(1-2\varepsilon_1\right)\frac{(1+a)}{4a}\right] \\
& \leq a\left[\frac{1}{1+\varepsilon}-\frac{1}{2}-\frac{1}{2}\right]<0 \text{ when } \varepsilon_1<\frac{1}{2}.
\end{aligned}
$$

By picking δ small enough, ε_1 becomes small. Equation (2.4.4) implies $\det\left(D+s^Ts\right)>0$. Since D only has only one negative eigenvalue and s^Ts is nonnegative, $D+s^Ts$ is positive definite. The proof is complete.

Remark 2.4.3. Alternatively, we can use Weyl's inequality, $0<\lambda_{n-m}(D)+\lambda_{n-m+1}\left(s^Ts\right)<\lambda_{n-m}\left(D+s^Ts\right)$, which implies $D+s^Ts$ has at most 1 negative eigenvalue.

□

2.5 Proof of the theorem

We will compute the second derivative for the following test function

$$Q=\log\left(\lambda_1+K\right)+h\left(|\nabla u|^2\right)+g(u),$$

where $-K$ is the lower bound of λ_n. At the maximum point, we pick a normal coordinate and diagonalize $\left(u_{i\bar{j}}\right)$ such that $u_{1\bar{1}}\geq u_{2\bar{2}}\geq\ldots\geq u_{n\bar{n}}$. In particular, $\lambda_1=1+u_{1\bar{1}}$. By direct computation, we obtain

$$
\begin{aligned}
& F^{m\bar{m}} Q_{m\bar{m}} \\
& \geq F^{m\bar{m}} \frac{(\lambda_1)_{m\bar{m}}}{\lambda_1 + K} - F^{m\bar{m}} \frac{|\nabla_m \lambda_1|^2}{(\lambda_1 + K)^2} + h'' F^{m\bar{m}} \left|\nabla_m |\nabla u|^2\right|^2 + h' F^{m\bar{m}} |\nabla u|^2_{m\bar{m}} \\
& + g' F^{m\bar{m}} \lambda_m - g' \sum F^{m\bar{m}} + g'' F^{m\bar{m}} |u_m|^2 .
\end{aligned}
$$

Note that above inequality is in viscosity sense (see Brendle-Choi-Daskalopoulos [1, Lemma 5]). By taking the second derivatives of the equation. We have

$$
F^{m\bar{m}} u_{m\bar{m}1\bar{1}} + F^{i\bar{j}.k\bar{l}} u_{i\bar{j}1} u_{k\bar{l}\bar{1}} = \psi_{1\bar{1}}.
$$

By commuting the covariant derivatives, we have

$$
\begin{aligned}
u_{i\bar{j}l} &= u_{il\bar{j}} - u_a R_{i\ l\bar{j}}^{a}, \\
u_{i\bar{j}l\bar{m}} &= u_{l\bar{m}i\bar{j}} + u_{a\bar{j}} R_{i\ l\bar{m}}^{a} - u_{b\bar{m}} R_{i\ l\bar{j}}^{b}.
\end{aligned}
$$

By applying Tosatti-Weinkove's formula [17, Lemma 3.2] on the second derivative of eigenvalues, we have the following inequalities.

$$
F^{m\bar{m}} (\lambda_1)_{m\bar{m}} \geq \psi_{1\bar{1}} - F^{i\bar{j}.k\bar{l}} u_{i\bar{j}1} u_{k\bar{l}\bar{1}} + F^{m\bar{m}} (u_{1\bar{1}} - u_{m\bar{m}}) R_{m\bar{m}i\bar{i}} + F^{i\bar{i}} \sum_{p>m} \frac{\left|u_{p\bar{1}i}\right|^2 + \left|u_{p\bar{1}\bar{i}}\right|^2}{\lambda_1 - \lambda_p}.
$$

Also, by direct computation, we have

$$
\begin{aligned}
\psi_{1\bar{1}} &\geq -C - C \sum_k \left(|u_{k1}|^2 + |u_{k\bar{1}}|^2\right) + \sum_l \left(\psi_{u_l} u_{1\bar{1}l} + \psi_{u_{\bar{l}}} u_{1\bar{1}\bar{l}} + \psi_{u_l} u_a R_{l1\bar{1}}^{a}\right). \\
F^{m\bar{m}} |\nabla u|^2_{m\bar{m}} &\geq \sum_l \left(\psi_{u_l} \nabla_l |\nabla u|^2 + \psi_{u_{\bar{l}}} \nabla_{\bar{l}} |\nabla u|^2\right) - C \left(1 + \sum F^{m\bar{m}}\right) \\
&\quad + \sum_a F^{m\bar{m}} \left(|u_{am}|^2 + |u_{\bar{a}m}|^2\right).
\end{aligned}
$$

Lastly, when $k=2$, we may always assume Lemma 2.3.1 holds. Otherwise, we have $\lambda_2 \geq \delta\lambda_1$. When $k=n-1$, we may always assume either Lemma 2.2.1 or 2.4.2 holds. Otherwise, we have $\lambda_{n-1} \geq \delta\lambda_1$.

$$
\begin{aligned}
\sigma_k &\geq \lambda_1 ... \lambda_k - C(n)\lambda_1 ... \lambda_{k-1} K \\
&\geq \lambda_1 ... \lambda_{k-1}(\lambda_k - C(n) K) \\
&\geq \lambda_1 ... \lambda_{k-1}(\delta\lambda_1 - C(n) K) \\
&\geq C(\delta, n)\lambda_1^k,
\end{aligned}
$$

which implies λ_1 is bounded above.

2.5.1 Third order terms

We first consider third order terms in $F^{m\bar{m}}(\lambda_1)_{m\bar{m}}$. We only care about the terms that has $u_{1\bar{1}p}$ or $u_{p\bar{p}1}$ for some p. By equation (2.2.2), third order terms in $F^{m\bar{m}}(\lambda_1)_{m\bar{m}}$ are

$$
\begin{aligned}
&- F^{i\bar{j}.k\bar{l}} u_{i\bar{j}1} u_{k\bar{l}\bar{1}} + F^{i\bar{i}} \sum_{p>m} \frac{\left|u_{p\bar{1}i}\right|^2 + \left|u_{p\bar{1}\bar{i}}\right|^2}{\lambda_1 - \lambda_p} \\
&= -f_{i\bar{p}} u_{i\bar{i}1} u_{p\bar{p}\bar{1}} - \sum_{p\neq q} \frac{f_p - f_q}{\lambda_p - \lambda_q} \left|u_{p\bar{q}1}\right|^2 + f_i \sum_{p>m} \frac{\left|u_{p\bar{1}i}\right|^2 + \left|u_{p\bar{1}\bar{i}}\right|^2}{\lambda_1 - \lambda_p} \\
&\geq -f_{i\bar{p}} u_{i\bar{i}1} u_{p\bar{p}\bar{1}} + \frac{f_p - f_1}{\lambda_1 - \lambda_p} \left|u_{p\bar{1}1}\right|^2 + f_1 \sum_{p>m} \frac{\left|u_{p\bar{1}1}\right|^2}{\lambda_1 - \lambda_p} + \sum_{p>m} f_p \frac{\left|u_{p\bar{1}\bar{p}}\right|^2}{\lambda_1 - \lambda_p} \\
&\geq -f_{i\bar{p}} u_{i\bar{i}1} u_{p\bar{p}\bar{1}} + f_p \sum_{p>m} \frac{\left|u_{p\bar{1}1}\right|^2}{\lambda_1 - \lambda_p} + \sum_{p>m} f_p \frac{\left|u_{p\bar{p}1}\right|^2}{\lambda_1 - \lambda_p} = I.
\end{aligned}
\tag{2.5.1}
$$

The terms in $-F^{m\bar{m}} \frac{\left|\nabla_m \lambda_1\right|^2}{\lambda_1 + K}$ are

$$
-\frac{1}{\lambda_1 + K} F^{m\bar{m}} \left|\nabla_m \lambda_1\right|^2 = -\frac{1}{\lambda_1 + K} f_1 \left|u_{1\bar{1}1}\right|^2 - \frac{1}{\lambda_1 + K} \sum_{p>m} f_p \left|u_{1\bar{1}p}\right|^2 = II. \tag{2.5.2}
$$

where we use the fact that if $\lambda_1 = \lambda_p$, then $u_{p\bar{1}k} = 0$ for $p \neq 1$ [17, Lemma 3.2] and $u_{p\bar{1}k} = u_{k\bar{1}p}$. Combining the third order terms in (2.5.1) and (2.5.2), we obtain

$$\begin{aligned}
& I + II \\
& \geq -f_{i\bar{p}} u_{i\bar{i}1} u_{p\bar{p}\bar{1}} + \sum_{p>m} f_p \frac{|u_{p\bar{p}1}|^2}{\lambda_1 - \lambda_p} - \frac{1}{\lambda_1 + K} f_1 |u_{1\bar{1}1}|^2 \\
& = -f_{i\bar{p}} u_{i\bar{i}1} u_{p\bar{p}\bar{1}} + \sum_{p>m} f_p \frac{|u_{p\bar{p}1}|^2}{\lambda_1 - \lambda_p} - \frac{1}{\lambda_1 + K} f_1 |u_{1\bar{1}1}|^2 \\
& = \frac{|\sigma_{k-1;p} u_{p\bar{p}1}|^2}{\sigma_k^2} - \frac{\sigma_{k-2;pq} u_{p\bar{p}1} u_{q\bar{q}\bar{1}}}{\sigma_k} + \sum_{p>m} \frac{\sigma_{k-1;p}}{\sigma_k} \frac{|u_{p\bar{p}1}|^2}{\lambda_1 - \lambda_p} - \frac{1}{\lambda_1 + K} \frac{\sigma_{k-1;1}}{\sigma_k} |u_{1\bar{1}1}|^2 \\
& \geq \varepsilon \frac{\sigma_{k-1;1}}{(\lambda_1 + K)\,\sigma_k} |u_{1\bar{1}1}|^2
\end{aligned}$$

by Lemma 2.3.1, 2.2.1 or 2.4.2.

