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NUMERICAL STUDY OF THE DISCRETE MINIMAL SURFACE EQUATION IN A NONCONVEX DOMAIN

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Numerical study of the discrete minimal surface equation in a nonconvex domain

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ABSTRACT

A discrete approximation to the minimal surface equation is solved numerically on an L-shaped domain with Dirichlet boundary conditions. The nature of the solution discontinuity at the vertex of the reentrant corner is depicted graphically.

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1. In this study, a discrete approximation to the minimal surface equation

$$(1) \quad \frac{\partial}{\partial x}\left(\frac{u_x}{W}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{W}\right) = 0 \quad , \quad W = (1+u_x^2+u_y^2)^{\frac{1}{2}},$$

is solved numerically on an L-shaped domain D with Dirichlet boundary conditions (see Fig. 1a). The domain D is the unit square (0,1) x (0,1) with the smaller square $[\frac{1}{2}, 1) \times [\frac{1}{2}, 1)$ deleted from it, and the boundary conditions are that u = 0 on the outside border of D and that u increase linearly to the value c at $(\frac{1}{2}, \frac{1}{2})$ along the reentrant legs. That is,

$$(2) \quad \begin{aligned} & u = 0 \quad \text{on} \quad \begin{cases} x=0 \\ y=0 \\ x=1, \; 0 < y < \frac{1}{2} \\ y=1, \; 0 < x < \frac{1}{2} \end{cases} \\ & u = 2c(1-x) \quad \text{on} \quad y=\tfrac{1}{2}, \; \tfrac{1}{2} \le x \le 1, \\ \text{and} \quad & u = 2c(1-y) \quad \text{on} \quad x=\tfrac{1}{2}, \; \tfrac{1}{2} \le y \le 1. \end{aligned}$$

The analogous Dirichlet problem on the nonconvex quadrilateral considered by Radó [5, 6] is discussed by Nitsche [3, 4], who proved that a solution $u \in C^2(D) \cap C^0(\overline{D})$ does not exist for any value of $c > 0$. A solution does exist, however, if it is permitted to have a discontinuity at the vertex of the reentrant corner. It is the purpose of the present study to indicate the behavior of such a solution surface in the neighborhood of a reentrant corner. An L-shaped domain, rather than the quadrilateral one, is chosen here because of its practical convenience: it can be subdivided by means of a uniform square mesh with no special treatment required at the boundary.

2. The discrete approximation used is that given in [1, 2]. Let U_{ij} denote the approximating value to $u(x,y)$ at the node point $x_i = ih$, $y_j = jh$ on the uniform square mesh of width $h = \frac{1}{N}$ (N an integer); then in place of (1) one has the system of nonlinear algebaric equations

$$F_{ij} = \frac{1}{W_{\bar{i}\bar{j}}}(2U_{ij}-U_{i-1,j}-U_{i,j-1}) + \frac{1}{W_{\overline{i+1},\bar{j}}}(2U_{ij}-U_{i+1,j}-U_{i,j-1})$$

(3)

$$+ \frac{1}{W_{\bar{i},\overline{j+1}}}(2U_{ij}-U_{i-1,j}-U_{i,j+1}) + \frac{1}{W_{\overline{i+1},\overline{j+1}}}(2U_{ij}-U_{i+1,j}-U_{i,j+1}) = 0.$$

In (3)

(4) $$W_{\bar{i}\bar{j}} \equiv (1+[u_x^2+u_y^2]_{\bar{i}\bar{j}})^{\frac{1}{2}}$$

denotes W for the mesh cell with center $(i-\frac{1}{2}, j-\frac{1}{2})$, evaluated by use of

$$[u_x^2+u_y^2]_{\bar{i}\bar{j}} \doteq \frac{1}{2h^2}\Big[(U_{ij}-U_{i-1,j})^2 + (U_{ij}-U_{i,j-1})^2 + (U_{i,j-1}-U_{i-1,j-1})^2$$

$$+(U_{i-1,j}-U_{i-1,j-1})^2\Big].$$

The system of equations (3) can be derived directly from the variational integral $A = \iint_D W dxdy$ by using (4) to obtain the corresponding discrete sum and then by setting equal to zero the partial derivatives with respect to the unknown nodal values of U. Equation (3) is to be solved for the interior nodal values of U subject to (2) at the boundary nodes.

3. The numerical solution of (3) was carried out using the technique of block nonlinear successive overrelaxation described in [2]. Approximate solutions, accurate to within $10^{-5}c$, were obtained for $c = 0.1(0.1)1.0$ and for mesh spacings $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$. The results are displayed graphically in Figs. 2-9. (The automatic plotter that prepared the figures interpolated linearly between the data points.)