2.5.2 C^2 estimates

Note that by critical equations, we have

$$\frac{1}{\lambda_1 + K} \psi_{u_l} u_{1\bar{1}l} + h' \psi_{u_l} \nabla_l |\nabla u|^2 = -\psi_{u_l} g' u_l \geq C g'.$$

Also

$$\frac{\varepsilon}{(\lambda_1 + K)^2} |u_{1\bar{1}1}|^2 \leq 2\varepsilon (h')^2 \left| \nabla_1 |\nabla u|^2 \right|^2 + 2\varepsilon (g')^2 |u_1|^2 .$$

$$
\begin{aligned}
&F^{m\bar{m}}\left(\log\left(\lambda_1+K\right)+h\left(|\nabla u|^2\right)+g\left(u\right)\right)_{m\bar{m}}\\
&\geq F^{m\bar{m}}\frac{\left(\lambda_1\right)_{m\bar{m}}}{\lambda_1+K}-F^{m\bar{m}}\frac{|\nabla_m\lambda_1|^2}{\left(\lambda_1+K\right)^2}-C\left(1+h'\right)\left(1+\sum F^{m\bar{m}}\right)\\
&+\frac{1}{\lambda_1+K}\left\{-C-C\sum_k\left(|u_{k1}|^2+|u_{k\bar{1}}|^2\right)+\sum_l\left(\psi_{u_l}u_{1\bar{1}l}+\psi_{u_{\bar{l}}}u_{1\bar{1}\bar{l}}\right)\right\}^2\\
&+h'\left\{\sum_l\left(\psi_{u_l}\nabla_l|\nabla u|^2+\psi_{u_{\bar{l}}}\nabla_{\bar{l}}|\nabla u|^2\right)+\sum_a F^{m\bar{m}}\left(|u_{am}|^2+|u_{\bar{a}m}|^2\right)\right\}\\
&+h''F^{m\bar{m}}\left|\nabla_m|\nabla u|^2\right|+g'F^{m\bar{m}}\lambda_m-g'\sum F^{m\bar{m}}+g''F^{m\bar{m}}|u_m|^2\\
&\geq-\frac{C}{\lambda_1}-h'C+Cg'+\left(C-h'C-g'\right)\sum F^{m\bar{m}}\\
&-\frac{C}{\lambda_1}\left(\sum_k\left(|u_{k1}|^2+|u_{k\bar{1}}|^2\right)\right)+h'\sum_a F^{m\bar{m}}\left(|u_{am}|^2+|u_{\bar{a}m}|^2\right)\\
&+\left(h''-2\varepsilon\left(h'\right)^2\right)F^{m\bar{m}}\left|\nabla_m|\nabla u|^2\right|^2+\left(g''-2\varepsilon\left(g'\right)^2\right)F^{m\bar{m}}|u_m|^2.
\end{aligned}
$$

Note that h has to satisfy the following two equations.

$$
h''-2\varepsilon\left(h'\right)^2\geq 0 \tag{2.5.3}
$$

$$
-\frac{C}{\lambda_1}\sum_k|u_{k1}|^2+h'\sum_a F^{m\bar{m}}|u_{am}|^2\geq 0 \tag{2.5.4}
$$

In particular, the most troubled term in equation (2.5.4) is u_{11}. Since $F^{1\bar{1}}\geq\frac{C_1}{\lambda_1}$, we have to satisfy

$$
\begin{aligned}
&-\frac{C}{\lambda_1}|u_{11}|^2+h'F^{1\bar{1}}|u_{11}|^2\\
&\geq\left(-\frac{C}{\lambda_1}+h'\frac{C_1}{\lambda_1}\right)|u_{11}|^2\geq 0
\end{aligned} \tag{2.5.5}
$$

Let $h(x) = -\frac{1}{\alpha}\log(M-x)$, where $M = \sup|\nabla u|^2 + 1$. Then

$$h'(x) = \frac{1}{\alpha}\frac{1}{M-x} \text{ and } h''(x) = \frac{1}{\alpha}\frac{1}{(M-x)^2}.$$

Pick α small such that

$$-\frac{C}{\lambda_1} + h'\frac{C_1}{\lambda_1} \geq \frac{1}{\lambda_1}.$$

Next, let $g(x) = -\frac{1}{\beta}\log(x-N)$, where $N = \inf u - 1$. Then

$$g'(x) = -\frac{1}{\beta}\frac{1}{x-N} \text{ and } g''(x) = \frac{1}{\beta}\frac{1}{(x-N)^2}.$$

Pick β small such that

$$(C - h'C - g')\sum F^{m\bar{m}} \geq 0.$$

Lastly, we fix ε small such that

$$h'' - 2\varepsilon(h')^2 \geq 0 \text{ and } g'' - 2\varepsilon(g')^2 \geq 0.$$

Therefore,

$$\begin{aligned}
0 &\geq F^{m\bar{m}}\left(\log(\lambda_1 + K) + h\left(|\nabla u|^2\right) + g(u)\right)_{m\bar{m}} \\
&\geq -\frac{C}{\lambda_1} - C(\alpha,\beta) + \frac{1}{\lambda_1}|u_{1\bar{1}}|^2.
\end{aligned}$$

It follows that λ_1 is bounded above.

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Chapter 3

Parabolic complex Hessian equations

3.1 Introduction

We consider the following parabolic flow on compact Kähler manifolds (M, α). χ is a given smooth closed real $(1,1)$ form, whose eigenvalues with respect to α belong to Γ_k cone in $\mathbb{R}^n$. In local coordinates, we have

$$\alpha(z) = \frac{\sqrt{-1}}{2}\alpha_{i\bar{j}}dz^i \wedge d\bar{z}^j \text{ and } \chi(z) = \frac{\sqrt{-1}}{2}\chi_{i\bar{j}}dz^i \wedge d\bar{z}^j.$$

We write $g_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$, and denote the eigenvalues of $\alpha^{-1}g$ by $\lambda(\alpha^{-1}g)$. We consider the following flow.

$$\begin{aligned} u_t &= F\left(\sigma_k\left(\lambda\left(\alpha^{-1}g\right)\right)\right) - \psi(z), \text{ where } F' > 0 \qquad (3.1.1) \\ u(z,0) &= u_0 \end{aligned}$$

Here $\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$. Equation (3.1.1) arises naturally from geometry. The most well-known one should be the Kähler-Ricci flow. By using Kähler-Ricci flow, Cao

[4] reproduces the celebrated results of Yau [20]. When F is concave, a general theory of fully nonlinear elliptic and parabolic equations are well-developed by Caffarelli- Nirenberg-Spruck [3]. In [14], Picard-Zhang study parabolic complex Monge-Ampère equations without concavity due to their related works to the anomaly flow [13]. Motivated by their works, our goal is to study similar problems on complex Hessian equations. Following Székelyhidi [16] and Phong-Tô [12], we require χ to satisfy the following C-subsolution condition.

$$\left(\lambda\left(\alpha^{-1}\chi\left(z\right)\right)+\Gamma_n\right)\cap\partial\Gamma^{\gamma}\text{ is bounded, where }\Gamma^{\sigma}=\left\{\lambda|F\left(\sigma_k\left(\lambda\right)\right)\geq\sigma\right\}\tag{$\Diamond$}$$

$$\begin{aligned}\gamma&=\psi\left(z\right)\text{ if }\sup\left[F\left(\sigma_k\left(\alpha^{-1}\chi\right)\right)-\psi\left(z\right)\right]\leq 0\\&=\sup_M\left[F\left(\sigma_k\left(\alpha^{-1}\chi\right)\right)-\psi\right]+\psi\left(z\right)\text{ otherwise.}\end{aligned}\tag{3.1.2}$$

The C-subsolution conditions here is slightly different from Phong-Tô [12]. Nonetheless, our condition is satisfied by other flows as well. For example, Sun considers the following equation in [15].

$$u_t=\log\frac{g^n}{g^{n-k}\wedge\alpha^k}-\log\psi\tag{3.1.3}$$

He fixes $\gamma=\psi\left(z\right)$. By studying J-functional, he imposes the following condition in order to do the uniform estimate.

$$\frac{\chi^n}{\chi^{n-k}\wedge\alpha^k}\leq\psi\tag{3.1.4}$$

Above condition is the same as saying

$$\sup\left(\log\frac{\chi^n}{\chi^{n-k}\wedge\alpha^k}-\log\psi\right)\leq 0,$$

which is the first case in our condition (3.1.2).

By taking derivatives with respect to t, we have the following linear heat equation for u_t.

$$\partial_t (u_t) = F' \frac{\partial \sigma_k}{\partial A^i_j} \alpha^{i\bar{k}} \nabla_j \nabla_{\bar{k}} (u_t), \text{ where } A^i_j = \left(\alpha^{-1} g\right)^i_j \tag{3.1.5}$$

Then maximum principle implies

$$\min_M u_t (z, 0) \leq u_t (t, \cdot) \leq \max_M u_t (z, 0) \tag{3.1.6}$$

To ensure the flow stays in Γ_k cone, we also require that

$$\min_M u_t (z, 0) + \min_M \psi (z) > \lim_{x \to 0^+} F (x) \tag{$\blacklozenge$}$$

We will **always** assume condition ($\lozenge$) and ($\blacklozenge$) hold below.

Remark: From our proof below, it is clear that we can replace (3.1.2) with the following condition.

$$\begin{aligned} \gamma &= \psi (z) \text{ if } \sup u_t \leq 0 \\ &= \sup_M u_t + \psi (z) \text{ otherwise.} \end{aligned} \tag{3.1.7}$$

If we use this condition instead, when $u_t = 0$, it reduces to the same C- subsolution condition for elliptic equations from [16]. However, this condition depends on the initial data due to the term $\sup_M u_t$. Phong-Tô also study the equation (3.1.3), and the condition [12, (4.43)] they impose by J-functional is the same as requiring $\sup u_t \leq 0$.

By adding a constant to $\psi (z)$, we may assume F is strictly positive or negative. We solve the flow if one of the following condition holds.

Condition 3.1.1. F is near to a concave operator in the following sense.

Let $G\left(\sigma_k\left(\alpha^{-1}g\right)\right)$ be a concave operator on $\left(\alpha^{-1}g\right)_j^i$ with $G' > 0$. We say $F\left(\sigma_k\right)$ is near to $G\left(\sigma_k\right)$ if

$$M_0 = \sup\left(\frac{F}{F'}\right)^2\left(\frac{F''}{F'} - \frac{G''}{G'}\right)\frac{n}{k\sigma_k} \leq \frac{1}{12}. \tag{3.1.8}$$

Condition 3.1.2. The given data u_0 and $\psi\left(z\right)$ has small oscillations.

$$\frac{\sqrt{12M_0 - 1}}{2}\left(\log\sup|F| - \log\inf|F|\right) < \pi. \tag{3.1.9}$$

Note that F is bounded by constants depending on u_0 and $\psi\left(z\right)$ by the maximum principle.

Observe that in equation (3.1.8), if $F = \sigma_k^p$ and $G = \sigma_k^{\frac{1}{k}}$, then M_0 doesn't depend on σ_k. We can find p such that the inequality holds. The following theorem is our main result.

Theorem 3.1.3. *Suppose that either (3.1.8) or (3.1.9) holds, then the flow admits a smooth solution $u\left(x,t\right)$ on $[0,\infty)$, and the normalized solution $\widetilde{u} = u - \frac{1}{vol(M)}\int_M u\alpha^n$ converges in C^∞ to a function υ satisfying*

$$F\left(\sigma_k\left(\alpha^{i\bar{j}}\left(\chi_{k\bar{j}} + \upsilon_{k\bar{j}}\right)\right)\right) = \psi\left(z\right) + C$$

for some constant C.

This chapter is organized as follows. In section 2, we study the C^0 estimate. We adapt an approach due to Blocki [1] and Székelyhidi [16]. However the Alexandroff-Bakelman-Pucci maximum principle [19] may be dependent on time t. To overcome this, we carefully analyze the C-subsolution condition (3.1.2) and apply Hamilton's estimate [9, Lemma 3.5].

In section 3, we modify the techniques from Székelyhidi [16] and Hou-Ma-Wu [11] to study the C^2 estimate. To overcome the loss of concavity, we carefully use the ODE comparison theorem to construct the test function. In section 4, we apply the blow up argument of

Dinew and Kolodziej [5]. In section 5, we obtain the $C^{2,\alpha}$ estimates by techniques from Phong-Picard-Zhang [13] and Tsai [18]. In the last section, the convergence follows from Gill's result [8].

3.2 $\mathbf{C}^0$ estimate

First Observe that

$$\left(\lambda\left(\alpha^{-1}\chi(z)\right)+\Gamma_n\right)\cap\partial\Gamma^\gamma \text{ is bounded}$$

implies

$$\lim_{\mu\to\infty} F\left(\sigma_k\left(\lambda+\mu e_i\right)\right)>\gamma,\text{ where } \lambda=\lambda\left(\alpha^{-1}\chi(z)\right)$$

Since M is compact, there exists uniform positive constants δ and ξ such that

$$F\left(\sigma_k\left(\lambda+\xi e_i-\delta I\right)\right)>\gamma+\delta,\ \forall z\in M \tag{3.2.1}$$

Therefore, if

$$F\left(\sigma_k\left(\lambda+a\right)\right)\le b,\ a+\delta I\in\Gamma_n \text{ and } b\le\gamma+\delta, \tag{3.2.2}$$

(3.2.1) implies that $|a|<\xi$, and b is bounded below by a uniform constant.

$$b\ge F\left(\sigma_k\left(\lambda+a\right)\right)\ge\inf_M F\left(\sigma_k\left(\lambda-\delta I\right)\right)=\xi_0$$

Lemma 3.2.1. *Suppose $u(z,t)$ is an admissible solution to the flow (3.1.1) on time interval*

$[0,T)$, *then there exists a constant* C *independent of* T *such that*

$$osc_M u(\cdot,t) \le C\left(M,n,F,\chi,\alpha,\delta,\|u_t\|_{L^\infty}\right) \text{ for all } t\in[0,T)$$

We will use an argument similar to Székelyhidi's [16], Picard-Zhang [14], and Phong-Tô [12]. Their method is based on Blocki's [1] approach to the complex Monge-Ampère equation.

Proof: Since u_t is bounded by equation (3.1.6), on a small time interval $[0,\varepsilon_0]$, u is uniformly bounded. Let

$$w(x,t) = \left(\sup_M u\right)(t) - u(x,t)$$

Fix $T' < T$, suppose

$$\sup_{M\times[0,T']} w(x,t) = w(x_0,t_0) = L,$$

We would like to show that L is independent of T'. We may assume $t_0 > \varepsilon_0$ and $L > 0$. If $t_0 \le \varepsilon_0$, L is determined by initial data. If $L \le 0$, w is bounded above by 0. At the time slice $T = t_0$, fix a coordinate ball $B_1(0)$ centered at x_0. We consider the following function. Here $\varepsilon < \varepsilon_0^2$ is a small constant to be fixed.

$$v(x,t) = w(x,t) - \varepsilon|z|^2 - (t-t_0)^2 - L + \varepsilon$$

In the cylinder $Q = B_1(0)\times(t_0-\sqrt{\varepsilon},t_0]$

$$\sup_Q v = \varepsilon,\ \ \sup_{\partial B_1} v \le 0, \text{ and } \sup_{t=t_0-\sqrt{\varepsilon}} v \le 0$$

We can then apply Krylov-Tso ABP estimate [19]. We have

$$\sup_{Q} v \leq C(n) \left(\int_{\Gamma^{+}(v)} \left| v_t \det\left(D^2 v\right) \right| \right)^{\frac{1}{n+1}},$$

where

$$\Gamma^{+}(v) = \{(x,t) \in Q | v(x,t) \geq 0,\ v(y,s) \leq v(x,t) + Dv(x,t) \cdot (y-x),\ \forall y \in B_1(0),\ s \leq t\}$$

Apply Blocki's adaptation of ABP estimate [1], we get

$$\varepsilon^{n+1} \leq C(n) \int_{\Gamma^{+}(v)} |v_t| \left| \det v_{i\bar{j}} \right|^2,$$

On $\Gamma^{+}(v)$, we have

$$D^2 v \leq 0 \text{ and } \partial_t v \geq 0 \ a.e$$

$(\sup_M u)(t)$ is Lipschitz, by Hamilton's estimate [9, Lemma 3.5], when $(\sup_M u)(t)$ is differentiable, we have,

$$\frac{d}{dt}\left(\sup_M u\right)(t) \leq \sup\left\{ \frac{\partial}{\partial t} u(x,t) : x \in X(t) \right\},\ \text{where } X(t) = \left\{ x : \left(\sup_M u\right)(t) = u(x,t) \right\}.$$

For $x \in X(t_\alpha)$, u is a local maximum at $t = t_\alpha$. Therefore,

$$\begin{aligned} u_t &= F\left(\sigma_k\left(\alpha^{l\bar{j}}\left(\chi_{i\bar{j}} + u_{i\bar{j}}\right)\right)\right) - \psi(x) \leq F\left(\sigma_k\left(\alpha^{-1}\chi\right)\right) - \psi(x),\ \forall x \in X(t) \\ \partial_t v &= \frac{d}{dt}\left(\sup_M u\right)(t) - u_t - 2(t - t_0) \leq \sup\left[F\left(\sigma_k\left(\alpha^{-1}\chi\right)\right) - \psi\right] - u_t + 2\sqrt{\varepsilon} \end{aligned} \quad (3.2.3)$$

It follows that v_t is bounded above by a constant independent of T' almost everywhere.

On the other hand, $D^2 v(x,t) \leq 0$ implies $u_{i\bar{j}}(x,t) \geq -\varepsilon \delta_{i\bar{j}}$. With δ chosen, we may fix ε

small, such that at $(x,t) \in \Gamma^+$,

$$\lambda\left(\alpha^{k\bar{j}}\left(\chi_{i\bar{j}}+u_{i\bar{j}}\right)\right) \in \lambda\left(\alpha^{k\bar{j}}\chi_{i\bar{j}}\right) - \delta I + \Gamma_n$$

From the equation, we have

$$F\left(\sigma_k\left(\alpha^{k\bar{j}}\left(\chi_{i\bar{j}}+u_{i\bar{j}}\right)\right)\right) = \psi(x) + u_t \tag{3.2.4}$$

Since $(x,t) \in \Gamma^+$, by (3.2.3), $0 \le v_t \le \sup\left[F\left(\sigma_k\left(\alpha^{-1}\chi\right)\right) - \psi\right] - u_t + 2\sqrt{\varepsilon}$. This implies

$$\psi(x) + u_t \le \sup\left[F\left(\sigma_k\left(\alpha^{-1}\chi\right)\right) - \psi\right] + \psi(x) + 2\sqrt{\varepsilon} \le \gamma + \delta \text{ when } \varepsilon \text{ is small.}$$

Now equation (3.2.4) can be viewed as a special case of (3.2.2), we then have

$$\left|\lambda\left(\alpha^{k\bar{j}}\left(\chi_{i\bar{j}}+u_{i\bar{j}}\right)\right) - \lambda\left(\alpha^{k\bar{j}}\chi_{i\bar{j}}\right)\right| < \xi$$