In Fig. 2 are plotted the values of U_{ij} for $c = 1.0$ along the line segment ℓ_1($x-y = 0$ - see Fig. 1b) for the four mesh spacings used. These curves indicate the behavior of the numerical solution as h is reduced. Figs. 3 and 4 depict the corresponding values of U_{ij} for $c = 1.0$ along the line segments ℓ_2 and ℓ_3 ($y = \frac{1}{2}$ and $x + y = 1$, resp. - see Fig. 1b). Note that, as one would expect for the original problem, the solutions of the discrete problems have the greatest jump in approaching $(\frac{1}{2}, \frac{1}{2})$ along the direction of ℓ_1.

Figs. 5-7 depict the analogous graphs for $c = 0.1$.

In Fig. 8 are depicted the values of U_{ij} for $h = \frac{1}{40}$ along the line ℓ_1 for $c = 0.1(0.1)1.0$. Note that, as in the previous graphs, the vertical scale is normalized to c. These curves illustrate the behavior of the jump discontinuity at $(\frac{1}{2}, \frac{1}{2})$ as a function of c.

Finally, in Fig. 9 is shown the extrapolated estimate as a function of c for the limiting value $u(\frac{1}{2}-, \frac{1}{2}-)$ along the line ℓ_1. The limiting values $u(\frac{1}{2}-, \frac{1}{2}-)$ in this figure were obtained for each c by passing a parabola through the computed approximation to $u(\frac{1}{2}-h, \frac{1}{2}-h)$ for the cases with $h = \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$ and extrapolating to $h = 0$. In addition to the values for $c = 0.1(0.1)1.0$, those for $c = 0.05$ and $c = 1.5$ were calculated as well.

4. A rigorous estimate for the accuracy with which Fig. 9 represents the solution of the original non-discrete problem (1, 2) would be very difficult to obtain, because of the complications introduced by the discontinuity at $(\frac{1}{2}, \frac{1}{2})$. However simple heuristic checks give evidence that the extrapolation of the numerical results to $h = 0$ can legitimately be carried out. One such check that was used was to include the value of $u(\frac{1}{2}-h, \frac{1}{2}-h)$ for $h = \frac{1}{10}$ in the extrapolation to $h = 0$, using a cubic polynomial to estimate the limit. These limits differed from those plotted in Fig. 9 by less than one percent.

The author expresses his appreciation to J.C.C. Nitsche, who, in conversations, suggested that these computations be carried out and to D. Johnson, who prepared the computer plots from the tabular data. This work was performed under the auspices of the U.S. Atomic Energy Commission.

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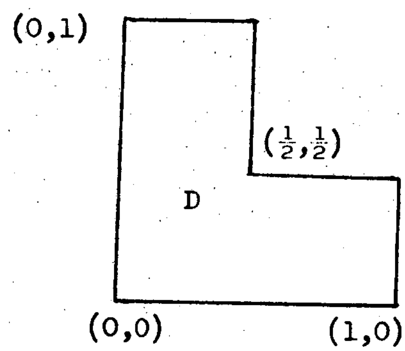