Therefore,

$$\left|\det v_{i\bar{j}}\right| = \left|\det\left(-u_{i\bar{j}} - \varepsilon\delta_{i\bar{j}}\right)\right| < C$$

And we have a lower bound on the volume of the contact set.

$$\varepsilon^{n+1} \le C\left|\Gamma^+(v)\right|, \text{ C is independent of } T'$$

Let $\bar{v} = v + L - \varepsilon$. On $\Gamma^+(v)$, $\bar{v} \ge L - \varepsilon$.

$$|L-\varepsilon|^p \le \frac{1}{|\Gamma^+(v)|}\int_{\Gamma^+(v)} |\bar{v}|^p \, dxdt \le C\int_{Q_1} |\bar{v}|^p \, dxdt \le C\int_{t_0-\sqrt{\varepsilon}}^{t_0} \|\bar{v}(x,t)\|_{L^p(M)}^p \, dt \tag{3.2.5}$$

$\forall\ t^* \in (t_0 - \sqrt{\varepsilon}, t_0]$

$$\|\bar{v}(x, t^*)\|_{L^p(B_1)} \leq \left\| \sup_M u(t^*) - u(x, t^*) \right\|_{L^p(B_1)} + C\varepsilon$$

Since $\lambda\left(\alpha^{k\bar{j}}\left(\chi_{i\bar{j}} + u_{i\bar{j}}\right)\right) \in \Gamma_k$, $\alpha^{k\bar{j}}\left(\chi_{k\bar{j}} + u_{k\bar{j}}\right) \geq 0$. Denote $(\sup_M u - u(x, t^*))$ as $\bar{u}$. Then $\bar{u}$ satisfies the following elliptic equation.

$$\alpha^{k\bar{j}} \bar{u}_{k\bar{j}} \leq \alpha^{k\bar{j}} \chi_{k\bar{j}}$$

Apply the weak Harnack inequality [7, Theorem 9.22] and a covering argument (e.g. see [14]), we have

$$\|\bar{u}(x, t^*)\|_{L^p(M)} \leq C(M, \alpha, \chi, p)$$

With the L^p bound, equation (3.2.5) implies L is bounded above independent of T'.

■

With the oscillation bound, we also have a bound on the normalized solution.

$$\|\hat{u}\|_{L^\infty(M\times[0,T))} \equiv \left\| u - \frac{1}{V}\int u\alpha^n \right\|_{L^\infty(M\times[0,T))} \leq C, \ C \text{ is independent of } T.$$

3.3 C^2 estimate

Since u_t is bounded by a uniform constant, it follows that

$$C^{-1} \leq \sigma_k \leq C \text{ and } -C \leq F \leq C$$

Therefore, there exists a constant C_0 such that either $\inf(F + C_0) > 0$ or $\sup(F + C_0) < 0$. Let $\tilde{G} = G(\sigma_k(\alpha^{-1}g))$ be a concave operator on $(\alpha^{-1}g)^i_j$ with $G' > 0$ and $H = F + C_0$. Both G and H are functions of σ_k. Define

$$M_0 = \sup\left(\frac{H}{H'}\right)^2\left(\frac{H''}{H'} - \frac{G''}{G'}\right)\frac{n}{k\sigma_k}$$

Lemma 3.3.1. *Suppose $u(z,t)$ is an admissible solution to the flow (3.1.1) on time interval $[0,T)$, then there exists a constant C independent of T such that*

$$\left|\partial\bar{\partial}u\right| \leq C\left(n, \alpha, F, \chi, \psi, osc_M u, \|u_t\|_{L^\infty}, M_0, C_0\right)\left(1 + \sup_{M\times[0,T)}\left|\alpha^{i\bar{j}}u_i u_{\bar{j}}\right|\right). \tag{3.3.1}$$

if one of the following conditions hold, (i)$\frac{\sqrt{12M_0-1}}{2}(\log\sup|H| - \log\inf|H|) < \pi$ *(ii)*$M_0 \leq \frac{1}{12}$

Diagonalize g and α at maximum point of the test function, and take a normal coordinate. We assume $g_{1\bar{1}}$ is the largest eigenvalue, and consider the following test function motivated by [11] and [14].

$$\log(g_{1\bar{1}}) + f(F) + g(F) + \varphi\left(|\nabla u|^2\right) + h(\hat{u})$$

f, g, φ, and h will be stated below. We have the linearized operator

$$L = F'\sigma_{k-1;s}\nabla_{\bar{s}}\nabla_s = a_{s\bar{s}}\nabla_{\bar{s}}\nabla_s$$

Apply the linearized operator to the equation, we obtain

$$u_{t1} = F'\nabla_1\sigma_k - \psi_1 = a_{s\bar{s}}g_{s\bar{s}1} - \psi_1$$

$$u_{tt} = a_{s\bar{s}}g_{s\bar{s}t}$$

We will denote

$$S=\sum_s a_{s\bar{s}},\ K=\sup|\nabla u|^2+1 \text{ and } R \text{ as the curvature bound}$$

Also

$$\begin{aligned}\sum_s a_{s\bar{s}}\lambda_s &= F'\sum_s \lambda_s\sigma_{k-1;s}\\ &= F'k\sigma_k\end{aligned}$$

Below, we state the key estimate for concave functions from Székelyhidi [16, Proposition 5] (see also Phong-Tô [12, Lemma 3]). The statement we give below is slightly different, but they are essentially the same.

Lemma 3.3.2. *Let $f(\lambda)$ be a strictly increasing concave function of $\lambda\in\Gamma_k$, $\mu\in\mathbb{R}^n$ and $\Gamma^\sigma=\{\lambda|f(\lambda)\geq\sigma\}$. Suppose there exists δ and $R>0$ such that*

$$\Omega=(\mu-\delta 1+\Gamma_n)\cap\partial\Gamma^{\sigma+\delta}\subset B_R(0)$$

Then there exists a constant $\kappa(n,\delta,\Omega)>0$ such that if $\lambda\in\partial\Gamma^{\sigma-\tau}$ and $|\lambda|>R$, then either

$$\sum_{i=1}^n f_i(\lambda)(\mu_i-\lambda_i)-\tau>\kappa\sum_{i=1}^n f_i(\lambda)$$

or

$$f_i(\lambda)>\kappa\sum_{i=1}^n f_i(\lambda)\ \textit{for all } i$$

Observe that $F(\sigma_k(\lambda))$ is a quasiconcave function. Rewrite it as $F\left(\left(\sigma_k^{\frac{1}{k}}(\lambda)\right)^k\right)$. $\sigma_k^{\frac{1}{k}}$ is a concave function of $\lambda\in\Gamma_k$. x^k and $F(x)$ are increasing functions on $\mathbb{R}^+$. So $\theta(\lambda)$ is a quasiconcave function. The proof of above lemma in [16] only relies on the fact that Γ^{σ} is convex. So **above lemma also works for quasiconcave functions**. By definition of C-subsolution, for $z \in M$, $(\lambda(\alpha^{-1}\chi(z)) + \Gamma_n) \cap \partial\Gamma^{\psi(z)}$ are bounded. Since M is compact, there exists δ and $R > 0$ such that

$$\left(\lambda\left(\alpha^{-1}\chi\right) - \delta 1 + \Gamma_n\right) \cap \partial\Gamma^{\psi(z)+\delta} \subset B_R(0), \ \forall z \in M$$

Let $\mu = \lambda(\alpha^{-1}\chi)$, $\tau = -u_t$, $\lambda = \lambda(\alpha^{-1}g(z))$. By above lemma, at any $z \in M$, there exists constant κ such that if $|\lambda(\alpha^{-1}g(z))| > R$, we have either

$$\sum_{s=1}^{n} a_{s\bar{s}}(-u_{s\bar{s}}) + u_t > \kappa \sum_{s=1}^{n} a_{s\bar{s}}$$

or

$$a_{s\bar{s}} > \kappa \sum_{s=1}^{n} a_{s\bar{s}} \text{ for all } s$$

Below, we will use C to denote a uniform constant. C may vary from line to line.

3.3.1 Computing $(\partial_t - L)\varphi\left(|\nabla u|^2\right)$

$$\begin{aligned}
&(\partial_t - L)\varphi\left(|\nabla u|^2\right) \\
&= \varphi'\partial_t|\nabla u|^2 - a_{s\bar{s}}\left[\varphi''\left|\nabla_s|\nabla u|^2\right|^2 + \varphi'\nabla_{\bar{s}}\nabla_s|\nabla u|^2\right] \\
&= \varphi'(\partial_t - L)|\nabla u|^2 - \varphi'' a_{s\bar{s}}\left|\partial_s|\nabla u|^2\right|^2
\end{aligned}$$

$$
\begin{aligned}
&(\partial_t - L)\left(\alpha^{p\bar{p}} u_p u_{\bar{p}}\right) \\
&= u_{tp} u_{\bar{p}} + u_p u_{t\bar{p}} - a_{s\bar{s}} \left(u_{ps\bar{s}} u_{\bar{p}} + u_p u_{\bar{p}s\bar{s}} + \sum_p |u_{ps}|^2 + \sum_p |u_{p\bar{s}}|^2 \right)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&u_{ps\bar{s}} = u_{s\bar{s}p} + u_q R_{p\bar{q}s\bar{s}} \\
&u_{\bar{p}s\bar{s}} = u_{s\bar{s}\bar{p}}
\end{aligned}
$$

Therefore, the evolution equation will be

$$
\begin{aligned}
&(\partial_t - L)|\nabla u|^2 \\
&= u_{\bar{p}}\left(u_{tp} - a_{s\bar{s}} u_{s\bar{s}p} - a_{s\bar{s}} u_q R_{p\bar{q}s\bar{s}}\right) + u_p\left(u_{t\bar{p}} - a_{s\bar{s}} u_{s\bar{s}\bar{p}}\right) - a_{s\bar{s}} \left(\sum_p |u_{ps}|^2 + |u_{p\bar{s}}|^2 \right) \\
&\leq u_{\bar{p}}\left(a_{s\bar{s}} \chi_{s\bar{s}p} - \psi_p\right) + u_p\left(a_{s\bar{s}} \chi_{s\bar{s}\bar{p}} - \psi_{\bar{p}}\right) - a_{s\bar{s}} u_{\bar{p}} u_q R_{p\bar{q}s\bar{s}} - a_{s\bar{s}} \left(\sum_p |u_{ps}|^2 + |u_{s\bar{s}}|^2 \right) \\
&\leq C|\nabla u| + C|\nabla u| S + R|\nabla u|^2 S - a_{s\bar{s}}\left(\lambda_s - \chi_{s\bar{s}}\right)^2 \\
&\leq C|\nabla u| + C|\nabla u| S + R|\nabla u|^2 S - a_{s\bar{s}}\left(\frac{1}{2}\lambda_s^2 - |\chi_{s\bar{s}}|^2\right) \leq CK + CKS - \frac{1}{2} a_{s\bar{s}} \lambda_s^2
\end{aligned}
$$

Suppose $\varphi' \geq 0$

$$
\begin{aligned}
&(\partial_t - L)\varphi\left(|\nabla u|^2\right) \\
&\leq \varphi'\left(CK + CKS - \frac{1}{2} a_{s\bar{s}} \lambda_s^2\right) - \varphi'' a_{s\bar{s}} \left|\partial_s |\nabla u|^2\right|^2
\end{aligned}
$$

3.3.2 Computing $(\partial_t - L)\, h(\hat{u})$

Here $\hat{u}$ is the normalization of u. Suppose $h' \leq 0$

$$u_x = \hat{u}_x$$
$$\hat{u}_t = u_t - \frac{1}{V}\int u_t \alpha^n$$

We have

$$
\begin{aligned}
&(\partial_t - L)\, h(\hat{u}) \\
&= h'\hat{u}_t - a_{s\bar{s}}\left[h''\left|\hat{u}_s\right|^2 + h'\hat{u}_{s\bar{s}}\right] \\
&\leq h'u_t - h''a_{s\bar{s}}\left|u_s\right|^2 - h'a_{s\bar{s}}u_{s\bar{s}} + h'C
\end{aligned}
$$

3.3.3 Computing $(\partial_t - L)\, f(F)$

$$(\partial_t - L)\, F = F_t - a_{s\bar{s}}F_{s\bar{s}} = u_{tt} - a_{s\bar{s}}\left(u_{ts\bar{s}} + \psi_{s\bar{s}}\right) = -a_{s\bar{s}}\psi_{s\bar{s}}$$

It follows that

$$
\begin{aligned}
(\partial_t - L)\, f(F) &= f'F_t - a_{s\bar{s}}\left[f''\left|F_s\right|^2 + f'F_{s\bar{s}}\right] \\
&= -f'a_{s\bar{s}}\psi_{s\bar{s}} - f''a_{s\bar{s}}\left|F_s\right|^2
\end{aligned}
$$

3.3.4 Evolution equation for test function

$$
\begin{aligned}
u_{t1\bar{1}} &= F''\left|\nabla_1\sigma_k\right|^2 + F'\left[\sum_s \sigma_{k-1;s}g_{s\bar{s}1\bar{1}} + \sum_{i\neq j}\sigma_{k-2;ij}\left(g_{j\bar{j}1}g_{i\bar{i}\bar{1}} - g_{i\bar{j}1}g_{j\bar{i}\bar{1}}\right)\right] - \psi_{1\bar{1}} \\
&= F''\left|\nabla_1\sigma_k\right|^2 + a_{s\bar{s}}g_{s\bar{s}1\bar{1}} + F'\sigma_{k-2;ij}(\ldots) - \psi_{1\bar{1}} = a_{s\bar{s}}g_{s\bar{s}1\bar{1}} + (*) - \psi_{1\bar{1}},
\end{aligned}
$$

where $(*)$ are all the third order terms. Note that

$$\begin{aligned}
&\sum_s a_{s\bar{s}} g_{s\bar{s}1\bar{1}} - a_{s\bar{s}} g_{1\bar{1}s\bar{s}} \\
&= a_{s\bar{s}} R_{1\bar{1}s\bar{s}} \left(g_{s\bar{s}} - g_{1\bar{1}}\right) \\
&\leq -R a_{s\bar{s}} \left(g_{s\bar{s}} - g_{1\bar{1}}\right) \\
&= -RF' k\sigma_k + R g_{1\bar{1}} S
\end{aligned}$$

Applying results from previous subsections, we obtain

$$\begin{aligned}
&(\partial_t - L)\left[\log\left(g_{1\bar{1}}\right) + f(F) + g(F) + \varphi\left(|\nabla u|^2\right) + h(\hat{u})\right] \\
&= \frac{1}{g_{1\bar{1}}} g_{1\bar{1}t} - \frac{1}{g_{1\bar{1}}} \sum_s a_{s\bar{s}} \left(g_{1\bar{1}s\bar{s}} - \frac{1}{g_{1\bar{1}}} \left|g_{1\bar{1}s}\right|^2\right) \\
&+ (\partial_t - L)\left[f(F) + g(F) + \varphi\left(|\nabla u|^2\right) + h(\hat{u})\right] \\
&\leq \frac{1}{g_{1\bar{1}}} \left(a_{s\bar{s}} g_{s\bar{s}1\bar{1}} + (*) - \psi_{1\bar{1}}\right) - \frac{1}{g_{1\bar{1}}} a_{s\bar{s}} g_{1\bar{1}s\bar{s}} + \sum_s a_{s\bar{s}} \frac{1}{\left(g_{1\bar{1}}\right)^2} \left|g_{1\bar{1}s}\right|^2 \\
&- f'' a_{s\bar{s}} |F_s|^2 - g'' a_{s\bar{s}} |F_s|^2 + \varphi' \left(CK + CKS - \frac{1}{2} a_{s\bar{s}} \lambda_s^2\right) - \varphi'' a_{s\bar{s}} \left|\partial_s |\nabla u|^2\right|^2 \\
&- f' a_{s\bar{s}} \psi_{s\bar{s}} - g' a_{s\bar{s}} \psi_{s\bar{s}} + h' u_t - h'' a_{s\bar{s}} |u_s|^2 - h' a_{s\bar{s}} u_{s\bar{s}} + h' C \\
&\leq -\frac{1}{g_{1\bar{1}}} RF' k\sigma_k - \frac{1}{g_{1\bar{1}}} \psi_{1\bar{1}} + C\varphi' K + h' u_t + h' C \\
&+ S\left(R + \varphi' CK + |f'| \|\psi\|_{C^2} + |g'| \|\psi\|_{C^2}\right) - \frac{1}{2} \varphi' a_{s\bar{s}} \lambda_s^2 - h'' a_{s\bar{s}} |u_s|^2 - h' a_{s\bar{s}} u_{s\bar{s}} \\
&+ \frac{1}{g_{1\bar{1}}} (*) + \sum_s a_{s\bar{s}} \frac{1}{\left(g_{1\bar{1}}\right)^2} \left|g_{1\bar{1}s}\right|^2 - f'' a_{s\bar{s}} |F_s|^2 - g'' a_{s\bar{s}} |F_s|^2 - \varphi'' a_{s\bar{s}} \left|\partial_s |\nabla u|^2\right|^2
\end{aligned}$$

Below following the method of Hou-Ma-Wu, we separate it into two cases. We set

$$\varphi(x) = -\frac{1}{3} \log\left(1 - \frac{x}{3K}\right)$$

We have

$$\varphi'' = 3\left(\varphi'\right)^2 \text{ and } \frac{1}{6K} \geq \varphi' \geq \frac{1}{9K}$$

Let $L = \sup|\hat{u}| + 1$

$$h(x) = -A\log\left(1 + \frac{x}{2L}\right)$$

$$\begin{aligned} \frac{A}{2L} &\geq -h' \geq \frac{A}{3L} \\ h'' &= \frac{1}{A}\left(h'\right)^2 \end{aligned}$$

A is a large constant to be fixed. f and g to be determined. Also, we require

$$3\left(f' + g'\right)^2 \leq g'' \text{ and } f'' \geq 0$$

3.3.4.1 $\lambda_n \geq -\delta\lambda_1$

δ is a constant to be fixed.

$$I = \left\{s|\sigma_{k-1;s} > \delta^{-1}\sigma_{k-1;1}\right\}$$

For $s \notin I$, $a_{s\bar{s}} \leq \delta^{-1}a_{1\bar{1}}$. From critical equation, we have

$$\frac{1}{g_{1\bar{1}}}g_{1\bar{1}s} + f'F_s + g'F_s + \varphi'\partial_s|\nabla u|^2 + h'u_s = 0$$

$$\left|\frac{1}{g_{1\bar{1}}}g_{1\bar{1}s}\right|^2 \leq 3\left(f' + g'\right)^2|F_s|^2 + 3\left(\varphi'\right)^2\left|\partial_s|\nabla u|^2\right|^2 + 3\left(h'\right)^2|u_s|^2$$

$$
\begin{aligned}
&\sum_{s\notin I} a_{s\bar{s}} \left|\frac{1}{g_{1\bar{1}}} g_{1\bar{1}s}\right|^2 \\
&\le 3\sum_{s\notin I} a_{s\bar{s}} \left(f'+g'\right)^2 \left|F_s\right|^2 + 3\sum_{s\notin I} a_{s\bar{s}} \left(\varphi'\right)^2 \left|\partial_s \left|\nabla u\right|^2\right|^2 + 3\sum_{s\notin I} a_{s\bar{s}} \left(h'\right)^2 \left|u_s\right|^2 \\
&\le \sum_{s\notin I} a_{s\bar{s}} g'' \left|F_s\right|^2 + \sum_{s\notin I} a_{s\bar{s}} \varphi'' \left|\partial_s \left|\nabla u\right|^2\right|^2 + 3\left(h'\right)^2 \delta^{-1} a_{1\bar{1}} K
\end{aligned}
$$