Figure 1a

The L-shaped domain

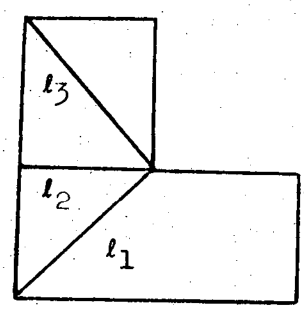

Figure 1b

Line segments ℓ_1, ℓ_2, ℓ_3 along which tabular data are plotted

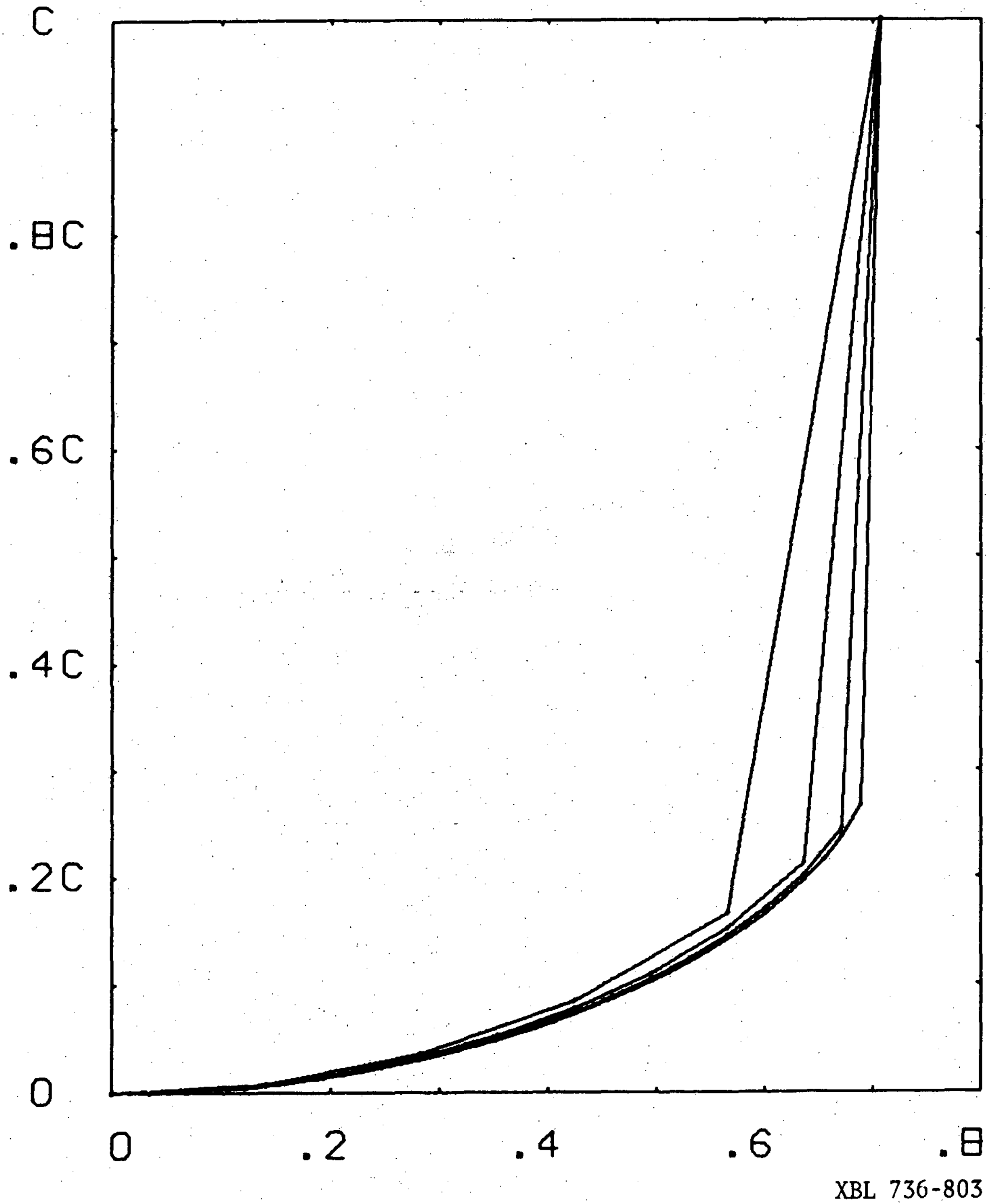

Figure 2. U as a function of distance along ℓ_1, for c = 1.0. Curves from top to bottom are for h = $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{40}$, and $\frac{1}{80}$.