We may assume $3\left(h'\right)^2 \delta^{-1} a_{1\bar{1}} K < \frac{1}{36K} a_{1\bar{1}} \lambda_1^2$. Otherwise we are done.

$$
\begin{aligned}
&-\frac{1}{2}\varphi' a_{s\bar{s}} \lambda_s^2 + 3\left(h'\right)^2 \delta^{-1} a_{1\bar{1}} K \\
&\le -\frac{1}{18K} a_{11} \lambda_1^2 + \frac{1}{36K} a_{11} \lambda_1^2
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\left(\partial_t - L\right)\left[\log\left(g_{1\bar{1}}\right) + f\left(F\right) + g\left(F\right) + \varphi\left(\left|\nabla u\right|^2\right) + h\left(\hat{u}\right)\right] \\
&\le h' u_t + h' C + C\left(f,g\right) S - \frac{1}{36K} a_{11} \lambda_1^2 - h' a_{s\bar{s}} u_{s\bar{s}} - h'' a_{s\bar{s}} \left|u_s\right|^2 \\
&- f'' a_{s\bar{s}} \left|F_s\right|^2 \\
&+ \frac{1}{g_{1\bar{1}}}\left(*\right) + \sum_{s\in I} a_{s\bar{s}} \frac{1}{\left(g_{1\bar{1}}\right)^2} \left|g_{1\bar{1}s}\right|^2 - \sum_{s\in I} g'' a_{s\bar{s}} \left|F_s\right|^2 - \sum_{s\in I} \varphi'' a_{s\bar{s}} \left|\partial_s \left|\nabla u\right|^2\right|^2
\end{aligned}
$$

Note that

$$
\left|a+b\right|^2 \ge \delta\left|a\right|^2 - \frac{\delta}{1-\delta}\left|b\right|^2
$$

From critical equation

$$
\begin{aligned}
-3\left|\varphi' \partial_s \left|\nabla u\right|^2\right|^2 &= -3\left|\frac{1}{g_{1\bar{1}}} g_{1\bar{1}s} + f' F_s + g' F_s + h' u_s\right|^2 \\
&\le -3\delta \frac{1}{\left(g_{1\bar{1}}\right)^2} \left|g_{1\bar{1}s}\right|^2 + \frac{3\delta}{1-\delta}\left|f' F_s + g' F_s + h' u_s\right|^2 \\
&\le -3\delta \frac{1}{\left(g_{1\bar{1}}\right)^2} \left|g_{1\bar{1}s}\right|^2 + \frac{3\delta}{1-\delta}\left[2\left(f'+g'\right)^2 \left|F_s\right|^2 + 2\left(h'\right)^2 \left|u_s\right|^2\right]
\end{aligned}
$$

We hope

$$-h'' + \frac{6\delta}{1-\delta}(h')^2 \leq 0 \text{ and } h'' = \frac{1}{A}(h')^2$$
$$\Rightarrow \frac{1}{6A+1} \geq \delta$$

Therefore,

$$\begin{aligned}
&- h'' a_{s\bar{s}} |u_s|^2 - f'' a_{s\bar{s}} |F_s|^2 \\
&+ \frac{1}{g_{1\bar{1}}}(*) + \sum_{s\in I} a_{s\bar{s}} \frac{1}{(g_{1\bar{1}})^2} |g_{1\bar{1}s}|^2 - \sum_{s\in I} g'' a_{s\bar{s}} |F_s|^2 - \sum_{s\in I} \varphi'' a_{s\bar{s}} \left|\partial_s |\nabla u|^2\right|^2 \\
&\leq -f'' a_{s\bar{s}} |F_s|^2 + \frac{1}{g_{1\bar{1}}}(*) + (1-3\delta) \sum_{s\in I} a_{s\bar{s}} \frac{1}{(g_{1\bar{1}})^2} |g_{1\bar{1}s}|^2
\end{aligned}$$

Now, we state Hou-Ma-Wu's key inequality (also Székelyhidi) to deal with the remaining terms.

Let $H = G(\sigma_k(\chi^{-1}g))$ be a concave operator on $(\chi^{-1}g)^i_j$ with $G' > 0$. If $\lambda_n \geq -\delta\lambda_1$, we have

$$H^{ij,kl} g_{i\bar{j}1} g_{k\bar{l}\bar{1}} + \frac{(1-3\delta)}{(g_{1\bar{1}})} \sum_{s\in I} H^{s\bar{s}} |g_{1\bar{1}s}|^2 \leq 0$$

Let $|\nabla_1 \sigma_k|^2 = X$, $\sum_{i\neq j} \sigma_{k-2;ij} \left(g_{j\bar{j}1} g_{i\bar{i}\bar{1}} - g_{i\bar{j}1} g_{j\bar{i}\bar{1}}\right) = Y$ and $\sum_{s\in I} \sigma_{k-1;s} |g_{1\bar{1}s}|^2 = Z$. Then we have

$$G''X + G'Y + \frac{(1-3\delta)}{(g_{1\bar{1}})} G'Z \leq 0$$
$$\frac{1}{g_{1\bar{1}}}\left[F'\frac{G''}{G'}X + F'Y\right] + \frac{(1-3\delta)}{(g_{1\bar{1}})^2} F'Z \leq 0$$

$$
\begin{aligned}
& - f'' a_{s\bar{s}} \left| F_s \right|^2 + \frac{1}{g_{1\bar{1}}} (*) + (1 - 3\delta) \sum_{s \in I} a_{s\bar{s}} \frac{1}{(g_{1\bar{1}})^2} \left| g_{1\bar{1}s} \right|^2 \\
& \leq - f'' (F')^3 \sigma_{k-1;1} X + \frac{1}{g_{1\bar{1}}} (F'' X + F' Y) + \frac{(1 - 3\delta)}{(g_{1\bar{1}})^2} F' Z
\end{aligned}
$$

We hope

$$
\begin{aligned}
- f'' (F')^3 \sigma_{k-1;1} X + \frac{1}{g_{1\bar{1}}} F'' X & \leq \frac{1}{g_{1\bar{1}}} F' \frac{G''}{G'} X \\
\frac{F''}{F'} - \frac{G''}{G'} & \leq f'' (F')^2 g_{1\bar{1}} \sigma_{k-1;1}
\end{aligned}
$$

This will be true if

$$
\frac{F''}{F'} - \frac{G''}{G'} \leq f'' (F')^2 \frac{k}{n} \sigma_k \leq f'' (F')^2 g_{1\bar{1}} \sigma_{k-1;1}
$$

$$
f'' F^2 \geq \left(\frac{F}{F'} \right)^2 \left(\frac{F''}{F'} - \frac{G''}{G'} \right) \frac{n}{k \sigma_k}
$$

The right hand side is a bounded function once σ_k is bounded. We multiply F^2 to make the right hand side a constant when $F = \sigma_k^p$. Now, instead of considering f and g as a function of F, we consider it as a function of $x = F + C_0$. Here we fix a constant C_0 such that $\inf (F + C_0) > 0$ (or $\sup (F + C_0) < 0$). Therefore, we have to solve the following equation for $f(x)$ and $g(x)$

$$
\begin{aligned}
f'' x^2 \geq M_0 = Max \left\{ 0, \sup \left(\frac{x}{F'} \right)^2 \left(\frac{F''}{F'} - \frac{G''}{G'} \right) \frac{n}{k \sigma_k} \right\} \\
3 (f' + g')^2 \leq g''
\end{aligned}
$$

Suppose $0 < m_1 \leq x = F + C_0 \leq M_1$. Let's consider two cases based on the value of M_0. The case for $x < 0$ is similar.

3.3.4.1.1 Case 1: If $M_0 \leq \frac{1}{12}$

We have We have the following solution for arbitrary m_1 and M_1

$$f(x) = -M_0 \log x$$
$$g(x) = (M_0 - \alpha_0) \log x$$

where α_0 is the larger root of $3x^2 - x + M_0 = 0$.

3.3.4.1.2 Case 2: If $M_0 > \frac{1}{12}$

$$f'' x^2 \geq M_0$$
$$3 (f' + g')^2 \leq g''$$

Let $z(x) = g'$

$$3 (f' + z)^2 \leq z'$$

Let $y(x) = z + f'$

$$3y^2 \leq y' - f'' \leq y' - \frac{M_0}{x^2}$$

Therefore

$$y' \geq 3y^2 + \frac{M_0}{x^2}$$
$$u' = 3u^2 + \frac{M_0}{x^2}$$

We can solve u

$$u=\frac{1}{6x}\left[\sqrt{12M_0-1}\tan\left(\frac{\sqrt{12M_0-1}}{2}\log x+\beta_0\right)-1\right] \text{ for some constant } \beta_0.$$

Suppose there exists y satisfing the ode inequality from $m_1 \leq x \leq M_1$. Let's denote $\max_{m_1\leq x\leq M_1} y(x)=M_2$, and $y(m_1)=y_0$. If there exists u with $u(m_1)=y_0$, but $u(x)>M_2$ for some $x\in[m_1,M_1]$. Then this contradicts to comparison principle. Since $\tan(x)$ can be arbitrary large, there is no solution if

$$\frac{\sqrt{12M_0-1}}{2}\log\frac{M_1}{m_1}\geq\pi$$

It follows that it is not solvable for arbitrary m_1 and M_1. When $\frac{\sqrt{12M_0-1}}{2}\log\frac{M_1}{m_1}<\pi$, we can solve the ode with

$$f(x)=-M_0\log x$$

$$g(x)=\int_{m_1}^{x}\left[u(t)+\frac{M_0}{t}\right]dt$$

Here we fix the constant β such that

$$-\frac{\pi}{2}<\frac{\sqrt{12M_0-1}}{2}\log m_1+\beta\leq\frac{\sqrt{12M_0-1}}{2}\log M_1+\beta<\frac{\pi}{2}$$

From above two cases, it follows that

$$\begin{aligned}&(\partial_t-L)\left[\log(g_{1\bar{1}})+f(F)+g(F)+\varphi\left(|\nabla u|^2\right)+h(\hat{u})\right]\\&\leq h'u_t+h'C+C(f,g)S-\frac{1}{36K}a_{11}\lambda_1^2-h'a_{s\bar{s}}u_{s\bar{s}}\end{aligned}$$

By the concave lemma 3.3.2, we have two cases.