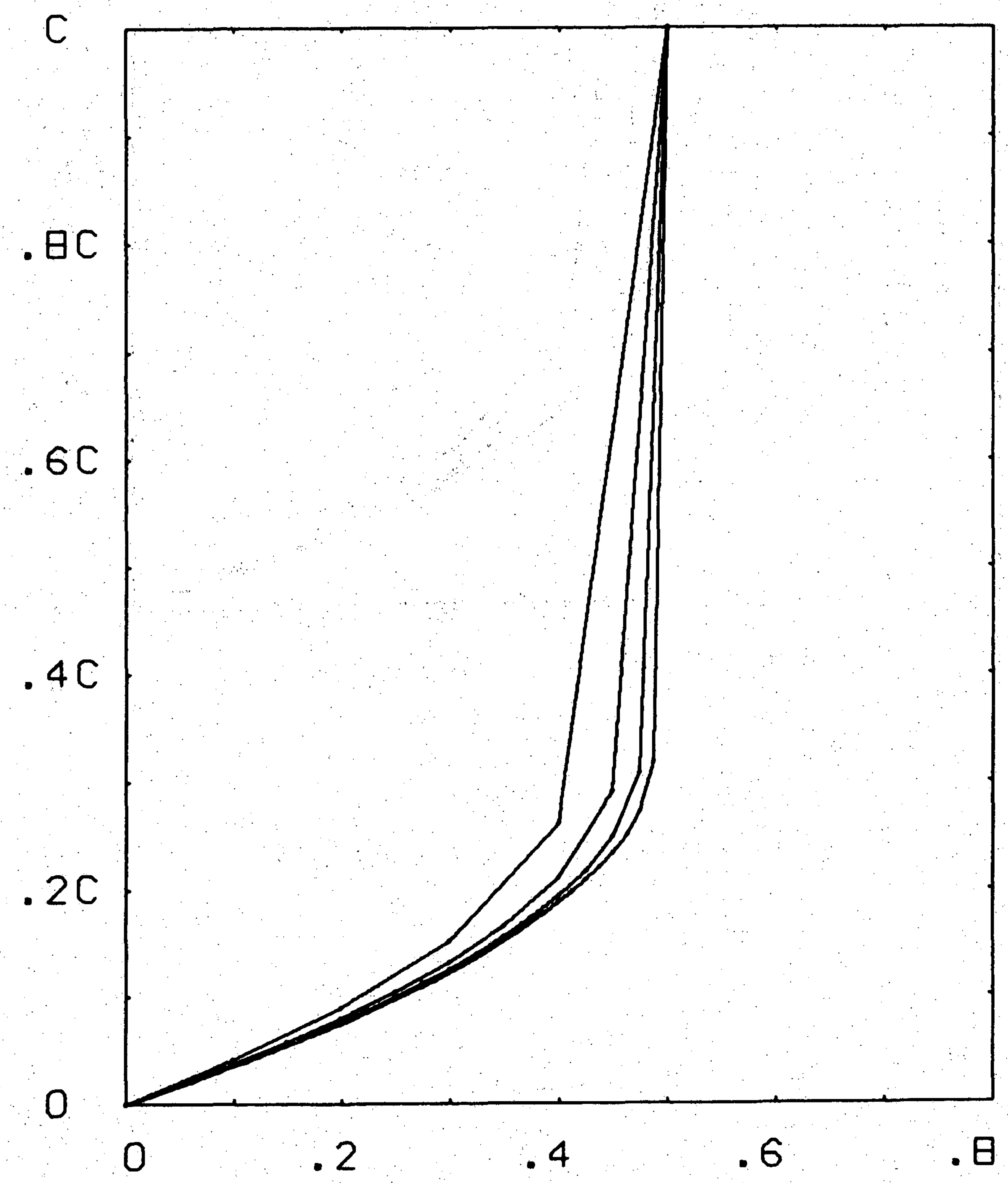

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Figure 3. U as a function of distance along ℓ_2 for $c = 1.0$. Curves from top to bottom are for $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}$, and $\frac{1}{80}$.