3.3.4.1.3 Case 1 If $\sum_{s=1}^{n} a_{s\bar{s}} (-u_{s\bar{s}}) + u_t > \kappa \sum_{s=1}^{n} a_{s\bar{s}} = \kappa S$. Once we fix f and g

$$
\begin{aligned}
&(\partial_t - L) \left[\log (g_{1\bar{1}}) + f(F) + g(F) + \varphi\left(|\nabla u|^2\right) + h(\hat{u})\right] \\
&\leq h' u_t + h' C + C(f,g) S - \frac{1}{36K} a_{11} \lambda_1^2 - h' a_{s\bar{s}} u_{s\bar{s}} \\
&\leq h' C + C(f,g) S - \frac{1}{36K} a_{11} \lambda_1^2 + h' \kappa S
\end{aligned}
$$

Since $\frac{A}{2L} \geq -h' \geq \frac{A}{3L}$, we may pick A large such that $C(f,g) + h'\kappa < -1$

$$
0 \leq C - S - \frac{1}{36K} a_{11} \lambda_1^2
$$

It follows that S is bounded above, so σ_{k-1} is bounded above. By Lemma 2.2 from Hou-Ma-Wu [11], this implies a_{11} is bounded below. So $\lambda_1 \leq CK$.

3.3.4.1.4 Case 2 If $a_{1\bar{1}} > \kappa \sum_{s=1}^{n} a_{s\bar{s}} = \kappa S$

$$
\begin{aligned}
&(\partial_t - L) \left[\log (g_{1\bar{1}}) + f(F) + g(F) + \varphi\left(|\nabla u|^2\right) + h(\hat{u})\right] \\
&\leq C(h') + C(f,g) S - \frac{\kappa}{36K} S \lambda_1^2 - h' a_{s\bar{s}} (\chi_{s\bar{s}} + u_{s\bar{s}}) + h' a_{s\bar{s}} \chi_{s\bar{s}} \\
&\leq C(h') + C(f,g) S - \frac{\kappa}{36K} S \lambda_1^2 - h' F' k \sigma_k + C h' S
\end{aligned}
$$

$$
\frac{\kappa}{36K} S \lambda_1^2 \leq C + CS
$$

Since S is bounded below, so $\lambda_1 \leq CK$.

3.3.4.2 $\lambda_n < -\delta\lambda_1$

From the evolution equation, we have

$$\begin{aligned}
&(\partial_t - L)\left[\log\left(g_{1\bar{1}}\right) + f\left(u_t\right) + g\left(u_t\right) + \varphi\left(|\nabla u|^2\right) + h\left(\hat{u}\right)\right] \\
&\leq C\left(h'\right) + C\left(f,g\right)S - \frac{1}{2}\varphi' a_{s\bar{s}}\lambda_s^2 - h'' a_{s\bar{s}}\left|u_s\right|^2 - h' a_{s\bar{s}} u_{s\bar{s}} \\
&+ \frac{1}{g_{1\bar{1}}}\left(*\right) + \sum_s a_{s\bar{s}}\frac{1}{\left(g_{1\bar{1}}\right)^2}\left|g_{1\bar{1}s}\right|^2 - f'' a_{s\bar{s}}\left|F_s\right|^2 - g'' a_{s\bar{s}}\left|F_s\right|^2 - \varphi'' a_{s\bar{s}}\left|\partial_s\left|\nabla u\right|^2\right|^2.
\end{aligned}$$

By critical equations,

$$\left|\frac{1}{g_{1\bar{1}}}g_{1\bar{1}s}\right|^2 \leq 3\left(f' + g'\right)^2\left|F_s\right|^2 + 3\left(\varphi'\right)^2\left|\partial_s\left|\nabla u\right|^2\right|^2 + 3\left(h'\right)^2\left|u_s\right|^2.$$

Now that f, g and h are fixed, by the same reasoning as before, all the third order terms will be canceled. Applying critical equations, the remaining terms are

$$\begin{aligned}
0 &\leq C\left(h'\right) + C\left(f,g\right)S - \frac{1}{2}\varphi' a_{s\bar{s}}\lambda_s^2 - h'' a_{s\bar{s}}\left|u_s\right|^2 - h' a_{s\bar{s}}u_{s\bar{s}} + 3\left(h'\right)^2 a_{s\bar{s}}\left|u_s\right|^2 \\
&\leq C + (CK)S - \frac{1}{2}\varphi' a_{n\bar{n}}\lambda_n^2
\end{aligned}$$

Since $a_{n\bar{n}} \geq \frac{S}{n}$

$$\frac{1}{18K}\frac{S}{n}\lambda_n^2 \leq C + (CK)S$$

By our assumption $\lambda_n < -\delta\lambda_1$, it implies $\lambda_n^2 > \delta^2\lambda_1^2$

$$\lambda_1^2 \leq \frac{18nK}{S\delta^2}\left[C + (CK)S\right]$$

Since S is bounded below, it follows that $\lambda_1 \leq CK$.

3.4 $\mathbf{C}^1$ estimate

Lemma 3.4.1. *There is a uniform constant $C > 0$ depending on $osc_{M\times[0,T)}u$, $\|u_t\|_{L^\infty}$, χ and constant C in equation (3.3.1) such that*

$$\sup_{M\times[0,T)} |\nabla u|^2 \leq C. \tag{3.4.1}$$

Proof : We follow the argument of Dinew and Kolodziej [5]. Below, we will use $\widehat{u}$ to denote the normalized solution. Suppose that the gradient estimate (3.4.1) does not hold. Then there exists a sequence $(z_m, t_m) \in M \times [0, T)$ with $t_m \to T$ such that $\lim_{m\to\infty} |\nabla\widehat{u}(z_m, t_m)| \to \infty$ and $|\nabla\widehat{u}(z_m, t_m)| = \sup_{M\times[0,t_m]} |\nabla\widehat{u}|$. We set $R_m := |\nabla\widehat{u}(z_m, t_m)|$. By passing to a subsequence, we assume that $z_m \to z \in M$. At z, we pick a normal coordinate centered at z such that $\alpha(0) = \beta := \sum_i dz^i \wedge d\bar{z}^i$. We may assume that the normal neighborhood contains $\overline{B_1(0)}$. On the ball $\overline{B_{R_m}(0)}$ in $\mathbb{C}^n$, we define

$$\tilde{u}_m(z) := \widehat{u}_m\left(\frac{z}{R_m} + z_k, t_k\right) \tag{3.4.2}$$

Since $|\nabla\tilde{u}_m| \leq 1$, by previous C^2 estimate,

$$|\tilde{u}_m|_{C^2(B_{R_m}(0))} < C$$

By passing to a subsequence, we assume that $\tilde{u}_m$ is $C^{1,\alpha}$ convergent to $\tilde{u} \in C^{1,\alpha}(\mathbb{C}^n)$, and $|\nabla\tilde{u}\,(0)| = 1$. To apply Liouville theorem due to Dinew and Kolodziej, it suffices to prove that $\tilde{u}$ is a maximal $k - sh$ function. From the equation, we have,

$$F^{-1}\left(u_t + \psi\left(z\right)\right) = \sigma_k\left(\alpha^{-1}g\right) = \binom{n}{k}\frac{g^k \wedge \alpha^{n-k}}{\alpha^n}$$

$$\binom{n}{k} g^k \wedge \alpha^{n-k} = \left(F^{-1}\left(u_t + \psi\right)\right) \alpha^n \tag{3.4.3}$$

Therefore, we have

$$\binom{n}{k} \frac{1}{R_m^{2n-2k}} \left[O\left(\frac{1}{R_m^2}\right) \chi\left(\frac{z}{R_m} + z_k\right) + \frac{\sqrt{-1}}{2} \partial\bar{\partial}\tilde{u}_m(z) \right]^k \wedge \left[\alpha\left(\frac{z}{R_m} + z_k\right) \right]^{n-k} \tag{3.4.4}$$

$$= \left(F^{-1}\left(u_t + \psi\right)\left(\frac{z}{R_m} + z_m\right)\right) \frac{1}{R_m^{2n}} \left[\alpha\left(\frac{z}{R_m} + z_k\right) \right]^n \tag{3.4.5}$$

Since $u_t + \psi$ is bounded, by equation (♦), $F^{-1}\left(u_t + \psi\right)$ is also bounded. Also, $a(z) = \beta + O\left(|z|^2\right)$ near zero and $\lim_{k\to\infty} z_k = 0$. Therefore,

$$\left[\frac{\sqrt{-1}}{2} \partial\bar{\partial}\tilde{u}(z) \right]^k \wedge \beta^{n-k} = 0, \tag{3.4.6}$$

which is in the pluripotential sense. A similar argument and the fact that our solution is in Γ_k cone implies that for any $1 \leq j < k$,

$$\left[\frac{\sqrt{-1}}{2} \partial\bar{\partial}\tilde{u}(z) \right]^j \wedge \beta^{n-j} \geq 0. \tag{3.4.7}$$

Due to Blocki's [2] result, equation (3.4.6) and (3.4.7) imply that $\tilde{u}$ is a maximal $k - sh$ function in $\mathbb{C}^n$. Then the Liouville theorem in [5] implies $\tilde{u}$ is a constant, which contradicts $|\nabla\tilde{u}(0)| = 1$.