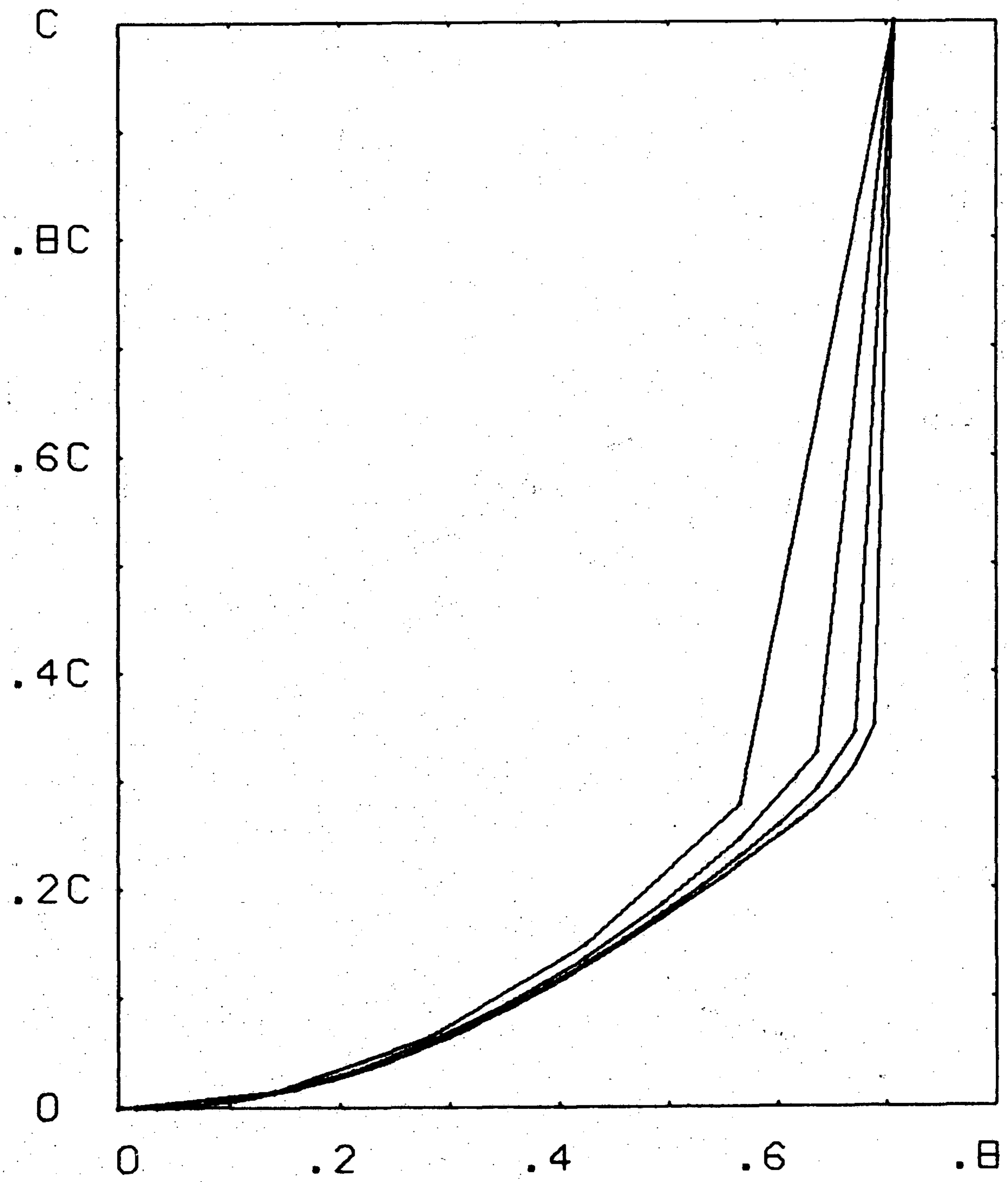

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Figure 4. U as a function of distance along ℓ_3 for c = 1.0. Curves from top to bottom are for h = $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{40}$, and $\frac{1}{80}$.

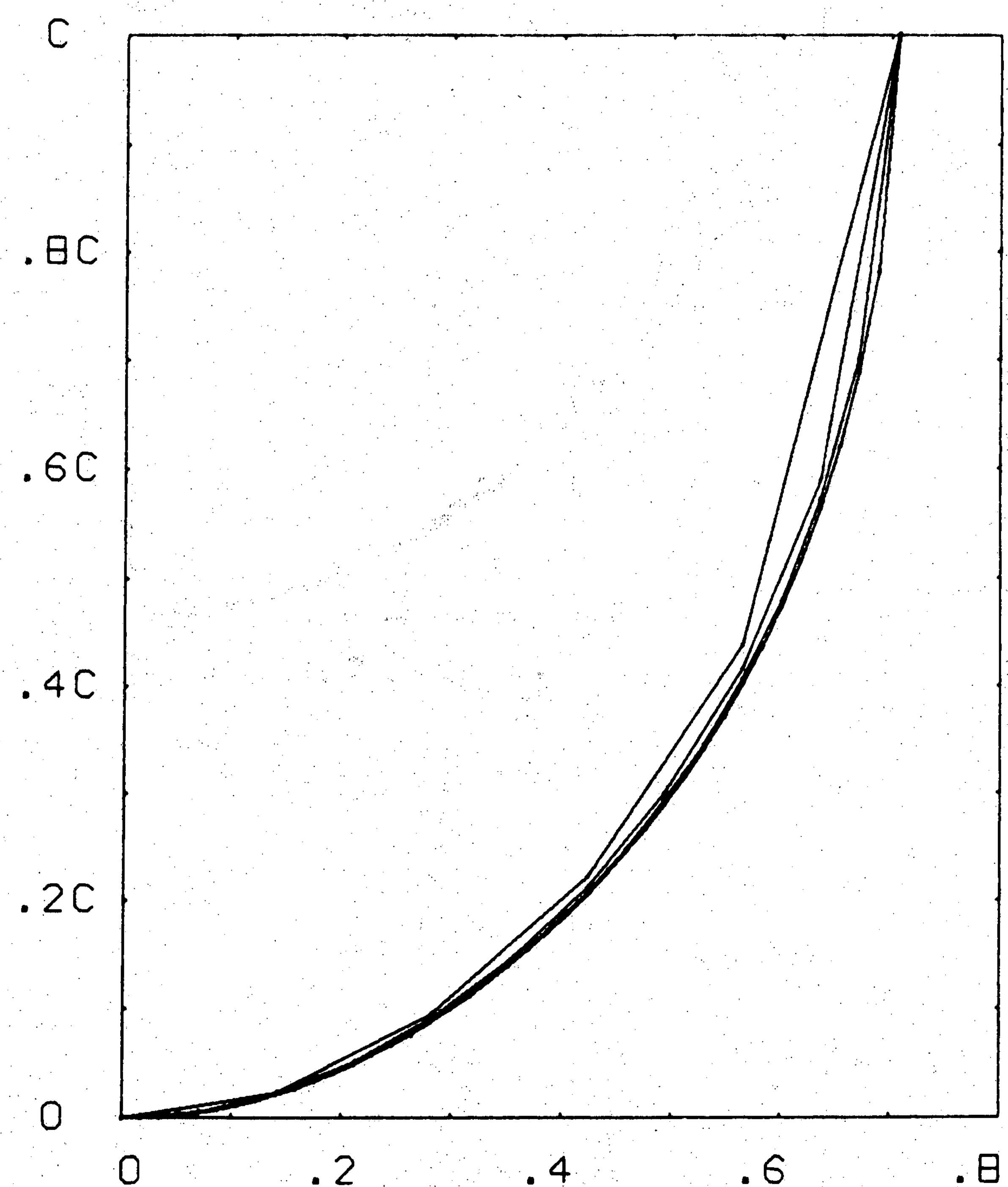

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Figure 5. Same as Fig. 2 except that c = 0.1