3.5 $\mathbf{C}^{2,\alpha}$ estimate

First observe that $\min_M u_t(z,0) + \min_M \psi(z) > \lim_{x\to 0^+} F(x)$ implies $\sigma_k(\lambda)$ is bounded below by a positive constant, so the eigenvalue λ cannot touch $\partial\Gamma_k$. Together with the bound in $\left|u_{i\bar{j}}\right|$ implies the eigenvalues are contained in a compact set. Therefore F is uniformly elliptic. With uniform ellipticity, we have the following lemma.

Lemma 3.5.1. *Let $u(z,t)$ be an admissible solution to the flow (3.1.1) on time interval $[0,\varepsilon)$. Suppose*

$$\|\hat{u}\|_{L^\infty(M\times[0,T))} + \|\partial_t u\|_{L^\infty(M\times[0,T))} + \left\|i\partial\bar{\partial}u\right\|_{L^\infty(M\times[0,T))} \le \Lambda$$

Let B_1 be a unit coordinate ball on M. $Q = B_{\frac{1}{2}} \times \left[\frac{\varepsilon}{2},\varepsilon\right)$. Then there exists $0<\alpha<1$ and C such that

$$\|\partial_t u\|_{C^{\alpha,\alpha/2}(Q)} + \left\|u_{i\bar{j}}\right\|_{C^{\alpha,\alpha/2}(Q)} \le C(\varepsilon, n, \Lambda, \alpha, F, \psi)$$

Proof: We follow the proof from Phong-Picard-Zhang [13]. The proof has two parts. For the spatial part, it comes from Krylov-Safanov Harnack inequality. For the time part, it follows by a difference quotients argument from Tsai [18].

Spatial part. Differentiate the equation in time, we have

$$\partial_t(u_t) = L(u_t)$$

By Krylov-Safanov theory, we have

$$\|\partial_t u\|_{C^{\alpha,\alpha/2}(Q)} \le C(n,\Lambda,\varepsilon)$$

Since ψ is smooth, above equation also implies $\|F\|_{C^{\alpha,\alpha/2}(Q)} \leq C$. Denote $\alpha^{-1}g\left(x,t\right)$ as $\eta\left(x,t\right)$. By mean value theorem,

$$\frac{1}{F'\left(\theta\right)}\frac{\left|F\left(\sigma_k\left(\eta\left(x,t\right)\right)\right)-F\left(\sigma_k\left(\eta\left(y,s\right)\right)\right)\right|}{\left(\left|x-y\right|+\left|t-s\right|^{\frac{1}{2}}\right)^{\alpha}}=\frac{\left|\sigma_k\left(\eta\left(x,t\right)\right)-\sigma_k\left(\eta\left(y,s\right)\right)\right|}{\left(\left|x-y\right|+\left|t-s\right|^{\frac{1}{2}}\right)^{\alpha}},$$

where θ is between $\sigma_k\left(\eta\left(x,t\right)\right)$ and $\sigma_k\left(\eta\left(y,s\right)\right)$. Since σ_k is bounded, we have

$$\|\sigma_k\|_{C^{\alpha,\alpha/2}(Q)} \leq C$$

Covering M with cooridnate balls, it follows that

$$\|\sigma_k\eta\left(\cdot,t\right)\|_{C^{\alpha}(M)} \leq C,\ \frac{\varepsilon}{2} \leq t \leq \varepsilon.$$

By Tosatti-Wang-Weinkove-Yang's [17] result, we get the estimate for the spatial part.

$$\left\|u_{i\bar{j}}\left(\cdot,t\right)\right\|_{C^{\beta}(M)} \leq C,\ 0<\beta<1,\ \frac{\varepsilon}{2} \leq t \leq \varepsilon.$$

Time part. Fix $t_0 \in \left(\frac{\varepsilon}{2},\varepsilon\right)$. Let $0<h<\varepsilon-t_0$, and $x \in B_1$. Denote $\alpha^{-1}g\left(x,t\right)$ as $\eta\left(x,t\right)$. We have

$$\begin{aligned}\sigma_k\eta\left(x,t_0\right)-\sigma_k\eta\left(x,t_0+h\right) &= \int_0^1 \frac{\partial}{\partial s}\sigma_k\left(s\eta\left(x,t_0\right)+\left(1-s\right)\eta\left(x,t_0+h\right)\right)ds \\ &= \int_0^1 \frac{\partial\sigma_k}{\partial A^i_j}\alpha^{i\bar{k}}ds\left[u_{j\bar{k}}\left(x,t_0\right)-u_{j\bar{k}}\left(x,t_0+h\right)\right]\end{aligned} \tag{3.5.1}$$

where $A^i_j = s\eta^i_j\left(x,t_0\right)+\left(1-s\right)\eta^i_j\left(x,t_0+h\right)$. And

$$\frac{\partial\sigma_k}{\partial s}=\frac{\partial\sigma_k}{\partial A^i_j}\alpha^{i\bar{k}}\left[u_{j\bar{k}}\left(x,t_0\right)-u_{j\bar{k}}\left(x,t_0+h\right)\right],\ x_{j\bar{k}} \text{ are canceled.}$$

Since Γ_k cone is convex, the eigenvalues of A^i_j are in Γ_k cone, and the eigenvalues lie in a compact set independent of x, s, h and t_0. Also each entry of A^i_j is Hölder continuous in x with bounded Hölder norm independent of s, h and t_0. Therefore, $\frac{\partial \sigma_k}{\partial A^i_j}$ is also Hölder continuous in x since it is just the product and sum of Hölder continuous functions. Denote

$$\int_0^1 \frac{\partial \sigma_k}{\partial A^i_j} \alpha^{i\bar{k}} ds = a_h^{j\bar{k}} (x, t_0)$$

It follows that there is $a_h^{j\bar{k}}$ is uniformly elliptic with elliptic constant independent of x, h and t_0. And

$$\left\| a_h^{j\bar{k}} (\cdot, t_0) \right\|_{C^{\beta}(B_1)} \leq C, \ C \text{ is independent of } h \text{ and } t_0.$$

Divide equation (3.5.1) by $h^{\frac{\beta}{4}}$, we have

$$a_h^{j\bar{k}} \partial_{\bar{k}} \partial_j u^h = \sigma_k^h,$$

where

$$u^h (x, t_0) = \frac{u (x, t_0) - u (x, t_0 + h)}{h^{\frac{\beta}{4}}}, \text{ and } \sigma_k^h (x, t_0) = \frac{\sigma_k \eta (x, t_0) - \sigma_k \eta (x, t_0 + h)}{h^{\frac{\beta}{4}}}$$

By direct computation [13, Lemma 6],

$$\left\| \sigma_k^h (\cdot, t_0) \right\|_{C^{\beta/4}(B_1)} \leq C, \ C \text{ is independent of } h \text{ and } t_0.$$
$$\left\| u^h (\cdot, t_0) \right\|_{L^{\infty}(B_1)} \leq C \text{ since } u_t \text{ is uniformly bounded.}$$

By Schauder estimate, we have

$$\left\| u^h (\cdot, t_0) \right\|_{C^2\left(B_{\frac{1}{2}}\right)} \leq C \left(\left\| \sigma_k^h (\cdot, t_0) \right\|_{C^{\beta/4}(B_1)} + \left\| u^h (\cdot, t_0) \right\|_{L^{\infty}(B_1)} \right) \leq C.$$

Therefore, for all $x \in B_{\frac{1}{2}}$

$$\frac{\left|u_{i\bar{j}}\left(x,t_0\right)-u_{i\bar{j}}\left(x,t_0+h\right)\right|}{|h|^{\frac{\beta}{4}}} \leq C,\ C \text{ is independent of } x, h \text{ and } t_0.$$

Combine the estimate of the spatial part and the time part, we have

$$\left\|u_{i\bar{j}}\right\|_{C^{\beta/2,\beta/4}(Q)} \leq C$$

3.6 Long time existence and convergence

Now that F is uniformly elliptic, and we have all the estimates up to $C^{2,\alpha}$. Long time existence and convergence follows from some general theory. We only briefly mention here. For the details, we refer to Cao [4], Gill [8], and Picard-Zhang [14]. First of all, higher order estimates can be done by bootstrap. Differentiate our equation, we get

$$\partial_t\left(\nabla_l u\right) = F' \frac{\partial \sigma_k}{\partial A_j^i} \alpha^{i\bar{k}} \nabla_j \nabla_{\bar{k}} \left(\nabla_l u\right) - \psi_l$$

By the estimates we get so far, this is a uniformly parabolic equation with Hölder continuous coefficients. Therefore, by Schauder theory, it follows that $\|\nabla_l u\|_{C^{2+\beta,\beta/2}(Q)} \leq C$. Differentiate the equation once again, we get another uniformly parabolic equation with Hölder continuous coefficients. Apply Schauder theory again, we get higher order estimates. It follows that we have estimated on all derivatives of u by repeating above procedures.

With all the estimates, if the solution only exist in the time interval $[0, T)$. We may take a subsequential limit of $\widehat{u}$, and extend the flow by the short time existence. Therefore, a smooth solution exists on $[0, \infty)$.

Convergence follows by studying the heat equation (3.1.5) of u_t. The key step in proving the convergence is to apply the Li-Yau Harnack inequality and get the oscillation decay. We refer to Gill [8, Section 7] for the detailed argument. Therefore, the main theorem 3.1.3 holds.

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