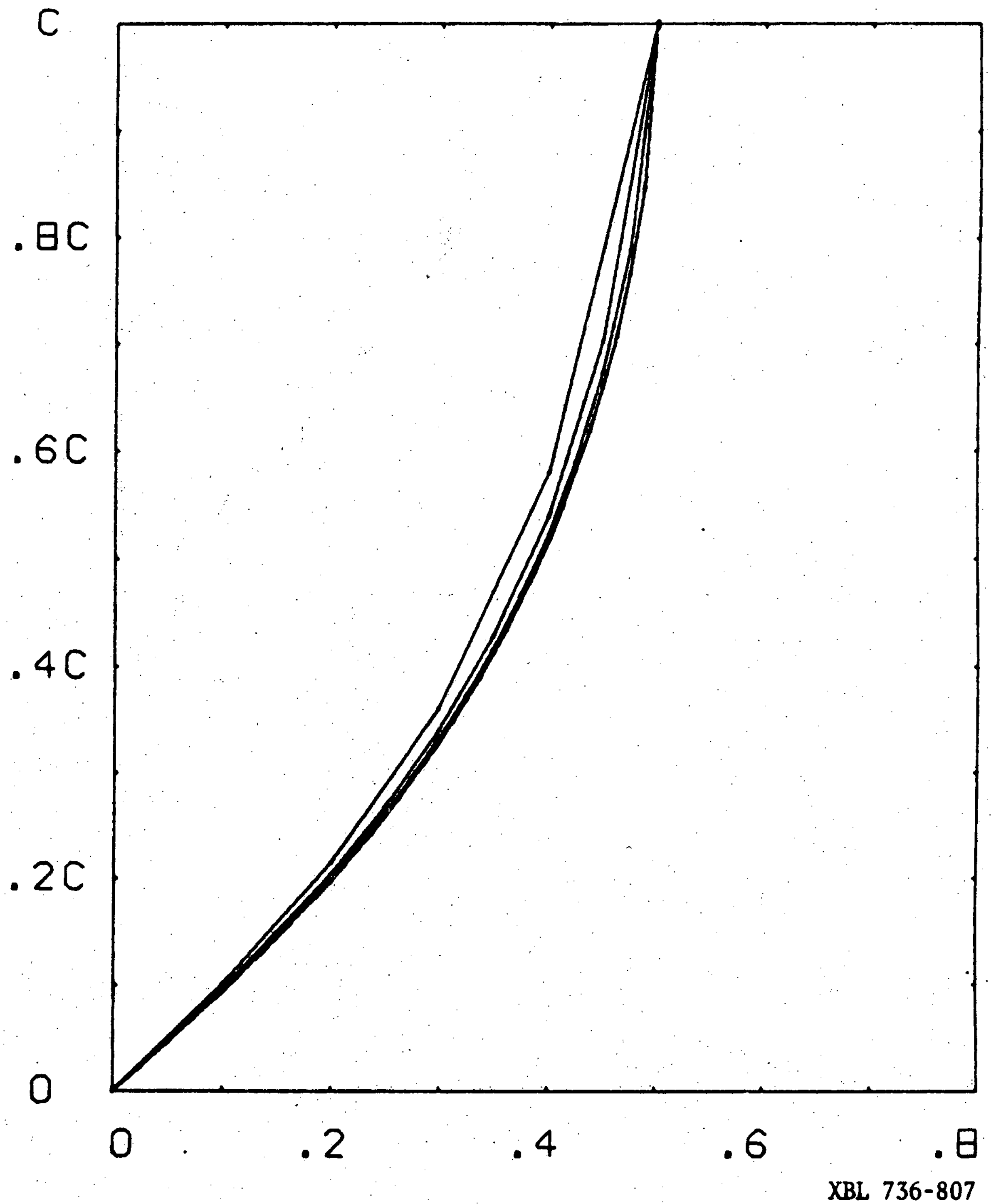

Figure 6. Same as Fig. 3 except that c = 0.1

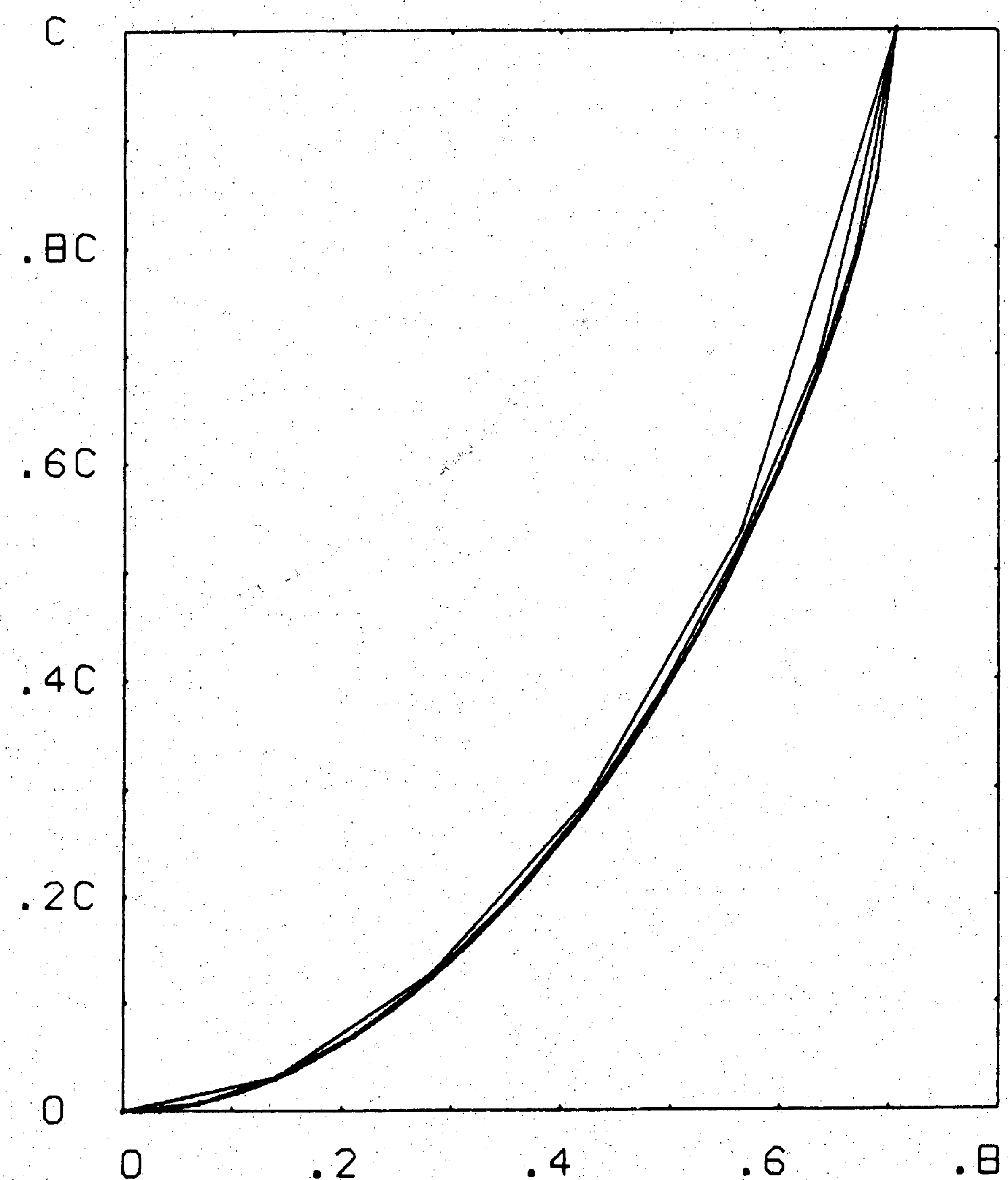

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Figure 7. Same as Fig. 4 except that c = 0.1

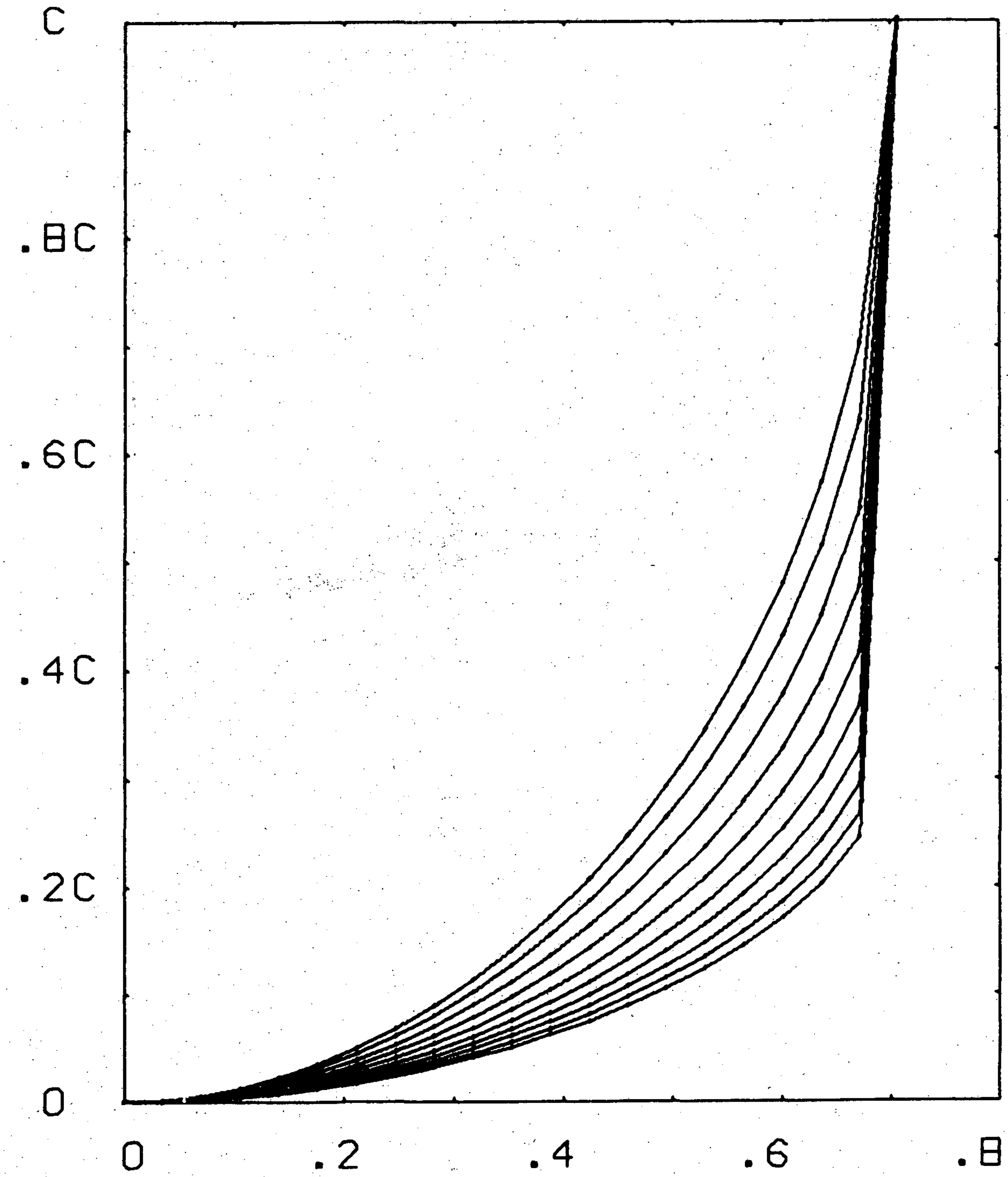

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Figure 8. U as a function of distance along ℓ_1 for $h = \frac{1}{40}$. Curves from top to bottom are for c = 0.1, 0.2, ..., 1.0.

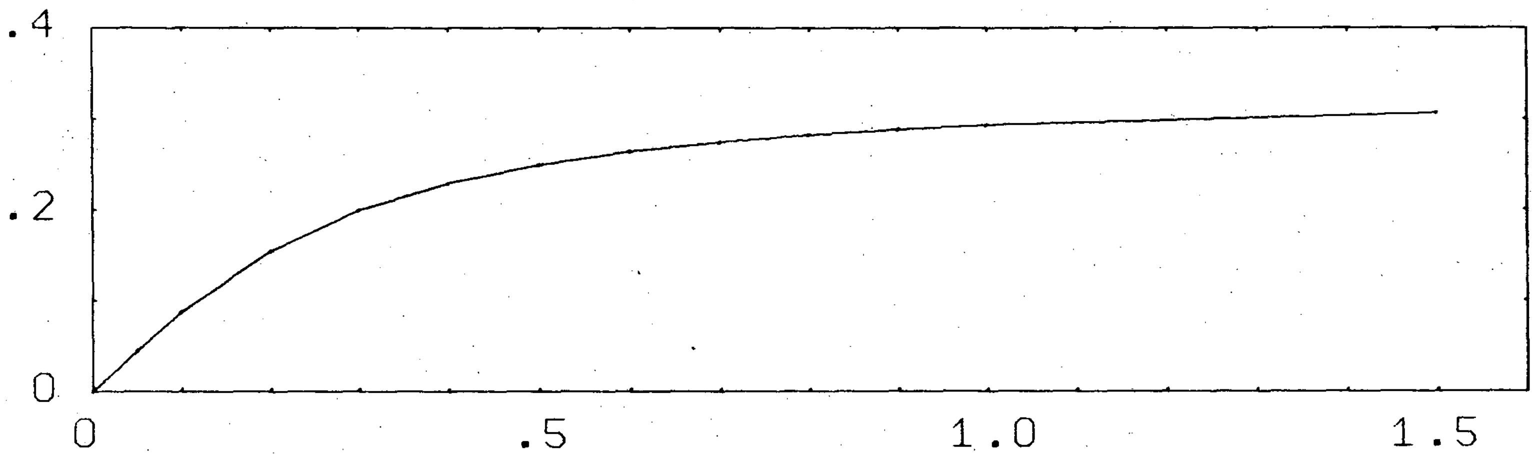

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Figure 9. Estimate of $\lim_{h \to 0} u(1/2-h,\ 1/2-h)$ as a function of c.

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